A NOTE ON GENERALIZED TRAVELING WAVE SOLUTIONS FOR MEAN CURVATURE FLOWS WITH MERELY L ∞ FORCING IN THE PLANE

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Abstract. In this note, we announce some results without proofs. The complete version with proofs is the object of a forthcoming article. We consider traveling wave solutions of curvature flows with periodic forcing R. The fixed front's profile is the supergraph of a function ψ. The latter satisfies a PDE for which the existence of bounded solutions typically holds when $\min R > 0$. This theory extends to generalized traveling wave solutions for $R \in W^{1,\infty}$ with positive mean value. The front's profile may then be unbounded and the new formulation is variational. Equivalently ψ solves a boundary value problem with infinite Dirichlet condition, on a certain domain to be determined. Our analysis concerns merely L^∞ forcing. In that case, the front's profile can have vertical lines. This amounts to consider discontinuous ψ with interface-like conditions at singularities. We then compute all possible fronts' profiles for a prototypical fibered medium, identifying necessary and sufficient conditions for their boundedness. We also shed light on the possible failure of a certain uniqueness property, which was typical of the $W^{1,\infty}$ theory.

1. Introduction

In this note, we are interested in traveling fronts of a periodic forced curvature flow equation,

$$V_n = -R + \kappa, \tag{1}$$

in the x-y plane. Here V_n is the normal velocity of a moving interface

$$t \in \mathbb{R} \mapsto \Gamma_t$$

with mean curvature κ, and R is a given forcing term having as much periodicity as is allowed by the medium. Throughout we assume that

$$R = R(x) \in L^\infty(\mathbb{T}) \quad \text{and} \quad \int_{\mathbb{T}} R > 0, \tag{2}$$

where $\mathbb{T}$ is the flat torus $\mathbb{R}/\mathbb{Z}$. A prototype example is a flame front propagating through a fibered medium composed of a periodic superposition of materials, where the combustion rate R is a step function; see e.g. [4] and the references therein.

We restrict our analysis to interfaces which can be represented as the graph of a function $u = u(x,t)$, i.e.

$$\Gamma_t = \{(x,y) \in \mathbb{T} \times \mathbb{R} \text{ s.t. } y = u(x,t)\}.$$

Considering that normals are oriented towards the negative y-direction, (1) becomes

$$u_t + R\sqrt{1+u_x^2} = \frac{u_{xx}}{1+u_x^2}, \quad t \in \mathbb{R},\ x \in \mathbb{T}. \tag{3}$$

Date: February 13, 2023.

2020 *Mathematics Subject Classification.* 35K55, 35B10.

Key words and phrases. Traveling wave solutions, bounded fronts, periodic fibered medium.

Now a *traveling front* is a particular solution of the form

$$u(x,t) = -ct + \psi(x),$$

for some speed $c > 0$ and stationary profile ψ. In that case (3) reduces further to

$$-c + R\sqrt{1+\psi_x^2} = \frac{\psi_{xx}}{1+\psi_x^2}, \quad x \in \mathbb{T}, \tag{4}$$

and the pair (c, ψ) is called a *traveling wave solution (TWS)*. The above frame is standard. For example if $\min R > 0$, then TWS of (3) exist, the speed is unique, the profile is unique up to an additive constant, and most importantly ψ is a global solution of (4) which is moreover bounded (cf. [4]). For other boundedness results on ψ, see [7, 2, 3].

Otherwise, the profile may be unbounded if R changes sign or just vanishes, and ψ may not even be globally defined. The correct notion of TWS of (3) is then the one of *generalized TWS* introduced in [3]. It is inspired from the variational theory of minimizing surfaces (cf. e.g. [5, 6]). The idea is to formally interpret (4) as the Euler-Lagrange equation of the functional

$$F_c(\psi) := \int_{\mathbb{T}} e^{-c\psi}\left(\sqrt{1+|\psi_x|^2} - \frac{R}{c}\right), \tag{5}$$

and define the front's profile as a nontrivial minimizer of F_c. This is rigorously done for the relaxed version of F_c, notably in the space

$$\left\{\psi : \mathbb{T} \to \mathbb{R} \cup \{+\infty\} \text{ s.t. } e^{-c\psi} \in BV(\mathbb{T})\right\}.$$

We refer to [3] for details, cf. also some reminders in Section 2. Let us roughly speaking recall here that the front's speed is the unique c for which there are nontrivial minimizers, and if moreover $R \in W^{1,\infty}$, then any minimizer is $C^{2,1}$ and solves (4) in the open set $E := \{\psi < +\infty\}$, while satisfying the Dirichlet boundary condition

$$\psi(x) \to +\infty \quad \text{as } \operatorname{dist}(x, \partial E) \to 0. \tag{6}$$

The maximal such possible set E is also uniquely determined. It may be disconnected, but ψ remains unique up to an additive constant on each connected component of E.

In this note, we focus on the case where R is no more regular than merely bounded. The front's profile can actually still be defined as a theoretical nontrivial minimizer of F_c, but ψ can now be discontinuous and the sense in which it satisfies (4) needs to be clarified. Notably, the definition of the front as the graph of ψ should be corrected as well. The most natural is to consider a geometric reformulation of the previous minimization problem, used also in [3]. The idea is just to see the front, not as a graph but, as the boundary of a supergraph. Its profile is then $\partial\Sigma_\psi$ where

$$\Sigma_\psi := \{(x,y) \in \mathbb{T} \times \mathbb{R} \text{ s.t. } y > \psi(x)\},$$

whose boundary makes sense without ambiguity even for discontinuous ψ. Moreover $F_c(\psi)$ can be rewritten as a new functional $\mathcal{F}_c(\Sigma_\psi)$ with a certain weighted perimeter (cf. [3] or Proposition 5). It is then natural to define the speed and profile of the front as before when $R \in L^\infty$, especially since this extends the notion of generalized mean curvature when $c = 0$ (cf. Remark 3).

In [3], the geometric approach was used for the regularity of minimizers via arguments from minimizing surfaces [6], adapted to weighted perimeters. But ψ was not discontinuous at the end, whereas it is in our setting. We then need to identify the correct PDE problem at such singularities. Roughly speaking, we show that ψ is again regular and solves (4) on some open set E, but with two types of

boundary conditions: either a Neuman-type condition or an interface-like condition. This formulation moreover implies (6) if $R \in W^{1,\infty}$. For a precise statement, see Theorem 1. To achieve this, we interestingly do not rely on sophisticated ideas from minimizing surfaces, but have instead a very simple, short and selfcontained proof (which will be given in the forthcoming complete article). It only uses a rewritting of F_c where the Cantor and jump parts of $D\psi$ are, roughly speaking, separated from the square root in (5) (cf. Lemma 2). This is specific to dimension one.

In a second part, we illustrate the L^∞ theory on the most prototype example of R taking just two values (a fibered medium made of two materials). We explicitely compute all possible fronts' profiles, exhibiting exact thresholds for having only bounded classical TWS or pure GTWS, etc. We notably shed light on examples where ψ is unbounded and discontinuous, or where the previous uniqueness on connected components is lost. Details are given in Proposition 8, Corollary 1 and Remark 6.

In the rest of this note, we precisely state the results we claim. Let us recall that the proofs will be given in the forthcoming complete article.

2. Reminders and main results

2.1. **Reminders on the $W^{1,\infty}$ theory.** Let us briefly recall results from [3] in this subsection. First recall that (4) is the Euler-Lagrange equation associated to the functional F_c in (5), i.e.

$$\overline{\psi} \in C^2(\mathbb{T}) \text{ solves (4)} \Longleftrightarrow F_c(\overline{\psi}) = \min_{\psi \in C^1(\mathbb{T})} F_c(\psi),$$

in for example the smooth case where R is continuous. Now the use of the following change of variables

$$\phi := \frac{e^{-c\psi}}{c},$$

for $c > 0$ fixed, allows to rewrite F_c as follows:

$$F_c(\psi) = G_c(\phi) := \int_{\mathbb{T}} \left(\sqrt{c^2\phi^2 + |\phi_x|^2} - R\phi \right).$$

The relaxed version of G_c is given below (cf. e.g. [1]).

Lemma 1. *Assume $R \in L^\infty(\mathbb{T})$ and $c > 0$. Then*

$$\inf G_c := \inf \left\{ G_c(\phi) : 0 \leq \phi \in C^1(\mathbb{T}) \right\} = \inf \left\{ G_c(\phi) : 0 \leq \phi \in BV(\mathbb{T}) \right\},$$

where G_c is extended to the Banach space $BV(\mathbb{T})$ by

$$G_c(\phi) := \sup \left\{ \int_{\mathbb{T}} \phi \left(c\varphi_1 + (\varphi_2)_x \right) : (\varphi_1, \varphi_2) \in C^1(\mathbb{T}; \mathbb{R}^2),\ \varphi_1^2 + \varphi_2^2 \leq 1 \right\} - \int_{\mathbb{T}} R\phi. \tag{7}$$

Note that $c \mapsto \inf G_c$ is nondecreasing while $\inf G_c$ can be only either 0 or $-\infty$, because G_c is positively 1-homogeneous. This allows to define the front's speed as below.

Proposition 1 (Existence of a unique speed)**.** *Under (2) there is a unique $\overline{c} > 0$ such that*

- *if $0 < c < \overline{c}$, then $\inf G_c = -\infty$,*
- *if $c \geq \overline{c}$, then $\inf G_c = 0$.*

Moreover $\overline{c}$ is such that $\int R \leq \overline{c} \leq \max R$.

Here are now first properties of minimizers.

Proposition 2 (Existence of minimizers)**.** *Under* (2)*, all minimizers of* G_c *are trivial (identically equal to* 0*) for any* $c > \overline{c}$*, whereas* $G_{\overline{c}}$ *admits a non trivial minimizer* $\overline{\phi}$*.*

This motivates the definition of generalized solutions as below.

Definition 1. Under (2), a *generalized traveling wave solution (GTWS)* of (3) is a pair $(\overline{c}, \overline{\psi})$ satisfying Propositions 1 and 2 with

$$\overline{\psi} = -\frac{1}{\overline{c}} \ln(\overline{c}\overline{\phi}).$$

In particular $\overline{\psi} : \mathbb{T} \to \mathbb{R} \cup \{+\infty\}$ is measurable and $e^{-\overline{c}\overline{\psi}} \in BV(\mathbb{T})$ (with usual convenient rules $e^{-\infty} = 0$, etc.) To say more, the authors of [3] assumed that

$$R \in W^{1,\infty}(\mathbb{T}) \quad \text{and} \quad \int_{\mathbb{T}} R > 0. \tag{8}$$

Proposition 3 (Regularity of minima and PDE formulation)**.** *Assume* (8) *and let* $(\overline{c}, \overline{\psi})$ *be a GTWS of* (3)*. Then there exists an open set* $E \subseteq \mathbb{T}$ *such that* $\overline{\psi} \in C^{2,1}(E)$ *and solves the Dirichlet problem:*

$$\begin{cases} -\overline{c} + R\sqrt{1+\overline{\psi}_x^2} = \dfrac{\overline{\psi}_{xx}}{1+\overline{\psi}_x^2}, & x \in E, \\ \overline{\psi}(x) \to +\infty, & \text{as } dist(x, \partial E) \to 0, \\ \overline{\psi}(x) = +\infty, & x \in \mathbb{T}\backslash E. \end{cases} \tag{9}$$

Remark 1. Conversely any $\overline{\psi}$ solving (9), for some open E, is such that $\frac{e^{-\overline{c}\overline{\psi}}}{\overline{c}}$ minimizes $G_{\overline{c}}$. This is easily verified by testing the equation with $\frac{e^{-\overline{c}\overline{\psi}}}{\sqrt{1+\overline{\psi}_x^2}}$, since the boundary terms cancel during integration by parts because of the infinite Dirichlet condition.

All possible such sets E and solutions $\overline{\psi}$ are in fact related to each others.

Proposition 4 (Uniqueness up to additive constants)**.** *Assume* (8) *and consider two GTWS of* (3) *with respective minimizers* $\overline{\psi}_i$ *and associated* E_i *as above. Then two arbitrary respective connected components of* E_1 *and* E_2 *are either the same or disjoint. Moreover* $\overline{\psi}_1 - \overline{\psi}_2$ *is constant on each common connected component.*

Remark 2 (Maximal PDE's domain). In particular there is a maximal set E such as above, whose associated front's profile $\{y = \overline{\psi}(x)\}$ is unique up to translating its connected components in the y-direction. More can be said on E, for example that it is a finite union of intervals.

The preceding regularity of GTWS was established in [3] via a geometric interpretation in terms of supergraphs $\Sigma_\psi = \{y > \psi(x)\}$. Let us recall it now since it will serve us later to define the front. Given $c > 0$ and a measurable $\psi : \mathbb{T} \to \mathbb{R} \cup \{+\infty\}$ such that $e^{-c\psi} \in BV(\mathbb{T})$, ψ is bounded from below and its indicator function

$$\mathbf{1}_{\Sigma_\psi}(x,y) := \begin{cases} 1 & \text{if } (x,y) \in \Sigma_\psi, \\ 0 & \text{otherwise,} \end{cases}$$

belongs to $BV_{loc}(\mathbb{T} \times \mathbb{R})$. Define then the new functional

$$\mathcal{F}_c(\Sigma_\psi) := \mathrm{Per}_c(\Sigma_\psi) - \int\limits_{\mathbb{T}\times\mathbb{R}} e^{-cy} R(x) \mathbf{1}_{\Sigma_\psi}(x,y)\, dxdy,$$

with the weighted perimeter

$$\mathrm{Per}_c(\Sigma_\psi) := \int_{\mathbb{T}\times\mathbb{R}} e^{-cy}\, d|D\mathbf{1}_{\Sigma_\psi}|(x,y)$$

and the total variation measure $|D\mathbf{1}_{\Sigma_\psi}|$, where D stands for the distribution gradient. We give below a reformulation of the previous minimization problem.

Proposition 5 (Geometric interpretation)**.** *We have*

$$G_c(\phi) = \mathcal{F}_c(\Sigma_\psi) \quad \forall c > 0,\, \forall 0 \le \phi \in BV(\mathbb{T}),$$

where $\psi = -\frac{1}{c}\ln(c\phi)$*. Moreover* $G_c(\phi) = \min G_c$ *if and only if*

$$\mathcal{F}_c(\Sigma_\psi) = \min\Big\{\mathcal{F}_c(B) : \textit{Borel } B \subseteq \mathbb{T}\times\mathbb{R} \textit{ s.t. } \Sigma_\psi \triangle B \textit{ is bounded}\Big\} \tag{10}$$

(with $A \triangle B := (A\backslash B)\cup(B\backslash A)$*).*

Remark 3. The minimization property (10) is local, i.e. taken wrt compact perturbations B of Σ_ψ. If $c = 0$, we recover the usual definition of generalized mean curvature (cf. e.g. [8, 5]). Let us indeed recall that $\partial\Sigma_\psi$ is said to have a mean curvature equal to $R \in L^\infty$ when (10) holds with the standard perimeter $\mathrm{Per}_{c=0}$.

2.2. **Main results: General L^∞ theory.** We are ready to state our results when R is merely bounded. Recall that Propositions 1 and 2 remain valid. In particular we continue to study GTWS $(\overline{c}, \overline{\psi})$ from Definition 1, which exist with a unique possible speed by the aforementioned propositions.

Remark 4 (The front's profile at singularities). Since $R \notin W^{1,\infty}$ as in [3], we only know at this stage that $e^{-\overline{c}\overline{\psi}} \in BV(\mathbb{T})$. We will moreover see that $\overline{\psi}$ can be discontinuous. Proposition 5 then suggests to define the front's profile, not anymore as the graph but, as the boundary of the supergraph of $\overline{\psi}$. In particular it will contain vertical lines at singularities of $\overline{\psi}$.

We first note that G_c can be rewritten in a more convenient way, when treating discontinuous minimizers as we will have. For any $\phi \in BV(\mathbb{T})$, the distribution gradient $D\phi$ is a finite measure and can be decomposed by Lebesgue-Radon-Nikodym Theorem. This reads

$$D\phi = \phi_x\lambda + \mu_C + \mu_J,$$

with the Lebesgue measure λ, the a.e. derivative ϕ_x, and the respective Cantor and jump measures μ_C and μ_J. We also continue to use $|\cdot|$ for total variation measures. We then claim:

Lemma 2 (Rewriting of G_c)**.** *For any* $c > 0$*, the functional in* (7) *satisfies*

$$G_c(\phi) = \int_{\mathbb{T}} \left(\sqrt{c^2\phi^2 + \phi_x^2} - R\phi\right) + |\mu_C|(\mathbb{T}) + |\mu_J|(\mathbb{T}) \quad \forall \phi \in BV(\mathbb{T}).$$

We now need some notation for the next result. We will have again an open E, outside which $\overline{\psi}$ equals $+\infty$ while satisfying (4) inside. The Dirichlet condition of (9) will become a Neuman condition. It will be convenient to define the outward unit normal ν_E over E itself. Given any connected component (a,b) of E and $x \in (a,b)$, we set

$$\nu_E(x) := \begin{cases} +1 & \text{if } b \text{ is the closest extremity to } x, \\ -1 & \text{if it is } a. \end{cases}$$

Here and throughout, an interval of the torus is defined as

$$(a,b) := \{x \in \mathbb{R}/\mathbb{Z} \text{ s.t. } (a,x,b) \text{ is ordered following the positive orientation}\}. \tag{11}$$

In the next result, we will also have an interface-like condition at some boundary points, which we classify in two parts:

$$\partial_{intf}E := \big\{x_0 \in \partial E \text{ s.t. } \overline{\psi}(x_0^-) \text{ and } \overline{\psi}(x_0^+) \text{ are finite}\big\}$$

and

$$\partial_{bdry}E := \partial E \setminus \partial_{intf}E.$$

In the above

$$\overline{\psi}(x_0^\pm) := \lim_{x_0^\pm} \overline{\psi}$$

are the limits of $\overline{\psi}(x)$ as $x \to x_0$ respectively from the right and the left, also in the sense of the positive orientation of $\mathbb{T}$. They exist when $e^{-\overline{c}\overline{\psi}} \in BV(\mathbb{T})$, being eventually $+\infty$. We then claim :

Theorem 1 (New PDE formulation)**.** *Assume* (2) *and let* $(\overline{c}, \overline{\psi})$ *be a GTWS of* (3)*. Then there exists an open set* $E \subseteq \mathbb{T}$ *such that* $\overline{\psi} \in C^{1,1}(E)$ *and solves the Neuman problem:*

$$\begin{cases} -\overline{c} + R\sqrt{1+\overline{\psi}_x^2} = \frac{\overline{\psi}_{xx}}{1+\overline{\psi}_x^2}, & \text{a.e. } x \in E, \\ \overline{\psi}_x(x)\nu_E(x) \to +\infty, & \text{as } dist(x, \partial_{bdry}E) \to 0, \\ \overline{\psi}(x) = +\infty, & x \in \mathbb{T}\setminus E, \end{cases}$$

together with the interface-like conditions

$$\begin{cases} \text{either } \overline{\psi}(x_0^-) = \overline{\psi}(x_0^+) \text{ and } \lim_{x_0^-} \overline{\psi}_x = \lim_{x_0^+} \overline{\psi}_x, \\ \text{or } \lim_{x_0} \overline{\psi}_x = \operatorname{sgn}(\overline{\psi}(x_0^+) - \overline{\psi}(x_0^-)) \cdot \infty, \end{cases} \qquad \forall x_0 \in \partial_{intf}E. \tag{12}$$

Remark 5. (1) The reciprocal assertion holds true, as in Remark 1.

(2) If R is $W^{1,\infty}$ at x_0, then $\lim_{x_0} \overline{\psi} = +\infty$.

(3) Since the equation also reads as

$$\frac{d}{dx}\left(\frac{\overline{\psi}_x}{\sqrt{1+\overline{\psi}_x^2}}\right) = -\frac{\overline{c}}{\sqrt{1+\overline{\psi}_x^2}} + R \in L^\infty(E),$$

$\lim_{x_0^\pm} \overline{\psi}_x$ exist in (12). They can eventually be infinite, and actually they should be. Indeed if not, then $\overline{\psi}$ is $C^{1,1}$ at x_0 which can then be included inside E up to redefining this domain.

(4) At the end, E can always be rewritten as a finite union of open intervals. But as we shall see in the next section, there is now no canonical front's profile, as in Remark 2, precisely due to discontinuities at $x = x_0$.

2.3. **Main results: A prototype example.** Now we consider a representative example of merely bounded R taking just two values, an example being a fibered medium with two materials.[1] We thus assume that

$$R(x) = \begin{cases} R_M & \text{if } x \in (x_0, x_1), \\ R_m & \text{if } x \in (x_1, x_0), \end{cases} \tag{13}$$

for some $R_M > R_m$ and $x_0, x_1 \in \mathbb{R}/\mathbb{Z}$, $x_0 \neq x_1$. The above intervals are understood as in (11). Denoting by α, $(0 < \alpha < 1)$, the length of (x_0, x_1), we assume more precisely that

$$\alpha R_M + (1-\alpha)R_m > 0 \tag{14}$$

[1]The case of a single value is trivial, having then $\overline{c} = R$ and a straight line as front's profile.

in order to fulfill (2). We proceed by defining threshold quantities which will serve to classify all possible fronts. For any (α, R_m, R_M) such as above, let

$$\Lambda_1 = \Lambda_1(\alpha, R_m) := 2 + (1-\alpha)R_m, \tag{15}$$

and

$$\Lambda_2 = \Lambda_2(\alpha, R_m, R_M) := \int_0^{+\infty} \frac{2\,d\tau}{(1+\tau^2)\left(-\min\left\{c(\alpha, R_m), R_M\right\} + R_M\sqrt{1+\tau^2}\right)} - \alpha, \tag{16}$$

where $c(\alpha, R_m) \in (0, +\infty]$ is defined through the equation

$$\int_0^{+\infty} \frac{2\,d\tau}{(1+\tau^2)\left(-c + R_m\sqrt{1+\tau^2}\right)} + 1 - \alpha = 0, \tag{17}$$

as

$$c(\alpha, R_m) := \begin{cases} \text{the unique } c > 0 \text{ solving (17)} & \text{if } R_m \le 0, \\ +\infty & \text{if } R_m > 0. \end{cases}$$

We then have the following three possible cases:

$$\text{either } [\Lambda_1 \ge 0 \text{ and } \Lambda_2 > 0] \text{ or } [\Lambda_1 < 0 \text{ or } \Lambda_2 < 0] \text{ or } [\Lambda_1 \ge 0 \text{ and } \Lambda_2 = 0].$$

We will decribe $\overline{\psi}$ in each of these cases. Here is our first claim.

Proposition 6 (Bounded profile, unique up to translation)**.** *Assume* (13)–(14) *with*

$$\Lambda_1 \ge 0 \quad \textit{and} \quad \Lambda_2 > 0,$$

and let $(\overline{c}, \overline{\psi})$ *be a GTWS of* (3)*. Then* $\overline{\psi}$ *is necessarily bounded and such that:*

$$\begin{cases} \overline{\psi} \in C^{1,1}(\mathbb{T}) \textit{ solves } (4) \textit{ on the whole torus,} \\ \overline{\psi} \textit{ is smooth, convex and symmetric on } (x_0, x_1), \\ \textit{as well as smooth, concave and symmetric on } (x_1, x_0). \end{cases}$$

Moreover $\overline{\psi}$ *is unique up to an arbitrary additive constant.*

In the above and below, the symmetry is wrt the centers of the respective intervals. We proceed with our second claim.

Proposition 7 (Unbounded profile, unique up to translation)**.** *Assume* (13)–(14) *with*

$$\Lambda_1 < 0 \quad \textit{or} \quad \Lambda_2 < 0,$$

and let $(\overline{c}, \overline{\psi})$ *be a GTWS of* (3)*. Then* $\overline{\psi}$ *is unbounded while satisfying:*

$$\begin{cases} \overline{\psi} \textit{ is smooth in } (x_0, x_1) \textit{ where it solves } (4), \\ \overline{\psi} \textit{ is convex and symmetric on } (x_0, x_1), \\ \overline{\psi}(x_0^+) = \overline{\psi}(x_1^-) < +\infty, \\ \lim_{x_0^+} \overline{\psi}_x = -\infty, \ \lim_{x_1^-} \overline{\psi}_x = +\infty, \textit{ and} \\ \overline{\psi} = +\infty \textit{ on } (x_1, x_0). \end{cases} \tag{18}$$

Moreover $\overline{\psi}$ *is unique up to an arbitrary additive constant.*

Although $\overline{\psi}$ is bounded on $[x_0, x_1]$, the front is not because it is represented by

$$\partial\{(x, y) \in \mathbb{T} \times \mathbb{R} \text{ s.t. } y > \overline{\psi}(x)\}.$$

In particular we have infinite vertical lines at $x = x_0$ and $x = x_1$. By the way, the above result also exhibits an example of a discontinuous $\overline{\psi}$, which was never the case in the $W^{1,\infty}$ theory. Finally our last claim goes as follows:

Proposition 8 (Many different profiles, bounded or not)**.** *Assume* (13)–(14) *with*

$$\Lambda_1 \geq 0 \quad \textit{and} \quad \Lambda_2 = 0,$$

and let $(\overline{c}, \overline{\psi})$ *be a GTWS of* (3)*. Then either* $\overline{\psi}$ *is unbounded and satisfies* (18) *as previously, or it may as well be bounded and such that:*

$$\begin{cases} \overline{\psi} \text{ is smooth in } (x_0, x_1) \cup (x_1, x_0) \text{ where it solves } (4), \\ \overline{\psi} \text{ is convex (resp. concave) and symmetric on } (x_0, x_1) \text{ (resp. } (x_1, x_0)), \\ \overline{\psi}(x_0^+) = \overline{\psi}(x_1^-) \leq \overline{\psi}(x_1^+) = \overline{\psi}(x_0^-), \\ \lim_{x_0} \overline{\psi}_x = -\infty \text{ and } \lim_{x_1} \overline{\psi}_x = +\infty. \end{cases}$$

Define moreover

$$h_j := \overline{\psi}(x_0^-) - \overline{\psi}(x_0^+) = \overline{\psi}(x_1^+) - \overline{\psi}(x_1^-)$$

the common height of the jumps at $x = x_0$ *and* $x = x_1$*. Then any* $h_j \in [0, +\infty]$ *corresponds to a possible GTWS. Finally, all admissible* $\overline{\psi}$ *are unique up to the choice of an* h_j *and arbitrary additive constant.*

Remark 6 (Lack of uniqueness). These fronts' profiles $\partial\{y > \overline{\psi}(x)\}$ are all connected, but not anymore unique up to translations if considering two different heights h_j. This is a notable difference with the $W^{1,\infty}$ theory.

As a corollary, we get necessary and sufficient conditions for having classical TWS.

Corollary 1. *Assume* (13)–(14)*. Let moreover* Λ_1 *and* Λ_2 *be as in* (15) *and* (16)*, respectively. There then exists* (c, ψ) *such that* $c > 0$ *and* $\psi \in C^{1,1}(\mathbb{T})$ *solves* (3)*, a.e., if and only if* $\Lambda_1 \geq 0$ *and* $\Lambda_2 > 0$*.*

Remark 7.
(1) We refer to [4, 7, 2, 3] for other sufficient conditions which are however not necessary.
(2) These TWS are bounded but they are not the only ones. Indeed, there are also bounded GTWS when $\Lambda_1 \geq 0$ and $\Lambda_2 = 0$, by Proposition 8. However such $\overline{\psi}$ are not $C^{1,1}$ at $x = x_0$ and $x = x_1$, thus $D^2\overline{\psi}$ are singular distributions, and (4) does not hold in the same sense. The correct PDE formulation requires instead the interface-like condition (12), even for $h_j = 0$ where $\overline{\psi}$ is continuous everywhere.

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