



HAL
open science

Rates of convergence in the central limit theorem for the elephant random walk with random step sizes

Jérôme Dedecker, Xiequan Fan, Haijuan Hu, Florence Merlevède

► To cite this version:

Jérôme Dedecker, Xiequan Fan, Haijuan Hu, Florence Merlevède. Rates of convergence in the central limit theorem for the elephant random walk with random step sizes. 2023. hal-03984155

HAL Id: hal-03984155

<https://hal.science/hal-03984155>

Preprint submitted on 12 Feb 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Rates of convergence in the central limit theorem for the elephant random walk with random step sizes

Jérôme Dedecker*, Xiequan Fan†, Haijuan Hu‡, Florence Merlevède§

Abstract

In this paper, we consider a generalization of the elephant random walk model. Compared to the usual elephant random walk, an interesting feature of this model is that the step sizes form a sequence of positive independent and identically distributed random variables instead of a fixed constant. For this model, we establish the law of the iterated logarithm, the central limit theorem, and we obtain rates of convergence in the central limit theorem with respect to the Kologmorov, Zolotarev and Wasserstein distances. We emphasize that, even in case of the usual elephant random walk, our results concerning the rates of convergence in the central limit theorem are new.

Keywords. Elephant random walk, law of the iterated logarithm, normal approximations, Wassertein's distance, Berry-Esseen bounds, central limit theorem

Mathematics Subject Classification (2010): 60G42, 60F05, 60E15, 82C41

1 Introduction

The elephant random walk (ERW) is a type of one-dimensional random walk on integers, which has a complete memory of its whole history. The ERW was introduced in 2004 by Schütz and Trimper [24] in order to study the memory effects of a non-Markovian random walk. The model has a link to a famous saying that elephants can always remember where they have been.

The ERW can be defined as follows. It starts at time $n = 0$, with position $T_0 = 0$. At time $n = 1$, the elephant moves to 1 with probability q and to -1 with probability $1 - q$, where $q \in (0, 1]$. So the position of the elephant at time $n = 1$ is given by $T_1 = X_1$, with X_1 a Rademacher $\mathcal{R}(q)$ random variable. At time $n + 1$, $n \geq 1$, the step X_{n+1} is

*Université Paris Cité, Laboratoire MAP5 and CNRS UMR 8145, 75016 Paris, France.

†Center for Applied Mathematics, Tianjin University, Tianjin 300072, China.

‡School of Mathematics and Statistics, Northeastern University at Qinhuangdao, Qinhuangdao, China.

§LAMA, Univ Gustave Eiffel, Univ Paris Est Créteil, UMR 8050 CNRS, F-77454 Marne-La-Vallée, France.

determined stochastically by the following rule. Let n' be an integer which is chosen from the set $\{1, 2, \dots, n\}$ uniformly at random. If $X_{n'} = 1$, then

$$X_{n+1} = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1 - p. \end{cases}$$

If $X_{n'} = -1$, then

$$X_{n+1} = \begin{cases} 1 & \text{with probability } 1 - p \\ -1 & \text{with probability } p. \end{cases}$$

Equivalently, X_{n+1} is determined stochastically by the following rule:

$$X_{n+1} = \begin{cases} X_{n'} & \text{with probability } p \\ -X_{n'} & \text{with probability } 1 - p. \end{cases}$$

The last equality suggests that at time $n + 1$, we reinforce $X_{n'}$ with probability p and reduce $X_{n'}$ with probability $1 - p$. Thus, for $n \geq 2$, the position of the elephant at time n is

$$T_n = \sum_{i=1}^n X_i, \text{ where } X_n = \alpha_n X_{\beta_n}, \quad (1.1)$$

with α_n has a Rademacher distribution $\mathcal{R}(p)$, $p \in (0, 1]$, and β_n is uniformly distributed over the integers $\{1, 2, \dots, n - 1\}$. Moreover, α_n is independent of $(X_i)_{1 \leq i \leq n}$, and the random variables $(\beta_i)_{i \geq 1}$ are independent. Here p is called the memory parameter. The ERW is respectively called diffusive, critical and superdiffusive according to $p \in (0, 3/4)$, $p = 3/4$ and $p \in (3/4, 1]$.

The description of the asymptotic behavior of the ERW has motivated many interesting works. By showing the connection of the ERW model with Polya-type urns, Baur and Bertoin obtained the functional form of the central limit theorem. Coletti, Gava and Schütz [9, 10] proved the central limit theorem (CLT) and a strong invariance principle for $p \in (0, 3/4]$ and a law of large numbers for $p \in (0, 1)$. Moreover, they also showed that if $p \in (3/4, 1]$, then the ERW converges to a non-degenerate random variable which is not normal. Vázquez Guevara [26] gave the almost sure CLT. Bercu [2] recovered the CLT via a martingale method. Bertoin [5] studied how memory impacts passages at the origin for the ERW. Recently, a number of variations for the ERW has been introduced. For instance, Bercu and Lucile [4] introduced a multi-dimensional type ERW, and established the multivariate CLT. Gut and Stadtmüller [18] investigated the case when the elephant remembers only a finite part of the first or last steps. Laulin [21] studied the asymptotic analysis for the reinforced ERW model. Recently, Bercu [3] considered the ERW with steps playing hide and seek with the Mittag-Leffler distribution.

In this paper, we consider a generalization of ERW, called ERW with random step sizes, such that the step sizes are random and varying in time. The ERW with random step sizes was introduced by Fan and Shao [16]. Let $(Z_i)_{i \geq 1}$ be a sequence of positive independent and identically distribution (i.i.d.) random variables, with finite mean $\nu =$

$\mathbf{E}Z_1$ and variance $\text{Var}(Z_1) = \sigma^2 \geq 0$. Moreover, $(Z_i)_{i \geq 1}$ is independent of $(X_i)_{i \geq 1}$. An ERW with random step sizes may be described as follows. At time $n = 1$, the elephant moves to Z_1 with probability q and to $-Z_1$ with probability $1 - q$. So the position Y_1 of the elephant at time $n = 1$ is given by the following rule:

$$Y_1 = \begin{cases} Z_1 & \text{with probability } q \\ -Z_1 & \text{with probability } 1 - q. \end{cases}$$

For $n \geq 2$, instead of (1.1), the position of the elephant with random step sizes at time n is

$$S_n = \sum_{i=1}^n Y_i, \text{ where } Y_n = \alpha_n X_{\beta_n} Z_n.$$

Notice that $|Y_n| = Z_n$ for all $n \geq 1$. Thus at time n , the step size is Z_n which is a random variable. Without loss of generality, we may assume that $\nu = 1$ (otherwise, we may consider the case S_n/ν instead of S_n). Clearly, when $\sigma^2 = 0$, we have $Z_n \equiv 1$ a.s. and then the ERW with random step sizes reduces to the usual ERW.

In this paper, we are interested in establishing the law of the iterated logarithm and convergence rates for normal approximations of S_n in terms of Kolmogorov distances and Zolotarev or Wasserstein distances. As we shall see, even in case of the usual ERW, our results concerning the rates of convergence in the central limit theorem are new.

Throughout the paper, C will denote a finite and positive constant that is allowed to depend on some fixed parameters such as $p, q, r, \rho, \gamma, \mathbf{E}Z_1^{2+\rho}$ and so on, but not on n . This constant may vary from line to line. For two sequences of positive numbers $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$, write $a_n \asymp b_n$ if there exists a constant $C > 0$ such that $a_n/C \leq b_n \leq Ca_n$ for all sufficiently large n . We shall also use the notation $a_n \ll b_n$ to mean that there exists a positive constant C not depending on n such that $a_n \leq Cb_n$. We also write $a_n \sim b_n$ if $\lim_{n \rightarrow \infty} a_n/b_n = 1$. We denote by $\mathcal{N}(0, \sigma^2)$ the normal distribution with mean 0 and variance σ^2 . \mathcal{N} will designate a standard normal random variable, and we will denote by $\Phi(\cdot)$ the cumulative distribution function of a standard normal random variable. We shall use the notations φ to denote the density of a $\mathcal{N}(0, 1)$ distribution and φ_{σ^2} to denote the density of a $\mathcal{N}(0, \sigma^2)$ distribution.

2 Law of iterated logarithm

The almost sure convergence for the ERW has been established by Bercu [2] and Bercu and Lucile [4] via a martingale method. In the next theorem, we give the corresponding results for the ERW with random step sizes.

Theorem 2.1. *Assume that $p \in (0, 1]$ and $\mathbf{E}Z_1^2 < \infty$.*

[i] *If $p \in (0, 3/4)$, then*

$$\limsup_{n \rightarrow \infty} \frac{|S_n|}{\sqrt{n \log \log n}} \leq \sqrt{\frac{2}{3-4p}} + \sqrt{2}\sigma \quad a.s. \quad (2.1)$$

[ii] If $p = 3/4$, then

$$\limsup_{n \rightarrow \infty} \frac{|S_n|}{\sqrt{n \log n \log \log n}} = \sqrt{2} \quad a.s. \quad (2.2)$$

[iii] If $p \in (3/4, 1]$, then

$$\lim_{n \rightarrow \infty} \frac{S_n}{n^{2p-1}} = L \quad a.s., \quad (2.3)$$

where L is a non-degenerate and non Gaussian random variable.

We immediately deduce from Theorem 2.1 that the law of large numbers holds. If $p \in (0, 1)$, then

$$\lim_{n \rightarrow \infty} \frac{S_n}{n} = 0 \quad a.s.$$

Hence, we find again the Kolmogorov strong law of large numbers given by Coletti, Gava and Schütz [9]. See also Bercu [2] and Bercu and Lucile [4] for closely related results. In particular, Bercu [3] gave the law of the iterated logarithm for ERW with stops playing hide and seek with the Mittag-Leffler distribution.

Remark 2.1. In [10] a strong invariance principle is proved for the usual ERW, from which the compact law of the iterated logarithm can be easily deduced. We would like to mention that the rate of approximation in this strong invariance principle can be improved by using Theorem 2.1 of Shao [25]. More precisely, letting $v_n = \sum_{i=1}^n a_i^2$ and applying Shao's results, one can prove that

[i] If $p \leq 5/8$, there exists a standard Wiener process $W(t)$ such that, for any $\delta > 0$,

$$a_n T_n - W(v_n) = o\left(a_n n^{1/4} (\log n)^{3/4} (\log \log n)^{\delta+1/4}\right) \quad a.s.$$

[ii] If $5/8 < p < 3/4$, then there exists a standard Wiener process $W(t)$ such that

$$a_n T_n - W(v_n) = o\left(u_n (\log n)^{1/2}\right) \quad a.s., \quad \text{where } u_n \text{ is any sequence increasing to } \infty.$$

[iii] If $p = 3/4$, then there exists a standard Wiener process $W(t)$ such that

$$a_n T_n - W(v_n) = o\left(u_n (\log \log n)^{1/2}\right) \quad a.s., \quad \text{where } u_n \text{ is any sequence increasing to } \infty.$$

3 Normal approximations

In this section, we consider the normal approximation for S_n . For $p \in (0, 3/4]$ and $n \geq 2$, denote

$$a_1 = 1, \quad a_n = \frac{\Gamma(n)\Gamma(2p)}{\Gamma(n+2p-1)} \quad \text{and} \quad v_n = \sum_{i=1}^n a_i^2.$$

Here $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$, $s > 0$, is the Gamma function. Notice that a_n and v_n are both positive. Moreover, by taking into account the relations (4.2) and (4.3) of Section 4, it is known that $v_n \rightarrow \infty$ as $n \rightarrow \infty$.

3.1 Central limit theorem

Theorem 3.1. *Assume that $\mathbf{E}Z_1^2 < \infty$. Then, for $p \in (0, 3/4]$,*

$$\frac{\tilde{S}_n}{\sqrt{v_n + na_n^2\sigma^2}} \xrightarrow{\mathbf{D}} \mathcal{N}, \quad (3.1)$$

where $\tilde{S}_n = a_n S_n - (2q - 1)$ and $\xrightarrow{\mathbf{D}}$ stands for convergence in distribution.

When $p \in (0, 3/4]$ and $\sigma^2 = 0$, the central limit theorem for the usual ERW has been established by Coletti, Gava and Schütz [9, 10] and Bercu [2]. Theorem 3.1 then extends the CLT to the case $\sigma^2 > 0$.

Note also that since $v_n \rightarrow \infty$ as $n \rightarrow \infty$, (3.1) also holds if we replace \tilde{S}_n by $a_n S_n$ (indeed the term $(2q - 1)/\sqrt{v_n + na_n^2\sigma^2}$ goes to 0 as $n \rightarrow \infty$). However, it is convenient to consider random variables which are centered as \tilde{S}_n . Indeed if we consider $a_n S_n$ rather than \tilde{S}_n then the corresponding rates of convergence in the CLT are slower (see for instance Remark 3.1 concerning the Kolmogorov distance).

Concerning the normalizing sequence, note that when $p \in (0, 3/4)$, then (4.1) combined with (4.2) entails that

$$b_n := a_n^{-1} \sqrt{v_n + na_n^2\sigma^2} \sim \sqrt{n(\sigma^2 + 1/(3 - 4p))},$$

whereas when $p = 3/4$, (4.1) combined with (4.2) implies that $b_n \sim \sqrt{n \log n}$.

Finally, notice that when $p \in (3/4, 1]$, $v_n + na_n^2\sigma^2$ is bounded and a_n is in order of n^{1-2p} (see (4.1)). Thus, by (2.3), $\tilde{S}_n/\sqrt{v_n + na_n^2\sigma^2}$ converges to a non-degenerate and non Gaussian random variable, and (3.1) does not hold. Hence, we shall only consider the case $p \in (0, 3/4]$ in the remaining of the paper.

3.2 Convergence rates in terms of Kolmogorov's distance

For a real-valued random variable X , denote

$$\mathbf{K}(X) = \sup_{u \in \mathbf{R}} \left| \mathbf{P}(X \leq u) - \Phi(u) \right|$$

the Kolmogorov's distance between the law of X and the standard normal distribution.

We start by giving the rates of convergence in the central limit theorem for the Kolmogorov distance and the usual ERW.

Theorem 3.2. *Let $p \in (0, 3/4]$. Then the following inequality holds:*

$$\mathbf{K}\left(\frac{a_n T_n - (2q - 1)}{\sqrt{v_n}}\right) \ll \frac{1}{\sqrt{n}} + \frac{1}{v_n}.$$

Taking into account (4.2) and (4.3) of the next section, it follows that when $p \leq 5/8$, the rate in Theorem 3.2 is $1/\sqrt{n}$ and when $p > 5/8$, the rate is $1/v_n$. Note also that Theorem 3.2 improves on the previous results obtained by Fan *et al.* [15] who obtained the rate $(\log n)/\sqrt{n}$ when $p \leq 1/2$, $(\log n)/\sqrt{v_n}$ when $p \in (1/2, 3/4)$ and $(\log \log n)/\sqrt{v_n}$ when $p = 3/4$.

Remark 3.1. Using Lemma 7.1, we then derive that for the non centered quantity $a_n T_n / \sqrt{v_n}$ we have the following rate of convergence in the CLT

$$\mathbf{K}\left(\frac{a_n T_n}{\sqrt{v_n}}\right) \ll \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{v_n}}.$$

The following theorem gives a Berry-Esseen type bound for the ERW with random step sizes. Theorem 3.2 can be viewed as a consequence of it (taking $\rho = 1$ and $\sigma = 0$.)

Theorem 3.3. Assume that $M(\rho) = \mathbf{E}Z_1^{2+\rho} < \infty$ for a constant $\rho \in (0, 1]$. The following inequality holds:

$$\mathbf{K}\left(\frac{a_n S_n - (2q - 1)}{\sqrt{v_n + na_n^2 \sigma^2}}\right) \ll \frac{1}{n^{\rho/2}} + \frac{1}{v_n}.$$

3.3 Convergence rates in terms of Zolotarev's distances

Zolotarev's distance of order r can be described as follows: for two probability measures μ and ν , and any $r > 0$,

$$\zeta_r(\mu, \nu) = \sup \left\{ \int f d\mu - \int f d\nu : f \in \Lambda_r \right\},$$

where Λ_r is defined as follows: denoting by l the natural integer satisfying $l < r \leq l + 1$, Λ_r is the class of real functions f which are l -times continuous differentiable and such that

$$|f^{(l)}(x) - f^{(l)}(y)| \leq |x - y|^{r-l} \quad \text{for any } (x, y) \in \mathbf{R}^2.$$

For two random variables X and Y , let μ_X and ν_Y be respectively the probability laws of X and Y . We denote $\zeta_r(X, Y) = \zeta_r(\mu_X, \nu_Y)$ to soothe the notations.

We start by giving the rates of convergence in the central limit theorem in terms of Zolotarev distances for the usual ERW.

Theorem 3.4. Let $r \in (0, 2]$. Then the following inequality holds:

$$\zeta_r\left(\frac{a_n T_n - (2q - 1)}{\sqrt{v_n}}, \mathcal{N}\right) \ll \frac{1}{n^{r/2}} + \frac{1}{v_n}.$$

Note that $\zeta_1 = W_1$, where W_1 is the Wasserstein distance of order 1 (see the next section for the definition). Hence, for $r = 1$, Theorem 3.4 improves on the previous results obtained by Ma *et al.* [22].

The following theorem gives some bounds for the ERW with random step sizes.

Theorem 3.5. Assume that $\mathbf{E}Z_1^{2+\rho} < \infty$ for a constant $\rho \in (0, 1]$. Let $r \in (0, 2]$. The following inequality holds:

$$\zeta_r\left(\frac{a_n S_n - (2q - 1)}{\sqrt{v_n + na_n^2 \sigma^2}}, \mathcal{N}\right) \ll \frac{1}{n^{(r \wedge \rho)/2}} + \frac{1}{v_n}.$$

3.4 Convergence rates in terms of Wasserstein's distances

We first recall the definitions of Wasserstein's distances. Let $\mathcal{L}(\mu, \nu)$ be a set of probability laws on \mathbf{R}^2 with marginals μ and ν . The Wasserstein distance of order $r > 0$ is defined as follows:

$$W_r(\mu, \nu) = \inf \left\{ \left(\int |x - y|^r \mathbf{P}(dx, dy) \right)^{1/\max(1, r)} : \mathbf{P} \in \mathcal{L}(\mu, \nu) \right\}.$$

For two random variables X and Y with respective laws μ_X and ν_Y , denote $W_r(X, Y) = W_r(\mu_X, \nu_Y)$ to soothe the notation.

For $r \in (0, 1]$, using Kantorovich-Rubinstein's theorem with the metric $d(x, y) = |x - y|^r$, we have $W_r(\mu, \nu) = \zeta_r(\mu, \nu)$. Hence, for $r \in (0, 1]$, the inequality of Theorem 3.5 holds for the distance W_r .

Next, for $r \in [1, 2]$, from [23], we know that there exists a positive constant c_r such that

$$W_r(\mu, \nu) \leq c_r (\zeta_r(\mu, \nu))^{1/r}. \quad (3.2)$$

Therefore, starting from Theorem 3.4, we derive for the usual ERW that for any $r \in [1, 2]$,

$$W_r \left(\frac{a_n T_n - (2q - 1)}{\sqrt{v_n}}, \mathcal{N} \right) \ll \frac{1}{\sqrt{n}} + \frac{1}{v_n^{1/r}}.$$

Combining this last estimate with some precise upper bounds for independent random variables given by Bobkov [7], we get the following result for the ERW with random step sizes.

Theorem 3.6. *Let $r \in [1, 2]$, and assume that $\mathbf{E}Z_1^{2+\rho} < \infty$ for a constant $\rho \in (0, r]$. The following inequalities hold:*

$$W_r \left(\frac{a_n S_n - (2q - 1)}{\sqrt{v_n}}, \mathcal{N} \right) \ll \frac{1}{n^{\rho/2r}} + \frac{1}{v_n^{1/r}}.$$

Note that when $\rho \in (0, 1]$, Theorem 3.6 is a direct consequence of Theorem 3.5 combined with inequality (3.2).

4 Preliminary considerations and lemmas

By the well-known Stirling formula

$$\log \Gamma(x) = \left(x - \frac{1}{2} \right) \log x - x + \frac{1}{2} \log 2\pi + O\left(\frac{1}{x}\right) \quad \text{as } x \rightarrow \infty,$$

we deduce that for $p \in (0, 1]$,

$$\lim_{n \rightarrow \infty} a_n n^{2p-1} = \Gamma(2p). \quad (4.1)$$

Moreover, for $p \in (0, 3/4)$, we have

$$\lim_{n \rightarrow \infty} \frac{v_n}{n^{3-4p}} = \frac{\Gamma(2p)^2}{3 - 4p}, \quad (4.2)$$

and, for $p = 3/4$,

$$\lim_{n \rightarrow \infty} \frac{v_n}{\log n} = \frac{\pi}{4}. \quad (4.3)$$

See also Bercu [2] for the equalities (4.1)-(4.3). Denote

$$\gamma_n = 1 + \frac{2p-1}{n}, \quad n \geq 1.$$

It is easy to see that $a_1 = 1$ and for $n \geq 2$,

$$a_n = \prod_{i=1}^{n-1} \frac{1}{\gamma_i}.$$

Notice that $a_{n+1} = a_n/\gamma_n$. Thus, a_n is increasing in n if $0 < p \leq 1/2$, and decreasing in n if $1/2 < p \leq 1$. Hence, we have

$$\max_{2 \leq i \leq n} a_i = \begin{cases} a_n & \text{if } 0 < p \leq 1/2 \\ (2p)^{-1} & \text{if } 1/2 < p \leq 1. \end{cases} \quad (4.4)$$

Define the filtration $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_n = \sigma\{\alpha_i, X_1, \beta_i, Z_i : 1 \leq i \leq n\}$, $n \geq 1$, and denote

$$a_n S_n - (2q-1) = a_n \sum_{i=1}^n \alpha_i X_{\beta_i} (Z_i - 1) + a_n T_n - (2q-1). \quad (4.5)$$

Let

$$M_0 = 0, \quad M_n = a_n T_n - (2q-1), \quad n \geq 1.$$

As noticed in [2], $(M_n, \mathcal{F}_n)_{n \geq 0}$ is a martingale. Indeed, we have

$$\mathbf{E}[M_1 | \mathcal{F}_0] = \mathbf{E}[a_1 T_1 - (2q-1)] = 0,$$

and for $n \geq 1$,

$$\begin{aligned} \mathbf{E}[M_{n+1} | \mathcal{F}_n] &= \mathbf{E}[a_{n+1} T_{n+1} - (2q-1) | \mathcal{F}_n] \\ &= a_{n+1} T_n + a_{n+1} \mathbf{E}[\alpha_{n+1}] \mathbf{E}[X_{\beta_{n+1}} | \mathcal{F}_n] - (2q-1) \\ &= a_{n+1} \left(T_n + (2p-1) \frac{T_n}{n} \right) - (2q-1) \\ &= a_n T_n - (2q-1) = M_n \quad a.s. \end{aligned}$$

Moreover, we can rewrite $(M_n, \mathcal{F}_n)_{n \geq 1}$ in the following additive form

$$M_n = \sum_{i=1}^n a_i \varepsilon_i := \sum_{i=1}^n \Delta M_i, \quad (4.6)$$

where $\varepsilon_i = T_i - \gamma_{i-1} T_{i-1}$ with the convention $\gamma_0 T_0 = 2q-1$. Note that $(\Delta M_k, \mathcal{F}_k)_{k \geq 0}$ is a sequence of martingale differences such that

$$\text{Var}(M_n) = \sum_{i=1}^n a_i^2 - (2p-1)^2 \mathbf{E} \zeta_n, \quad \text{where } \zeta_n = \sum_{k=1}^{n-1} a_{k+1}^2 \left(\frac{T_k}{k} \right)^2$$

(see (A.7) in [2]). In particular

$$\text{Var}(M_n) \leq v_n \quad \text{and} \quad \|a_n T_n\|_2^2 \leq 2(1 + v_n). \quad (4.7)$$

The following lemma shows that ΔM_k is bounded.

Lemma 4.1. *For $k \geq 1$ and $p \in (0, 1]$, it holds $|\Delta M_k| \leq 2a_k$.*

Proof. We have

$$|\Delta M_1| = |a_1 T_1 - (2q - 1)| \leq |X_1| + |2q - 1| \leq 2 = 2a_1,$$

and, for $k \geq 2$,

$$\Delta M_k = a_k T_k - a_{k-1} T_{k-1} = a_k X_k - a_k \frac{T_{k-1}}{k-1} (2p-1).$$

Since $|X_k| \leq 1$, we have $|T_k| \leq k$. Hence, for $k \geq 2$,

$$|\Delta M_k| \leq a_k |X_k| + a_k |2p-1| \leq 2a_k.$$

This completes the proof of the lemma. \square

Next lemma gives estimates of the conditional moments of order 3 and 4 of the sequence of martingale differences $(\Delta M_k, \mathcal{F}_k)_{k \geq 1}$.

Lemma 4.2. *For $k \geq 1$, the following equalities hold:*

$$\mathbf{E}[\Delta M_k^3 | \mathcal{F}_{k-1}] = 2(2p-1)a_k^3 \left(-\frac{T_{k-1}}{k-1} + (2p-1)^2 \frac{T_{k-1}^3}{(k-1)^3} \right)$$

and

$$\mathbf{E}[\Delta M_k^4 | \mathcal{F}_{k-1}] = a_k^4 + a_k^4 (2p-1)^2 \left(2 \frac{T_{k-1}^2}{(k-1)^2} - 3(2p-1)^2 \frac{T_{k-1}^4}{(k-1)^4} \right),$$

with the convention $T_0/0 = 1$. In particular,

$$\left| \mathbf{E}[\Delta M_k^3 | \mathcal{F}_{k-1}] \right| \leq 4a_k^3 \left| \frac{T_{k-1}}{k-1} \right| \quad \text{and} \quad \left| \mathbf{E}[\Delta M_k^4 | \mathcal{F}_{k-1}] - a_k^4 \right| \leq 5a_k^4 \frac{T_{k-1}^2}{(k-1)^2}.$$

Proof. The equality for the conditional moment of order 4 comes from [2, (A.5)]. It remains to prove the equality for the conditional moment of order 3. Note first that

$$\begin{aligned} \mathbf{E}[\Delta M_k^3 | \mathcal{F}_{k-1}] &= a_k^3 \mathbf{E}[(T_k - \gamma_{k-1} T_{k-1})^3 | \mathcal{F}_{k-1}] \\ &= a_k^3 \left(\mathbf{E}[T_k^3 | \mathcal{F}_{k-1}] - 3\gamma_{k-1} T_{k-1} \mathbf{E}[T_k^2 | \mathcal{F}_{k-1}] + 3\gamma_{k-1}^2 T_{k-1}^2 \mathbf{E}[T_k | \mathcal{F}_{k-1}] - \gamma_{k-1}^3 T_{k-1}^3 \right). \end{aligned}$$

For $\mathbf{E}[T_k^3 | \mathcal{F}_{k-1}]$, we have the following estimation

$$\begin{aligned} \mathbf{E}[T_k^3 | \mathcal{F}_{k-1}] &= \mathbf{E}[(T_{k-1} + \alpha_k X_{\beta_k})^3 | \mathcal{F}_{k-1}] \\ &= T_{k-1}^3 + 3T_{k-1}^2 \mathbf{E}[\alpha_k X_{\beta_k} | \mathcal{F}_{k-1}] \\ &\quad + 3T_{k-1} \mathbf{E}[(\alpha_k X_{\beta_k})^2 | \mathcal{F}_{k-1}] + \mathbf{E}[(\alpha_k X_{\beta_k})^3 | \mathcal{F}_{k-1}] \\ &= T_{k-1}^3 + 3(2p-1) \frac{T_{k-1}^3}{k-1} + 3T_{k-1} + (2p-1) \frac{T_{k-1}}{k-1} \\ &= (3\gamma_{k-1} - 2)T_{k-1}^3 + (\gamma_{k-1} + 2)T_{k-1}. \end{aligned}$$

Notice that $\mathbf{E}[T_k^2|\mathcal{F}_{k-1}] = (2\gamma_{k-1} - 1)T_{k-1}^2 + 1$ and $\mathbf{E}[T_k|\mathcal{F}_{k-1}] = \gamma_{k-1}T_{k-1}$ (see the relations (A.3) and (2.3) in [2]). Hence, we get

$$\mathbf{E}[\Delta M_k^3|\mathcal{F}_{k-1}] = 2a_k^3 \left(T_{k-1}(1 - \gamma_{k-1}) + T_{k-1}^3(\gamma_{k-1} - 1)^3 \right).$$

The first equality follows from the fact that $\gamma_n = 1 + \frac{2p-1}{n}$. To get the inequalities, it suffices to notice that $|T_k| \leq k$. \square

5 Proof of Theorem 2.1

We first give a proof of item [i]. For any $n \geq 1$, let

$$H_n = \sum_{i=1}^n \alpha_i X_{\beta_i} (Z_i - 1). \quad (5.1)$$

By (4.5), recall that $S_n = H_n + T_n$. Let $\mathcal{G}_n = \sigma(\alpha_i, \beta_i, X_1, Z_i, 1 \leq i \leq n)$. Note that $(H_n, \mathcal{G}_n)_{n \geq 1}$ is a zero-mean, square integrable martingale such that $\text{Var}(\alpha_i X_{\beta_i} (Z_i - 1)) = \sigma^2$ and $|\alpha_i X_{\beta_i} (Z_i - 1)| = |Z_i - 1|$. Then, by [20, Corollary 4.2],

$$\limsup_{n \rightarrow \infty} \frac{|H_n|}{\sqrt{n \log \log n}} = \sqrt{2}\sigma \quad a.s. \quad (5.2)$$

On another hand, by Theorem 3.2 of Bercu [2],

$$\limsup_{n \rightarrow \infty} \frac{|T_n|}{\sqrt{n \log \log n}} = \sqrt{\frac{2}{3-4p}} \quad a.s.$$

Therefore

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{|S_n|}{\sqrt{n \log \log n}} &\leq \limsup_{n \rightarrow \infty} \frac{|H_n|}{\sqrt{n \log \log n}} + \limsup_{n \rightarrow \infty} \frac{|T_n|}{\sqrt{n \log \log n}} \\ &\leq \sqrt{2}\sigma + \sqrt{\frac{2}{3-4p}} \quad a.s., \end{aligned}$$

which gives (2.1).

We turn to the proof of item [ii]. By Theorem 3.5 of Bercu [2], we have

$$\limsup_{n \rightarrow \infty} \frac{|T_n|}{\sqrt{n \log n \log \log n}} = \sqrt{2} \quad a.s.$$

Since $S_n = H_n + T_n$, we get from (5.2) that

$$\limsup_{n \rightarrow \infty} \left| \frac{|S_n| - |T_n|}{\sqrt{n \log n \log \log n}} \right| \leq \limsup_{n \rightarrow \infty} \frac{|H_n|}{\sqrt{n \log n \log \log n}} = 0 \quad a.s.,$$

and (2.2) follows.

We complete the proof by proving item [iii] that is when $3/4 < p \leq 1$. Since in this case $n^{2p-1}/\sqrt{n} \rightarrow 0$, (5.2) entails that $\lim_{n \rightarrow \infty} \frac{H_n}{n^{2p-1}} = 0$. Hence the result follows from Theorems 3.7 and 3.8 in Bercu [2] and the fact that $S_n = H_n + T_n$.

6 Proof of Theorem 3.1

By (5.1), we have

$$\frac{a_n S_n - (2q - 1)}{\sqrt{v_n + na_n^2 \sigma^2}} = \frac{a_n H_n}{\sqrt{v_n + na_n^2 \sigma^2}} + \frac{M_n}{\sqrt{v_n + na_n^2 \sigma^2}} =: U_n + V_n, \quad (6.1)$$

where H_n is defined in (5.1). Denote $\mathcal{F} = \sigma(\alpha_i, \beta_i, X_1, i \geq 1)$. For $s, t \in \mathbf{R}$, write

$$f_n(s, t) = \mathbf{E} \exp\{itU_n + isV_n\} = \mathbf{E}[\mathbf{E}[\exp\{itU_n\}|\mathcal{F}] \exp\{isV_n\}],$$

the joint characteristic function of (U_n, V_n) . Since $(Z_i)_{i \geq 1}$ is a sequence of i.i.d. r.v.'s,

$$\lim_{n \rightarrow \infty} \mathbf{E}[\exp\{itU_n\}|\mathcal{F}] = \varphi(\sigma_1 t) \quad \text{a.s., where } \sigma_1^2 = \lim_{n \rightarrow \infty} \frac{na_n^2 \sigma^2}{v_n + na_n^2 \sigma^2}$$

and $\varphi(t) = \exp\{-t^2/2\}$. Now

$$f_n(s, t) = \mathbf{E} \left[\left(\mathbf{E}[\exp\{itU_n\}|\mathcal{F}] - \varphi(\sigma_1 t) \right) \exp\{isV_n\} \right] + \varphi(\sigma_1 t) \mathbf{E}[\exp\{isV_n\}].$$

From the CLT for the usual ERW (see Theorems 3.3 and 3.4 in Bercu [2]), we know that

$$\lim_{n \rightarrow \infty} \mathbf{E}[\exp\{isV_n\}] = \varphi(\sigma_2 s), \quad \text{where } \sigma_2^2 = \lim_{n \rightarrow \infty} \frac{v_n}{v_n + na_n^2 \sigma^2}.$$

Consequently, it holds

$$\lim_{n \rightarrow \infty} f_n(s, t) = \varphi(\sigma_1 t) \varphi(\sigma_2 s).$$

In particular, since $\sigma_1^2 + \sigma_2^2 = 1$, we get $\lim_{n \rightarrow \infty} f_n(t, t) = \varphi(t)$, which implies that $(U_n + V_n)_{n \geq 1}$ converges in distribution to a standard normal random variable.

7 Proof of Theorem 3.2

The proof of Theorem 3.2 is a refinement on the argument of Lemma 3.3 of Grama and Haeusler [17], where the authors obtained the best possible Berry-Esseen bound for martingales with bounded differences. Compared to [17], the main challenge of our proof comes from the fact that the conditional variance of martingales does not converge to a constant in L^∞ -norm. Our proof will be based on Lindeberg's telescoping sums argument, see Bolthausen [8]. The following two technical lemmas due to Bolthausen [8] will be needed.

Lemma 7.1. [8, Lemma 1] *Let X and Y be random variables. Then*

$$\sup_{u \in \mathbf{R}} \left| \mathbf{P}(X \leq u) - \Phi(u) \right| \leq c_1 \sup_{u \in \mathbf{R}} \left| \mathbf{P}(X + Y \leq u) - \Phi(u) \right| + c_2 \|\mathbf{E}[Y^2|X]\|_\infty^{1/2},$$

where $c_1 = 2$ and $c_2 = 5/\sqrt{2\pi}$.

Lemma 7.2. [8, Lemma 2] Let G be an integrable function on \mathbf{R} of bounded variation $\|G\|_V$, X be a random variable and $a, b \neq 0$ be real numbers. Then

$$\mathbf{E} \left[G \left(\frac{X+a}{b} \right) \right] \leq \|G\|_V \sup_u \left| \mathbf{P}(X \leq u) - \Phi(u) \right| + \|G\|_1 |b|,$$

where $\|G\|_1$ is the $L_1(\mathbf{R})$ norm of G .

We shall divide the proof in two steps according to the values of p .

1) Case $p \in (0, 3/4)$. Let us introduce some notations that will be used all along the proof. Let β be a constant greater than one that will be specified later. Denote

$$\kappa_n^2 = \max_{[n/2] \leq i \leq n} a_i^2, \quad \delta_n^2 = \beta \kappa_n^2 / v_n, \quad (7.1)$$

and for $0 \leq k \leq n$,

$$t_{k,n} = (\beta \kappa_n^2 + v_n - v_k)^{1/2}, \quad A_k = t_{k,n}^2 / v_n, \quad u_{k,n} = \beta + \beta^{-1}(1 + \beta) \sum_{i=k+1}^n a_i^2 / \kappa_n^2. \quad (7.2)$$

Note that A_k is non-increasing in k , and satisfies $A_0 = \delta_n^2 + 1$ and $A_n = \delta_n^2$. Moreover, for $u, x \in \mathbf{R}$ and $y > 0$, set, for brevity,

$$\Phi_u(x, y) = \Phi \left(\frac{u-x}{\sqrt{y}} \right). \quad (7.3)$$

Let \mathcal{N}_1 be a standard normal random variable, which is also independent of T_n . For $1 \leq k \leq n$, denote

$$\hat{K}_0 = 0, \quad \hat{K}_k = \frac{1}{\sqrt{v_n}} \left(a_k T_k - (2q-1) \right), \quad \xi_k = \hat{K}_k - \hat{K}_{k-1} \quad \text{and} \quad \sigma_k^2 = \frac{a_k^2}{v_n}.$$

Then we have

$$\frac{a_n T_n - (2q-1)}{\sqrt{v_n}} = \hat{K}_n = \sum_{k=1}^n \xi_k.$$

Next, we estimate the upper bound of $\mathbf{K}(\hat{K}_n)$. By Lemma 7.1, we get

$$\mathbf{K}(\hat{K}_n) \leq c_1 \sup_{u \in \mathbf{R}} \left| \mathbf{P}(\hat{K}_n + \delta_n \mathcal{N}_1 \leq u) - \Phi(u) \right| + c_2 \delta_n = c_1 \sup_{u \in \mathbf{R}} \left| \mathbf{E}[\Phi_u(\hat{K}_n, A_n)] - \Phi(u) \right| + c_2 \delta_n.$$

Hence

$$\mathbf{K}(\hat{K}_n) \leq c_1 \sup_{u \in \mathbf{R}} \left| \mathbf{E}[\Phi_u(\hat{K}_n, A_n)] - \mathbf{E}[\Phi_u(\hat{K}_0, A_0)] \right| + c_1 \sup_{u \in \mathbf{R}} \left| \Phi \left(\frac{u}{\sqrt{\delta_n^2 + 1}} \right) - \Phi(u) \right| + c_2 \delta_n.$$

There exist positive constants c_3 and c_4 such that for any $n \geq 1$,

$$\left| \Phi \left(\frac{u}{\sqrt{\delta_n^2 + 1}} \right) - \Phi(u) \right| \leq c_3 \left| \frac{1}{\sqrt{\delta_n^2 + 1}} - 1 \right| \leq c_4 \delta_n^2. \quad (7.4)$$

Therefore, since $\delta_n^2 \leq \beta$,

$$\mathbf{K}(\hat{K}_n) \leq c_1 \sup_{u \in \mathbf{R}} \left| \mathbf{E}[\Phi_u(\hat{K}_n, A_n)] - \mathbf{E}[\Phi_u(\hat{K}_0, A_0)] \right| + c_5 \delta_n. \quad (7.5)$$

Next, we give an estimation for $\mathbf{E}[\Phi_u(\hat{K}_n, A_n)] - \mathbf{E}[\Phi_u(\hat{K}_0, A_0)]$. Write first

$$\mathbf{E}[\Phi_u(\hat{K}_n, A_n)] - \mathbf{E}[\Phi_u(\hat{K}_0, A_0)] = \sum_{k=1}^n \mathbf{E}[\Phi_u(\hat{K}_k, A_k) - \Phi_u(\hat{K}_{k-1}, A_{k-1})].$$

Using the fact that

$$\frac{\partial^2}{\partial x^2} \Phi_u(x, y) = 2 \frac{\partial}{\partial y} \Phi_u(x, y),$$

and that $\mathbf{E}[\xi_k | \mathcal{F}_{k-1}] = 0$, we obtain

$$\mathbf{E}[\Phi_u(\hat{K}_n, A_n)] - \mathbf{E}[\Phi_u(\hat{K}_0, A_0)] = I_1 + I_2 - I_3,$$

where

$$\begin{aligned} I_1 &= \sum_{k=1}^n \mathbf{E} \left[\Phi_u(\hat{K}_k, A_k) - \Phi_u(\hat{K}_{k-1}, A_k) \right. \\ &\quad \left. - \frac{\partial}{\partial x} \Phi_u(\hat{K}_{k-1}, A_k) \xi_k - \frac{1}{2} \frac{\partial^2}{\partial x^2} \Phi_u(\hat{K}_{k-1}, A_k) \xi_k^2 \right], \\ I_2 &= \frac{1}{2} \sum_{k=1}^n \mathbf{E} \left[\frac{\partial^2}{\partial x^2} \Phi_u(\hat{K}_{k-1}, A_k) \left(\mathbf{E}[\xi_k^2 | \mathcal{F}_{k-1}] - \sigma_k^2 \right) \right], \\ I_3 &= \sum_{k=1}^n \mathbf{E} \left[\Phi_u(\hat{K}_{k-1}, A_{k-1}) - \Phi_u(\hat{K}_{k-1}, A_k) - \frac{\partial}{\partial y} \Phi_u(\hat{K}_{k-1}, A_k) \sigma_k^2 \right]. \end{aligned}$$

From (7.5), we deduce that

$$\mathbf{K}(\hat{K}_n) \leq C \left(|I_1| + |I_2| + |I_3| + \delta_n \right). \quad (7.6)$$

In the sequel, we give some estimates for I_1 , I_2 and I_3 . The notation ϑ_i stands for some values or random variables satisfying $0 \leq \vartheta_i \leq 1$.

a) Control of I_1 . To shorten notations, denote $\hat{H}_{k-1}(u) = \frac{u - \hat{K}_{k-1}}{\sqrt{A_k}}$. For $1 \leq k \leq n$, we have

$$\begin{aligned} R_k &:= \Phi_u(\hat{K}_k, A_k) - \Phi_u(\hat{K}_{k-1}, A_k) - \frac{\partial}{\partial x} \Phi_u(\hat{K}_{k-1}, A_k) \xi_k - \frac{1}{2} \frac{\partial^2}{\partial x^2} \Phi_u(\hat{K}_{k-1}, A_k) \xi_k^2 \\ &= \Phi \left(\hat{H}_{k-1}(u) - \frac{\xi_k}{\sqrt{A_k}} \right) - \Phi(\hat{H}_{k-1}(u)) + \Phi'(\hat{H}_{k-1}(u)) \frac{\xi_k}{\sqrt{A_k}} \\ &\quad - \frac{1}{2} \Phi''(\hat{H}_{k-1}(u)) \left(\frac{\xi_k}{\sqrt{A_k}} \right)^2. \end{aligned}$$

By Taylor expansion at order 4, we deduce that

$$I_1 = \sum_{k=1}^n \mathbf{E} R_{k,1} + \sum_{k=1}^n \mathbf{E} R_{k,2}, \quad (7.7)$$

where

$$R_{k,1} = -\frac{1}{6} \Phi'''(\hat{H}_{k-1}(u)) \left(\frac{\xi_k}{\sqrt{A_k}} \right)^3 \quad \text{and} \quad R_{k,2} = \frac{1}{24} \Phi^{(4)} \left(\hat{H}_{k-1}(u) - \vartheta_k \frac{\xi_k}{\sqrt{A_k}} \right) \left(\frac{\xi_k}{\sqrt{A_k}} \right)^4.$$

Next, we handle the term $|\mathbf{E}R_{k,1}|$. As $\hat{H}_{k-1}(u)$ is \mathcal{F}_{k-1} -measurable, we have

$$|\mathbf{E}R_{k,1}| \leq \frac{1}{A_k^{3/2}} \mathbf{E} \left[\left| \Phi'''(\hat{H}_{k-1}(u)) \right| |\mathbf{E}[\xi_k^3 | \mathcal{F}_{k-1}]| \right] \leq C \frac{1}{A_k^{3/2}} \mathbf{E} |\mathbf{E}[\xi_k^3 | \mathcal{F}_{k-1}]|.$$

By Lemma 4.2, we have $|\mathbf{E}\xi_1^3| \leq 4a_1^3/v_n^{3/2}$, and for $2 \leq k \leq n$,

$$\mathbf{E} |\mathbf{E}[\xi_k^3 | \mathcal{F}_{k-1}]| \ll \left(\frac{a_k}{\sqrt{v_n}} \right)^3 \mathbf{E} \left| \frac{T_{k-1}}{k-1} \right| \ll \frac{a_k^3 (\mathbf{E} |a_{k-1} T_{k-1}|^2)^{1/2}}{a_{k-1} v_n^{3/2} (k-1)}.$$

Taking into account (4.7), it follows that for $2 \leq k \leq n$,

$$\mathbf{E} |\mathbf{E}[\xi_k^3 | \mathcal{F}_{k-1}]| \ll \frac{a_k^3 (1 + \sqrt{v_{k-1}})}{a_{k-1} v_n^{3/2} (k-1)}.$$

Hence, we get

$$\sum_{k=1}^n |\mathbf{E}R_{k,1}| \ll \frac{1}{v_n^{3/2}} + \sum_{k=2}^n \frac{1}{A_k^{3/2}} \frac{a_k^3 \sqrt{v_{k-1}}}{a_{k-1} v_n^{3/2} (k-1)}. \quad (7.8)$$

Next we give an estimation for the last bound. By (4.1) and (4.2), and since $p \in (0, 3/4)$, $\frac{\sqrt{v_{k-1}}}{a_{k-1}(k-1)} = O\left(\frac{1}{\sqrt{k}}\right)$, $k \rightarrow \infty$. Note also that $A_k v_n = t_{k,n}^2$ and, since $p \in (0, 3/4)$, $\max_{1 \leq k \leq [n/2]} t_{k,n}^{-2} = t_{[n/2],n}^{-2} \ll v_n^{-1}$. Therefore, we get

$$\sum_{k=2}^n \frac{1}{A_k^{3/2}} \frac{a_k^3 \sqrt{v_{k-1}}}{a_{k-1} v_n^{3/2} (k-1)} \ll v_n^{-3/2} \sum_{k \in [1, n/2]} \frac{a_k^3}{\sqrt{k}} + \frac{\kappa_n}{\sqrt{n}} \sum_{k \in (n/2, n]} t_{k,n}^{-3} a_k^2. \quad (7.9)$$

For the first term on the right-hand side of (7.9), we infer that

$$v_n^{-3/2} \sum_{k \in [1, n/2]} \frac{a_k^3}{\sqrt{k}} \ll v_n^{-3/2} \mathbf{1}_{p > 7/12} + v_n^{-3/2} (\log n) \mathbf{1}_{p = 7/12} + n^{-1} \mathbf{1}_{p < 7/12}. \quad (7.10)$$

To deal with the second term on the right-hand side of (7.9), we first notice that for any integer k such that $[n/2] \leq k \leq n$,

$$\frac{a_k^2}{t_{k,n}^2} \leq \log(u_{k-1,n}/u_{k,n}) \text{ and } \frac{\kappa_n^2 u_{k-1,n}}{3} \leq t_{k,n}^2 \leq \kappa_n^2 u_{k,n} \quad (7.11)$$

(see inequalities (5.2) and (5.4) in [12]). Therefore, for any $\alpha \geq 0$,

$$\begin{aligned} \mathcal{T}_n(\alpha) &:= \sum_{k=[n/2]+1}^n t_{k,n}^\alpha \frac{a_k^2}{t_{k,n}^2} \leq \kappa_n^\alpha \sum_{k=[n/2]+1}^n u_{k,n}^{\alpha/2} \int_{u_{k,n}}^{u_{k-1,n}} \frac{1}{x} dx \\ &\leq \kappa_n^\alpha \sum_{k=[n/2]+1}^n \int_{u_{k,n}}^{u_{k-1,n}} x^{\alpha/2-1} dx \end{aligned}$$

and, for any $\alpha < 0$,

$$\mathcal{T}_n(\alpha) \leq \kappa_n^\alpha 3^{-\alpha/2} \sum_{k=[n/2]+1}^n u_{k-1,n}^{\alpha/2} \int_{u_{k,n}}^{u_{k-1,n}} \frac{1}{x} dx \leq \kappa_n^\alpha 3^{-\alpha/2} \sum_{k=[n/2]+1}^n \int_{u_{k,n}}^{u_{k-1,n}} x^{\alpha/2-1} dx.$$

So, overall, for any real α , using the fact that $p \in (0, 3/4)$, we get

$$\begin{aligned} \mathcal{T}_n(\alpha) &= \sum_{k=[n/2]+1}^n t_{k,n}^\alpha \frac{a_k^2}{t_{k,n}^2} \leq \kappa_n^\alpha (1 + 3^{-\alpha/2}) \int_\beta^{u_{[n/2],n}} x^{\alpha/2-1} dx \\ &\ll \kappa_n^\alpha \beta^{\alpha/2} \mathbf{1}_{\alpha < 0} + (\log n) \mathbf{1}_{\alpha=0} + v_n^{\alpha/2} \mathbf{1}_{\alpha > 0}. \end{aligned} \quad (7.12)$$

Therefore, by (7.12) applied with $\alpha = -1$, we get

$$\frac{\kappa_n}{\sqrt{n}} \sum_{k \in (n/2, n]} t_{k,n}^{-3} a_k^2 \ll n^{-1/2}. \quad (7.13)$$

So, overall, starting from (7.8) and taking into account (7.9), (7.10), (7.13) and that $(\log n) v_n^{-3/2} \mathbf{1}_{p=7/12} \ll n^{-1} \log n$, we derive that, for $p \in (0, 3/4)$,

$$\sum_{k=1}^n |\mathbf{E}R_{k,1}| \ll \frac{1}{v_n^{3/2}} + \frac{1}{\sqrt{n}}. \quad (7.14)$$

Next, we give an estimate of $|\mathbf{E}R_{k,2}|$. By Lemma 4.1, note that $\left| \frac{\xi_k}{\sqrt{A_k}} \right| \leq \frac{2a_k}{t_{k,n}}$. Hence, if $k \geq [n/2] + 1$, $\left| \frac{\xi_k}{\sqrt{A_k}} \right| \leq 2/\sqrt{\beta}$. Next, when $k \leq [n/2]$ and $p \in [0, 1/2]$, we still have $\left| \frac{\xi_k}{\sqrt{A_k}} \right| \leq 2/\sqrt{\beta}$. On another hand, when $k \leq [n/2]$ and $p > 1/2$, we have $\left| \frac{\xi_k}{\sqrt{A_k}} \right| \leq \min(2/\sqrt{\beta \kappa_n^2}, 2/\sqrt{v_n - v_{[n/2]}})$. So, in each case, we can select β large enough in such a way that $\left| \frac{\xi_k}{\sqrt{A_k}} \right| \leq 1$. From now on, β is selected this way. Thus we have

$$\begin{aligned} |R_{k,2}| &\leq \frac{1}{24} \left| \Phi^{(4)} \left(\hat{H}_{k-1}(u) - \vartheta_k \frac{\xi_k}{\sqrt{A_k}} \right) \right| \left(\frac{\xi_k}{\sqrt{A_k}} \right)^4 \\ &\leq G(\hat{H}_{k-1}(u)) \left(\frac{\xi_k}{\sqrt{A_k}} \right)^4, \end{aligned}$$

where $G(z) = \sup_{|t-z| \leq 1} |\Phi^{(4)}(t)|$. By Lemma 4.1, $\xi_k^4 \leq 2^4 a_k^4 v_n^{-2}$. Therefore,

$$\sum_{k=1}^n |\mathbf{E}R_{k,2}| \ll \sum_{k=1}^n \frac{\mathbf{E}[G(\hat{H}_{k-1}(u))] \left(\frac{a_k}{\sqrt{v_n}} \right)^4}{A_k^2}.$$

By the definition of $\hat{H}_{k-1}(u)$ and Lemma 7.2, it follows that there exists a positive constant C such that for any $k \geq 1$ and any $u \in \mathbf{R}$,

$$\mathbf{E}[G(\hat{H}_{k-1}(u))] \leq C \mathbf{K}(\hat{K}_{k-1}) + C \sqrt{A_k}.$$

Now, by Lemma 7.1,

$$\mathbf{K}(\hat{K}_{k-1}) \leq C \mathbf{K}(\hat{K}_n) + C \|\mathbf{E}[(\hat{K}_n - \hat{K}_{k-1})^2 | \hat{K}_{k-1}]\|_\infty^{1/2},$$

and, by the martingale property of $(M_n, \mathcal{F}_n)_{n \geq 1}$ and Lemma 4.1,

$$\mathbf{E}[(\hat{K}_n - \hat{K}_{k-1})^2 | \hat{K}_{k-1}] = v_n^{-1} \sum_{i=k}^n \mathbf{E}[(\Delta M_i)^2 | \hat{K}_{k-1}] \leq 4v_n^{-1}(v_n - v_{k-1}).$$

Therefore, it holds

$$\mathbf{E}[G(\hat{H}_{k-1}(u))] \ll \mathbf{K}(\hat{K}_n) + \sqrt{A_{k-1}}. \quad (7.15)$$

So, overall, we have

$$\sum_{k=1}^n |\mathbf{E}R_{k,2}| \ll \sum_{k=1}^n \frac{1}{A_k^2} \left(\frac{a_k}{\sqrt{v_n}} \right)^4 \mathbf{K}(\hat{K}_n) + \sum_{k=1}^n \frac{1}{A_{k-1}^{3/2}} \left(\frac{a_k}{\sqrt{v_n}} \right)^4. \quad (7.16)$$

Next, we give some estimations for the right-hand side of the last inequality. Recall that $\max_{1 \leq k \leq [n/2]} (A_k v_n)^{-1} = t_{[n/2],n}^{-2} \ll v_n^{-1}$. Therefore

$$\sum_{k=1}^n \frac{1}{A_k^2} \left(\frac{a_k}{\sqrt{v_n}} \right)^4 \ll \frac{1}{v_n^2} \sum_{k=1}^{[n/2]} a_k^4 + \kappa_n^2 \sum_{k=[n/2]+1}^n t_{k,n}^{-2} \frac{a_k^2}{t_{k,n}^2}.$$

Now $\sum_{k=1}^n a_k^4 \ll \mathbf{1}_{p>5/8} + (\log n) \mathbf{1}_{p=5/8} + a_n^2 v_n \mathbf{1}_{p<5/8}$. On another hand, by taking into account (7.12) with $\alpha = -2$, we get

$$\sum_{k=[n/2]+1}^n t_{k,n}^{-2} \frac{a_k^2}{t_{k,n}^2} \ll \kappa_n^{-2} \beta^{-1}.$$

It follows that

$$\sum_{k=1}^n \frac{1}{A_k^2} \left(\frac{a_k}{\sqrt{v_n}} \right)^4 \ll \beta^{-1} + n^{-1} + v_n^{-2} \log n.$$

Using similar arguments, we infer that

$$\sum_{k=1}^n \frac{1}{A_k^{3/2}} \left(\frac{a_k}{\sqrt{v_n}} \right)^4 \ll v_n^{-2} \sum_{k=1}^n a_k^4 + v_n^{-1/2} \kappa_n \ll v_n^{-2} (\mathbf{1}_{p>5/8} + (\log n) \mathbf{1}_{p=5/8}) + n^{-1/2}.$$

It follows that, for n large enough,

$$\sum_{k=1}^n |\mathbf{E}R_{k,2}| \ll \beta^{-1} \mathbf{K}(\hat{K}_n) + v_n^{-2} \mathbf{1}_{p>5/8} + n^{-1/2}. \quad (7.17)$$

Starting from (7.7) and taking into account (7.14) and (7.17), we derive that

$$|I_1| \ll \beta^{-1} \mathbf{K}(\hat{K}_n) + \frac{1}{v_n^{3/2}} + \frac{1}{\sqrt{n}}. \quad (7.18)$$

b) Control of I_2 . We have

$$|I_2| \leq \left| \sum_{k=1}^n \frac{1}{2A_k} \mathbf{E} \left[\varphi'(\hat{H}_{k-1}(u)) \left(\mathbf{E}[\xi_k^2 | \mathcal{F}_{k-1}] - \sigma_k^2 \right) \right] \right| \ll \sum_{k=1}^n \frac{1}{A_k} \mathbf{E} \left| \mathbf{E}[\xi_k^2 | \mathcal{F}_{k-1}] - \sigma_k^2 \right|.$$

Clearly, we have $\mathbf{E}[\xi_1^2 | \mathcal{F}_0] = \sigma_1^2$. On another hand, by relation (A.4) in [2], we get, for $2 \leq k \leq n$,

$$\mathbf{E}[\Delta M_k^2 | \mathcal{F}_{k-1}] = a_k^2 - a_k^2 (2p-1)^2 \left(\frac{T_{k-1}}{k-1} \right)^2, \quad (7.19)$$

so that

$$|\mathbf{E}[\xi_k^2 | \mathcal{F}_{k-1}] - \sigma_k^2| \leq (2p-1)^2 \frac{a_k^2}{v_n} \left(\frac{T_{k-1}}{k-1} \right)^2.$$

Thus, by (4.7),

$$\begin{aligned} |I_2| &\ll \sum_{k=2}^n \frac{1}{A_k} \frac{a_k^2}{v_n a_{k-1}^2 (k-1)^2} \mathbf{E}(a_{k-1} T_{k-1})^2 \\ &\ll \sum_{k \in [2, n/2]} \frac{1}{A_k} \frac{a_k^2}{v_n} \frac{v_{k-1}}{a_{k-1}^2 (k-1)^2} + \sum_{k \in (n/2, n]} \frac{1}{A_k} \frac{a_k^2}{v_n} \frac{v_{k-1}}{a_{k-1}^2 (k-1)^2}. \end{aligned}$$

By (4.1) and (4.2), we have for $p \in (0, 3/4)$, $\frac{v_{k-1}}{a_{k-1}^2 (k-1)^2} \asymp \frac{1}{k}$, $k \rightarrow \infty$, and

$$\max_{1 \leq k \leq [n/2]} (A_k v_n)^{-1} \ll v_n^{-1}.$$

Hence

$$\sum_{k \in [2, n/2]} \frac{1}{A_k} \frac{a_k^2}{v_n} \frac{v_{k-1}}{a_{k-1}^2 (k-1)^2} \ll v_n^{-1} \sum_{k \in [2, n/2]} \frac{a_k^2}{k} \ll n^{-1} + v_n^{-1} + n^{-1} (\log n) \mathbf{1}_{p=1/2}.$$

On another hand, by (7.12) with $\alpha = 0$, we get

$$\sum_{k \in (n/2, n]} \frac{1}{A_k} \frac{a_k^2}{v_n} \frac{v_{k-1}}{a_{k-1}^2 (k-1)^2} \ll n^{-1} \sum_{k \in (n/2, n]} \frac{a_k^2}{t_{k,n}^2} \ll n^{-1} \log n.$$

So, overall,

$$|I_2| \ll \frac{\log n}{n} + \frac{1}{v_n}. \quad (7.20)$$

c) Control of I_3 . Note first that

$$\Phi_u(\hat{K}_{k-1}, A_{k-1}) - \Phi_u(\hat{K}_{k-1}, A_k) - \frac{\partial}{\partial y} \Phi_u(\hat{K}_{k-1}, A_k) \sigma_k^2 = \frac{1}{2} \frac{\partial^2}{\partial y^2} \Phi_u(\hat{K}_{k-1}, A_k + \vartheta_k \sigma_k^2) \sigma_k^4,$$

for some $\vartheta_k \in [0, 1]$. Now,

$$\frac{\partial^2}{\partial y^2} \Phi_u(x, z) = \frac{3}{4} \frac{u-x}{z^{5/2}} \varphi\left(\frac{u-x}{\sqrt{z}}\right) + \frac{1}{4} \frac{(u-x)^2}{z^3} \varphi'\left(\frac{u-x}{\sqrt{z}}\right)$$

and, for any $\vartheta \in [0, 1]$, $A_{k-1}/5 \leq A_k + \vartheta \sigma_k^2 \leq A_{k-1}$. The right-hand side of this inequality is trivial. To prove the left hand side, we note that $a_k \leq 2a_{k+1}$ which entails that $a_k^2 \leq 4 \sum_{\ell=k+1}^n a_\ell^2$. Therefore, $v_n A_{k-1} \leq \beta \kappa_n^2 + 5 \sum_{\ell=k+1}^n a_\ell^2 \leq 5v_n A_k$. Let $g(t) = (|t| + |t|^3) \varphi(t)$. For $t = (u - \hat{K}_{k-1}) / \sqrt{A_k + \vartheta \sigma_k^2}$ and $\tilde{H}_{k-1}(u) = (u - \hat{K}_{k-1}) / \sqrt{A_{k-1}}$, we then get

$$g(t) \leq \left(\sqrt{5} |\tilde{H}_{k-1}(u)| + 5^{3/2} |\tilde{H}_{k-1}(u)|^3 \right) \varphi(\tilde{H}_{k-1}(u)) := \hat{G}(\tilde{H}_{k-1}(u)).$$

So, overall,

$$|I_3| \leq \sum_{k=1}^n \frac{1}{A_k^2} \mathbf{E}[\hat{G}(\tilde{H}_{k-1}(u))] \sigma_k^4.$$

Proceeding as to get (7.15), we infer that $\mathbf{E}[\widehat{G}(\widetilde{H}_{k-1}(u))] \ll \mathbf{K}(\hat{K}_n) + \sqrt{A_{k-1}}$. Therefore

$$|I_3| \ll \sum_{k=1}^n \frac{1}{A_k^2} \sigma_k^4 \mathbf{K}(\hat{K}_n) + \sum_{k=1}^n \frac{1}{A_k^{3/2}} \sigma_k^4.$$

Hence the right-hand side of the above inequality is the same as the one of inequality (7.16). Therefore, according to (7.17), we get, for n large enough, that

$$|I_3| \ll \beta^{-1} \mathbf{K}(\hat{K}_n) + v_n^{-2} \mathbf{1}_{p>5/8} + n^{-1/2}. \quad (7.21)$$

Starting from (7.6) and taking into account the estimates (7.18), (7.20), (7.21) and the fact that $\delta_n \ll v_n^{-1} + n^{-1}$, it follows that there exists a positive constant C (not depending on n) such that, for n large enough,

$$(1 - C\beta^{-1}) \mathbf{K}(\hat{K}_n) \ll n^{-1/2} + v_n^{-1},$$

and Theorem 3.2 (for $p \in (0, 3/4)$) follows by taking β large enough so that $1 - C\beta^{-1} \geq 1/2$.

2) *Case $p = 3/4$.* Recall that in this case, $v_n/(\log n) \rightarrow \pi/4$. Compared to the previous case, the differences are as follows. First we fix a $\gamma \in (0, 1)$ and we define

$$\kappa_n^2 = \max_{[n^\gamma] \leq i \leq n} a_i^2 = a_{[n^\gamma]}^2. \quad (7.22)$$

Next, δ_n^2 , $t_{k,n}$, A_k and $u_{k,n}$ are still defined as in (7.1) and (7.2). It follows that

$$\max_{1 \leq k \leq [n^\gamma]} (A_k v_n)^{-1} = \max_{1 \leq k \leq [n^\gamma]} t_{k,n}^{-2} = t_{[n^\gamma],n}^{-2} \ll v_n^{-1}.$$

On another hand, for any real α ,

$$\begin{aligned} \mathcal{T}_n(\alpha) &= \sum_{k=[n^\gamma]+1}^n t_{k,n}^\alpha \frac{a_k^2}{t_{k,n}^2} \leq \kappa_n^\alpha (1 + 3^{-\alpha/2}) \int_{\beta}^{t_{[n^\gamma],n}} x^{\alpha/2-1} dx \\ &\ll \kappa_n^\alpha \beta^{\alpha/2} \mathbf{1}_{\alpha < 0} + (\log \log n) \mathbf{1}_{\alpha=0} + v_n^{\alpha/2} \mathbf{1}_{\alpha > 0}. \end{aligned} \quad (7.23)$$

Moreover we shall use in case $p = 3/4$ that $\frac{\sqrt{v_{k-1}}}{a_{k-1}(k-1)} = O\left(\frac{\sqrt{\log k}}{\sqrt{k}}\right)$, $k \rightarrow \infty$. Taking into account all the differences pointed above, separating the sums from 1 to n into a sum from 1 to $[n^\gamma]$ plus a sum from $[n^\gamma] + 1$ to n , and proceeding as for the case $p \in (0, 3/4)$ we infer that

$$|I_1| + |I_2| + |I_3| \ll \beta^{-1} \mathbf{K}(\hat{K}_n) + v_n^{-1},$$

which completes the proof of Theorem 3.2 in case $p = 3/4$.

8 Proof of Theorem 3.3

Now we are in the position to prove Theorem 3.3. Set any integer k in $[1, n]$, set $\tau_k^2 = na_n^2 \sigma^2 + v_k$. For $1 \leq k \leq n$, denote

$$\widetilde{H}_k = \frac{1}{\tau_n} \sum_{i=1}^k \alpha_i X_{\beta_i} (Z_i - 1) \text{ and } \widetilde{K}_k = \frac{1}{\tau_n} \left(a_k T_k - (2q - 1) \right).$$

Then we have

$$\frac{a_n S_n - (2q - 1)}{\sqrt{v_n + n a_n^2 \sigma^2}} = a_n \tilde{H}_n + \tilde{K}_n.$$

By the triangle inequality, we get

$$\mathbf{K}\left(\frac{a_n S_n - (2q - 1)}{\sqrt{v_n + n a_n^2 \sigma^2}}\right) \leq P_1 + P_2, \quad (8.1)$$

where

$$P_1 = \sup_{u \in \mathbf{R}} \left| \mathbf{P}\left(a_n \tilde{H}_n + \tilde{K}_n \leq u\right) - \mathbf{P}\left(\mathcal{N}_1 \frac{\sqrt{n} a_n \sigma}{\tau_n} + \tilde{K}_n \leq u\right) \right|$$

and

$$P_2 = \sup_{u \in \mathbf{R}} \left| \mathbf{P}\left(\mathcal{N}_1 \frac{\sqrt{n} a_n \sigma}{\tau_n} + \tilde{K}_n \leq u\right) - \Phi(u) \right|,$$

where \mathcal{N}_1 is a standard normal random variable independent of $(\alpha_k, \beta_k, X_k, Z_k, k \geq 0)$. We first give an upper bound of P_1 . Let $\mathcal{F} = \sigma\{\alpha_k, \beta_k, X_1, k \geq 0\}$. By the classical Berry-Esseen bound for independent random variables (see [14]), we have for $\rho \in (0, 1]$,

$$\begin{aligned} & \left| \mathbf{P}\left(a_n \tilde{H}_n + \tilde{K}_n \leq u \mid \mathcal{F}\right) - \mathbf{P}\left(\mathcal{N}_1 \frac{\sqrt{n} a_n \sigma}{\tau_n} + \tilde{K}_n \leq u \mid \mathcal{F}\right) \right| \\ &= \left| \mathbf{P}\left(\frac{\tilde{H}_n \tau_n}{\sigma \sqrt{n}} \leq \frac{(u - \tilde{K}_n) \tau_n}{\sigma \sqrt{n} a_n} \mid \mathcal{F}\right) - \mathbf{P}\left(\mathcal{N}_1 \leq \frac{(u - \tilde{K}_n) \tau_n}{\sigma \sqrt{n} a_n} \mid \mathcal{F}\right) \right| \leq \frac{C}{n^{\rho/2}}, \end{aligned}$$

where C is a positive constant depending on σ^2 and $\mathbf{E}Z_1^{2+\rho}$. Hence, for $\rho \in (0, 1]$,

$$P_1 \leq \frac{C}{n^{\rho/2}}. \quad (8.2)$$

On another hand, since \mathcal{N}_1 is independent of \tilde{K}_n , we have

$$P_2 = \sup_{u \in \mathbf{R}} \left| \mathbf{P}\left(\tilde{K}_n \leq u\right) - \Phi\left(\frac{u \tau_n}{\sqrt{v_n}}\right) \right| = \mathbf{K}\left(\frac{\tilde{K}_n \tau_n}{\sqrt{v_n}}\right) \ll \frac{1}{\sqrt{n}} + \frac{1}{v_n}, \quad (8.3)$$

where the last inequality comes from Theorem 3.2. Starting from (8.1) and considering the upper bounds (8.2) and (8.3), we obtain the desired estimate.

9 Proof of Theorem 3.4

Let $(N_i)_{1 \leq i \leq n}$ be a sequence of $\mathcal{N}(0, a_i^2)$ -distributed independent random variables. Assume moreover that $(N_i)_{1 \leq i \leq n}$ is independent of $(\Delta M_i)_{1 \leq i \leq n}$ (recall (4.6) for the definition of ΔM_i). For $n \geq 3$, set $U_k = \sum_{j=1}^k N_j$, $1 \leq k \leq n$.

As in the proof of Theorem 3.2, we shall divide the proof in two cases according to the values of p .

1) *Case* $p \in (0, 3/4)$. We shall use the same notations as those defined in (7.1) and (7.2), selecting $\beta = 1$. So

$$\kappa_n^2 = \max_{[n/2] \leq i \leq n} a_i^2, \quad t_{k,n} = (\kappa_n^2 + v_n - v_k)^{1/2}, \quad u_{k,n} = 1 + 2 \sum_{i=k+1}^n a_i^2 / \kappa_n^2, \quad 0 \leq k \leq n. \quad (9.1)$$

For any $r > 0$ and any $p \in [0, 1]$, let also

$$B_n(r, p) = \frac{1}{n^{r/2}} + \frac{1}{v_n}.$$

Let N be a $\mathcal{N}(0, \kappa_n^2)$ -distributed random variable independent of $(\Delta M_i)_{1 \leq i \leq n}$ and $(N_i)_{1 \leq i \leq n}$. Using Lemma 5.1 in Dedecker *et al.* [11], for any $r \in (0, 2]$, we first write

$$\zeta_r(\mathbf{P}_{M_n}, \mathbf{P}_{U_n}) \leq 2\zeta_r(\mathbf{P}_{M_n+N}, \mathbf{P}_{U_n+N}) + 4\sqrt{2}\kappa_n^r. \quad (9.2)$$

Note that $\kappa_n^r \ll a_n^r \ll v_n^{r/2}/n^{r/2}$. It remains to give an estimation for

$$\zeta_r(\mathbf{P}_{M_n+N}, \mathbf{P}_{U_n+N}) = \sup_{f \in \Lambda_r} \mathbf{E}[f(M_n + N) - f(U_n + N)].$$

With this aim, we shall use the Lindeberg method and denote

$$f_{n,k}(x) = \mathbf{E}\left[f\left(x + U_n - U_k + N\right)\right], \quad 1 \leq k \leq n.$$

By independence of the sequences,

$$\mathbf{E}\left[f(M_n + N) - f\left(\sum_{j=1}^n N_j + N\right)\right] = \sum_{k=1}^n D_k, \quad (9.3)$$

where

$$D_k = \mathbf{E}\left[f_{n,k}(M_{k-1} + \Delta M_k) - f_{n,k}(M_{k-1} + N_k)\right].$$

Using twice a Taylor's expansion at order 5, we get

$$D_k = I_{k,1} + I_{k,2} + I_{k,3} + I_{k,4} + I_{k,5}, \quad (9.4)$$

where, for any integer $i \in [1, 4]$,

$$I_{k,i} = \frac{1}{i!} \mathbf{E}\left[f_{n,k}^{(i)}(M_{k-1})\Delta M_k^i - f_{n,k}^{(i)}(M_{k-1})N_k^i\right],$$

and

$$I_{k,5} = \frac{1}{120} \mathbf{E}\left[f_{n,k}^{(5)}(M_{k-1} + \vartheta_1 \Delta M_k)\Delta M_k^5 - f_{n,k}^{(5)}(M_{k-1} + \vartheta_2 N_k)N_k^5\right],$$

with ϑ_1, ϑ_2 some random variables with values in $[0, 1]$. In the sequel, we give some estimations for $\sum_{k=1}^n I_{k,l}, l = 1, 2, \dots, 5$.

Since (M_k, \mathcal{F}_k) is a martingale, $I_{k,1} = \mathbf{E}[f'_{n,k}(M_{k-1})\mathbf{E}[\Delta M_k - N_k | \mathcal{F}_{k-1}]] = 0$. Therefore,

$$\sum_{k=1}^n I_{k,1} = 0. \quad (9.5)$$

We handle now the sum of the $|I_{k,2}|$'s. By Lemma 6.1 in [11], for any positive integer i such that $i \geq r$, there exists a positive constant $C_{r,i}$ such that for any integer $n \geq 1$ and any $k \in [1, n]$,

$$\|f_{n,k}^{(i)}(\cdot)\|_\infty \leq C_{r,i} t_{k,n}^{r-i}. \quad (9.6)$$

For $k = 1$, we have

$$I_{1,2} = \mathbf{E} \left[\frac{1}{2} f''_{n,1}(M_0) \left(\mathbf{E}[\Delta M_1^2 | \mathcal{F}_0] - a_1^2 \right) \right] = 0.$$

On another hand, by taking into account (7.19), (9.6) and (4.7), we get for $2 \leq k \leq n$,

$$\begin{aligned} |I_{k,2}| &= \left| \mathbf{E} \left[\frac{1}{2} f''_{n,k}(M_{k-1}) \left(\mathbf{E}[\Delta M_k^2 | \mathcal{F}_{k-1}] - a_k^2 \right) \right] \right| \\ &\ll (2p-1)^2 t_{k,n}^{r-2} \frac{a_k^2}{a_{k-1}^2} \mathbf{E} \left[\left(\frac{a_{k-1} T_{k-1}}{k-1} \right)^2 \right] \\ &\ll (2p-1)^2 t_{k,n}^{r-2} \frac{a_k^2}{a_{k-1}^2} \frac{(1+v_{k-1})}{(k-1)^2}. \end{aligned} \quad (9.7)$$

Clearly, when $p = 1/2$, $|I_{k,2}| = 0$ for any positive integer k . In what follows we handle $\sum_{k=1}^n |I_{k,2}|$ when $p \in (0, 3/4) \setminus \{1/2\}$. In this case, by (4.1) and (4.2), we have $\frac{v_{k-1}}{a_{k-1}^2 (k-1)^2} \asymp \frac{1}{k}, k \rightarrow \infty$. Therefore

$$\sum_{k=1}^n |I_{k,2}| \ll \max_{1 \leq k \leq n/2} t_{k,n}^{r-2} \sum_{k \in [1, n/2]} \frac{a_k^2}{k} + \frac{1}{n} \sum_{k \in (n/2, n]} t_{k,n}^r \frac{a_k^2}{t_{k,n}^2}.$$

Clearly $\max_{1 \leq k \leq n/2} t_{k,n}^{r-2} \leq C v_n^{(r-2)/2}$. Moreover, by (4.1),

$$\sum_{k \in [1, n]} k^{-1} a_k^2 \ll 1 + (\log n) \mathbf{1}_{p=1/2} + a_n^2 \mathbf{1}_{p < 1/2}.$$

Hence, by taking in addition into account (4.4), we derive that, when $p \neq 1/2$,

$$\sum_{k=1}^n |I_{k,2}| \ll v_n^{(r-2)/2} (1 + a_n^2 \mathbf{1}_{p < 1/2}) + \frac{1}{n} \sum_{k \in (n/2, n]} t_{k,n}^r \frac{a_k^2}{t_{k,n}^2}.$$

Hence, taking into account (7.12) with $\alpha = r$, we derive that, for any p in $(0, 1/2) \cup (1/2, 3/4)$,

$$\sum_{k=1}^n |I_{k,2}| \ll v_n^{r/2} (v_n^{-1} + n^{-1}) + n^{-1} \kappa_n^r u_{[n/2], n}^{r/2}.$$

But $\kappa_n^r u_{[n/2], n}^{r/2} \ll v_n^{r/2}$. So, overall, when $p \in (0, 3/4)$ and $r \in (0, 2]$,

$$\sum_{k=1}^n |I_{k,2}| \ll v_n^{r/2} B_n(r, p). \quad (9.8)$$

We handle now the sum of the $|I_{k,3}|$'s. By (9.6), Lemma 4.2 and (4.7), we have

$$\begin{aligned} |I_{k,3}| &\leq \frac{1}{6} \left| \mathbf{E} \left[f'''_{n,k}(M_{k-1}) \mathbf{E}[\Delta M_k^3 | \mathcal{F}_{k-1}] - f'''_{n,k}(M_{k-1}) \mathbf{E}[N_k^3 | \mathcal{F}_{k-1}] \right] \right| \\ &\ll t_{k,n}^{r-3} \mathbf{E} \left| \mathbf{E}[\Delta M_k^3 | \mathcal{F}_{k-1}] \right| \ll t_{k,n}^{r-3} a_k^3 \mathbf{E} \left| \frac{T_{k-1}}{k-1} \right| \\ &\ll t_{k,n}^{r-3} \frac{a_k^3}{(k-1)a_{k-1}} \sqrt{\mathbf{E}(a_{k-1} T_{k-1})^2} \ll t_{k,n}^{r-3} a_k^3 \frac{\sqrt{v_{k-1}}}{(k-1)a_{k-1}}, \end{aligned}$$

with the convention $\sqrt{v_0}/0a_0 = 1$. By (4.1) and (4.2), and since $p \in (0, 3/4)$, we have $\frac{\sqrt{v_{k-1}}}{(k-1)a_{k-1}} = O\left(\frac{1}{\sqrt{k}}\right)$, $k \rightarrow \infty$. Hence

$$\sum_{k=1}^n |I_{k,3}| \ll A_n + \frac{1}{\sqrt{n}} B_n, \quad (9.9)$$

where

$$A_n = \sum_{k \in [1, n/2]} \frac{t_{k,n}^{r-3} a_k^3}{\sqrt{k}} \quad \text{and} \quad B_n = \sum_{k \in (n/2, n]} t_{k,n}^{r-3} a_k^3.$$

By (4.1), it holds

$$A_n \ll v_n^{(r-3)/2} \sum_{k=1}^n \frac{a_k^3}{\sqrt{k}} \ll \begin{cases} v_n^{(r-3)/2} & \text{if } p > 7/12 \\ v_n^{(r-3)/2} \log n & \text{if } p = 7/12 \\ v_n^{(r-3)/2} \sqrt{n} a_n^3 & \text{if } p < 7/12. \end{cases}$$

By (4.1) and (4.2), when $p \in (0, 3/4)$, $v_n^{-3/2} \sqrt{n} a_n^3 \ll n^{-1}$ and $v_n^{-3/2} \log n \ll v_n^{-1}$. Therefore, for any $p \in (0, 3/4)$,

$$A_n \ll v_n^{r/2} B_n(r, p). \quad (9.10)$$

We handle now the term B_n in (9.9). By taking into account (7.12), we derive

$$B_n \leq \kappa_n \mathcal{T}_n(r-1) \ll \kappa_n^r \mathbf{1}_{(0,1)}(r) + \kappa_n (\log n) \mathbf{1}_{r=1} + \kappa_n v_n^{(r-1)/2} \mathbf{1}_{(1,2]}(r).$$

When, p in $(0, 3/4)$, we have $\kappa_n^r \ll v_n^{r/2} n^{-r/2}$, $\kappa_n v_n^{(r-1)/2} \ll v_n^{r/2} n^{-1/2}$ and $\kappa_n (\log n) \ll v_n^{1/2} n^{-1/2} (\log n)$. Therefore, for any p in $(0, 3/4)$,

$$\frac{B_n}{\sqrt{n}} \ll v_n^{r/2} B_n(r, p). \quad (9.11)$$

So, starting from (9.9) and taking into account (9.10) and (9.11), we derive that, for any r in $(0, 2]$ and any p in $(0, 3/4)$,

$$\sum_{k=1}^n |I_{k,3}| \ll v_n^{r/2} B_n(r, p). \quad (9.12)$$

Before dealing with the sum of the $|I_{k,4}|$'s, let us give an upper bound for $\sum_{k=1}^n |I_{k,5}|$. By (9.6) and Lemma 4.1 again, we have

$$|I_{k,5}| \ll t_{k,n}^{r-5} \mathbf{E}[\mathbf{E}[|\Delta M_k|^5 | \mathcal{F}_{k-1}] + \mathbf{E}|N_k|^5] \ll t_{k,n}^{r-5} a_k^5.$$

Thus, by taking into account (7.12), we have for $p \in (0, 3/4)$ and $r \in (0, 2]$,

$$\sum_{k=1}^n |I_{k,5}| \ll v_n^{(r-5)/2} \sum_{k=1}^{[n/2]} a_k^5 + \kappa_n^3 \sum_{k=[n/2]+1}^n t_{k,n}^{r-5} a_k^2 \ll v_n^{(r-5)/2} \sum_{k=1}^{[n/2]} a_k^5 + \kappa_n^r.$$

Hence, by simple computations,

$$\sum_{k=1}^n |I_{k,5}| \ll v_n^{r/2} B_n(r, p). \quad (9.13)$$

We handle now the term $\sum_{k=1}^n |I_{k,4}|$. By (9.6) and Lemma 4.1, we have

$$|I_{k,4}| = \frac{1}{24} \left| \mathbf{E} \left[f_{n,k}^{(4)}(M_{k-1}) \mathbf{E}[\Delta M_k^4 | \mathcal{F}_{k-1}] - f_{n,k}^{(4)}(M_{k-1}) \mathbf{E}[N_k^4] \right] \right| \ll t_{k,n}^{r-4} a_k^4.$$

Thus, by taking into account (7.12), we have for $p \in (0, 3/4]$ and $r \in (0, 2]$,

$$\begin{aligned} \sum_{k=1}^n |I_{k,4}| &\ll v_n^{(r-4)/2} \sum_{k=1}^{\lfloor n/2 \rfloor} a_k^4 + \kappa_n^2 \sum_{k=\lfloor n/2 \rfloor+1}^n t_{k,n}^{r-4} a_k^2 \\ &\ll v_n^{(r-4)/2} \sum_{k=1}^{\lfloor n/2 \rfloor} a_k^4 + \kappa_n^r (\mathbf{1}_{r \in (0,2)} + (\log n) \mathbf{1}_{r=2}). \end{aligned}$$

Thus, for $p \in (0, 3/4]$ and $r \in (0, 2]$, we infer that

$$\sum_{k=1}^n |I_{k,4}| \ll v_n^{r/2} B_n(r, p) \mathbf{1}_{r \neq 2} + \kappa_n^2 (\log n) \mathbf{1}_{r=2}. \quad (9.14)$$

Therefore, starting from (9.3) and taking into account (9.4), (9.5) and the upper bounds (9.8), (9.12), (9.13) and (9.14), the result is proved for $r \in (0, 2)$. It remains to prove it when $r = 2$. With this aim, it is enough to prove that when $r = 2$, $\sum_{k=1}^n |I_{k,4}| \leq C v_n B_n(2, p)$. This means to delete the additional logarithmic term $\log n$ in the right-hand side of (9.14). From now on, we assume that $r = 2$. Note first that the previous computations proved that $\sum_{k=1}^{\lfloor n/2 \rfloor} |I_{k,4}| \ll v_n B_n(2, p)$. So it remains only to prove that

$$\sum_{k=\lfloor n/2 \rfloor+1}^n |I_{k,4}| \ll v_n B_n(2, p). \quad (9.15)$$

With this aim, we first write the following decomposition: Since $\mathbf{E}[N_k^4] = 3a_k^4$,

$$\begin{aligned} |I_{k,4}| &= \frac{1}{24} \left| \mathbf{E} \left[f_{n,k}^{(4)}(M_{k-1}) (\mathbf{E}[\Delta M_k^4 | \mathcal{F}_{k-1}] - \mathbf{E}[N_k^4]) \right] \right| \\ &\leq I_{k,4}^{(1)} + I_{k,4}^{(2)}, \end{aligned}$$

where

$$I_{k,4}^{(1)} := \frac{1}{24} \left| \mathbf{E} \left[f_{n,k}^{(4)}(M_{k-1}) (\mathbf{E}[\Delta M_k^4 | \mathcal{F}_{k-1}] - a_k^4) \right] \right| \quad \text{and} \quad I_{k,4}^{(2)} := \frac{a_k^4}{12} \left| \mathbf{E} \left[f_{n,k}^{(4)}(M_{k-1}) \right] \right|.$$

Since $r = 2$, by using (9.6) and Lemma 4.2, we get that

$$\sum_{k=\lfloor n/2 \rfloor+1}^n I_{k,4}^{(1)} \ll \sum_{k=\lfloor n/2 \rfloor+1}^n t_{k,n}^{-2} a_k^4 \frac{\mathbf{E}(T_{k-1}^2)}{(k-1)^2} \ll \sum_{k=\lfloor n/2 \rfloor+1}^n t_{k,n}^{-2} \frac{a_k^4}{a_{k-1}^2} \frac{v_{k-1}}{(k-1)^2}.$$

As $k \rightarrow \infty$, recall that, when $p \neq 3/4$, $\frac{v_{k-1}}{a_{k-1}^2(k-1)^2} \asymp \frac{1}{k}$. Therefore

$$\sum_{k=[n/2]+1}^n I_{k,4}^{(1)} \ll \sum_{k=[n/2]+1}^n t_{k,n}^{-2} \frac{a_k^4}{k} \ll n^{-1} \kappa_n^2 \sum_{k \in [n/2, n]} \frac{a_k^2}{t_{k,n}^2}.$$

Hence, taking into account (7.12), we derive that

$$\sum_{k=[n/2]+1}^n I_{k,4}^{(1)} \ll n^{-1} \kappa_n^2 (\log n) \ll \kappa_n^2.$$

Note that when $p \in (0, 3/4)$, $\kappa_n^2 \ll v_n/n \ll v_n B_n(2, p)$. We then infer that for $r = 2$ and any $p \in (0, 3/4)$,

$$\sum_{k=[n/2]+1}^n I_{k,4}^{(1)} \ll v_n B_n(2, p).$$

Therefore to end the proof of (9.15) and then of the theorem, we need to prove that

$$\sum_{k=[n/2]+1}^n I_{k,4}^{(2)} \ll v_n B_n(2, p). \quad (9.16)$$

To achieve this, we need to have a better control of the term $\mathbf{E}[|f_{n,k}^{(4)}(M_{k-1})|]$ and we write

$$\mathbf{E}[f_{n,k}^{(4)}(M_{k-1})] = \mathbf{E}[f_{n,k}^{(4)}(M_{k-1})] - \mathbf{E}[f_{n,k}^{(4)}(G_{k-1})] + \mathbf{E}[f_{n,k}^{(4)}(G_{k-1})],$$

where G_{k-1} is a normal random variable with variance $a_1^2 + \dots + a_{k-1}^2$. By definition of $f_{n,k}$, we have

$$\left| \mathbf{E}[f_{n,k}^{(4)}(N_{k-1})] \right| = \left| g_{n,k}^{(4)}(0) \right|,$$

where

$$g_{n,k}(x) = \mathbf{E} \left[f \left(x + \sum_{i=1, i \neq k}^n N_i + Z \right) \right].$$

Since, for any integer k such that $[n/2] + 1 \leq k \leq n$, $\kappa_n^2 + \sum_{i=1, i \neq k}^n a_i^2 \geq v_n$, it follows from (9.6) that $|g_{n,k}^{(4)}(0)| \ll v_n^{-1}$. Therefore,

$$\left| \mathbf{E}[f_{n,k}^{(4)}(G_{k-1})] \right| \ll v_n^{-1}.$$

Hence, since $p \in (0, 3/4)$,

$$\sum_{k=[n/2]+1}^n a_k^4 \left| \mathbf{E}[f_{n,k}^{(4)}(G_{k-1})] \right| \ll \frac{1}{v_n} \sum_{k=[n/2]+1}^n a_k^4 \ll a_n^2 \ll n^{-1} v_n. \quad (9.17)$$

We now deal with the term

$$\begin{aligned} & \mathbf{E}[f_{n,k}^{(4)}(M_{k-1})] - \mathbf{E}[f_{n,k}^{(4)}(G_{k-1})] \\ &= \int \left(\mathbf{E}[f'(M_{k-1} - u)] - \mathbf{E}[f'(G_{k-1} - u)] \right) \varphi_{t_{n,k}^2}^{(3)}(u) du. \end{aligned}$$

Hence, we have

$$\begin{aligned} & \left| \mathbf{E}[f_{n,k}^{(4)}(M_{k-1})] - \mathbf{E}[f_{n,k}^{(4)}(G_{k-1})] \right| \\ & \leq c \sup_{u \in \mathbf{R}} \left| \mathbf{E}[f'(M_{k-1} - u)] - \mathbf{E}[f'(G_{k-1} - u)] \right| \int \left| \varphi_{t_{n,k}^2}^{(3)}(u) \right| du. \end{aligned}$$

Recall that we have proved Theorem 3.4 for any $r \in (0, 2)$. Hence applying it for $r = 1$, we have, for any integer k such that $[n/2] + 1 \leq k \leq n$,

$$\sup_{u \in \mathbf{R}} \left| \mathbf{E}[f'(M_{k-1} - u)] - \mathbf{E}[f'(G_{k-1} - u)] \right| \ll \sqrt{\frac{v_n}{n}} + \frac{1}{\sqrt{v_n}}.$$

On another hand,

$$\int \left| \varphi_{t_{n,k}^2}^{(3)}(u) \right| du \ll t_{k,n}^{-3}.$$

Consequently, by taking into account (7.12),

$$\begin{aligned} \sum_{k=[n/2]+1}^n a_k^4 \left| \mathbf{E}[f_{n,k}^{(4)}(M_{k-1})] - \mathbf{E}[f_{n,k}^{(4)}(G_{k-1})] \right| & \ll \kappa_n^2 \left(\sqrt{\frac{v_n}{n}} + \frac{1}{\sqrt{v_n}} \right) \sum_{k=[n/2]+1}^n \frac{a_k^2}{t_{k,n}^3} \\ & \ll \kappa_n \left(\sqrt{\frac{v_n}{n}} + \frac{1}{\sqrt{v_n}} \right) \ll v_n \left(\frac{1}{n} + \frac{1}{v_n \sqrt{n}} \right). \end{aligned}$$

Hence, for any $p \in (0, 3/4)$,

$$\sum_{k=[n/2]+1}^n a_k^4 \left| \mathbf{E}[f_{n,k}^{(4)}(M_{k-1})] - \mathbf{E}[f_{n,k}^{(4)}(G_{k-1})] \right| \ll v_n B_n(2, p). \quad (9.18)$$

Combining the inequalities (9.17) and (9.18) together (9.16) follows. This completes the proof of the theorem when $p \in (0, 3/4)$.

2) Case $p = 3/4$. As in the proof of Theorem 3.2 in case $p = 3/4$, we fix $\gamma \in (0, 1)$ and we select κ_n^2 as in (7.22). The definitions of $t_{k,n}$ and $u_{k,n}$ are as in (9.1). Recall also that $v_n / \log n \rightarrow \pi/4$. We start again from the inequality (9.2) and use the decompositions (9.3) and (9.4). We still have to get upper bounds for the quantities $|I_{k,i}|$ for $i = 2, \dots, 5$. For $|I_{k,2}|$ compared to the previous case, the difference is that $\frac{v_{k-1}}{a_{k-1}^2(k-1)^2} \asymp \frac{\log k}{k}, k \rightarrow \infty$. Hence starting from (9.7), we get in this case that

$$\sum_{k=1}^n |I_{k,2}| \ll \max_{1 \leq k \leq n^\gamma} t_{k,n}^{r-2} \sum_{k \in [1, n^\gamma]} \frac{\log k}{k^2} + \frac{\log n}{n^\gamma} \sum_{k \in (n^\gamma, n]} t_{k,n}^r \frac{a_k^2}{t_{k,n}^2}.$$

We then infer that when $r \in (0, 2]$ and $p = 3/4$,

$$\sum_{k=1}^n |I_{k,2}| \ll v_n^{(r-2)/2} + v_n^{r/2} (\log n) / n^\gamma \ll v_n^{(r-2)/2}.$$

We handle now the term $\sum_{k=1}^n |I_{k,3}|$. When, $p = 3/4$, the only difference with the previous case is that $\frac{\sqrt{v_{k-1}}}{(k-1)a_{k-1}} = O\left(\frac{\sqrt{\log k}}{\sqrt{k}}\right)$, $k \rightarrow \infty$. Hence, (9.9) has to be replaced by

$$\sum_{k=1}^n |I_{k,3}| \ll \left(A'_n + \frac{\sqrt{\log n}}{n^{\gamma/2}} B'_n\right),$$

where $A'_n = \sum_{k \in [1, [n^\gamma]]} \frac{t_{k,n}^{r-3} a_k^3 \sqrt{\log k}}{\sqrt{k}}$ and $B'_n = \sum_{k \in (n^\gamma, n]} t_{k,n}^{r-3} a_k^3$. Since $p = 3/4$, $A'_n \ll v_n^{(r-3)/2}$. On another hand, when $p = 3/4$, by using (7.23), we infer that

$$B'_n \ll \kappa_n^r \mathbf{1}_{(0,1)}(r) + \kappa_n (\log \log n) \mathbf{1}_{r=1} + \kappa_n v_n^{(r-1)/2} \mathbf{1}_{(1,2]}(r),$$

implying that $\frac{\sqrt{\log n}}{n^{\gamma/2}} B'_n \ll v_n^{(r-2)/2}$. Therefore, when $p = 3/4$, for any r in $(0, 2]$,

$$\sum_{k=1}^n |I_{k,3}| \ll v_n^{(r-2)/2}.$$

We handle now the term $\sum_{k=1}^n |I_{k,4}|$. We write this time

$$\sum_{k=1}^n |I_{k,4}| \ll v_n^{(r-4)/2} \sum_{k=1}^{[n^\gamma]} a_k^4 + \kappa_n^2 \sum_{k=[n^\gamma]+1}^n t_{k,n}^{r-4} a_k^2.$$

By using (7.23), it follows that, for any $r \in (0, 2]$,

$$\sum_{k=1}^n |I_{k,4}| \ll v_n^{(r-4)/2} + \kappa_n^r + \kappa_n^2 (\log \log n) \mathbf{1}_{r=2} \ll v_n^{(r-2)/2}.$$

Finally, let us handle the term $\sum_{k=1}^n |I_{k,5}|$. We write this time

$$\sum_{k=1}^n |I_{k,5}| \ll v_n^{(r-5)/2} \sum_{k=1}^{[n^\gamma]} a_k^5 + \kappa_n^3 \sum_{k=[n^\gamma]+1}^n t_{k,n}^{r-5} a_k^2.$$

Thus, by taking into account (7.23), we have for $0 < r \leq 2$,

$$\sum_{k=1}^n |I_{k,5}| \ll v_n^{(r-5)/2} \sum_{k=1}^{[n/2]} a_k^5 + \kappa_n^r \ll v_n^{(r-2)/2}.$$

Considering all these upper bounds, Theorem 3.2 is then proved in case $p = 3/4$.

10 Proof of Theorem 3.5

We start from the decomposition (6.1) of Section 6. Let also

$$\sigma_{1,n}^2 = \frac{na_n^2 \sigma^2}{v_n + na_n^2 \sigma^2} \quad \text{and} \quad \sigma_{2,n}^2 = \frac{v_n}{v_n + na_n^2 \sigma^2}. \quad (10.1)$$

Let then $G_{1,n}$ and $G_{2,n}$ be two independent normal random variables with respective variances $\sigma_{1,n}^2$ and $\sigma_{2,n}^2$, and independent of $(\alpha_i, \beta_i, X_1, Z_i)_{i \geq 1}$. By the triangle inequality,

$$\zeta_r(U_n + V_n, \mathcal{N}) \leq \zeta_r(U_n + V_n, G_{1,n} + V_n) + \zeta_r(G_{1,n} + V_n, G_{1,n} + G_{2,n}). \quad (10.2)$$

We first deal with the first term on right hand in (10.2). Let $\mathcal{F} = \sigma(\alpha_i, \beta_i, X_1, i \geq 1)$, and note that

$$\zeta_r(U_n + V_n, G_{1,n} + V_n) \leq \mathbf{E}[\zeta_r(P_{U_n+V_n|\mathcal{F}}, P_{G_{1,n}+V_n|\mathcal{F}})]. \quad (10.3)$$

Now, since V_n is \mathcal{F} -measurable and $G_{1,n}$ is independent of \mathcal{F} , we have

$$\zeta_r(P_{U_n+V_n|\mathcal{F}}, P_{G_{1,n}+V_n|\mathcal{F}}) = \zeta_r(P_{U_n|\mathcal{F}}, P_{G_{1,n}}) \quad a.s. \quad (10.4)$$

Conditionally to \mathcal{F} , U_n is a sum of independent random variables with variance $a_n^2 \sigma^2 / (v_n + na_n^2 \sigma^2)$ and absolute moment of order $2 + \rho$ bounded by

$$\mathbf{E}[|Z_1 - 1|^{2+\rho}] a_n^{2+\rho} / (v_n + na_n^2 \sigma^2)^{(2+\rho)/2}.$$

Applying Theorem 2.1 in [12] if $(r, \rho) \neq (1, 1)$ and the result of Bikelis [6] if $(r, \rho) = (1, 1)$, we infer that, for any $r \in (0, 2]$ and $\rho \in (0, 1]$,

$$\zeta_r(P_{U_n|\mathcal{F}}, P_{G_{1,n}}) \leq C \left(\frac{\sqrt{na_n^2 \sigma^2}}{\sqrt{v_n + na_n^2 \sigma^2}} \right)^r \frac{1}{n^{(r \wedge \rho)/2}} \quad a.s. \quad (10.5)$$

where C is a constant depending only on r, σ^2 and $\mathbf{E}Z_1^{2+\rho}$. Hence, by (10.3), (10.4) and (10.5), we get

$$\zeta_r(U_n + V_n, G_{1,n} + V_n) \ll \frac{1}{n^{(r \wedge \rho)/2}}. \quad (10.6)$$

We now deal with the second term on the right hand side of (10.2). Since $G_{1,n}$ is independent of $(V_n, G_{2,n})$, it is easy to see that

$$\zeta_r(G_{1,n} + V_n, G_{1,n} + G_{2,n}) \leq \zeta_r(V_n, G_{2,n}).$$

Notice that

$$\begin{aligned} \zeta_r(V_n, G_{2,n}) &= \left(\frac{\sqrt{v_n}}{\sqrt{v_n + na_n^2 \sigma^2}} \right)^r \zeta_r \left(\frac{a_n T_n - (2q - 1)}{\sqrt{v_n}}, \mathcal{N} \right) \\ &\leq \zeta_r \left(\frac{a_n T_n - (2q - 1)}{\sqrt{v_n}}, \mathcal{N} \right). \end{aligned}$$

Thus from Theorem 3.4, we get for $r \in (0, 2]$,

$$\zeta_r(G_{1,n} + V_n, G_{1,n} + G_{2,n}) \ll \frac{1}{n^{r/2}} + \frac{1}{v_n}. \quad (10.7)$$

Starting from (10.2) and considering the upper bounds (10.6) and (10.7), the result follows.

11 Proof of Theorem 3.6

We start again from the decomposition (6.1) of Section 6 and use the notations of Section 10. By the triangle inequality

$$W_r(U_n + V_n, \mathcal{N}) \leq W_r(U_n + V_n, G_{1,n} + V_n) + W_r(G_{1,n} + V_n, G_{1,n} + G_{2,n}). \quad (11.1)$$

We first deal with the first term on the right hand side of (11.1). By Fact 1.1 in [13], note that

$$W_r(U_n + V_n, G_{1,n} + V_n) \leq \mathbf{E}[W_r(P_{U_n+V_n|\mathcal{F}}, P_{G_{1,n}+V_n|\mathcal{F}})]. \quad (11.2)$$

Now, since V_n is \mathcal{F} -measurable and $G_{1,n}$ is independent of \mathcal{F} , we get

$$W_r(P_{U_n+V_n|\mathcal{F}}, P_{G_{1,n}+V_n|\mathcal{F}}) = W_r(P_{U_n|\mathcal{F}}, P_{G_{1,n}}) \quad a.s. \quad (11.3)$$

Conditionally on \mathcal{F} , U_n is a sum of independent random variables with variance $a_n^2\sigma^2/(v_n + na_n^2\sigma^2)$ and absolute moment of order $2 + \rho$ bounded by

$$\mathbf{E}[|Z_1 - 1|^{2+\rho}] a_n^{2+\rho}/(v_n + na_n^2\sigma^2)^{(2+\rho)/2}.$$

Applying Corollary 1.2 in [7], we get for any $r \in [1, 2]$ and $\rho \in (0, r]$,

$$\begin{aligned} W_r(P_{U_n|\mathcal{F}}, P_{G_{1,n}}) &\leq C \frac{\sqrt{na_n^2\sigma^2}}{\sqrt{v_n + na_n^2\sigma^2}} \left(\frac{\mathbf{E}[|Z_1 - 1|^{2+\rho}]}{\sigma^{2+\rho}} \right)^{1/r} \frac{1}{n^{\rho/2r}} \\ &\leq \frac{C}{n^{\rho/2r}} \quad a.s. \end{aligned}$$

where C is a constant depending only on r , σ^2 and $\mathbf{E}Z_1^{2+\rho}$. From (11.2) and (11.3), we get

$$W_r(U_n + V_n, G_{1,n} + V_n) \ll \frac{1}{n^{\rho/2r}}. \quad (11.4)$$

We now deal with the second term on the right hand side of (11.1). Since $G_{1,n}$ is independent of $(V_n, G_{2,n})$, it is easy to see that

$$W_r(G_{1,n} + V_n, G_{1,n} + G_{2,n}) \leq W_r(V_n, G_{2,n}).$$

Now

$$W_r(V_n, G_{2,n}) = \frac{\sqrt{v_n}}{\sqrt{v_n + na_n^2\sigma^2}} W_r\left(\frac{a_n T_n - (2q - 1)}{\sqrt{v_n}}, \mathcal{N}\right).$$

By using inequality (3.2), we get that

$$W_r(V_n, G_{2,n}) \leq c_r \left(\zeta_r \left(\frac{a_n T_n - (2q - 1)}{\sqrt{v_n}}, \mathcal{N} \right) \right)^{1/r}.$$

Hence, from Theorem 3.4, it follows that, for any $r \in [1, 2]$,

$$W_r(G_{1,n} + V_n, G_{1,n} + G_{2,n}) \ll \frac{1}{\sqrt{n}} + \frac{1}{v_n^{1/r}}. \quad (11.5)$$

The result follows by considering (11.1) together with (11.4) and (11.5).

References

- [1] Baur, E. and Bertoin, J. 2016. Elephant random walks and their connection to Pólya-type urns. *Phys. Rev. E* **94**(5) 052134
- [2] Bercu, B. 2018. A martingale approach for the elephant random walk. *J. Phys. A: Math. Theor.* **51** 015201
- [3] Bercu, B. 2022. On the elephant random walk with stops playing hide and seek with the Mittag-Leffler distribution. *J. Statist. Phys.* **189**(12)
- [4] Bercu, B. and Lucile, L. 2019. On the multi-dimensional elephant random walk. *J. Statist. Phys.* **175**(6) 1146-1163
- [5] Bertoin, J. 2022. Counting the zeros of an elephant random walk. *Trans. Amer. Math. Soc.* **375** 5539-5560
- [6] Bikelis, A. 1966. Estimates of the remainder term in the central limit theorem. *Litovsk. Mat. Sb.* **6** 323-346.
- [7] Bobkov, S. G. 2018. Berry-Esseen bounds and Edgeworth expansions in the central limit theorem for transport distances. *Probab. Theory Related Fields* **170**(1-2) 229-262.
- [8] Bolthausen, E. 1982. Exact convergence rates in some martingale central limit theorems. *Ann. Probab.* **10** 672-688.
- [9] Coletti, C. F., Gava, R. and Schütz, G. M. 2017. Central limit theorem and related results for the elephant random walk. *J. Math. Phys.* **58**(5) 053303
- [10] Coletti, C. F., Gava, R. and Schütz, G. M. 2017. A strong invariance principle for the elephant random walk. *J. Stat. Mech. Theory E.* **12** 123207
- [11] Dedecker, J., Merlevède, F. and Rio, E. 2009. Rates of convergence for minimal distances in the central limit theorem under projective criteria. *Electron. J. Probab.* **14** 978-1011
- [12] Dedecker, J., Merlevède, F. and Rio, E. 2022. Rates of convergence in the central limit theorem for martingales in the non stationary setting. *Ann. Inst. H. Poincaré Probab. Statist.* **58**(2) 945-966.
- [13] Dedecker, J., Merlevède, F. and Rio, E. 2022. Quadratic transportation cost in the conditional central limit theorem for dependent sequences. *hal-03890107*.
- [14] Esseen, C.G. 1942. On the Liapounoff limit of error in the theory of probability. *Ark. Mat. Astr. Fys.* 28A, no. 9, 19 pp.
- [15] Fan, X., Hu, H. and Ma, X. 2021. Cramér moderate deviations for the elephant random walk. *J. Stat. Mech. Theory E.* 023402.
- [16] Fan, X. and Shao, Q., 2023. Cramér’s moderate deviations for martingales with applications. *Ann. Inst. Henri Poincaré Probab. Stat.*, to appear.
- [17] Grama, I. and Haeusler E. 2000. Large deviations for martingales via Cramér’s method. *Stochastic Process. Appl.* **85** 279-293.
- [18] Gut, A. and Stadtmüller, U. 2021. Variations of the elephant random walk. *J. Appl. Probab.* **58**(3) 805-829.
- [19] Haeusler, E. 1988. On the rate of convergence in the central limit theorem for martingales with discrete and continuous time. *Ann. Probab.* **16** (1) 275-299.

- [20] Hall, P. and Heyde, C. C. *Martingale Limit Theory and its Applications*, Academic, New York, 1980.
- [21] Laulin, L. 2022. New insights on the reinforced elephant random walk using a martingale approach. *J. Stat. Phys.* **186**(1): 1-23.
- [22] Ma, X., El Machkouri, M. and Fan, X. 2022. On Wasserstein-1 distance in the central limit theorem for elephant random walk. *J. Math. Phys.* **63**(1): 013301
- [23] Rio, E. 2009. Upper bounds for minimal distances in the central limit theorem. *Ann. Inst. Henri Poincaré Probab. Stat.* **45**(3) 802–817.
- [24] Schütz, G. M. and Trimper, S. 2004. Elephants can always remember: Exact long-range memory effects in a non-Markovian random walk. *Phys. Rev. E* **70**(4) 045101.
- [25] Shao, Q. M. 1993. Almost sure invariance principles for mixing sequences of random variables. *Stochastic Process. Appl.* **48**(2): 319–334.
- [26] Vázquez Guevara, V. H. 2019. On the almost sure central limit theorem for the elephant random walk. *J. Phys. A: Math. Theor.* **52**(47) 475201