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# Recognizability in $S$-adic shifts 

Marie-Pierre Béal©, Dominique Perrin, Antonio Restivo, and Wolfgang Steiner

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#### Abstract

We investigate questions related to the notion of recognizability of sequences of morphisms, a generalization of Mossé's Theorem. We consider the most general class of morphisms including ones with erasable letters. The main result states that a sequence of morphisms with bounded alphabet size is eventually recognizable for aperiodic points, improving and simplifying a result of Berthé et al. (2019). This also provides a new simple proof for the recognizability of a single morphism on its shift space. The main ingredient of the proof are elementary morphisms.


## 1 Introduction

Given a bi-infinite sequence in $A^{\mathbb{Z}}$ and a morphism (also called a substitution) $\sigma: A^{*} \rightarrow A^{*}$, recognizability is a form of injectivity of $\sigma$ that allows one to uniquely desubstitute $y$ to another sequence $x$, i.e., to express $y$ as a concatenation of substitution words dictated by the letters in $x$. The sequences $y$ and $x$ are traditionally required to be in the shift space $X(\sigma)$, which is the set of biinfinite sequences (also called points) whose finite factors are factors of $\sigma^{n}(a)$ for some integer $n$ and some letter $a$ in $A$.

By Mossé's Theorem [11, 12, every aperiodic primitive morphism $\sigma$ is recognizable in the shift $X(\sigma)$; see the precise definitions in Section 3. This surprising result was initially formulated (in an incorrect way) by [10]; see [8] on the genesis of the theorem and its possible variants. It was further generalized by Bezuglyi, Kwiatkowski, and Medynets [3], who proved that every aperiodic non-erasing morphism $\sigma$ is recognizable in $X(\sigma)$. Next, it was proved by Berthé, Thuswaldner, Yassawi, and the fourth author [2] that every non-erasing morphism $\sigma$ is recognizable in $X(\sigma)$ for aperiodic points, and the first three authors proved in [1] that every morphism $\sigma$ is recognizable in $X(\sigma)$ for aperiodic points.

There is a strong link between recognizability and automata theory due to a translation of the property of recognizability in terms of finite monoids. For instance there is a quadratic-time algorithm to check whether an injective morphism is recognizable in the full shift for aperiodic points [1].

In this paper, we investigate recognizability in the context of sequences of morphisms $\sigma=\left(\sigma_{n}: A_{n+1}^{*} \rightarrow A_{n}^{*}\right)_{n \geq 0}$. Such a sequence defines an $S$-adic shift, generated by iterations of the form $\sigma_{0} \circ \sigma_{1} \circ \cdots \circ \sigma_{n}$; see Section 6 for the definition. In fact, $\sigma$ defines a sequence $X^{(n)}(\sigma)$ of shift spaces and here, by recognizability of $\sigma_{n}$, we mean that any sequence in $X^{(n)}(\sigma)$ can be desubstituted in at most one (and usually exactly one) way as sequences in $X^{(n+1)}(\sigma)$ using $\sigma_{n}$. We distinguish between recognizability of $\sigma$, where each $\sigma$ is recognizable, and eventual recognizability, where all but finitely many morphisms $\sigma_{n}$ are recognizable. We consider the most general class of sequences of morphisms, including ones with morphisms with erasable letters.

Recognizability of $S$-adic shifts has been studied in [2], where it is proved that a morphism $\sigma$ is eventually recognizable for aperiodic points in its shift spaces under some mild conditions: the morphisms $\sigma_{n}$ are non-erasing, the sequence $\operatorname{Card}\left(A_{n}\right)$ contains a bounded subsequence, and $\sigma$ is everywhere growing (or the points in each $X^{(n)}(\sigma)$ generate a bounded number of different languages). This means that, for large enough $n$, every aperiodic point in $X^{(n)}(\sigma)$ has a unique centered $\sigma_{n}$-representation as a shift of the image by $\sigma_{n}$ of some $x \in X^{(n+1)}(\sigma)$. This recognizability property implies a natural representation of an $S$-adic shift as a Bratteli-Vershik system [2]. A weaker notion, called quasi-recognizability, is studied in [4].

The main result of this paper is a generalization to possibly erasing sequences of morphisms of the result of [2 concerning the recognizability of a sequence of morphisms in its shift spaces for aperiodic points.

When a morphism $\sigma_{n}$ erases a letter, it is possible that a sequence in $X^{(n)}(\sigma)$ cannot be desubstituted as a sequence in $X^{(n+1)}(\sigma)$ using $\sigma_{n}$. We are therefore not only concerned with recognizability but also with representability, which means that $X^{(n)}(\sigma)$ is the shift closure of the image of $X^{(n+1)}(\sigma)$ by $\sigma_{n}$; see Section 5 for details.

We prove the following result, with a more precise statement in Section 7
Main Theorem Let $\sigma=\left(\sigma_{n}\right)_{n \geq 0}$ with $\sigma_{n}: A_{n+1}^{*} \rightarrow A_{n}^{*}$ be a sequence of morphisms such that $\lim _{\inf _{n \rightarrow \infty}} \operatorname{Card}\left(A_{n}\right)$ is finite. Then $\sigma$ is eventually recognizable for aperiodic points and eventually representable.

Our proof is much simpler than that of 2, and we we do not require that $\operatorname{Card}\left(\left\{\mathcal{L}_{x} \mid x \in X^{(n)}(\sigma)\right\}\right)$ is bounded, where $\mathcal{L}_{x}$ is the set of factors of a point $x$. Moreover our proof gives a bound $\liminf _{n \rightarrow \infty} \operatorname{Card}\left(A_{n}\right)-2$ on the number of levels at which $\sigma$ is not recognizable for aperiodic points, improving the bound of order $K(K+L \log K)$ obtained in [2] for alphabets of size at most $K$ and $\operatorname{Card}\left(\left\{\mathcal{L}_{x} \mid x \in X^{(n)}(\sigma)\right\}\right) \leq L$. We also show that this bound is tight.

Our result allows one also to get a new simpler proof of the recognizability of a (possibly erasable) morphism $\sigma$ on $X(\sigma)$ for aperiodic points obtained in [1].

As in [1], our proof relies on the notion of elementary morphism, due to Ehrenfeucht and Rozenberg [6]. By a result of Karhumäki, Maňuch and Plandowski [7, every elementary morphism is recognizable for aperiodic points; see also [2]. We use this result to prove eventual recognizability.

The paper is organized as follows. After an introductory section on basic notions of symbolic dynamics, we formulate the precise definition of a morphism recognizable on a shift space and prove some elementary properties of recognizable morphisms. In Section 4 we introduce elementary morphisms and recall that every elementary morphism is recognizable for aperiodic points (Proposition 4.3). The main results are proved in Section 6.

## 2 Symbolic dynamics

We briefly recall some basic definitions of symbolic dynamics. For a more complete presentation, see [9] or the recent (5].

### 2.1 Words

Let $A$ be a finite alphabet. We denote by $A^{*}$ the free monoid on $A$, i.e., the set of finite words over the alphabet $A$. The empty word is denoted by $\varepsilon$. We denote by $|u|$ the length of the word $u$.

A word $s \in A^{*}$ is a factor of $w \in A^{*}$ if $w=r s t$; the word $r$ is called a prefix of $w$, and it is proper if $r \neq w$.

### 2.2 Shift spaces

We consider the set $A^{\mathbb{Z}}$ of two-sided infinite sequences (also called points) on $A$. For $x=\left(x_{n}\right)_{n \in \mathbb{Z}}$, and $i \leq j$, we denote $x_{[i, j)}$ the word $x_{i} x_{i+1} \cdots x_{j-1}$, where $x_{[i, i)}$ is the empty word; the word $x_{[i, j)}$ is called a factor of $x$.

The set $A^{\mathbb{Z}}$ is a compact metric space for the distance defined for $x \neq y$ by $d(x, y)=2^{-\min \left\{|n| \mid n \in \mathbb{Z}, x_{n} \neq y_{n}\right\}}$. The shift transformation $T: A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ is defined by $T\left(\left(x_{n}\right)_{n \in \mathbb{Z}}\right)=\left(x_{n+1}\right)_{n \in \mathbb{Z}}$. A shift space $X$ on a finite alphabet $A$ is a closed and shift-invariant subset of $A^{\mathbb{Z}}$.

A point $x \in A^{\mathbb{Z}}$ is periodic if there is an $n \geq 1$ such that $T^{n}(x)=x$. Otherwise, it is aperiodic. A periodic point has the form $w^{\infty}=\cdots w w \cdot w w \cdots$ (the letter of index 0 of $w^{\infty}$ is the first letter of $w$ ).

### 2.3 Morphisms

A morphism $\sigma: A^{*} \rightarrow B^{*}$ is a monoid morphism from $A^{*}$ to $B^{*}$. It is erasing if $\sigma(a)$ is the empty word for some $a \in A$, non-erasing otherwise.

The morphism $\sigma$ is extended to a map from $A^{\mathbb{Z}}$ to $B^{\mathbb{Z}}$ by

$$
\sigma\left(\cdots x_{-2} x_{-1} \cdot x_{0} x_{1} \cdots\right)=\cdots \sigma\left(x_{-2}\right) \sigma\left(x_{-1}\right) \cdot \sigma\left(x_{0}\right) \sigma\left(x_{1}\right) \cdots
$$

i.e., $\sigma\left(\left(x_{n}\right)_{n \in \mathbb{Z}}\right)=\left(y_{n}\right)_{n \in \mathbb{Z}}$ with $y_{\left[\left|\sigma\left(x_{[0, n)}\right)\right|,\left|\sigma\left(x_{[0, n+1)}\right)\right|\right)}=\sigma\left(x_{n}\right)$ for all $n \geq 0$ and $y_{\left[\left|\sigma\left(x_{[n, 0)}\right)\right|,\left|\sigma\left(x_{[n+1,0)}\right)\right|\right)}=\sigma\left(x_{n}\right)$ for all $n<0$; this map is defined only for points in $A^{\mathbb{Z}}$ containing infinitely many letters on the left and infinitely many letters on the right that are not erased.

Let $\sigma: A^{*} \rightarrow A^{*}$ be a morphism from $A^{*}$ to itself. For $n \geq 0$, we denote by $\sigma^{n}$ the morphism obtained with $n$ iterations of $\sigma$. The language of $\sigma$, denoted $\mathcal{L}(\sigma)$, is the set of factors of the words $\sigma^{n}(a)$ for some $n \geq 0$ and $a \in A$. The shift defined by $\sigma$, denoted by $X(\sigma)$, is the set of sequences with all their factors in $\mathcal{L}(\sigma)$. The morphism $\sigma$ is primitive if there exists $n \geq 1$ such that the word $\sigma^{n}(a)$ contains the letter $b$ for all $a, b \in A$.

## $2.4 \quad S$-adic shifts

Let $\sigma=\left(\sigma_{n}\right)_{n \geq 0}$ be a sequence of morphisms $\sigma_{n}: A_{n+1}^{*} \rightarrow A_{n}^{*}$, where $A_{n}$ are finite alphabets:

$$
A_{0}^{*} \stackrel{\sigma_{0}}{\longleftarrow} A_{1}^{*} \stackrel{\sigma_{1}}{\longleftarrow} A_{2}^{*} \stackrel{\sigma_{2}}{\longleftarrow} \cdots \stackrel{\sigma_{n-1}}{\longleftarrow} A_{n}^{*} \stackrel{\sigma_{n}}{\longleftarrow} A_{n+1}^{*} \stackrel{\sigma_{n+1}}{\longleftarrow} \cdots
$$

For $0 \leq n \leq m$, we define the morphism $\sigma_{[n, m)}: A_{m}^{*} \rightarrow A_{n}^{*}$ by

$$
\sigma_{[n, m)}=\sigma_{n} \circ \sigma_{n+1} \circ \cdots \circ \sigma_{m-1}
$$

where $\sigma_{[n, n)}$ is the identity. For $n \geq 0$, the language of $\mathcal{L}^{(n)}(\sigma)$ is the subset of $A_{n}^{*}$ of factors of the words $\sigma_{[n, m)}(a), a \in A_{m}, m \geq n$, and the shift $X^{(n)}(\sigma)$ is the set of sequences with all their factors in $\mathcal{L}^{(n)}(\sigma)$. The $S$-adic shift defined by $\sigma$ is $X^{(0)}(\sigma)$.

A sequence of morphisms $\sigma=\left(\sigma_{n}\right)_{n \geq 0}$ is non-erasing if all $\sigma_{n}$ are nonerasing. It is primitive if for each $n \geq 0$ there exists $m>n$ such that the word $\sigma_{[n, m)}(a)$ contains the letter $b$ for all $a \in A_{m}, b \in A_{n}$.

## 3 Recognizable morphisms

Let $\sigma: A^{*} \rightarrow B^{*}$ be a morphism. A $\sigma$-representation of $y \in B^{\mathbb{Z}}$ is a pair $(x, k)$ of a sequence $x \in A^{\mathbb{Z}}$ and an integer $k$ such that

$$
\begin{equation*}
y=T^{k}(\sigma(x)) \tag{3.1}
\end{equation*}
$$

where $T$ is the shift transformation. The $\sigma$-representation $(x, k)$ is centered if $0 \leq k<\left|\sigma\left(x_{0}\right)\right|$. In particular, a centered $\sigma$-representation $(x, k)$ satisfies $\sigma\left(x_{0}\right) \neq \varepsilon$. We say that the $\sigma$-representation $(x, k)$ is in $X$ if $x \in X$.

Note that, if $y$ has a $\sigma$-representation $(x, k)$, then it has a centered $\sigma$ representation $\left(x^{\prime}, k^{\prime}\right)$ with $x^{\prime}$ a shift of $x$.

Definition 3.1 Let $X$ be a shift space on $A$. A morphism $\sigma: A^{*} \rightarrow B^{*}$ is recognizable in $X$ (respectively recognizable in $X$ for aperiodic points) if every point in $B^{\mathbb{Z}}$ (respectively every aperiodic point in $B^{\mathbb{Z}}$ ) has at most one centered $\sigma$ representation in $X$. A morphism $\sigma: A^{*} \rightarrow B^{*}$ is fully recognizable (respectively fully recognizable for aperiodic points) if it is recognizable in $A^{\mathbb{Z}}$ (respectively recognizable in $A^{\mathbb{Z}}$ for aperiodic points).

Note that an equivalent definition of recognizability in $X$ is that, for every $x, x^{\prime} \in X$ and $0 \leq k<\left|\left|\sigma\left(x_{0}^{\prime}\right)\right|-\left|\sigma\left(x_{0}\right)\right|\right|$ such that $\sigma(x)=T^{k}\left(\sigma\left(x^{\prime}\right)\right)$, one has $k=0$ and $x=x^{\prime}$.

Example 3.2 The Fibonacci morphism $\sigma: a \mapsto a b, b \mapsto a$ is fully recognizable.
Example 3.3 The Thue-Morse morphism $\sigma: a \mapsto a b, b \mapsto b a$ is not fully recognizable since $(a b)^{\infty}$ can be obtained as $\sigma\left(a^{\infty}\right)$ and as $T\left(\sigma\left(b^{\infty}\right)\right)$. However, it is fully recognizable for aperiodic points since any sequence containing $a a$ or $b b$ has at most one factorization in $\{a b, b a\}$.

Example 3.4 The morphism $\sigma: a \rightarrow a a, b \mapsto a b, c \mapsto b a$ is not fully recognizable for aperiodic points. Indeed, every sequence without occurrence of $b b$ has two factorizations in words of $\{a a, a b, b a\}$.

By [2, 1], the family of morphisms recognizable for aperiodic points is closed under composition.

## 4 Elementary morphisms

Definition 4.1 A morphism $\sigma: A^{*} \rightarrow C^{*}$ is elementary if for every alphabet $B$ and every pair of morphisms $\alpha: B^{*} \rightarrow C^{*}$ and $\beta: A^{*} \rightarrow B^{*}$ such that $\sigma=\alpha \circ \beta$, one has $\operatorname{Card}(B) \geq \operatorname{Card}(A)$.

If $\sigma: A^{*} \rightarrow C^{*}$ is elementary, one has in particular $\operatorname{Card}(C) \geq \operatorname{Card}(A)$ and moreover $\sigma$ is non-erasing.

Example 4.2 The Thue-Morse morphism $\sigma: a \mapsto a b, b \mapsto b a$ is elementary.
The notion of elementary morphism appears for the first time in [6]. The following result is from [7]. It also appears in [2] with the stronger hypothesis that $\sigma: A^{*} \rightarrow B^{*}$ is such that the incidence matrix of $\sigma$ has rank $\operatorname{Card}(A)$. An independent proof is given in [1].
Proposition 4.3 Any elementary morphism is fully recognizable for aperiodic points.

## 5 Representable $S$-adic shifts

Contrary to sequences of non-erasing morphisms $\sigma$, a point in $X^{(n)}(\sigma)$ need not have a $\sigma_{n}$-representation in $X^{(n+1)}(\sigma)$ when $\sigma_{n}$ is erasing, as the following example shows.

Example 5.1 Let the sequence of morphisms $\sigma$ be defined by

$$
\begin{aligned}
\sigma_{0}: a \mapsto a, b \mapsto \varepsilon, \\
\sigma_{1}: a \mapsto a, b \mapsto b b, c \mapsto a b, \\
\sigma_{n}: a \mapsto a, b \mapsto b b, c \mapsto c a b, \quad \text { for all } n \geq 2
\end{aligned}
$$

Since $\sigma_{[1, n)}(a)=a, \sigma_{[1, n)}(b)=b^{2^{n-1}}, \sigma_{[1, n)}(c)=a b a b^{2} \cdots a b^{2^{n-2}}$ for all $n \geq 2$, we have $X^{(1)}(\sigma)=\left\{b^{\infty}\right\}$ and $X^{(0)}(\sigma)=\left\{a^{\infty}\right\}$, and $a^{\infty}$ has no $\sigma_{0}$-representation in $X^{(1)}(\sigma)$.

We say that a sequence of morphisms $\sigma$ is representable at level $n$ if every point in $X^{(n)}(\sigma)$ has at least one $\sigma_{n}$-representation in $X^{(n+1)}(\sigma)$. It is representable if it is representable at each level. We say that a sequence of morphisms is eventually representable if there is an integer $M$ such that it is representable at each level at least equal to $M$.

Note that $X^{(n)}(\sigma)$ is the shift-closure of $\sigma_{n}\left(X^{(n+1)}(\sigma)\right)$ if and only if $\sigma$ is representable at level $n$.

The following lemma is proved in [2, Lemma 4.2] for sequences of non-erasing morphisms. We recall its proof to make clear where the non-erasing property is used.

Lemma 5.2 Let $\sigma=\left(\sigma_{n}\right)_{n \geq 0}$ with $\sigma_{n}: A_{n+1}^{*} \rightarrow A_{n}^{*}$ be a sequence of morphisms. If $\sigma_{[n, m)}$ is non-erasing, $0 \leq n<m$, then every point in $X^{(n)}(\sigma)$ has at least one $\sigma_{[n, m)}$-representation in $X^{(m)}(\sigma)$. In particular, if $\sigma_{n}$ is non-erasing, then $\sigma$ is representable at level $n$.

Proof. Let $y \in X^{(n)}(\sigma)$. Then each word $y_{[-\ell, \ell)}$ is a factor of $\sigma_{[n, N)}(a)$ for some $a \in A_{N}, N \geq m$, hence $y_{[-\ell+i, \ell-j)}=\sigma_{[n, m)}(w)$ for some $w \in \mathcal{L}^{(m)}(\sigma), 0 \leq i, j<$ $\max _{a \in A_{m}}\left|\sigma_{[n, m)}(a)\right|$. Since $|w| \rightarrow \infty$ as $\ell \rightarrow \infty$, a Cantor diagonal argument gives a word $x \in X^{(m)}(\sigma)$ and $0 \leq k<\left|\sigma_{n}\left(x_{0}\right)\right|$ such that $\sigma_{[n, m)}\left(x_{[-\ell, \ell)}\right)=$ $y_{\left[\left|\sigma_{[n, m)}\left(x_{[-\ell, 0)}\right)\right|-k,\left|\sigma_{[n, m)}\left(x_{[0, \ell)}\right)\right|-k\right)}$ for all $\ell \geq 1$. Since $\sigma_{[n, m)}$ is non-erasing, $(k, x)$ is a $\sigma_{[n, m)}$-representation of $y$.

Lemma 5.3 If $\sigma$ is not representable at level $n$, then $\sigma_{[n, m)}$ is erasing for all $m>n$.

Proof. If $\sigma_{[n, m)}$ is non-erasing, then, by Lemma5.2, each $y \in X^{(n)}(\sigma)$ has $\sigma_{[n, m)^{-}}$ representation in $X^{(m)}(\sigma)$, thus it also has a $\sigma_{n}$-representation in $X^{(n+1)}(\sigma)$, i.e., $\sigma$ is representable at level $n$.

## 6 Recognizable $S$-adic shifts

A sequence of morphisms $\sigma=\left(\sigma_{n}\right)_{n \geq 0}$ with $\sigma_{n}: A_{n+1}^{*} \rightarrow A_{n}^{*}$ is recognizable at level $n$ (respectively recognizable at level $n$ for aperiodic points) if $\sigma_{n}$ is recognizable (respectively recognizable for aperiodic points) in $X^{(n+1)}(\sigma)$. We say that $\sigma$ is recognizable (respectively recognizable for aperiodic points) if it is recognizable (respectively recognizable for aperiodic points) at each nonnegative level $n$, and $\sigma$ is eventually recognizable (respectively eventually recognizable for aperiodic points) if there is a nonnegative integer $M$ such that $\sigma$ is recognizable (respectively recognizable for aperiodic points) at level $n$ for each $n \geq M$.

We show that recognizability at level $n$ and representability between levels $n+1$ and $m$ implies recognizability between levels $n$ and $m$.

Lemma 6.1 If $\sigma$ is not recognizable at level $n$ and each point in $X^{(n+1)}(\sigma)$ has a $\sigma_{[n+1, m)}$-representation in $X^{(m)}(\sigma)$, then $\sigma_{[n, m)}$ is not recognizable in $X^{(m)}(\sigma)$. The same statement holds for recognizability for aperiodic points.

Proof. This is proved in [2, Lemma 3.5] for non-erasing morphisms; we recall the proof. If $\sigma$ is not recognizable at level $n$, then there exists $z \in X^{(n)}(\sigma)$ with two centered $\sigma_{n}$-representations $(y, \ell) \neq\left(y, \ell^{\prime}\right)$ in $X^{(n+1)}(\sigma)$. Let $(x, k)$ and $\left(x^{\prime}, k^{\prime}\right)$ be centered $\sigma_{[n+1, m)}$-representations in $X^{(m)}(\sigma)$ of $y$ and $y^{\prime}$, respectively. Then $\left(x,\left|\sigma_{n}\left(y_{[-k, 0)}\right)\right|+\ell\right)$ and $\left(x^{\prime},\left|\sigma_{n}\left(y_{\left[-k^{\prime}, 0\right)}^{\prime}\right)\right|+\ell^{\prime}\right)$ are centered $\sigma_{[n, m)^{-}}$ representations of $z$. To see that the two representations are different, note that $y_{[-k, 0)}, z_{[-\ell, 0)}$, and $\sigma_{n}\left(y_{[-k, 0)}\right) z_{[-\ell, 0)}$ are proper prefixes of $\sigma_{[n+1, m)}\left(x_{0}\right)$, $\sigma_{n}\left(y_{0}\right)$, and $\sigma_{[n, m)}\left(x_{0}\right)$ respectively. Since each proper prefix of $\sigma_{[n, m)}\left(x_{0}\right)$ has a unique decomposition as $\sigma_{n}(u) v$ with $u \in A_{n+1}^{*}, v \in A_{n}^{*}$, such that $u a$ is a prefix of $\sigma_{[n+1, m)}\left(x_{0}\right)$ and $v$ is a proper prefix of $\sigma_{n}(a)$ for some $a \in A_{n+1}$, $\left(x,\left|\sigma_{n}\left(y_{[-k, 0)}\right)\right|+\ell\right)=\left(x^{\prime},\left|\sigma_{n}\left(y_{\left[-k^{\prime}, 0\right)}^{\prime}\right)\right|+\ell^{\prime}\right)$ would imply that $k=k^{\prime}$ and $\ell=\ell^{\prime}$, thus $y=y^{\prime}$, contradicting that $(y, \ell) \neq\left(y, \ell^{\prime}\right)$. Therefore, $\sigma_{[n, m)}$ is not recognizable on $X^{(m)}(\sigma)$.

Taking aperiodic points $y, y^{\prime}$ proves the statement for aperiodic points.

## 7 Levels of recognizability and representability

We can now state and prove our main results, which give bounds for the number of levels where a sequence of morphisms can be non-recognizable for aperiodic points or non-representable, in terms of the size of the alphabets.

Proposition 7.1 Let $\sigma=\left(\sigma_{n}\right)_{n \geq 0}$ with $\sigma_{n}: A_{n+1}^{*} \rightarrow A_{n}^{*}$ be a sequence of morphisms. Let $m>n_{1}>n_{2}>\cdots>n_{K} \geq 0$, $K \geq 0$, be such that, for each $1 \leq k \leq K, \sigma$ is not recognizable at level $n_{k}$ for aperiodic points or $\sigma$ is not representable at level $n_{k}$. Then we have $K<\operatorname{Card}\left(A_{m}\right)$. Moreover, $K=\operatorname{Card}\left(A_{m}\right)-1 \geq 1$ implies that $X^{\left(n_{K}\right)}(\sigma)$ has no aperiodic points.

Proof. Let $m>n_{1}>n_{2}>\cdots>n_{K} \geq 0$ be as in the statement of the proposition. Since the proposition is trivial for $K=0$, we assume that $K \geq 1$.

We define $\alpha_{0}: A_{m}^{*} \rightarrow A_{m}^{*}$ as the identity morphism and set $n_{0}=m, B_{0}=$ $A_{m}$. For each $1 \leq k<K$, we show inductively that the morphism $\sigma_{\left[n_{k}, n_{k-1}\right)} \circ$ $\alpha_{k-1}$ is not elementary and admits therefore a decomposition $\sigma_{\left[n_{k}, n_{k-1}\right)} \circ \alpha_{k-1}=$ $\alpha_{k} \circ \beta_{k}$ (see Figure 7.1) with morphisms $\alpha_{k}: B_{k}^{*} \rightarrow A_{n_{k}}^{*}, \beta_{k}: B_{k-1}^{*} \rightarrow B_{k}^{*}$, for some alphabet $B_{k}$ satisfying $\operatorname{Card}\left(B_{k}\right)<\operatorname{Card}\left(B_{k-1}\right)$.

Indeed, consider the sequence of morphisms

$$
\sigma^{\prime}=\left(\sigma_{0}, \sigma_{1}, \ldots, \sigma_{n_{k}}, \ldots, \sigma_{n_{k-1}-1}, \alpha_{k-1}, \beta_{k-1}, \beta_{k-2}, \ldots, \beta_{1}, \sigma_{m}, \sigma_{m+1}, \ldots\right)
$$

Since $\alpha_{k-1} \circ \beta_{k-1} \circ \beta_{k-2} \circ \cdots \circ \beta_{1}=\sigma_{\left[n_{k-1}, m\right)}$, we have $X^{(h)}(\sigma)=X^{(h)}\left(\sigma^{\prime}\right)$ for all $0 \leq h \leq n_{k-1}$. If $\sigma$ is not representable at level $n_{k}$, then $\sigma^{\prime}$ is also not representable at level $n_{k}$ and, by Lemma 5.3, $\sigma_{\left[n_{k}, n_{k-1}\right)} \circ \alpha_{k-1}$ is erasing, thus non-elementary. If $\sigma$ is not recognizable at level $n_{k}$ for aperiodic points and each


Figure 7.1: Alphabets and morphisms in the proof of Proposition 7.1
point in $X^{\left(n_{k}+1\right)}(\sigma)$ has a $\sigma_{\left[n_{k}+1, n_{k-1}\right)} \circ \alpha_{k-1}$-representation in $X^{\left(n_{k-1}+1\right)}\left(\sigma^{\prime}\right)$, then $\sigma_{\left[n_{k}, n_{k-1}\right)} \circ \alpha_{k-1}$ is not recognizable for aperiodic points in $X^{\left(n_{k-1}+1\right)}\left(\sigma^{\prime}\right)$ by Lemma 6.1 thus it is non-elementary. Finally, if there exists a point in $X^{\left(n_{k}+1\right)}(\sigma)$ without $\sigma_{\left[n_{k}+1, n_{k-1}\right)} \circ \alpha_{k-1}$-representation in $X^{\left(n_{k-1}+1\right)}\left(\sigma^{\prime}\right)$, then $\sigma_{\left[n_{k}+1, n_{k-1}\right)} \circ \alpha_{k-1}$ is erasing by Lemma 5.3, hence $\sigma_{\left[n_{k}, n_{k-1}\right)} \circ \alpha_{k-1}$ is erasing, thus non-elementary.

We get that $\operatorname{Card}\left(A_{m}\right)>\operatorname{Card}\left(B_{1}\right)>\operatorname{Card}\left(B_{2}\right)>\cdots>\operatorname{Card}\left(B_{K}\right) \geq 1$, thus $K<\operatorname{Card}\left(A_{m}\right)$. If $K=\operatorname{Card}\left(A_{m}\right)-1$, then $\operatorname{Card}\left(B_{K}\right)=1$, hence $X^{\left(n_{K}\right)}(\sigma)$ consists of a single periodic orbit, thus $\sigma$ is recognizable at level $n_{K}$ for aperiodic points and not representable at level $n_{K}$.

Theorem 7.2 Let $\sigma=\left(\sigma_{n}\right)_{n \geq 0}$ with $\sigma_{n}: A_{n+1}^{*} \rightarrow A_{n}^{*}$ be a sequence of morphisms such that $\lim \inf _{n \rightarrow \infty} \operatorname{Card}\left(A_{n}\right)$ is finite. Then $\sigma$ is eventually recognizable for aperiodic points and eventually representable.

Moreover, the number of levels at which $\sigma$ is not recognizable for aperiodic points is bounded by $\liminf _{n \rightarrow \infty} \operatorname{Card}\left(A_{n}\right)-2$ and the number of levels at which $\sigma$ is not representable is bounded by $\lim \inf _{n \rightarrow \infty} \operatorname{Card}\left(A_{n}\right)-1$.

Proof. Suppose that $\sigma$ is not representable at $K=\lim _{\inf }^{n \rightarrow \infty} \boldsymbol{\operatorname { C a r d }}\left(A_{n}\right)<\infty$ levels, then applying Proposition 7.1 for some $m$ which is larger than these levels and satisfies $\operatorname{Card}\left(A_{m}\right)=K$ gives a contradiction. Similarly, we cannot have $K=\liminf _{n \rightarrow \infty} \operatorname{Card}\left(A_{n}\right)-1$ levels where $\sigma$ is not recognizable for aperiodic points by Proposition 7.1 because the level $n_{K}$ in Proposition 7.1 can only be non-representable for this $K$.

Note that the condition $\liminf _{n \rightarrow \infty} \operatorname{Card}\left(A_{n}\right)<\infty$ holds in particular when the sizes of the alphabets are bounded.

In the particular case of a constant sequence of morphisms $\sigma^{\prime}=(\sigma, \sigma, \ldots)$, all shifts $X^{(n)}\left(\sigma^{\prime}\right)$ are equal to the shift space $X(\sigma)$ of the morphism $\sigma$, and non-recognizability of $\sigma$ in $X(\sigma)$ for aperiodic points means that $\sigma^{\prime}$ is nonrecognizable at aperiodic points at all levels. Since this is not possible by Theorem [7.2] this proves the main result of [1, in a way that is simpler than all the previous proofs of recognizability.
Corollary 7.3 Any morphism $\sigma$ is recognizable for aperiodic points in $X(\sigma)$.

In the same way, we have a simple proof of [1, Proposition 5.1] concerning the representability of $\sigma$.

Corollary 7.4 For any morphism $\sigma$, any point in $X(\sigma)$ has a $\sigma$-representation in $X(\sigma)$.

Finally, we consider the tightness of the bounds in Proposition 7.1 and Theorem 7.2. We have already seen in Example 5.1 that a sequence can be nonrepresentable at level 0 with $\operatorname{Card}\left(A_{1}\right)=2$. An example of a primitive sequence of morphisms with $\operatorname{Card}\left(A_{n}\right)=3$ for all $n \geq 1$ that is not recognizable for aperiodic points at level 0 is given in [2, Example 4.3]. The following example shows that we can have $\lim \inf _{n \rightarrow \infty} \operatorname{Card}\left(A_{n}\right)-2$ levels of non-recognizability for aperiodic points.

Example 7.5 Let $K \geq 1, A_{n}=\left\{a_{0}, a_{1}, \ldots, a_{n}\right\}$ for $0 \leq n \leq K, A_{n}=$ $\left\{a_{0}, a_{1}, \ldots, a_{K+1}\right\}$ for $n>K$, and

$$
\begin{array}{ll}
\sigma_{n}: a_{i} \mapsto a_{0} a_{i} a_{0} \text { for all } 0 \leq i \leq n, a_{n+1} \mapsto a_{n}, & \text { for } 0 \leq n \leq K, \\
\sigma_{n}: a_{i} \mapsto a_{0} a_{i} a_{0} \text { for all } 0 \leq i \leq K+1, & \text { for } n>K
\end{array}
$$

Then, for all $0 \leq n \leq K+1, X^{(n)}(\sigma)$ consists of the shift orbits of the sequences $\cdots a_{0} a_{0} a_{i} a_{0} a_{0} \cdots, 0 \leq i \leq n$. For $0 \leq n \leq K$, we have

$$
\sigma_{n}\left(\cdots a_{0} a_{0} a_{n} a_{0} a_{0} \cdots\right)=\cdots a_{0} a_{0} a_{n} a_{0} a_{0} \cdots=\sigma_{n}\left(\cdots a_{0} a_{0} a_{n+1} a_{0} a_{0} \cdots\right),
$$

thus $\sigma$ is not recognizable at level $n$ for aperiodic points for all $1 \leq n \leq K$ (and not recognizable at level 0).

Example 7.5 can be easily modified to obtain a sequence of morphisms that is not recognizable at any level $n$ for aperiodic points.

Example 7.6 Let $A_{n}=\left\{a_{0}, a_{1}, \ldots, a_{n+1}\right\}$ for all $n \geq 0$, and

$$
\sigma_{n}: a_{i} \mapsto a_{0} a_{i} a_{0} \text { for all } 0 \leq i \leq n+1, a_{n+2} \mapsto a_{n+1}, \quad \text { for all } n \geq 0
$$

Then $\sigma$ is not recognizable at level $n$ for aperiodic points for all $n \geq 0$ because

$$
\sigma_{n}\left(\cdots a_{0} a_{0} a_{n+1} a_{0} a_{0} \cdots\right)=\cdots a_{0} a_{0} a_{n+1} a_{0} a_{0} \cdots=\sigma_{n}\left(\cdots a_{0} a_{0} a_{n+2} a_{0} a_{0} \cdots\right)
$$

We do not know whether the bound of $\liminf _{n \rightarrow \infty} \operatorname{Card}\left(A_{n}\right)-1$ levels of non-representability is tight.

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