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## Gradient estimates for a nonlinear elliptic equation on a smooth metric measure space

*Xiaoshan Wang, Linfen Cao*

**Abstract.** Let  $(M, g, e^{-f} dv)$  be a smooth metric measure space. We consider local gradient estimates for positive solutions to the following elliptic equation

$$\Delta_f u + au \log u + bu = 0$$

where  $a, b$  are two real constants and  $f$  be a smooth function defined on  $M$ . As an application, we obtain a Liouville type result for such equation in the case  $a < 0$  under the  $m$ -dimensions Bakry-Émery Ricci curvature.

In this paper, we study the local gradient estimate for the positive solution to the following weighted nonlinear elliptic equation

$$\Delta_f u + au \log u + bu = 0 \tag{1}$$

on a smooth metric measure space  $(M, g, e^{-f} dv)$ , where  $a, b$  are two real constants and  $f$  be a smooth function defined on  $M$ . The motivation to study (1) comes from understanding the Ricci flow. Moreover, the (1) is closely related to the famous Gross Logarithmic Sobolev inequality, see [2]. It is well known that Yau has proved in [14] that every positive or bounded harmonic function is constant if  $M$  has nonnegative Ricci curvature by establishing gradient estimates for the solutions to Laplacian equation, see also [6], [8], [11].

It is natural to consider similar Liouville type results for positive solutions to the nonlinear elliptic equation (1). In [10], Qian considered positive solutions to

$$\Delta u + au \log u = 0 \tag{2}$$

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and proved the following

**Theorem 0.1.** (B. Qian) *Let  $(M, g)$  be an  $n$ -dimensional complete Riemannian manifold with the Ricci curvature  $\text{Ric}(B(x, R)) \geq -K$ , where  $K \geq 0$  is a constant. Let  $u$  be a positive solution to (2) on  $B(x, R)$ , then for any  $\alpha > 0$ ,*

$$\sup_{y \in B(x, \frac{R}{2})} \frac{|\nabla u|}{u} \leq C(n) \sqrt{(1 + \alpha)(a + K) + \frac{(1 + \alpha)}{R^2}} + \sqrt{\frac{1 + \alpha}{\alpha}} |a| L(x, R), \quad (3)$$

where  $L(x, R) = \sup_{y \in B(x, R)} |\log u| < \infty$  and  $C(n)$  is a constant depending only on the dimension  $n$ .

In particular, by letting  $R \rightarrow \infty$  in (3), we obtain the following gradient estimates on complete non-compact Riemannian manifolds:

$$\frac{|\nabla u|}{u} \leq C(n) \sqrt{(1 + \alpha)(a + K) + \frac{1 + \alpha}{\alpha}} |a| L, \quad (4)$$

where  $L = \sup_M |\log u|$ .

**Remark 0.2.** Clearly, from 4, it is easy to see that if  $a + K < 0$  and  $a < 0$ , then any bounded positive solution to (2) must be a constant  $u \equiv 1$ . On the other hand, in [5], Huang and Ma also obtained the similar Liouville type result by a different method.

Let  $(M, g)$  be an  $n$ -dimensional complete Riemannian manifold and  $f$  be a smooth function defined on  $M$ . In general, the triple  $(M, g, e^{-f} dv)$  is called a smooth metric measure space. The  $f$ -Laplacian operator is defined by

$$\Delta_f = e^f \text{div}(e^{-f} \nabla) = \Delta - \nabla f \nabla,$$

which is symmetric in  $L^2(M, g, e^{-f} dv)$ .

It is well-known that the  $m$ -dimensions Bakry-Émery Ricci curvature associated with the  $f$ -Laplacian is defined by (see [3], [4], [13] )

$$\text{Ric}_f^m = \text{Ric} + \nabla^2 f - \frac{df \otimes df}{m - n},$$

where  $m \geq n$  is a constant and  $m = n$  if and only if  $f$  is a constant. Define

$$\text{Ric}_f = \text{Ric} + \nabla^2 f.$$

Then  $\text{Ric}_f$  can be seen as  $\infty$ -dimensions Bakry-Émery Ricci curvature. Recently, the  $\infty$ -dimensions Bakry-Émery Ricci curvature has become an important object of study in Riemannian geometry. The equation  $\text{Ric}_f = \rho \langle, \rangle$  for some constant  $\rho$  is just the gradient Ricci soliton equation, which plays an important role in the study of Ricci flow (see [1]).

The aim of this paper is to generalize the results of Qian in [10] to the weighted nonlinear elliptic equation (1) under the assumption that the  $m$ -dimensions Bakry-Émery Ricci curvature is bounded from blow. Our main results are as follows:

**Theorem 0.3.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional complete smooth metric measure space with  $m$ -dimensions Bakry-Émery Ricci curvature  $Ric_f^m(B(x, R)) \geq -K$  for some constant  $K \geq 0$ . Let  $u$  be a positive solution to the nonlinear equation (1) on  $B(x, R)$ , then there exists a constant  $C = C(m)$ , such that*

$$\sup_{y \in B(x, \frac{R}{2})} \frac{|\nabla u|}{u} \leq C \sqrt{a + K + \frac{1}{R^2} + |b| + |a|L}. \quad (5)$$

where  $L(x, R) := \sup_{y \in B(x, R)} |\log u| < \infty$ .

In particular, by letting  $R \rightarrow \infty$  in (5), we obtain the following gradient estimates on complete non-compact Riemannian manifolds:

$$\frac{|\nabla u|}{u} \leq C \sqrt{a + K + |b| + |a|L}, \quad (6)$$

where  $L = \sup_M |\log u|$ .

From (6), it is easy to obtain the following results:

**Corollary 0.4.** *Let  $(M, g, e^{-f} dv)$  be an  $n$ -dimensional complete smooth metric measure space with  $Ric_f^m \geq -K$ . If  $u$  is a bounded positive solution to (1) with  $a + K < 0$  and  $a < 0$ , then  $u \equiv 1$ . Furthermore, if  $Ric_f^m \geq 0$  and  $a \leq 0$ , then any bounded positive solution to (1) must be  $u \equiv 1$ .*

**Remark 0.5.** When  $m = n$  in Theorem 0.3, we have  $Ric_f^m = Ric$  and  $\Delta_f = \Delta$ . Hence, in this case, our Theorem 0.3 becomes Theorem 0.1 of Qian. That is, our results of this paper generalize those of Qian in [10].

## 1 Proof of Theorem 0.3

**Lemma 1.1.** *Let  $u$  be a bounded positive solution to the nonlinear equation (1), then we have,*

$$|\nabla u| \Delta_f |\nabla u| \geq \frac{|\nabla(|\nabla u|)|^2}{m} - (au \log u + bu)^2 + Ric_f^m(\nabla u, \nabla u) - (a + a \log u + b)|\nabla u|^2. \quad (7)$$

*Proof.* Since we have the Bochner-weitzenböck formula with respect to  $f$ -Laplacian, for any  $u \in C^2(M)$ , we have

$$\frac{1}{2} \Delta_f |\nabla u|^2 = |\nabla^2 u|^2 + Ric_f(\nabla u, \nabla u) + (\nabla u, \nabla \Delta_f u).$$

On the other hand,

$$\Delta_f |\nabla u|^2 = 2|\nabla u| \Delta_f |\nabla u| + 2|\nabla(|\nabla u|)|^2,$$

hence

$$|\nabla u| \Delta_f |\nabla u| = |\nabla^2 u|^2 + Ric_f(\nabla u, \nabla u) + (\nabla u, \nabla \Delta_f u) - |\nabla(|\nabla u|)|^2.$$

Since  $u$  is a solution to (1), we obtain

$$|\nabla u| \Delta_f |\nabla u| = |\nabla^2 u|^2 + Ric_f(\nabla u, \nabla u) - (a + a \log u + b) |\nabla u|^2 - |\nabla(|\nabla u|)|^2.$$

If we consider a local normal chart at  $x$  in which  $u_1(x) = |\nabla u|(x)$  and  $u_j(x) = 0$  for  $j \geq 2$ , then  $\nabla_i(|\nabla u|) = u_{1i}$ , hence  $|\nabla(|\nabla u|)|^2 = \sum_i u_{1i}^2$ . Since  $u$  is a solution to (1), in the above local chart we have at  $x$

$$\sum_{i \geq 2} u_{ii} = -u_{11} - au \log u - bu + \nabla f \cdot \nabla u.$$

Therefore,

$$\begin{aligned} |\nabla^2 u|^2 - |\nabla(|\nabla u|)|^2 &= \sum_{i \geq 1, j \geq 1} u_{ij}^2 - \sum_{j \geq 1} u_{1j}^2 = \sum_{i \geq 2, j \geq 1} u_{ij}^2 \geq \sum_{i \geq 2} u_{i1}^2 + \sum_{i \geq 2} u_{ii}^2 \\ &\geq \sum_{i \geq 2} u_{i1}^2 + \frac{1}{n-1} \left( \sum_{i \geq 2} u_{ii} \right)^2 \\ &= \sum_{i \geq 2} u_{i1}^2 + \frac{1}{n-1} (-u_{11} - au \log u - bu + \nabla f \cdot \nabla u)^2 \\ &\geq \frac{1}{(n-1)(1+\alpha)} \sum_{i \geq 1} u_{i1}^2 - \frac{1}{(n-1)\alpha} (au \log u + bu - \nabla f \cdot \nabla u)^2 \\ &\geq -\frac{1}{(n-1)\alpha} \left[ \left(1 + \frac{1}{\epsilon}\right) (au \log u + bu)^2 + (1+\epsilon) (\nabla f \cdot \nabla u)^2 \right] \\ &\quad + \frac{1}{(n-1)(1+\alpha)} \sum_{i \geq 1} u_{i1}^2 \\ &= \frac{1}{m} \sum_{i \geq 1} u_{i1}^2 - (au \log u + bu)^2 - \frac{(\nabla f \cdot \nabla u)^2}{m-n}, \end{aligned}$$

where we use the elementary inequalities (see [12]):

$$(a+b)^2 \geq \frac{1}{1+\alpha} a^2 - \frac{1}{\alpha} b^2 \quad \text{and} \quad (a+b)^2 \leq (1+\epsilon) a^2 + \left(1 + \frac{1}{\epsilon}\right) b^2,$$

which holds for any  $\alpha > 0$ ,  $\epsilon > 0$ . The last equality we choose  $\alpha = \frac{m-n+1}{n-1}$  and  $\epsilon = \frac{1}{m-n}$ .

Hence

$$\begin{aligned} |\nabla u| \Delta_f |\nabla u| &\geq \frac{|\nabla(|\nabla u|)|^2}{m} - (au \log u + bu)^2 - \frac{(\nabla f \cdot \nabla u)^2}{m-n} \\ &\quad + Ric_f(\nabla u, \nabla u) - (a + a \log u + b) |\nabla u|^2 \\ &= \frac{|\nabla(|\nabla u|)|^2}{m} - (au \log u + bu)^2 + \left( Ric_f - \frac{df \otimes df}{m-n} \right) u_i u_j \\ &\quad - (a + a \log u + b) |\nabla u|^2 \\ &= \frac{|\nabla(|\nabla u|)|^2}{m} - (au \log u + bu)^2 + Ric_f^m(\nabla u, \nabla u) \\ &\quad - (a + a \log u + b) |\nabla u|^2. \end{aligned} \tag{8}$$

This completes the prove of Lemma 1.1.  $\square$

*Proof of Theorem 0.3.* Denote  $\psi = |\nabla \log u| = \frac{|\nabla u|}{u}$ , i.e.,

$$|\nabla u| = \psi u.$$

Direct computation gives

$$\nabla \psi = \frac{\nabla(|\nabla u|)}{u} - \frac{|\nabla u| \nabla u}{u^2}. \quad (9)$$

At the point where  $\nabla u \neq 0$ , we have

$$\begin{aligned} \Delta_f |\nabla u| &= \Delta_f(\psi u) = u \Delta_f \psi + \psi \Delta_f u + 2 \nabla \psi \cdot \nabla u \\ &= u \Delta_f \psi - \psi (a u \log u + b u) + 2 \nabla \psi \cdot \nabla u. \end{aligned}$$

This yields

$$\begin{aligned} \Delta_f \psi &= \frac{\Delta_f |\nabla u|}{u} + \psi (a \log u + b) - 2 \frac{\nabla \psi \cdot \nabla u}{u} \\ &= \frac{|\nabla u| \Delta_f |\nabla u|}{|\nabla u| u} + \psi (a \log u + b) - 2 \frac{\nabla \psi \cdot \nabla u}{u}. \end{aligned}$$

By Lemma 1.1, we can derive,

$$\begin{aligned} \Delta_f \psi &\geq \frac{\frac{|\nabla(|\nabla u|)|^2}{m} - (a u \log u + b u)^2 + Ric_f^m(\nabla u, \nabla u) - (a + a \log u + b) |\nabla u|^2}{|\nabla u| u} \\ &\quad + \psi (a \log u + b) - 2 \frac{\nabla \psi \cdot \nabla u}{u} \\ &\geq \frac{|\nabla(|\nabla u|)|^2}{m |\nabla u| u} - \frac{(a \log u + b)^2}{\psi} - (a + K) \psi - 2 \frac{\nabla \psi \cdot \nabla u}{u}. \end{aligned} \quad (10)$$

For any  $\delta > 0$ , by (9)

$$\begin{aligned} 2 \frac{\nabla \psi \cdot \nabla u}{u} &= (2 - \delta) \frac{\nabla \psi \cdot \nabla u}{u} + \delta \frac{\nabla \psi \cdot \nabla u}{u} \\ &= (2 - \delta) \frac{\nabla \psi \cdot \nabla u}{u} + \delta \frac{\nabla u}{u} \left( \frac{\nabla(|\nabla u|)}{u} - \frac{|\nabla u| \nabla u}{u^2} \right) \\ &= (2 - \delta) \frac{\nabla \psi \cdot \nabla u}{u} + \delta \frac{\nabla(|\nabla u|) \cdot \nabla u}{u^2} - \delta \psi^3 \\ &\leq (2 - \delta) \frac{\nabla \psi \cdot \nabla u}{u} + \delta \frac{|\nabla(|\nabla u|)| \cdot |\nabla u|}{u^2} - \delta \psi^3 \\ &\leq (2 - \delta) \frac{\nabla \psi \cdot \nabla u}{u} + \frac{\delta}{2} \left( \frac{|\nabla(|\nabla u|)|^2}{|\nabla u| u} + \frac{|\nabla u|^3}{u^3} \right) - \delta \psi^3 \\ &= (2 - \delta) \frac{\nabla \psi \cdot \nabla u}{u} + \frac{\delta}{2} \frac{|\nabla(|\nabla u|)|^2}{|\nabla u| u} - \frac{\delta}{2} \psi^3. \end{aligned}$$

Choosing  $\delta = \frac{2}{m}$  and substituting into (10), we obtain

$$\begin{aligned} \Delta_f \psi \geq & -(a+K)\psi - \left(2 - \frac{2}{m}\right) \frac{\nabla \psi \cdot \nabla u}{u} + \frac{\psi^3}{m} \\ & - \frac{(a \log u + b)^2}{\psi}. \end{aligned} \quad (11)$$

Now we define

$$F(y) := (R^2 - d^2(x, y)) \frac{|\nabla u|(y)}{u(y)} = (R^2 - d^2)\psi(y),$$

and

$$\psi(y) = \frac{|\nabla u|(y)}{u(y)}, \quad y \in B(x, R).$$

Since  $F|_{\partial B(x, R)} = 0$ , if  $\nabla u = 0$ , then  $F$  can only achieve its maximum at some point  $x_0 \in B(x, R)$ , if  $|\nabla u|(x_0) = 0$ , the desired result holds. Then, without loss of generality, we can suppose  $|\nabla u|(x_0) \neq 0$ . Assume  $x_0 \notin \text{cut}(x)$ , by the maximum principle we have  $\Delta_f F(x_0) \leq 0$  and  $\nabla F(x_0) = 0$ . It yields, at  $x_0$ ,

$$\nabla F = -\psi \nabla d^2 + (R^2 - d^2) \nabla \psi = 0.$$

It holds,

$$\frac{\nabla \psi}{\psi} = \frac{\nabla d^2}{R^2 - d^2} = \frac{2d \nabla d}{R^2 - d^2} \quad (12)$$

and

$$\Delta_f F = \Delta_f((R^2 - d^2)\psi) = (R^2 - d^2)\Delta_f \psi - \psi \Delta_f d^2 - 2\nabla d^2 \nabla \psi \leq 0. \quad (13)$$

Hence, dividing by  $(R^2 - d^2)\psi$  to both sides of (13) and combining (12), we have at  $x_0$ ,

$$0 \geq \frac{\Delta_f \psi}{\psi} - \frac{\Delta_f d^2}{R^2 - d^2} - \frac{8d^2}{(R^2 - d^2)^2}.$$

By the  $f$ -Laplacian comparison theorem in [12] (see also [7] or [9]), we have

$$\Delta_f d^2 \leq C\sqrt{K}d \coth(\sqrt{K}d) \leq C\sqrt{K}d,$$

where  $C$  only depends on  $m$ . Together with (11), we have at  $x_0$ ,

$$\begin{aligned} 0 \geq & -(a+K) - \left(2 - \frac{2}{m}\right) \frac{\nabla \psi \cdot \nabla u}{\psi u} + \frac{\psi^2}{m} \\ & - \frac{(a \log u + b)^2}{\psi^2} - \frac{C\sqrt{K}d}{R^2 - d^2} - \frac{8d^2}{(R^2 - d^2)^2}. \end{aligned} \quad (14)$$

By (12),

$$\frac{\nabla \psi \cdot \nabla u}{\psi u} = \frac{2d}{R^2 - d^2} \frac{\nabla d \cdot \nabla u}{u} \leq \frac{2d\psi}{R^2 - d^2},$$

then multiplying both sides of (14) by  $\psi^2(R^2 - d^2)^4$ , combining with  $\psi = \frac{F}{R^2 - d^2}$ , we can derive

$$\begin{aligned} 0 &\geq \frac{1}{m}F^4 - \frac{4(m-1)}{m}dF^3 - (R^2 - d^2)^4(a \log u + b)^2 \\ &\quad - ((a+K)(R^2 - d^2)^2 + 8d^2 + C\sqrt{K}d(R^2 - d^2))F^2 \\ &\geq \frac{1}{m}F^4 - 4RF^3 - ((a+K)R^2 + C\sqrt{K}R + 8)R^2F^2 \\ &\quad - R^8(a \log u + b)^2. \end{aligned}$$

Since  $\frac{1}{2m}F^4 - 4RF^3 \geq -8mR^2F^2$ , it holds

$$\begin{aligned} \frac{1}{2m}F^4 - ((a+K)R^2 + C\sqrt{K}R + C)R^2F^2 \\ - R^8(a \log u + b)^2 \leq 0. \end{aligned}$$

It follows

$$\begin{aligned} \sup_{y \in B(x, R)} (R^2 - d^2(x, y)) |\nabla \log u| &\leq F(x_0) \\ &\leq \sqrt{2m((a+K)R^2 + C\sqrt{K}R + C)R^2 + \sqrt{2m}(|b| + |a|L)R^4}. \end{aligned}$$

Restricting on the ball  $B(x, \frac{R}{2})$ , we have

$$\begin{aligned} \sup_{y \in B(x, \frac{R}{2})} \frac{3R^2}{4} |\nabla \log u| &\leq \sup_{y \in B(x, \frac{R}{2})} (R^2 - d^2(x, y)) |\nabla \log u| \leq F(x_0) \\ &\leq \sqrt{2m((a+K)R^2 + C\sqrt{K}R + C)R^2 + \sqrt{2m}(|b| + |a|L)R^4}. \end{aligned}$$

Therefor, we can derive

$$\sup_{y \in B(x, \frac{R}{2})} \frac{|\nabla u|}{u} \leq \sqrt{4m(a+K + C\frac{\sqrt{K}}{R} + \frac{C}{R^2}) + \sqrt{8m}(|b| + |a|L)}.$$

Then use the Cauchy-Schwarz inequality, we obtain

$$\sup_{y \in B(x, \frac{R}{2})} \frac{|\nabla u|}{u} \leq C\sqrt{a+K + \frac{1}{R^2} + |b| + |a|L}.$$

Now let  $R \rightarrow \infty$ , this yields, for any  $x \in M$ ,

$$\psi(x) \leq \psi(x_0) \leq C\sqrt{a+K + |b| + |a|L}.$$

This completes the prove of Theorem 0.3. □



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