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# Limit shape for regularisation of large partitions under the Plancherel measure

Salim ROSTAM\*

## Abstract

A celebrated result of Kerov–Vershik and Logan–Shepp gives an asymptotic shape for large partitions under the Plancherel measure. We prove that when we consider  $\epsilon$ -regularisations of such partitions we still have a limit shape, which is given by a shaking of the Kerov–Vershik–Logan–Shepp curve. We deduce an explicit form for the first asymptotics of the length of the first rows and the first columns for the  $\epsilon$ -regularisation.

## 1 Introduction

Partitions of a given integer  $n$  are the different way to decompose  $n$  as an unordered sum of positive integers. In other words, a **partition** of  $n \in \mathbb{Z}_{\geq 0}$  is a non-increasing sequence  $\lambda = (\lambda_1 \geq \dots)$  of non-negative integers with sum  $n$ . This mathematical object appears for instance in the study of the symmetric group  $\mathfrak{S}_n$  of permutations of  $\{1, \dots, n\}$ , since partitions of  $n$  index the conjugacy classes (via the cycle decomposition).

A *representation* of  $\mathfrak{S}_n$  of dimension  $N$  over a field  $k$  is a group homomorphism  $\rho : \mathfrak{S}_n \rightarrow \mathrm{GL}_N(k)$ . We focus for the moment at the case where  $k$  is the field  $\mathbb{C}$  of complex numbers, in which case we say that we have a *complex representation*. As any integer decomposes into a product of primes, any complex representation decomposes into a sum of *irreducible* complex representations. It turns out that the fact that the set  $\mathcal{P}_n$  of partitions of  $n$  index the conjugacy classes implies that  $\mathcal{P}_n$  also index the set of irreducible complex representations. If we denote by  $\rho_\lambda$  the irreducible complex representation associated with  $\lambda \in \mathcal{P}_n$ , a standard result of complex representation theory shows that:

$$\#\mathfrak{S}_n = n! = \sum_{\lambda \in \mathcal{P}_n} (\dim \rho_\lambda)^2. \quad (1.1)$$

A remarkable fact is that we are able to explicitly compute the numbers  $\dim \rho_\lambda$  (namely with the famous *hook length formula*). We refer for instance to Sagan [Sa] or James–Kerber [JaKe] for more details on the complex representations of  $\mathfrak{S}_n$ .

Now if we want to study the representations of  $\mathfrak{S}_n$  over the field  $k = \mathbb{F}_p$  with  $p$  elements (with  $p$  a prime number; we will say that we have a  *$p$ -representation*), almost everything that is known for complex representations falls apart. A fundamental difference is that some representations may *not* decompose into a sum of irreducible ones. Nevertheless, the study of irreducible representations is still interesting since any representation can always be decomposed into irreducible constituents via a *composition* (or *Jordan–Hölder series*).

A general theorem of Brauer says that the irreducible  $p$ -representations are in bijection with the  $p$ -regular conjugation classes, that is, with conjugation classes formed by elements whose order does not divide  $p$ . For the symmetric group, one can see that the set of  $p$ -regular conjugation classes is in one-to-one correspondence with the set of  **$p$ -regular** partitions, that is, partitions with no  $p$  (or more) consecutive equal parts. Note that the dual notion of  *$p$ -restricted* partitions is also present in the literature, where the difference between two

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\*Univ Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France

consecutive parts of a  $p$ -restricted partition it at most  $p - 1$ . If  $\lambda$  is a  $p$ -regular partition, we will denote by  $\rho_\lambda^p$  the associated irreducible  $p$ -representation. Contrary to the complex case, there is no formula expressing the dimension of  $\rho_\lambda^p$  (yet). We refer for instance to James–Kerber [JaKe] for more details on the  $p$ -representations of  $\mathfrak{S}_n$ .

A way to understand  $p$ -irreducible representations is to study the  $p$ -irreducible representations that appear in the decomposition series of an irreducible complex representation reduced modulo  $p$ . More explicitly, it can be shown that any irreducible complex representation can be realised over  $\mathbb{Z}$ , that is, we can assume that  $\rho_\lambda : \mathfrak{S}_n \rightarrow \mathrm{GL}_N(\mathbb{Z})$ . The reduction modulo  $p$ , that we denote by  $\bar{\rho}_p : \mathfrak{S}_n \rightarrow \mathrm{GL}_N(\mathbb{F}_p)$ , is then simply the reduction modulo  $p$  of the matrix entries. Now if we return to the problem of determining which irreducible  $p$ -representations appear in a decomposition series of  $\bar{\rho}_p$ , James [Ja] gave an explicit combinatorial construction of a  $p$ -regular partition  $\mathrm{reg}_e(\lambda)$  such that  $\rho_\lambda^p$  appears in a decomposition series of  $\bar{\rho}_p$ . This partition  $\mathrm{reg}_e(\lambda)$  is the  **$p$ -regularisation** of  $\lambda$ . The  $p$ -regularisation operation has in fact a meaning also when  $p$  is not prime, in the context of the *Iwahori–Hecke algebra* of  $\mathfrak{S}_n$  (see, for instance, Mathas [Ma]). In particular, for an integer  $e \geq 2$  we will also use the terms  $e$ -regular and  $e$ -regularisation. Note that the  $e$ -regularisation map was recently generalised by Millan Berdasco [Mi] to an  $(e, i)$ -regularisation map on partitions: it is likely that our results generalise to this setting.

We now go back to (1.1). This equation shows that  $\mathrm{Pl}_n(\lambda) := \frac{(\dim \rho_\lambda)^2}{n!}$  is a probability measure on the set  $\mathcal{P}_n$  of partitions of  $n$ , called the *Plancherel measure*. Via some calculations involving the *hook integral* (defined in the spirit of the hook length formula), Kerov–Vershik [KeVe] and Logan–Shepp [LoSh] proved independently that there is a curve  $\Omega$  so that the upper rim  $\tilde{\omega}_\lambda$  of the Young diagram (in the Russian convention) of a large partition  $\lambda$  converges uniformly in probability to  $\Omega$ . An illustration of this convergence is given in Figure 1 (note<sup>1</sup>). This limit shape theorem allowed to determine the first asymptotics of the length of the first row (and first column) of a Young diagram, taken under the Plancherel measure. Note that, via the Robinson–Schensted correspondence (which provides a bijective proof of (1.1)), this provides a solution to the *Ulam problem* on the length of a longest increasing subsequence of a word in  $\mathfrak{S}_n$  chosen uniformly. We refer for instance to Romik [Rom] for more details on the Plancherel measure and related asymptotics.

In the previous context of regularisation of partitions, the following question is thus natural: for an integer  $e \geq 2$ , what can be said about the partition  $\mathrm{reg}_e(\lambda)$  when  $\lambda$  is a large partition taken under the Plancherel measure? The aim of this paper is to give a first answer to this question.

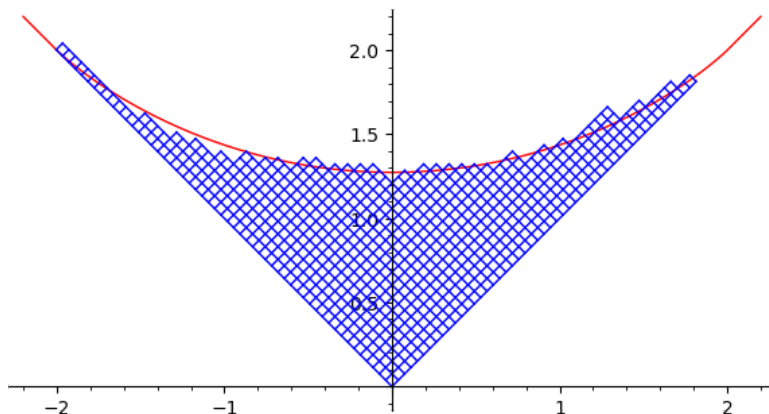


Figure 1: The limit shape theorem for large partitions under the Plancherel measure

<sup>1</sup>All the computations were made using SAGEMATH [SM].

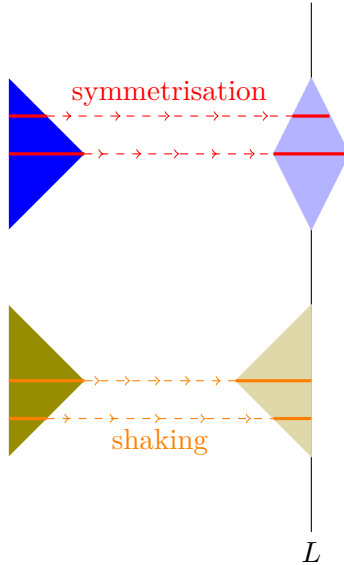


Figure 2: Examples of symmetrisation and shaking with respect to a line  $L$

Steiner symmetrisation is a useful tool to study (namely) the isoperimetric problem. More precisely, if  $K$  is a compact convex subset of  $\mathbb{R}^2$  then its *symmetrisation* with respect to the line  $L$  is the set  $S_L(K)$  that we obtain after sliding the different slices of  $K$  (with respect to the orthogonal direction of  $L$ ) until the midpoint of the slice is on  $L$ . An example of symmetrisation with respect to a vertical line is given in Figure 2. The link with the isoperimetric problem is that  $S_L(K)$  has the same area as  $K$  but a smaller perimeter. Note the following result, known as the *sphericity theorem* of Gross: from any compact convex subset of  $\mathbb{R}^2$  one can obtain the unit disk after a succession of (possibly infinite number of) symmetrisations. Note that this concept of symmetrisation can also be extended to maps, with for example Schwarz *symmetric rearrangement*, which is a powerful tool to study functional inequalities. We refer for instance to Gruber [Gr] or Krantz–Parks [KrPa] for more details on the Steiner symmetrisation.

A variation of Steiner symmetrisation is the notion of **shaking**. This notion was first introduced by Blaschke [Bl] to solve Sylvester’s “four points problem”. The difference with Steiner symmetrisation is that we slide the slices until we *meet*  $L$ . An example is given in Figure 2. Shaking and Steiner symmetrisation share many properties, for instance, Gross theorem holds with “unit disk” replaced by “simplex”. We refer to [CCG] for more details on the shaking operation. We will in fact use the shaking operation in a context where the slices are not orthogonal to the line  $L$ ; such an operation is for instance used in [FrLu].

We can now state our main result (Theorems 4.7 and 6.1).

**Theorem A.** *Let  $e \geq 2$ . Under the Plancherel measure  $\text{Pl}_n$ , the upper rim  $\tilde{\omega}_{\text{reg}_e(\lambda)}$  of the Young diagram (in the Russian convention) of  $\text{reg}_e(\lambda)$  converges uniformly in probability to the shape  $\Omega_\alpha$  as  $n \rightarrow +\infty$ , in other words, for any  $\epsilon > 0$  we have*

$$\text{Pl}_n \left( \sup_{\mathbb{R}} |\tilde{\omega}_{\text{reg}_e(\lambda)} - \Omega_\alpha| > \epsilon \right) \xrightarrow{n \rightarrow +\infty} 0.$$

*The shape  $\Omega_\alpha$  is obtained by shaking the part  $\mathbb{Y}(\Omega)$  of the graph of  $\Omega$  that is above the graph of the absolute value, with respect to the line of equation  $y = -x$  and angle  $\alpha := 1 - 2e^{-1}$ ,*

and is given by:

$$\begin{aligned} \Omega_\alpha(x) &= \Omega(x), & \text{for all } x \leq x_\alpha^-, \\ \Omega_\alpha(x + \delta_x) &= \Omega(x) + \alpha\delta_x, & \text{for all } x \in (x_\alpha^-, x_\alpha^+), \\ \Omega_\alpha(x) &= x, & \text{for all } x \geq s_\alpha, \end{aligned}$$

where:

- $x_\alpha^+ = \Omega'^{-1}(\alpha) \in [0, a)$ ,
- $\delta_x = (1 - \alpha)^{-1}F_\alpha(x) - F_\alpha^{-1}(F_\alpha(x))$ , where  $F_\alpha : x \mapsto \Omega(x) - \alpha x$  and  $F_\alpha^{-1}$  denotes its inverse on  $(x_\alpha^+, a)$ ,
- $x_\alpha^- \in (-a, x_\alpha^+)$  (is the unique point that) satisfies  $F_\alpha(x_\alpha^-) = (1 - \alpha)a$ ,
- $(1 - \alpha)s_\alpha = F_\alpha(x_\alpha^+) = \Omega(\Omega'^{-1}(\alpha)) - \alpha\Omega'^{-1}(\alpha) \in [\Omega(0) - \alpha\Omega(x_\alpha^+), \Omega(0)]$ .

We illustrate the convergence in Figure 3 and compare the limit shapes  $\Omega_0$  and  $\Omega$ . The fact that  $\Omega_0$  is obtained via the horizontal shaking of  $\mathbb{Y}(\Omega)$  (with respect to the line of equation  $y = -x$ ) is highlighted in Figure 4. In the particular case  $e = 2$ , the statement of Theorem A becomes explicit (Corollary 4.14 and Theorem 6.1).

**Corollary B.** *Under the Plancherel measure  $\text{Pl}_n$ , the upper rim  $\tilde{\omega}_{\text{reg}_2(\lambda)}$  of the Young diagram (in the Russian convention) of  $\text{reg}_2(\lambda)$  converges uniformly in probability to the shape  $\Omega_0$  given by:*

$$\begin{aligned} \Omega_0(x) &= \Omega(x), & \text{for all } x \leq -2, \\ \Omega_0(2x + \Omega(x)) &= \Omega(x), & \text{for all } x \in (-2, 0), \\ \Omega_0(x) &= x, & \text{for all } x \geq \Omega(0). \end{aligned}$$

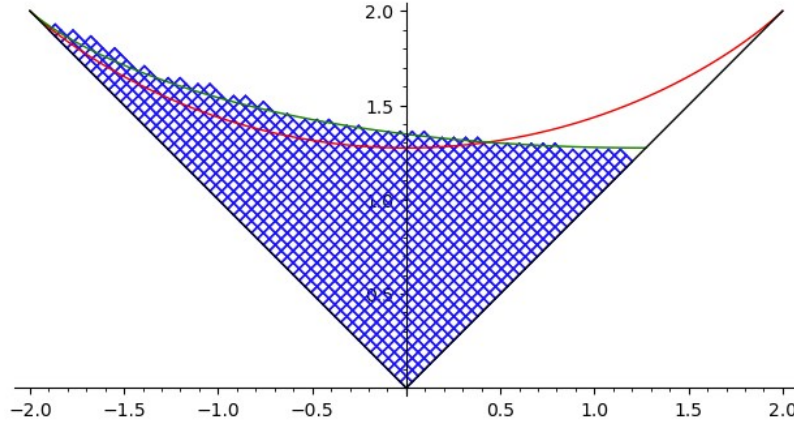


Figure 3: Example of 2-regularisation of a large partition, with in green the limit shape  $\Omega_0$  of Corollary B and in red the limit shape  $\Omega$  of Kerov-Vershik-Logan-Shepp

Despite the fact that Theorem A is not fully explicit, we are able to deduce the asymptotic behaviour of the length of the first line and of the first column of  $\text{reg}_e(\lambda)$  (Corollary 6.5).

**Corollary C.** *Let  $e \geq 2$ . Under the Plancherel measure  $\text{Pl}_n$ :*

- the rescaled size  $\frac{1}{\sqrt{n}}\text{reg}_e(\lambda)_1$  of the first row of  $\text{reg}_e(\lambda)$  converges as  $n \rightarrow +\infty$  in probability to 2;
- the rescaled size  $\frac{1}{\sqrt{n}}\text{reg}_e(\lambda)'_1$  of the first column of  $\text{reg}_e(\lambda)$  converges as  $n \rightarrow +\infty$  in probability to  $\frac{2e}{\pi} \sin \frac{\pi}{e}$ .

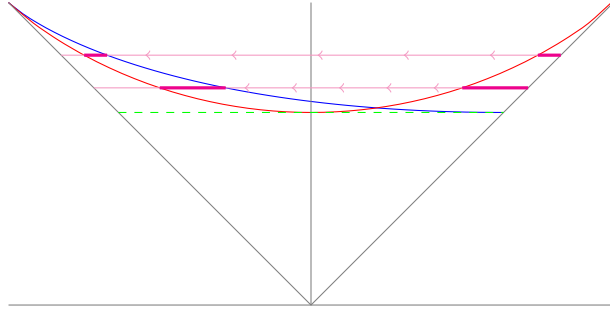


Figure 4: The maps  $\Omega_0$  (in blue) and  $\Omega$  (in red). In green the tangent for  $\Omega$  at 0. The left magenta segments are what is “added” to the red curve, and comes from the right magenta segments by shaking  $\mathbb{Y}(\Omega)$ .

We now give the outline of the paper. In Section 2 we recall the definition of the  $\epsilon$ -regularisation map on the set of partitions of  $n$  and we state the limit shape result of Kerov–Vershik and Logan–Shepp for large partitions taken under the Plancherel measure. The short Section 3 is devoted to the (non-orthogonal) shaking operation on compact subsets of  $\mathbb{R}^2$ . Section 4 contains the first main result of the paper, Theorem 4.7, which compute the shape associated with the shaking of the graph of a convex function (more precisely, the part of the graph that is above the graph of the absolute value). We give in Corollary 4.14 a particular case when the shaking is made by horizontal slices, where the result is explicit. In Section 5 we introduce two approximations  $\rho^\pm(\lambda)$  for the upper rim  $\omega_\lambda$  of the Young diagram (in the Russian convention) of a partition  $\lambda$ , in order to facilitate the shakings computations. More precisely, in §5.2 we define the *outer regularisation*  $\rho^+(\lambda)$  of an  $\epsilon$ -regular partition  $\lambda$  (Definition 5.4) and in Proposition 5.5 we prove that  $\rho^+(\lambda)$  is both close to  $\omega_\lambda$  and that its graph  $\rho^+(\lambda)$  is stable under the shaking operation. A similar construction is made in §5.3 for the *inner regularisation*  $\rho^-(\lambda)$ . Finally, Section 6 is devoted to the proof of the second main result, Theorem 6.1, which states that the (rescaled) upper rim  $\tilde{\omega}_{\text{reg}_\epsilon(\lambda)}$  of the Young diagram (in the Russian convention) of the  $\epsilon$ -regularisation of a partition taken under the Plancherel measure converges uniformly in probability to the shape defined in Theorem 4.7. The proof uses the results of Sections 4 and 5. We also prove the convergence of the support of  $\tilde{\omega}_\lambda - |\cdot|$  (Theorem 6.4), and we deduce the asymptotic length of the first row and the first column of  $\text{reg}_\epsilon(\lambda)$  (Corollary 6.5).

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## 2 Background on partitions


We recall in §2.1 the combinatorial notion of  $\epsilon$ -regularisation of a partition. In §2.2 we recall the limit shape result of Kerov–Vershik and Logan–Shepp for large partitions under the Plancherel measure.

## 2.1 Regularisation

A *partition* is a finite non-increasing sequence  $\lambda = (\lambda_1 \geq \dots \geq \lambda_h > 0)$  of positive integers. If  $|\lambda| := \sum_{i=1}^h \lambda_i = n$  then we say that  $\lambda$  is a partition of  $n$ . We denote by  $\mathcal{P}_n$  the set of partitions of  $n$ . The *Young diagram* of a partition  $\lambda = (\lambda_1 \geq \dots \geq \lambda_h > 0)$  is the subset of  $\mathbb{Z}_{\geq 1}^2$  given by:

$$\mathcal{Y}(\lambda) = \{(a, b) \in \mathbb{Z}_{\geq 1}^2 : 1 \leq a \leq h \text{ and } 1 \leq b \leq \lambda_a\}.$$

Note that we will consider that the  $a$ -coordinate in the Young diagrams goes downwards. A *node* is an element of  $\mathbb{Z}_{\geq 1}^2$ .

*Example 2.1.* The Young diagram of the partition  $\lambda = (4, 4, 1)$  is . The node in blue

has coordinates  $(1, 2)$ .

**Definition 2.2.** Let  $e \geq 2$ . A partition  $\lambda = (\lambda_1, \dots, \lambda_h)$  is  *$e$ -regular* if no parts repeat  $e$  times or more, that is, if  $\lambda_i > \lambda_{i+e-1}$  for any  $i \in \{1, \dots, h-e\}$ .

As we mentioned in the introduction, the notion of  $e$ -regular partitions appears for instance in the context of the representation theory of the *Iwahori–Hecke algebra*  $\mathcal{H}_q(\mathfrak{S}_n)$  of  $\mathfrak{S}_n$  (a certain deformation of  $\mathbb{C}\mathfrak{S}_n$ ), with  $q \in \mathbb{C}^\times$  having order  $e$ , see, for instance, [Ma] (note that in [Ma] the dual notion of  *$e$ -restricted* partition is used). The next two definitions are due to James [Ja].

**Definition 2.3.** Let  $e \geq 2$ .

- The  *$e$ -ladder number* (or simply *ladder number*) of a node  $\gamma = (a, b) \in \mathbb{Z}_{\geq 1}^2$  is:

$$\text{lad}_e(\gamma) := a + (e-1)(b-1) \in \mathbb{Z}_{\geq 1}.$$

- Let  $\ell \geq 1$ . The  *$(e, \ell)$ -th ladder* (or simply  *$\ell$ -th ladder*) is the (finite) set of all nodes of  $\mathbb{Z}_{\geq 1}^2$  with  $e$ -ladder  $\ell$ . The  *$e$ -ladder* of a node  $\gamma$  is the  $(e, \text{lad}_e(\gamma))$ -th ladder.

*Example 2.4.* In the Young diagram of  $\lambda = (4, 4, 3, 3, 3, 3, 1)$ , in each node we write the corresponding 4-ladder numbers:

1	4	7	10
2	5	8	11
3	6	9	
4	7	10	
5	8	11	
6	9	12	
7	10	13	
8			

**Definition 2.5.** Let  $e \geq 2$  and let  $\lambda$  be a partition. The  *$e$ -regularisation* of  $\lambda$  is the partition  $\text{reg}_e(\lambda)$  that we obtain after moving each node of  $\mathcal{Y}(\lambda)$  as high as possible in its  $e$ -ladder.

Note that  $\text{reg}_e(\lambda)$  is an  $e$ -regular partition, and if  $\lambda$  is  $e$ -regular then  $\text{reg}_e(\lambda) = \lambda$ . As we have mentioned in the introduction, the  $e$ -regularisation map has a significance in terms of modular representation theory of the symmetric group (or its associated Iwahori–Hecke algebra).

*Example 2.6.* The 4-regularisation of the partition of Example 2.4 is  $(5, 4, 4, 3, 3, 2, 2, 1)$ . The

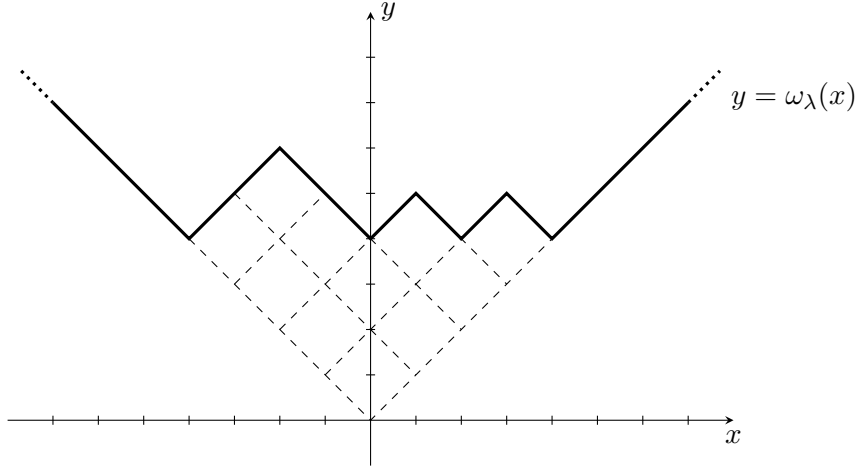


Figure 5: Russian convention for the Young diagram of  $\lambda = (4, 4, 2, 1)$

4-ladders of the added (respectively, deleted) nodes are in green (resp. red).

1	4	7	10	13
2	5	8	11	
3	6	9	12	
4	7	10		
5	8	11		
6	9	12		
7	10	13		
8				

## 2.2 Limit shape

**Russian convention** Rotating the Young diagram of  $\lambda = (\lambda_1 \geq \dots \geq \lambda_h > 0) \in \mathcal{P}_n$  by an angle of  $\frac{3\pi}{4}$  and embedding it inside  $\mathbb{R} \times \mathbb{R}_{\geq 0}$  so that the box  $(1, 1)$  has bottom vertex at  $(0, 0)$  and each box has area 2 (*i.e.* semi-diagonal length 1) gives the *Russian convention* for the Young diagram of  $\lambda$ . Note that the node  $(a, b) \in \mathcal{Y}(\lambda)$  corresponds to the (square) box with top vertex  $(a - b, a + b)$  in  $\mathbb{R} \times \mathbb{R}_{\geq 0}$ . We denote by  $\omega_\lambda : \mathbb{R} \rightarrow \mathbb{R}$  the upper rim of the resulting diagram, extending  $\omega_\lambda$  by  $\omega_\lambda(x) := |x|$  outside the diagram. Then  $\omega_\lambda$  is a continuous piecewise linear function such that:

- for each  $k \in \mathbb{Z}$  we have

$$\omega'_\lambda|_{(k, k+1)} = \pm 1, \quad (2.7)$$

- we have  $\omega_\lambda(x) = |x|$  for  $|x| \gg 0$  (more precisely, for  $x \leq -\lambda_1$  or  $x \geq h$ ),
- we have  $\int_{\mathbb{R}} [\omega_\lambda(x) - |x|] dx = 2n$ .

An illustration of the construction of  $\omega_\lambda$  is given in Figure 5. (We warn the reader that in the literature the convention is sometimes reversed, that is, our  $\omega_\lambda$  is sometimes reflected with respect to the axis  $\{0\} \times \mathbb{R}$ .) We will use a particular rescaling  $\tilde{\omega}_\lambda : \mathbb{R} \rightarrow \mathbb{R}$  of  $\omega_\lambda$ , given by  $\tilde{\omega}_\lambda(s) := \frac{1}{\sqrt{n}} \omega_\lambda(s\sqrt{n})$ . Note that the area between the graphs of  $\tilde{\omega}_\lambda$  and  $|\cdot|$  is 2.

**Plancherel measure** Let  $\lambda \in \mathcal{P}_n$ . A *standard tableau of shape  $\lambda$*  is a bijection  $\mathfrak{t} : \mathcal{Y}(\lambda) \rightarrow \{1, \dots, n\}$  such that  $\mathfrak{t}$  increases along the rows and down the columns, in other words for  $(a, b) \in \mathcal{Y}(\lambda)$  we have  $\mathfrak{t}(a, b) < \mathfrak{t}(a + 1, b)$  if  $(a + 1, b) \in \mathcal{Y}(\lambda)$  and  $\mathfrak{t}(a, b) < \mathfrak{t}(a, b + 1)$



if  $(a, b + 1) \in \mathcal{Y}(\lambda)$ . We denote by  $\text{Std}(\lambda)$  the set of standard Young tableaux of shape  $\lambda$ . We have the following standard identity:

$$n! = \sum_{\lambda \in \mathcal{P}_n} \#\text{Std}(\lambda)^2.$$

**Definition 2.8.** The *Plancherel measure* on the set  $\mathcal{P}_n$  of the partitions of  $n$  is given by

$$\text{Pl}_n(\lambda) := \frac{\#\text{Std}(\lambda)^2}{n!},$$

for all  $\lambda \in \mathcal{P}_n$ .

The next result describes the Plancherel measure  $\text{Pl}_n$  for large  $n$ .

**Theorem 2.9** ([LoSh], [KeVe], [Rom, Theorem 1.26]). *Let  $\Omega : \mathbb{R} \rightarrow \mathbb{R}$  be defined by*

$$\Omega(s) := \begin{cases} \frac{2}{\pi} \left( s \arcsin\left(\frac{s}{2}\right) + \sqrt{4 - s^2} \right), & \text{if } |s| \leq 2, \\ |s|, & \text{otherwise.} \end{cases}$$

*Then, under the Plancherel measure  $\text{Pl}_n$ , the function  $\tilde{\omega}_\lambda$  converges uniformly in probability to  $\Omega$  as  $n \rightarrow +\infty$ . In other words, for any  $\epsilon > 0$  we have*

$$\text{Pl}_n \left( \sup_{\mathbb{R}} |\tilde{\omega}_\lambda - \Omega| > \epsilon \right) \xrightarrow{n \rightarrow +\infty} 0.$$

*Moreover, we also have convergence of the supports, that is:*

$$\inf \{s \in \mathbb{R} : \tilde{\omega}_\lambda(s) \neq |s|\} \longrightarrow -2,$$

*and*

$$\sup \{s \in \mathbb{R} : \tilde{\omega}_\lambda(s) \neq |s|\} \longrightarrow 2,$$

*in probability under  $\text{Pl}_n$  as  $n \rightarrow +\infty$ .*

An illustration of Theorem 2.9 for  $n = 1000$  is given in Figure 1. Note that the limit shape  $\Omega$  has a much simpler form after derivation.

**Lemma 2.10.** *The map  $\Omega$  is an antiderivative on  $\mathbb{R}$  of:*

$$s \mapsto \begin{cases} \arcsin\left(\frac{s}{2}\right), & \text{if } |s| \leq 2, \\ \text{sgn}(s), & \text{if } |s| > 2. \end{cases}$$

*In particular, the minimum of  $\Omega$  on  $\mathbb{R}$  is  $\Omega(0) = \frac{4}{\pi}$ .*

### 3 Shakings

The aim of this paper is to put together the notions of §2.1 and §2.2. As a first step, we show in Figure 6 what do the 4-ladders of Example 2.4 look like in the Russian convention.

**Proposition 3.1.** *In the Russian convention, the  $e$ -regularisation makes the nodes going as left as possible in the direction of  $y = (1 - 2e^{-1})x$ .*

*Proof.* In the Young diagram we know that the node  $(e, 1) \in \mathbb{Z}_{\geq 1}^2$  goes to  $(1, 2)$ . In the Russian convention it means that the box with top vertex at  $(e - 1, e + 1)$  goes to the box with top vertex  $(-1, 3)$ . The corresponding slope is thus  $\frac{e+1-3}{e-1-(-1)} = \frac{e-2}{e} = 1 - 2e^{-1}$  as announced (note that the slope do not change when both axes are rescaled by a same constant).  $\square$

We now define the shaking operation that will be of interest for us. As we mentioned in the introduction, this is a variation of Steiner symmetrisation first introduced by Blaschke [Bl].

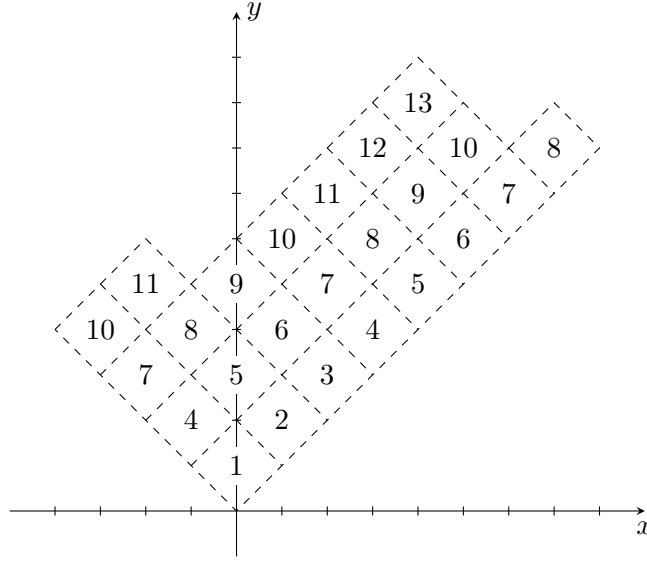


Figure 6: 4-ladders in the Russian convention for  $\lambda = (4, 4, 3, 3, 3, 3, 3, 1)$

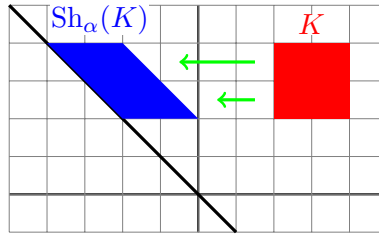


Figure 7: Example of the map  $\text{Sh}_\alpha$  with  $\alpha = 0$

**Definition 3.2.** Let  $\alpha \in \mathbb{R}_{\geq 0}$  and let  $v_\alpha$  be the unit vector positively collinear to  $(1, \alpha)^\top$ . If  $K$  is a compact subset of  $\mathbb{R}^2$ , its *shaking* with direction  $\alpha$  (against  $y = -x$ ) is the compact subset  $\text{Sh}_\alpha(K)$  of  $\mathbb{R}^2$  that we obtain by sliding the set  $K$  along the direction of  $(L_\alpha) : y = \alpha x$  until we meet the line  $(L) : y = -x$ , that is:

$$\text{Sh}_\alpha(K) := \bigsqcup_{x \in L} K_v,$$

where  $K_v$  is the segment with extreme points  $x$  and  $x + |K \cap (x + L_\alpha)|v_\alpha$  with  $|\cdot|$  denoting the one-dimensional Lebesgue measure.

In the usual definition of shaking the direction  $L_\alpha$  is orthogonal to the the line  $L$  against which we shake. This non-orthogonal variation is for instance considered in [FrLu, §5].

*Example 3.3.* Take  $\alpha = 0$  and let  $K$  be the unit square with bottom left corner at  $(1, 1)$ . Then, as shown on Figure 7, the set  $\text{Sh}_\alpha(K)$  is the parallelogram with vertices  $(-2, 2)$ ,  $(-1, 2)$ ,  $(0, 1)$  and  $(-1, 1)$ .

The following property is a standard property of Steiner symmetrisations and shakings. The proof is immediate from the definition.

**Proposition 3.4.** *If  $K \subseteq K'$  are compact subsets of  $\mathbb{R}^2$  then  $\text{Sh}_\alpha(K) \subseteq \text{Sh}_\alpha(K')$ .*

*Proof.* See [FrLu, Lemma 5.3]. Note that the proof is in fact the same as for the usual (orthogonal) shakings.  $\square$

Recalling Proposition 3.1, we are interested in special cases of shakings.

**Definition 3.5.** For any  $e \geq 2$ , we denote by  $\text{sh}_e$  the shaking  $\text{Sh}_\alpha$  with direction  $\alpha_e := 1 - 2e^{-1}$ .

## 4 Shaking the limit shape

Let  $a \in \mathbb{R}_{>0}$ . We define  $\mathfrak{Y}_a$  to be the set of all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  of class  $\mathcal{C}^1$  such that:

$$f \text{ is even,} \quad (4.1a)$$

$$\text{for any } |s| \geq a \text{ we have } f(s) = |s|, \quad (4.1b)$$

$$f \text{ is strictly convex on } [-a, a]. \quad (4.1c)$$

Note that by Lemma 2.10 we have  $\Omega \in \mathfrak{Y}_2$ .

*Example 4.2.* Let  $\Sigma : \mathbb{R} \rightarrow \mathbb{R}$  be given by  $\Sigma(x) = |x|$  if  $|x| \geq 1$  and  $\Sigma(x) = \frac{1}{2}(x^2 + 1)$  otherwise. Then  $\Sigma \in \mathfrak{Y}_1$ .

Note that the last two equations in (4.1) imply that for all  $|s| < a$  we have:

$$f(s) > |s|, \quad (4.3a)$$

since  $f$  is above its tangent at  $s = a$ , and:

$$|f'(s)| < 1, \quad (4.3b)$$

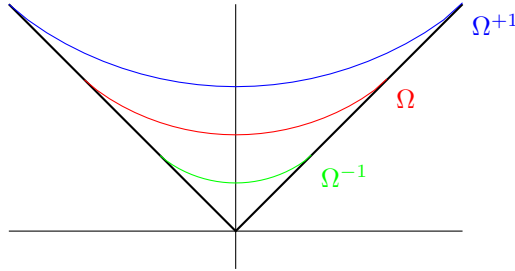
for all  $s \in (-a, a)$ , since  $f'$  is strictly increasing on  $[-a, a]$ .

Before studying the shaking operation on the elements of  $\mathfrak{Y}_a$ , we first define a map between some sets  $\mathfrak{Y}_a$  via double scaling.

**Definition 4.4.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be any map and let  $\epsilon \in (0, 1)$ . We define the two maps  $f^{\pm\epsilon} : \mathbb{R} \rightarrow \mathbb{R}$  by, for any  $s \in \mathbb{R}$ ,

$$f^{\pm\epsilon}(s) := (1 \pm \frac{\epsilon}{2}) f\left(\frac{s}{1 \pm \frac{\epsilon}{2}}\right).$$

For instance, the maps  $\Omega^{\pm 1}$  are as follows:



We now gather some informations about  $f^{\pm\epsilon}$ .

**Lemma 4.5.** Let  $f \in \mathfrak{Y}_2$  and  $\epsilon \in (0, 1)$ .

- (i) We have  $f^{\pm\epsilon} \in \mathfrak{Y}_{2 \pm \epsilon}$ .
- (ii) For any  $s \in \mathbb{R}$  we have  $(f^{\pm\epsilon})'(s) = f'\left(\frac{s}{1 \pm \frac{\epsilon}{2}}\right)$ .
- (iii) The maps  $g^{+\epsilon} := f^{+\epsilon} - f$  and  $g^{-\epsilon} := f - f^{-\epsilon}$  are decreasing on  $\mathbb{R}_{\geq 0}$ .
- (iv) For any  $s \in \mathbb{R}$  we have  $f^{-\epsilon}(s) \leq f(s) \leq f^{+\epsilon}(s)$ , and:
  - (a) we have  $f^{-\epsilon}(s) = f(s)$  if and only if  $|s| \geq 2$ ,
  - (b) we have  $f^{+\epsilon}(s) = f(s)$  if and only if  $|s| \geq 2 + \epsilon$ ,
- (v) We have  $\|f - f^{\pm\epsilon}\|_\infty = \frac{\epsilon}{2} f(0)$ .

*Proof.* (i) Clear.

(ii) Clear.

(iii) Follows from the previous point and from the fact that  $f'$  is increasing.

(iv) Since the functions are even, it suffices to prove the inequality on  $\mathbb{R}_{\geq 0}$ . We prove only the relations for  $f^{+\epsilon}$ , the ones for  $f^{-\epsilon}$  being similar. By the previous point, we know that  $g^{+\epsilon}$  is decreasing on  $\mathbb{R}_{\geq 0}$ , thus we deduce that  $g \geq 0$  since  $g(s) = |s| - |s| = 0$  for  $s \gg 0$ . For the equality case, first note that the necessary condition holds. For the sufficient condition, since  $f$  is strictly convex on  $[-2, 2]$  we have in fact  $g'(s) < 0$  for all  $s \in (0, 2]$  thus as before we have  $g(s) > 0$  for all  $s \in [0, 2]$ . Now for  $s \in [2, 2 + 2\epsilon)$  we have  $g'(s) = f'(\frac{s}{1+\epsilon}) - 1 < 0$  since  $1 = f'(2)$  and  $f'$  is strictly increasing on  $[0, 2]$  and we conclude the proof as before.

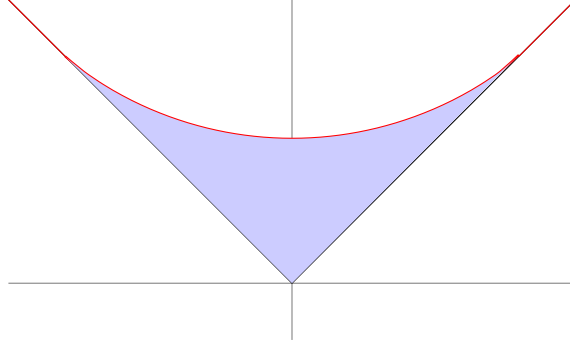
(v) Again we only prove the result for  $f^{+\epsilon}$ . With the previous notation, we know that  $g = f^{+\epsilon} - f$  is non-negative and decreases on  $\mathbb{R}_{\geq 0}$  thus its maximum is reached at  $s = 0$ . □

We need one more definition before giving the main result of this section.

**Definition 4.6.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a map. We denote by  $\mathbb{Y}(f)$  the part of the graph of  $f$  that is above the graph of the absolute value, that is:

$$\mathbb{Y}(f) := \{(x, y) \in \mathbb{R}^2 : |x| \leq y \leq f(x)\}.$$

For instance, we picture here in blue the set  $\mathbb{Y}(\Omega)$ :



**Theorem 4.7.** Let  $\alpha \in [0, 1)$  and  $a \in \mathbb{R}_{>0}$ . For any  $f \in \mathfrak{Y}_a$  we have:

$$\text{Sh}_\alpha(\mathbb{Y}(f)) \cup \mathbb{Y}(\text{id}_{\mathbb{R}}) = \mathbb{Y}(f_\alpha),$$

where  $f_\alpha : \mathbb{R} \rightarrow \mathbb{R}$  is given by:

$$\begin{aligned} f_\alpha(x) &= f(x), & \text{for all } x \leq x_\alpha^-, \\ f(x + \delta_x) &= f(x) + \alpha\delta_x, & \text{for all } x \in (x_\alpha^-, x_\alpha^+), \\ f_\alpha(x) &= x, & \text{for all } x \geq s_\alpha := x_\alpha^+ + \delta_{x_\alpha^+}, \end{aligned}$$

where:

- $x_\alpha^+ = f'^{-1}(\alpha) \in [0, a)$ ,
- $\delta_x = (1 - \alpha)^{-1}F_\alpha(x) - F_\alpha^{-1}(F_\alpha(x))$ , where  $F_\alpha : x \mapsto f(x) - \alpha x$  and  $F_\alpha^{-1}$  denotes its inverse on  $(x_\alpha^+, a)$ ;
- $x_\alpha^- \in (-a, x_\alpha^+)$  (is the unique point that) satisfies  $F_\alpha(x_\alpha^-) = (1 - \alpha)a$ .

Note that  $\mathbb{Y}(\text{id}_{\mathbb{R}}) = \{(x, x) : x \in \mathbb{R}_{\geq 0}\}$ .

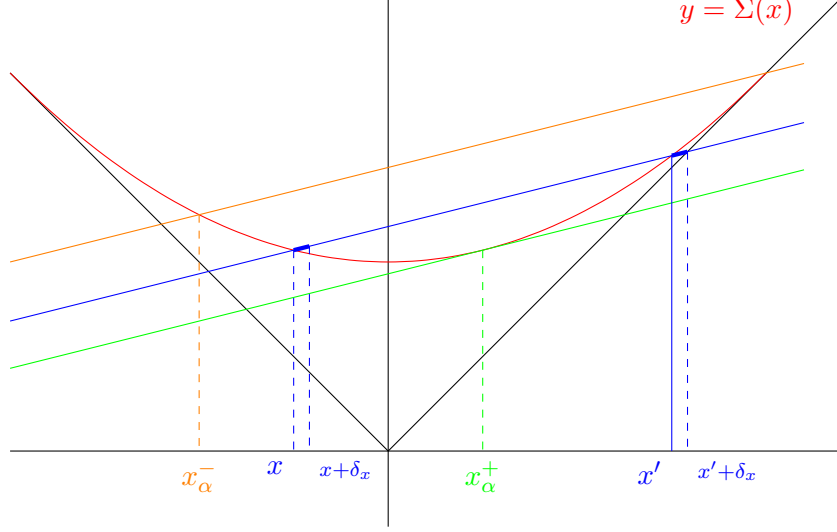


Figure 8: Illustration of the proof of Theorem 4.7 with  $f = \Sigma$  and  $\alpha = \frac{1}{4}$

*Proof.* As in Definition 3.2, let  $L_\alpha$  be the line of equation  $y = \alpha x$ . For  $M \in \mathbb{R}^2$ , the line  $L_\alpha^M := M + L_\alpha$  cuts the curve of  $f$  at most twice since  $f$  is strictly convex. Note that we can ignore the parts of  $\mathbb{Y}(f)$  outside  $(-a, a)$  since it has measure 0 there. Note also that there is only one intersection point if and only if  $\mathbb{Y}(f) \cap L_\alpha^M$  is stable under  $\text{Sh}_\alpha$ . Moreover, it suffices to choose  $M$  of the form  $(0, y_M)$ , and if  $y_{M_0}$  is the unique real number such that  $L_\alpha^{M_0}$  contains  $(a, a)$  then:

for any  $y \leq y_{M_0}$  then  $L_\alpha^M$  intersects the curve of  $f$  in  $(-a, a)$  only once.

Let  $x_\alpha^- < a$  be such that  $L_\alpha^{M_0}$  intersects the curve of  $f$  at  $(x_\alpha^-, f(x_\alpha^-))$ . This means that:

$$f_\alpha(x) = f(x), \quad \text{for all } x \leq x_\alpha^-.$$

Note that  $x_\alpha^- \geq -a$  since  $f(-a) = a$  and  $\alpha \geq 0$ . Now there is also a real number  $y_{M_1}$  such that  $L_\alpha^{M_1}$  does not intersect the curve of  $f$  apart from  $(y_{M_1}, f(y_{M_1}))$ . Again by convexity, this  $y_{M_1}$  is given by the unique point such that  $L_\alpha^{M_1}$  is tangent to the curve of  $f$  at some point  $M_\alpha = (x_\alpha^+, f(x_\alpha^+))$ . By (4.1) we know that such an  $M_\alpha$  is unique and is determined by  $f'(x_\alpha^+) = \alpha$ , in other words, by:

$$x_\alpha^+ = f'^{-1}(\alpha)$$

(where  $f'^{-1}$  denotes the inverse of  $f'|_{(-a, a)}$ ). Note that  $x_\alpha^+ \geq 0$  since  $\alpha \geq 0$  and  $f$  is even, thus  $f'(0) = 0 \leq \alpha$ , and  $f'$  is increasing by strict convexity. Similarly, we have  $x_\alpha^+ < a$  since  $f'(a) = 1 > \alpha$ , moreover the line between  $(x_\alpha^+, f(x_\alpha^+))$  and  $(a, a)$  has slope  $> \alpha$  thus  $x_\alpha^+ > x_\alpha^-$ .

Hence, to determine  $f_\alpha$ , it suffices to determine for each  $x \in (x_\alpha^-, x_\alpha^+)$  the measure  $m$  of  $(M + L_\alpha) \cap (\mathbb{Y}(f) \cap ((x, a) \times \mathbb{R}_{\geq 0}))$ , with  $M = (x, f(x))$ , and to add it to  $f(x)$  in the direction  $\alpha$ . More precisely, if:

- $x' > x$  is the unique other abscissa where  $L_\alpha^x := M + L_\alpha$  crosses the curve of  $f$ ,
- $\delta_x > 0$  is the unique positive real number such that  $x' + \delta_x$  is the abscissa of the intersection between  $L_\alpha^x$  and the line with equation  $y = x$  for  $x \geq 0$ ,

then the point  $(x, f(x))$  becomes:

$$(x + \delta_x, f(x) + x' + \delta_x - f(x')). \quad (4.8)$$

In Figure 8 we give an example of the various quantities that we have introduced so far for  $f$  being the map  $\Sigma$  of Example 4.2.

We will now determine  $x'$  and  $\delta_x$  as functions of  $x$ . We first determine  $x'$ . The line  $L_\alpha^x$  has equation  $(s - x)\alpha + f(x)$  for  $s \in \mathbb{R}$ , thus  $x'$  is given by:

$$f(x') = (x' - x)\alpha + f(x),$$

thus:

$$f(x') - \alpha x' = f(x) - \alpha x. \quad (4.9)$$

Now the function  $F_\alpha : s \mapsto f(s) - \alpha s$  has derivative  $f'(s) - \alpha$ , thus since  $\alpha \in (0, 1)$  we know that  $s \mapsto f(s) - \alpha s$  is decreasing for  $s \leq x_\alpha^+$  and increases (to  $+\infty$  by (4.1b)) by for  $s \geq x_\alpha^+$ . Hence, we deduce that there is indeed a unique  $x' > x_\alpha^+$  such that (4.9) is satisfied. Note that  $F_\alpha$  is invertible on  $(x_\alpha^+, +\infty)$  and if  $F_\alpha^{-1}$  denotes its inverse on this interval then:

$$x' = F_\alpha^{-1}(F_\alpha(x)).$$

Note that (4.9) writes:

$$F_\alpha(x') = F_\alpha(x).$$

Note also that  $x < x_\alpha^+ < x'$  so that  $x' \neq x$ . Finally, note that  $x_\alpha^-$  satisfies:

$$F_\alpha(x_\alpha^-) = F_\alpha(a) = f(a) - \alpha a = (1 - \alpha)a,$$

as announced.

We now determine  $\delta_x$ . First note that the equation of  $L_\alpha^x$  becomes  $\alpha s + F_\alpha(x')$ . We thus find that  $\delta_x$  satisfies

$$x' + \delta_x = \alpha(x' + \delta_x) + F_\alpha(x'), \quad (4.10)$$

$$= f(x') + \alpha\delta_x, \quad (4.11)$$

in particular (4.10) gives:

$$x' + \delta_x = (1 - \alpha)^{-1}F_\alpha(x'). \quad (4.12)$$

Recalling (4.9) and (4.11), the second coordinate in equation (4.8) becomes:

$$f(x) + x' + \delta_x - f(x') = f(x) + \alpha\delta_x,$$

as announced.

To prove that the set of points that we obtain is the curve of a function, it suffices to prove that  $x \mapsto x + \delta_x$  is increasing on  $(x_\alpha^-, x_\alpha^+)$ , thus it suffices to prove that  $x \mapsto \delta_x$  is increasing. By (4.12) we have:

$$\delta_x = (1 - \alpha)^{-1}F_\alpha(x') - x'.$$

Note that  $x \mapsto x' = F_\alpha^{-1}(F_\alpha(x))$  is decreasing on  $(x_\alpha^-, x_\alpha^+)$ . Indeed, we have  $F_\alpha'(x) = f'(x) - \alpha$  that is increasing since  $f$  is strictly convex, and by definition  $f'(x_\alpha^+) = 0$  thus  $F_\alpha'(x) < 0$  for  $x < x_\alpha^+$  thus  $F_\alpha$  is decreasing on  $(x_\alpha^-, x_\alpha^+)$ . Now  $F_\alpha$  is increasing on  $(x_\alpha^+, +\infty)$  thus  $F_\alpha^{-1}$  is also increasing. By composition, we deduce that  $x \mapsto x' = F_\alpha^{-1}(F_\alpha(x))$  is decreasing on  $(x_\alpha^-, x_\alpha^+)$ , with values in  $(x_\alpha^+, a)$ .

Hence, to prove that  $x \mapsto \delta_x = (1 - \alpha)^{-1}F_\alpha(x') - x'$  is increasing on  $(x_\alpha^-, x_\alpha^+)$ , it suffices to prove that  $g : t \mapsto (1 - \alpha)^{-1}F_\alpha(t) - t$  is decreasing in  $t \in (x_\alpha^+, a)$ . But this is clear since the derivative of  $g$  is, after multiplication by  $1 - \alpha$ ,

$$\begin{aligned} F_\alpha'(t) - (1 - \alpha) &= (f'(t) - \alpha) - (1 - \alpha) \\ &= f'(t) - 1, \end{aligned}$$

which is strictly negative since  $f'$  is increasing (by strict convexity) and  $f'(t) < 1$  for all  $t < a$  by (4.1b).  $\square$

In Figure 4 we represent the curve of  $f_\alpha$  for  $f = \Omega$  and  $\alpha = 0$ .

**Proposition 4.13.** *Let  $\alpha \in [0, 1)$  and  $a \in \mathbb{R}_{>0}$ . For any  $f \in \mathfrak{Y}_a$  we have:*

$$(1 - \alpha)s_\alpha = f(f'^{-1}(\alpha)) - \alpha f'^{-1}(\alpha) \in [f(0) - \alpha f(x_\alpha^+), f(0)].$$

*Proof.* Recall that from Theorem 4.7 and its proof that  $s_\alpha = x + \delta_x$  for  $x = x_\alpha^+$ . In this setting we have  $x' = x$  thus (4.12) gives:

$$\begin{aligned} (1 - \alpha)s_\alpha &= F_\alpha(x_\alpha^+) \\ &= f(x_\alpha^+) - \alpha x_\alpha^+ \\ &= f(f'^{-1}(\alpha)) - \alpha f'^{-1}(\alpha), \end{aligned}$$

which gives the desired equality. Now by the mean value theorem, there exists  $c \in (0, x_\alpha^+)$  such that  $F_\alpha(x_\alpha^+) = F_\alpha(0) + F'_\alpha(c)x_\alpha^+$ . Hence, since  $f$  is strictly convex we know that  $F_\alpha$  is also strictly convex, thus we have  $F'_\alpha(0) = f'(0) - \alpha = -\alpha < F'_\alpha(c) < F'_\alpha(x_\alpha) = 0$  (where  $f'(0) = 0$  since  $f$  is even). Recalling that  $F_\alpha(0) = f(0)$  gives the announced inequality.  $\square$

We have the following particular case of Theorem 4.7 and Proposition 4.13 for  $\alpha = 0$  (as pictured in Figure 4).

**Corollary 4.14.** *Let  $a \in \mathbb{R}_{>0}$ . For any  $f \in \mathfrak{Y}_a$ , the map  $f_0 : \mathbb{R} \rightarrow \mathbb{R}$  is given by:*

$$\begin{aligned} f_0(x) &= f(x), & \text{for all } x \leq -a, \\ f_0(2x + f(x)) &= f(x), & \text{for all } x \in (-a, 0), \\ f_0(x) &= x, & \text{for all } x \geq f(0). \end{aligned}$$

We conclude this part by relating the shapes  $f_\alpha$  and  $f_\alpha^{\pm\epsilon} := (f^{\pm\epsilon})_\alpha$ .

**Proposition 4.15.** *Let  $\alpha \in [0, 1)$  and  $a \in \mathbb{R}_{>0}$ . Let  $f \in \mathfrak{Y}_a$  and  $\epsilon \in (0, 1)$ . With the notation of Theorem 4.7 we have:*

$$(f^{\pm\epsilon})_\alpha = (f_\alpha)^{\pm\epsilon} =: f_\alpha^{\pm\epsilon}.$$

*Proof.* We denote by  $\delta_x^{\pm\epsilon}$  and  $F_\alpha^{\pm\epsilon}$  the quantities that appear in Theorem 4.7 for  $f^{\pm\epsilon}$ . Note that:

$$\begin{aligned} F_\alpha^{\pm\epsilon}(x) &= f^{\pm\epsilon}(x) - \alpha x \\ &= (1 \pm \frac{\epsilon}{2})f\left(\frac{x}{1 \pm \frac{\epsilon}{2}}\right) - (1 \pm \frac{\epsilon}{2})\alpha \frac{x}{1 \pm \frac{\epsilon}{2}} \\ &= (1 \pm \frac{\epsilon}{2})F_\alpha\left(\frac{x}{1 \pm \frac{\epsilon}{2}}\right) \\ &= (F_\alpha)^{\pm\epsilon}(x), \end{aligned}$$

and we deduce that  $(F_\alpha^{\pm\epsilon})^{-1} = (F_\alpha^{-1})^{\pm\epsilon}$ . We deduce that:

$$\delta_x^{\pm\epsilon} = (1 \pm \frac{\epsilon}{2})\delta_{\frac{x}{1 \pm \frac{\epsilon}{2}}}. \quad (4.16)$$

Now note that the equality  $f(x + \delta_x) = f(x) + \alpha\delta_x$  holds in fact for any  $x \in \mathbb{R}$ . If  $g$  denotes the inverse of  $x \mapsto x + \delta_x$  on  $\mathbb{R}$  (which exists since  $x \mapsto x + \delta_x$  is strictly increasing by the proof of Theorem 4.7), if  $y = x + \delta_x$  we thus have  $x = g(y)$  and thus  $\delta_x = y - g(y)$ . We deduce that:

$$f_\alpha(y) = f(g(y)) - \alpha(y - g(y)) = F_\alpha(g(y)) - \alpha y. \quad (4.17)$$

Now we deduce from (4.16) that the inverse of  $x \mapsto x + \delta_x^{\pm\epsilon}$  is  $g^{\pm\epsilon}$  (in the sense of Defini-

tion 4.4), thus by (4.17) we obtain:

$$\begin{aligned}
f_\alpha^{\pm\epsilon}(y) &= F_\alpha^{\pm\epsilon}(g^{\pm\epsilon}(y)) - \alpha y \\
&= (1 \pm \frac{\epsilon}{2})F_\alpha\left(g\left(\frac{y}{1 \pm \frac{\epsilon}{2}}\right)\right) - \alpha y \\
&= (1 \pm \frac{\epsilon}{2})\left[F_\alpha\left(g\left(\frac{x}{1 \pm \frac{\epsilon}{2}}\right)\right) - \alpha \frac{y}{1 \pm \frac{\epsilon}{2}}\right] \\
&= (1 \pm \frac{\epsilon}{2})f_\alpha\left(\frac{y}{1 \pm \frac{\epsilon}{2}}\right) \\
&= (f_\alpha)^{\pm\epsilon}(y),
\end{aligned}$$

which is the announced result.  $\square$

**Proposition 4.18.** *Let  $\alpha \in [0, 1)$  and  $a \in \mathbb{R}_{>0}$ . Let  $f \in \mathfrak{Y}_a$ . We have  $0 < \|f_\alpha - f_\alpha^{\pm\epsilon}\|_{\infty, \mathbb{R}} \rightarrow 0$  as  $\epsilon \rightarrow 0$ .*

*Proof.* First, note that the infinite norm is positive indeed since  $f_\alpha$  and  $f_\alpha^{\pm\epsilon}$  are continuous (even of class  $\mathcal{C}^1$ ) by (4.17), by Lemma 4.5 and since the shaking preserve the area (this is a simple consequence of Fubini theorem, see, for instance, [FrLu, Lemma 3.2]). To prove that it goes to 0 as  $\epsilon \rightarrow 0$ , it suffices to prove that  $f_\alpha$  is Lipschitz. Indeed, if  $h : I \rightarrow \mathbb{R}$  is  $L$ -Lipschitz and bounded then:

$$\begin{aligned}
h^{\pm\epsilon}(x) - h(x) &= (1 \pm \frac{\epsilon}{2})h\left(\frac{x}{1 \pm \frac{\epsilon}{2}}\right) - h(x) \\
&= h\left(\frac{x}{1 \pm \frac{\epsilon}{2}}\right) - h(x) + \frac{\epsilon}{2}h\left(\frac{x}{1 \pm \frac{\epsilon}{2}}\right),
\end{aligned}$$

thus:

$$\begin{aligned}
|h^{+\epsilon}(x) - h(x)| &\leq L|x|\left|\frac{1}{1 + \frac{\epsilon}{2}} - 1\right| + \frac{\epsilon}{2}\|h\|_\infty \\
&\leq (L|x| + \|h\|_\infty)\epsilon,
\end{aligned}$$

with a similar calculation for  $h^{-\epsilon}$ . Note that it suffices to study  $f - f_\alpha$  on a segment since this function vanishes for  $|x| \gg 0$ . Recalling (4.17), we have:

$$\begin{aligned}
f'_\alpha(y) &= g'(y)F'_\alpha(g(y)) - \alpha \\
&= g'(y)\left[f'(g(y)) - \alpha\right] - \alpha.
\end{aligned}$$

We know that  $f'$  is bounded by (4.1). For  $g'$ , recall from the proof of Theorem 4.7 that  $x \mapsto \delta_x$  is increasing thus the derivative of  $x \mapsto x + \delta_x$  is greater than 1, thus  $\|g'\|_\infty \leq 1$ . This concludes the proof.  $\square$

## 5 Shaking partitions

Let  $\lambda$  be a partition. Proposition 3.1 implies the following result.

**Lemma 5.1.** *We have  $\text{sh}_e(\mathbb{Y}(\omega_\lambda)) = \text{sh}_e(\mathbb{Y}(\omega_{\text{reg}_e(\lambda)}))$  and  $\text{sh}_e(\mathbb{Y}(\tilde{\omega}_\lambda)) = \text{sh}_e(\mathbb{Y}(\tilde{\omega}_{\text{reg}_e(\lambda)}))$ .*

Hence, to study the shaking operations on partitions it suffices to study the shaking operations on regular partitions. In fact, the shaking of  $\mathbb{Y}(\tilde{\omega}_{\text{reg}_e(\lambda)})$  for  $\lambda$  an  $e$ -regular partition is a bit delicate to determine. Instead, we will bound the latter graph by two close graphs that are stable under the shaking operation.



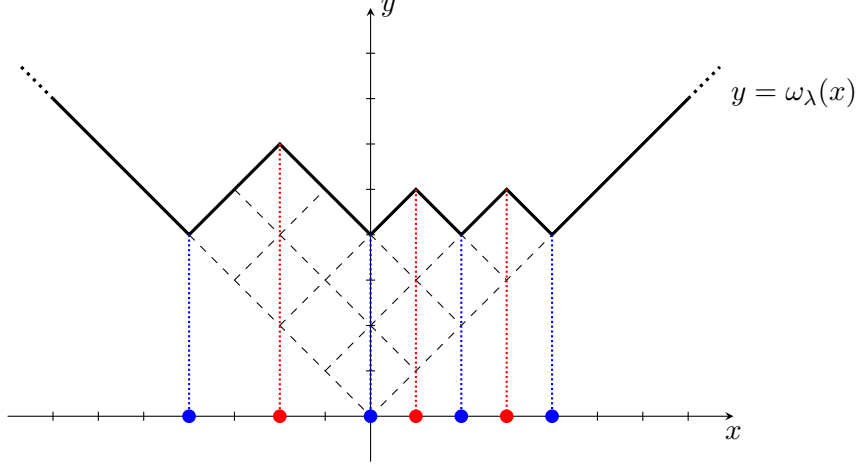


Figure 9: Outer corners (in red) and inner corners (in blue) for  $\lambda = (4, 4, 2, 1)$

## 5.1 Corners

We say that  $c \in \mathbb{Z}$  is an *outer corner* (resp. *inner corner*) of  $\lambda$  if  $\omega'_\lambda|_{(c-1,c)} = 1$  (resp.  $-1$ ) and  $\omega'_\lambda|_{(c,c+1)} = -1$  (resp.  $1$ ). We give in Figure 9 an example of outer and inner corners.

The following standard result follows from the definition of inner and outer corners and from the fact that  $\omega_\lambda(x) = |x|$  for  $x \gg 0$ .

**Lemma 5.2.** *Let  $\lambda$  be a partition. If  $\{o_1 < \dots < o_k\}$  (resp.  $\{i_1 < \dots < i_\ell\}$ ) is the set of outer (resp. inner) corners then  $\ell = k + 1$  and  $i_m < o_m < i_{m+1}$  for all  $m \in \{1, \dots, k\}$ .*

In the sequel, fix  $e \geq 2$  and let  $L_e$  be the linear subspace of  $\mathbb{R}^2$  with slope  $\alpha_e = 1 - 2e^{-1}$ .

## 5.2 Outer regularisation

If  $\lambda$  has no outer corners then we define  $\rho^+(\lambda) = \rho_0^+(\lambda) := \omega_\lambda$ , otherwise let  $\{c_1 < \dots < c_r\}$  with  $r \geq 1$  the set of outer corners. Assuming by induction on  $k \in \{1, \dots, r\}$  that we have constructed the piecewise linear function  $\rho_{k-1}^+(\lambda) : \mathbb{R} \rightarrow \mathbb{R}$ , then we construct the piecewise linear function  $\rho_k^+(\lambda) : \mathbb{R} \rightarrow \mathbb{R}$  as follows. Let  $L_e^{(k)}$  be the affine line  $(c_k, \omega_\lambda(c_k)) + L_e$ .

- (i) For  $s \geq c_k$  then  $\rho_k^+(\lambda)(s) := \rho_{k-1}^+(\lambda)(s)$ ;
- (ii) at  $s = c_k$ , we follow the line  $L_e^{(k)}$  (in the negative direction) until we meet the curve of  $\rho_{k-1}^+(\lambda)$ , at point of abscissa  $h_k$ ;
- (iii) for  $s \leq h_k$  then  $\rho_k^+(\lambda)(s) := \rho_{k-1}^+(\lambda)(s)$  again.

**Proposition 5.3.** *Assume that the partition  $\lambda$  is  $e$ -regular. With the above notation we have  $h_k \in [c_{k-1}, c_k)$ . In particular, the functions  $\rho_{k-1}^+(\lambda)$  and  $\rho_k^+(\lambda)$  coincide on (at least)  $\mathbb{R} \setminus (c_{k-1}, c_k)$ , and for any  $s \in (c_{k-1}, c_k)$  we have  $0 \leq \rho_k^+(\lambda)(s) - \rho_{k-1}^+(\lambda)(s) \leq e$ .*

*Proof.* Since  $\alpha_e < 1$ , the curve of  $\omega_\lambda$  can cross  $L_e^{(k)}$  only after an inner corner, that is, if  $c' < c_k$  is the inner corner preceding  $c_k$  (cf. Lemma 5.2) then  $h < c'$ . Moreover, by definition of an inner corner we have  $\omega_\lambda(c' - 1) = \omega_\lambda(c') + 1$ , thus the line joining  $(c_k, \omega_\lambda(c_k))$  and  $(c' - 1, \omega_\lambda(c' - 1))$  has slope:

$$\alpha := \frac{\omega_\lambda(c_k) - \omega_\lambda(c') - 1}{c_k - c' + 1}.$$

Since  $\omega_\lambda(c') = \omega_\lambda(c_k) - c_k + c'$ , we obtain:

$$\alpha = \frac{c_k - c' - 1}{c_k - c' + 1} = 1 - \frac{2}{c_k - c' + 1}.$$

But now  $\lambda$  is  $e$ -regular thus  $c' \geq c_k - e + 1$ . We thus have  $c_k - c' + 1 \leq e$  and thus, since  $c_k - c' + 1 > 0$  (since  $c' \leq c_k$ ):

$$\alpha \leq 1 - 2e^{-1} = \alpha_e.$$

Finally, we have  $\alpha \leq \alpha_e < 1$  thus  $c' - 1 \leq h_k < c_k$ , in other words, the line  $L_e^{(k)}$  meet the curve of  $\omega_\lambda$  somewhere between the points of abscissa  $c'$  (excluded) and  $c' - 1$  (included) thus this concludes the first part of the proof.

Now for any  $s \in (c_{k-1}, h)$  we have  $\rho_k^+(\lambda)(s) = \rho_k^+(\lambda)(s)$ , and by construction for any  $s \in (h, c_k)$  we have  $\rho_k^+(\lambda)(s) > \rho_k^+(\lambda)$  thus we obtain the first member of the announced inequality. The second one follows from the fact that  $\alpha_e \in [0, 1]$  and from the fact that on  $(c_{k-1}, c_k)$ , the function  $s \mapsto \rho_k^+(\lambda)(s) - \rho_k^+(\lambda)(s)$  reaches its minimum at  $s = h_k$ , where:

$$\omega_\lambda(c_k) \geq \rho_k^+(\lambda)(h_k) \geq \rho_{k-1}^+(\lambda)(h_k) \geq \omega_\lambda(c').$$

Now we already saw that:

$$\begin{cases} \omega_\lambda(c') = \omega_\lambda(c_k) - c_k + c', \\ c_k - c' + 1 \leq e, \end{cases}$$

thus we obtain:

$$\omega_\lambda(c_k) - \omega_\lambda(c') = c_k - c' \leq e - 1,$$

whence the result.  $\square$

**Definition 5.4.** Let  $e \geq 2$  and let  $\lambda$  be an  $e$ -regular partition. Let  $r \geq 0$  be the number of outer corners of  $\lambda$ . We define  $\rho^+(\lambda) := \rho_r^+(\lambda)$ .

Note that, by Proposition 5.3, the map  $\rho^+(\lambda)$  does not in fact depend on the order that we chose on the outer corners. An example of a map  $\rho^+(\lambda)$  is given in Figure 10.

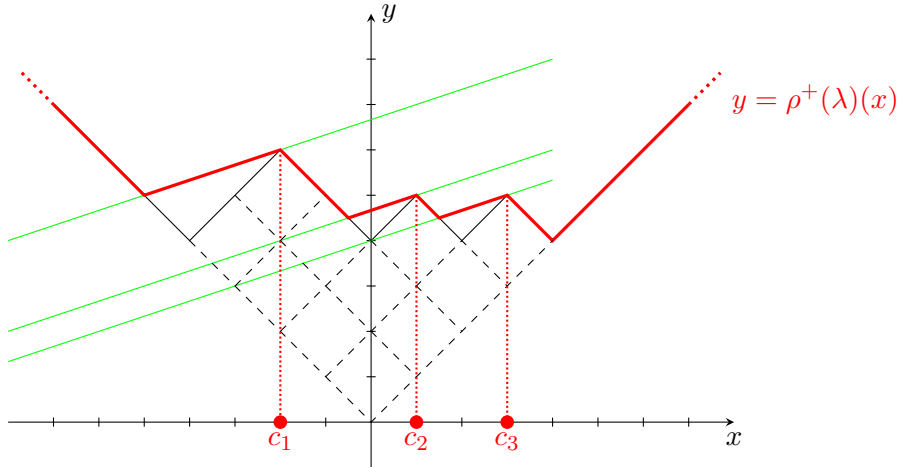


Figure 10: The map  $\rho^+(\lambda)$  (in thick red) for the 3-regular partition  $\lambda = (4, 4, 2, 1)$ . In green are the different lines  $L_e^{(k)}$ .

**Proposition 5.5.** Recall that  $\lambda$  is  $e$ -regular.

- (i) For all  $s \in \mathbb{R}$  we have  $0 \leq \rho^+(\lambda) - \omega_\lambda(s) \leq e$ .
- (ii) The graph  $G_\lambda^+ := \mathbb{Y}(\rho^+(\lambda))$  is stable under the shaking operation with slope  $\alpha_e$ , that is, we have  $\text{sh}_e(G_\lambda^+) = G_\lambda^+$ .

*Proof.* The first point follows directly from Proposition 5.3 since the intervals  $(c_{k-1}, c_k)$  are pairwise disjoint. For the second point, it follows from the simple observation that the slopes of  $\rho^+(\lambda)$  are either  $-1$  or  $\alpha_e$  (except the part  $\rho^+(\lambda)(s) = s$  for  $s \gg 0$ ). As a consequence, if

for  $x \in \mathbb{R}$  the line  $L' := x + L_e$  intersects the curve of  $\rho^+(\lambda)$  at a point  $s \in \mathbb{R}$  then  $L'$  remains below the curve  $\rho^+(\lambda)$  on  $(-\infty, s)$ . Observing that  $L'$  intersects the curve of  $|\cdot|$  on  $(-\infty, 0)$  since  $\alpha_e < 1$  gives the result.  $\square$

### 5.3 Inner regularisation

We now define a similar construction as  $\rho^+(\lambda)$  but for inner corners. We give the statements without proofs since they are entirely similar.

Define  $\rho_{r+1}^-(\lambda) := \omega_\lambda$  and let  $\{c_1 < \dots < c_{r+1}\}$  with  $r \geq 0$  the set of inner corners (in particular, note that a partition has always at least one inner corner). Assuming by decreasing induction on  $k \in \{1, \dots, r\}$  that we have constructed the piecewise linear function  $\rho_{k+1}^-(\lambda) : \mathbb{R} \rightarrow \mathbb{R}$ , then we construct the piecewise linear function  $\rho_k^-(\lambda) : \mathbb{R} \rightarrow \mathbb{R}$  as follows. Let  $L_e^{(k)}$  be the affine line  $(c_k, \omega_\lambda(c_k)) + L_e$ .

- (i) For  $s \leq c_k$  then  $\rho_k^-(\lambda)(s) := \rho_{k+1}^-(\lambda)(s)$ ;
- (ii) at  $s = c_k$ , we follow the line  $L_e^{(k)}$  (in the *positive* direction) until we meet the curve of  $\rho_{k+1}^-(\lambda)$ , at point of abscissa  $h_k$ ;
- (iii) for  $s \geq h_k$  then  $\rho_k^-(\lambda)(s) := \rho_{k+1}^-(\lambda)(s)$  again.

**Proposition 5.6.** *Assume that the partition  $\lambda$  is  $e$ -regular. With the above notation we have  $h_k \in [c_k, c_{k+1})$ . In particular, the functions  $\rho_{k+1}^-(\lambda)$  and  $\rho_k^-(\lambda)$  coincide on (at least)  $\mathbb{R} \setminus (c_k, c_{k+1})$ , and for any  $s \in (c_k, c_{k+1})$  we have  $-e \leq \rho_k^-(\lambda)(s) - \rho_{k+1}^-(\lambda)(s) \leq 0$ .*

An example of a map  $\rho^-(\lambda)$  is given in Figure 11.

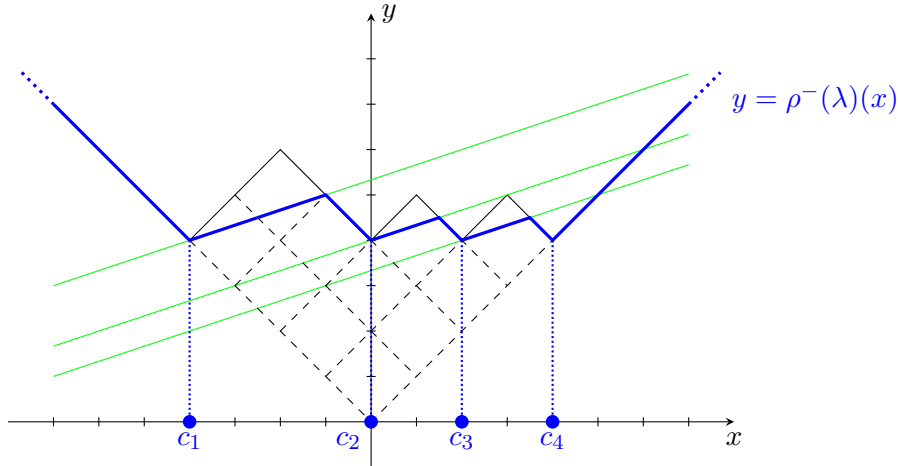


Figure 11: The map  $\rho^-(\lambda)$  (in thick blue) for the 3-regular partition  $\lambda = (4, 4, 2, 1)$ . In green are the different lines  $L_e^{(k)}$ .

**Definition 5.7.** Let  $e \geq 2$  and let  $\lambda$  be an  $e$ -regular partition. We define  $\rho^-(\lambda) := \rho_1^-(\lambda)$ .

**Proposition 5.8.** *Recall that  $\lambda$  is  $e$ -regular.*

- (i) For all  $s \in \mathbb{R}$  we have  $-e \leq \rho^-(\lambda) - \omega_\lambda(s) \leq 0$ .
- (ii) The graph  $G_\lambda^- := \mathbb{Y}(\rho^-(\lambda))$  is stable under the shaking operation with slope  $\alpha_e$ , that is, we have  $\text{sh}_e(G_\lambda^-) = G_\lambda^-$ .

## 6 Limit shape for regularisation of large partitions

We can now state our main theorem. Recall from Theorem 4.7 the definition of  $\Omega_\alpha$  for  $\alpha \geq 0$ .

**Theorem 6.1.** *Let  $e \geq 2$  and take  $\alpha = \alpha_e = 1 - 2e^{-1}$ . Under the Plancherel measure  $\text{Pl}_n$ , the function  $\tilde{\omega}_{\text{reg}_e(\lambda)}$  converges uniformly in probability to  $\Omega_\alpha$  as  $n \rightarrow +\infty$ . In other words, for any  $\epsilon > 0$  we have*

$$\text{Pl}_n \left( \sup_{\mathbb{R}} |\tilde{\omega}_{\text{reg}_e(\lambda)} - \Omega_\alpha| > \epsilon \right) \xrightarrow{n \rightarrow +\infty} 0.$$

*Proof.* Let  $\epsilon > 0$ . Recalling the notation of Definition 4.4, define:

$$m^\pm := \inf \{ |\Omega^{\pm\epsilon}(s) - \Omega(s)| : |s| \leq 2 \pm \frac{\epsilon}{2} \}.$$

By Lemma 4.5 applied with  $f = \Omega$ , we have  $m^\pm > 0$ . Now let  $M_n$  be the set of partitions  $\lambda \in \mathcal{P}_n$  such that:

$$\begin{aligned} \|\tilde{\omega}_\lambda - \Omega\|_{\mathbb{R}} &< \min(m^+, m^-), \\ \inf \{ s \in \mathbb{R} : \tilde{\omega}(s) \neq |s| \} &> -2 - \frac{\epsilon}{2}, \\ \sup \{ s \in \mathbb{R} : \tilde{\omega}(s) \neq |s| \} &< 2 + \frac{\epsilon}{2}. \end{aligned}$$

Note that by Theorem 2.9 we have  $\text{Pl}_n(M_n) \rightarrow 1$  as  $n \rightarrow +\infty$ . By Lemma 4.5, for all  $\lambda \in M_n$  we have:

$$\Omega^{-\epsilon}(s) \leq \tilde{\omega}_\lambda(s) \leq \Omega^{+\epsilon}(s),$$

for all  $s \in \mathbb{R}$ . In other words, for all  $\lambda \in M_n$  we have:

$$\mathbb{Y}(\Omega^{-\epsilon}) \subseteq \mathbb{Y}(\tilde{\omega}_\lambda) \subseteq \mathbb{Y}(\Omega^{+\epsilon}).$$

By Proposition 3.4, Theorem 4.7 together with Lemma 5.1, we deduce that:

$$\mathbb{Y}(\Omega_\alpha^{-\epsilon}) \subseteq \text{sh}_e(\mathbb{Y}(\tilde{\omega}_{\text{reg}_e(\lambda)})) \subseteq \mathbb{Y}(\Omega_\alpha^{+\epsilon}). \quad (6.2)$$

In particular, note that we can apply Theorem 4.7 for  $\Omega^{\pm\epsilon}$  indeed by Lemma 4.5, and  $\Omega_\alpha^{\pm\epsilon} := (\Omega^{\pm\epsilon})_\alpha$ . Now by Propositions 5.5 and 5.8, for all  $s \in \mathbb{R}$  we have:

$$\tilde{\rho}^-(\lambda)(s) \leq \tilde{\omega}_\lambda(s) \leq \tilde{\rho}^+(\lambda)(s)$$

(where  $\tilde{\rho}^\pm(\lambda)(s) = \frac{1}{\sqrt{n}} \rho^\pm(\lambda)(\sqrt{n}s)$  is rescaled as  $\tilde{\omega}_\lambda$ , see §2.2), thus:

$$\mathbb{Y}(\tilde{\rho}^-(\lambda)) \subseteq \mathbb{Y}(\tilde{\omega}_{\text{reg}_e(\lambda)}) \subseteq \mathbb{Y}(\tilde{\rho}^+(\lambda)).$$

By Propositions 5.5 and 5.8 again and Proposition 3.4, we deduce that:

$$\mathbb{Y}(\tilde{\rho}^-(\lambda)) \subseteq \text{sh}_e(\mathbb{Y}(\tilde{\omega}_{\text{reg}_e(\lambda)})) \subseteq \mathbb{Y}(\tilde{\rho}^+(\lambda)). \quad (6.3)$$

By (6.2) and (6.3), we deduce that:

$$\mathbb{Y}(\Omega_\alpha^{-\epsilon}) \subseteq \mathbb{Y}(\tilde{\rho}^+(\lambda)), \quad \text{and} \quad \mathbb{Y}(\tilde{\rho}^-(\lambda)) \subseteq \mathbb{Y}(\Omega_\alpha^{+\epsilon}).$$

Thus, we conclude that for any  $s \in \mathbb{R}$  we have:

$$\Omega_\alpha^{-\epsilon}(s) \leq \tilde{\rho}^+(\lambda)(s), \quad \text{and} \quad \tilde{\rho}^-(\lambda)(s) \leq \Omega_\alpha^{+\epsilon}(s).$$

We deduce that for any  $s \in \mathbb{R}$  we have:

$$\begin{aligned} \tilde{\rho}^-(\lambda)(s) &\leq \Omega_\alpha^{+\epsilon}(s) \\ &\leq \Omega_\alpha^{-\epsilon}(s) + \|\Omega_\alpha^{-\epsilon} - \Omega_\alpha^{+\epsilon}\|_\infty \\ &\leq \tilde{\rho}^+(\lambda)(s) + \|\Omega_\alpha^{-\epsilon} - \Omega_\alpha^{+\epsilon}\|_\infty \\ &\leq \tilde{\rho}^-(\lambda)(s) + \frac{2e}{\sqrt{n}} + \|\Omega_\alpha^{-\epsilon} - \Omega_\alpha^{+\epsilon}\|_\infty \quad \text{by Propositions 5.5 and 5.8.} \end{aligned}$$

Thus, again using Proposition 5.8, we deduce that for any  $s \in \mathbb{R}$  we have:

$$\begin{aligned} |\tilde{\omega}_{\text{reg}_e(\lambda)}(s) - \Omega_\alpha(s)| &\leq |\tilde{\omega}_{\text{reg}_e(\lambda)}(s) - \rho^-(\lambda)(s)| + |\rho^-(\lambda)(s) - \Omega_\alpha^{+\epsilon}(s)| + |\Omega_\alpha^{+\epsilon}(s) - \Omega_\alpha(s)| \\ &\leq \frac{2e}{\sqrt{n}} + \left( \frac{2e}{\sqrt{n}} + \|\Omega_\alpha^{-\epsilon} - \Omega_\alpha^{+\epsilon}\|_\infty \right) + \|\Omega_\alpha^{+\epsilon} - \Omega_\alpha\|_\infty, \end{aligned}$$

whence:

$$\sup_{\mathbb{R}} |\tilde{\omega}_{\text{reg}_e(\lambda)} - \Omega_\alpha| \leq \frac{4e}{\sqrt{n}} + 3\|\Omega_\alpha^{+\epsilon} - \Omega_\alpha\|_\infty.$$

Thus, by Proposition 4.18 deduce that for  $n \gg 0$  we have:

$$\sup_{\mathbb{R}} |\tilde{\omega}_{\text{reg}_e(\lambda)} - \Omega_\alpha| \leq 4\|\Omega_\alpha^{+\epsilon} - \Omega_\alpha\|_\infty.$$

Recalling that  $\lambda \in M_n$  with  $\text{Pl}_n(M_n) \rightarrow 1$  as  $n \rightarrow +\infty$ , we deduce that:

$$\text{Pl}_n \left( \sup_{\mathbb{R}} |\tilde{\omega}_{\text{reg}_e(\lambda)} - \Omega_\alpha| > 4\|\Omega_\alpha^{+\epsilon} - \Omega_\alpha\|_\infty \right) \xrightarrow{n \rightarrow +\infty} 0,$$

which, again by Proposition 4.18, proves the desired convergence.  $\square$

We illustrate the convergence of Theorem 6.1 for  $e = 2$  in Figure 12 and for  $e \in \{3, 4\}$  in Figure 13, with in red the usual limit shape  $\Omega$  and in blue the map  $\tilde{\omega}_{\text{reg}_e(\lambda)}$  for  $\lambda$  a partition of  $3 \times 10^4$  taken under the Plancherel measure.

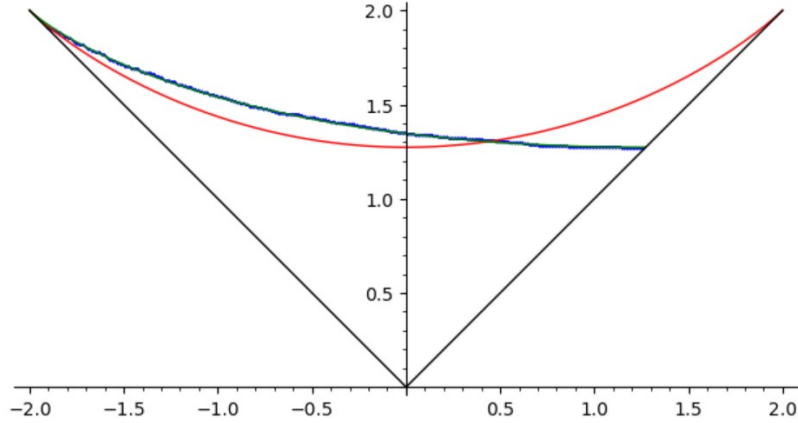


Figure 12: Example of 2-regularisation of a large partition, with in green the limit shape of Corollary B

**Theorem 6.4.** *The convergence of Theorem 6.1 also holds for the supports, that is:*

$$\inf\{s \in \mathbb{R} : \tilde{\omega}_{\text{reg}_e(\lambda)}(s) \neq |s|\} \longrightarrow -2,$$

and

$$\sup\{s \in \mathbb{R} : \tilde{\omega}_{\text{reg}_e(\lambda)}(s) \neq |s|\} \longrightarrow s_\alpha,$$

in probability under  $\text{Pl}_n$  as  $n \rightarrow +\infty$ , where  $s_\alpha$  is the constant on Proposition 4.13 for  $f = \Omega$ .

*Proof.* We use the same set  $M_n$  and the same strategy as in the proof of Theorem 6.1.  $\square$

Finally, we deduce the asymptotic behaviour of the first part and the first column.

**Corollary 6.5.** *Let  $e \geq 2$ . Under the Plancherel measure  $\text{Pl}_n$ :*

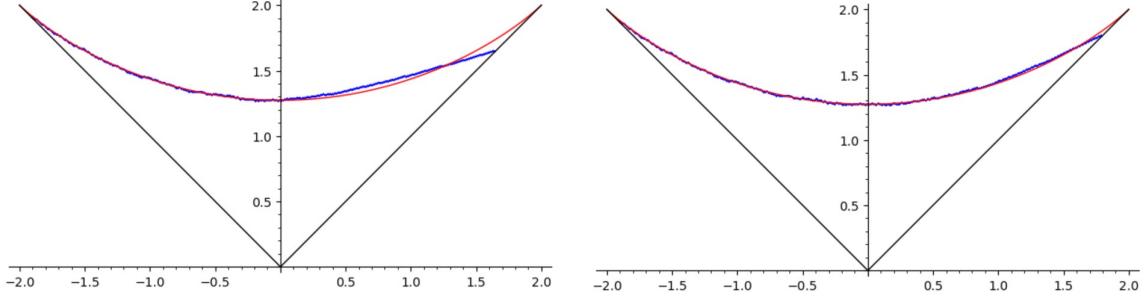


Figure 13: Example of 3- and 4-regularisation of a large partition, respectively on the left and right.

- (i) the rescaled size  $\frac{1}{\sqrt{n}}\text{reg}_e(\lambda)_1$  of the first row of  $\text{reg}_e(\lambda)$  converges as  $n \rightarrow +\infty$  in probability to 2;
- (ii) the rescaled size  $\frac{1}{\sqrt{n}}\text{reg}_e(\lambda)'_1$  of the first column of  $\text{reg}_e(\lambda)$  converges as  $n \rightarrow +\infty$  in probability to  $\frac{2e}{\pi} \sin \frac{\pi}{e}$ .

*Proof.* The first point is clear by Theorem 6.4. For the second one, by the same theorem it suffices to prove that the announced value is equal to  $s_\alpha$ . Proposition 4.13 applied for  $f = \Omega$  we have:

$$s_\alpha = (1 - \alpha)^{-1} \left[ \Omega(\Omega'^{-1}(\alpha)) - \alpha \Omega'^{-1}(\alpha) \right].$$

Recall that  $\alpha = 1 - 2e^{-1}$  and  $\Omega'(s) = \frac{2}{\pi} \arcsin(\frac{s}{2})$ . We have  $\Omega'(s) = t \iff \arcsin(\frac{s}{2}) = \frac{\pi t}{2} \iff s = 2 \sin \frac{\pi t}{2}$  thus  $\Omega'^{-1}(t) = 2 \sin \frac{\pi t}{2}$ . We thus have:

$$\begin{aligned} \Omega'^{-1}(\alpha) &= 2 \sin \frac{\pi \alpha}{2} \\ &= 2 \sin \frac{\pi(1 - 2e^{-1})}{2} \\ &= 2 \sin \left( \frac{\pi}{2} - \frac{\pi}{e} \right) \\ &= 2 \cos \frac{\pi}{e}. \end{aligned}$$

We obtain:

$$\begin{aligned} s_\alpha &= \frac{e}{2} \left[ \Omega(\Omega'^{-1}(\alpha)) - \alpha \Omega'^{-1}(\alpha) \right] \\ &= \frac{e}{2} \left[ \Omega\left(2 \cos \frac{\pi}{e}\right) - 2\alpha \cos \frac{\pi}{e} \right]. \end{aligned}$$

We have, recalling the identity  $\arcsin(\cos x) = \frac{\pi}{2} - x$  for  $x \in [-1, 1]$ :

$$\begin{aligned} \Omega\left(2 \cos \frac{\pi}{e}\right) &= \frac{2}{\pi} \left[ \arcsin\left(\cos \frac{\pi}{e}\right) 2 \cos \frac{\pi}{e} + \sqrt{4 - 4 \cos^2 \frac{\pi}{e}} \right] \\ &= \frac{2}{\pi} \left[ \left(\frac{\pi}{2} - \frac{\pi}{e}\right) 2 \cos \frac{\pi}{e} + 2 \sin \frac{\pi}{e} \right] \\ &= 2\left(1 - \frac{2}{e}\right) \cos \frac{\pi}{e} + \frac{4}{\pi} \sin \frac{\pi}{e} \\ &= 2\alpha \cos \frac{\pi}{e} + \frac{4}{\pi} \sin \frac{\pi}{e}, \end{aligned}$$

thus we finally obtain:

$$s_\alpha = \frac{2e}{\pi} \sin \frac{\pi}{e}.$$

□

*Example 6.6.* Here are approximations of the first values of the limit for  $\frac{1}{\sqrt{n}}\text{reg}_e(\lambda)_1$ :

$e$	$\frac{2e}{\pi} \sin \frac{\pi}{e}$
2	1.27
3	1.65
4	1.80

The reader can check that they match the corresponding values of Figures 12 and 13.

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