On coprime percolation, the visibility graphon, and the local limit of the GCD profile

Sébastien Martineau

On coprime percolation, the visibility graphon, and the local limit of the GCD profile

Sébastien Martineau*

June 22, 2018

Abstract

Colour an element of $\mathbb{Z}^d$ white if its coordinates are coprime and black otherwise. What does this colouring look like when seen from a "uniformly chosen" point of $\mathbb{Z}^d$? Our answer to this question opens up further fields of investigation. One can generalise this problem in several directions (look from a point uniformly sampled in a subgroup, study the $\mathbb{N}$-valued colouring by the GCD, etc.): we solve the generalised problems, provide results of graphon convergence, as well as a "local/graphon" convergence. One can also study the percolative properties of the colouring under study.

Colour an element of $\mathbb{Z}^d$ white if its coordinates are coprime and black otherwise. What does this colouring look like? For $d = 2$, this question was investigated in [Var99]. The initial purpose of the present paper was to give a meaning and an answer to the following variation of this question: what does this colouring look like *when seen from a "uniformly chosen" point of* $\mathbb{Z}^d$? An answer has already been formulated in [PH13] but, from the perspective adopted in the current paper, our vocabulary, techniques and results are more satisfactory. See Figure 1.

Our answer to this question opens up further fields of investigation. One can generalise this problem in several directions (look from a point uniformly sampled in a subgroup, study the $\mathbb{N}$-valued colouring by the GCD, etc.): we solve the generalised problems, provide results of graphon convergence, as well as a "local/graphon" convergence[1]. By using previous work of Vardi [Var99], we can also study the percolative properties of the colouring under study for $d = 2$.

*Université Paris-Sud, sebastien.martineau@u-psud.fr

[1]Essentially, the same vertex-set will at the same time be endowed with some structure of sparse graph and some structure of dense graph.

1 Introduction

In this paper, the set $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$ is taken to contain 0. The set of positive integers will be denoted by $\mathbb{N}^\star = \{1, 2, 3, 4, \dots\}$.

Let d denote a positive integer. It is well-known that the probability that d numbers chosen independently and uniformly in $[\![1, N]\!]$ are globally coprime converges to $1/\zeta(d)$ when N goes to infinity [Dir51, Ces81, Ces83, Syl83]. Recall that on $[1, \infty)$, the **Euler-Riemann** ζ **function** is defined by

$$\forall s \in [1, \infty),\ \zeta(s) := \sum_{n \geq 1} n^{-s} = \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{-s}} \in [1, \infty],$$

where $\mathcal{P} = \{2, 3, 5, 7, 11, \dots\}$ denotes the set of prime numbers. More generally, one has the following result.

THEOREM 1.1 (THEOREM 459 IN [HW79]). *Let F be a bounded subset of $\mathbb{R}^d$. For every $r \in (0, \infty)$, set $F_r := \{x \in \mathbb{Z}^d \ :\ r^{-1}x \in F\}$. Assume that $\frac{|F_r|}{r^d}$ converges to a non-zero limit when r tends to infinity.*

Then, one has $\lim_{r\to\infty} \frac{|\{x \in F_r \ :\ \mathrm{GCD}(x_1,\dots,x_d)=1\}|}{|F_r|} = 1/\zeta(d)$.

The study of **coprime** vectors of $\mathbb{Z}^d$, i.e. of the x's such that $\mathrm{GCD}(x_1, \dots, x_d) = 1$, can be performed for its own sake. It may also be motivated by the reducibility of fractions (the probability that a random fraction cannot be reduced is $1/\zeta(2) = \frac{6}{\pi^2}$) for $d = 2$ or by the visibility problem for arbitrary d. If x and y denote two distinct points of $\mathbb{Z}^d$, one says that x is **visible** from y if the line segment $[x, y]$ intersects $\mathbb{Z}^d$ only at x and y. This condition is equivalent to $x - y$ having a GCD equal to 1. The set of visible points has been studied in various ways: see e.g. [BCZ00, BH15, BMP00, CFF, Gar15, GHKM, HS71, PH13, Var99].

In this paper, we are interested in taking limits of probability measures defined as follows. Let $\Omega := \{0, 1\}^{\mathbb{Z}^d}$, which is identified with $\mathsf{Subsets}(\mathbb{Z}^d)$ via $\omega \longleftrightarrow \omega^{-1}(\{1\})$. One element of interest is

$$\omega_0 := \{x \in \mathbb{Z}^d \ :\ \mathrm{GCD}(x_1, \dots, x_d) = 1\}.$$

For any $y \in \mathbb{Z}^d$ and $\omega \in \Omega$, one may define $\tau_y\omega$ by

$$\forall x \in \mathbb{Z}^d,\ x \in \tau_y\omega \iff x - y \in \omega.$$

Let F denote a nonempty finite subset of $\mathbb{Z}^d$, and let Y denote a uniformly chosen element of F. We denote by μ_F the distribution of $\tau_{-Y}\omega_0$. This is the distribution of "ω_0 seen from Y".

We want to describe the limit of μ_{F_n} for natural sequences (F_n), such as $([\![1, N]\!]^d)$, $([\![-N, N]\!]^d)$, or $(\{x \in \mathbb{Z}^d : \|x\|_2 \leq n\})$. In order to specify the meaning of "natural" and "limit" in the previous sentence, we need some

additional vocabulary. A **Følner sequence** of $\mathbb{Z}^d$ is a sequence (F_n) of finite nonempty subsets of $\mathbb{Z}^d$ such that for every $y \in \mathbb{Z}^d$, one has $|F_n \Delta (F_n + y)| = o(|F_n|)$. It suffices to check this condition for a family of y's generating $\mathbb{Z}^d$ as a group. This condition is also equivalent to $\lim_n \frac{|\partial F_n|}{|F_n|} = 0$, where the **boundary** ∂F of $F \subset \mathbb{Z}^d$ is the set of the elements of $\mathbb{Z}^d \backslash F$ which are adjacent to an element of F for the usual (hypercubic) graph structure of $\mathbb{Z}^d$.

Let (μ_n) be a sequence of probability measures on Ω. Let μ be a probability measure on Ω. One says that μ_n **converges** to μ if for every cylindric event E, the quantity $\mu_n(E)$ converges to $\mu(E)$. Recall that a **cylindric event** is an event of the form $\{\omega \in \Omega \ : \ \omega_{|F} \in A\}$, where F is a finite subset of $\mathbb{Z}^d$, $\omega_{|F}$ stands for the restriction of ω to F, and A is some subset of $\{0,1\}^F$.

Remark. This notion of convergence agrees with weak convergence of probability measures on Ω, where Ω is endowed with the product of discrete topologies. Every sequence of probability measures on Ω converges to at most one probability measure. By using the finiteness of $\{0,1\}$, it is not hard to prove via a diagonal argument that every sequence of probability measures on Ω admits a converging subsequence. This notion of convergence is very closely related to local convergence and local weak convergence (also called Benjamini-Schramm convergence): see [Bab91, BS01, DL01].

Let us now define μ_∞, which will be the limiting probability measure on Ω. For every $p \in \mathcal{P}$, let W_p denote a uniformly chosen coset of $p\mathbb{Z}^d$ in $\mathbb{Z}^d$, i.e. one of the p^d sets of the form $x + p\mathbb{Z}^d$. Do all these choices independently. Colour each vertex that belongs to at least one W_p black, and set the remaining vertices to be white. The distribution of the set of white vertices is denoted by μ_∞. See Figure 1.

We will prove the following theorem.

THEOREM 1.2. *Let $d \geq 1$ and let (F_n) be a Følner sequence of $\mathbb{Z}^d$. Assume that $\mu_{F_n}(\{\omega \ : \ (0,\dots,0) \in \omega\})$ converges to $1/\zeta(d)$.*

Then, μ_{F_n} converges to μ_∞.

Remark. The convergence of μ_{F_n} to μ_∞ for some sequences (F_n) of balls was conjectured by Vardi and obtained by Pleasants and Huck: see Conjecture 1 in [Var99] and Theorem 1 in [PH13]. However, Pleasants and Huck were not phrasing this in the vocabulary of convergence of probability measures: they had countably many results of convergence of probabilities (one per cylindric event) instead of a unique limiting object. This is not merely cosmetic: with μ_∞ at hand, we can now ask new questions, as we can try to compute the probability of some non-cylindric events (see Section 2). Besides, compared with that in [PH13], our proof is quite computation-free: for a given Følner sequence (F_n), the only computations needed to obtain the convergence to μ_∞ are those required to prove the convergence of the density to $\zeta(d)^{-1}$.

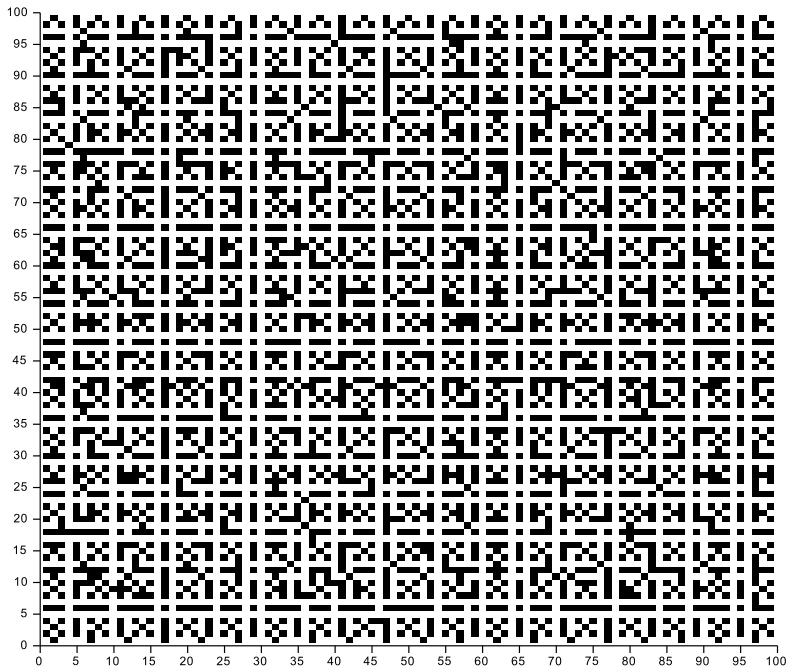

Figure 1: A sample of μ_∞ for $d = 2$. By Theorem 1.2, this corresponds to the coprime colouring "seen from a uniform point in $\mathbb{Z}^2$".

We will see that, in Theorem 1.2, none of the Følner condition and the $1/\zeta(d)$-condition can be removed. This theorem will be obtained as a corollary of the more informative Theorem 1.3. Before stating it, we need to introduce the notion of stochastic domination. Let μ and ν denote two probability distributions. A **coupling** of (μ, ν) is the data of a couple of random variables (X, Y) defined on a same probability space such that X has distribution μ and Y has distribution ν. If μ and ν are two probability distributions on Ω, we say that μ is **stochastically dominated** by ν if there is a coupling (W, W') of (μ, ν) such that $W \subset W'$ almost surely.

THEOREM 1.3. *Let $d \geq 1$ and let (F_n) be a Følner sequence of $\mathbb{Z}^d$. Assume that μ_{F_n} converges to some probability measure μ.*

Then, μ is stochastically dominated by μ_∞.

Instead of looking at the coprime colouring, one can try to perform the same study by colouring each point of $\mathbb{Z}^d$ by its GCD. Let us first recall the analog of the $1/\zeta(d)$-result for this question, which is a direct corollary of

Theorem 1.1. Call **zeta distribution** of parameter $s > 1$ the probability distribution on $\mathbb{N}^\star$ giving weight $\frac{n^{-s}}{\zeta(s)}$ to each $n \in \mathbb{N}^\star$.

THEOREM 1.4. *Let $d \geq 2$. Let F be a bounded subset of $\mathbb{R}^d$. For every $r \in (0, \infty)$, set $F_r := \{x \in \mathbb{Z}^d \ : \ r^{-1}x \in F\}$. Assume that $\frac{|F_r|}{r^d}$ converges to a non-zero limit when r tends to infinity. Let Y_r denote a uniform element in F_r.*

Then, $\mathrm{GCD}(Y_r)$ converges in distribution to zeta distribution of parameter d, as r goes to infinity.

Let $\Omega_{\mathbb{N}} := \mathbb{N}^{\mathbb{Z}^d}$. We denote by ω_{GCD} the element of $\Omega_{\mathbb{N}}$ defined by $\omega_{\mathrm{GCD}} : x \mapsto \mathrm{GCD}(x)$. For any $y \in \mathbb{Z}^d$ and $\omega \in \Omega_{\mathbb{N}}$, one may define $\tau_y\omega$ by $\tau_y\omega : x \mapsto \omega(x - y)$. Let F denote a nonempty finite subset of $\mathbb{Z}^d$, and let Y denote a uniformly chosen element in F. The probability measure ν_F is defined to be the distribution of $\tau_{-Y}\omega_{\mathrm{GCD}}$.

Let (ν_n) denote a sequence of probability measures on $\Omega_{\mathbb{N}}$. Let ν be a probability measure on $\Omega_{\mathbb{N}}$. One says that ν_n **converges** to ν if for *every* cylindric event E, the quantity $\nu_n(E)$ converges to $\nu(E)$ when n goes to infinity.

Remark. Any sequence of probability measures on $\Omega_{\mathbb{N}}$ admits at most one limit. However, it is not the case anymore that any sequence of probability measures on $\Omega_{\mathbb{N}}$ admits a limit up to extraction: you may set ν_n to be the Dirac mass on the constant map of value n.

We want to prove that for natural sequences (F_n), the sequence (ν_{F_n}) converges to some probability measure ν_∞ which does not depend on the choice of (F_n). Before stating the corresponding theorem, let us define the relevant measure ν_∞.

For every prime p, set $W_0^p := \mathbb{Z}^d$, pick a uniform coset W_1^p of $p\mathbb{Z}^d$ in W_0^p, and conditionally on that a uniform coset W_2^p of $p^2\mathbb{Z}^d$ in W_1^p, and conditionally on that a uniform coset W_3^p of $p^3\mathbb{Z}^d$ in W_2^p, etc. Do this independently for every p. We set the random p-adic valuation of a vertex x in $\mathbb{Z}^d$ to be $V_p(x) := \sup\{n \in \mathbb{N} \ : \ x \in W_n^p\} \in [\![0, \infty]\!]$. We define the random GCD to be the random map $x \mapsto \prod_{p \in \mathcal{P}} p^{V_p(x)}$. (This occurs almost surely nowhere, but one should set $\prod_{p \in \mathcal{P}} p^{V_p(x)}$ to be 0 whenever $\forall p,\ V_p(x) = \infty$.) The distribution of the random GCD is denoted by ν_∞. It is a priori a probability distribution on $\Omega_{[\![0,\infty]\!]} := [\![0, \infty]\!]^{\mathbb{Z}^d}$, but by the Borel-Cantelli Lemma, for every $d \geq 2$, it is also a probability distribution on $\Omega_{\mathbb{N}}$. Notice that in Theorem 1.5, the assumptions imply that $d \geq 2$.

Remark. The data of $(W_n^p)_{n \in \mathbb{N},\ p \in \mathcal{P}}$ is the same as that of d independent random elements in $\hat{\mathbb{Z}} := \varprojlim_n \mathbb{Z}/n\mathbb{Z}$, each distributed according to the Haar measure of $\hat{\mathbb{Z}}$. The Chinese Remainder Theorem guarantees that a Haar-distributed element of $\hat{\mathbb{Z}}$ is the same as choosing independently a Haar-distributed p-adic integer for every prime p.

A sequence (η_n) of probability distributions on $\mathbb{N}$ is **tight** if

$$\forall \varepsilon > 0,\ \exists K \geq 0,\ \forall n \geq 0,\ \eta_n([\![0, K]\!]) > 1 - \varepsilon.$$

THEOREM 1.5. *Let $d \geq 1$ and let (F_n) be a Følner sequence of $\mathbb{Z}^d$. For every n, let Y_n denote a uniformly chosen element of F_n. Assume that the distribution of $\mathrm{GCD}(Y_n)$ is tight.*

Then, ν_{F_n} converges to ν_∞.

Structure of the paper The remaining of the paper is organised as follows. Section 2 makes a few remarks on Theorems 1.2, 1.3 and 1.5. In particular, we make a few comments regarding the percolative properties of the measue μ_∞: even though such questions cannot be asked without the formalism of convergence of measures, their answers are derived from the study of Vardi [Var99], which was devoid of this formalism. Section 3 proves Theorems 1.2, 1.3 and 1.5. Finally, Section 4 provides several generalisations of these results.

2 Several remarks

We insist that in none of our results, we ask for the sequence (F_n) to be monotone, or for $\bigcup_n F_n$ to be equal to $\mathbb{Z}^d$. The fact that $|F_n|$ tends to infinity is a consequence of being Følner.

Even though the conclusion of Theorem 1.5 implies that of Theorem 1.2, it is not the case that Theorem 1.2 can be derived as a direct corollary of Theorem 1.5. Indeed, one can find a Følner sequence (F_n) such that,

- if Y_n denotes a uniform element of F_n, the distribution of $\mathrm{GCD}(Y_n)$ is not tight,
- the proportion of coprime vectors in F_n converges to $1/\zeta(d)$.

This can be derived from the Chinese Remainder Theorem, or from the proof of Theorem 1.5. Let us establish this fact. We will use the probability measure ν'_∞ introduced in Section 3.1.

Proof. For $d = 1$, any Følner sequence satisfies automatically the desired properties. Let us assume that $d \geq 2$, so that ν_∞-almost every ω takes only finite non-zero values[2]. Let $\varepsilon > 0$ and let $K \in \mathbb{N}$.

Since $\nu_\infty(\{\omega \in \Omega_\mathbb{N} : \omega(0) = 1\}) = \prod_{p \in \mathcal{P}}(1 - p^{-d}) = 1/\zeta(d)$, for every $N \in \mathbb{N}$, the ν'_∞-expected proportion of $x \in [\![-N, N]\!]^d$ satisfying $\min_p \omega'(x, p) = 1$

[2]Actually, infinite values would help us. One could also not make the distinction between finite and infinite values by considering supernatural numbers, see Section 3.3.

is $1/\zeta(d)$. Either by ergodicity[3] or because of Theorem 1.3, one can pick N such that this proportion has positive probability to be ε-close to $1/\zeta(d)$ — let us call this event E. One can pick some prime $P \geq \max(K, 2N+1)$ such that with positive probability, this proportion is ε-close to $1/\zeta(d)$ and for every $x \in [\![-N, N]\!]^d$ and every prime $p \geq P$, one has $p \nmid \omega(x)$, i.e. "not $p|\omega(x)$". Let $\mathfrak{p}$ denote an injective map from $[\![-N, N]\!]^d$ to $\{p \in \mathcal{P} : p \geq P\}$. For every $x \in [\![-N, N]\!]^d$ such that $\omega(x) \neq 1$, one can use the prime $\mathfrak{p}(x)$ to make its GCD at least K without changing the points of $[\![-N, N]\!]^d$ with a GCD equal to 1: conditionally on E, there is a positive probability that every $x \in [\![-N, N]\!]^d$ satisfies $\omega(x) \neq 1 \implies \omega(x) \geq \mathfrak{p}(x) \geq K$. Call E' the event that this occurs (together with E).

Let (F_n^0) be a Følner sequence such that, if Y_n denotes a uniform point in F_n^0, then $\mathrm{GCD}(Y_n)$ is tight. Such sequences exist: for instance, by Theorem 1.4, one may take $F_n^0 := [\![0, n]\!]^d$. By Theorem 1.5, the proportion of $x \in F_n^0$ such that $\tau_{-Y_n}\omega_{\mathrm{GCD}}$ satisfies E' converges to a positive number, hence is positive for n large enough. In particular, there is some $y \in \mathbb{Z}^d$ such that $\tau_{-y}\omega_{\mathrm{GCD}}$ satisfies E'. We say that any such y is an (N, K, ε)-counterexample. For every n, pick some y^n that is an $(n, n, 1/n)$-counterexample: the sequence $F_n = \prod_{i=1}^d [\![y_i^n - n, y_i^n + n]\!]$ satisfies the desired properties. □

Likewise, the $1/\zeta(d)$-condition cannot be removed from Theorem 1.2: it is well-known that the Chinese Remainder Theorem implies that there are arbitrarily large boxes $[\![x, x+N]\!]^d$ in $\mathbb{Z}^d$ devoid of coprime point. This also follows from Theorems 1.1 and 1.2, as μ_∞ has a positive probability to colour black the whole box $[\![0, N]\!]^d$ — use one prime per point of the box. The Følner condition cannot be removed from Theorem 1.2 either.

All the results in this paper concerning visibility extend readily to lattices in $\mathbb{R}^d$, as any such lattice may be mapped to $\mathbb{Z}^d$ by a linear automorphism of $\mathbb{R}^d$, which preserves visibility.

Notice that μ_∞ and ν_∞ are translation-invariant probability measures, but that they are also $\mathsf{GL}_d(\mathbb{Z})$-invariant. Even though ω_0 and ω_{GCD} are indeed $\mathsf{GL}_d(\mathbb{Z})$-invariant, the $\mathsf{GL}_d(\mathbb{Z})$-invariance of the measures does not follow from Theorems 1.2 and 1.5. One way to understand why goes as follows. Every orbit of the group G generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ contains a unique point inside $A := \{x \in \mathbb{Z}^2 : 0 \leq x_1 < |x_2|\} \cup (\mathbb{Z} \times \{0\})$. Consider the colouring of $\mathbb{Z}^2$ which is defined as a the chessboard colouring on A (say white if the sum of coordinates is odd and black otherwise), and take its unique G-invariant extension. For $F_n = A \cap [\![-n, n]\!]^2$, this colouring seen from a uniform point in F_n converges to the unbiased choice of one of the two chessboard colourings of the plane: this probability measure is not G-

[3]Later in this section, we will see that ν_∞ is ergodic, i.e. that any translation-invariant event has probability 0 or 1. One can then use an ergodic theorem such as Theorem 1.1 in [Lin01].

invariant.

Finally, let us mention a few questions that can be asked only with the formalism of local limits, i.e. with μ_∞ instead of just one convergence result per cylindric event. We can ask whether the measure μ_∞ (resp. ν_∞) is ergodic — i.e. if every translation-invariant measurable subset of Ω (resp. $\Omega_{\mathbb{N}}$) has μ_∞-probability (resp. ν_∞-probability) 0 or 1. And indeed, these measures are ergodic, as well as the measures μ'_∞ and ν'_∞ of Sections 3.1 and 3.3. For simplicity, we expose the argument for μ_∞. The Chinese Remainder Theorem guarantees the following fact: if $p_1, \dots, p_n$ denote the first n primes, and if for every $i \leq n$, C_i denotes a coset of $p_i\mathbb{Z}^d$ in $\mathbb{Z}^d$, then there is a translation mapping every C_i to $p_i\mathbb{Z}^d$. As a result, whenever we consider only finitely many primes, choosing one coset per prime yields a *deterministic* outcome up to translation. One concludes by noting that if a translation-invariant probability measure on $\{0,1\}^{\mathbb{Z}^d \times \mathcal{P}}$ yields ergodic measures in projection to any $\{0,1\}^{\mathbb{Z}^d \times \{p_1,\dots,p_n\}}$, then the measure under study is itself ergodic. This is easily proved by martingale theory, and similar reasonings are classical in the study of profinite actions.

Considering $\mathbb{Z}^d$ to be endowed with its usual (hypercubic) graph structure, one may also ask questions of percolation theory [Gri99, LP16]: how many infinite white (resp. black) connected components does the colouring μ_∞ yield? By ergodicity, these numbers have to be deterministic outside some event of probability zero. One can derive from Theorem 3.3 in [Var99] that, for $d = 2$ hence for any $d \geq 2$, there is at least one infinite white connected component almost surely. One can derive from Theorem 3.4 in [Var99] that, for $d = 2$, there is almost surely at most one infinite white connected component and no infinite black component.

3 Proofs of Theorems 1.2, 1.3 and 1.5

3.1 Proof of Theorem 1.3

LEMMA 3.1. *Let $d \geq 1$, let N be a positive integer, and let $\pi : \mathbb{Z}^d \to (\mathbb{Z}/N\mathbb{Z})^d$ denote reduction modulo N. Let (F_n) be a Følner sequence of $\mathbb{Z}^d$, and let Y_n denote a uniformly chosen element of F_n.*

Then, $\pi(Y_n)$ converges in distribution to the uniform measure on $(\mathbb{Z}/N\mathbb{Z})^d$.

Remark. Since $(\mathbb{Z}/N\mathbb{Z})^d$ is a discrete set, convergence in distribution coincides with that in total variation.

Proof. Partition $\mathbb{Z}^d$ into boxes of the form $\prod_{i=1}^d [\![Nx_i, N(x_i+1)-1]\!]$. For every n, say that an element x of F_n is n-good if the box B containing it satisfies $B \subset F_n$. Let Y_n denote a uniformly chosen element of F_n. Because (F_n) is a Følner sequence, the probability that Y_n is n-good converges to 1 as n goes to infinity. But if Y'_n denotes a uniformly chosen *n-good* element

of F_n, then $\pi(Y'_n)$ is precisely uniform in $(\mathbb{Z}/N\mathbb{Z})^d$: one may generate Y'_n by picking independently a uniform element Y in $[\![0, N-1]\!]^d$ and a uniform element Y''_n in $\{x \in N\mathbb{Z}^d : x \text{ is } n\text{-good}\}$, and then writing $Y'_n = Y''_n + Y$. The result follows. □

Let $\Omega' := \{0,1\}^{\mathbb{Z}^d \times \mathcal{P}}$. For every $p \in \mathcal{P}$, pick a uniform coset W_p of $p\mathbb{Z}^d$ among the p^d different ones. Do this independently for every p. One defines the random element W' of Ω' via $W'(x,p) := \mathbb{1}_{x \notin W_p}$. The probability measure μ'_∞ is defined as the distribution of the random variable W'. Notice that the random element W of Ω defined by $W(x) := \min_p W'(x,p)$ has distribution μ_∞.

Once again, for every $y \in \mathbb{Z}^d$ and $\omega \in \Omega'$, one may define $\tau_y\omega$ as $(x,p) \mapsto \omega(x-y,p)$. If F is a nonempty finite subset of $\mathbb{Z}^d$ and if Y denotes a uniformly chosen element of F, the distribution of $\tau_{-Y}\omega'_0$ is denoted by μ'_F, where $\omega'_0 : (x,p) \mapsto \mathbb{1}_{p \nmid \mathrm{GCD}(x)}$. Convergence for probability measures on Ω' is again concerned with the convergence of the probabilities of cylindric events, i.e. involving finitely many (x,p)'s.

PROPOSITION 3.2. *Let $d \geq 1$ and let (F_n) be a Følner sequence of $\mathbb{Z}^d$. Then, μ'_{F_n} converges to μ'_∞.*

Proof. Let k be a positive integer, and let $p_1, \ldots, p_k$ denote the k smallest primes. Let $N := p_1 \ldots p_k$. Let $\pi_i : \mathbb{Z}^d \to (\mathbb{Z}/p_i\mathbb{Z})^d$ denote reduction modulo p_i, and let $\pi : \mathbb{Z}^d \to (\mathbb{Z}/N\mathbb{Z})^d$ denote reduction modulo N. Let Y_n denote a uniformly chosen element of F_n. It is enough to show that $(\pi_1(Y_n), \ldots, \pi_k(Y_n))$ converges in distribution to the product of uniform distributions on $(\mathbb{Z}/p_i\mathbb{Z})^d$, namely the uniform distribution on $\prod_{i=1}^k (\mathbb{Z}/p_i\mathbb{Z})^d$. By the Chinese Remainder Theorem, it suffices to prove that $\pi(Y_n)$ converges in distribution to the uniform distribution on $(\mathbb{Z}/N\mathbb{Z})^d$, which is guaranteed by Lemma 3.1. □

Proof of Theorem 1.3. For Y_n uniformly chosen in F_n, let us consider $(\tau_{-Y}\omega'_0, \tau_{-Y}\omega_0) \in \Omega' \times \Omega$. Up to taking a subsequence, we may assume that the distribution of $(\tau_{-Y_n}\omega'_0, \tau_{-Y_n}\omega_0)$ converges to some probability measure ρ on $\Omega' \times \Omega$. Notice that for every $x \in \mathbb{Z}^d$,

$$\omega'_0(x,p) = 0 \implies \omega_0(x) = 0.$$

Besides, for every x and p (which range over countable sets), the event $\{(\omega', \omega) : \omega'(x,p) = 0 \implies \omega(x) = 0\}$ is cylindric. As a result, for ρ-almost every (ω', ω), for every $x \in \mathbb{Z}^d$, one has $\omega(x) \leq \min_p \omega'(x,p)$. But recall that if (W', W) denotes a random variable with distribution ρ, then W has distribution μ and $\min_p W'(x,p)$ has distribution μ_∞. Theorem 1.3 follows.

□

3.2 Proof of Theorem 1.2

Let us make the assumptions of Theorem 1.2. Up to taking a subsequence of (F_n), we may assume that μ_{F_n} converges to some μ. We want to prove that $\mu = \mu_\infty$. By Theorem 1.3, there is a monotone coupling of (μ, μ_∞), i.e. some coupling ρ of (μ, μ_∞) such that for ρ-almost every (ω, ω_∞), one has $\forall x,\ \omega(x) \leq \omega_\infty(x)$. But by the $1/\zeta(d)$-condition, for every $x \in \mathbb{Z}^d$, one has

$$\mu(\{\omega : \omega(x) = 1\}) = 1/\zeta(d) = \prod_{p \in \mathcal{P}} (1 - p^{-d}) = \mu_\infty(\{\omega : \omega(x) = 1\}).$$

As a result, for ρ-almost every (ω, ω_∞), one has $\omega = \omega_\infty$. Thus, μ is equal to μ_∞ and Theorem 1.2 follows.

3.3 Proof of Theorem 1.5

Let us make the assumptions of Theorem 1.2. Let $\Omega_\dagger := [\![0, \infty]\!]^{\mathbb{Z}^d \times \mathcal{P}}$. The prime factorisation of integers yields an injection of $\mathbb{N}$ into the set $[\![0, \infty]\!]^{\mathcal{P}}$ of the so-called **supernatural numbers**. The definition of ν_∞ is readily adapted to a corresponding probability measure ν'_∞ on $\Omega_\dagger$ — and actually, since tightness guarantees $d \geq 2$, these measures almost surely assign to every point of $\mathbb{Z}^d$ a positive (finite) integer, which corresponds to an element of $\mathbb{N}^{\mathcal{P}}$ with finite support in the supernatural setting.

Let $\omega_\dagger$ be the element of $\Omega_\dagger$ mapping (x, p) to the p-adic valuation of $\textsc{gcd}(x)$. Let Y_n denote a uniformly chosen element of F_n. As in Proposition 3.2, it is the case that the distribution of $\tau_{-Y_n}\omega_\dagger$ converges to ν'_∞. Besides, as $\textsc{gcd}(Y_n)$ is tight and (F_n) is a Følner sequence, it is the case that every subsequence of $\tau_{-Y_n}\omega_{\textsc{gcd}}$ admits a converging subsequence. We may thus assume that the distribution of $\tau_{-Y_n}\omega_{\textsc{gcd}}$ converges to some ν, and we want to prove that $\nu = \nu_\infty$.

Actually, we may further assume that the distribution of the random variable $(\tau_{-Y_n}\omega_\dagger, \tau_{-Y_n}\omega_{\textsc{gcd}})$ converges to some probability measure ρ on $\Omega_\dagger \times \Omega_{\mathbb{N}}$. For every $x \in \mathbb{Z}^d$ and every $n \in \mathbb{N}$, it is the case that

$$\omega_\dagger(x, p) \geq n \iff p^n | \omega_{\mathsf{gcd}}(x).$$

Therefore, for ρ-almost every $(\omega^\dagger, \omega)$, for every $x \in \mathbb{Z}^d$ and every $n \in \mathbb{N}$, it is the case that $\omega^\dagger(x, p) \geq n \iff p^n | \omega(x)$, hence that $\omega(x) = \prod_{p \in P} p^{\omega^\dagger(x,p)}$. It results that $\nu = \nu_\infty$ and Theorem 1.5 follows.

4 Beyond Theorems 1.2, 1.3 and 1.5

4.1 Sampling along affine subspaces

Set $\Omega_\infty = [\![0, \infty]\!]^{\mathbb{Z}^d}$. The set Ω_∞ thus contains $\Omega_{\mathbb{N}}$. When ν_F is seen as a probability measure on Ω_∞ instead of $\Omega_{\mathbb{N}}$, we denote it by $\nu_F^\star$, so that there

is no ambiguity about which notion of convergence is being considered. We will also adopt the notation $\nu^\star$ for arbitrary probability measures on Ω_∞.

We say that a subset E of Ω_∞ is **suitable** if there is some M such that for any $(\omega, \omega') \in \Omega_\infty^2$, if one has

$$\forall x \in \mathbb{Z}^d,\ (\omega(x) \neq \omega'(x) \implies \min(\omega(x), \omega'(x)) \geq M),$$

then $\omega \in E \iff \omega' \in E$.

Let $(\nu_n^\star)$ denotes a sequence of probability measures on Ω_∞, and let $\nu^\star$ be a probability measure on Ω_∞. We say that $\nu_n^\star$ **converges** to $\nu^\star$ if for every *suitable* cylindric event E, the $\nu_n^\star$-probability of E converges to its $\nu^\star$-probability. From every sequence of probability measures on Ω_∞, one can extract a converging subsequence. If $V \subset \mathbb{Z}^d$, we say that $\nu_n^\star$ converges to $\nu^\star$ **on** V if the convergence holds for every suitable cylindric event that "only looks at the labels in V", i.e. every suitable cylindric event such that

$$\forall (\omega, \omega') \in \Omega_\infty^2,\ \omega_{|V} = \omega'_{|V} \implies (\omega \in E \iff \omega' \in E).$$

Given Γ an infinite subgroup of $\mathbb{Z}^d$, one defines the probability measure $\nu_{\infty,\Gamma}^\star$ by taking the definition of ν_∞ but asking furthermore that every W_n^p intersects Γ. This corresponds to taking a Haar-distributed element in the closure of Γ in $\hat{\mathbb{Z}}^d$.

Finally, if Γ is an infinite subgroup of $\mathbb{Z}^d$, we say that a sequence (F_n) of nonempty finite subsets of Γ is a **Følner sequence for** Γ if for every $y \in \Gamma$, one has $|F_n \Delta (F_n + y)| = o(|F_n|)$.

THEOREM 4.1. *Let Γ denote a subgroup of $\mathbb{Z}^d$ of rank $k \geq 1$ which is maximal among subgroups of rank k. Let (F_n) be a Følner sequence for Γ. Let Y_n denote a uniform element of F_n.*

If $k = 1$, then $\nu_{F_n}^\star$ converges to $\nu_{\infty,\Gamma}^\star$ as n goes to infinity.

If $k \geq 2$, then $\nu_{F_n}^\star$ converges to $\nu_{\infty,\Gamma}^\star$ on $\mathbb{Z}^d \backslash \Gamma$.

If $\mathrm{GCD}(Y_n)$ is tight, then $\nu_{F_n}^\star$ converges to $\nu_{\infty,\Gamma}^\star$.

Remark. The view from a "uniform point" in an *affine* subspace $\Gamma + y$ is just the view seen from a "uniform point" in Γ shifted by $-y$. One may also notice that if one starts with a group Γ that is not maximal given its rank, then it lies in a unique such group, which is the intersection of its (rational or real) linear span with $\mathbb{Z}^d$: denote it by $\overline{\Gamma}$. It follows from the proof of Theorem 4.1 that the following holds. Let (F_n) be a Følner sequence for Γ. Let Y_n denote a uniform element of F_n. Assume that for every $y \in \overline{\Gamma}$, $\mathrm{GCD}(Y_n)$ is tight. Then $\nu_{F_n}^\star$ converges to $\nu_{\infty,\Gamma}^\star$. Actually, it suffices to make the assumption for a system of representatives of $y \in \overline{\Gamma}$ for the equivalence relation "being equal modulo Γ", i.e. for finitely many y's.

COROLLARY 4.2. *Let Γ denote a subgroup of $\mathbb{Z}^d$ of rank $k \geq 1$. Let F denote a bounded subset of the linear span of Γ. For every $r \in (0, \infty)$, set*

$F_r := \{x \in \Gamma \ : \ r^{-1}x \in F\}$. *Assume that* $\frac{|F_r|}{r^k}$ *converges to a non-zero limit when* r *tends to infinity and that* (F_n) *is a Følner sequence for* Γ.

Then, $\nu^\star_{F_n}$ *converges to* $\nu^\star_{\infty,\Gamma}$ *as* n *goes to infinity.*

Proof of Theorem 4.1. As in the proof of Theorem 1.5, the corresponding convergence is straightforward for supernatural numbers. For every n, let Y_n denote a uniform element in F_n. Assume that $(F_{n(i)})$ is a subsequence of (F_n) such that the distribution of $(\tau_{-Y_n}\omega_{\mathrm{GCD}}, \tau_{-Y_n}\omega_\dagger)$ converges to some probability ρ, in the sense that for every *suitable* cylindric event $E^\star$ of Ω_∞ and every cylindric event $E^\dagger$ of $\Omega_\dagger$, the probability of the event $E^\star \times E^\dagger$ converges accordingly. Notice that for ρ-almost every $(\omega^\star, \omega^\dagger)$, for every $x \in \mathbb{Z}^d$, one has $\omega^\star(x) \neq 0$ and $\forall p,\ \omega^\dagger(x,p) < \infty$. In order to prove the convergence of $\nu^\star_{F_n}$ to $\nu^\star_\infty$, it now suffices to prove that for ρ-almost every $(\omega^\star, \omega^\dagger)$, for every $x \in \mathbb{Z}^d$,

$$\omega^\star(x) < \infty \iff \exists P,\ \forall p \geq P,\ \omega^\dagger(x,p) = 0.$$

For x outside Γ, this results from Lemma 4.3: both sides of the equivalence are true, and P can be taken not to depend on $(\omega^\star, \omega^\dagger)$.

If $k = 1$, it is easy to see that, for ρ-almost every $(\omega^\star, \omega^\dagger)$, for every $x \in \Gamma$, one has $\omega^\star(x) = \infty$. Besides, the second Borel-Cantelli Lemma and $\sum_{p\in\mathcal{P}} 1/p = \infty$ guarantee that, still for $k = 1$, almost surely, no $x \in \Gamma$ satisfies $\exists P,\ \forall p \geq P,\ \omega^\dagger(x,p) = 0$.

If $\mathrm{GCD}(Y_n)$ is tight, then for ρ-almost every $(\omega^\star, \omega^\dagger)$, for every $x \in \Gamma$, one has $\omega^\star(x) < \infty$. This implies that $k \geq 2$. By the first Borel-Cantelli Lemma and because $\sum_{p\in\mathcal{P}} p^{-k} < \infty$, almost surely, every $x \in \Gamma$ satisfies $\exists P,\ \forall p \geq P,\ \omega^\dagger(x,p) = 0$. □

Lemma 4.3. *Let* $d \geq 1$. *Let* $k \leq d$. *Let* Γ *be a subgroup of* $\mathbb{Z}^d$ *of rank* k *and maximal with this property. Then, there is a norm* $\|\ \|$ *on* $\mathbb{R}^d$ *such that for every* $N \geq 1$ *and every* $x \in N\mathbb{Z}^d$, *one has*

$$d(x,\Gamma) > 0 \implies d(x,\Gamma) \geq N,$$

where $d(x,\Gamma) := \min\{\|x - y\| : y \in \Gamma\}$.

Proof. By a change of coordinate and by maximality of Γ, one may assume that $\Gamma = \mathbb{Z}^k \times \{0_{d-k}\}$, in which case the lemma is clear. Notice that any element of $\mathsf{GL}_d(\mathbb{Z})$ maps $N\mathbb{Z}^d$ precisely to itself. □

Proof of Corollary 4.2. Let Y_n denote a uniform element of F_n. By Theorem 4.1 and the remark following it, it suffices to assume that Γ is maximal given its rank and to show that $\mathrm{GCD}(Y_n)$ is tight. Since GCD's are unchanged by $\mathsf{GL}_d(\mathbb{Z})$, we may assume that $\Gamma = \mathbb{Z}^k \times \{0_{d-k}\}$, and tightness results from Theorem 1.4. □

4.2 Graphons and local-graphon limits

A **"graphon"** is the data of a standard probability space $(\mathfrak{X}, \mathbb{P})$ together with a measurable function $f : \mathfrak{P}^2 \to [0,1]$ that is symmetric, i.e. satisfies $\forall (x_1, x_2) \in \mathfrak{X}^2,\ f(x_1, x_2) = f(x_2, x_1)$. Two "graphons" $(\mathfrak{X}, \mathbb{P}_{\mathfrak{X}}, f)$ and $(\mathfrak{Y}, \mathbb{P}_{\mathfrak{Y}}, g)$ are said to **induce the same graphon** if, up to throwing away sets of measure zero, there is a measure-preserving isomorphism $\pi : (\mathfrak{X}, \mathbb{P}_{\mathfrak{X}}) \to (\mathfrak{Y}, \mathbb{P}_{\mathfrak{Y}})$ such that $\forall (x_1, x_2) \in \mathfrak{X},\ f(x_1, x_2) = g(\pi(x_1), \pi(x_2))$. A **graphon** is an equivalence class of "graphons" for the relation "inducing the same graphon". We say that a graphon is **represented** by any "graphon" that induces it. See [BCL$^+$08].

Let $\mathcal{G}_n = (V_n, E_n)$ denote a sequence of random[4] finite graphs such that $|V_n|$ converges in probability to infinity. It is said to **converge** to the (deterministic) graphon represented by $(\mathfrak{X}, \mathbb{P}, f)$ if the following holds: for every k, if $(X_1^n, \dots, X_k^n)$ denotes a uniform element of V_n^k, then the random variable $(\mathbb{1}_{\{X_i^n, X_j^n\} \in E_n})_{1 \le i < j \le k}$ converges in distribution to (the distribution of) the random variable $(f(X_i^\infty, X_j^\infty))_{1 \le i < j \le k}$, where the X_i^∞'s are independent random variables of distribution $\mathbb{P}$. See [DJ08].

Consider the following standard probability space $\mathfrak{X}_0 := \prod_{p \in \mathcal{P}} (\mathbb{Z}/p\mathbb{Z})^d$, endowed with the product of uniform measures. Consider the measurable function $\delta : \mathfrak{X}_0^2 \to [0,1]$ defined by $\delta(x_1, x_2) := \mathbb{1}_{\forall p,\ x_1(p) \neq x_2(p)}$. See Figure 2.

By the same arguments as in Section 3, one can prove the following result.

THEOREM 4.4. *Let $d \ge 1$ and let (F_n) be a Følner sequence of $\mathbb{Z}^d$. Assume that $\mu_{F_n}(\{\omega : (0, \dots, 0) \in \omega\})$ converges to $1/\zeta(d)$. Let $\mathcal{G}_n$ denote the random graph with vertex set F_n and an edge between two distinct vertices if and only if the one if visible from the other.*

Then, $\mathcal{G}_n$ converges to the graphon represented by $(\mathfrak{X}_0, \delta)$.

Actually, we can even get a result combining both Theorem 1.2 and Theorem 4.4.

THEOREM 4.5. *Let $d \ge 1$ and let (F_n) be a Følner sequence of $\mathbb{Z}^d$. Assume that $\mu_{F_n}(\{\omega : (0, \dots, 0) \in \omega\})$ converges to $1/\zeta(d)$. Let (X_n) denote a sequence of independent uniform elements of $\mathfrak{X}_0$.*

Let $M \ge 1$ and $R \ge 1$. For every n, let $(Y_m^n)_{1 \le m \le M}$ denote M independent uniform elements in F_n. Consider the following random maps:

$$\begin{array}{rccc} \psi_n : & ([\![1, M,]\!] \times [\![-R, R]\!]^d)^2 & \longrightarrow & \{0,1\} \\ & ((m_0, y_0), (m_1, y_1)) & \longmapsto & \mathbb{1}_{Y_{m_0}^n + y_0 \text{ is visible from } Y_{m_1}^n + y_1}, \end{array}$$

$$\begin{array}{rccc} \psi_\infty : & ([\![1, M,]\!] \times [\![-R, R]\!]^d)^2 & \longrightarrow & \{0,1\} \\ & ((m_0, y_0), (m_1, y_1)) & \longmapsto & \mathbb{1}_{\forall p,\ X_{m_0}(p) + y_0 \neq X_{m_1}(p) + y_1}. \end{array}$$

Then, the distribution of ψ_n converges to that of ψ_∞.

[4] One does not need $\mathcal{G}_n$ and $\mathcal{G}_m$ to be defined on the same probability space.

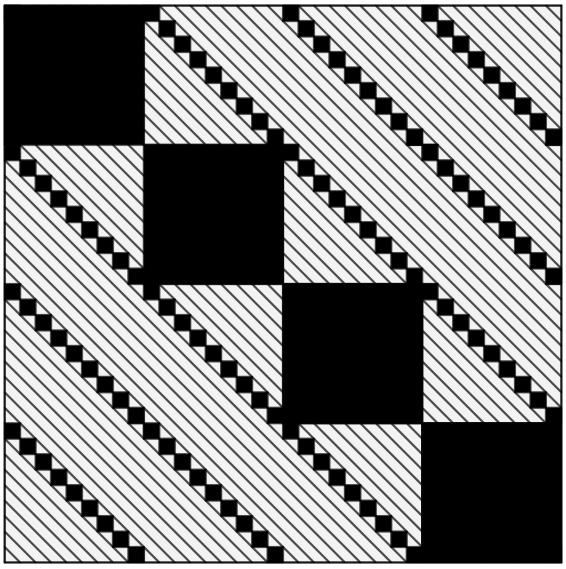

Figure 2: Visualisation of the graphon $(\mathfrak{X}_0, \delta)$ for $d = 2$. After suitable identification of $\mathfrak{X}_0$ with $[0,1]$, the set $\delta^{-1}(\{0\})$ is represented in black. Each square at scale n is divided into $p_n^d \times p_n^d$ smaller squares, where (p_n) denotes the sequence of prime numbers.

This theorem can be readily adapted to the whole GCD profile (one assumes tightness, considers maps to $[\![0,\infty]\!]$ rather than to $\{0,1\}$, and predicts the GCD of $(Y^n_{m_0} + y_0) - (Y^n_{m_1} + y_1)$) and the case of affine subspaces.

Let us conclude with a last generalisation. One may be interested in other arithmetic conditions than coprimality: for example, saying that a number is k-**free** if no p^k divide it, one may colour in white the k-free points of $\mathbb{Z}$, or the points in $\mathbb{Z}^d$ with k-free GCD. See [BMP00, Ces85, Mir47, Mir48, PH13]. One can generalise Theorem 4.5 to such contexts as follows.

Let $\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \{x \in \prod_{n\geq 1} \mathbb{Z}/n\mathbb{Z} : \forall (m,n),\ m|n \implies \pi_{m,n}(x_n) = x_m\}$, where $\pi_{m,n}$ denotes the morphism of reduction modulo m from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$. One can see $\mathbb{Z}$ as a dense subgroup of $\hat{\mathbb{Z}}$ via the injection

$$N \mapsto (n \mapsto N + n\mathbb{Z}).$$

Let $\mathfrak{X}'_0 := \hat{\mathbb{Z}}^d$, and endow it with its Haar-measure. The product topology on $\prod_{n\geq 1} \mathbb{Z}/n\mathbb{Z}$ induces on $\hat{\mathbb{Z}}$ and on $\mathbb{Z} \subset \hat{\mathbb{Z}}$ the so-called **profinite topology**. One can endow the groups $\hat{\mathbb{Z}}^d$ and $\mathbb{Z}^d$ with the product of these topologies, which one calls the **profinite topology** on $\hat{\mathbb{Z}}^d$ or $\mathbb{Z}^d$.

Example. For every $k \geq 0$, the set $\{x \in \mathbb{Z}^d : \text{GCD}(x) \text{ is } k\text{-free}\}$ is profinitely closed in $\mathbb{Z}^d$. For $k = 1$, this coincides with the set $\{x \in \mathbb{Z}^d : \text{GCD}(x) = 1\}$.

If ω_1 denotes a subset of $\mathbb{Z}^d$, let $\mu_F^{\omega_1}$ denote the measure constructed as μ_F with ω_1 instead of ω_0. Denote by $\mu_\infty^{\omega_1}$ the distribution of $x \mapsto \mathbb{1}_{W(x)\in\overline{\omega_1}}$, where W is Haar-distributed in $\hat{\mathbb{Z}}^d$ and $\overline{\omega_1}$ denotes the closure of the set ω_1 *in* $\hat{\mathbb{Z}}^d$.

THEOREM 4.6. *Let $d \geq 1$ and let (F_n) be a Følner sequence of $\mathbb{Z}^d$. Let ω_1 denote a subset of $\mathbb{Z}^d$. Assume that ω_1 is profinitely closed* in *$\mathbb{Z}^d$. Assume that $\mu_{F_n}(\{\omega : (0,\dots,0) \in \omega\})$ converges to $\mu_\infty^{\omega_1}(\{\omega : (0,\dots,0) \in \omega\})$. Let (X_n) denote a sequence of independent Haar-distributed elements of $\mathfrak{X}_0'$.*

Let $M \geq 1$ and $R \geq 1$. For every n, let $(Y_m^n)_{1\leq m\leq M}$ denote M independent uniform elements in F_n. Consider the following random maps:

$$\begin{array}{rccl} \psi_n : & ([\![1,M,]\!]\times[\![-R,R]\!]^d)^2 & \longrightarrow & \{0,1\} \\ & ((m_0,y_0),(m_1,y_1)) & \longmapsto & \mathbb{1}_{(Y^n_{m_0}+y_0)-(Y^n_{m_1}+y_1)\in\omega_1}, \end{array}$$

$$\begin{array}{rccl} \psi_\infty : & ([\![1,M,]\!]\times[\![-R,R]\!]^d)^2 & \longrightarrow & \{0,1\} \\ & ((m_0,y_0),(m_1,y_1)) & \longmapsto & \mathbb{1}_{(X_{m_0}+y_0)-(X_{m_1}+y_1)\in\overline{\omega_1}}. \end{array}$$

Then, the distribution of ψ_n converges to that of ψ_∞.

Acknowledgements At the end of a talk given by Nathanaël Enriquez, Bálint Virág asked him about the local limit of visible points: I would like to thank Bálint Virág for asking this question and Nathanaël Enriquez for letting me know of it. I am also grateful to Nathanaël Enriquez for many enthusiastic discussions about this project. I am thankful to Subhajit Goswami and Aran Raoufi for comments on an earlier version of this paper. Finally, I am grateful to my postdoctoral advisors Nicolas Curien and Jean-François Le Gall, as well as to the ERC grant GeoBrown and the Université Paris-Sud, for providing me with an excellent working environment.

References

[Bab91] László Babai. Vertex-transitive graphs and vertex-transitive maps. *Journal of graph theory*, 15(6):587–627, 1991.

[BCL+08] Christian Borgs, Jennifer Chayes, László Lovász, Vera Sós, and Katalin Vesztergombi. Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing. *Advances in Mathematics*, 219(6):1801–1851, 2008.

[BCZ00] Florin Boca, Cristian Cobeli, and Alexandru Zaharescu. Distribution of lattice points visible from the origin. *Communications in Mathematical Physics*, 213(2):433–470, 2000.

[BH15] Michael Baake and Christian Huck. Ergodic properties of visible lattice points. *Proceedings of the Steklov Institute of Mathematics*, 288(1):165–188, 2015.

[BMP00] Michael Baake, Robert Moody, and Peter Pleasants. Diffraction from visible lattice points and kth power free integers. *Discrete Mathematics*, 221(1-3):3–42, 2000.

[BS01] Itai Benjamini and Oded Schramm. Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.*, 6(23):1–13, 2001.

[Ces81] Ernesto Cesàro. Question proposée 75. *Mathesis*, 1:184, 1881.

[Ces83] Ernesto Cesàro. Question 75 (solution). *Mathesis*, 3:224–225, 1883.

[Ces85] Ernesto Cesàro. Sur le plus grand commun diviseur de plusieurs nombres. *Annali di Matematica Pura ed Applicata (1867-1897)*, 13(1):291–294, 1885.

[CFF] Javier Cilleruelo, José Fernández, and Pablo Fernández. Visible lattice points in random walks. *ArXiv:1512.04722*.

[Dir51] Gustav Lejeune Dirichlet. *Über die Bestimmung der mittleren Werthe in der Zahlentheorie*. Dümmler in Komm., 1851.

[DJ08] Persi Diaconis and Svante Janson. Graph limits and exchangeable random graphs. *Rendiconti di Matematica, Serie VII*, 28:33–61, 2008.

[DL01] Reinhard Diestel and Imre Leader. A conjecture concerning a limit of non-Cayley graphs. *Journal of Algebraic Combinatorics*, 14(1):17–25, 2001.

[Gar15] Olivier Garet. Les lois Zêta pour l'arithmétique. *Quadrature*, 96, 2015.

[GHKM] Edray Goins, Pamela Harris, Bethany Kubik, and Aba Mbirika. Lattice point visibility on generalized lines of sight. *ArXiv:1710.04554*.

[Gri99] Geoffrey Grimmett. *Percolation*, volume 321 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, second edition, 1999.

[HS71] Fritz Herzog and Bonnie Stewart. Patterns of visible and nonvisible lattice points. *The American Mathematical Monthly*, 78(5):487–496, 1971.

[HW79] Godfrey Hardy and Edward Wright. *An introduction to the theory of numbers*. Oxford University Press, 1979.

[Lin01] Elon Lindenstrauss. Pointwise theorems for amenable groups. *Inventiones mathematicae*, 146(2):259–295, 2001.

[LP16] Russell Lyons and Yuval Peres. *Probability on trees and networks*, volume 42. Cambridge University Press, 2016.

[Mir47] Leonid Mirsky. Note on an asymptotic formula connected with r-free integers. *The Quarterly Journal of Mathematics*, (1):178–182, 1947.

[Mir48] Leonid Mirsky. Arithmetical pattern problems relating to divisibility by rth powers. *Proceedings of the London Mathematical Society*, 2(1):497–508, 1948.

[PH13] Peter Pleasants and Christian Huck. Entropy and diffraction of the k-free points in n-dimensional lattices. *Discrete & Computational Geometry*, 50(1):39–68, 2013.

[Syl83] James Sylvester. Sur le nombre de fractions ordinaires inégales qu'on peut exprimer en se servant de chiffres qui n'excèdent pas un nombre donné. *CR Acad. Sci. Paris*, 96:409–413, 1883.

[Var99] Ilan Vardi. Deterministic percolation. *Communications in mathematical physics*, 207(1):43–66, 1999.