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# Limit shape for regularisation of large partitions under the Plancherel measure 

Salim Rostam*


#### Abstract

A celebrated result of Kerov-Vershik and Logan-Shepp gives an asymptotic shape for large partitions under the Plancherel measure. We prove that when we consider e-regularisations of such partitions we still have a limit shape, which is given by a shaking of the Kerov-Vershik-Logan-Shepp curve. We deduce an explicit form for the first asymptotics of the length of the first rows and the first columns for the $e$-regularisation.


## 1 Introduction

Partitions of a given integer $n$ are the different way to decompose $n$ as an unordered sum of positive integers. In other words, a partition of $n \in \mathbb{Z}_{\geq 0}$ is a non-increasing sequence $\lambda=\left(\lambda_{1} \geq \ldots\right)$ of non-negative integers with sum $n$. This mathematical object appears for instance in the study of the symmetric group $\mathfrak{S}_{n}$ of permutations of $\{1, \ldots, n\}$, since partitions of $n$ index the conjugacy classes (via the cycle decomposition).

A representation of $\mathfrak{S}_{n}$ of dimension $N$ over a field $k$ is a group homomorphism $\rho: \mathfrak{S}_{n} \rightarrow$ $\mathrm{GL}_{N}(k)$. We focus for the moment at the case where $k$ is the field $\mathbb{C}$ of complex numbers, in which case we say that we have a complex representation. As any integer decomposes into a product of primes, any complex representation decomposes into a sum of irreducible complex representations. It turns out that the fact that the set $\mathcal{P}_{n}$ of partitions of $n$ index the conjugacy classes implies that $\mathcal{P}_{n}$ also index the set of irreducible complex representations. If we denote by $\rho_{\lambda}$ the irreducible complex representation associated with $\lambda \in \mathcal{P}_{n}$, a standard result of complex representation theory shows that:

$$
\begin{equation*}
\# \mathfrak{S}_{n}=n!=\sum_{\lambda \in \mathcal{P}_{n}}\left(\operatorname{dim} \rho_{\lambda}\right)^{2} . \tag{1.1}
\end{equation*}
$$

A remarkable fact is that we are able to explicitly compute the numbers $\operatorname{dim} \rho_{\lambda}$ (namely with the famous hook length formula). We refer for instance to Sagan [Sa] or James-Kerber [JaKe] for more details on the complex representations of $\mathfrak{S}_{n}$.

Now if we want to study the representations of $\mathfrak{S}_{n}$ over the field $k=\mathbb{F}_{p}$ with $p$ elements (with $p$ a prime number; we will say that we have a $p$-representation), almost everything that is known for complex representations falls apart. A fundamental difference is that some representations may not decompose into a sum of irreducible ones. Nevertheless, the study of irreducible representations is still interesting since any representation can always be decomposed into irreducible constituents via a composition (or Jordan-Hölder) series.

A general theorem of Brauer says that the irreducible $p$-representations are in bijection with the $p$-regular conjugation classes, that is, with conjugation classes formed by elements whose order does not divide $p$. For the symmetric group, one can see that the set of $p$-regular conjugation classes is in one-to-one correspondence with the set of $\boldsymbol{p}$-regular partitions, that is, partitions with no $p$ (or more) consecutive equal parts. Note that the dual notion of $p$-restricted partitions is also present in the literature, where the difference between two

[^0]consecutive parts of a $p$-restricted partition it at most $p-1$. If $\lambda$ is a $p$-regular partition, we will denote by $\rho_{\lambda}^{p}$ the associated irreducible $p$-representation. Contrary to the complex case, there is no formula expressing the dimension of $\rho_{\lambda}^{p}$ (yet). We refer for instance to James-Kerber [JaKe] for more details on the $p$-representations of $\mathfrak{S}_{n}$.

A way to understand $p$-irreducible representations is to study the $p$-irreducible representations that appear in the decomposition series of an irreducible complex representation reduced modulo $p$. More explicitly, it can be shown that any irreducible complex representation can be realised over $\mathbb{Z}$, that is, we can assume that $\rho_{\lambda}: \mathfrak{S}_{n} \rightarrow \mathrm{GL}_{N}(\mathbb{Z})$. The reduction modulo $p$, that we denote by $\bar{\rho}_{p}: \mathfrak{S}_{n} \rightarrow \mathrm{GL}_{N}\left(\mathbb{F}_{p}\right)$, is then simply the reduction modulo $p$ of the matrix entries. Now if we return to the problem of determining which irreducible $p$-representations appear in a decomposition series of $\bar{\rho}_{\lambda}$, James [Ja] gave an explicit combinatorial construction of a $p$-regular partition $\operatorname{reg}_{e}(\lambda)$ such that $\rho_{\lambda}^{p}$ appears in a decomposition series of $\bar{\rho}_{\lambda}$. This partition $\operatorname{reg}_{e}(\lambda)$ is the $\boldsymbol{p}$-regularisation of $\lambda$. The $p$-regularisation operation has in fact a meaning also when $p$ is not prime, in the context of the Iwahori-Hecke algebra of $\mathfrak{S}_{n}$ (see, for instance, Mathas [Ma]). In particular, for an integer $e \geq 2$ we will also use the terms $e$-regular and $e$-regularisation. Note that the $e$-regularisation map was recently generalised by Millan Berdasco [Mi] to an $(e, i)$-regularisation map on partitions: it is likely that our results generalise to this setting.

We now go back to (1.1). This equation shows that $\mathrm{Pl}_{n}(\lambda):=\frac{\left(\operatorname{dim} \rho_{\lambda}\right)^{2}}{n!}$ is a probability measure on the set $\mathcal{P}_{n}$ of partitions of $n$, called the Plancherel measure. Via some calculations involving the hook integral (defined in the spirit of the hook length formula), Kerov-Vershik [KeVe] and Logan-Shepp [LoSh] proved independently that there is a curve $\Omega$ so that the upper rim $\widetilde{\omega}_{\lambda}$ of the Young diagram (in the Russian convention) of a large partition $\lambda$ converges uniformly in probability to $\Omega$. An illustration of this convergence is given in Figure 1 (note ${ }^{1}$ ). This limit shape theorem allowed to determine the first asymptotics of the length of the first row (and first column) of a Young diagram, taken under the Plancherel measure. Note that, via the Robinson-Schensted correspondence (which provides a bijective proof of (1.1)), this provides a solution to the Ulam problem on the length of a longest increasing subsequence of a word in $\mathfrak{S}_{n}$ chosen uniformly. We refer for instance to Romik [Rom] for more details on the Plancherel measure and related asymptotics.

In the previous context of regularisation of partitions, the following question in thus natural: for an integer $e \geq 2$, what can be said about the $\operatorname{partition}^{\operatorname{reg}}{ }_{e}(\lambda)$ when $\lambda$ is a large partition taken under the Plancherel measure? The aim of this paper is to give a first answer to this question.


Figure 1: The limit shape theorem for large partitions under the Plancherel measure

[^1]

Figure 2: Examples of symmetrisation and shaking with respect to a line $L$

Steiner symmetrisation is a useful tool to study (namely) the isoperimetric problem. More precisely, if $K$ is a compact convex subset of $\mathbb{R}^{2}$ then its symmetrisation with respect to the line $L$ is the set $S_{L}(K)$ that we obtain after sliding the different slices of $K$ (with respect to the orthogonal direction of $L$ ) until the midpoint of the slice is on $L$. An example of symmetrisation with respect to a vertical line in given in Figure 2. The link with the isoperimetric problem is that $S_{L}(K)$ has the same area as $K$ but a smaller perimeter. Note the following result, known as the sphericity theorem of Gross: from any compact convex subset of $\mathbb{R}^{2}$ one can obtain the unit disk after a succession of (possibly infinite number of) symmetrisations. Note that this concept of symmetrisation can also be extended to maps, with for example Schwarz symmetric rearrangment, which is a powerful tool to study functional inequalities. We refer for instance to Gruber [Gr] or Krantz-Parks [KrPa] for more details on the Steiner symmetrisation.

A variation of Steiner symmetrisation is the notion of shaking. This notion was first introduced by Blaschke [B1] to solve Sylvester's "four points problem". The difference with Steiner symmetrisation is that we slide the slices until we meet $L$. An example is given in Figure 2. Shaking and Steiner symmetrisation share many properties, for instance, Gross theorem holds with "unit disk" replaced by "simplex". We refer to [CCG] for fore details on the shaking operation. We will in fact use the shaking operation in a context where the slices are not orthogonal to the line $L$; such an operation is for instance used in [FrLu].

We can now state our main result (Theorems 4.7 and 6.1).
Theorem A. Let $e \geq 2$. Under the Plancherel measure $\mathrm{Pl}_{n}$, the upper rim $\widetilde{\omega}_{\mathrm{reg}_{e}(\lambda)}$ of the Young diagram (in the Russian convention) of $\mathrm{reg}_{e}(\lambda)$ converges uniformly in probability to the shape $\Omega_{\alpha}$ as $n \rightarrow+\infty$, in other words, for any $\epsilon>0$ we have

$$
\mathrm{Pl}_{n}\left(\sup _{\mathbb{R}}\left|\widetilde{\omega}_{\operatorname{reg}_{e}(\lambda)}-\Omega_{\alpha}\right|>\epsilon\right) \xrightarrow{n \rightarrow+\infty} 0
$$

The shape $\Omega_{\alpha}$ is obtained by shaking the part $\mathbb{Y}(\Omega)$ of the graph of $\Omega$ that is above the graph of the absolute value, with respect to the line of equation $y=-x$ and angle $\alpha:=1-2 e^{-1}$,
and is given by:

$$
\begin{aligned}
\Omega_{\alpha}(x) & =\Omega(x), & & \text { for all } x \leq x_{\alpha}^{-} \\
\Omega_{\alpha}\left(x+\delta_{x}\right) & =\Omega(x)+\alpha \delta_{x}, & & \text { for all } x \in\left(x_{\alpha}^{-}, x_{\alpha}^{+}\right) \\
\Omega_{\alpha}(x) & =x, & & \text { for all } x \geq s_{\alpha}
\end{aligned}
$$

where:

- $x_{\alpha}^{+}=\Omega^{\prime-1}(\alpha) \in[0, a)$,
- $\delta_{x}=(1-\alpha)^{-1} F_{\alpha}(x)-F_{\alpha}^{-1}\left(F_{\alpha}(x)\right)$, where $F_{\alpha}: x \mapsto \Omega(x)-\alpha x$ and $F_{\alpha}^{-1}$ denotes its inverse on $\left(x_{\alpha}^{+}, a\right)$,
- $x_{\alpha}^{-} \in\left(-a, x_{\alpha}^{+}\right)$(is the unique point that) satisfies $F_{\alpha}\left(x_{\alpha}^{-}\right)=(1-\alpha) a$,
- $(1-\alpha) s_{\alpha}=F_{\alpha}\left(x_{\alpha}^{+}\right)=\Omega\left(\Omega^{\prime-1}(\alpha)\right)-\alpha \Omega^{\prime-1}(\alpha) \in\left[\Omega(0)-\alpha \Omega\left(x_{\alpha}^{+}\right), \Omega(0)\right]$.

We illustrate the convergence in Figure 3 and compare the limit shapes $\Omega_{0}$ and $\Omega$. The fact that $\Omega_{0}$ is obtained via the horizontal shaking of $\mathbb{Y}(\Omega)$ (with respect to the line of equation $y=-x$ ) is highlighted in Figure 4. In the particular case $e=2$, the statement of Theorem A becomes explicit (Corollary 4.14 and Theorem 6.1).

Corollary B. Under the Plancherel measure $\mathrm{Pl}_{n}$, the upper rim $\widetilde{\omega}_{\mathrm{reg}_{2}(\lambda)}$ of the Young diagram (in the Russian convention) of $\operatorname{reg}_{2}(\lambda)$ converges uniformly in probability to the shape $\Omega_{0}$ given by:

$$
\begin{aligned}
\Omega_{0}(x) & =\Omega(x), & & \text { for all } x \leq-2, \\
\Omega_{0}(2 x+\Omega(x)) & =\Omega(x), & & \text { for all } x \in(-2,0), \\
\Omega_{0}(x) & =x, & & \text { for all } x \geq \Omega(0) .
\end{aligned}
$$

|  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |  |  |  |

Figure 3: Example of 2-regularisation of a large partition, with in green the limit shape $\Omega_{0}$ of Corollary B and in red the limit shape $\Omega$ of Kerov-Vershik-Logan-Shepp

Despite the fact that Theorem A is not fully explicit, we are able to deduce the asymptotic behaviour of the length of the first line and of the first column of $\operatorname{reg}_{e}(\lambda)$ (Corollary 6.5).

Corollary C. Let $e \geq 2$. Under the Plancherel measure $\mathrm{Pl}_{n}$ :
(i) the rescaled size $\frac{1}{\sqrt{n}} \operatorname{reg}_{e}(\lambda)_{1}$ of the first row of $\operatorname{reg}_{e}(\lambda)$ converges as $n \rightarrow+\infty$ in probability to 2 ;
(ii) the rescaled size $\frac{1}{\sqrt{n}} \operatorname{reg}_{e}(\lambda)_{1}^{\prime}$ of the first column of $\operatorname{reg}_{e}(\lambda)$ converges as $n \rightarrow+\infty$ in probability to $\frac{2 e}{\pi} \sin \frac{\pi}{e}$.


Figure 4: The maps $\Omega_{0}$ (in blue) and $\Omega$ (in red). In green the tangent for $\Omega$ at 0 . The left magenta segments are what is "added" to the red curve, and comes from the right magenta segments by shaking $\mathbb{Y}(\Omega)$.

We now give the outline of the paper. In Section 2 we recall the definition of the $e$-regularisation map on the set of partitions of $n$ and we state the limit shape result of Kerov-Vershik and Logan-Shepp for large partitions taken under the Plancherel measure. The short Section 3 is devoted to the (non-orthogonal) shaking operation on compact subsets of $\mathbb{R}^{2}$. Section 4 contains the first main result of the paper, Theorem 4.7, which compute the shape associated with the shaking of the graph of a convex function (more precisely, the part of the graph that is above the graph of the absolute value). We give in Corollary 4.14 a particular case when the shaking is made by horizontal slices, where the result is explicit. In Section 5 we introduce two approximations $\rho^{ \pm}(\lambda)$ for the upper rim $\omega_{\lambda}$ of the Young diagram (in the Russian convention) of a partition $\lambda$, in order to facilitate the shakings computations. More precisely, in $\S 5.2$ we define the outer regularisation $\rho^{+}(\lambda)$ of an $e$-regular partition $\lambda$ (Definition 5.4) and in Proposition 5.5 we prove that $\rho^{+}(\lambda)$ is both close to $\omega_{\lambda}$ and that its graph $\rho^{+}(\lambda)$ is stable under the shaking operation. A similar construction is made in $\S 5.3$ for the inner regularisation $\rho^{-}(\lambda)$. Finally, Section 6 is devoted to the proof of the second main result, Theorem 6.1, which states that the (rescaled) upper rim $\widetilde{\omega}_{\text {reg }_{e}(\lambda)}$ of the Young diagram (in the Russian convention) of the e-regularisation of a partition taken under the Plancherel measure converges uniformly in probability to the shape defined in Theorem 4.7. The proof uses the results of Sections 4 and 5 . We also prove the convergence of the support of $\widetilde{\omega}_{\lambda}-|\cdot|$ (Theorem 6.4), and we deduce the asymptotic length of the first row and the first column of $\operatorname{reg}_{e}(\lambda)$ (Corollary 6.5).

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## 2 Background on partitions

We recall in $\S 2.1$ the combinatorial notion of $e$-regularisation of a partition. In $\S 2.2$ we recall the limit shape result of Kerov-Vershik and Logan-Shepp for large partitions under the Plancherel measure.

### 2.1 Regularisation

A partition is a finite non-increasing sequence $\lambda=\left(\lambda_{1} \geq \cdots \geq \lambda_{h}>0\right)$ of positive integers. If $|\lambda|:=\sum_{i=1}^{h} \lambda_{i}=n$ then we say that $\lambda$ is a partition of $n$. We denote by $\mathcal{P}_{n}$ the set of partitions of $n$. The Young diagram of a partition $\lambda=\left(\lambda_{1} \geq \cdots \geq \lambda_{h}>0\right)$ is the subset of $\mathbb{Z}_{\geq 1}^{2}$ given by:

$$
\mathcal{Y}(\lambda)=\left\{(a, b) \in \mathbb{Z}_{\geq 1}^{2}: 1 \leq a \leq h \text { and } 1 \leq b \leq \lambda_{a}\right\} .
$$

Note that we will consider that the $a$-coordinate in the Young diagrams goes downwards. A node is an element of $\mathbb{Z}_{\geq 1}^{2}$.
Example 2.1. The Young diagram of the partition $\lambda=(4,4,1)$ is
 The node in blue has coordinates $(1,2)$.
Definition 2.2. Let $e \geq 2$. A partition $\lambda=\left(\lambda_{1}, \ldots, \lambda_{h}\right)$ is $e$-regular if no parts repeat $e$ times or more, that is, if $\lambda_{i}>\lambda_{i+e-1}$ for any $i \in\{1, \ldots, h-e\}$.

As we mentioned in the introduction, the notion of $e$-regular partitions appears for instance in the context of the representation theory of the Iwahori-Hecke algebra $\mathcal{H}_{q}\left(\mathfrak{S}_{n}\right)$ of $\mathfrak{S}_{n}$ (a certain deformation of $\mathbb{C} \mathfrak{S}_{n}$ ), with $q \in \mathbb{C}^{\times}$having order $e$, see, for instance, [Ma] (note that in [Ma] the dual notion of e-restricted partition is used). The next two definitions are due to James [Ja].

Definition 2.3. Let $e \geq 2$.

- The e-ladder number (or simply ladder number) of a node $\gamma=(a, b) \in \mathbb{Z}_{\geq 1}^{2}$ is:

$$
\operatorname{lad}_{e}(\gamma):=a+(e-1)(b-1) \in \mathbb{Z}_{\geq 1} .
$$

- Let $\ell \geq 1$. The $(e, \ell)$-th ladder (or simply $\ell$-th ladder) is the (finite) set of all nodes of $\mathbb{Z}_{\geq 1}^{2}$ with $e$-ladder $\ell$. The $e$-ladder of a node $\gamma$ is the $\left(e, \operatorname{lad}_{e}(\gamma)\right)$-th ladder.

Example 2.4. In the Young diagram of $\lambda=(4,4,3,3,3,3,3,1)$, in each node we write the corresponding 4-ladder numbers:

| 1 | 4 | 7 | 10 |
| :---: | :---: | :---: | :---: |
| 2 | 5 | 8 | 11 |
| 3 | 6 | 9 |  |
| 4 | 7 | 10 |  |
| 5 | 8 | 11 |  |
| 6 | 9 | 12 |  |
| 7 | 10 | 13 |  |
| 8 |  |  |  |
|  |  |  |  |

Definition 2.5. Let $e \geq 2$ and let $\lambda$ be a partition. The $e$-regularisation of $\lambda$ is the partition $\operatorname{reg}_{e}(\lambda)$ that we obtain after moving each node of $\mathcal{Y}(\lambda)$ as high as possible in its $e$-ladder.

Note that $\operatorname{reg}_{e}(\lambda)$ is an $e$-regular partition, and if $\lambda$ is $e$-regular then $\operatorname{reg}_{e}(\lambda)=\lambda$. As we have mentioned in the introduction, the $e$-regularisation map has a significance in terms of modular representation theory of the symmetric group (or its associated Iwahori-Hecke algebra).
Example 2.6. The 4-regularisation of the partition of Example 2.4 is $(5,4,4,3,3,2,2,1)$. The


Figure 5: Russian convention for the Young diagram of $\lambda=(4,4,2,1)$

4-ladders of the added (respectively, deleted) nodes are in green (resp. red).

| 1 | 4 | 7 | 10 | 13 |
| :---: | :---: | :---: | :---: | :---: |
| 2 | 5 | 8 | 11 |  |
| 3 | 6 | 9 | 12 |  |
| 4 | 7 | 10 |  |  |
| 5 | 8 | 11 |  |  |
| 6 | 9 | 12 |  |  |
| 7 | 10 | 13 |  |  |
| 8 |  |  |  |  |

### 2.2 Limit shape

Russian convention Rotating the Young diagram of $\lambda=\left(\lambda_{1} \geq \cdots \geq \lambda_{h}>0\right) \in \mathcal{P}_{n}$ by an angle of $\frac{3 \pi}{4}$ and embedding it inside $\mathbb{R} \times \mathbb{R}_{\geq 0}$ so that the box $(1,1)$ has bottom vertex at $(0,0)$ and each box has area 2 (i.e. semi-diagonal length 1) gives the Russian convention for the Young diagram of $\lambda$. Note that the node $(a, b) \in \mathcal{Y}(\lambda)$ corresponds to the (square) box with top vertex $(a-b, a+b)$ in $\mathbb{R} \times \mathbb{R}_{\geq 0}$. We denote by $\omega_{\lambda}: \mathbb{R} \rightarrow \mathbb{R}$ the upper rim of the resulting diagram, extending $\omega_{\lambda}$ by $\omega_{\lambda}(x):=|x|$ outside the diagram. Then $\omega_{\lambda}$ is a continuous piecewise linear function such that:

- for each $k \in \mathbb{Z}$ we have

$$
\begin{equation*}
\left.\omega_{\lambda}^{\prime}\right|_{(k, k+1)}= \pm 1 \tag{2.7}
\end{equation*}
$$

- we have $\omega_{\lambda}(x)=|x|$ for $|x| \gg 0$ (more precisely, for $x \leq-\lambda_{1}$ or $x \geq h$ ),
- we have $\int_{\mathbb{R}}\left[\omega_{\lambda}(x)-|x|\right] d x=2 n$.

An illustration of the construction of $\omega_{\lambda}$ is given in Figure 5. (We warn the reader that in the literature the convention is sometimes reversed, that is, our $\omega_{\lambda}$ is sometimes reflected with respect to the axis $\{0\} \times \mathbb{R}$.) We will use a particular rescaling $\widetilde{\omega}_{\lambda}: \mathbb{R} \rightarrow \mathbb{R}$ of $\omega_{\lambda}$, given by $\widetilde{\omega}_{\lambda}(s):=\frac{1}{\sqrt{n}} \omega_{\lambda}(s \sqrt{n})$. Note that the area between the graphs of $\widetilde{\omega}_{\lambda}$ and $|\cdot|$ is 2 .

Plancherel measure Let $\lambda \in \mathcal{P}_{n}$. A standard tableau of shape $\lambda$ is a bijection $\mathfrak{t}$ : $\mathcal{Y}(\lambda) \rightarrow\{1, \ldots, n\}$ such that $\mathfrak{t}$ increases along the rows and down the columns, in other words for $(a, b) \in \mathcal{Y}(\lambda)$ we have $\mathfrak{t}(a, b)<\mathfrak{t}(a+1, b)$ if $(a+1, b) \in \mathcal{Y}(\lambda)$ and $\mathfrak{t}(a, b)<\mathfrak{t}(a, b+1)$
if $(a, b+1) \in \mathcal{Y}(\lambda)$. We denote by $\operatorname{Std}(\lambda)$ the set of standard Young tableaux of shape $\lambda$. We have the following standard identity:

$$
n!=\sum_{\lambda \in \mathcal{P}_{n}} \# \operatorname{Std}(\lambda)^{2}
$$

Definition 2.8. The Plancherel measure on the set $\mathcal{P}_{n}$ of the partitions of $n$ is given by

$$
\mathrm{Pl}_{n}(\lambda):=\frac{\# \operatorname{Std}(\lambda)^{2}}{n!}
$$

for all $\lambda \in \mathcal{P}_{n}$.
The next result describes the Plancherel measure $\mathrm{Pl}_{n}$ for large $n$.
Theorem 2.9 ([LoSh], [KeVe], [Rom, Theorem 1.26]). Let $\Omega: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$
\Omega(s):= \begin{cases}\frac{2}{\pi}\left(s \arcsin \left(\frac{s}{2}\right)+\sqrt{4-s^{2}}\right), & \text { if }|s| \leq 2 \\ |s|, & \text { otherwise }\end{cases}
$$

Then, under the Plancherel measure $\mathrm{Pl}_{n}$, the function $\widetilde{\omega}_{\lambda}$ converges uniformly in probability to $\Omega$ as $n \rightarrow+\infty$. In other words, for any $\epsilon>0$ we have

$$
\mathrm{Pl}_{n}\left(\sup _{\mathbb{R}}\left|\widetilde{\omega}_{\lambda}-\Omega\right|>\epsilon\right) \xrightarrow{n \rightarrow+\infty} 0
$$

Moreover, we also have convergence of the supports, that is:

$$
\inf \left\{s \in \mathbb{R}: \widetilde{\omega}_{\lambda}(s) \neq|s|\right\} \longrightarrow-2
$$

and

$$
\sup \left\{s \in \mathbb{R}: \widetilde{\omega}_{\lambda}(s) \neq|s|\right\} \longrightarrow 2
$$

in probability under $\mathrm{Pl}_{n}$ as $n \rightarrow+\infty$.
An illustration of Theorem 2.9 for $n=1000$ is given in Figure 1. Note that the limit shape $\Omega$ has a much simpler form after derivation.

Lemma 2.10. The map $\Omega$ is an antiderivative on $\mathbb{R}$ of:

$$
s \mapsto \begin{cases}\arcsin \left(\frac{s}{2}\right), & \text { if }|s| \leq 2, \\ \operatorname{sgn}(s), & \text { if }|s|>2\end{cases}
$$

In particular, the minimum of $\Omega$ on $\mathbb{R}$ is $\Omega(0)=\frac{4}{\pi}$.

## 3 Shakings

The aim of this paper is to put together the notions of $\S 2.1$ and $\S 2.2$. As a first step, we show in Figure 6 what do the 4-ladders of Example 2.4 look like in the Russian convention.

Proposition 3.1. In the Russian convention, the e-regularisation makes the nodes going as left as possible in the direction of $y=\left(1-2 e^{-1}\right) x$.

Proof. In the Young diagram we know that the node $(e, 1) \in \mathbb{Z}_{\geq 1}^{2}$ goes to $(1,2)$. In the Russian convention it means that the box with top vertex at $(e-1, e+1)$ goes to the box with top vertex $(-1,3)$. The corresponding slope is thus $\frac{e+1-3}{e-1-(-1)}=\frac{e-2}{e}=1-2 e^{-1}$ as announced (note that the slope do not change when both axes are rescaled by a same constant).

We now define the shaking operation that will be of interest for us. As we mentioned in the introduction, this is a variation of Steiner symmetrisation first introduced by Blaschke [Bl].


Figure 6: 4-ladders in the Russian convention for $\lambda=(4,4,3,3,3,3,3,1)$


Figure 7: Example of the map $\mathrm{Sh}_{\alpha}$ with $\alpha=0$

Definition 3.2. Let $\alpha \in \mathbb{R}_{\geq 0}$ and let $v_{\alpha}$ be the unit vector positively collinear to $(1, \alpha)^{\top}$. If $K$ is a compact subset of $\mathbb{R}^{2}$, its shaking with direction $\alpha$ (against $y=-x$ ) is the compact subset $\mathrm{Sh}_{\alpha}(K)$ of $\mathbb{R}^{2}$ that we obtain by sliding the set $K$ along the direction of $\left(L_{\alpha}\right): y=\alpha x$ until we meet the line $(L): y=-x$, that is:

$$
\mathrm{Sh}_{\alpha}(K):=\bigsqcup_{x \in L} K_{v}
$$

where $K_{v}$ is the segment with extreme points $x$ and $x+\left|K \cap\left(x+L_{\alpha}\right)\right| v_{\alpha}$ with $|\cdot|$ denoting the one-dimensional Lebesgue measure.

In the usual definition of shaking the direction $L_{\alpha}$ is orthogonal to the the line $L$ against which we shake. This non-orthogonal variation is for instance considered in [FrLu, §5].
Example 3.3. Take $\alpha=0$ and let $K$ be the unit square with bottom left corner at $(1,1)$. Then, as shown on Figure 7, the set $\operatorname{Sh}_{\alpha}(K)$ is the parallelogram with vertices $(-2,2),(-1,2),(0,1)$ and $(-1,1)$.

The following property is a standard property of Steiner symmetrisations and shakings. The proof is immediate from the definition.

Proposition 3.4. If $K \subseteq K^{\prime}$ are compact subsets of $\mathbb{R}^{2}$ then $\operatorname{Sh}_{\alpha}(K) \subseteq \operatorname{Sh}_{\alpha}\left(K^{\prime}\right)$.
Proof. See [FrLu, Lemma 5.3]. Note that the proof is in fact the same as for the usual (orthogonal) shakings.

Recalling Proposition 3.1, we are interested in special cases of shakings.
Definition 3.5. For any $e \geq 2$, we denote by $\operatorname{sh}_{e}$ the shaking $\operatorname{Sh}_{\alpha}$ with direction $\alpha_{e}:=$ $1-2 e^{-1}$.

## 4 Shaking the limit shape

Let $a \in \mathbb{R}_{>0}$. We define $\mathfrak{Y}_{a}$ to be the set of all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ of class $\mathcal{C}^{1}$ such that:
$f$ is even,

$$
\begin{equation*}
\text { for any }|s| \geq a \text { we have } f(s)=|s|, \tag{4.1a}
\end{equation*}
$$

$$
\begin{equation*}
f \text { is strictly convex on }[-a, a] \text {. } \tag{4.1b}
\end{equation*}
$$

Note that by Lemma 2.10 we have $\Omega \in \mathfrak{Y}_{2}$.
Example 4.2. Let $\Sigma: \mathbb{R} \rightarrow \mathbb{R}$ be given by $\Sigma(x)=|x|$ if $|x| \geq 1$ and $\Sigma(x)=\frac{1}{2}\left(x^{2}+1\right)$ otherwise. Then $\Sigma \in \mathfrak{Y}_{1}$.

Note that the last two equations in (4.1) imply that for all $|s|<a$ we have:

$$
\begin{equation*}
f(s)>|s| \tag{4.3a}
\end{equation*}
$$

since $f$ is above its tangent at $s=a$, and:

$$
\begin{equation*}
\left|f^{\prime}(s)\right|<1 \tag{4.3b}
\end{equation*}
$$

for all $s \in(-a, a)$, since $f^{\prime}$ is strictly increasing on $[-a, a]$.
Before studying the shaking operation on the elements of $\mathfrak{Y}_{a}$, we first define a map between some sets $\mathfrak{Y}_{a}$ via double scaling.

Definition 4.4. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be any map and let $\epsilon \in(0,1)$. We define the two maps $f^{ \pm \epsilon}: \mathbb{R} \rightarrow \mathbb{R}$ by, for any $s \in \mathbb{R}$,

$$
f^{ \pm \epsilon}(s):=\left(1 \pm \frac{\epsilon}{2}\right) f\left(\frac{s}{1 \pm \frac{\epsilon}{2}}\right) .
$$

For instance, the maps $\Omega^{ \pm 1}$ are as follows:


We now gather some informations about $f^{ \pm \epsilon}$.
Lemma 4.5. Let $f \in \mathfrak{Y}_{2}$ and $\epsilon \in(0,1)$.
(i) We have $f^{ \pm \epsilon} \in \mathfrak{Y}_{2 \pm \epsilon}$.
(ii) For any $s \in \mathbb{R}$ we have $\left(f^{ \pm \epsilon}\right)^{\prime}(s)=f^{\prime}\left(\frac{s}{1 \pm \frac{\epsilon}{2}}\right)$.
(iii) The maps $g^{+\epsilon}:=f^{+\epsilon}-f$ and $g^{-\epsilon}:=f-f^{-\epsilon}$ are decreasing on $\mathbb{R}_{\geq 0}$.
(iv) For any $s \in \mathbb{R}$ we have $f^{-\epsilon}(s) \leq f(s) \leq f^{+\epsilon}(s)$, and:
(a) we have $f^{-\epsilon}(s)=f(s)$ if and only if $|s| \geq 2$,
(b) we have $f^{+\epsilon}(s)=f(s)$ if and only if $|s| \geq 2+\epsilon$,
(v) We have $\left\|f-f^{ \pm \epsilon}\right\|_{\infty}=\frac{\epsilon}{2} f(0)$.

Proof. (i) Clear.
(ii) Clear.
(iii) Follows from the previous point and from the fact that $f^{\prime}$ is increasing.
(iv) Since the functions are even, it suffices to prove the inequality on $\mathbb{R}_{\geq 0}$. We prove only the relations for $f^{+\epsilon}$, the ones for $f^{-\epsilon}$ being similar. By the previous point, we know that $g^{+\epsilon}$ is decreasing on $\mathbb{R}_{\geq 0}$, thus we deduce that $g \geq 0$ since $g(s)=|s|-|s|=0$ for $s \gg 0$. For the equality case, first note that the necessary condition holds. For the sufficient condition, since $f$ is strictly convex on $[-2,2]$ we have in fact $g^{\prime}(s)<0$ for all $s \in(0,2]$ thus as before we have $g(s)>0$ for all $s \in[0,2]$. Now for $s \in[2,2+2 \epsilon)$ we have $g^{\prime}(s)=f^{\prime}\left(\frac{s}{1+\epsilon}\right)-1<0$ since $1=f^{\prime}(2)$ and $f^{\prime}$ is strictly increasing on $[0,2]$ and we conclude the proof as before.
$(v)$ Again we only prove the result for $f^{+\epsilon}$. With the previous notation, we know that $g=f^{+\epsilon}-f$ is non-negative and decreases on $\mathbb{R}_{\geq 0}$ thus its maximum is reached at $s=0$.

We need one more definition before giving the main result of this section.
Definition 4.6. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a map. We denote by $\mathbb{Y}(f)$ the part of the graph of $f$ that is above the graph of the absolute value, that is:

$$
\mathbb{Y}(f):=\left\{(x, y) \in \mathbb{R}^{2}:|x| \leq y \leq f(x)\right\}
$$

For instance, we picture here in blue the set $\mathbb{Y}(\Omega)$ :


Theorem 4.7. Let $\alpha \in[0,1)$ and $a \in \mathbb{R}_{>0}$. For any $f \in \mathfrak{Y}_{a}$ we have:

$$
\mathrm{Sh}_{\alpha}(\mathbb{Y}(f)) \cup \mathbb{Y}\left(\operatorname{id}_{\mathbb{R}}\right)=\mathbb{Y}\left(f_{\alpha}\right)
$$

where $f_{\alpha}: \mathbb{R} \rightarrow \mathbb{R}$ is given by:

$$
\begin{aligned}
f_{\alpha}(x) & =f(x), & & \text { for all } x \leq x_{\alpha}^{-} \\
f\left(x+\delta_{x}\right) & =f(x)+\alpha \delta_{x}, & & \text { for all } x \in\left(x_{\alpha}^{-}, x_{\alpha}^{+}\right) \\
f_{\alpha}(x) & =x, & & \text { for all } x \geq s_{\alpha}:=x_{\alpha}^{+}+\delta_{x_{\alpha}^{+}}
\end{aligned}
$$

where:

- $x_{\alpha}^{+}=f^{\prime-1}(\alpha) \in[0, a)$,
- $\delta_{x}=(1-\alpha)^{-1} F_{\alpha}(x)-F_{\alpha}^{-1}\left(F_{\alpha}(x)\right)$, where $F_{\alpha}: x \mapsto f(x)-\alpha x$ and $F_{\alpha}^{-1}$ denotes its inverse on $\left(x_{\alpha}^{+}, a\right)$;
- $x_{\alpha}^{-} \in\left(-a, x_{\alpha}^{+}\right)$(is the unique point that) satisfies $F_{\alpha}\left(x_{\alpha}^{-}\right)=(1-\alpha) a$.

Note that $\mathbb{Y}\left(\mathrm{id}_{\mathbb{R}}\right)=\left\{(x, x): x \in \mathbb{R}_{\geq 0}\right\}$.


Figure 8: Illustration of the proof of Theorem 4.7 with $f=\Sigma$ and $\alpha=\frac{1}{4}$

Proof. As in Definition 3.2, let $L_{\alpha}$ be the line of equation $y=\alpha x$. For $M \in \mathbb{R}^{2}$, the line $L_{\alpha}^{M}:=M+L_{\alpha}$ cuts the curve of $f$ at most twice since $f$ is strictly convex. Note that we can ignore the parts of $\mathbb{Y}(f)$ outside $(-a, a)$ since it has measure 0 there. Note also that there is only one intersection point if and only if $\mathbb{Y}(f) \cap L_{\alpha}^{M}$ is stable under $\mathrm{Sh}_{\alpha}$. Moreover, it suffices to choose $M$ of the form $\left(0, y_{M}\right)$, and if $y_{M_{0}}$ is the unique real number such that $L_{\alpha}^{M_{0}}$ contains ( $a, a$ ) then:
for any $y \leq y_{M_{0}}$ then $L_{\alpha}^{M}$ intersects the curve of $f$ in $(-a, a)$ only once.
Let $x_{\alpha}^{-}<a$ be such that $L_{\alpha}^{M_{0}}$ intersects the curve of $f$ at $\left(x_{\alpha}^{-}, f\left(x_{\alpha}^{-}\right)\right)$. This means that:

$$
f_{\alpha}(x)=f(x), \quad \text { for all } x \leq x_{\alpha}^{-}
$$

Note that $x_{\alpha}^{-} \geq-a$ since $f(-a)=a$ and $\alpha \geq 0$. Now there is also a real number $y_{M_{1}}$ such that $L_{\alpha}^{M_{1}}$ does not intersects the curve of $f$ apart from $\left(y_{M_{1}}, f\left(y_{M_{1}}\right)\right)$. Again by convexity, this $y_{M_{1}}$ is given by the unique point such that $L_{\alpha}^{M_{1}}$ is tangent to the curve of $f$ at some point $M_{\alpha}=\left(x_{\alpha}^{+}, f\left(x_{\alpha}^{+}\right)\right)$. By (4.1) we know that such an $M_{\alpha}$ is unique and is determined by $f^{\prime}\left(x_{\alpha}^{+}\right)=\alpha$, in other words, by:

$$
x_{\alpha}^{+}=f^{\prime-1}(\alpha)
$$

(where $f^{\prime-1}$ denotes the inverse of $\left.f^{\prime}\right|_{(-a, a)}$ ). Note that $x_{\alpha}^{+} \geq 0$ since $\alpha \geq 0$ and $f$ is even, thus $f^{\prime}(0)=0 \leq \alpha$, and $f^{\prime}$ is increasing by strict convexity. Similarly, we have $x_{\alpha}^{+}<a$ since $f^{\prime}(a)=1>\alpha$, moreover the line between $\left(x_{\alpha}^{+}, f\left(x_{\alpha}^{+}\right)\right)$and ( $a, a$ ) has slope $>\alpha$ thus $x_{\alpha}^{+}>x_{\alpha}^{-}$.

Hence, to determine $f_{\alpha}$, it suffices to determine for each $x \in\left(x_{\alpha}^{-}, x_{\alpha}^{+}\right)$the measure $m$ of $\left(M+L_{\alpha}\right) \cap\left(\mathbb{Y}(f) \cap\left((x, a) \times \mathbb{R}_{\geq 0}\right)\right.$, with $M=(x, f(x))$, and to add it to $f(x)$ in the direction $\alpha$. More precisely, if:

- $x^{\prime}>x$ is the unique other abscissa where $L_{\alpha}^{x}:=M+L_{\alpha}$ crosses the curve of $f$,
- $\delta_{x}>0$ is the unique positive real number such that $x^{\prime}+\delta_{x}$ is the abscissa of the intersection between $L_{\alpha}^{x}$ and the line with equation $y=x$ for $x \geq 0$,
then the point $(x, f(x))$ becomes:

$$
\begin{equation*}
\left(x+\delta_{x}, f(x)+x^{\prime}+\delta_{x}-f\left(x^{\prime}\right)\right) . \tag{4.8}
\end{equation*}
$$

In Figure 8 we give an example of the various quantities that we have introduced so far for $f$ being the map $\Sigma$ of Example 4.2.

We will now determine $x^{\prime}$ and $\delta_{x}$ as functions of $x$. We first determine $x^{\prime}$. The line $L_{\alpha}^{x}$ has equation $(s-x) \alpha+f(x)$ for $s \in \mathbb{R}$, thus $x^{\prime}$ is given by:

$$
f\left(x^{\prime}\right)=\left(x^{\prime}-x\right) \alpha+f(x)
$$

thus:

$$
\begin{equation*}
f\left(x^{\prime}\right)-\alpha x^{\prime}=f(x)-\alpha x . \tag{4.9}
\end{equation*}
$$

Now the function $F_{\alpha}: s \mapsto f(s)-\alpha s$ has derivative $f^{\prime}(s)-\alpha$, thus since $\alpha \in(0,1)$ we know that $s \mapsto f(s)-\alpha s$ is decreasing for $s \leq x_{\alpha}^{+}$and increases (to $+\infty$ by (4.1b)) by for $s \geq x_{\alpha}^{+}$. Hence, we deduce that there is indeed a unique $x^{\prime}>x_{\alpha}^{+}$such that (4.9) is satisfied. Note that $F_{\alpha}$ is invertible on $\left(x_{\alpha}^{+},+\infty\right)$ and if $F_{\alpha}^{-1}$ denotes its inverse on this interval then:

$$
x^{\prime}=F_{\alpha}^{-1}\left(F_{\alpha}(x)\right) .
$$

Note that (4.9) writes:

$$
F_{\alpha}\left(x^{\prime}\right)=F_{\alpha}(x)
$$

Note also that $x<x_{\alpha}^{+}<x^{\prime}$ so that $x^{\prime} \neq x$. Finally, note that $x_{\alpha}^{-}$satisfies:

$$
F_{\alpha}\left(x_{\alpha}^{-}\right)=F_{\alpha}(a)=f(a)-\alpha a=(1-\alpha) a,
$$

as announced.
We now determine $\delta_{x}$. First note that the equation of $L_{\alpha}^{x}$ becomes $\alpha s+F_{\alpha}\left(x^{\prime}\right)$. We thus find that $\delta_{x}$ satisfies

$$
\begin{align*}
x^{\prime}+\delta_{x} & =\alpha\left(x^{\prime}+\delta_{x}\right)+F_{\alpha}\left(x^{\prime}\right),  \tag{4.10}\\
& =f\left(x^{\prime}\right)+\alpha \delta_{x} \tag{4.11}
\end{align*}
$$

in particular (4.10) gives:

$$
\begin{equation*}
x^{\prime}+\delta_{x}=(1-\alpha)^{-1} F_{\alpha}\left(x^{\prime}\right) . \tag{4.12}
\end{equation*}
$$

Recalling (4.9) and (4.11), the second coordinate in equation (4.8) becomes:

$$
f(x)+x^{\prime}+\delta_{x}-f\left(x^{\prime}\right)=f(x)+\alpha \delta_{x},
$$

as announced.
To prove that the set of points that we obtain is the curve of a function, it suffices to prove that $x \mapsto x+\delta_{x}$ is increasing on $\left(x_{\alpha}^{-}, x_{\alpha}^{+}\right)$, thus it suffices to prove that $x \mapsto \delta_{x}$ is increasing. By (4.12) we have:

$$
\delta_{x}=(1-\alpha)^{-1} F_{\alpha}\left(x^{\prime}\right)-x^{\prime} .
$$

Note that $x \mapsto x^{\prime}=F_{\alpha}^{-1}\left(F_{\alpha}(x)\right)$ is decreasing on $\left(x_{\alpha}^{-}, x_{\alpha}^{+}\right)$. Indeed, we have $F_{\alpha}^{\prime}(x)=f^{\prime}(x)-\alpha$ that is increasing since $f$ is strictly convex, and by definition $f^{\prime}\left(x_{\alpha}^{+}\right)=0$ thus $F_{\alpha}^{\prime}(x)<0$ for $x<x_{\alpha}^{+}$thus $F_{\alpha}$ is decreasing on $\left(x_{\alpha}^{-}, x_{\alpha}^{+}\right)$. Now $F_{\alpha}$ is increasing on $\left(x_{\alpha}^{+},+\infty\right)$ thus $F_{\alpha}^{-1}$ is also increasing. By composition, we deduce that $x \mapsto x^{\prime}=F_{\alpha}^{-1}\left(F_{\alpha}(x)\right)$ is decreasing on $\left(x_{\alpha}^{-}, x_{\alpha}^{+}\right)$, with values in $\left(x_{\alpha}^{+}, a\right)$.

Hence, to prove that $x \mapsto \delta_{x}=(1-\alpha)^{-1} F_{\alpha}\left(x^{\prime}\right)-x^{\prime}$ is increasing on $\left(x_{\alpha}^{-}, x_{\alpha}^{+}\right)$, it suffices to prove that $g: t \mapsto(1-\alpha)^{-1} F_{\alpha}(t)-t$ is decreasing in $t \in\left(x_{\alpha}^{+}, a\right)$. But this is clear since the derivative of $g$ is, after multiplication by $1-\alpha$,

$$
\begin{aligned}
F_{\alpha}^{\prime}(t)-(1-\alpha) & =\left(f^{\prime}(t)-\alpha\right)-(1-\alpha) \\
& =f^{\prime}(t)-1,
\end{aligned}
$$

which is strictly negative since $f^{\prime}$ is increasing (by strict convexity) and $f^{\prime}(t)<1$ for all $t<a$ by (4.1b).

In Figure 4 we represent the curve of $f_{\alpha}$ for $f=\Omega$ and $\alpha=0$.

Proposition 4.13. Let $\alpha \in[0,1)$ and $a \in \mathbb{R}_{>0}$. For any $f \in \mathfrak{Y}_{a}$ we have:

$$
(1-\alpha) s_{\alpha}=f\left(f^{\prime-1}(\alpha)\right)-\alpha f^{\prime-1}(\alpha) \in\left[f(0)-\alpha f\left(x_{\alpha}^{+}\right), f(0)\right]
$$

Proof. Recall that from Theorem 4.7 and its proof that $s_{\alpha}=x+\delta_{x}$ for $x=x_{\alpha}^{+}$. In this setting we have $x^{\prime}=x$ thus (4.12) gives:

$$
\begin{aligned}
(1-\alpha) s_{\alpha} & =F_{\alpha}\left(x_{\alpha}^{+}\right) \\
& =f\left(x_{\alpha}^{+}\right)-\alpha x_{\alpha}^{+} \\
& =f\left(f^{\prime-1}(\alpha)\right)-\alpha f^{\prime-1}(\alpha)
\end{aligned}
$$

which gives the desired equality. Now by the mean value theorem, there exists $c \in\left(0, x_{\alpha}^{+}\right)$ such that $F_{\alpha}\left(x_{\alpha}^{+}\right)=F_{\alpha}(0)+F_{\alpha}^{\prime}(c) x_{\alpha}^{+}$. Hence, since $f$ is strictly convex we know that $F_{\alpha}$ is also strictly convex, thus we have $F_{\alpha}^{\prime}(0)=f^{\prime}(0)-\alpha=-\alpha<F_{\alpha}^{\prime}(c)<F_{\alpha}^{\prime}\left(x_{\alpha}\right)=0$ (where $f^{\prime}(0)=0$ since $f$ is even). Recalling that $F_{\alpha}(0)=f(0)$ gives the announced inequality.

We have the following particular case of Theorem 4.7 and Proposition 4.13 for $\alpha=0$ (as pictured in Figure 4).
Corollary 4.14. Let $a \in \mathbb{R}_{>_{0}}$. For any $f \in \mathfrak{Y}_{a}$, the map $f_{0}: \mathbb{R} \rightarrow \mathbb{R}$ is given by:

$$
\begin{aligned}
f_{0}(x) & =f(x), & & \text { for all } x \leq-a \\
f_{0}(2 x+f(x)) & =f(x), & & \text { for all } x \in(-a, 0) \\
f_{0}(x) & =x, & & \text { for all } x \geq f(0)
\end{aligned}
$$

We conclude this part by relating the shapes $f_{\alpha}$ and $f_{\alpha}^{ \pm \epsilon}:=\left(f^{ \pm \epsilon}\right)_{\alpha}$.
Proposition 4.15. Let $\alpha \in[0,1)$ and $a \in \mathbb{R}_{>0}$. Let $f \in \mathfrak{Y}_{a}$ and $\epsilon \in(0,1)$. With the notation of Theorem 4.7 we have:

$$
\left(f^{ \pm \epsilon}\right)_{\alpha}=\left(f_{\alpha}\right)^{ \pm \epsilon}=: f_{\alpha}^{ \pm \epsilon} .
$$

Proof. We denote by $\delta_{x}^{ \pm \epsilon}$ and $F_{\alpha}^{ \pm \epsilon}$ the quantities that appear in Theorem 4.7 for $f^{ \pm \epsilon}$. Note that:

$$
\begin{aligned}
F_{\alpha}^{ \pm \epsilon}(x) & =f^{ \pm \epsilon}(x)-\alpha x \\
& =\left(1 \pm \frac{\epsilon}{2}\right) f\left(\frac{x}{1 \pm \frac{\epsilon}{2}}\right)-\left(1 \pm \frac{\epsilon}{2}\right) \alpha \frac{x}{1 \pm \frac{\epsilon}{2}} \\
& =\left(1 \pm \frac{\epsilon}{2}\right) F_{\alpha}\left(\frac{x}{1 \pm \frac{\epsilon}{2}}\right) \\
& =\left(F_{\alpha}\right)^{ \pm \epsilon}(x)
\end{aligned}
$$

and we deduce that $\left(F_{\alpha}^{ \pm \epsilon}\right)^{-1}=\left(F_{\alpha}^{-1}\right)^{ \pm \epsilon}$. We deduce that:

$$
\begin{equation*}
\delta_{x}^{ \pm \epsilon}=\left(1 \pm \frac{\epsilon}{2}\right) \delta_{\frac{x}{1 \pm \epsilon / 2}} . \tag{4.16}
\end{equation*}
$$

Now note that the equality $f\left(x+\delta_{x}\right)=f(x)+\alpha \delta_{x}$ holds in fact for any $x \in \mathbb{R}$. If $g$ denotes the inverse of $x \mapsto x+\delta_{x}$ on $\mathbb{R}$ (which exists since $x \mapsto x+\delta_{x}$ is strictly increasing by the proof of Theorem 4.7), if $y=x+\delta_{x}$ we thus have $x=g(y)$ and thus $\delta_{x}=y-g(y)$. We deduce that:

$$
\begin{equation*}
f_{\alpha}(y)=f(g(y))-\alpha(y-g(y))=F_{\alpha}(g(y))-\alpha y \tag{4.17}
\end{equation*}
$$

Now we deduce from (4.16) that the inverse of $x \mapsto x+\delta_{x}^{ \pm \epsilon}$ is $g^{ \pm \epsilon}$ (in the sense of Defini-
tion 4.4), thus by (4.17) we obtain:

$$
\begin{aligned}
f_{\alpha}^{ \pm \epsilon}(y) & =F_{\alpha}^{ \pm \epsilon}\left(g^{ \pm \epsilon}(y)\right)-\alpha y \\
& =\left(1 \pm \frac{\epsilon}{2}\right) F_{\alpha}\left(g\left(\frac{y}{1 \pm \frac{\epsilon}{2}}\right)\right)-\alpha y \\
& =\left(1 \pm \frac{\epsilon}{2}\right)\left[F_{\alpha}\left(g\left(\frac{x}{1 \pm \frac{\epsilon}{2}}\right)\right)-\alpha \frac{y}{1 \pm \frac{\epsilon}{2}}\right] \\
& =\left(1 \pm \frac{\epsilon}{2}\right) f_{\alpha}\left(\frac{y}{1 \pm \frac{\epsilon}{2}}\right) \\
& =\left(f_{\alpha}\right)^{ \pm \epsilon}(y)
\end{aligned}
$$

which is the announced result.
Proposition 4.18. Let $\alpha \in[0,1)$ and $a \in \mathbb{R}_{>0}$. Let $f \in \mathfrak{Y} a$. We have $0<\left\|f_{\alpha}-f_{\alpha}^{ \pm \epsilon}\right\|_{\infty, \mathbb{R}} \rightarrow$ 0 as $\epsilon \rightarrow 0$.

Proof. First, note that the infinite norm is positive indeed since $f_{\alpha}$ and $f_{\alpha}^{ \pm \epsilon}$ are continuous (even of class $\mathcal{C}^{1}$ ) by (4.17), by Lemma 4.5 and since the shaking preserve the area (this is a simple consequence of Fubini theorem, see, for instance, [FrLu, Lemma 3.2]). To prove that is goes to 0 as $\epsilon \rightarrow 0$, it suffices to prove that $f_{\alpha}$ is Lipschitz. Indeed, if $h: I \rightarrow \mathbb{R}$ is $L$-Lipschitz and bounded then:

$$
\begin{aligned}
h^{ \pm \epsilon}(x)-h(x) & =\left(1+\frac{\epsilon}{2}\right) h\left(\frac{x}{1+\frac{\epsilon}{2}}\right)-h(x) \\
& =h\left(\frac{x}{1+\frac{\epsilon}{2}}\right)-h(x)+\frac{\epsilon}{2} h\left(\frac{x}{1+\frac{\epsilon}{2}}\right),
\end{aligned}
$$

thus:

$$
\begin{aligned}
\left|h^{+\epsilon}(x)-h(x)\right| & \leq L|x|\left|\frac{1}{1+\frac{\epsilon}{2}}-1\right|+\frac{\epsilon}{2}\|h\|_{\infty} \\
& \leq\left(L|x|+\|h\|_{\infty}\right) \epsilon
\end{aligned}
$$

with a similar calculation for $h^{-\epsilon}$. Note that it suffices to study $f-f_{\alpha}$ on a segment since this function vanishes for $|x| \gg 0$. Recalling (4.17), we have:

$$
\begin{aligned}
f_{\alpha}^{\prime}(y) & =g^{\prime}(y) F_{\alpha}^{\prime}(g(y))-\alpha \\
& =g^{\prime}(y)\left[f^{\prime}(g(y))-\alpha\right]-\alpha .
\end{aligned}
$$

We know that $f^{\prime}$ is bounded by (4.1). For $g^{\prime}$, recall from the proof of Theorem 4.7 that $x \mapsto \delta_{x}$ is increasing thus the derivative of $x \mapsto x+\delta_{x}$ is greater than 1 , thus $\left\|g^{\prime}\right\|_{\infty} \leq 1$. This concludes the proof.

## 5 Shaking partitions

Let $\lambda$ be a partition. Proposition 3.1 implies the following result.
Lemma 5.1. We have $\operatorname{sh}_{e}\left(\mathbb{Y}\left(\omega_{\lambda}\right)\right)=\operatorname{sh}_{e}\left(\mathbb{Y}\left(\omega_{\operatorname{reg}_{e}(\lambda)}\right)\right)$ and $\operatorname{sh}_{e}\left(\mathbb{Y}\left(\widetilde{\omega}_{\lambda}\right)\right)=\operatorname{sh}_{e}\left(\mathbb{Y}\left(\widetilde{\omega}_{\operatorname{reg}_{e}(\lambda)}\right)\right)$.
Hence, to study the shaking operations on partitions it suffices to study the shaking operations on regular partitions. In fact, the shaking of $\mathbb{Y}\left(\widetilde{\omega}_{\mathrm{reg}_{e}(\lambda)}\right)$ for $\lambda$ an $e$-regular partition is a bit delicate to determine. Instead, we will bound the latter graph by two close graphs that are stable under the shaking operation.


Figure 9: Outer corners (in red) and inner corners (in blue) for $\lambda=(4,4,2,1)$

### 5.1 Corners

We say that $c \in \mathbb{Z}$ is an outer corner (resp. inner corner) of $\lambda$ if $\left.\omega_{\lambda}^{\prime}\right|_{(c-1, c)}=1$ (resp. - 1 ) and $\left.\omega_{\lambda}^{\prime}\right|_{(c, c+1)}=-1$ (resp. 1). We give in Figure 9 an example of outer and inner corners.

The following standard result follows from the definition of inner and outer corners and from the fact that $\omega_{\lambda}(x)=|x|$ for $x \gg 0$.
Lemma 5.2. Let $\lambda$ be a partition. If $\left\{o_{1}<\cdots<o_{k}\right\}$ (resp. $\left\{i_{1}<\cdots<i_{\ell}\right\}$ ) is the set of outer (resp. inner) corners then $\ell=k+1$ and $i_{m}<o_{m}<i_{m+1}$ for all $m \in\{1, \ldots, k\}$.

In the sequel, fix $e \geq 2$ and let $L_{e}$ be the linear subspace of $\mathbb{R}^{2}$ with slope $\alpha_{e}=1-2 e^{-1}$.

### 5.2 Outer regularisation

If $\lambda$ has no outer corners then we define $\rho^{+}(\lambda)=\rho_{0}^{+}(\lambda):=\omega_{\lambda}$, otherwise let $\left\{c_{1}<\cdots<c_{r}\right\}$ with $r \geq 1$ the set of outer corners. Assuming by induction on $k \in\{1, \ldots, r\}$ that we have constructed the piecewise linear function $\rho_{k-1}^{+}(\lambda): \mathbb{R} \rightarrow \mathbb{R}$, then we construct the piecewise linear function $\rho_{k}^{+}(\lambda): \mathbb{R} \rightarrow \mathbb{R}$ as follows. Let $L_{e}^{(k)}$ be the affine line $\left(c_{k}, \omega_{\lambda}\left(c_{k}\right)\right)+L_{e}$.
(i) For $s \geq c_{k}$ then $\rho_{k}^{+}(\lambda)(s):=\rho_{k-1}^{+}(\lambda)(s)$;
(ii) at $s=c_{k}$, we follow the line $L_{e}^{(k)}$ (in the negative direction) until we meet the curve of $\rho_{k-1}^{+}(\lambda)$, at point of abscissa $h_{k}$;
(iii) for $s \leq h_{k}$ then $\rho_{k}^{+}(\lambda)(s):=\rho_{k-1}^{+}(\lambda)(s)$ again.

Proposition 5.3. Assume that the partition $\lambda$ is e-regular. With the above notation we have $h_{k} \in\left[c_{k-1}, c_{k}\right)$. In particular, the functions $\rho_{k-1}^{+}(\lambda)$ and $\rho_{k}^{+}(\lambda)$ coincide on (at least) $\mathbb{R} \backslash\left(c_{k-1}, c_{k}\right)$, and for any $s \in\left(c_{k-1}, c_{k}\right)$ we have $0 \leq \rho_{k}^{+}(\lambda)(s)-\rho_{k-1}^{+}(\lambda)(s) \leq e$.
Proof. Since $\alpha_{e}<1$, the curve of $\omega_{\lambda}$ can cross $L_{e}^{(k)}$ only after an inner corner, that is, if $c^{\prime}<c_{k}$ is the inner corner preceding $c_{k}$ (cf. Lemma 5.2) then $h<c^{\prime}$. Moreover, by definition of an inner corner we have $\omega_{\lambda}\left(c^{\prime}-1\right)=\omega_{\lambda}\left(c^{\prime}\right)+1$, thus the line joining $\left(c_{k}, \omega_{\lambda}\left(c_{k}\right)\right)$ and $\left(c^{\prime}-1, \omega_{\lambda}\left(c^{\prime}-1\right)\right)$ has slope:

$$
\alpha:=\frac{\omega_{\lambda}\left(c_{k}\right)-\omega_{\lambda}\left(c^{\prime}\right)-1}{c_{k}-c^{\prime}+1} .
$$

Since $\omega_{\lambda}\left(c^{\prime}\right)=\omega_{\lambda}\left(c_{k}\right)-c_{k}+c^{\prime}$, we obtain:

$$
\alpha=\frac{c_{k}-c^{\prime}-1}{c_{k}-c^{\prime}+1}=1-\frac{2}{c_{k}-c^{\prime}+1} .
$$

But now $\lambda$ is $e$-regular thus $c^{\prime} \geq c_{k}-e+1$. We thus have $c_{k}-c^{\prime}+1 \leq e$ and thus, since $c_{k}-c^{\prime}+1>0\left(\right.$ since $\left.c^{\prime} \leq c_{k}\right)$ :

$$
\alpha \leq 1-2 e^{-1}=\alpha_{e} .
$$

Finally, we have $\alpha \leq \alpha_{e}<1$ thus $c^{\prime}-1 \leq h_{k}<c_{k}$, in other words, the line $L_{e}^{(k)}$ meet the curve of $\omega_{\lambda}$ somewhere between the points of abscissa $c^{\prime}$ (excluded) and $c^{\prime}-1$ (included) thus this concludes the first part of the proof.

Now for any $s \in\left(c_{k-1}, h\right)$ we have $\rho_{k}^{+}(\lambda)(s)=\rho_{k}^{+}(\lambda)(s)$, and by construction for any $s \in\left(h, c_{k}\right)$ we have $\rho_{k}^{+}(\lambda)(s)>\rho_{k}^{+}(\lambda)$ thus we obtain the first member of the announced inequality. The second one follows from the fact that $\alpha_{e} \in[0,1]$ and from the fact that on $\left(c_{k-1}, c_{k}\right)$, the function $s \mapsto \rho_{k}^{+}(\lambda)(s)-\rho_{k}^{+}(\lambda)(s)$ reaches its minimum at $s=h_{k}$, where:

$$
\omega_{\lambda}\left(c_{k}\right) \geq \rho_{k}^{+}(\lambda)\left(h_{k}\right) \geq \rho_{k-1}^{+}(\lambda)\left(h_{k}\right) \geq \omega_{\lambda}\left(c^{\prime}\right)
$$

Now we already saw that:

$$
\left\{\begin{array}{l}
\omega_{\lambda}\left(c^{\prime}\right)=\omega_{\lambda}\left(c_{k}\right)-c_{k}+c^{\prime} \\
c_{k}-c^{\prime}+1 \leq e
\end{array}\right.
$$

thus we obtain:

$$
\omega_{\lambda}\left(c_{k}\right)-\omega_{\lambda}\left(c^{\prime}\right)=c_{k}-c^{\prime} \leq e-1
$$

whence the result.
Definition 5.4. Let $e \geq 2$ and let $\lambda$ be an $e$-regular partition. Let $r \geq 0$ be the number of outer corners of $\lambda$. We define $\rho^{+}(\lambda):=\rho_{r}^{+}(\lambda)$.

Note that, by Proposition 5.3, the map $\rho^{+}(\lambda)$ does not in fact depend on the order that we chose on the outer corners. An example of a map $\rho^{+}(\lambda)$ is given in Figure 10.


Figure 10: The map $\rho^{+}(\lambda)$ (in thick red) for the 3-regular partition $\lambda=(4,4,2,1)$. In green are the different lines $L_{e}^{(k)}$.

Proposition 5.5. Recall that $\lambda$ is e-regular.
(i) For all $s \in \mathbb{R}$ we have $0 \leq \rho^{+}(\lambda)-\omega_{\lambda}(s) \leq e$.
(ii) The graph $G_{\lambda}^{+}:=\mathbb{Y}\left(\rho^{+}(\lambda)\right)$ is stable under the shaking operation with slope $\alpha_{e}$, that is, we have $\operatorname{sh}_{e}\left(G_{\lambda}^{+}\right)=G_{\lambda}^{+}$.

Proof. The first point follows directly from Proposition 5.3 since the intervals ( $c_{k-1}, c_{k}$ ) are pairwise disjoints. For the second point, it follows from the simple observation that the slopes of $\rho^{+}(\lambda)$ are either -1 or $\alpha_{e}$ (except the part $\rho^{+}(\lambda)(s)=s$ for $s \gg 0$ ). As a consequence, if
for $x \in \mathbb{R}$ the line $L^{\prime}:=x+L_{e}$ intersects the curve of $\rho^{+}(\lambda)$ at a point $s \in \mathbb{R}$ then $L^{\prime}$ remains below the curve $\rho^{+}(\lambda)$ on $(-\infty, s)$. Observing that $L^{\prime}$ intersects the curve of $|\cdot|$ on $(-\infty, 0)$ since $\alpha_{e}<1$ gives the result.

### 5.3 Inner regularisation

We now define a similar construction as $\rho^{+}(\lambda)$ but for inner corners. We give the statements without proofs since they are entirely similar.

Define $\rho_{r+1}^{-}(\lambda):=\omega_{\lambda}$ and let $\left\{c_{1}<\cdots<c_{r+1}\right\}$ with $r \geq 0$ the set of inner corners (in particular, note that a partition has always at least one inner corner). Assuming by decreasing induction on $k \in\{1, \ldots, r\}$ that we have constructed the piecewise linear function $\rho_{k+1}^{-}(\lambda): \mathbb{R} \rightarrow \mathbb{R}$, then we construct the piecewise linear function $\rho_{k}^{-}(\lambda): \mathbb{R} \rightarrow \mathbb{R}$ as follows. Let $L_{e}^{(k)}$ be the affine line $\left(c_{k}, \omega_{\lambda}\left(c_{k}\right)\right)+L_{e}$.
(i) For $s \leq c_{k}$ then $\rho_{k}^{-}(\lambda)(s):=\rho_{k+1}^{-}(\lambda)(s)$;
(ii) at $s=c_{k}$, we follow the line $L_{e}^{(k)}$ (in the positive direction) until we meet the curve of $\rho_{k+1}^{-}(\lambda)$, at point of abscissa $h_{k}$;
(iii) for $s \geq h_{k}$ then $\rho_{k}^{-}(\lambda)(s):=\rho_{k+1}^{-}(\lambda)(s)$ again.

Proposition 5.6. Assume that the partition $\lambda$ is e-regular. With the above notation we have $h_{k} \in\left[c_{k}, c_{k+1}\right)$. In particular, the functions $\rho_{k+1}^{-}(\lambda)$ and $\rho_{k}^{-}(\lambda)$ coincide on (at least) $\mathbb{R} \backslash\left(c_{k}, c_{k+1}\right)$, and for any $s \in\left(c_{k}, c_{k+1}\right)$ we have $-e \leq \rho_{k}^{-}(\lambda)(s)-\rho_{k+1}^{-}(\lambda)(s) \leq 0$.

An example of a map $\rho^{-}(\lambda)$ is given in Figure 11.


Figure 11: The map $\rho^{-}(\lambda)$ (in thick blue) for the 3-regular partition $\lambda=(4,4,2,1)$. In green are the different lines $L_{e}^{(k)}$.

Definition 5.7. Let $e \geq 2$ and let $\lambda$ be an $e$-regular partition. We define $\rho^{-}(\lambda):=\rho_{1}^{-}(\lambda)$.
Proposition 5.8. Recall that $\lambda$ is e-regular.
(i) For all $s \in \mathbb{R}$ we have $-e \leq \rho^{-}(\lambda)-\omega_{\lambda}(s) \leq 0$.
(ii) The graph $G_{\lambda}^{-}:=\mathbb{Y}\left(\rho^{-}(\lambda)\right)$ is stable under the shaking operation with slope $\alpha_{e}$, that is, we have $\operatorname{sh}_{e}\left(G_{\lambda}^{-}\right)=G_{\lambda}^{-}$.

## 6 Limit shape for regularisation of large partitions

We can now state our main theorem. Recall from Theorem 4.7 the definition of $\Omega_{\alpha}$ for $\alpha \geq 0$.

Theorem 6.1. Let $e \geq 2$ and take $\alpha=\alpha_{e}=1-2 e^{-1}$. Under the Plancherel measure $\mathrm{Pl}_{n}$, the function $\widetilde{\omega}_{\text {reg }_{e}(\lambda)}$ converges uniformly in probability to $\Omega_{\alpha}$ as $n \rightarrow+\infty$. In other words, for any $\epsilon>0$ we have

$$
\mathrm{Pl}_{n}\left(\sup _{\mathbb{R}}\left|\widetilde{\omega}_{\operatorname{reg}_{e}(\lambda)}-\Omega_{\alpha}\right|>\epsilon\right) \xrightarrow{n \rightarrow+\infty} 0
$$

Proof. Let $\epsilon>0$. Recalling the notation of Definition 4.4, define:

$$
m^{ \pm}:=\inf \left\{\left|\Omega^{ \pm \epsilon}(s)-\Omega(s)\right|:|s| \leq 2 \pm \frac{\epsilon}{2}\right\}
$$

By Lemma 4.5 applied with $f=\Omega$, we have $m^{ \pm}>0$. Now let $M_{n}$ be the set of partitions $\lambda \in \mathcal{P}_{n}$ such that:

$$
\begin{gathered}
\left\|\widetilde{\omega}_{\lambda}-\Omega\right\|_{\mathbb{R}}<\min \left(m^{+}, m^{-}\right), \\
\inf \{s \in \mathbb{R}: \widetilde{\omega}(s) \neq|s|\}>-2-\frac{\epsilon}{2} \\
\sup \{s \in \mathbb{R}: \widetilde{\omega}(s) \neq|s|\}<2+\frac{\epsilon}{2}
\end{gathered}
$$

Note that by Theorem 2.9 we have $\mathrm{Pl}_{n}\left(M_{n}\right) \rightarrow 1$ as $n \rightarrow+\infty$. By Lemma 4.5, for all $\lambda \in M_{n}$ we have:

$$
\Omega^{-\epsilon}(s) \leq \widetilde{\omega}_{\lambda}(s) \leq \Omega^{+\epsilon}(s),
$$

for all $s \in \mathbb{R}$. In other words, for all $\lambda \in M_{n}$ we have:

$$
\mathbb{Y}\left(\Omega^{-\epsilon}\right) \subseteq \mathbb{Y}\left(\widetilde{\omega}_{\lambda}\right) \subseteq \mathbb{Y}\left(\Omega^{+\epsilon}\right)
$$

By Proposition 3.4, Theorem 4.7 together with Lemma 5.1, we deduce that:

$$
\begin{equation*}
\mathbb{Y}\left(\Omega_{\alpha}^{-\epsilon}\right) \subseteq \operatorname{sh}_{e}\left(\mathbb{Y}\left(\widetilde{\omega}_{\mathrm{reg}_{e}(\lambda)}\right) \subseteq \mathbb{Y}\left(\Omega_{\alpha}^{+\epsilon}\right)\right. \tag{6.2}
\end{equation*}
$$

In particular, note that we can apply Theorem 4.7 for $\Omega^{ \pm \epsilon}$ indeed by Lemma 4.5, and $\Omega_{\alpha}^{ \pm \epsilon}:=\left(\Omega^{ \pm \epsilon}\right)_{\alpha}$. Now by Propositions 5.5 and 5.8 , for all $s \in \mathbb{R}$ we have:

$$
\widetilde{\rho^{-}}(\lambda)(s) \leq \widetilde{\omega}_{\lambda}(s) \leq \widetilde{\rho^{+}}(\lambda)(s)
$$

(where $\widetilde{\rho^{ \pm}}(\lambda)(s)=\frac{1}{\sqrt{n}} \rho^{ \pm}(\lambda)(\sqrt{n} s)$ is rescaled as $\widetilde{\omega}_{\lambda}$, see $\left.\S 2.2\right)$, thus:

$$
\mathbb{Y}\left(\widetilde{\rho^{-}}(\lambda)\right) \subseteq \mathbb{Y}\left(\widetilde{\omega}_{\operatorname{reg}_{e}}(\lambda)\right) \subseteq \mathbb{Y}\left(\widetilde{\rho^{+}}(\lambda)\right)
$$

By Propositions 5.5 and 5.8 again and Proposition 3.4, we deduce that:

$$
\begin{equation*}
\mathbb{Y}\left(\widetilde{\rho^{-}}(\lambda)\right) \subseteq \operatorname{sh}_{e}\left(\mathbb{Y}\left(\widetilde{\omega}_{\mathrm{reg}_{e}}(\lambda)\right)\right) \subseteq \mathbb{Y}\left(\widetilde{\rho^{+}}(\lambda)\right) \tag{6.3}
\end{equation*}
$$

By (6.2) and (6.3), we deduce that:

$$
\mathbb{Y}\left(\Omega_{\alpha}^{-\epsilon}\right) \subseteq \mathbb{Y}\left(\widetilde{\rho^{+}}(\lambda)\right), \quad \text { and } \quad \mathbb{Y}\left(\widetilde{\rho^{-}}(\lambda)\right) \subseteq \mathbb{Y}\left(\Omega^{+\epsilon}\right)
$$

Thus, we conclude that for any $s \in \mathbb{R}$ we have:

$$
\Omega_{\alpha}^{-\epsilon}(s) \leq \widetilde{\rho^{+}}(\lambda)(s), \quad \text { and } \quad \widetilde{\rho^{-}}(\lambda)(s) \leq \Omega_{\alpha}^{+\epsilon}(s)
$$

We deduce that for any $s \in \mathbb{R}$ we have:

$$
\begin{aligned}
\widetilde{\rho^{-}}(\lambda)(s) & \leq \Omega_{\alpha}^{+\epsilon}(s) \\
& \leq \Omega_{\alpha}^{-\epsilon}(s)+\left\|\Omega_{\alpha}^{-\epsilon}-\Omega_{\alpha}^{+\epsilon}\right\|_{\infty} \\
& \leq \widetilde{\rho^{+}}(\lambda)(s)+\left\|\Omega_{\alpha}^{-\epsilon}-\Omega_{\alpha}^{+\epsilon}\right\|_{\infty} \\
& \leq \widetilde{\rho^{-}}(\lambda)(s)+\frac{2 e}{\sqrt{n}}+\left\|\Omega_{\alpha}^{-\epsilon}-\Omega_{\alpha}^{+\epsilon}\right\|_{\infty} \quad \text { by Propositions 5.5 and 5.8. }
\end{aligned}
$$

Thus, again using Proposition 5.8, we deduce that for any $s \in \mathbb{R}$ we have:

$$
\begin{aligned}
\left|\widetilde{\omega}_{\operatorname{reg}_{e}(\lambda)}(s)-\Omega_{\alpha}(s)\right| & \leq\left|\widetilde{\omega}_{\operatorname{reg}_{e}(\lambda)}(s)-\rho^{-}(\lambda)(s)\right|+\left|\rho^{-}(\lambda)(s)-\Omega_{\alpha}^{+\epsilon}(s)\right|+\left|\Omega_{\alpha}^{+\epsilon}(s)-\Omega_{\alpha}(s)\right| \\
& \leq \frac{2 e}{\sqrt{n}}+\left(\frac{2 e}{\sqrt{n}}+\left\|\Omega_{\alpha}^{-\epsilon}-\Omega_{\alpha}^{+\epsilon}\right\|_{\infty}\right)+\left\|\Omega_{\alpha}^{+\epsilon}-\Omega_{\alpha}\right\|_{\infty},
\end{aligned}
$$

whence:

$$
\sup _{\mathbb{R}}\left|\widetilde{\omega}_{\mathrm{reg}_{e}(\lambda)}-\Omega_{\alpha}\right| \leq \frac{4 e}{\sqrt{n}}+3\left\|\Omega_{\alpha}^{+\epsilon}-\Omega_{\alpha}\right\|_{\infty}
$$

Thus, by Proposition 4.18 deduce that for $n \gg 0$ we have:

$$
\sup _{\mathbb{R}}\left|\widetilde{\omega}_{\operatorname{reg}_{e}(\lambda)}-\Omega_{\alpha}\right| \leq 4\left\|\Omega_{\alpha}^{+\epsilon}-\Omega_{\alpha}\right\|_{\infty}
$$

Recalling that $\lambda \in M_{n}$ with $\mathrm{Pl}_{n}\left(M_{n}\right) \rightarrow 1$ as $n \rightarrow+\infty$, we deduce that:

$$
\mathrm{Pl}_{n}\left(\sup _{\mathbb{R}}\left|\widetilde{\omega}_{\mathrm{reg}_{e}(\lambda)}-\Omega_{\alpha}\right|>4\left\|\Omega_{\alpha}^{+\epsilon}-\Omega_{\alpha}\right\|_{\infty}\right) \xrightarrow{n \rightarrow+\infty} 0,
$$

which, again by Proposition 4.18, proves the desired convergence.
We illustrate the convergence of Theorem 6.1 for $e=2$ in Figure 12 and for $e \in\{3,4\}$ in Figure 13, with in red the usual limit shape $\Omega$ and in blue the map $\widetilde{\omega}_{\operatorname{reg}_{e}(\lambda)}$ for $\lambda$ a partition of $3 \times 10^{4}$ taken under the Plancherel measure.


Figure 12: Example of 2-regularisation of a large partition, with in green the limit shape of Corollary B

Theorem 6.4. The convergence of Theorem 6.1 also holds for the supports, that is:

$$
\inf \left\{s \in \mathbb{R}: \widetilde{\omega}_{\operatorname{reg}_{e}(\lambda)}(s) \neq|s|\right\} \longrightarrow-2
$$

and

$$
\sup \left\{s \in \mathbb{R}: \widetilde{\omega}_{\operatorname{reg}_{e}(\lambda)}(s) \neq|s|\right\} \longrightarrow s_{\alpha}
$$

in probability under $\mathrm{Pl}_{n}$ as $n \rightarrow+\infty$, where $s_{\alpha}$ is the constant on Proposition 4.13 for $f=\Omega$.
Proof. We use the same set $M_{n}$ and the same strategy as in the proof of Theorem 6.1.
Finally, we deduce the asymptotic behaviour of the first part and the first column.
Corollary 6.5. Let $e \geq 2$. Under the Plancherel measure $\mathrm{Pl}_{n}$ :


Figure 13: Example of 3- and 4-regularisation of a large partition, respectively on the left and right.
(i) the rescaled size $\frac{1}{\sqrt{n}} \operatorname{reg}_{e}(\lambda)_{1}$ of the first row of $\operatorname{reg}_{e}(\lambda)$ converges as $n \rightarrow+\infty$ in probability to 2;
(ii) the rescaled size $\frac{1}{\sqrt{n}} \mathrm{reg}_{e}(\lambda)_{1}^{\prime}$ of the first column of $\mathrm{reg}_{e}(\lambda)$ converges as $n \rightarrow+\infty$ in probability to $\frac{2 e}{\pi} \sin \frac{\pi}{e}$.

Proof. The first point is clear by Theorem 6.4. For the second one, by the same theorem it suffices to prove that the announced value is equal to $s_{\alpha}$. Proposition 4.13 applied for $f=\Omega$ we have:

$$
s_{\alpha}=(1-\alpha)^{-1}\left[\Omega\left(\Omega^{\prime-1}(\alpha)\right)-\alpha \Omega^{\prime-1}(\alpha)\right] .
$$

Recall that $\alpha=1-2 e^{-1}$ and $\Omega^{\prime}(s)=\frac{2}{\pi} \arcsin \left(\frac{s}{2}\right)$. We have $\Omega^{\prime}(s)=t \Longleftrightarrow \arcsin \left(\frac{s}{2}\right)=$ $\frac{\pi t}{2} \Longleftrightarrow s=2 \sin \frac{\pi t}{2}$ thus $\Omega^{\prime-1}(t)=2 \sin \frac{\pi t}{2}$. We thus have:

$$
\begin{aligned}
\Omega^{\prime-1}(\alpha) & =2 \sin \frac{\pi \alpha}{2} \\
& =2 \sin \frac{\pi\left(1-2 e^{-1}\right)}{2} \\
& =2 \sin \left(\frac{p i}{2}-\frac{\pi}{e}\right) \\
& =2 \cos \frac{\pi}{e}
\end{aligned}
$$

We obtain:

$$
\begin{aligned}
s_{\alpha} & =\frac{e}{2}\left[\Omega\left(\Omega^{\prime-1}(\alpha)\right)-\alpha \Omega^{\prime-1}(\alpha)\right] \\
& =\frac{e}{2}\left[\Omega\left(2 \cos \frac{\pi}{e}\right)-2 \alpha \cos \frac{\pi}{e}\right] .
\end{aligned}
$$

We have, recalling the identity $\arcsin (\cos x)=\frac{\pi}{2}-x$ for $x \in[-1,1]$ :

$$
\begin{aligned}
\Omega\left(2 \cos \frac{\pi}{e}\right) & =\frac{2}{\pi}\left[\arcsin \left(\cos \frac{\pi}{e}\right) 2 \cos \frac{\pi}{e}+\sqrt{4-4 \cos ^{2} \frac{\pi}{e}}\right] \\
& =\frac{2}{\pi}\left[\left(\frac{\pi}{2}-\frac{\pi}{e}\right) 2 \cos \frac{\pi}{e}+2 \sin \frac{\pi}{e}\right] \\
& =2\left(1-\frac{2}{e}\right) \cos \frac{\pi}{e}+\frac{4}{\pi} \sin \frac{\pi}{e} \\
& =2 \alpha \cos \frac{\pi}{e}+\frac{4}{\pi} \sin \frac{\pi}{e}
\end{aligned}
$$

thus we finally obtain:

$$
s_{\alpha}=\frac{2 e}{\pi} \sin \frac{\pi}{e} .
$$

Example 6.6. Here are approximations of the first values of the limit for $\frac{1}{\sqrt{n}} \operatorname{reg}_{e}(\lambda)_{1}^{\prime}$ :

| $e$ | $\frac{2 e}{\pi} \sin \frac{\pi}{e}$ |
| :---: | :---: |
| 2 | 1.27 |
| 3 | 1.65 |
| 4 | 1.80 |

The reader can check that they match the corresponding values of Figures 12 and 13.

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[^1]:    ${ }^{1}$ All the computations were made using SAGEMATH [SM].

