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Limit shape for regularisation of large partitions under the Plancherel measure

Salim ROSTAM*

Abstract

A celebrated result of Kerov–Vershik and Logan–Shepp gives an asymptotic shape for large partitions under the Plancherel measure. We prove that when we consider ϵ -regularisations of such partitions we still have a limit shape, which is given by a shaking of the Kerov–Vershik–Logan–Shepp curve. We deduce an explicit form for the first asymptotics of the length of the first rows and the first columns for the ϵ -regularisation.

1 Introduction

Partitions of a given integer n are the different way to decompose n as an unordered sum of positive integers. In other words, a **partition** of $n \in \mathbb{Z}_{\geq 0}$ is a non-increasing sequence $\lambda = (\lambda_1 \geq \dots)$ of non-negative integers with sum n . This mathematical object appears for instance in the study of the symmetric group \mathfrak{S}_n of permutations of $\{1, \dots, n\}$, since partitions of n index the conjugacy classes (via the cycle decomposition).

A *representation* of \mathfrak{S}_n of dimension N over a field k is a group homomorphism $\rho : \mathfrak{S}_n \rightarrow \mathrm{GL}_N(k)$. We focus for the moment at the case where k is the field \mathbb{C} of complex numbers, in which case we say that we have a *complex representation*. As any integer decomposes into a product of primes, any complex representation decomposes into a sum of *irreducible* complex representations. It turns out that the fact that the set \mathcal{P}_n of partitions of n index the conjugacy classes implies that \mathcal{P}_n also index the set of irreducible complex representations. If we denote by ρ_λ the irreducible complex representation associated with $\lambda \in \mathcal{P}_n$, a standard result of complex representation theory shows that:

$$\#\mathfrak{S}_n = n! = \sum_{\lambda \in \mathcal{P}_n} (\dim \rho_\lambda)^2. \quad (1.1)$$

A remarkable fact is that we are able to explicitly compute the numbers $\dim \rho_\lambda$ (namely with the famous *hook length formula*). We refer for instance to Sagan [Sa] or James–Kerber [JaKe] for more details on the complex representations of \mathfrak{S}_n .

Now if we want to study the representations of \mathfrak{S}_n over the field $k = \mathbb{F}_p$ with p elements (with p a prime number; we will say that we have a *p -representation*), almost everything that is known for complex representations falls apart. A fundamental difference is that some representations may *not* decompose into a sum of irreducible ones. Nevertheless, the study of irreducible representations is still interesting since any representation can always be decomposed into irreducible constituents via a *composition* (or *Jordan–Hölder series*).

A general theorem of Brauer says that the irreducible p -representations are in bijection with the p -regular conjugation classes, that is, with conjugation classes formed by elements whose order does not divide p . For the symmetric group, one can see that the set of p -regular conjugation classes is in one-to-one correspondence with the set of **p -regular** partitions, that is, partitions with no p (or more) consecutive equal parts. Note that the dual notion of *p -restricted* partitions is also present in the literature, where the difference between two

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consecutive parts of a p -restricted partition it at most $p - 1$. If λ is a p -regular partition, we will denote by ρ_λ^p the associated irreducible p -representation. Contrary to the complex case, there is no formula expressing the dimension of ρ_λ^p (yet). We refer for instance to James–Kerber [JaKe] for more details on the p -representations of \mathfrak{S}_n .

A way to understand p -irreducible representations is to study the p -irreducible representations that appear in the decomposition series of an irreducible complex representation reduced modulo p . More explicitly, it can be shown that any irreducible complex representation can be realised over \mathbb{Z} , that is, we can assume that $\rho_\lambda : \mathfrak{S}_n \rightarrow \mathrm{GL}_N(\mathbb{Z})$. The reduction modulo p , that we denote by $\bar{\rho}_p : \mathfrak{S}_n \rightarrow \mathrm{GL}_N(\mathbb{F}_p)$, is then simply the reduction modulo p of the matrix entries. Now if we return to the problem of determining which irreducible p -representations appear in a decomposition series of $\bar{\rho}_p$, James [Ja] gave an explicit combinatorial construction of a p -regular partition $\mathrm{reg}_e(\lambda)$ such that ρ_λ^p appears in a decomposition series of $\bar{\rho}_p$. This partition $\mathrm{reg}_e(\lambda)$ is the **p -regularisation** of λ . The p -regularisation operation has in fact a meaning also when p is not prime, in the context of the *Iwahori–Hecke algebra* of \mathfrak{S}_n (see, for instance, Mathas [Ma]). In particular, for an integer $e \geq 2$ we will also use the terms e -regular and e -regularisation. Note that the e -regularisation map was recently generalised by Millan Berdasco [Mi] to an (e, i) -regularisation map on partitions: it is likely that our results generalise to this setting.

We now go back to (1.1). This equation shows that $\mathrm{Pl}_n(\lambda) := \frac{(\dim \rho_\lambda)^2}{n!}$ is a probability measure on the set \mathcal{P}_n of partitions of n , called the *Plancherel measure*. Via some calculations involving the *hook integral* (defined in the spirit of the hook length formula), Kerov–Vershik [KeVe] and Logan–Shepp [LoSh] proved independently that there is a curve Ω so that the upper rim $\tilde{\omega}_\lambda$ of the Young diagram (in the Russian convention) of a large partition λ converges uniformly in probability to Ω . An illustration of this convergence is given in Figure 1 (note¹). This limit shape theorem allowed to determine the first asymptotics of the length of the first row (and first column) of a Young diagram, taken under the Plancherel measure. Note that, via the Robinson–Schensted correspondence (which provides a bijective proof of (1.1)), this provides a solution to the *Ulam problem* on the length of a longest increasing subsequence of a word in \mathfrak{S}_n chosen uniformly. We refer for instance to Romik [Rom] for more details on the Plancherel measure and related asymptotics.

In the previous context of regularisation of partitions, the following question is thus natural: for an integer $e \geq 2$, what can be said about the partition $\mathrm{reg}_e(\lambda)$ when λ is a large partition taken under the Plancherel measure? The aim of this paper is to give a first answer to this question.

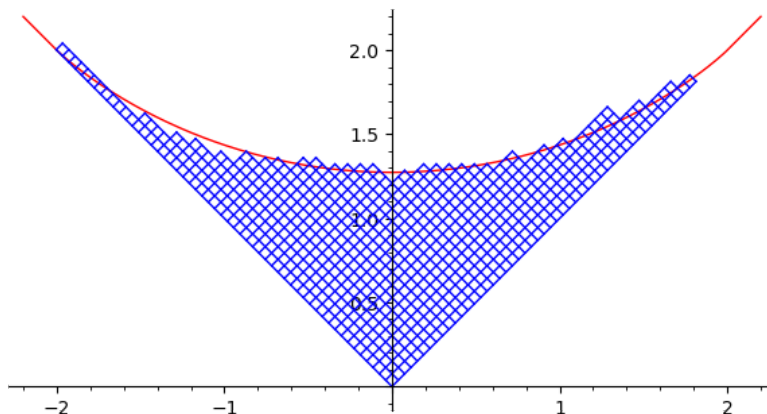


Figure 1: The limit shape theorem for large partitions under the Plancherel measure

¹All the computations were made using SAGEMATH [SM].

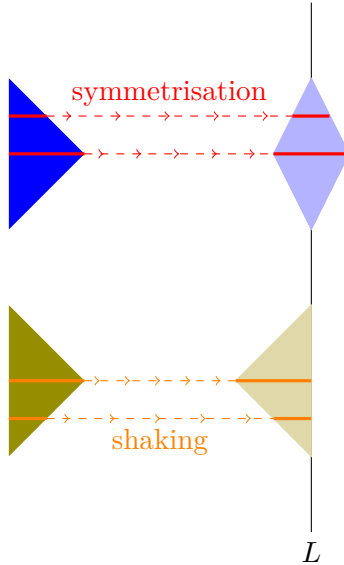


Figure 2: Examples of symmetrisation and shaking with respect to a line L

Steiner symmetrisation is a useful tool to study (namely) the isoperimetric problem. More precisely, if K is a compact convex subset of \mathbb{R}^2 then its *symmetrisation* with respect to the line L is the set $S_L(K)$ that we obtain after sliding the different slices of K (with respect to the orthogonal direction of L) until the midpoint of the slice is on L . An example of symmetrisation with respect to a vertical line is given in Figure 2. The link with the isoperimetric problem is that $S_L(K)$ has the same area as K but a smaller perimeter. Note the following result, known as the *sphericity theorem* of Gross: from any compact convex subset of \mathbb{R}^2 one can obtain the unit disk after a succession of (possibly infinite number of) symmetrisations. Note that this concept of symmetrisation can also be extended to maps, with for example Schwarz *symmetric rearrangement*, which is a powerful tool to study functional inequalities. We refer for instance to Gruber [Gr] or Krantz–Parks [KrPa] for more details on the Steiner symmetrisation.

A variation of Steiner symmetrisation is the notion of **shaking**. This notion was first introduced by Blaschke [Bl] to solve Sylvester’s “four points problem”. The difference with Steiner symmetrisation is that we slide the slices until we *meet* L . An example is given in Figure 2. Shaking and Steiner symmetrisation share many properties, for instance, Gross theorem holds with “unit disk” replaced by “simplex”. We refer to [CCG] for more details on the shaking operation. We will in fact use the shaking operation in a context where the slices are not orthogonal to the line L ; such an operation is for instance used in [FrLu].

We can now state our main result (Theorems 4.7 and 6.1).

Theorem A. *Let $e \geq 2$. Under the Plancherel measure Pl_n , the upper rim $\tilde{\omega}_{\text{reg}_e(\lambda)}$ of the Young diagram (in the Russian convention) of $\text{reg}_e(\lambda)$ converges uniformly in probability to the shape Ω_α as $n \rightarrow +\infty$, in other words, for any $\epsilon > 0$ we have*

$$\text{Pl}_n \left(\sup_{\mathbb{R}} |\tilde{\omega}_{\text{reg}_e(\lambda)} - \Omega_\alpha| > \epsilon \right) \xrightarrow{n \rightarrow +\infty} 0.$$

The shape Ω_α is obtained by shaking the part $\mathbb{Y}(\Omega)$ of the graph of Ω that is above the graph of the absolute value, with respect to the line of equation $y = -x$ and angle $\alpha := 1 - 2e^{-1}$,

and is given by:

$$\begin{aligned} \Omega_\alpha(x) &= \Omega(x), & \text{for all } x \leq x_\alpha^-, \\ \Omega_\alpha(x + \delta_x) &= \Omega(x) + \alpha\delta_x, & \text{for all } x \in (x_\alpha^-, x_\alpha^+), \\ \Omega_\alpha(x) &= x, & \text{for all } x \geq s_\alpha, \end{aligned}$$

where:

- $x_\alpha^+ = \Omega'^{-1}(\alpha) \in [0, a)$,
- $\delta_x = (1 - \alpha)^{-1}F_\alpha(x) - F_\alpha^{-1}(F_\alpha(x))$, where $F_\alpha : x \mapsto \Omega(x) - \alpha x$ and F_α^{-1} denotes its inverse on (x_α^+, a) ,
- $x_\alpha^- \in (-a, x_\alpha^+)$ (is the unique point that) satisfies $F_\alpha(x_\alpha^-) = (1 - \alpha)a$,
- $(1 - \alpha)s_\alpha = F_\alpha(x_\alpha^+) = \Omega(\Omega'^{-1}(\alpha)) - \alpha\Omega'^{-1}(\alpha) \in [\Omega(0) - \alpha\Omega(x_\alpha^+), \Omega(0)]$.

We illustrate the convergence in Figure 3 and compare the limit shapes Ω_0 and Ω . The fact that Ω_0 is obtained via the horizontal shaking of $\mathbb{Y}(\Omega)$ (with respect to the line of equation $y = -x$) is highlighted in Figure 4. In the particular case $e = 2$, the statement of Theorem A becomes explicit (Corollary 4.14 and Theorem 6.1).

Corollary B. *Under the Plancherel measure Pl_n , the upper rim $\tilde{\omega}_{\text{reg}_2(\lambda)}$ of the Young diagram (in the Russian convention) of $\text{reg}_2(\lambda)$ converges uniformly in probability to the shape Ω_0 given by:*

$$\begin{aligned} \Omega_0(x) &= \Omega(x), & \text{for all } x \leq -2, \\ \Omega_0(2x + \Omega(x)) &= \Omega(x), & \text{for all } x \in (-2, 0), \\ \Omega_0(x) &= x, & \text{for all } x \geq \Omega(0). \end{aligned}$$

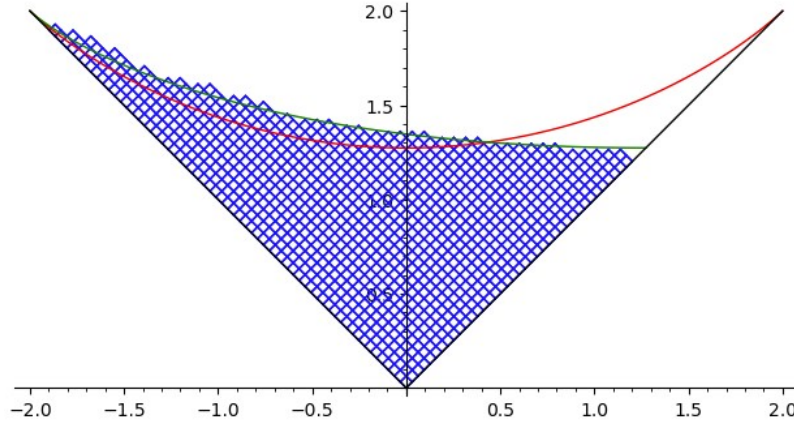


Figure 3: Example of 2-regularisation of a large partition, with in green the limit shape Ω_0 of Corollary B and in red the limit shape Ω of Kerov-Vershik-Logan-Shepp

Despite the fact that Theorem A is not fully explicit, we are able to deduce the asymptotic behaviour of the length of the first line and of the first column of $\text{reg}_e(\lambda)$ (Corollary 6.5).

Corollary C. *Let $e \geq 2$. Under the Plancherel measure Pl_n :*

- the rescaled size $\frac{1}{\sqrt{n}}\text{reg}_e(\lambda)_1$ of the first row of $\text{reg}_e(\lambda)$ converges as $n \rightarrow +\infty$ in probability to 2;
- the rescaled size $\frac{1}{\sqrt{n}}\text{reg}_e(\lambda)'_1$ of the first column of $\text{reg}_e(\lambda)$ converges as $n \rightarrow +\infty$ in probability to $\frac{2e}{\pi} \sin \frac{\pi}{e}$.

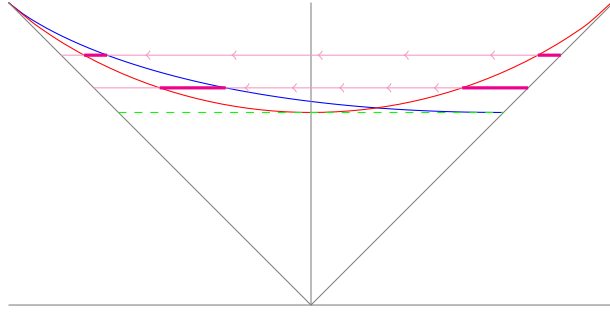


Figure 4: The maps Ω_0 (in blue) and Ω (in red). In green the tangent for Ω at 0. The left magenta segments are what is “added” to the red curve, and comes from the right magenta segments by shaking $\mathbb{Y}(\Omega)$.

We now give the outline of the paper. In Section 2 we recall the definition of the ϵ -regularisation map on the set of partitions of n and we state the limit shape result of Kerov–Vershik and Logan–Shepp for large partitions taken under the Plancherel measure. The short Section 3 is devoted to the (non-orthogonal) shaking operation on compact subsets of \mathbb{R}^2 . Section 4 contains the first main result of the paper, Theorem 4.7, which compute the shape associated with the shaking of the graph of a convex function (more precisely, the part of the graph that is above the graph of the absolute value). We give in Corollary 4.14 a particular case when the shaking is made by horizontal slices, where the result is explicit. In Section 5 we introduce two approximations $\rho^\pm(\lambda)$ for the upper rim ω_λ of the Young diagram (in the Russian convention) of a partition λ , in order to facilitate the shakings computations. More precisely, in §5.2 we define the *outer regularisation* $\rho^+(\lambda)$ of an ϵ -regular partition λ (Definition 5.4) and in Proposition 5.5 we prove that $\rho^+(\lambda)$ is both close to ω_λ and that its graph $\rho^+(\lambda)$ is stable under the shaking operation. A similar construction is made in §5.3 for the *inner regularisation* $\rho^-(\lambda)$. Finally, Section 6 is devoted to the proof of the second main result, Theorem 6.1, which states that the (rescaled) upper rim $\tilde{\omega}_{\text{reg}_\epsilon(\lambda)}$ of the Young diagram (in the Russian convention) of the ϵ -regularisation of a partition taken under the Plancherel measure converges uniformly in probability to the shape defined in Theorem 4.7. The proof uses the results of Sections 4 and 5. We also prove the convergence of the support of $\tilde{\omega}_\lambda - |\cdot|$ (Theorem 6.4), and we deduce the asymptotic length of the first row and the first column of $\text{reg}_\epsilon(\lambda)$ (Corollary 6.5).

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2 Background on partitions


We recall in §2.1 the combinatorial notion of ϵ -regularisation of a partition. In §2.2 we recall the limit shape result of Kerov–Vershik and Logan–Shepp for large partitions under the Plancherel measure.

2.1 Regularisation

A *partition* is a finite non-increasing sequence $\lambda = (\lambda_1 \geq \dots \geq \lambda_h > 0)$ of positive integers. If $|\lambda| := \sum_{i=1}^h \lambda_i = n$ then we say that λ is a partition of n . We denote by \mathcal{P}_n the set of partitions of n . The *Young diagram* of a partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_h > 0)$ is the subset of $\mathbb{Z}_{\geq 1}^2$ given by:

$$\mathcal{Y}(\lambda) = \{(a, b) \in \mathbb{Z}_{\geq 1}^2 : 1 \leq a \leq h \text{ and } 1 \leq b \leq \lambda_a\}.$$

Note that we will consider that the a -coordinate in the Young diagrams goes downwards. A *node* is an element of $\mathbb{Z}_{\geq 1}^2$.

Example 2.1. The Young diagram of the partition $\lambda = (4, 4, 1)$ is . The node in blue

has coordinates $(1, 2)$.

Definition 2.2. Let $e \geq 2$. A partition $\lambda = (\lambda_1, \dots, \lambda_h)$ is *e -regular* if no parts repeat e times or more, that is, if $\lambda_i > \lambda_{i+e-1}$ for any $i \in \{1, \dots, h-e\}$.

As we mentioned in the introduction, the notion of e -regular partitions appears for instance in the context of the representation theory of the *Iwahori–Hecke algebra* $\mathcal{H}_q(\mathfrak{S}_n)$ of \mathfrak{S}_n (a certain deformation of $\mathbb{C}\mathfrak{S}_n$), with $q \in \mathbb{C}^\times$ having order e , see, for instance, [Ma] (note that in [Ma] the dual notion of *e -restricted* partition is used). The next two definitions are due to James [Ja].

Definition 2.3. Let $e \geq 2$.

- The *e -ladder number* (or simply *ladder number*) of a node $\gamma = (a, b) \in \mathbb{Z}_{\geq 1}^2$ is:

$$\text{lad}_e(\gamma) := a + (e-1)(b-1) \in \mathbb{Z}_{\geq 1}.$$

- Let $\ell \geq 1$. The *(e, ℓ) -th ladder* (or simply *ℓ -th ladder*) is the (finite) set of all nodes of $\mathbb{Z}_{\geq 1}^2$ with e -ladder ℓ . The *e -ladder* of a node γ is the $(e, \text{lad}_e(\gamma))$ -th ladder.

Example 2.4. In the Young diagram of $\lambda = (4, 4, 3, 3, 3, 3, 1)$, in each node we write the corresponding 4-ladder numbers:

1	4	7	10
2	5	8	11
3	6	9	
4	7	10	
5	8	11	
6	9	12	
7	10	13	
8			

Definition 2.5. Let $e \geq 2$ and let λ be a partition. The *e -regularisation* of λ is the partition $\text{reg}_e(\lambda)$ that we obtain after moving each node of $\mathcal{Y}(\lambda)$ as high as possible in its e -ladder.

Note that $\text{reg}_e(\lambda)$ is an e -regular partition, and if λ is e -regular then $\text{reg}_e(\lambda) = \lambda$. As we have mentioned in the introduction, the e -regularisation map has a significance in terms of modular representation theory of the symmetric group (or its associated Iwahori–Hecke algebra).

Example 2.6. The 4-regularisation of the partition of Example 2.4 is $(5, 4, 4, 3, 3, 2, 2, 1)$. The

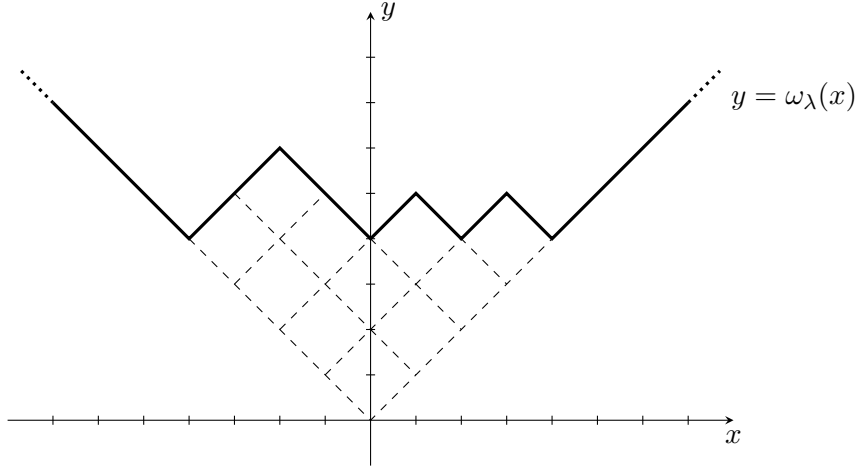


Figure 5: Russian convention for the Young diagram of $\lambda = (4, 4, 2, 1)$

4-ladders of the added (respectively, deleted) nodes are in green (resp. red).

1	4	7	10	13
2	5	8	11	
3	6	9	12	
4	7	10		
5	8	11		
6	9	12		
7	10	13		
8				

2.2 Limit shape

Russian convention Rotating the Young diagram of $\lambda = (\lambda_1 \geq \dots \geq \lambda_h > 0) \in \mathcal{P}_n$ by an angle of $\frac{3\pi}{4}$ and embedding it inside $\mathbb{R} \times \mathbb{R}_{\geq 0}$ so that the box $(1, 1)$ has bottom vertex at $(0, 0)$ and each box has area 2 (*i.e.* semi-diagonal length 1) gives the *Russian convention* for the Young diagram of λ . Note that the node $(a, b) \in \mathcal{Y}(\lambda)$ corresponds to the (square) box with top vertex $(a - b, a + b)$ in $\mathbb{R} \times \mathbb{R}_{\geq 0}$. We denote by $\omega_\lambda : \mathbb{R} \rightarrow \mathbb{R}$ the upper rim of the resulting diagram, extending ω_λ by $\omega_\lambda(x) := |x|$ outside the diagram. Then ω_λ is a continuous piecewise linear function such that:

- for each $k \in \mathbb{Z}$ we have

$$\omega'_\lambda|_{(k, k+1)} = \pm 1, \quad (2.7)$$

- we have $\omega_\lambda(x) = |x|$ for $|x| \gg 0$ (more precisely, for $x \leq -\lambda_1$ or $x \geq h$),
- we have $\int_{\mathbb{R}} [\omega_\lambda(x) - |x|] dx = 2n$.

An illustration of the construction of ω_λ is given in Figure 5. (We warn the reader that in the literature the convention is sometimes reversed, that is, our ω_λ is sometimes reflected with respect to the axis $\{0\} \times \mathbb{R}$.) We will use a particular rescaling $\tilde{\omega}_\lambda : \mathbb{R} \rightarrow \mathbb{R}$ of ω_λ , given by $\tilde{\omega}_\lambda(s) := \frac{1}{\sqrt{n}} \omega_\lambda(s\sqrt{n})$. Note that the area between the graphs of $\tilde{\omega}_\lambda$ and $|\cdot|$ is 2.

Plancherel measure Let $\lambda \in \mathcal{P}_n$. A *standard tableau of shape λ* is a bijection $\mathfrak{t} : \mathcal{Y}(\lambda) \rightarrow \{1, \dots, n\}$ such that \mathfrak{t} increases along the rows and down the columns, in other words for $(a, b) \in \mathcal{Y}(\lambda)$ we have $\mathfrak{t}(a, b) < \mathfrak{t}(a + 1, b)$ if $(a + 1, b) \in \mathcal{Y}(\lambda)$ and $\mathfrak{t}(a, b) < \mathfrak{t}(a, b + 1)$

if $(a, b + 1) \in \mathcal{Y}(\lambda)$. We denote by $\text{Std}(\lambda)$ the set of standard Young tableaux of shape λ . We have the following standard identity:

$$n! = \sum_{\lambda \in \mathcal{P}_n} \#\text{Std}(\lambda)^2.$$

Definition 2.8. The *Plancherel measure* on the set \mathcal{P}_n of the partitions of n is given by

$$\text{Pl}_n(\lambda) := \frac{\#\text{Std}(\lambda)^2}{n!},$$

for all $\lambda \in \mathcal{P}_n$.

The next result describes the Plancherel measure Pl_n for large n .

Theorem 2.9 ([LoSh], [KeVe], [Rom, Theorem 1.26]). *Let $\Omega : \mathbb{R} \rightarrow \mathbb{R}$ be defined by*

$$\Omega(s) := \begin{cases} \frac{2}{\pi} \left(s \arcsin\left(\frac{s}{2}\right) + \sqrt{4 - s^2} \right), & \text{if } |s| \leq 2, \\ |s|, & \text{otherwise.} \end{cases}$$

Then, under the Plancherel measure Pl_n , the function $\tilde{\omega}_\lambda$ converges uniformly in probability to Ω as $n \rightarrow +\infty$. In other words, for any $\epsilon > 0$ we have

$$\text{Pl}_n \left(\sup_{\mathbb{R}} |\tilde{\omega}_\lambda - \Omega| > \epsilon \right) \xrightarrow{n \rightarrow +\infty} 0.$$

Moreover, we also have convergence of the supports, that is:

$$\inf \{s \in \mathbb{R} : \tilde{\omega}_\lambda(s) \neq |s|\} \longrightarrow -2,$$

and

$$\sup \{s \in \mathbb{R} : \tilde{\omega}_\lambda(s) \neq |s|\} \longrightarrow 2,$$

in probability under Pl_n as $n \rightarrow +\infty$.

An illustration of Theorem 2.9 for $n = 1000$ is given in Figure 1. Note that the limit shape Ω has a much simpler form after derivation.

Lemma 2.10. *The map Ω is an antiderivative on \mathbb{R} of:*

$$s \mapsto \begin{cases} \arcsin\left(\frac{s}{2}\right), & \text{if } |s| \leq 2, \\ \text{sgn}(s), & \text{if } |s| > 2. \end{cases}$$

In particular, the minimum of Ω on \mathbb{R} is $\Omega(0) = \frac{4}{\pi}$.

3 Shakings

The aim of this paper is to put together the notions of §2.1 and §2.2. As a first step, we show in Figure 6 what do the 4-ladders of Example 2.4 look like in the Russian convention.

Proposition 3.1. *In the Russian convention, the e -regularisation makes the nodes going as left as possible in the direction of $y = (1 - 2e^{-1})x$.*

Proof. In the Young diagram we know that the node $(e, 1) \in \mathbb{Z}_{\geq 1}^2$ goes to $(1, 2)$. In the Russian convention it means that the box with top vertex at $(e - 1, e + 1)$ goes to the box with top vertex $(-1, 3)$. The corresponding slope is thus $\frac{e+1-3}{e-1-(-1)} = \frac{e-2}{e} = 1 - 2e^{-1}$ as announced (note that the slope do not change when both axes are rescaled by a same constant). \square

We now define the shaking operation that will be of interest for us. As we mentioned in the introduction, this is a variation of Steiner symmetrisation first introduced by Blaschke [Bl].

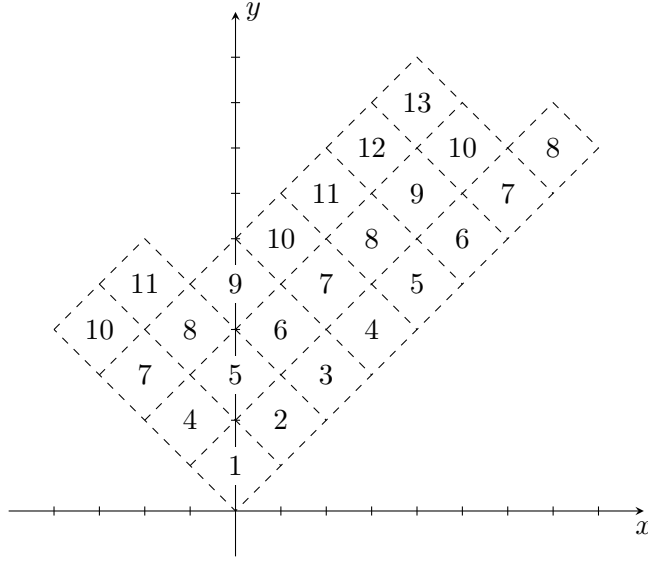


Figure 6: 4-ladders in the Russian convention for $\lambda = (4, 4, 3, 3, 3, 3, 3, 1)$

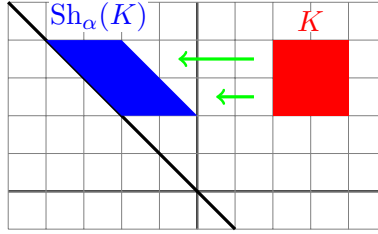


Figure 7: Example of the map Sh_α with $\alpha = 0$

Definition 3.2. Let $\alpha \in \mathbb{R}_{\geq 0}$ and let v_α be the unit vector positively collinear to $(1, \alpha)^\top$. If K is a compact subset of \mathbb{R}^2 , its *shaking* with direction α (against $y = -x$) is the compact subset $\text{Sh}_\alpha(K)$ of \mathbb{R}^2 that we obtain by sliding the set K along the direction of $(L_\alpha) : y = \alpha x$ until we meet the line $(L) : y = -x$, that is:

$$\text{Sh}_\alpha(K) := \bigsqcup_{x \in L} K_v,$$

where K_v is the segment with extreme points x and $x + |K \cap (x + L_\alpha)|v_\alpha$ with $|\cdot|$ denoting the one-dimensional Lebesgue measure.

In the usual definition of shaking the direction L_α is orthogonal to the the line L against which we shake. This non-orthogonal variation is for instance considered in [FrLu, §5].

Example 3.3. Take $\alpha = 0$ and let K be the unit square with bottom left corner at $(1, 1)$. Then, as shown on Figure 7, the set $\text{Sh}_\alpha(K)$ is the parallelogram with vertices $(-2, 2)$, $(-1, 2)$, $(0, 1)$ and $(-1, 1)$.

The following property is a standard property of Steiner symmetrisations and shakings. The proof is immediate from the definition.

Proposition 3.4. *If $K \subseteq K'$ are compact subsets of \mathbb{R}^2 then $\text{Sh}_\alpha(K) \subseteq \text{Sh}_\alpha(K')$.*

Proof. See [FrLu, Lemma 5.3]. Note that the proof is in fact the same as for the usual (orthogonal) shakings. \square

Recalling Proposition 3.1, we are interested in special cases of shakings.

Definition 3.5. For any $e \geq 2$, we denote by sh_e the shaking Sh_α with direction $\alpha_e := 1 - 2e^{-1}$.

4 Shaking the limit shape

Let $a \in \mathbb{R}_{>0}$. We define \mathfrak{Y}_a to be the set of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ of class \mathcal{C}^1 such that:

$$f \text{ is even,} \quad (4.1a)$$

$$\text{for any } |s| \geq a \text{ we have } f(s) = |s|, \quad (4.1b)$$

$$f \text{ is strictly convex on } [-a, a]. \quad (4.1c)$$

Note that by Lemma 2.10 we have $\Omega \in \mathfrak{Y}_2$.

Example 4.2. Let $\Sigma : \mathbb{R} \rightarrow \mathbb{R}$ be given by $\Sigma(x) = |x|$ if $|x| \geq 1$ and $\Sigma(x) = \frac{1}{2}(x^2 + 1)$ otherwise. Then $\Sigma \in \mathfrak{Y}_1$.

Note that the last two equations in (4.1) imply that for all $|s| < a$ we have:

$$f(s) > |s|, \quad (4.3a)$$

since f is above its tangent at $s = a$, and:

$$|f'(s)| < 1, \quad (4.3b)$$

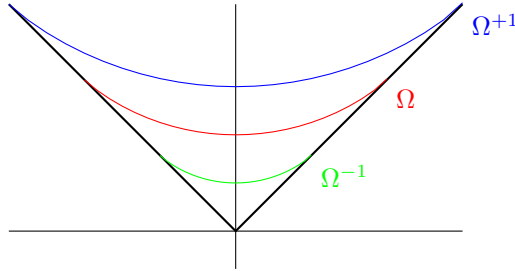
for all $s \in (-a, a)$, since f' is strictly increasing on $[-a, a]$.

Before studying the shaking operation on the elements of \mathfrak{Y}_a , we first define a map between some sets \mathfrak{Y}_a via double scaling.

Definition 4.4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be any map and let $\epsilon \in (0, 1)$. We define the two maps $f^{\pm\epsilon} : \mathbb{R} \rightarrow \mathbb{R}$ by, for any $s \in \mathbb{R}$,

$$f^{\pm\epsilon}(s) := (1 \pm \frac{\epsilon}{2}) f\left(\frac{s}{1 \pm \frac{\epsilon}{2}}\right).$$

For instance, the maps $\Omega^{\pm 1}$ are as follows:



We now gather some informations about $f^{\pm\epsilon}$.

Lemma 4.5. Let $f \in \mathfrak{Y}_2$ and $\epsilon \in (0, 1)$.

- (i) We have $f^{\pm\epsilon} \in \mathfrak{Y}_{2 \pm \epsilon}$.
- (ii) For any $s \in \mathbb{R}$ we have $(f^{\pm\epsilon})'(s) = f'\left(\frac{s}{1 \pm \frac{\epsilon}{2}}\right)$.
- (iii) The maps $g^{+\epsilon} := f^{+\epsilon} - f$ and $g^{-\epsilon} := f - f^{-\epsilon}$ are decreasing on $\mathbb{R}_{\geq 0}$.
- (iv) For any $s \in \mathbb{R}$ we have $f^{-\epsilon}(s) \leq f(s) \leq f^{+\epsilon}(s)$, and:
 - (a) we have $f^{-\epsilon}(s) = f(s)$ if and only if $|s| \geq 2$,
 - (b) we have $f^{+\epsilon}(s) = f(s)$ if and only if $|s| \geq 2 + \epsilon$,
- (v) We have $\|f - f^{\pm\epsilon}\|_\infty = \frac{\epsilon}{2} f(0)$.

Proof. (i) Clear.

(ii) Clear.

(iii) Follows from the previous point and from the fact that f' is increasing.

(iv) Since the functions are even, it suffices to prove the inequality on $\mathbb{R}_{\geq 0}$. We prove only the relations for $f^{+\epsilon}$, the ones for $f^{-\epsilon}$ being similar. By the previous point, we know that $g^{+\epsilon}$ is decreasing on $\mathbb{R}_{\geq 0}$, thus we deduce that $g \geq 0$ since $g(s) = |s| - |s| = 0$ for $s \gg 0$. For the equality case, first note that the necessary condition holds. For the sufficient condition, since f is strictly convex on $[-2, 2]$ we have in fact $g'(s) < 0$ for all $s \in (0, 2]$ thus as before we have $g(s) > 0$ for all $s \in [0, 2]$. Now for $s \in [2, 2 + 2\epsilon)$ we have $g'(s) = f'(\frac{s}{1+\epsilon}) - 1 < 0$ since $1 = f'(2)$ and f' is strictly increasing on $[0, 2]$ and we conclude the proof as before.

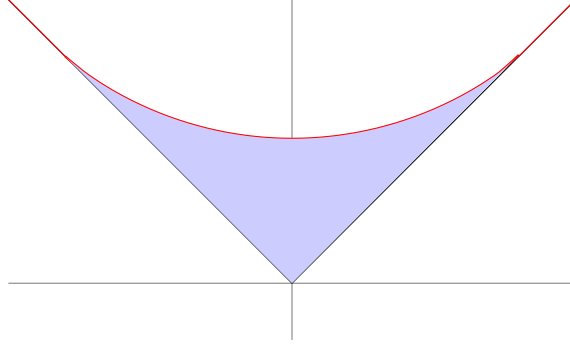
(v) Again we only prove the result for $f^{+\epsilon}$. With the previous notation, we know that $g = f^{+\epsilon} - f$ is non-negative and decreases on $\mathbb{R}_{\geq 0}$ thus its maximum is reached at $s = 0$. □

We need one more definition before giving the main result of this section.

Definition 4.6. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a map. We denote by $\mathbb{Y}(f)$ the part of the graph of f that is above the graph of the absolute value, that is:

$$\mathbb{Y}(f) := \{(x, y) \in \mathbb{R}^2 : |x| \leq y \leq f(x)\}.$$

For instance, we picture here in blue the set $\mathbb{Y}(\Omega)$:



Theorem 4.7. Let $\alpha \in [0, 1)$ and $a \in \mathbb{R}_{>0}$. For any $f \in \mathfrak{Y}_a$ we have:

$$\text{Sh}_\alpha(\mathbb{Y}(f)) \cup \mathbb{Y}(\text{id}_{\mathbb{R}}) = \mathbb{Y}(f_\alpha),$$

where $f_\alpha : \mathbb{R} \rightarrow \mathbb{R}$ is given by:

$$\begin{aligned} f_\alpha(x) &= f(x), & \text{for all } x \leq x_\alpha^-, \\ f(x + \delta_x) &= f(x) + \alpha\delta_x, & \text{for all } x \in (x_\alpha^-, x_\alpha^+), \\ f_\alpha(x) &= x, & \text{for all } x \geq s_\alpha := x_\alpha^+ + \delta_{x_\alpha^+}, \end{aligned}$$

where:

- $x_\alpha^+ = f'^{-1}(\alpha) \in [0, a)$,
- $\delta_x = (1 - \alpha)^{-1}F_\alpha(x) - F_\alpha^{-1}(F_\alpha(x))$, where $F_\alpha : x \mapsto f(x) - \alpha x$ and F_α^{-1} denotes its inverse on (x_α^+, a) ;
- $x_\alpha^- \in (-a, x_\alpha^+)$ (is the unique point that) satisfies $F_\alpha(x_\alpha^-) = (1 - \alpha)a$.

Note that $\mathbb{Y}(\text{id}_{\mathbb{R}}) = \{(x, x) : x \in \mathbb{R}_{\geq 0}\}$.

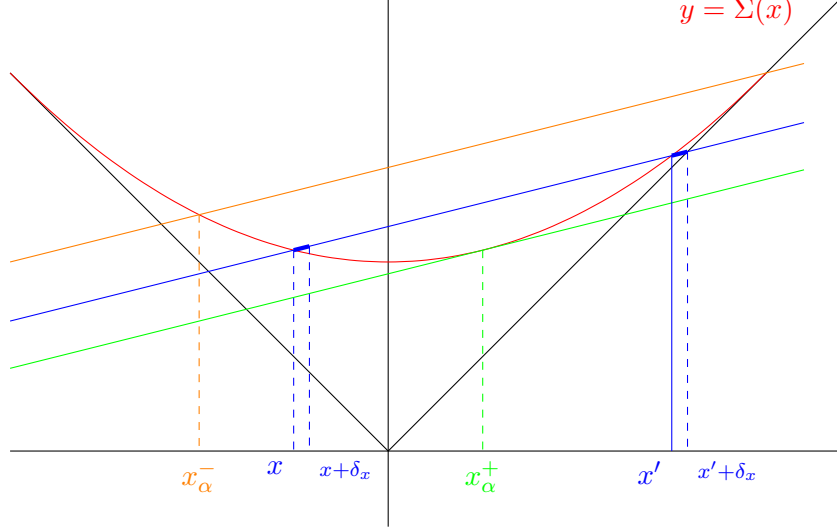


Figure 8: Illustration of the proof of Theorem 4.7 with $f = \Sigma$ and $\alpha = \frac{1}{4}$

Proof. As in Definition 3.2, let L_α be the line of equation $y = \alpha x$. For $M \in \mathbb{R}^2$, the line $L_\alpha^M := M + L_\alpha$ cuts the curve of f at most twice since f is strictly convex. Note that we can ignore the parts of $\mathbb{Y}(f)$ outside $(-a, a)$ since it has measure 0 there. Note also that there is only one intersection point if and only if $\mathbb{Y}(f) \cap L_\alpha^M$ is stable under Sh_α . Moreover, it suffices to choose M of the form $(0, y_M)$, and if y_{M_0} is the unique real number such that $L_\alpha^{M_0}$ contains (a, a) then:

for any $y \leq y_{M_0}$ then L_α^M intersects the curve of f in $(-a, a)$ only once.

Let $x_\alpha^- < a$ be such that $L_\alpha^{M_0}$ intersects the curve of f at $(x_\alpha^-, f(x_\alpha^-))$. This means that:

$$f_\alpha(x) = f(x), \quad \text{for all } x \leq x_\alpha^-.$$

Note that $x_\alpha^- \geq -a$ since $f(-a) = a$ and $\alpha \geq 0$. Now there is also a real number y_{M_1} such that $L_\alpha^{M_1}$ does not intersect the curve of f apart from $(y_{M_1}, f(y_{M_1}))$. Again by convexity, this y_{M_1} is given by the unique point such that $L_\alpha^{M_1}$ is tangent to the curve of f at some point $M_\alpha = (x_\alpha^+, f(x_\alpha^+))$. By (4.1) we know that such an M_α is unique and is determined by $f'(x_\alpha^+) = \alpha$, in other words, by:

$$x_\alpha^+ = f'^{-1}(\alpha)$$

(where f'^{-1} denotes the inverse of $f'|_{(-a, a)}$). Note that $x_\alpha^+ \geq 0$ since $\alpha \geq 0$ and f is even, thus $f'(0) = 0 \leq \alpha$, and f' is increasing by strict convexity. Similarly, we have $x_\alpha^+ < a$ since $f'(a) = 1 > \alpha$, moreover the line between $(x_\alpha^+, f(x_\alpha^+))$ and (a, a) has slope $> \alpha$ thus $x_\alpha^+ > x_\alpha^-$.

Hence, to determine f_α , it suffices to determine for each $x \in (x_\alpha^-, x_\alpha^+)$ the measure m of $(M + L_\alpha) \cap (\mathbb{Y}(f) \cap ((x, a) \times \mathbb{R}_{\geq 0}))$, with $M = (x, f(x))$, and to add it to $f(x)$ in the direction α . More precisely, if:

- $x' > x$ is the unique other abscissa where $L_\alpha^x := M + L_\alpha$ crosses the curve of f ,
- $\delta_x > 0$ is the unique positive real number such that $x' + \delta_x$ is the abscissa of the intersection between L_α^x and the line with equation $y = x$ for $x \geq 0$,

then the point $(x, f(x))$ becomes:

$$(x + \delta_x, f(x) + x' + \delta_x - f(x')). \quad (4.8)$$

In Figure 8 we give an example of the various quantities that we have introduced so far for f being the map Σ of Example 4.2.

We will now determine x' and δ_x as functions of x . We first determine x' . The line L_α^x has equation $(s - x)\alpha + f(x)$ for $s \in \mathbb{R}$, thus x' is given by:

$$f(x') = (x' - x)\alpha + f(x),$$

thus:

$$f(x') - \alpha x' = f(x) - \alpha x. \quad (4.9)$$

Now the function $F_\alpha : s \mapsto f(s) - \alpha s$ has derivative $f'(s) - \alpha$, thus since $\alpha \in (0, 1)$ we know that $s \mapsto f(s) - \alpha s$ is decreasing for $s \leq x_\alpha^+$ and increases (to $+\infty$ by (4.1b)) by for $s \geq x_\alpha^+$. Hence, we deduce that there is indeed a unique $x' > x_\alpha^+$ such that (4.9) is satisfied. Note that F_α is invertible on $(x_\alpha^+, +\infty)$ and if F_α^{-1} denotes its inverse on this interval then:

$$x' = F_\alpha^{-1}(F_\alpha(x)).$$

Note that (4.9) writes:

$$F_\alpha(x') = F_\alpha(x).$$

Note also that $x < x_\alpha^+ < x'$ so that $x' \neq x$. Finally, note that x_α^- satisfies:

$$F_\alpha(x_\alpha^-) = F_\alpha(a) = f(a) - \alpha a = (1 - \alpha)a,$$

as announced.

We now determine δ_x . First note that the equation of L_α^x becomes $\alpha s + F_\alpha(x')$. We thus find that δ_x satisfies

$$x' + \delta_x = \alpha(x' + \delta_x) + F_\alpha(x'), \quad (4.10)$$

$$= f(x') + \alpha\delta_x, \quad (4.11)$$

in particular (4.10) gives:

$$x' + \delta_x = (1 - \alpha)^{-1}F_\alpha(x'). \quad (4.12)$$

Recalling (4.9) and (4.11), the second coordinate in equation (4.8) becomes:

$$f(x) + x' + \delta_x - f(x') = f(x) + \alpha\delta_x,$$

as announced.

To prove that the set of points that we obtain is the curve of a function, it suffices to prove that $x \mapsto x + \delta_x$ is increasing on (x_α^-, x_α^+) , thus it suffices to prove that $x \mapsto \delta_x$ is increasing. By (4.12) we have:

$$\delta_x = (1 - \alpha)^{-1}F_\alpha(x') - x'.$$

Note that $x \mapsto x' = F_\alpha^{-1}(F_\alpha(x))$ is decreasing on (x_α^-, x_α^+) . Indeed, we have $F_\alpha'(x) = f'(x) - \alpha$ that is increasing since f is strictly convex, and by definition $f'(x_\alpha^+) = 0$ thus $F_\alpha'(x) < 0$ for $x < x_\alpha^+$ thus F_α is decreasing on (x_α^-, x_α^+) . Now F_α is increasing on $(x_\alpha^+, +\infty)$ thus F_α^{-1} is also increasing. By composition, we deduce that $x \mapsto x' = F_\alpha^{-1}(F_\alpha(x))$ is decreasing on (x_α^-, x_α^+) , with values in (x_α^+, a) .

Hence, to prove that $x \mapsto \delta_x = (1 - \alpha)^{-1}F_\alpha(x') - x'$ is increasing on (x_α^-, x_α^+) , it suffices to prove that $g : t \mapsto (1 - \alpha)^{-1}F_\alpha(t) - t$ is decreasing in $t \in (x_\alpha^+, a)$. But this is clear since the derivative of g is, after multiplication by $1 - \alpha$,

$$\begin{aligned} F_\alpha'(t) - (1 - \alpha) &= (f'(t) - \alpha) - (1 - \alpha) \\ &= f'(t) - 1, \end{aligned}$$

which is strictly negative since f' is increasing (by strict convexity) and $f'(t) < 1$ for all $t < a$ by (4.1b). \square

In Figure 4 we represent the curve of f_α for $f = \Omega$ and $\alpha = 0$.

Proposition 4.13. *Let $\alpha \in [0, 1)$ and $a \in \mathbb{R}_{>0}$. For any $f \in \mathfrak{Y}_a$ we have:*

$$(1 - \alpha)s_\alpha = f(f'^{-1}(\alpha)) - \alpha f'^{-1}(\alpha) \in [f(0) - \alpha f(x_\alpha^+), f(0)].$$

Proof. Recall that from Theorem 4.7 and its proof that $s_\alpha = x + \delta_x$ for $x = x_\alpha^+$. In this setting we have $x' = x$ thus (4.12) gives:

$$\begin{aligned} (1 - \alpha)s_\alpha &= F_\alpha(x_\alpha^+) \\ &= f(x_\alpha^+) - \alpha x_\alpha^+ \\ &= f(f'^{-1}(\alpha)) - \alpha f'^{-1}(\alpha), \end{aligned}$$

which gives the desired equality. Now by the mean value theorem, there exists $c \in (0, x_\alpha^+)$ such that $F_\alpha(x_\alpha^+) = F_\alpha(0) + F'_\alpha(c)x_\alpha^+$. Hence, since f is strictly convex we know that F_α is also strictly convex, thus we have $F'_\alpha(0) = f'(0) - \alpha = -\alpha < F'_\alpha(c) < F'_\alpha(x_\alpha) = 0$ (where $f'(0) = 0$ since f is even). Recalling that $F_\alpha(0) = f(0)$ gives the announced inequality. \square

We have the following particular case of Theorem 4.7 and Proposition 4.13 for $\alpha = 0$ (as pictured in Figure 4).

Corollary 4.14. *Let $a \in \mathbb{R}_{>0}$. For any $f \in \mathfrak{Y}_a$, the map $f_0 : \mathbb{R} \rightarrow \mathbb{R}$ is given by:*

$$\begin{aligned} f_0(x) &= f(x), & \text{for all } x \leq -a, \\ f_0(2x + f(x)) &= f(x), & \text{for all } x \in (-a, 0), \\ f_0(x) &= x, & \text{for all } x \geq f(0). \end{aligned}$$

We conclude this part by relating the shapes f_α and $f_\alpha^{\pm\epsilon} := (f^{\pm\epsilon})_\alpha$.

Proposition 4.15. *Let $\alpha \in [0, 1)$ and $a \in \mathbb{R}_{>0}$. Let $f \in \mathfrak{Y}_a$ and $\epsilon \in (0, 1)$. With the notation of Theorem 4.7 we have:*

$$(f^{\pm\epsilon})_\alpha = (f_\alpha)^{\pm\epsilon} =: f_\alpha^{\pm\epsilon}.$$

Proof. We denote by $\delta_x^{\pm\epsilon}$ and $F_\alpha^{\pm\epsilon}$ the quantities that appear in Theorem 4.7 for $f^{\pm\epsilon}$. Note that:

$$\begin{aligned} F_\alpha^{\pm\epsilon}(x) &= f^{\pm\epsilon}(x) - \alpha x \\ &= (1 \pm \frac{\epsilon}{2})f\left(\frac{x}{1 \pm \frac{\epsilon}{2}}\right) - (1 \pm \frac{\epsilon}{2})\alpha \frac{x}{1 \pm \frac{\epsilon}{2}} \\ &= (1 \pm \frac{\epsilon}{2})F_\alpha\left(\frac{x}{1 \pm \frac{\epsilon}{2}}\right) \\ &= (F_\alpha)^{\pm\epsilon}(x), \end{aligned}$$

and we deduce that $(F_\alpha^{\pm\epsilon})^{-1} = (F_\alpha^{-1})^{\pm\epsilon}$. We deduce that:

$$\delta_x^{\pm\epsilon} = (1 \pm \frac{\epsilon}{2})\delta_{\frac{x}{1 \pm \frac{\epsilon}{2}}}. \quad (4.16)$$

Now note that the equality $f(x + \delta_x) = f(x) + \alpha\delta_x$ holds in fact for any $x \in \mathbb{R}$. If g denotes the inverse of $x \mapsto x + \delta_x$ on \mathbb{R} (which exists since $x \mapsto x + \delta_x$ is strictly increasing by the proof of Theorem 4.7), if $y = x + \delta_x$ we thus have $x = g(y)$ and thus $\delta_x = y - g(y)$. We deduce that:

$$f_\alpha(y) = f(g(y)) - \alpha(y - g(y)) = F_\alpha(g(y)) - \alpha y. \quad (4.17)$$

Now we deduce from (4.16) that the inverse of $x \mapsto x + \delta_x^{\pm\epsilon}$ is $g^{\pm\epsilon}$ (in the sense of Defini-

tion 4.4), thus by (4.17) we obtain:

$$\begin{aligned}
f_\alpha^{\pm\epsilon}(y) &= F_\alpha^{\pm\epsilon}(g^{\pm\epsilon}(y)) - \alpha y \\
&= (1 \pm \frac{\epsilon}{2})F_\alpha\left(g\left(\frac{y}{1 \pm \frac{\epsilon}{2}}\right)\right) - \alpha y \\
&= (1 \pm \frac{\epsilon}{2})\left[F_\alpha\left(g\left(\frac{x}{1 \pm \frac{\epsilon}{2}}\right)\right) - \alpha \frac{y}{1 \pm \frac{\epsilon}{2}}\right] \\
&= (1 \pm \frac{\epsilon}{2})f_\alpha\left(\frac{y}{1 \pm \frac{\epsilon}{2}}\right) \\
&= (f_\alpha)^{\pm\epsilon}(y),
\end{aligned}$$

which is the announced result. \square

Proposition 4.18. *Let $\alpha \in [0, 1)$ and $a \in \mathbb{R}_{>0}$. Let $f \in \mathfrak{Y}_a$. We have $0 < \|f_\alpha - f_\alpha^{\pm\epsilon}\|_{\infty, \mathbb{R}} \rightarrow 0$ as $\epsilon \rightarrow 0$.*

Proof. First, note that the infinite norm is positive indeed since f_α and $f_\alpha^{\pm\epsilon}$ are continuous (even of class \mathcal{C}^1) by (4.17), by Lemma 4.5 and since the shaking preserve the area (this is a simple consequence of Fubini theorem, see, for instance, [FrLu, Lemma 3.2]). To prove that it goes to 0 as $\epsilon \rightarrow 0$, it suffices to prove that f_α is Lipschitz. Indeed, if $h : I \rightarrow \mathbb{R}$ is L -Lipschitz and bounded then:

$$\begin{aligned}
h^{\pm\epsilon}(x) - h(x) &= (1 \pm \frac{\epsilon}{2})h\left(\frac{x}{1 \pm \frac{\epsilon}{2}}\right) - h(x) \\
&= h\left(\frac{x}{1 \pm \frac{\epsilon}{2}}\right) - h(x) + \frac{\epsilon}{2}h\left(\frac{x}{1 \pm \frac{\epsilon}{2}}\right),
\end{aligned}$$

thus:

$$\begin{aligned}
|h^{+\epsilon}(x) - h(x)| &\leq L|x|\left|\frac{1}{1 + \frac{\epsilon}{2}} - 1\right| + \frac{\epsilon}{2}\|h\|_\infty \\
&\leq (L|x| + \|h\|_\infty)\epsilon,
\end{aligned}$$

with a similar calculation for $h^{-\epsilon}$. Note that it suffices to study $f - f_\alpha$ on a segment since this function vanishes for $|x| \gg 0$. Recalling (4.17), we have:

$$\begin{aligned}
f'_\alpha(y) &= g'(y)F'_\alpha(g(y)) - \alpha \\
&= g'(y)\left[f'(g(y)) - \alpha\right] - \alpha.
\end{aligned}$$

We know that f' is bounded by (4.1). For g' , recall from the proof of Theorem 4.7 that $x \mapsto \delta_x$ is increasing thus the derivative of $x \mapsto x + \delta_x$ is greater than 1, thus $\|g'\|_\infty \leq 1$. This concludes the proof. \square

5 Shaking partitions

Let λ be a partition. Proposition 3.1 implies the following result.

Lemma 5.1. *We have $\text{sh}_e(\mathbb{Y}(\omega_\lambda)) = \text{sh}_e(\mathbb{Y}(\omega_{\text{reg}_e(\lambda)}))$ and $\text{sh}_e(\mathbb{Y}(\tilde{\omega}_\lambda)) = \text{sh}_e(\mathbb{Y}(\tilde{\omega}_{\text{reg}_e(\lambda)}))$.*

Hence, to study the shaking operations on partitions it suffices to study the shaking operations on regular partitions. In fact, the shaking of $\mathbb{Y}(\tilde{\omega}_{\text{reg}_e(\lambda)})$ for λ an e -regular partition is a bit delicate to determine. Instead, we will bound the latter graph by two close graphs that are stable under the shaking operation.

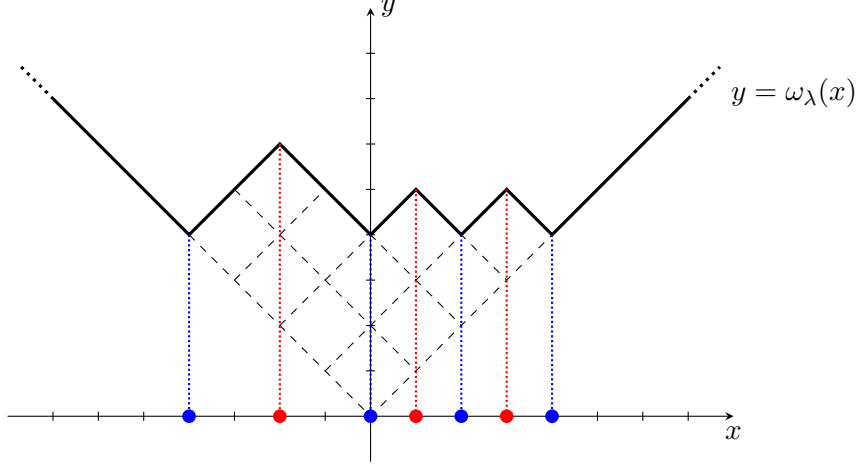


Figure 9: Outer corners (in red) and inner corners (in blue) for $\lambda = (4, 4, 2, 1)$

5.1 Corners

We say that $c \in \mathbb{Z}$ is an *outer corner* (resp. *inner corner*) of λ if $\omega'_\lambda|_{(c-1,c)} = 1$ (resp. -1) and $\omega'_\lambda|_{(c,c+1)} = -1$ (resp. 1). We give in Figure 9 an example of outer and inner corners.

The following standard result follows from the definition of inner and outer corners and from the fact that $\omega_\lambda(x) = |x|$ for $x \gg 0$.

Lemma 5.2. *Let λ be a partition. If $\{o_1 < \dots < o_k\}$ (resp. $\{i_1 < \dots < i_\ell\}$) is the set of outer (resp. inner) corners then $\ell = k + 1$ and $i_m < o_m < i_{m+1}$ for all $m \in \{1, \dots, k\}$.*

In the sequel, fix $e \geq 2$ and let L_e be the linear subspace of \mathbb{R}^2 with slope $\alpha_e = 1 - 2e^{-1}$.

5.2 Outer regularisation

If λ has no outer corners then we define $\rho^+(\lambda) = \rho_0^+(\lambda) := \omega_\lambda$, otherwise let $\{c_1 < \dots < c_r\}$ with $r \geq 1$ the set of outer corners. Assuming by induction on $k \in \{1, \dots, r\}$ that we have constructed the piecewise linear function $\rho_{k-1}^+(\lambda) : \mathbb{R} \rightarrow \mathbb{R}$, then we construct the piecewise linear function $\rho_k^+(\lambda) : \mathbb{R} \rightarrow \mathbb{R}$ as follows. Let $L_e^{(k)}$ be the affine line $(c_k, \omega_\lambda(c_k)) + L_e$.

- (i) For $s \geq c_k$ then $\rho_k^+(\lambda)(s) := \rho_{k-1}^+(\lambda)(s)$;
- (ii) at $s = c_k$, we follow the line $L_e^{(k)}$ (in the negative direction) until we meet the curve of $\rho_{k-1}^+(\lambda)$, at point of abscissa h_k ;
- (iii) for $s \leq h_k$ then $\rho_k^+(\lambda)(s) := \rho_{k-1}^+(\lambda)(s)$ again.

Proposition 5.3. *Assume that the partition λ is e -regular. With the above notation we have $h_k \in [c_{k-1}, c_k)$. In particular, the functions $\rho_{k-1}^+(\lambda)$ and $\rho_k^+(\lambda)$ coincide on (at least) $\mathbb{R} \setminus (c_{k-1}, c_k)$, and for any $s \in (c_{k-1}, c_k)$ we have $0 \leq \rho_k^+(\lambda)(s) - \rho_{k-1}^+(\lambda)(s) \leq e$.*

Proof. Since $\alpha_e < 1$, the curve of ω_λ can cross $L_e^{(k)}$ only after an inner corner, that is, if $c' < c_k$ is the inner corner preceding c_k (cf. Lemma 5.2) then $h < c'$. Moreover, by definition of an inner corner we have $\omega_\lambda(c' - 1) = \omega_\lambda(c') + 1$, thus the line joining $(c_k, \omega_\lambda(c_k))$ and $(c' - 1, \omega_\lambda(c' - 1))$ has slope:

$$\alpha := \frac{\omega_\lambda(c_k) - \omega_\lambda(c') - 1}{c_k - c' + 1}.$$

Since $\omega_\lambda(c') = \omega_\lambda(c_k) - c_k + c'$, we obtain:

$$\alpha = \frac{c_k - c' - 1}{c_k - c' + 1} = 1 - \frac{2}{c_k - c' + 1}.$$

But now λ is e -regular thus $c' \geq c_k - e + 1$. We thus have $c_k - c' + 1 \leq e$ and thus, since $c_k - c' + 1 > 0$ (since $c' \leq c_k$):

$$\alpha \leq 1 - 2e^{-1} = \alpha_e.$$

Finally, we have $\alpha \leq \alpha_e < 1$ thus $c' - 1 \leq h_k < c_k$, in other words, the line $L_e^{(k)}$ meet the curve of ω_λ somewhere between the points of abscissa c' (excluded) and $c' - 1$ (included) thus this concludes the first part of the proof.

Now for any $s \in (c_{k-1}, h)$ we have $\rho_k^+(\lambda)(s) = \rho_k^+(\lambda)(s)$, and by construction for any $s \in (h, c_k)$ we have $\rho_k^+(\lambda)(s) > \rho_k^+(\lambda)$ thus we obtain the first member of the announced inequality. The second one follows from the fact that $\alpha_e \in [0, 1]$ and from the fact that on (c_{k-1}, c_k) , the function $s \mapsto \rho_k^+(\lambda)(s) - \rho_k^+(\lambda)(s)$ reaches its minimum at $s = h_k$, where:

$$\omega_\lambda(c_k) \geq \rho_k^+(\lambda)(h_k) \geq \rho_{k-1}^+(\lambda)(h_k) \geq \omega_\lambda(c').$$

Now we already saw that:

$$\begin{cases} \omega_\lambda(c') = \omega_\lambda(c_k) - c_k + c', \\ c_k - c' + 1 \leq e, \end{cases}$$

thus we obtain:

$$\omega_\lambda(c_k) - \omega_\lambda(c') = c_k - c' \leq e - 1,$$

whence the result. \square

Definition 5.4. Let $e \geq 2$ and let λ be an e -regular partition. Let $r \geq 0$ be the number of outer corners of λ . We define $\rho^+(\lambda) := \rho_r^+(\lambda)$.

Note that, by Proposition 5.3, the map $\rho^+(\lambda)$ does not in fact depend on the order that we chose on the outer corners. An example of a map $\rho^+(\lambda)$ is given in Figure 10.

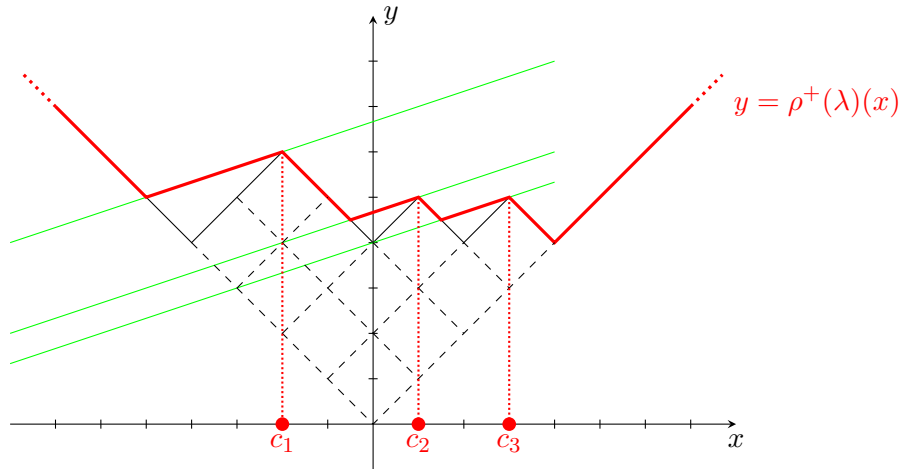


Figure 10: The map $\rho^+(\lambda)$ (in thick red) for the 3-regular partition $\lambda = (4, 4, 2, 1)$. In green are the different lines $L_e^{(k)}$.

Proposition 5.5. Recall that λ is e -regular.

- (i) For all $s \in \mathbb{R}$ we have $0 \leq \rho^+(\lambda) - \omega_\lambda(s) \leq e$.
- (ii) The graph $G_\lambda^+ := \mathbb{Y}(\rho^+(\lambda))$ is stable under the shaking operation with slope α_e , that is, we have $\text{sh}_e(G_\lambda^+) = G_\lambda^+$.

Proof. The first point follows directly from Proposition 5.3 since the intervals (c_{k-1}, c_k) are pairwise disjoint. For the second point, it follows from the simple observation that the slopes of $\rho^+(\lambda)$ are either -1 or α_e (except the part $\rho^+(\lambda)(s) = s$ for $s \gg 0$). As a consequence, if

for $x \in \mathbb{R}$ the line $L' := x + L_e$ intersects the curve of $\rho^+(\lambda)$ at a point $s \in \mathbb{R}$ then L' remains below the curve $\rho^+(\lambda)$ on $(-\infty, s)$. Observing that L' intersects the curve of $|\cdot|$ on $(-\infty, 0)$ since $\alpha_e < 1$ gives the result. \square

5.3 Inner regularisation

We now define a similar construction as $\rho^+(\lambda)$ but for inner corners. We give the statements without proofs since they are entirely similar.

Define $\rho_{r+1}^-(\lambda) := \omega_\lambda$ and let $\{c_1 < \dots < c_{r+1}\}$ with $r \geq 0$ the set of inner corners (in particular, note that a partition has always at least one inner corner). Assuming by decreasing induction on $k \in \{1, \dots, r\}$ that we have constructed the piecewise linear function $\rho_{k+1}^-(\lambda) : \mathbb{R} \rightarrow \mathbb{R}$, then we construct the piecewise linear function $\rho_k^-(\lambda) : \mathbb{R} \rightarrow \mathbb{R}$ as follows. Let $L_e^{(k)}$ be the affine line $(c_k, \omega_\lambda(c_k)) + L_e$.

- (i) For $s \leq c_k$ then $\rho_k^-(\lambda)(s) := \rho_{k+1}^-(\lambda)(s)$;
- (ii) at $s = c_k$, we follow the line $L_e^{(k)}$ (in the *positive* direction) until we meet the curve of $\rho_{k+1}^-(\lambda)$, at point of abscissa h_k ;
- (iii) for $s \geq h_k$ then $\rho_k^-(\lambda)(s) := \rho_{k+1}^-(\lambda)(s)$ again.

Proposition 5.6. *Assume that the partition λ is e -regular. With the above notation we have $h_k \in [c_k, c_{k+1})$. In particular, the functions $\rho_{k+1}^-(\lambda)$ and $\rho_k^-(\lambda)$ coincide on (at least) $\mathbb{R} \setminus (c_k, c_{k+1})$, and for any $s \in (c_k, c_{k+1})$ we have $-e \leq \rho_k^-(\lambda)(s) - \rho_{k+1}^-(\lambda)(s) \leq 0$.*

An example of a map $\rho^-(\lambda)$ is given in Figure 11.

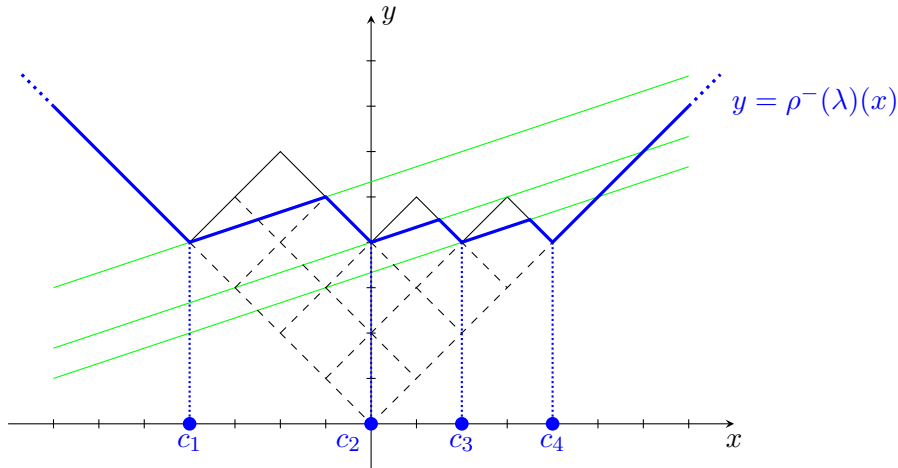


Figure 11: The map $\rho^-(\lambda)$ (in thick blue) for the 3-regular partition $\lambda = (4, 4, 2, 1)$. In green are the different lines $L_e^{(k)}$.

Definition 5.7. Let $e \geq 2$ and let λ be an e -regular partition. We define $\rho^-(\lambda) := \rho_1^-(\lambda)$.

Proposition 5.8. *Recall that λ is e -regular.*

- (i) For all $s \in \mathbb{R}$ we have $-e \leq \rho^-(\lambda) - \omega_\lambda(s) \leq 0$.
- (ii) The graph $G_\lambda^- := \mathbb{Y}(\rho^-(\lambda))$ is stable under the shaking operation with slope α_e , that is, we have $\text{sh}_e(G_\lambda^-) = G_\lambda^-$.

6 Limit shape for regularisation of large partitions

We can now state our main theorem. Recall from Theorem 4.7 the definition of Ω_α for $\alpha \geq 0$.

Theorem 6.1. *Let $e \geq 2$ and take $\alpha = \alpha_e = 1 - 2e^{-1}$. Under the Plancherel measure Pl_n , the function $\tilde{\omega}_{\text{reg}_e(\lambda)}$ converges uniformly in probability to Ω_α as $n \rightarrow +\infty$. In other words, for any $\epsilon > 0$ we have*

$$\text{Pl}_n \left(\sup_{\mathbb{R}} |\tilde{\omega}_{\text{reg}_e(\lambda)} - \Omega_\alpha| > \epsilon \right) \xrightarrow{n \rightarrow +\infty} 0.$$

Proof. Let $\epsilon > 0$. Recalling the notation of Definition 4.4, define:

$$m^\pm := \inf \{ |\Omega^{\pm\epsilon}(s) - \Omega(s)| : |s| \leq 2 \pm \frac{\epsilon}{2} \}.$$

By Lemma 4.5 applied with $f = \Omega$, we have $m^\pm > 0$. Now let M_n be the set of partitions $\lambda \in \mathcal{P}_n$ such that:

$$\begin{aligned} \|\tilde{\omega}_\lambda - \Omega\|_{\mathbb{R}} &< \min(m^+, m^-), \\ \inf \{ s \in \mathbb{R} : \tilde{\omega}(s) \neq |s| \} &> -2 - \frac{\epsilon}{2}, \\ \sup \{ s \in \mathbb{R} : \tilde{\omega}(s) \neq |s| \} &< 2 + \frac{\epsilon}{2}. \end{aligned}$$

Note that by Theorem 2.9 we have $\text{Pl}_n(M_n) \rightarrow 1$ as $n \rightarrow +\infty$. By Lemma 4.5, for all $\lambda \in M_n$ we have:

$$\Omega^{-\epsilon}(s) \leq \tilde{\omega}_\lambda(s) \leq \Omega^{+\epsilon}(s),$$

for all $s \in \mathbb{R}$. In other words, for all $\lambda \in M_n$ we have:

$$\mathbb{Y}(\Omega^{-\epsilon}) \subseteq \mathbb{Y}(\tilde{\omega}_\lambda) \subseteq \mathbb{Y}(\Omega^{+\epsilon}).$$

By Proposition 3.4, Theorem 4.7 together with Lemma 5.1, we deduce that:

$$\mathbb{Y}(\Omega_\alpha^{-\epsilon}) \subseteq \text{sh}_e(\mathbb{Y}(\tilde{\omega}_{\text{reg}_e(\lambda)})) \subseteq \mathbb{Y}(\Omega_\alpha^{+\epsilon}). \quad (6.2)$$

In particular, note that we can apply Theorem 4.7 for $\Omega^{\pm\epsilon}$ indeed by Lemma 4.5, and $\Omega_\alpha^{\pm\epsilon} := (\Omega^{\pm\epsilon})_\alpha$. Now by Propositions 5.5 and 5.8, for all $s \in \mathbb{R}$ we have:

$$\tilde{\rho}^-(\lambda)(s) \leq \tilde{\omega}_\lambda(s) \leq \tilde{\rho}^+(\lambda)(s)$$

(where $\tilde{\rho}^\pm(\lambda)(s) = \frac{1}{\sqrt{n}} \rho^\pm(\lambda)(\sqrt{n}s)$ is rescaled as $\tilde{\omega}_\lambda$, see §2.2), thus:

$$\mathbb{Y}(\tilde{\rho}^-(\lambda)) \subseteq \mathbb{Y}(\tilde{\omega}_{\text{reg}_e(\lambda)}) \subseteq \mathbb{Y}(\tilde{\rho}^+(\lambda)).$$

By Propositions 5.5 and 5.8 again and Proposition 3.4, we deduce that:

$$\mathbb{Y}(\tilde{\rho}^-(\lambda)) \subseteq \text{sh}_e(\mathbb{Y}(\tilde{\omega}_{\text{reg}_e(\lambda)})) \subseteq \mathbb{Y}(\tilde{\rho}^+(\lambda)). \quad (6.3)$$

By (6.2) and (6.3), we deduce that:

$$\mathbb{Y}(\Omega_\alpha^{-\epsilon}) \subseteq \mathbb{Y}(\tilde{\rho}^+(\lambda)), \quad \text{and} \quad \mathbb{Y}(\tilde{\rho}^-(\lambda)) \subseteq \mathbb{Y}(\Omega_\alpha^{+\epsilon}).$$

Thus, we conclude that for any $s \in \mathbb{R}$ we have:

$$\Omega_\alpha^{-\epsilon}(s) \leq \tilde{\rho}^+(\lambda)(s), \quad \text{and} \quad \tilde{\rho}^-(\lambda)(s) \leq \Omega_\alpha^{+\epsilon}(s).$$

We deduce that for any $s \in \mathbb{R}$ we have:

$$\begin{aligned} \tilde{\rho}^-(\lambda)(s) &\leq \Omega_\alpha^{+\epsilon}(s) \\ &\leq \Omega_\alpha^{-\epsilon}(s) + \|\Omega_\alpha^{-\epsilon} - \Omega_\alpha^{+\epsilon}\|_\infty \\ &\leq \tilde{\rho}^+(\lambda)(s) + \|\Omega_\alpha^{-\epsilon} - \Omega_\alpha^{+\epsilon}\|_\infty \\ &\leq \tilde{\rho}^-(\lambda)(s) + \frac{2e}{\sqrt{n}} + \|\Omega_\alpha^{-\epsilon} - \Omega_\alpha^{+\epsilon}\|_\infty \quad \text{by Propositions 5.5 and 5.8.} \end{aligned}$$

Thus, again using Proposition 5.8, we deduce that for any $s \in \mathbb{R}$ we have:

$$\begin{aligned} |\tilde{\omega}_{\text{reg}_e(\lambda)}(s) - \Omega_\alpha(s)| &\leq |\tilde{\omega}_{\text{reg}_e(\lambda)}(s) - \rho^-(\lambda)(s)| + |\rho^-(\lambda)(s) - \Omega_\alpha^{+\epsilon}(s)| + |\Omega_\alpha^{+\epsilon}(s) - \Omega_\alpha(s)| \\ &\leq \frac{2e}{\sqrt{n}} + \left(\frac{2e}{\sqrt{n}} + \|\Omega_\alpha^{-\epsilon} - \Omega_\alpha^{+\epsilon}\|_\infty \right) + \|\Omega_\alpha^{+\epsilon} - \Omega_\alpha\|_\infty, \end{aligned}$$

whence:

$$\sup_{\mathbb{R}} |\tilde{\omega}_{\text{reg}_e(\lambda)} - \Omega_\alpha| \leq \frac{4e}{\sqrt{n}} + 3\|\Omega_\alpha^{+\epsilon} - \Omega_\alpha\|_\infty.$$

Thus, by Proposition 4.18 deduce that for $n \gg 0$ we have:

$$\sup_{\mathbb{R}} |\tilde{\omega}_{\text{reg}_e(\lambda)} - \Omega_\alpha| \leq 4\|\Omega_\alpha^{+\epsilon} - \Omega_\alpha\|_\infty.$$

Recalling that $\lambda \in M_n$ with $\text{Pl}_n(M_n) \rightarrow 1$ as $n \rightarrow +\infty$, we deduce that:

$$\text{Pl}_n \left(\sup_{\mathbb{R}} |\tilde{\omega}_{\text{reg}_e(\lambda)} - \Omega_\alpha| > 4\|\Omega_\alpha^{+\epsilon} - \Omega_\alpha\|_\infty \right) \xrightarrow{n \rightarrow +\infty} 0,$$

which, again by Proposition 4.18, proves the desired convergence. \square

We illustrate the convergence of Theorem 6.1 for $e = 2$ in Figure 12 and for $e \in \{3, 4\}$ in Figure 13, with in red the usual limit shape Ω and in blue the map $\tilde{\omega}_{\text{reg}_e(\lambda)}$ for λ a partition of 3×10^4 taken under the Plancherel measure.

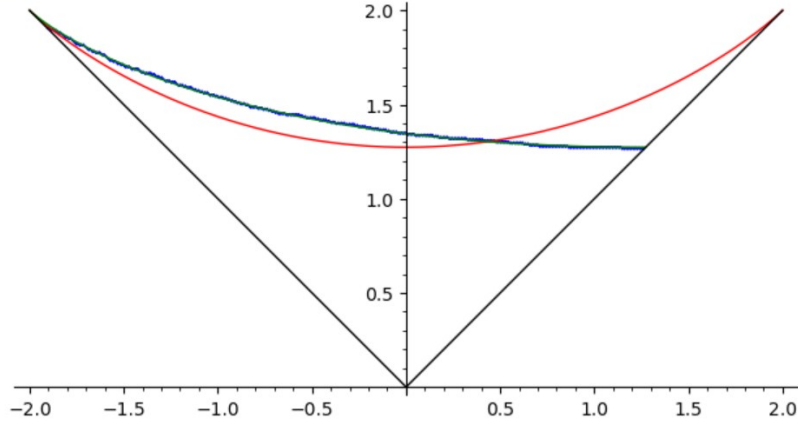


Figure 12: Example of 2-regularisation of a large partition, with in green the limit shape of Corollary B

Theorem 6.4. *The convergence of Theorem 6.1 also holds for the supports, that is:*

$$\inf\{s \in \mathbb{R} : \tilde{\omega}_{\text{reg}_e(\lambda)}(s) \neq |s|\} \longrightarrow -2,$$

and

$$\sup\{s \in \mathbb{R} : \tilde{\omega}_{\text{reg}_e(\lambda)}(s) \neq |s|\} \longrightarrow s_\alpha,$$

in probability under Pl_n as $n \rightarrow +\infty$, where s_α is the constant on Proposition 4.13 for $f = \Omega$.

Proof. We use the same set M_n and the same strategy as in the proof of Theorem 6.1. \square

Finally, we deduce the asymptotic behaviour of the first part and the first column.

Corollary 6.5. *Let $e \geq 2$. Under the Plancherel measure Pl_n :*

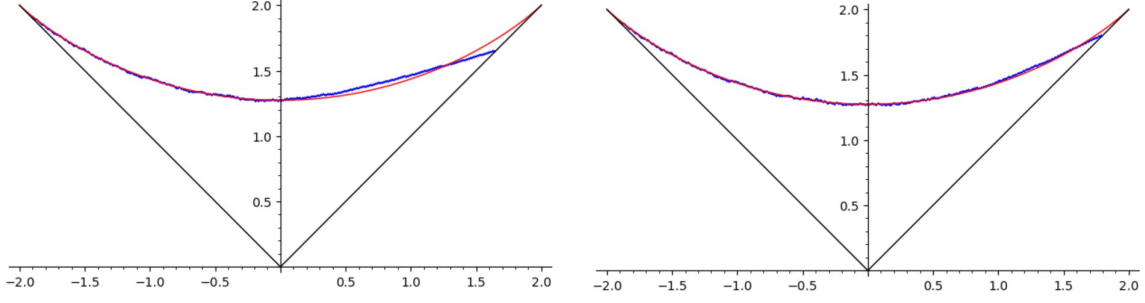


Figure 13: Example of 3- and 4-regularisation of a large partition, respectively on the left and right.

- (i) the rescaled size $\frac{1}{\sqrt{n}}\text{reg}_e(\lambda)_1$ of the first row of $\text{reg}_e(\lambda)$ converges as $n \rightarrow +\infty$ in probability to 2;
- (ii) the rescaled size $\frac{1}{\sqrt{n}}\text{reg}_e(\lambda)'_1$ of the first column of $\text{reg}_e(\lambda)$ converges as $n \rightarrow +\infty$ in probability to $\frac{2e}{\pi} \sin \frac{\pi}{e}$.

Proof. The first point is clear by Theorem 6.4. For the second one, by the same theorem it suffices to prove that the announced value is equal to s_α . Proposition 4.13 applied for $f = \Omega$ we have:

$$s_\alpha = (1 - \alpha)^{-1} \left[\Omega(\Omega'^{-1}(\alpha)) - \alpha \Omega'^{-1}(\alpha) \right].$$

Recall that $\alpha = 1 - 2e^{-1}$ and $\Omega'(s) = \frac{2}{\pi} \arcsin(\frac{s}{2})$. We have $\Omega'(s) = t \iff \arcsin(\frac{s}{2}) = \frac{\pi t}{2} \iff s = 2 \sin \frac{\pi t}{2}$ thus $\Omega'^{-1}(t) = 2 \sin \frac{\pi t}{2}$. We thus have:

$$\begin{aligned} \Omega'^{-1}(\alpha) &= 2 \sin \frac{\pi \alpha}{2} \\ &= 2 \sin \frac{\pi(1 - 2e^{-1})}{2} \\ &= 2 \sin \left(\frac{\pi}{2} - \frac{\pi}{e} \right) \\ &= 2 \cos \frac{\pi}{e}. \end{aligned}$$

We obtain:

$$\begin{aligned} s_\alpha &= \frac{e}{2} \left[\Omega(\Omega'^{-1}(\alpha)) - \alpha \Omega'^{-1}(\alpha) \right] \\ &= \frac{e}{2} \left[\Omega\left(2 \cos \frac{\pi}{e}\right) - 2\alpha \cos \frac{\pi}{e} \right]. \end{aligned}$$

We have, recalling the identity $\arcsin(\cos x) = \frac{\pi}{2} - x$ for $x \in [-1, 1]$:

$$\begin{aligned} \Omega\left(2 \cos \frac{\pi}{e}\right) &= \frac{2}{\pi} \left[\arcsin\left(\cos \frac{\pi}{e}\right) 2 \cos \frac{\pi}{e} + \sqrt{4 - 4 \cos^2 \frac{\pi}{e}} \right] \\ &= \frac{2}{\pi} \left[\left(\frac{\pi}{2} - \frac{\pi}{e}\right) 2 \cos \frac{\pi}{e} + 2 \sin \frac{\pi}{e} \right] \\ &= 2\left(1 - \frac{2}{e}\right) \cos \frac{\pi}{e} + \frac{4}{\pi} \sin \frac{\pi}{e} \\ &= 2\alpha \cos \frac{\pi}{e} + \frac{4}{\pi} \sin \frac{\pi}{e}, \end{aligned}$$

thus we finally obtain:

$$s_\alpha = \frac{2e}{\pi} \sin \frac{\pi}{e}.$$

□

Example 6.6. Here are approximations of the first values of the limit for $\frac{1}{\sqrt{n}}\text{reg}_e(\lambda)_1$:

e	$\frac{2e}{\pi} \sin \frac{\pi}{e}$
2	1.27
3	1.65
4	1.80

The reader can check that they match the corresponding values of Figures 12 and 13.

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