

Large deviations for a stochastic Cahn-Hilliard equation in Hölder norm

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Résumé

We consider a stochastic Cahn-Hilliard partial differential equation driven by space-time white noise. We prove the Large Deviations Principle (LDP) for the law of the solutions in the Hölder norm. We use the weak convergence approach that reduces the proof to establishing basic qualitative properties for controlled analogues of the original stochastic system..

Keywords : Stochastic Cahn-Hilliard equation ; Space-time white noise ; Stochastic partial differential equations ; Large deviations principle ; Weak convergence method ; Green function.

AMS Subject Classification : 60F10, 60H15, 60G15.

1 Introduction

In this paper we consider the following Stochastic Cahn-Hilliard equation with multiplicative space-time white noise, indexed by $\varepsilon > 0$, given by

$$\begin{cases} \frac{\partial u^\varepsilon}{\partial t}(t, x) = -\Delta(\Delta u^\varepsilon(t, x) - f(u^\varepsilon(t, x))) + \sqrt{\varepsilon}\sigma(u^\varepsilon(t, x))\dot{W}(t, x), & (t, x) \in [0, T] \times D, \\ u^\varepsilon(0, x) = u_0(x), \\ \frac{\partial u^\varepsilon}{\partial \nu}(t, x) = \frac{\partial \Delta u^\varepsilon}{\partial \nu}(t, x) = 0, & \text{on } [0, T] \times \partial \mathcal{D}, \end{cases} \quad (1.1)$$

where $T > 0$, $\mathcal{D} = [0, \pi]^d$ with $d = 1, 2, 3$, f is a polynomial of degree 3 with positive dominant coefficient such as $f = F'$, where $F(u) = (1 - u^2)^2$ is a double equal-well potential. The noise diffusion coefficient σ is a bounded and Lipschitzian function, W is a space-time Brownian sheet defined on some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, and ν is the outward normal vector. The initial condition u_0 is a real-valued function satisfying some assumptions that will be specified later.

The deterministic Cahn-Hilliard equation (i.e., $\sigma \equiv 0$ in (1.1)) was introduced by Cahn and Hilliard in 1958 as a mathematical model of spinodal decomposition for a binary alloy in order to determine the comprising species concentrations when the separation phase take place, see [11]. In this model, the function f is the derivative of the homogeneous free energy F that was given in its original form by

$$F(u) = -\frac{\theta_c}{2}u^2 + \frac{\theta}{2}((1+u)\ln(1+u) - (1-u)\ln(1-u)), \quad -1 < u < 1,$$

where θ_c and θ are respectively proportional to the critical and the quenching temperatures. One can see [17] where it is rigorously justified that F can be replaced in some circumstances by a polynomial of even degree

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with a strictly positive dominant coefficient. For more details on physical aspects of this equation one can see e.g. ([11], [17], [20], [26], [28]).

Over the past three decades, different questions and properties related to the equation (1.1) have been the subject of many works. Indeed, the existence of the solution and of its density was established by Cardon-Weber in [13], a support theorem was showed in [5] by Bo and its co-authors, a Freidlin-Wentzell large deviations principle was obtained by Shi et al. in [32]. Recently, Antonopoulou et al. [1] have attempted to go beyond bounded coefficient noisy term in order to improve the results of [13]. Also, the Cahn-Hilliard equation driven by non Gaussian perturbations was studied in a multitude of setting, we can cite e.g. [4] and [25].

Inspired by the pioneering works [35] and [24] of Freidlin and Wentzell on large deviations for diffusion stochastic processes, growing interest has been paid to this topic during the last three decades. This was thanks to its various applications in many scientific areas. Also, its nonlinear character and its connection with several mathematic theories make it an active field of theoretical researches. And besides a considerable literature about the large deviations for stochastic differential equations (SDE), this aspect has been investigated for the most popular stochastic partial differential equations (SPDEs) and we here cite e.g. [27] for the stochastic heat equation, [12] for the stochastic Burgers equation, [15] for the stochastic wave equation of degree two and [33] for a reaction diffusion equation with non- Gaussian perturbation. Note that in all these works authors used the classical approach of Freidlin and Wentzell that was developed essentially in [2], [30] and [3]. For a complete and deep exposition of the topic of large deviations theory we refer to [18].

Recently, the weak convergence approach introduced by Ellis and Dupuis in [21] and developed in [6], [9] and [10] have gave a new impetus to the study of large deviations both to investigate new random dynamic systems or to revisit and improve anterior results of the point of relaxing assumptions or simplifying the proof. And taking advantage of this approach, many works on various SPDEs has been appeared in last few years. See for a short list e.g. ([8], [19], [23], [31], [29], [34]). The present paper fits into this optic.

It is worth mentioning that the weak convergence approach consists to use a Laplace principle and some variational representations for exponential functionals of infinite dimensional Brownian motion. The proofs are based on showing qualitative properties for controlled versions of the origin processes. This fact unable one to avoid well known difficulties of the classical approach when one wants establish exponential estimates that use approximation and discretization procedure.

In this work we show a large deviations principle for the stochastic Cahn-Hilliard equation in the Höder norm. Thereby, we improve the result of [32] that was given in terms of the uniform convergence topology. Moreover, our proofs are less difficult of that used in [32].

The present paper is organized as follows. Coming section contains basic backgrounds of large deviations theory and well known results about the solution of the equation (1.1). Section 3 gives the general framework of our work. And in the last section, we announce and prove our main result.

2 Preliminaries and main assumptions

In this section we present some assumptions, preliminaries and standard definitions which are needed for the formulation of the problem.

2.1 Large deviations

For a family of random variables $\{X^\varepsilon; \varepsilon > 0\}$ defined on a probability space (Ω, \mathcal{F}, P) and taking values in a Polish space \mathcal{E} , the LDP is concerned with events A for which probabilities $P(X^\varepsilon \in A)$ converges to zero exponentially fast as $\varepsilon \rightarrow 0$. The exponential decay rate of such probabilities are typically expressed in terms of a "rate function" I mapping \mathcal{E} into $[0, \infty]$.

Definition 2.1 *The family of random variables $\{X^\varepsilon; \varepsilon > 0\}$ is said to satisfy the LDP with the good rate function (or action functional) $I : \mathcal{E} \rightarrow [0, \infty]$, on \mathcal{E} , if*

1. For each $M < \infty$ the level set $\{x \in \mathcal{E}; I(x) \leq M\}$ is a compact subset of \mathcal{E} .
2. Large deviation upper bound : for any closed subset F of \mathcal{E}

$$\limsup_{\varepsilon \rightarrow 0^+} \varepsilon \log P(X^\varepsilon \in F) \leq -I(F).$$

3. Large deviation lower bound : for any open subset O of \mathcal{E}

$$\liminf_{\varepsilon \rightarrow 0^+} \varepsilon \log P(X^\varepsilon \in O) \geq -I(O).$$

Where, for $A \subset \mathcal{E}$, we define $I(A) = \inf_{x \in A} I(x)$.

The Freidlin-Wentzell theory [22] describes the path asymptotics, as $\varepsilon \rightarrow 0$, of probabilities of the large deviations of the solutions of small noise finite dimensional Stochastic differential equations (SDEs), away from its law if large number limite. For the case where the driving brownian motion is infinite dimensional, that covers the stochastic partial differential equations (SPDEs), Budhiraja et al. [10] use certain variational representations for infinite dimensional brownian motions (from Boué et al. [6] or also Budhiraja et al. [9]) to give a framework for proving large deviations for a variety of infinite dimensionals systems.

In a many problems one is interested in obtaining exponential estimates on functions which are more general than indicator functions of closed or open sets. This leads to the study of the, so called, Laplace principle.

Definition 2.2 (*Laplace principle*) *The family of random variables $\{X^\varepsilon; \varepsilon > 0\}$ defined on the Polish space \mathcal{E} , is said to satisfy the Laplace principle with rate function I if for any bounded continuous function $h : \mathcal{E} \rightarrow \mathbb{R}$,*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log \mathbf{E} \left(\exp \left[-\frac{1}{\varepsilon} h(X^\varepsilon) \right] \right) = - \inf_{f \in \mathcal{E}} \{h(f) + I(f)\},$$

where \mathbf{E} is the expectation with respect to P .

In [35] and [7], Varadhan and Bryc established an equivalence between LDP and Laplace principle (LP) on a Polish space. In a view of this equivalence, the rest of this paper will be concerned with the study of the Laplace principle.

In the weak convergence approach which is a suitable for the evaluation of integrals appearing in the Laplace principle, the integrals are associated to a variational representation through a family of minimal cost functions. The asymptotic behavior of these minimal cost functions are in turn determined by the weak convergence approach [21]. The classical approach to large deviation used in [32] for this equation, require more technicalities. For instance, the time discretizations required in proving the regularity of the skeleton and the exponential inequalities for the stochastic integral in Hölder norms are also needed. These are usually the most difficult parts if the large deviations analysis based on the standard approximation method.

2.2 Assumptions and mild solution

Letting $q \geq 1$, we define $\|\cdot\|_q$ as the usual norm in $L^q(\mathcal{D})$. Assume the following assumptions :

- (H1) f is a polynomial function of degree 3 with positive dominant coefficient.
- (H2) σ : is a bounded and Lipschitz function.
- (H3) $u_0 \in L^p(\mathcal{D})$ (for some $p \geq 4$) is continuous on \mathcal{D} .
- (H3') u_0 is an γ -Hölder continuous function on \mathcal{D} , $\gamma \in]0, 1]$.

Following J. B. Walsh for parabolic problems [36], and C. Cardon-Weber [13], a rigorous meaning for solution of equation (1.1) can be given by means of the following definition.

Definition 2.3 (*Mild solution*) A jointly measurable and adapted process $\{u(t, x); (t, x) \in [0, T] \times \mathcal{D}\}$ is called a mild solution of (1.1) with initial condition u_0 if it satisfies, for each $t \geq 0$ and a.s. for almost all $x \in \mathcal{D}$ the following evolution equation :

$$\begin{aligned} u^\varepsilon(t, x) &= \int_{\mathcal{D}} G_t(x, y) u_0(y) dy + \sqrt{\varepsilon} \int_0^t \int_{\mathcal{D}} G_{t-s}(x, y) \sigma(u^\varepsilon(s, y)) W(ds, dy) \\ &\quad + \int_0^t \int_{\mathcal{D}} \Delta G_{t-s}(x, y) f(u^\varepsilon(s, y)) ds dy, \end{aligned} \quad (2.2)$$

where $G_t(\cdot, \cdot)$ denotes the Green kernel corresponding to the operator $\frac{\partial}{\partial t} + \Delta^2$ with the Neumann boundary conditions. Note that some useful estimates concerning $G_t(\cdot, \cdot)$ are given in [13].

The following result of C. Cadon-Weber ([13], Theorem 1.3) asserts the existence and uniqueness of a solution to (1.1).

Theorem 2.1 (*Existence, uniqueness and the regularity of solutions*) Under the assumptions (H1)-(H3), there exists a unique solution (in the Walsh's sense) of the equation (1.1) which satisfies

$$\sup_{0 \leq t \leq T} (\mathbf{E} \|u(t, \cdot)\|_p^q)^{1/q} < \infty, \quad (2.3)$$

for $q \geq p$ if $d \in \{1, 2\}$ and for $p \leq q < \frac{6p}{(6-p)^+}$ if $d = 3$. Moreover, under (H1)- (H3'), ([13], Theorem 4.1) gives the a.s. β -Hölder continuous property for the trajectories of the solution with $\beta \leq \frac{\gamma}{4}$ and $\beta < \frac{1}{2}(1 - \frac{d}{4})$.

Theorem 2.2 (*The solution mapping of equation (1.1)*). Assuming (H1)-(H3'). Let $\alpha \leq \frac{\gamma}{4} \wedge \frac{1}{2}(1 - \frac{d}{4})$ and set $\mathcal{E}_0 = L^p(\mathcal{D}) \cap C^\gamma(\mathcal{D})$, for some $p \geq 4, \gamma \in]0, 1]$. There exists a measurable function

$$\mathcal{G}^\varepsilon : \mathcal{E}_0 \times \mathcal{C}([0, T] \times \mathcal{D} : \mathbb{R}) \rightarrow C^\alpha([0, T], L^p(\mathcal{D})),$$

such that, for any filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with a Brownian sheet W as above and $u_0 \in \mathcal{E}_0$, $u^\varepsilon = \mathcal{G}^\varepsilon(u_0, \sqrt{\varepsilon}W)$, is the unique mild solution of (1.1) (with initial condition u_0) and satisfies (2.3).

In order to be able to apply the weak convergence approach for large deviations theory, we need a Polish space setting carrying the probability laws of the family $\{u^\varepsilon(t, x); \varepsilon \in (0, 1], (t, x) \in [0, T] \times \mathcal{D}\}$. And regarding the Hölder property of u we introduce the space $C^\alpha([0, T], L^p(\mathcal{D}))$ endowed with the following norm

$$\|u\|_{\alpha, p} = \sup_{t \in [0, T]} \|u(t, \cdot)\|_p + \sup_{t \neq t', t, t' \in [0, T]} \frac{\|u(t, \cdot) - u(t', \cdot)\|_p}{|t - t'|^\alpha}, \quad (2.4)$$

for $p \geq 4$ and $\alpha \in]0, 1[$. And because our setting requires a Polish space state, we recall that if $\alpha' < \alpha$, then the separable space $C^{\alpha', 0}([0, T], L^p(\mathcal{D}))$ of functions belonging to $C^{\alpha'}([0, T], L^p(\mathcal{D}))$ and satisfying

$$\lim_{\delta \rightarrow 0} \sup_{0 < |t - t'| < \delta, t \neq t'} \frac{\|u(t, \cdot) - u(t', \cdot)\|_p}{|t - t'|^{\alpha'}} = 0$$

is a polish space containing $C^\alpha([0, T], L^p(\mathcal{D}))$. Hence, we can restrict ourselves in all the sequel to the space $\mathcal{E}^\alpha := C^{\alpha, 0}([0, T], L^p(\mathcal{D}))$ for $\alpha < \frac{\gamma}{4} \wedge \frac{1}{2}(1 - \frac{d}{4})$.

3 Framework for the Laplace Principle

The obtention of Laplace principle by the weak convergence approach is based on an important result given by Budhiraja et al. [10].

3.1 Laplace principle of functionals of Brownian sheet.

Consider the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ defined in the introduction, and let \mathcal{E}_0 and \mathcal{E} be Polish spaces such that the initial condition u_0 takes values in a compact subspace of \mathcal{E}_0 . Moreover, let $\{\mathcal{G}^\varepsilon : \mathcal{E}_0 \times \mathcal{C}([0, T] \times \mathcal{D}; \mathbb{R}) \rightarrow \mathcal{E}, \varepsilon > 0\}$ be a family of measurable maps.

For $u_0 \in \mathcal{E}_0$, Define

$$X^{\varepsilon, u_0} := \mathcal{G}^\varepsilon(u_0, \sqrt{\varepsilon}W). \quad (3.5)$$

In the sequel we will give sufficient conditions for the Laplace principle for X^{ε, u_0} to hold uniformly in u_0 for compact subsets of \mathcal{E}_0 .

For $N \in \mathbb{N}$, consider the following

$$S^N = \left\{ \phi \in L^2([0, T] \times \mathcal{D}) : \int_0^T \int_{\mathcal{D}} \phi^2(s, y) ds dy \leq N \right\},$$

which is a compact metric space, equipped with the weak topology on $L^2([0, T] \times \mathcal{D})$.

Let \mathcal{P}_2 be the class of all predictable processes ϕ such that $\int_0^T \int_{\mathcal{D}} \phi^2(s, y) ds dy < \infty$, *a.s.* Also, define

$$\mathcal{P}_2^N = \{v(\omega) \in \mathcal{P}_2 : v(\omega) \in S^N, P - a.s.\},$$

the space of controls.

The following condition is the standing assumption of Theorem 3.1 which states the Laplace principle for the family $\{X^{\varepsilon, u_0}\}_{\varepsilon > 0}$ defined by (3.5). For $u \in L^2([0, T] \times \mathcal{D})$, define $\mathcal{I}(u) \in \mathcal{C}([0, T] \times \mathcal{D} : \mathbb{R})$ as

$$\mathcal{I}(u)(t, x) := \int_0^t \int_0^x u(s, y) ds dy.$$

Assumption (A) : There exists a measurable map $\mathcal{G}^0 : \mathcal{E}_0 \times \mathcal{C}([0, T] \times \mathcal{D}; \mathbb{R}) \rightarrow \mathcal{E}$ such that the following hold :

(A1) For every $M < \infty$ and a compact set $K \subset \mathcal{E}_0$, the set

$$\Gamma_{M, K} := \{\mathcal{G}^0(u_0, \mathcal{I}(u)); u \in S^M, u_0 \in K\}$$

is a compact subset of \mathcal{E} .

(A2) Consider $M < \infty$ and a family $\{v^\varepsilon : \varepsilon > 0\} \subset \mathcal{P}_2^M$, and $\{u_0^\varepsilon\} \subset \mathcal{E}_0$ such that $v^\varepsilon \rightarrow v$ and $u_0^\varepsilon \rightarrow u_0$ in distribution (as S^M -valued random elements) as $\varepsilon \rightarrow 0$. Then

$$\mathcal{G}^\varepsilon(u_0^\varepsilon, \sqrt{\varepsilon}W + \mathcal{I}(v^\varepsilon)) \rightarrow \mathcal{G}^0(u_0, \mathcal{I}(v)),$$

in distribution as $\varepsilon \rightarrow 0$.

For $h \in \mathcal{E}$, and $u_0 \in \mathcal{E}_0$, define the rate function

$$I_{u_0}(h) := \inf_{\{v \in L^2([0, T] \times \mathcal{D}); h = \mathcal{G}^0(u_0, \mathcal{I}(v))\}} \left\{ \frac{1}{2} \int_0^T \int_{\mathcal{D}} v^2(s, y) dy ds \right\}, \quad (3.6)$$

where the infimum over an empty set is taken to be ∞ .

The following theorem is due to proved by Budhiraja et al. ([10], Theorem 7) and states the Laplace principle for the family X^{ε, u_0} .

Theorem 3.1 (Theorem 7 in [10]) *Let $\mathcal{G}^0 : \mathcal{E}_0 \times \mathcal{C}([0, T] \times \mathcal{D}; \mathbb{R}) \rightarrow \mathcal{E}$ be a measurable map satisfying assumption (A). Suppose that for all $h \in \mathcal{E}$, $u_0 \rightarrow I_{u_0}(h)$ is a lower semi-continuous map from \mathcal{E}_0 to $[0, \infty]$. Then for every $u_0 \in \mathcal{E}_0$, $I_{u_0}(h) : \mathcal{E} \rightarrow [0, \infty]$, is a rate function on \mathcal{E} and the family $\{I_{u_0}, u_0 \in \mathcal{E}\}$ of rate functions has compact level sets on compacts. Furthermore, the family X^{ε, u_0} satisfies the Laplace principle on \mathcal{E} with rate function I_{u_0} defined by (3.6), uniformly in u_0 on compact subsets of \mathcal{E}_0 .*

3.2 The controlled and limiting equations for the spde (1.1)

In the context of the spde under our study, $\mathcal{E}_0 = L^p(\mathcal{D}) \cap \mathcal{C}^\gamma(\mathcal{D})$, for some $p \geq 4, \gamma \in]0, 1]$ is the space of the initial condition, and $\mathcal{E} = \mathcal{E}^\alpha := C^{\alpha, 0}([0, T], L^p(\mathcal{D}))$ for $\alpha < \frac{\gamma}{4} \wedge \frac{1}{2}(1 - \frac{d}{4})$ the space of solutions.

The solution map of equation (1.1) is $u^\varepsilon = \mathcal{G}^\varepsilon(u_0, \sqrt{\varepsilon}W)$. Then, for $v \in \mathcal{P}_2^N$, $u^{\varepsilon, v} := \mathcal{G}^\varepsilon(u_0, \sqrt{\varepsilon}W + \mathcal{I}(v))$ is the solution map of the stochastic controlled equation for the spde (1.1) :

$$\frac{\partial u^{\varepsilon, v}}{\partial t}(t, x) = -\Delta(\Delta u^{\varepsilon, v}(t, x) - f(u^{\varepsilon, v}(t, x))) + \sqrt{\varepsilon}\sigma(u^{\varepsilon, v}(t, x))\frac{\partial^2 W}{\partial t \partial x} + \sigma(u^{\varepsilon, v}(t, x))v(t, x),$$

whose mild solution is

$$\begin{aligned} u^{\varepsilon, v}(t, x) &= \int_{\mathcal{D}} G_t(x, y)u_0(y)dy + \sqrt{\varepsilon} \int_0^t \int_{\mathcal{D}} G_{t-s}(x, y)\sigma(u^{\varepsilon, v}(s, y))W(ds, dy) \\ &\quad + \int_0^t \int_{\mathcal{D}} \Delta G_{t-s}(x, y)f(u^{\varepsilon, v}(s, y))dsdy + \int_0^t \int_{\mathcal{D}} G_{t-s}(x, y)\sigma(u^{\varepsilon, v}(s, y))v(s, y)dsdy. \end{aligned} \quad (3.7)$$

Also, define the map $\mathcal{G}^0(u_0, \mathcal{I}(v)) := u^v$, where u^v is the solution of the following zero-noise equation :

$$\begin{aligned} u^v(t, x) &= \int_{\mathcal{D}} G_t(x, y)u_0(y)dy + \int_0^t \int_{\mathcal{D}} G_{t-s}(x, y)\sigma(u^v(s, y))v(s, y)dsdy \\ &\quad + \int_0^t \int_{\mathcal{D}} \Delta G_{t-s}(x, y)f(u^v(s, y))dsdy. \end{aligned} \quad (3.8)$$

The following theorem gives a statement of existence and uniqueness for the solution of the stochastic controlled equation given by (3.7).

Theorem 3.2 (Existence and uniqueness of controlled process) *Assuming (H1) – (H3). Let \mathcal{G}^ε denote the solution mapping, and let $v \in \mathcal{P}_2^N$ for some $N \in \mathbb{N}$. Define*

$$u^{\varepsilon, v} = \mathcal{G}^\varepsilon(u_0, \sqrt{\varepsilon}W + \mathcal{I}(v)),$$

then $u^{\varepsilon, v}$ is the unique solution of equation (3.7), which satisfies

$$\sup_{\varepsilon \leq 1} \sup_{v \in \mathcal{P}_2^N} \sup_{0 \leq t \leq T} \mathbf{E} \left(\|u^{\varepsilon, v}(t, \cdot)\|_p^q \right) < \infty, \quad (3.9)$$

for $q \geq p$ if $d = 1, 2$, and $p \leq q < \frac{6p}{(6-p)^+}$ in the case $d = 3$.

Proof. For $v \in \mathcal{P}_2^N$, set

$$dQ^{\varepsilon,v} := \exp \left\{ -\frac{1}{\sqrt{\varepsilon}} \int_0^t \int_D v(s,y) W(ds, dy) - \frac{1}{2\varepsilon} \int_0^t \int_D v(s,y)^2 ds dy \right\} dP.$$

Since is defined by an exponential martingale, $Q^{\varepsilon,v}$ is a probability measure on Ω . And by Girsanov theorem the process

$$\tilde{W}(dt, dx) = W(dt, dx) + \frac{1}{\sqrt{\varepsilon}} \int_0^t \int_D v(s,y) ds dy$$

is a space-time white noise on the space Ω under the probability measure $Q^{\varepsilon,v}$. Rewriting (3.7) using $\tilde{W}(dt, dx)$ we obtain (2.2) with $\tilde{W}(dt, dx)$ in place of $W(dt, dx)$. Let \mathbf{u} be the unique solution of (2.2) with $\tilde{W}(dt, dx)$ on the space $(\Omega, \mathcal{F}, Q^{\varepsilon,v})$. Then \mathbf{u} satisfies (3.7), $Q^{\varepsilon,v}$ a.s. And by equivalence of probabilities, then \mathbf{u} satisfies (3.7), \mathbb{P} a.s.

For the uniqueness, if \mathbf{u}_1 and \mathbf{u}_2 are two solutions of (3.7) on $(\Omega, \mathcal{F}, \mathbb{P})$, then \mathbf{u}_1 and \mathbf{u}_2 are solutions of (2.2) governed by $\tilde{W}(dt, dx)$ on $(\Omega, \mathcal{F}, Q^{\varepsilon,v})$. By the uniqueness of the solution of (2.2), we obtain $\mathbf{u}_1 = \mathbf{u}_2$, $Q^{\varepsilon,v}$ a.s. And by equivalence of probabilities, we obtain $\mathbf{u}_1 = \mathbf{u}_2$, \mathbb{P} a.s.

Concerning the estimate (3.9), it holds true for the three first terms by the estimations (2.16), (2.17) and (2.35) in [13]. It remains to show it for the last term. Indeed, by the estimation (1.11) in [13], there exists a constant $c > 0$ such that

$$\left\| \int_0^t \int_D G_{t-s}(\cdot, y) \sigma(u^{\varepsilon,v}(s, y)) v(s, y) ds dy \right\|_p \leq c \int_0^t (t-s)^{\frac{d}{4}(\frac{1}{r}-1)} \|\sigma(u^{\varepsilon,v}(s, \cdot)) v(s, \cdot)\|_\rho ds,$$

where $\rho \in [1, p]$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{\rho} + 1$. Using the boundedness of σ , taking $\rho = 2$ in the last inequality and applying Cauchy Schwarz inequality we get a.s.

$$\begin{aligned} \left\| \int_0^t \int_D G_{t-s}(\cdot, y) \sigma(u^{\varepsilon,v}(s, y)) v(s, y) ds dy \right\|_p &\leq \frac{c}{\frac{d}{2}(\frac{1}{r}-1)+1} T^{\frac{d}{2}(\frac{1}{r}-1)+1} \|v\|_{\mathcal{H}_T} \\ &\leq \frac{c}{\frac{d}{2}(\frac{1}{r}-1)+1} T^{\frac{d}{2}(\frac{1}{r}-1)+1} N. \end{aligned} \quad (3.10)$$

Note that, with the condition $p \geq 4$, there exists r satisfying (3.10) that can be taken in $[\frac{4}{3}, 3]$. Then

$$\mathbf{E} \left(\left\| \int_0^t \int_D G_{t-s}(\cdot, y) \sigma(u^{\varepsilon,v}(s, y)) v(s, y) ds dy \right\|_p^q \right) < \infty. \quad (3.11)$$

Hence, (3.9) holds. \square

Remark 3.1 (*Hölder regularity of controlled and limiting processes*) Assuming (H1) – (H3'). Both processes $\{u^{\varepsilon,v}(t, \cdot); t \in [0, T]\}$ and $\{u^v(t, \cdot); t \in [0, T]\}$, defined by (3.7) and (3.8) respectively, live in the space \mathcal{E}^α .

Proof.

The Hölder regularity for these two processes can be obtained by arguing as in the point *ii*) of the proof of Theorem 4.1. \square

For $h \in \mathcal{E}^\alpha$, and $u_0 \in \mathcal{E}_0$, define the rate function

$$I_{u_0}(h) := \inf_v \left\{ \frac{1}{2} \int_0^T \int_D v^2(s, y) dy ds \right\}, \quad (3.12)$$

where the infimum is taken over all $v \in L^2([0, T] \times \mathcal{D})$ such that

$$\begin{aligned}
h(t, x) &= \int_{\mathcal{D}} G_t(x, y) u_0(y) dy + \int_0^t \int_{\mathcal{D}} G_{t-s}(x, y) \sigma(h(s, y)) v(s, y) ds dy \\
&+ \int_0^t \int_{\mathcal{D}} \Delta G_{t-s}(x, y) f(h(s, y)) ds dy.
\end{aligned} \tag{3.13}$$

Note that under assumptions (H1)-(H3), for every $v \in \mathcal{P}_2^N$, the equation (3.13) admits a unique solution which belongs to $C([0, T], L^p(D))$, and moreover

$$\sup_{t \in [0, T]} \|u^v(t, \cdot)\|_p^q < \infty, \tag{3.14}$$

for $q \geq p$ if $d \in \{1, 2\}$ and for $p \leq q \leq \frac{6p}{(6-p)_+}$ if $d = 3$. The proof is omitted since is similar to that of Theorem 3.1 of [13] but by replacing the stochastic integral by the integral containing v .

4 The main result

The main result of this paper is the following :

Theorem 4.1 *Under the assumptions (H1)-(H3'), the law of the solution $\{u^\varepsilon; \varepsilon \in (0, 1]\}$, defined by (2.2), satisfies, on \mathcal{E}^α , a large deviation principle with the rate function I_{u_0} , define by (3.12), uniformly for u_0 in compact subsets of \mathcal{E}_0 .*

In view of Theorem 3.1, to prove Theorem 4.1 it suffices to verify conditions (A1) and (A2).

Remark 4.1 *This result improve that of Shi and al. [32] where the LDP was established by the classical approach in the space $C([0, T]; L^p(\mathcal{D}))$ equipped with the topology of uniform convergence.*

Proof of Theorem 4.1 As mentioned above, here we will show that the conditions (A1) and (A2) hold. In a first time we deal with (A2). That is, we need to show that for all $q \geq p$ we have

$$\|u^{\varepsilon, v^\varepsilon}(t) - u^v(t)\|_{\alpha, p}^q \longrightarrow 0 \text{ in probability as } \varepsilon \longrightarrow 0. \tag{4.15}$$

To do it, we will use a localization argument introduced in [14]. For $M > 0$, define the following event

$$A_\varepsilon^M(t) = \{w \in \Omega; \sup_{s \in [0, t]} \|u^{\varepsilon, v^\varepsilon}(s)\|_p \vee \sup_{s \in [0, t]} \|u^v(s)\|_p \leq M\}.$$

and set

$$Y_\varepsilon(t) := u^{\varepsilon, v^\varepsilon}(t) - u^v(t).$$

Owing to (3.9) and (3.14), we have $\mathbb{P}(A_\varepsilon^M(T)^c) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ and $M \longrightarrow \infty$. Then, by using Lemma A.1 in [14], it suffices to show

i) for all $t \in [0, T]$;

$$\lim_{\varepsilon \rightarrow 0} \mathbf{E} [\mathbf{1}_{A_\varepsilon^M(t)} \|Y_\varepsilon(t)\|_p^q] = 0 \tag{4.16}$$

ii) there exists $\beta > 0$ such that for all $t, t' \in [0, T]$,

$$\sup_{\varepsilon \in]0, 1[} \mathbf{E} [\mathbf{1}_{A_\varepsilon^M(T)} \|Y_\varepsilon(t) - Y_\varepsilon(t')\|_p^q] \leq c |t - t'|^{\beta q}. \tag{4.17}$$

To prove *i*), we write

$$\begin{aligned}
Y_\varepsilon(t) &= \sqrt{\varepsilon} \int_0^t \int_D G_{t-s}(\cdot, y) \sigma(u^{\varepsilon, v^\varepsilon}(s, y)) W(ds, dy) + \int_0^t \int_D \Delta G_{t-s}(\cdot, y) \left[f(u^{\varepsilon, v^\varepsilon}(s, y)) - f(u^v(s, y)) \right] ds dy, \\
&+ \int_0^t \int_D G_{t-s}(\cdot, y) \sigma(u^{\varepsilon, v^\varepsilon}(s, y)) [v^\varepsilon(s, y) - v(s, y)] ds dy \\
&+ \int_0^t \int_D G_{t-s}(\cdot, y) \left[\sigma(u^{\varepsilon, v^\varepsilon}(s, y)) - \sigma(u^v(s, y)) \right] v(s, y) ds dy \\
&= \sum_{i=1}^4 J_i^\varepsilon(t).
\end{aligned}$$

Then

$$\begin{aligned}
\mathbf{E} \left[\mathbf{1}_{A_\varepsilon^M(t)} \|Y_\varepsilon(t)\|_p^q \right] &\leq c \sum_{i=1}^4 \mathbf{E} \left(\mathbf{1}_{A_\varepsilon^M(t)} \|J_i(t)\|_p^q \right) \\
&\leq c \sum_{i=1, i \neq 2}^4 \mathbf{E} \left(\|J_i(t)\|_p^q \right) + c \mathbf{E} \left(\mathbf{1}_{A_\varepsilon^M(t)} \|J_2(t)\|_p^q \right).
\end{aligned} \tag{4.18}$$

For $J_1^\varepsilon(t)$, first we apply Hölder inequality and we get

$$\mathbf{E} \left(\|J_1^\varepsilon(t)\|_p^q \right) = \mathbf{E} \left(\int_D |J_1^\varepsilon(t, x)|^p dx \right)^{\frac{q}{p}} \leq c \int_D \mathbf{E} |J_1^\varepsilon(t, x)|^q dx. \tag{4.19}$$

Later we use Burkholder inequality, the boundedness of σ and estimation (4.16)

$$\begin{aligned}
\mathbf{E} |J_1^\varepsilon(t, x)|^q &\leq c \varepsilon^{\frac{q}{2}} \left(\int_0^t \int_D G_{t-s}^2(x, y) \sigma^2(u^{\varepsilon, v^\varepsilon}(s, y)) ds dy \right)^{q/2} \\
&\leq c \varepsilon^{\frac{q}{2}} \left(\int_0^t \int_D G_{t-s}^2(x, y) ds dy \right)^{q/2} \\
&< c \varepsilon^{\frac{q}{2}}.
\end{aligned} \tag{4.20}$$

Concerning $J_2^\varepsilon(t)$, using (3.16) in [4] and Hölder inequality we get for $1 \leq \rho \leq p$ and $1 < \gamma \leq q$

$$\begin{aligned}
\mathbf{E} \left(\mathbf{1}_{A_\varepsilon^M(t)} \|J_2^\varepsilon(t)\|_p^q \right) &\leq c \mathbf{E} \left(\mathbf{1}_{A_\varepsilon^M(t)} \int_0^t \|f(u^{\varepsilon, v^\varepsilon}(s, \cdot)) - f(u^v(s, \cdot))\|_\rho^\gamma ds \right)^{\frac{q}{\gamma}}, \\
&\leq c \mathbf{E} \left(\int_0^t \mathbf{1}_{A_\varepsilon^M(s)} \|f(u^{\varepsilon, v^\varepsilon}(s, \cdot)) - f(u^v(s, \cdot))\|_\rho^q ds \right).
\end{aligned}$$

Note that, for the last inequality, taking in account the fact that $A_\varepsilon^M(t) \subset A_\varepsilon^M(s)$ for $0 \leq s \leq t$, we have used the following upper estimate

$$\left| \mathbf{1}_{A_\varepsilon^M(t)} \int_0^t \int_D \phi(s, y) dy ds \right| \leq \left| \int_0^t \int_D \mathbf{1}_{A_\varepsilon^M(s)} \phi(s, y) dy ds \right|,$$

for $t \in [0, T]$ and for a measurable function $\phi : \Omega \times [0, T] \times D \rightarrow \mathbb{R}$, one can see Remark 3.2. in [14]. Since f is a polynomial function of degree 3, we can write

$$\begin{aligned}
\|f(u^{\varepsilon, v^\varepsilon}(s, \cdot)) - f(u^v(s, \cdot))\|_\rho^q &\leq c \left[\|u^{\varepsilon, v^\varepsilon}(s, \cdot) - u^v(s, \cdot)\|_\rho^q + \|u^{\varepsilon, v^\varepsilon}(s, \cdot)^2 - u^v(s, \cdot)^2\|_\rho^q \right. \\
&\quad \left. + \|(u^{\varepsilon, v^\varepsilon}(s, \cdot))^3 - u^v(s, \cdot)^3\|_\rho^q \right].
\end{aligned}$$

Taking $\rho = \frac{p}{3}$, we have

$$\begin{aligned} \|u^{\varepsilon, v^\varepsilon}(s, \cdot) - u^v(s, \cdot)\|_\rho^q &\leq c \|u^{\varepsilon, v^\varepsilon}(s, \cdot) - u^v(s, \cdot)\|_p^q; \\ \|u^{\varepsilon, v^\varepsilon}(s, \cdot)^2 - u^v(s, \cdot)^2\|_\rho^q &\leq c \|u^{\varepsilon, v^\varepsilon}(s, \cdot) - u^v(s, \cdot)\|_p^q \left(\|u^{\varepsilon, v^\varepsilon}(s, \cdot)\|_p^q + \|u^v(s, \cdot)\|_p^q \right); \\ \|u^{\varepsilon, v^\varepsilon}(s, \cdot)^3 - u^v(s, \cdot)^3\|_\rho^q &\leq c \|u^{\varepsilon, v^\varepsilon}(s, \cdot) - u^v(s, \cdot)\|_p^q \\ &\quad \times \left(\|u^{\varepsilon, v^\varepsilon}(s, \cdot)\|_p^q + \|u^{\varepsilon, v^\varepsilon}(s, \cdot)\|_p^q \|u^v(s, \cdot)\|_p^q + \|u^v(s, \cdot)\|_p^q \right). \end{aligned}$$

Then

$$\mathbf{E}(\mathbf{1}_{A_\varepsilon^M}(t) \|J_2^\varepsilon(t)\|_p^q) \leq c \int_0^T \mathbf{E}(\mathbf{1}_{A_\varepsilon^M}(s) \|Y_\varepsilon(s)\|_p^q) ds. \quad (4.21)$$

For $J_3^\varepsilon(t)$, Holder inequality, the boundedness of σ and the Cauchy schwarz inequality yield

$$\begin{aligned} \mathbf{E}(\|J_3^\varepsilon(t)\|_p^q) &\leq c \int_D \mathbf{E}|J_3(t, x)|^q dx \\ &\leq c \int_D \mathbf{E} \left(\int_0^t \int_D G_{t-s}(x, y) |v^\varepsilon(s, y) - v(s, y)| ds dy \right)^q dx \\ &\leq c \sup_{x \in D} \left(\int_0^t \int_D G_{t-s}^2(x, y) dy \right)^q \mathbf{E}(\|v^\varepsilon - v\|_{\mathcal{H}_T}^q) \\ &\leq c \mathbf{E}(\|v^\varepsilon - v\|_{\mathcal{H}_T}^q). \end{aligned} \quad (4.22)$$

For $J_4^\varepsilon(t)$, the same arguments as before yield that a.s we have

$$\begin{aligned} \|J_4^\varepsilon(t)\|_p^q &\leq \|v\|_{\mathcal{H}_T}^q \left(\int_D \left(\int_0^t \int_D G_{t-s}^2(x, y) [\sigma(u^{\varepsilon, v^\varepsilon}(s, y)) - \sigma(u^v(s, y))]^2 ds dy \right)^{\frac{p}{2}} dx \right)^{q/p} \\ &\leq N^q \left(\int_D \left[\left(\int_0^t \int_D G_s^2(x, y) ds dy \right)^{\frac{p}{2}-1} \int_0^t \int_D G_{t-s}^2(x, y) |Y_\varepsilon(s, y)|^p ds dy \right] dx \right)^{q/p} \\ &\leq c \left(\int_D \left[\int_0^t \int_D G_{t-s}^2(x, y) |u^{\varepsilon, v^\varepsilon}(s, y) - u^v(s, y)|^p ds dy \right] dx \right)^{q/p} \\ &\leq c \left(\int_0^t \left(\int_D G_{t-s}^2(x, y) dx \right) \int_D |Y_\varepsilon(s, y)|^p dy ds \right)^{q/p} \\ &\leq c \left(\int_0^t (t-s)^{-\frac{d}{4}} \|Y_\varepsilon(s, y)\|_p^p ds \right)^{q/p} \\ &\leq c \left(\int_0^t (t-s)^{-\frac{d}{4}} \right)^{\frac{q}{p}-1} \int_0^t (t-s)^{-\frac{d}{4}} \|Y_\varepsilon(s, y)\|_p^q ds \\ &\leq c \int_0^t (t-s)^{-\frac{d}{4}} \|Y_\varepsilon(s)\|_p^q ds. \end{aligned} \quad (4.23)$$

Hence, combining (4.18)-(4.23) we obtain

$$\mathbf{E}(\mathbf{1}_{A_\varepsilon^M}(t) \|Y_\varepsilon(t)\|_p^q) \leq c \left(\varepsilon^{\frac{q}{2}} + \mathbf{E}(\|v^\varepsilon - v\|_{\mathcal{H}_T}^q) + \int_0^T \left(1 + (t-s)^{-\frac{d}{4}} \right) \mathbf{E}(\mathbf{1}_{A_\varepsilon^M}(s) \|Y_\varepsilon(s)\|_p^q) ds \right).$$

We obtain (4.16) by applying a version of Gronwall lemma given by (Lemma 15, [16]).

To prove *ii*), consider $t; t' \in [0, T]$ such that $t < t'$. We have

$$Y_\varepsilon(t) - Y_\varepsilon(t') = (u^{\varepsilon, v^\varepsilon}(t) - u^{\varepsilon, v^\varepsilon}(t')) - (u^v(t) - u^v(t')).$$

Then

$$\mathbf{E} \left(\mathbf{1}_{A_\varepsilon^M(T)} \|Y_\varepsilon(t) - Y_\varepsilon(t')\|_p^q \right) \leq \mathbf{E} \left(\mathbf{1}_{A_\varepsilon^M(T)} \|u^{\varepsilon, v^\varepsilon}(t) - u^{\varepsilon, v^\varepsilon}(t')\|_p^q \right) + \mathbb{P}(A_\varepsilon^M(T)) \|u^v(t) - u^v(t')\|_p^q.$$

At beginning we deal with the first term and we write (3.7) as

$$u^{\varepsilon, v^\varepsilon}(t, x) = I_0^{\varepsilon, v^\varepsilon}(t, x) + I_1^{\varepsilon, v^\varepsilon}(t, x) + I_2^{\varepsilon, v^\varepsilon}(t, x) + I_3^{\varepsilon, v^\varepsilon}(t, x),$$

where $I_i^{\varepsilon, v^\varepsilon}(t, x)$ stands for the i -th term in (3.7). By Lemma 2.2. in [13], there exists a constant $c > 0$ such that

$$\|I_0^{\varepsilon, v^\varepsilon}(t) - I_0^{\varepsilon, v^\varepsilon}(t')\|_p^q \leq c|t - t'|^{\frac{q\gamma}{4}}. \quad (4.24)$$

The same reference gives the existence of $\beta > 0$ such that $0 < \beta < \frac{1}{2} \left(1 - \frac{d}{4}\right)$ such that

$$\mathbf{E} \left(\|I_1^{\varepsilon, v^\varepsilon}(t) - I_1^{\varepsilon, v^\varepsilon}(t')\|_p^q \right) \leq c\sqrt{\varepsilon}|t - t'|^{\beta q}. \quad (4.25)$$

Concerning $I_2^{\varepsilon, v^\varepsilon}$ we have

$$\begin{aligned} I_2^{\varepsilon, v^\varepsilon}(t') - I_2^{\varepsilon, v^\varepsilon}(t) &= \int_0^{t'} \int_D \Delta[G_{t'-s}(\cdot, y) - G_{t-s}(\cdot, y)]f(u^{\varepsilon, v^\varepsilon}(s, y))dsdy \\ &\quad + \int_t^{t'} \int_D \Delta G_{t'-s}(\cdot, y)f(u^{\varepsilon, v^\varepsilon}(s, y))dsdy \\ &\equiv I_{2,1}^{\varepsilon, v^\varepsilon}(t, t') + I_{2,2}^{\varepsilon, v^\varepsilon}(t, t'). \end{aligned}$$

By (1.12) in [13], the Hö inequality and the estimation (3.9) there exists $1 \leq \rho \leq p$ and $\kappa \in [0, 1]$ such that

$$\begin{aligned} \mathbf{E} \left(\|I_{2,2}^{\varepsilon, v^\varepsilon}(t, t')\|_p^q \right) &\leq c\mathbf{E} \left(\int_0^{t'-t} (t' - t - s)^{-\frac{1}{2} + \frac{d}{4}(\kappa-1)} \|f(u^{\varepsilon, v^\varepsilon}(t+s, \cdot))\|_\rho ds \right)^q \\ &\leq c|t' - t|^{q\left(\frac{1}{2} + \frac{d}{4}(\kappa-1)\right)} \int_0^{t'-t} (t' - t - s)^{-\frac{1}{2} + \frac{d}{4}(\kappa-1)} \mathbf{E} \left(\|f(u^{\varepsilon, v^\varepsilon}(t+s, \cdot))\|_\rho^q \right) ds \\ &\leq c|t' - t|^{(q+1)\left(\frac{1}{2} + \frac{d}{4}(\kappa-1)\right)}. \end{aligned} \quad (4.26)$$

Using (3.14) in [4], Hölder inequality and the estimation (3.9), there exist $\theta \in]0, \frac{1}{2} + \frac{d}{4}(\kappa-1)[$ and $\gamma \in \left] \frac{1}{\frac{1}{2} + \frac{d}{4}(\kappa-1)} - \theta, q \right[$ such that

$$\begin{aligned} \mathbf{E} \left(\|I_{2,1}^{\varepsilon, v^\varepsilon}(t, t')\|_p^q \right) &\leq c|t' - t|^{\theta q} \mathbf{E} \left(\|f(u^{\varepsilon, v^\varepsilon}(\cdot, *))\|_{L^\gamma([0, T], L^\rho(D))}^q \right) \\ &\leq c|t' - t|^{\theta q}. \end{aligned} \quad (4.27)$$

Concerning $I_3^{\varepsilon, v^\varepsilon}$ we have

$$\begin{aligned} I_3^{\varepsilon, v^\varepsilon}(t') - I_3^{\varepsilon, v^\varepsilon}(t) &= \int_0^t \int_D [G_{t'-s}(\cdot, y) - G_{t-s}(\cdot, y)]\sigma(u^{\varepsilon, v^\varepsilon}(s, y))v^\varepsilon(s, y)dsdy \\ &\quad + \int_t^{t'} \int_D G_{t'-s}(\cdot, y)\sigma(u^{\varepsilon, v^\varepsilon}(s, y))v^\varepsilon(s, y)dsdy \\ &\equiv I_{3,1}^{\varepsilon, v^\varepsilon}(t, t') + I_{3,2}^{\varepsilon, v^\varepsilon}(t, t'). \end{aligned}$$

By Cauchy-Schwarz inequality, the fact that $\|v^\varepsilon\|_{\mathcal{H}_T} \leq N$ a.s. and by Lemma 1.8. in [13] we obtain the existence of $\eta > 0$ such that

$$\mathbf{E} \left(\|I_{3,i}^{\varepsilon, v^\varepsilon}(t, t')\|_p^q \right) \leq c|t' - t|^{\eta q}, \quad (4.28)$$

for $i = 1, 2$.

Therefore, by (4.24), (4.25), (4.27) and (4.28) we obtain (4.17) for the first term. And arguing similarly and using the estimation (3.14) we obtain (4.17) for the second term. Hence, the condition (A2) is checked.

Concerning (A1), it will be a consequence of the continuity of the mapping $h : \mathcal{H}_T^N \rightarrow \mathcal{E}^\alpha$ with respect to the weak topology. It consists to consider $v, (v_n) \subset \mathcal{H}_T^N$ such that for any $g \in \mathcal{H}_T^N$,

$$\lim_{n \rightarrow +\infty} \langle v - v_n, g \rangle_{\mathcal{H}_T^N} = 0,$$

and to prove

$$\lim_{n \rightarrow +\infty} \|u^{v_n} - u^v\|_{\alpha, p} = 0. \quad (4.29)$$

The proof will be omitted since we can proceed as in (A2) and by using the following estimate

$$\sup_{\|v\| \leq N} \sup_{t \in [0, T]} \|u^v(t)\|_p < \infty, \quad (4.30)$$

which follows from Lemma 3.1. in [32].

Finally, since the conditions (A1) and (A2) are held, the proof of the Theorem 4.1 is completed. \square

5 Appendix

We recall here some useful results that we have used in the proofs of our result. The following lemma gives well-known estimates on space and time increments for the Green function G associated to the Cahn-Hilliard operator. For the proof, we refer to [13].

Lemma 5.1 *There exists positive constants c, γ and γ' satisfying $\gamma < 4 - d$, $\gamma \leq 2$ and $\gamma' < 1 - \frac{d}{4}$ such that for all $y, z \in \mathcal{D}$, $0 \leq s < t \leq T$ and $0 \leq h \leq t$ we have*

1.

$$\int_0^t \int_{\mathcal{D}} |G_r(x, y) - G_r(x, z)|^2 dr dy \leq c|y - z|^\gamma \quad (5.31)$$

2.

$$\int_0^t \int_{\mathcal{D}} |G_{r+h}(x, y) - G_r(x, y)|^2 dr dy \leq c|h|^{\gamma'} \quad (5.32)$$

3.

$$\int_s^t \int_{\mathcal{D}} |G_r(x, y)|^2 ds dy \leq c|t - s|^{\gamma'} \quad (5.33)$$

The following lemma is a version of the Garsia-Rademich-Rumsay lemma. For the proof, we refer to [14] and references therein.

Lemma 5.2 *let $(Y_n)_n$ be a sequence of $C^\alpha([0, T]; L^p([0, 1]))$ valued stochastic processes, let (τ_n) be a sequence of stopping times and let $p \in]1, +\infty[$ such that*

1. For any $t \in [0, T]$,

$$\lim_{n \rightarrow +\infty} \mathbb{E} (1_{\{t \leq \tau_n\}} \|Y_n(t, \cdot)\|_p^q) = 0.$$

2. There exists $\gamma > 0$ such that for any $(t, t') \in [0, T]$

$$\sup_n \mathbb{E} \left(\mathbf{1}_{\{t \vee t' \leq \tau_n\}} \|Y_n(t, \cdot) - Y_n(t', \cdot)\|_p^q \right) \leq c |t - t'|^{\gamma+d}.$$

Then, for $1 \leq r < p$ and $\theta < \frac{\gamma}{p}$ one has

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left(\mathbf{1}_{\{t \leq \tau_n\}} \|Y_n(t, \cdot)\|_{\theta, p, \tau_n}^r \right) = 0;$$

where

$$\|u\|_{\theta, p, \tau} := \sup_{t \in [0, T \wedge \tau]} \|u(t)\|_p + \sup_{t \neq t', t, t' \in [0, T \wedge \tau]} \frac{\|u(t) - u(t')\|_p}{|t - t'|^\theta},$$

for a stopping time τ .

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