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# LOCAL CONTROLLABILITY OF THE KORTEWEG-DE VRIES EQUATION WITH THE RIGHT DIRICHLET CONTROL

HOAI-MINH NGUYEN

ABSTRACT. The Korteweg-de Vries (KdV) equation with the right Dirichlet control was initially investigated by Glass and Guerrero. They showed that this system is small time locally exactly controllable for all non critical lengths and its linearized system is not controllable for all critical lengths. Even the controllability of the KdV system has been studied extensively in the last two decades, the local controllability of this system for critical lengths remains an open question. In this paper, we give a definitive answer to this question. More precisely, we show that the KdV equation with the right Dirichlet control is *finite time* locally exactly controllable but *not small time* locally null controllable for *all* critical lengths.

**Key words:** Controllability, KdV equations, critical lengths, unreachable space, power series expansion, Hilbert uniqueness method

**AMS subject classification:** 35C20, 35Q53, 93B05, 93B07, 93C20.

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## 1. INTRODUCTION AND THE STATEMENT OF THE MAIN RESULTS

This paper is devoted to the local controllability of the Korteweg-de Vries (KdV) system using the right Dirichlet control. More precisely, we consider the following control problem, for  $T > 0$ ,

$$(1.1) \quad \begin{cases} y_t + y_x + y_{xxx} + yy_x = 0 & \text{in } (0, T) \times (0, L), \\ y(\cdot, 0) = y_x(\cdot, L) = 0 & \text{in } (0, T), \\ y(\cdot, L) = u & \text{in } (0, T), \\ y(0, \cdot) = y_0 & \text{in } (0, L). \end{cases}$$

Here  $y$  is the state,  $y_0 \in L^2(0, L)$  is an initial datum, and  $u$  is a control, which belongs to an appropriate functional space. The KdV equation has been introduced by Boussinesq [11] and Korteweg and de Vries [27] as a model for propagation of surface water waves along a channel. This equation also furnishes a very useful nonlinear approximation model including a balance between a weak nonlinearity and weak dispersive effects, see e.g. [45, 33, 25]. The KdV equation has been investigated from various aspects of mathematics, including the well-posedness, the existence and stability of solitary waves, the integrability, the long-time behavior, etc., see e.g. [45, 33, 25, 43, 29].

**1.1. State of the art.** The local controllability for the KdV equation has been studied extensively in the literature, see, e.g., the surveys [40, 14] and the references therein. We briefly review here some results concerning boundary controls. When the controls are  $y(\cdot, 0)$ ,  $y(\cdot, L)$ ,  $y_x(\cdot, L)$ , Russell and Zhang [42] proved that the KdV equation is small time locally controllable. The case of left boundary control ( $y(\cdot, L) = y_x(\cdot, L) = 0$ ) was investigated by Rosier [39] (see also [22]). The local null controllability holds for small time in this case. The exact controllability does not hold for initial and final data in the  $L^2(0, L)$  due to the regularizing effect but holds for a subclass of infinitely smooth initial and final data [31].

A very close setting with the one considered here is the setting in which one controls the right Neumann boundary, i.e.,  $y(\cdot, 0) = y(\cdot, L) = 0$  and  $y_x(\cdot, L)$  is a control. For initial and final data in  $L^2(0, L)$ , and controls in  $L^2(0, T)$ , Rosier [38] proved that the KdV system with the right Neumann control is small time locally exactly controllable provided that the length  $L$  is not critical, i.e.,  $L \notin \mathcal{N}_N$ <sup>1</sup>, where<sup>2</sup>

$$(1.2) \quad \mathcal{N}_N := \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}}; k, l \in \mathbb{N}_* \right\}.$$

To this end, Rosier studied the controllability of the corresponding linearized system and showed that the linearized system is exactly controllable if  $L \notin \mathcal{N}_N$ . He as well established that when  $L \in \mathcal{N}_N$ , the linearized system is not controllable. More precisely, Rosier showed that there exists a non-trivial finite-dimensional subspace  $\mathcal{M}_N$  of  $L^2(0, L)$  such that its orthogonal space is reachable from 0 for small time whereas  $\mathcal{M}_N$  is not for any time. To tackle the control problem for a critical length  $L \in \mathcal{N}_N$  with initial and final data in  $L^2(0, L)$  and controls in  $L^2(0, T)$ , Coron and Crépeau introduced the power series expansion method [19]. The idea is to take into account the effect of the nonlinear term  $yy_x$  absent in the corresponding linearized system. Using this method, Coron and Crépeau showed [19] (see also [18, section 8.2]) that the KdV system is small-time locally exactly controllable when  $\dim \mathcal{M}_N = 1$ . Cerpa [13] developed the analysis in [19] to prove that the KdV system is *finite time* locally exactly controllable in the case  $\dim \mathcal{M}_N = 2$ . Later, Crépeau and Cerpa [15] succeeded to extend the ideas in [13] to obtain the local exact controllability in finite time for all other critical lengths. Recently, with Coron and Koenig [21], we prove that such a system is not small time locally controllable for a class of critical lengths. This fact is surprising when

<sup>1</sup>The letter  $N$  stands for the Neumann boundary control.

<sup>2</sup> $\mathbb{N}_* = \mathbb{N} \setminus \{0\}$ .

compared with known results on internal controls for the KdV system. It is known, see [12, 32, 37], that the KdV system (1.1) with  $u = 0$  is small time locally controllable using internal controls *whenever* the control region contains an *arbitrary* open subset of  $(0, L)$ . The power expansion method is also a starting point of the analysis in [21]. Part of the of the analysis there is to characterize all controls which brings 0 at 0 to 0 at time T for the corresponding linearized system. This idea is then used in the study of water tank problem [20]. It is interesting to note that there are other types of boundary controls for the KdV equation for which there is no critical length, see [38, 39, 23, 14]. There are also results on internal controllability for the KdV equation, see [42, 28, 12] and the references therein. A minimal time of the null-controllability is also required for some linear partial differential equations. This is obviously the case for equations with a finite speed of propagation, such as the transport equation, the wave equation, or hyperbolic systems, see, e.g., [18] and the references therein. But this can also happen for equations with infinite speed of propagation, such as some parabolic systems [7], Grushin-type equations [2, 6, 26, 3], Kolmogorov-type equations [4], and the references therein.

We now turn back to the control problem (1.1). This control problem was first investigated by Glass and Guerrero [23]. To this end, in the spirit of Rosier's work mentioned above, they introduced the corresponding set of critical lengths<sup>3</sup>

$$(1.3) \quad \mathcal{N}_D = \left\{ L \in \mathbb{R}_+; \exists z_1, z_2 \in \mathbb{C} : (1.4) \text{ holds} \right\},$$

where

$$(1.4) \quad z_1 e^{z_1} = z_2 e^{z_2} = -(z_1 + z_2) e^{-(z_1 + z_2)} \quad \text{and} \quad L^2 = -(z_1^2 + z_1 z_2 + z_2^2).$$

They proved that the set  $\mathcal{N}_D$  is infinite and has no accumulation point. Concerning (1.1), Glass and Guerrero proved that the corresponding linearized KdV system is small time exactly controllable with initial and final data in  $H^{-1}(0, L)$  using controls in  $L^2(0, T)$  if  $L \notin \mathcal{N}_D$ . Developing this result, they also established that the KdV system (1.1) is small time locally controllable for initial and final data in  $L^2(0, L)$  and controls in  $H^{1/6-}$ <sup>4</sup> for non-critical lengths, i.e.,  $L \notin \mathcal{N}_D$ .

Even the local controllability of the KdV system using boundary controls has been investigated extensively in the last two decades, to our knowledge, there is no result on the local controllability of the system (1.1) for critical lengths. In particular, the answer to the question whether or not the KdV system (1.1) is locally controllable for a critical length  $L \in \mathcal{N}_D$  raised from the work of Glass and Guerrero is open (see, e.g., [14, Open question 4]). In comparison with the set of critical lengths  $\mathcal{N}_N$ , the set  $\mathcal{N}_D$  is less explicit. Moreover, the unreachable space for the linearized system related to (1.1) has not been determined due to the lack of an appropriate observability inequality for the linearized system in a suitable functional setting.

**1.2. Statement of the main results.** The main goal of this paper is to give a rather complete picture of the local controllability of the control KdV system (1.1).

Set

$$(1.5) \quad X_T := C([0, T]; L^2(0, L)) \cap L^2((0, T); H^1([0, L]))$$

equipped with the corresponding norm:

$$\|y\|_{X_T} = \|y\|_{C([0, T]; L^2(0, L))} + \|y\|_{L^2((0, T); H^1([0, L]))}.$$

The first main result is on the finite time local controllability property of (1.1).

<sup>3</sup>The letter  $D$  stands for the Dirichlet boundary control.

<sup>4</sup>Controls in  $H^{1/6-}$  means controls in  $H^{1/6-\varepsilon}$  for all  $\varepsilon > 0$

**Theorem 1.1.** *Let  $L \in \mathcal{N}_D$ . There exist two constants  $T_L > 0$  and  $C_L > 0$  depending only on  $L$  such that for every  $T > T_L$ , there exists  $\varepsilon_0 = \varepsilon_0(L, T) > 0$  such that for all  $y_0, y_1 \in L^2(0, L)$  with  $\|y_0\|_{L^2(0, L)} \leq \varepsilon_0$  and  $\|y_1\|_{L^2(0, L)} \leq \varepsilon_0$ , there exists  $u \in H^{1/3}(0, T)$  with  $\|u\|_{H^{1/3}(0, T)} \leq C_{L, T}(\|y_0\|_{L^2(0, L)} + \|y_1\|_{L^2(0, L)})$  such that*

$$y(T, \cdot) = y_1,$$

where  $y \in X_T$  is the unique solution of (1.1).

Recall that  $\mathcal{N}_D$  is defined in (1.3).

The second main result is on the local controllability property of (1.1) in small time.

**Theorem 1.2.** *Let  $L \in \mathcal{N}_D$ . There exist  $T_* > 0$ ,  $\varepsilon_0 > 0$ , and  $\psi \in H_0^1(0, L) \cap C^\infty([0, L])$  such that for all  $0 < T < T_*$  and for all solutions  $y \in X_T$  of*

$$(1.6) \quad \begin{cases} y_t + y_x + y_{xxx} + yy_x = 0 & \text{in } (0, T) \times (0, L), \\ y(\cdot, 0) = y_x(\cdot, L) = 0 & \text{in } (0, T), \\ y(\cdot, L) = u & \text{in } (0, T), \\ y(0, \cdot) = y_0 := \varepsilon\psi & \text{in } (0, L), \end{cases}$$

with  $0 < \varepsilon < \varepsilon_0$  and  $\|u\|_{H^{1/3}(0, T)} < \varepsilon_0$ , we have

$$y(T, \cdot) \neq 0.$$

**Remark 1.1.** Taking controls in  $H^{1/3}(0, T)$  is very natural for the control system (1.1) with initial and final data in  $L^2(0, L)$ . In fact, this is the right functional space if one considers the solutions in the (energy) space  $X_T$ .

As a consequence of Theorem 1.1 and Theorem 1.2, the KdV system (1.1) is *finite time* locally exactly controllable but *not small time* locally controllable for initial and final data in  $L^2(0, L)$  using controls in  $H^{1/3}(0, T)$  for *all* critical lengths. We thus give a complete answer to [14, Open question 4]. These results are surprising when compared with known results on internal controls for the KdV system. The system (1.1) with  $u = 0$  is locally controllable using internal controls *whenever* the control region contains an *arbitrary* open subset of  $(0, L)$  as mentioned previously. These results are also distinct with the control properties of the KdV system with the Neumann control on the right for which the system is small time locally controllable when  $\dim \mathcal{M}_N = 1$ , see [19]. Nevertheless, these facts are consistent in the sense that the dimension of the unreachable space for the linearized system of (1.1) is always even (a fact proved later in this paper, see (6.4)).

In comparison the results given in this paper with the ones in [21], both works deal with the local controllability of the KdV equation using boundary controls. The right Dirichlet boundary control is considered in this paper. We prove here that the local null controllability in small time does not hold for solutions with small norms in the functional *energy space*  $X_T$  and for *all* critical lengths. In [21], with Coron and Koenig, we considered the right Neumann boundary control. We established there that the small time local null controllability does not hold in a more regular space than  $X_T$  and for a class of critical lengths whose unreachable space is of dimension 2. The local controllability in finite time with the right Neumann control holds for small solutions in  $X_T$  for all critical lengths [19, 13, 15]. Nevertheless, whether or not the small time local controllability of the KdV system with the right Neumann control hold either for small solutions in  $X_T$  or (and) for all corresponding critical lengths whose dimension of the unreachable space is strictly greater than 1 is still open (see the list of open questions in [21]).

Finally, in the case  $L \notin \mathcal{N}_D$ , we can prove the following result, which sharpens the results in [23] mentioned previously.

**Theorem 1.3.** *Let  $L \notin \mathcal{N}_D$  and  $T > 0$ . There exist  $C > 0$  and  $\varepsilon_0 > 0$  depending only on  $L$  and  $T$  such that for all  $y_0, y_1 \in L^2(0, L)$  with  $\|y_0\|_{L^2(0, L)} \leq \varepsilon_0$  and  $\|y_1\|_{L^2(0, L)} \leq \varepsilon_0$ , there exists  $u \in H^{1/3}(0, T)$  with  $\|u\|_{H^{1/3}(0, T)} \leq C(\|y_0\|_{L^2(0, L)} + \|y_1\|_{L^2(0, L)})$  such that*

$$y(T, \cdot) = y_1,$$

where  $y \in X_T$  is the unique solution of (1.1).

**1.3. Ideas of the proofs.** Our approach is inspired by the power series expansion method introduced by Coron and Crépeau [19]. The idea of this method is to search/understand a control  $u$  of the form

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \dots$$

The corresponding solution then formally has the form

$$y = \varepsilon y_1 + \varepsilon^2 y_2 + \dots,$$

and the non-linear term  $yy_x$  can be written as

$$yy_x = \varepsilon^2 y_1 y_{1,x} + \dots$$

One then obtains the following systems for  $y_1$  and  $y_2$ :

$$(1.7) \quad \begin{cases} y_{1,t} + y_{1,x} + y_{1,xxx} = 0 & \text{in } (0, T) \times (0, L), \\ y_1(\cdot, 0) = y_{1,x}(\cdot, L) = 0 & \text{in } (0, T), \\ y_1(\cdot, L) = u_1 & \text{in } (0, T), \end{cases}$$

$$(1.8) \quad \begin{cases} y_{2,t} + y_{2,x} + y_{2,xxx} + y_1 y_{1,x} = 0 & \text{in } (0, T) \times (0, L), \\ y_2(\cdot, 0) = y_{2,x}(\cdot, L) = 0 & \text{in } (0, T), \\ y_2(\cdot, L) = u_2 & \text{in } (0, T). \end{cases}$$

The idea is then to find the corresponding controls  $u_1$  and  $u_2$  such that, if  $y_1(0) = y_2(0) = 0$ , then  $y_1(T) = 0$  and the  $L^2$ -orthogonal projection of  $y_2(T)$  on  $\mathcal{M}_D$  is a given (non-zero) element in  $\mathcal{M}_D$ . To this end, in [19, 13, 15], the authors used delicate contradiction arguments to capture the structure of the KdV systems.

We here follow and extend the analysis in [21]. The starting point of the analysis is also the power series expansion method. The strategy is to characterize all possible  $u_1$  which steers 0 at time 0 to 0 at time  $T$  (see Proposition 4.1). This is done by taking the Fourier transform with respect to time of the solution  $y_1$  and applying Paley-Wiener's theorem. We then investigate the projection of  $y_2$  into the unreachable space  $\mathcal{M}_D$  (in Section 4). As a result, for all critical lengths  $L \in \mathcal{N}_D$ , if the control time is sufficiently large, then one can reach all directions in  $\mathcal{M}_D$  via  $y_2$  and if the control time is sufficiently small, then there are directions in  $\mathcal{M}_D$  which cannot be reached via  $y_2$ . Using these facts, we then implement arguments to prove the local exact controllability in finite time and disprove the local exact controllability in small time.

Here are some other comments on the analysis. First, the arguments used to establish the fact that one can reach all directions in  $\mathcal{M}_D$  via  $y_2$  for time large enough is as in [13] when  $\dim \mathcal{M}_D = 2$  using the rotation idea. When  $\dim \mathcal{M}_D > 2$ , the arguments are more involved and different since the phases might be complex. Second, the arguments used in this paper to disprove the small time local controllability improve the ones in [21] and yield the result in the energy space  $X_T$ . The proof also relies on the positivity of a scalar product which comes naturally in the study of the controllability of the KdV system for small time. The positivity of a scalar product for small time was also the crucial point of several lack of small time local controllability results for systems with infinite propagation speed, nevertheless the previous ways to derive it are different, see, e.g., [17, 30, 5]. Finally, the analysis for the well-posedness of (1.1) is in the spirit of the work of Coron,

Koenig, and Nguyen [21]. This partly involves a connection between the linear KdV equation and the linear KdV-Burgers equation as previously used by Bona et al. [9] and inspired by the work of Bourgain [10], and Molinet and Ribaud [34]. Nevertheless, new estimates requiring new insights are derived in order to allow us to get the observability inequality for the critical lengths, which are useful in determining the unreachable space for the linearized KdV system (see Remark 2.3).

**1.4. Organisation of the paper.** The paper is organized as follows. In Section 2, we establish several properties for the linearized KdV equations equipped with various boundary conditions. The main results there are Proposition 2.1 and Proposition 2.2. In Section 3, we present the unreachable space and establish the corresponding observability inequality (Lemma 3.1). The main result of this section is Proposition 3.1. In Section 4, we study properties of the controls which steer 0 at time 0 to 0 at time  $T$ . Using results in Section 4, we study attainable directions for small time in Section 5. The main result of this section is Proposition 5.1, which is one of the most important ingredients of the proof of Theorem 1.1 and Theorem 1.2. Theorem 1.1, Theorem 1.2, and Theorem 1.3 are given in Section 6, Section 7, and Section 8, respectively. Several technical results are given and proved in the appendix.

## 2. ON THE LINEARIZED KdV EQUATIONS

In this section, we establish two well-posedness results (Proposition 2.1 and Proposition 2.2) for the linearized KdV equation equipped with two different types of boundary conditions in the energy space  $X_T$  defined in (1.5). These results are the starting point of our analysis and play a role in deriving the unreachable space and establishing the corresponding observability inequality in Section 3 (Proposition 3.1).

Here is the first main result of this section.

**Proposition 2.1.** *Let  $T > 0$ ,  $(h_1, h_2, h_3) \in H^{1/3}(0, T) \times H^{1/3}(0, T) \times L^2(0, T)$ ,  $f \in L^1((0, T); L^2(0, L))$ , and  $y_0 \in L^2(0, L)$ . There exists a unique solution  $y \in X_T$  of the system*

$$(2.1) \quad \begin{cases} y_t + y_x + y_{xxx} = f & \text{in } (0, T) \times (0, L), \\ y(\cdot, 0) = h_1, y(\cdot, L) = h_2, y_x(\cdot, L) = h_3 & \text{in } (0, T), \\ y(0, \cdot) = y_0 & \text{in } (0, L). \end{cases}$$

Moreover, for  $x \in [0, L]$ ,

$$(2.2) \quad \begin{aligned} & \|y\|_{X_T} + \|y(\cdot, x)\|_{H^{1/3}(0, T)} + \|y_x(\cdot, x)\|_{L^2(0, T)} + \|y_{xx}(\cdot, x)\|_{[H^{1/3}(0, T)]^*} \\ & \leq C_{T, L} \left( \|y_0\|_{L^2(0, L)} + \|f\|_{L^1((0, T); L^2(0, L))} + \|(h_1, h_2)\|_{H^{1/3}(0, T)} + \|h_3\|_{L^2(0, T)} \right), \end{aligned}$$

where  $C_{T, L}$  denotes a positive constant independent of  $x$ ,  $y_0$ ,  $f$ , and  $h_1, h_2, h_3$ .

**Remark 2.1.** In Proposition 2.1, we implicitly admit that

$$y \in C([0, L]; H^{1/3}(0, T)), \quad y_x \in C([0, L]; L^2(0, T)), \quad \text{and} \quad y_{xx} \in C([0, L]; H^{-1/3}(0, T)).$$

These facts are derived from the proof. Results and their analysis which are related to Proposition 2.1 will be discussed in Remark 2.3. The estimates for  $\|y_x(\cdot, x)\|_{L^2(0, T)}$  and  $\|y_{xx}(\cdot, x)\|_{[H^{1/3}(0, T)]^*}$  are used in establishing the observability inequality.

The next result is on the well-posedness of the linearized KdV system for which the Dirichlet condition and the second derivative in  $x$  on the right are described.

**Proposition 2.2.** *Let  $T > 0$ ,  $(h_1, h_2, h_3) \in H^{1/3}(0, T) \times H^{1/3}(0, T) \times [H^{1/3}(0, T)]^*$ ,  $f \in L^1((0, T); L^2(0, L))$ , and  $y_0 \in L^2(0, L)$ . Then there exists a unique solution  $y \in X_T$  of the system*

$$(2.3) \quad \begin{cases} y_t + y_x + y_{xxx} = f & \text{in } (0, T) \times (0, L), \\ y(\cdot, 0) = h_1, y(\cdot, L) = h_2, y_{xx}(\cdot, L) = h_3 & \text{in } (0, T), \\ y(0, \cdot) = y_0 & \text{in } (0, L). \end{cases}$$

Moreover, for  $0 \leq x \leq L$ ,

$$(2.4) \quad \|y\|_{X_T} + \|y(\cdot, x)\|_{H^{1/3}(0, T)} + \|y_x(\cdot, x)\|_{L^2(0, T)} + \|y_{xx}(\cdot, x)\|_{[H^{1/3}(0, T)]^*} \\ \leq C_{T, L} \left( \|y_0\|_{L^2(0, L)} + \|f\|_{L^1((0, T); L^2(0, L))} + \|(h_1, h_2)\|_{H^{1/3}(0, T)} + \|h_3\|_{[H^{1/3}(0, T)]^*} \right),$$

for some positive constant  $C_{T, L}$  independent of  $x$ ,  $y_0$ ,  $f$ , and  $(h_1, h_2, h_3)$ .

As a consequence of Proposition 2.1, we obtain the following well-posedness result for (1.1).

**Proposition 2.3.** *Let  $L > 0$  and  $T > 0$ . There exists  $\varepsilon_0 > 0$  such that for  $y_0 \in L^2(0, L)$ ,  $u \in H^{1/3}(0, T)$ , and  $f \in L^1((0, T); L^2(0, L))$  with  $\|y_0\|_{L^2(0, L)} + \|f\|_{L^1((0, T); L^2(0, L))} + \|u\|_{H^{1/3}(0, T)} < \varepsilon_0$ , there exists a unique solution  $y \in X_T$  of the system*

$$(2.5) \quad \begin{cases} y_t + y_x + y_{xxx} + yy_x = f & \text{in } (0, T) \times (0, L), \\ y(\cdot, 0) = y_x(\cdot, L) = 0 & \text{in } (0, T), \\ y(\cdot, L) = u & \text{in } (0, T), \\ y(0, \cdot) = y_0 & \text{in } (0, L). \end{cases}$$

Moreover, we have

$$(2.6) \quad \|y\|_{X_T} + \|y(\cdot, x)\|_{H^{1/3}(0, T)} + \|y_x(\cdot, x)\|_{L^2(0, T)} + \|y_{xx}(\cdot, x)\|_{[H^{1/3}(0, T)]^*} \\ \leq C_{T, L} \left( \|y_0\|_{L^2(0, L)} + \|f\|_{L^1((0, T); L^2(0, L))} + \|u\|_{H^{1/3}(0, T)} \right).$$

The rest of this section consisting of four subsections is organized as follows. In the first subsection, we present several results on the KdV-Burgers equations, which are used in the proof of Proposition 2.1 and Proposition 2.2. The proof of Proposition 2.1, Proposition 2.2, and Proposition 2.3 are given in the last three subsections, respectively.

**2.1. On the linear KdV-Burgers equations.** This section is devoted to the study of the linear KdV-Burgers equations with the periodic boundary condition. Here is the main result of this section.

**Proposition 2.4.** *Let  $g \in L^1(\mathbb{R}_+; L^2(0, L))$  with  $\int_0^L g(t, x) dx = 0$  for a.e.  $t > 0$ , and let  $y_0 \in L^2(0, L)$  be such that  $\int_0^L y_0(x) dx = 0$ . There exists a unique solution  $y \in C([0, +\infty); L^2(0, L)) \cap L^2([0, +\infty); H^1([0, L]))$  which is periodic in space of the system*

$$(2.7) \quad \begin{cases} y_t + 4y_x + y_{xxx} - 3y_{xx} = g & \text{in } \mathbb{R}_+ \times (0, L), \\ y(0, \cdot) = y_0 & \text{in } (0, L). \end{cases}$$



Moreover, for  $t \in \mathbb{R}_+$  and  $x \in [0, L]$ ,

$$(2.8) \quad \|y(t, \cdot)\|_{L^2(0,L)} + \|y(\cdot, x)\|_{H^{1/3}(\mathbb{R}_+)} + \|y_x(\cdot, x)\|_{L^2(\mathbb{R}_+)} + \|y_{xx}(\cdot, x)\|_{[H^{1/3}(\mathbb{R})]^*} \\ \leq C \left( \|y_0\|_{L^2(0,L)} + \|g\|_{L^1(\mathbb{R}_+; L^2(0,L))} \right),$$

for some positive constant  $C$  depending only on  $L$ .

*Proof.* We only derive the estimates. The uniqueness follows from the estimates and the existence follows the proof of these estimates as well.

Multiplying the equation by  $y$  and integrating by parts, we have

$$\frac{1}{2} \frac{d}{dt} \int_0^L |y(t, x)|^2 dx + 3 \int_0^L |y_x(t, x)|^2 dx = \int_0^L g(t, x) y(t, x) dx.$$

This yields

$$\frac{1}{2} \frac{d}{dt} \int_0^L |y(t, x)|^2 dx \leq \|g(t, \cdot)\|_{L^2(0,L)} \|y(t, \cdot)\|_{L^2(0,L)}.$$

Applying the Grönwall lemma, we obtain the desired estimate for  $\|y(t, \cdot)\|_{L^2(0,L)}$ .

We next establish the estimates for  $\|y(\cdot, x)\|_{H^{1/3}(\mathbb{R}_+)}$ ,  $\|y_x(\cdot, x)\|_{L^2(\mathbb{R}_+)}$ , and  $\|y_{xx}(\cdot, x)\|_{[H^{1/3}(\mathbb{R})]^*}$ . For notational ease, we assume that  $L = 2\pi$ . It suffices to consider the case  $g \equiv 0$  and the case  $y_0 \equiv 0$  separately.

We first consider the case  $g \equiv 0$ . Write the solution under the form

$$y(t, x) = \sum_{n \in \mathbb{Z}} a_n(t) e^{inx} \text{ in } \mathbb{R}_+ \times (0, L) = \mathbb{R}_+ \times (0, 2\pi).$$

We then derive that

$$y_0(x) = \sum_{n \in \mathbb{Z}} a_n(0) e^{inx} \text{ for } x \in [0, 2\pi]$$

and

$$a'_n(t) = (-3n^2 - i(4n - n^3)) a_n(t) \text{ for } t \in \mathbb{R}_+.$$

We thus have

$$a_n(t) = e^{(-3n^2 - i(4n - n^3))t} a_n(0) \text{ for } t \in \mathbb{R}_+, n \in \mathbb{Z}.$$

Since  $\int_0^L y_0 = 0$ , it follows that  $a_0 = 0$ . This in turn implies that  $a_0(t) = 0$  for  $t \in \mathbb{R}_+$ . We thus obtain

$$y(t, x) = \sum_{n \in \mathbb{Z} \setminus \{0\}} e^{inx} e^{(-3n^2 - i(4n - n^3))t} a_n(0) \text{ in } \mathbb{R}_+ \times (0, 2\pi).$$

Extend  $y(t, x)$  for  $t < 0$  by

$$(2.9) \quad y(t, x) = \sum_{n \in \mathbb{Z} \setminus \{0\}} e^{inx} e^{(3n^2 - i(4n - n^3))t} a_n(0) \text{ in } \mathbb{R}_- \times (0, 2\pi),$$

and still denote this extension by  $y(t, x)$ . We have, for  $z \in \mathbb{R}$ ,

$$(2.10) \quad \int_{\mathbb{R}} y(t, x) e^{-itz} dt = \int_0^\infty e^{(-3n^2 - i(4n - n^3))t} e^{-itz} dt + \int_{-\infty}^0 e^{(3n^2 - i(4n - n^3))t} e^{-itz} dt \\ = \frac{1}{3n^2 + i(4n - n^3 + z)} + \frac{1}{3n^2 - i(4n - n^3 + z)} = \frac{6n^2}{9n^4 + (4n - n^3 + z)^2}.$$

This implies

$$(2.11) \quad \|y(\cdot, x)\|_{H^{1/3}(\mathbb{R}_+)}^2 \leq C \int_{z \in \mathbb{R}} (1 + |z|)^{2/3} \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{n^2 |a_n(0)|}{n^4 + (z + 4n - n^3)^2} \right)^2.$$

Applying Lemma 2.1 below with  $s = 1$ , we obtain

$$(2.12) \quad \|y(\cdot, x)\|_{H^{1/3}(\mathbb{R}_+)}^2 \leq C \|y_0\|_{L^2(0, 2\pi)}^2.$$

Similarly, we have

$$(2.13) \quad \|y_x(\cdot, x)\|_{L^2(\mathbb{R}_+)}^2 \leq C \int_{z \in \mathbb{R}} \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{|n|^3 |a_n(0)|}{(z + 4n - n^3)^2 + n^4} \right)^2$$

and

$$(2.14) \quad \|y_{xx}(\cdot, x)\|_{H^{-1/3}(\mathbb{R}_+)}^2 \leq C \int_{z \in \mathbb{R}} (1 + |z|)^{-2/3} \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{n^4 |a_n(0)|}{(z + 4n - n^3)^2 + n^4} \right)^2.$$

Applying Lemma 2.1 below with  $s = 0$  and  $s = -1$ , we get

$$(2.15) \quad \|y_x(\cdot, x)\|_{L^2(\mathbb{R}_+)}^2 \leq C \|y_0\|_{L^2(0, 2\pi)}^2.$$

and

$$(2.16) \quad \|y_{xx}(\cdot, x)\|_{H^{-1/3}(\mathbb{R}_+)}^2 \leq C \|y_0\|_{L^2(0, 2\pi)}^2.$$

The proof in the case  $g \equiv 0$  is complete.

We next deal with the case  $y_0 \equiv 0$ . The proof in this case can be derived from the previous case as follows. For  $t > 0$ , let  $W(t)$  be the mapping which maps  $y_0$  to  $y(t, \cdot)$  with  $g \equiv 0$ . We then have

$$(2.17) \quad y(t, x) = \int_0^t W(t-s)g(s, x) ds.$$

This implies

$$(2.18) \quad y_x(t, x) = \int_0^t \partial_x(W(t-s)g(s, x)) ds = \int_0^\infty \mathbf{1}_{(0,t)}(s) \partial_x(W(t-s)g(s, x)) ds.$$

Hence

$$\begin{aligned} \|y_x(\cdot, x)\|_{L_t^2(\mathbb{R}_+)} &\leq \int_0^{+\infty} \|\mathbf{1}_{(0,t)}(s) \partial_x(W(t-s)g(s, x))\|_{L_t^2(\mathbb{R}_+)} ds \\ &= \int_0^{+\infty} \left( \int_0^\infty |\mathbf{1}_{(0,t)}(s) \partial_x(W(t-s)g(s, x))|^2 dt \right)^{1/2} ds \\ &= \int_0^{+\infty} \left( \int_s^\infty |\partial_x(W(t-s)g(s, x))|^2 dt \right)^{1/2} ds. \end{aligned}$$

By the previous case, we have

$$\left( \int_s^\infty |\partial_x(W(t-s)g(s, x))|^2 dt \right)^{1/2} \leq C \|g(s, x)\|_{L_x^2(0, L)}.$$

We thus obtain

$$(2.19) \quad \|y_x(\cdot, x)\|_{L_t^2(\mathbb{R}_+)} \leq C \int_0^{+\infty} \|g(s, x)\|_{L_x^2} ds.$$

By the same arguments, we have

$$(2.20) \quad \begin{aligned} \|y_{xx}(\cdot, x)\|_{H_t^{-1/3}(\mathbb{R}_+)} &\leq \int_0^{+\infty} \|\mathbb{1}_{(0,t)}(s)\partial_{xx}(W(t-s)g(s,x))\|_{H_t^{-1/3}(\mathbb{R}_+)} ds \\ &= \int_0^{+\infty} \|\partial_{xx}(W(t-s)g(s,x))\|_{H_t^{-1/3}(s,+\infty)} ds \leq C \int_0^{+\infty} \|g(s,x)\|_{L_x^2(0,L)} ds. \end{aligned}$$

Similarly, we obtain

$$(2.21) \quad \begin{aligned} \|y(\cdot, x)\|_{H_t^{1/3}(\mathbb{R}_+)} &\leq \int_0^{+\infty} \|\mathbb{1}_{(0,t)}(s)W(t-s)g(s,x)\|_{H_t^{1/3}(\mathbb{R}_+)} ds \\ &= \int_0^{+\infty} \|W(t-s)g(s,x)\|_{H_t^{1/3}(s,+\infty)} ds \leq C \int_0^{+\infty} \|g(s,x)\|_{L_x^2(0,L)} ds. \end{aligned}$$

The conclusion in the case  $y_0 \equiv 0$  now follows from (2.19), (2.20), and (2.21).

The proof is complete.  $\square$

**Remark 2.2.** The proof gives as well that

$$y \in C([0, L]; H^{1/3}(0, +\infty)), \quad y_x \in C([0, L]; L^2(0, +\infty)), \quad \text{and} \quad y_{xx} \in C([0, L]; H^{-1/3}(0, +\infty)).$$

In the proof of Proposition 2.4, we used the following result in the spirit of Bourgain's.

**Lemma 2.1.** *Let  $(a_n)_{n \in \mathbb{Z} \setminus \{0\}} \subset \mathbb{R}$  be such that  $\sum_{n \in \mathbb{Z} \setminus \{0\}} a_n^2 < +\infty$ . For  $-5/2 < s \leq 3$ , there exists a positive constant  $C = C_s$  such that*

$$(2.22) \quad \int_{z \in \mathbb{R}} (1 + |z|)^{2s/3} \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{|a_n| |n|^{3-s}}{(z + 4n - n^3)^2 + n^4} \right)^2 dz \leq C \sum_{n \in \mathbb{Z} \setminus \{0\}} a_n^2.$$

*Proof.* For  $s > -5/2$  and for  $-1 \leq z \leq 1$ , one has

$$\sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{|a_n| |n|^{3-s}}{(z + 4n - n^3)^2 + n^4} \leq C \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} a_n^2 \right)^{1/2}.$$

It hence suffices to prove that, for  $-5/2 < s \leq 3$ ,

$$(2.23) \quad \int_{z \in \mathbb{R}; |z| > 1} (1 + |z|)^{2s/3} \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{|a_n| |n|^{3-s}}{(z + 4n - n^3)^2 + n^4} \right)^2 dz \leq C \sum_{n \in \mathbb{Z} \setminus \{0\}} a_n^2.$$

We have

$$(2.24) \quad \begin{aligned} \int_{z > 1} (1 + |z|)^{2s/3} \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{|a_n| |n|^{3-s}}{(z + 4n - n^3)^2 + n^4} \right)^2 dz \\ \leq C \sum_{m \in \mathbb{N}_*} \int_{m^3}^{(m+1)^3} m^{2s} \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{|a_n| |n|^{3-s}}{(z + 4n - n^3)^2 + n^4} \right)^2 dz. \end{aligned}$$

For  $m^3 \leq z \leq (m+1)^3$ , one gets

$$\begin{aligned}
(2.25) \quad & \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{|a_n| |n|^{3-s}}{(z+4n-n^3)^2 + n^4} \\
&= \sum_{k; m+k \in \mathbb{Z} \setminus \{0\}} \frac{|m+k|^{3-s} |a_{m+k}|}{(z+4(m+k)-(m+k)^3)^2 + (m+k)^4} \\
&\leq C \sum_{\substack{k; |k| \leq 2m \\ m+k \in \mathbb{Z} \setminus \{0\}}} \frac{|m+k|^{3-s} |a_{m+k}|}{m^4 (|k|+1)^2} + C \sum_{\substack{k; |k| \geq 2m+1 \\ m+k \in \mathbb{Z} \setminus \{0\}}} \frac{|k|^{3-s} |a_{m+k}|}{k^4 (|k|+1)^2},
\end{aligned}$$

Combining (2.24) and (2.25) yields

$$\begin{aligned}
(2.26) \quad & \int_{z>1} (1+|z|)^{2s/3} \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{|a_n| |n|^{3-s}}{(z+4n-n^3)^2 + n^4} \right)^2 dz \\
&\leq C \sum_{m \in \mathbb{N}_*} \left( \sum_{\substack{k; |k| \leq 2m \\ m+k \in \mathbb{Z} \setminus \{0\}}} \frac{|m+k|^{3-s} |a_{m+k}|}{m^{3-s} (|k|+1)^2} + C \sum_{\substack{k; |k| \geq 2m+1 \\ m+k \in \mathbb{Z} \setminus \{0\}}} \frac{|m|^{1+s} |a_{m+k}|}{|k|^{1+s} (|k|+1)^2} \right)^2 \\
&\leq C \sum_{m \in \mathbb{N}_*} \left( \sum_{\substack{k \\ m+k \in \mathbb{Z} \setminus \{0\}}} \frac{|a_{m+k}|^2}{(|k|+1)^{1+\varepsilon}} \right) \left( \sum_{\substack{k; |k| \leq 2m \\ m+k \in \mathbb{Z} \setminus \{0\}}} \frac{|m+k|^{6-2s}}{m^{6-2s} (|k|+1)^{3-\varepsilon}} + \sum_{\substack{k; |k| \geq 2m+1 \\ m+k \in \mathbb{Z} \setminus \{0\}}} \frac{m^{2+2s}}{|k|^{2+2s} (|k|+1)^{3-\varepsilon}} \right).
\end{aligned}$$

Here  $0 < \varepsilon \leq 1$  is fixed such that  $s > (\varepsilon - 5)/2$ . We have, for  $m \in \mathbb{N}_*$ ,  $|k| \leq 2m$ , and  $m+k \in \mathbb{Z} \setminus \{0\}$ ,

$$\frac{|m+k|^{6-2s}}{m^{6-2s}} \leq C \text{ for } s \leq 3,$$

and, for  $m \in \mathbb{N}_*$ ,

$$\sum_{\substack{k; |k| \geq 2m+1 \\ m+k \in \mathbb{Z} \setminus \{0\}}} \frac{m^{2+2s}}{|k|^{2+2s} (|k|+1)^{3-\varepsilon}} = m^{2+2s} \sum_{\substack{k; |k| \geq 2m+1 \\ m+k \in \mathbb{Z} \setminus \{0\}}} \frac{1}{k^{2+2s} (|k|+1)^{3-\varepsilon}} \leq \frac{C}{m^{2-\varepsilon}} \leq C \text{ for } s > (\varepsilon - 5)/2.$$

Using the fact, for  $m \in \mathbb{N}_*$ ,

$$\sum_{k; m+k \in \mathbb{Z} \setminus \{0\}} \frac{1}{(|k|+1)^2} < +\infty,$$

we derive from (2.26) that

$$\begin{aligned}
(2.27) \quad & \int_{z>1} (1+|z|)^{2s/3} \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{|a_n| |n|^{3-s}}{(z+4n-n^3)^2 + n^4} \right)^2 dz \\
&\leq C \sum_{m \in \mathbb{N}_*} \sum_{k; m+k \in \mathbb{Z} \setminus \{0\}} \frac{|a_{m+k}|^2}{(|k|+1)^2} \leq C \sum_{k \in \mathbb{Z}} \frac{1}{(|k|+1)^2} \sum_{m; m+k \in \mathbb{Z} \setminus \{0\}} |a_{m+k}|^2,
\end{aligned}$$

which yields

$$(2.28) \quad \int_{z>1} (1+|z|)^{2s/3} \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{|a_n| |n|^{3-s}}{(z+4n-n^3)^2 + n^4} \right)^2 dz \leq C \sum_{n \in \mathbb{Z} \setminus \{0\}} |a_n|^2.$$

Similarly, we have

$$(2.29) \quad \int_{z<-1} (1+|z|)^{2s/3} \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{|a_n| |n|^{3-s}}{(z+4n-n^3)^2 + n^4} \right)^2 dz \leq C \sum_{n \in \mathbb{Z} \setminus \{0\}} |a_n|^2.$$

Estimate (2.23) now follows from (2.28) and (2.29). The proof is complete.  $\square$

**2.2. Proof of Proposition 2.1.** The uniqueness is standard. The existence follows from the construction of a solution given below.

We next deal with the estimates. Extend  $f$  and  $h_1, h_2,$  and  $h_3$  by 0 for  $t > T$  and still denote  $y$  the corresponding solution. The proof of (2.2) is given into two steps.

*Step 1:* We first consider the case where  $y_0 = 0$  and  $f = 0$ . By the linearity and the uniqueness of the system, it suffices to consider the three cases  $(h_1, h_2, h_3) = (0, 0, h_3)$ ,  $(h_1, h_2, h_3) = (h_1, 0, 0)$ , and  $(h_1, h_2, h_3) = (0, h_2, 0)$  separately.

In what follows, for an appropriate function  $v$  defined on  $\mathbb{R}_+ \times (0, L)$ , we extend  $v$  by 0 on  $\mathbb{R}_- \times (0, L)$  and we denote by  $\hat{v}$  its Fourier transform with respect to  $t$ , i.e., for  $z \in \mathbb{C}$ ,

$$\hat{v}(z, x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} v(t, x) e^{-izt} dt.$$

Extend  $y$  and  $f$  by 0 for  $t < 0$  and still denote these extension by  $y$  and  $f$ . Then

$$(2.30) \quad y_t + y_x + y_{xxx} = 0 \text{ in } \mathbb{R} \times (0, L).$$

Taking the Fourier transform with respect to  $t$ , we obtain, for  $z \in \mathbb{R}$ ,

$$(2.31) \quad iz\hat{y} + \hat{y}_x + \hat{y}_{xxx} = 0 \text{ in } (0, L).$$

For  $z \in \mathbb{C}$ , let  $\lambda_1, \lambda_2, \lambda_3$  be the three solutions of the equation  $\lambda^3 + \lambda + iz = 0$ . Set

$$P = P(z) = \sum_{j=1}^3 \lambda_j (e^{\lambda_j+2L} - e^{\lambda_j+L}) = \det Q_D(z), \text{ where } Q_D(z) := \begin{pmatrix} 1 & 1 & 1 \\ e^{\lambda_1 L} & e^{\lambda_2 L} & e^{\lambda_3 L} \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix},$$

and

$$(2.32) \quad \Xi = \Xi(z) := -(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_3) = \det \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix},$$

with the convention  $\lambda_{j+3} = \lambda_j$  for  $j \geq 1$ . It is useful to note that  $P(z)\Xi(z)$  is an analytic function in  $z$ . Denote

$$(2.33) \quad H(z) = \frac{P(z)}{\Xi(z)} \text{ in } \mathbb{C}.$$

Then  $H$  is an analytic function and  $H$  only has a finite number of zeros on the real line, see [21, Lemma B1].

We first consider the case  $(h_1, h_2, h_3) = (0, 0, h_3)$ . We extend  $h_3$  by 0 for  $t > T$  and still denote this extension by  $h_3$ . By Lemma C.1 in the appendix, there exists  $g_3 \in C^\infty(\mathbb{R})$  with

supp  $g_3 \subset [T, 3T]$  such that if  $z$  is a real solution of the equation  $H(z) = 0$  of order  $m$  then  $z$  is also a real solution of order  $m$  of  $\hat{h}_3(z) - \hat{g}_3(z)$ , and, for  $k \geq 1$ ,

$$(2.34) \quad \|g_3\|_{H^k(\mathbb{R})} \leq C_k \|h_3\|_{L^2(\mathbb{R})}.$$

We now establish (2.2). Let  $y_3$  be the solution of (2.44) where  $(h_1, h_2, h_3)$  are  $(0, 0, h_3 - g_3)$ , and  $f = 0$ , and  $y_0 = 0$ . We have, by taking into account the boundary conditions,

$$(2.35) \quad \hat{y}_3(z, x) = \frac{\hat{h}_3(z) - \hat{g}_3(z)}{\det Q_D(z)} \sum_{j=1}^3 (e^{\lambda_{j+2}L} - e^{\lambda_{j+1}L}) e^{\lambda_j x} \text{ for a.e. } x \in (0, L).$$

We have, by the choice of  $g_3$ , for  $z \in \mathbb{R}$  and  $|z| \leq \gamma$ ,

$$(2.36) \quad \left| \frac{\hat{h}_3(z) - \hat{g}_3(z)}{H(z)} \right| \left| \frac{1}{\Xi(z)} \sum_{j=1}^3 (e^{\lambda_{j+2}L} - e^{\lambda_{j+1}L}) e^{\lambda_j x} \right| \leq C_{T,\gamma} \|h_3 - g_3\|_{L^2(\mathbb{R})},$$

and, by Lemma 2.2 below, for  $z \in \mathbb{R}$ ,  $|z| \geq \gamma$  with sufficiently large  $\gamma$ ,

$$(2.37) \quad \left| \frac{1}{\det Q_D} \sum_{j=1}^3 (e^{\lambda_{j+2}L} - e^{\lambda_{j+1}L}) e^{\lambda_j x} \right| \leq \frac{C}{(1 + |z|)^{1/3}}.$$

Combining (2.36) and (2.37) yields

$$\|y_3(\cdot, x)\|_{H^{1/3}(\mathbb{R})} \leq C_{T,L} \|h_3 - g_3\|_{L^2(\mathbb{R})}.$$

Similarly, we have

$$\|y_{3,x}(\cdot, x)\|_{L^2(\mathbb{R})} + \|y_{3,xx}(\cdot, x)\|_{H^{-1/3}(\mathbb{R})} \leq C_{T,L} \|h_3 - g_3\|_{L^2(\mathbb{R})}.$$

The estimates for  $\|y(\cdot, x)\|_{H^{1/3}(0,T)}$ ,  $\|y_{3,x}(\cdot, x)\|_{L^2(0,T)}$ , and  $\|y_{3,xx}(\cdot, x)\|_{[H^{1/3}(0,T)]^*}$  follow by noting that  $y = y_3$  in  $(0, T) \times (0, L)$ .

We also have, by an integration by parts, for  $0 \leq \tau_1 < \tau_2 \leq T$ ,

$$\begin{aligned} & \frac{1}{2} \int_0^L |y(\tau_2, x)|^2 dx - \frac{1}{2} \int_0^L |y(\tau_1, x)|^2 dx + \frac{1}{2} \int_{\tau_1}^{\tau_2} (|y(t, L)|^2 - |y(t, 0)|^2) dt \\ & + \int_{\tau_1}^{\tau_2} (y_{xx}(t, L)y(t, L) - y_{xx}(t, 0)y(t, 0)) dt - \frac{1}{2} \int_{\tau_1}^{\tau_2} (|y_x(t, L)|^2 - |y_x(t, 0)|^2) dt = 0. \end{aligned}$$

Using the estimates for  $\|y(\cdot, x)\|_{H^{1/3}(0,T)}$ ,  $\|y_{3,x}(\cdot, x)\|_{L^2(0,T)}$ , and  $\|y_{3,xx}(\cdot, x)\|_{[H^{1/3}(0,T)]^*}$ , we obtain the one for  $\|y\|_{X_T}$ .

The proof in the case  $(h_1, h_2, h_3) = (h_1, 0, 0)$  or in the case  $(h_1, h_2, h_3) = (0, h_2, 0)$  is similar after noting Lemma A.1 in the appendix. We mention here that the solution corresponding to the triple  $(h_1, 0, 0)$  is given by

$$\hat{y}(z, x) = \frac{\hat{h}_1(z)}{\det Q_D} \sum_{j=1}^3 (\lambda_{j+2} - \lambda_{j+1}) e^{\lambda_j(x-L)} \text{ for a.e. } x \in (0, L),$$

and the solution corresponding to the triple  $(0, h_2, 0)$  is given by

$$\hat{y}(z, x) = \frac{\hat{h}_2(z)}{\det Q_D} \sum_{j=1}^3 (\lambda_{j+1} e^{\lambda_{j+1}L} - \lambda_{j+2} e^{\lambda_{j+2}L}) e^{\lambda_j x} \text{ for a.e. } x \in (0, L).$$

The details are left to the reader.

*Step 2:* We now deal with the general case. The starting point of the proof is a connection between the linearized KdV equation and the linear KdV-Burgers equation. Set  $v(t, x) = e^{-2t+x}y(t, x)$ , which is equivalent to  $y(t, x) = e^{2t-x}v(t, x)$ . One can check that if  $y$  satisfies the equation

$$y_t + y_x + y_{xxx} = f \text{ in } \mathbb{R}_+ \times (0, L),$$

then it holds

$$v_t + 4v_x + v_{xxx} - 3v_{xx} = fe^{-2t+x} \text{ in } \mathbb{R}_+ \times (0, L).$$

Set, in  $\mathbb{R}_+ \times (0, L)$ ,

$$(2.38) \quad \psi(t, x) = \psi(t) := \frac{1}{L} \int_0^L f(t, \xi) e^{-2t+\xi} d\xi \quad \text{and} \quad g(t, x) := f(t, x) e^{-2t+x} - \psi(t, x).$$

Then

$$\int_0^L g(t, x) dx = 0.$$

Let  $y_1 \in C([0, +\infty); L^2(0, L)) \cap L_{\text{loc}}^2([0, +\infty); H^1(0, L))$  be the unique solution which is periodic in space of the system

$$(2.39) \quad y_{1,t} + 4y_{1,x} + y_{1,xxx} - 3y_{1,xx} = g \text{ in } (0, +\infty) \times (0, L),$$

and

$$(2.40) \quad y_1(0, \cdot) = y_0 e^x \text{ in } (0, L).$$

Set

$$\alpha = \frac{1}{L} \int_0^L y_1(0, x) dx$$

By Proposition 2.4, we have, for  $x \in [0, L]$ ,

$$(2.41) \quad \|y_1(\cdot, x) - \alpha\|_{H^{1/3}(\mathbb{R}_+)} + \|y_{1,x}(\cdot, x)\|_{L^2(\mathbb{R}_+)} + \|y_{1,xx}(\cdot, x)\|_{H^{-1/3}(\mathbb{R})} \leq C \|f\|_{L^1(\mathbb{R}_+; L^2(0, L))}.$$

Let  $y_2 \in X_T$  be the unique solution of

$$\begin{cases} y_{2,t} + y_{2,x} + y_{2,xxx} = 0 & \text{in } (0, T) \times (0, L), \\ y_2(t, 0) = h_1(t) - e^{2t} \left( y_1(t, 0) + \int_0^t \psi(s) ds \right) & \text{in } (0, T), \\ y_2(t, L) = h_2(t) - e^{2t-L} \left( y_1(t, L) + \int_0^t \psi(s) ds \right) & \text{in } (0, T), \\ y_{2,x}(t, L) = h_3(t) - \left( e^{2t-\cdot} \left( y_1(t, \cdot) + \int_0^t \psi(s) ds \right) \right)_x(t, L) & \text{in } (0, T), \end{cases}$$

and

$$y_2(t = 0, \cdot) = 0 \text{ in } (0, L).$$

Applying the results of Step 1 to  $y_2$ , and using (2.41), we derive that, for  $x \in [0, L]$ ,

$$\begin{aligned} \|y_2\|_{X_T} + \|y_2(\cdot, x)\|_{H^{1/3}(0, T)} + \|y_{2,x}(\cdot, x)\|_{L^2(0, T)} + \|y_{2,xx}(\cdot, x)\|_{[H^{1/3}(0, T)]^*} \\ \leq C_T \left( \|(h_1, h_2)\|_{H^{1/3}(\mathbb{R}_+)} + \|h_3\|_{L^2(\mathbb{R}_+)} + \|f\|_{L^1(\mathbb{R}_+; L^2(0, L))} \right). \end{aligned}$$

The conclusion follows by noting that  $y = e^{2t-x} \left( y_1 + \int_0^t \psi(s) ds \right) + y_2$  in  $(0, T) \times (0, L)$ .

The proof is complete.  $\square$

In the proof of Proposition 2.1, we used the following elementary result, see, e.g., [21].

**Lemma 2.2.** For  $p \in \mathbb{C}$  and  $z$  in a sufficiently small conic neighborhood of  $\mathbb{R}_+$ , let  $\lambda_j$  and  $\tilde{\lambda}_j$  with  $j = 1, 2, 3$  be the three solutions of the equation  $\lambda^3 + \lambda + iz = 0$  and  $\lambda^3 + \lambda + i(p - z) = 0$ , respectively. Consider the convention  $\Re(\lambda_1) < \Re(\lambda_2) < \Re(\lambda_3)$  and similarly for  $\tilde{\lambda}_j$ . We have, in the limit  $|z| \rightarrow \infty$ ,

$$(2.42) \quad \lambda_j = \mu_j z^{1/3} - \frac{1}{3\mu_j} z^{-1/3} + O(z^{-2/3}) \quad \text{with } \mu_j = e^{-i\pi/6 - 2ij\pi/3},$$

$$(2.43) \quad \tilde{\lambda}_j = \tilde{\mu}_j z^{1/3} - \frac{1}{3\tilde{\mu}_j} z^{-1/3} + O(z^{-2/3}) \quad \text{with } \tilde{\mu}_j = e^{i\pi/6 + 2ij\pi/3}$$

Here  $z^{1/3}$  denotes the cube root of  $z$  with the real part positive.

Here and in what follows, for  $s \in \mathbb{R}$ ,  $O(z^s)$  denotes a quantity bounded by  $Cz^s$  for large positive  $z$ . Similar convention is used for  $O(|z|^s)$  for  $z \in \mathbb{C}$ .

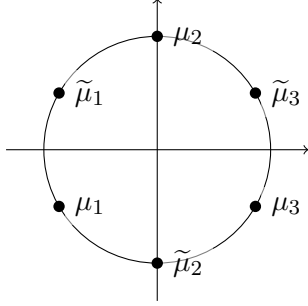


FIGURE 1. The roots  $\lambda_j$  of  $\lambda^3 + \lambda + iz = 0$  satisfy, when  $z > 0$  is large,  $\lambda_j \sim \mu_j z^{1/3}$  where  $\mu_j^3 = -i$ . When  $z < 0$  and  $|z|$  is large, then the corresponding roots  $\hat{\lambda}_j$  satisfy  $\hat{\lambda}_j \sim \tilde{\mu}_j |z|^{1/3}$  with  $\tilde{\mu}_j = \bar{\mu}_j$ . We also have  $\tilde{\lambda}_j \sim \hat{\lambda}_j$ .

The same proof as in Step 1 of the proof of Proposition 2.1 gives the following result.

**Lemma 2.3.** Let  $T > 0$  and  $(h_1, h_2, h_3) \in H^{1/3}(0, T) \times H^{1/3}(0, T) \times L^2(0, T)$ . Then the unique solution  $y \in X_T$  of the system

$$(2.44) \quad \begin{cases} y_t + y_x + y_{xxx} = 0 & \text{in } (0, T) \times (0, L), \\ y(\cdot, 0) = h_1, y(\cdot, L) = h_2, y_x(\cdot, L) = h_3 & \text{in } (0, T), \\ y(0, \cdot) = 0 & \text{in } (0, L). \end{cases}$$

satisfy, for  $x \in [0, L]$ ,

$$(2.45) \quad \|y(\cdot, x)\|_{H^{-2/3}(0, T)} \leq C_{T, L} \left( \|(h_1, h_2)\|_{H^{-2/3}(0, T)} + \|h_3\|_{H^{-1}(0, T)} \right),$$

where  $C_{T, L}$  denotes a positive constant independent of  $x, y_0$ , and  $h_1, h_2, h_3$ .

**Remark 2.3.** We end this section with some comments on Proposition 2.1 and its proof.

- i) The well-posedness of (2.44) in  $X_T$  is proved in [8, Theorem 2.10 and Proposition 2.16] for  $L = 1$  when  $(0, T) = \mathbb{R}_+$  and the estimate for  $\|y_x(\cdot, x)\|_{L^2(0, T)}$  in this case is a consequence of their results. For  $L = 1$ , their results imply that, when  $(h_1, h_2, h_3) \equiv (0, 0, 0)$ ,

$$(2.46) \quad \|y_x(\cdot, x)\|_{L^2(\mathbb{R}_+)} \leq C \|y_0\|_{L^2(0, 1)} \quad \text{for } x \in [0, 1].$$

Note that  $L = 1$  is smaller than the smallest critical length in  $\mathcal{N}_N$  which is  $2\pi$ . The estimate as (2.46) cannot hold for arbitrary  $L$ . In fact, such an estimate is not valid for any critical length by considering a non-zero initial datum in  $\mathcal{M}_N$ .



ii) Similar estimates for  $\|y(\cdot, x)\|_{H^{1/3}(0, T)}$  and  $\|y_x(\cdot, x)\|_{L^2(0, T)}$  as in (2.2) in the real line space setting can be found in [16, 24]. To our knowledge, variants of the estimate for  $\|y_{xx}(\cdot, x)\|_{[H^{1/3}(0, T)]^*}$  in (2.2) are not known even in the real line space setting. Our proof of Proposition 2.1 is in the spirit of [21], which involves the Fourier transform with respect to time of the solution, as in [8], and a connection between the linearized KdV and the linear KdV-Burger equations. However, in the study of the the linear KdV-Burger equation with periodic boundary conditions, the singularity of the kernel is appropriately *compensated* (see the proof of Proposition 2.4), which is the novelty of the analysis. The proof given here is self-contained and is different from the ones in [16, 24] which are based on the Riemann-Liouville fractional integrals and the theory of Airy functions.

**2.3. Proof of Proposition 2.2.** Using Proposition 2.1, without loss of generality, one might assume that  $y_0 = 0$ ,  $h_1 = 0$ ,  $h_2 = 0$ , and  $f = 0$ . This will be assumed from later on.

Extend  $h_3$  by 0 for  $t \notin [0, T]$  and still denote this extension by  $h_3$ . Then  $h_3 \in H^{-1/3}(\mathbb{R})$ . Denote  $y$  be the corresponding solution for  $t \geq 0$  and extend  $y$  by 0 for  $t < 0$ . Still denote this extension by  $y$ . One then has

$$(2.47) \quad \begin{cases} y_t + y_x + y_{xxx} = 0 & \text{in } \mathbb{R} \times (0, L), \\ y(\cdot, 0) = 0, y(\cdot, L) = 0, y_{xx}(\cdot, L) = h_3 & \text{in } \mathbb{R}. \end{cases}$$

Taking the Fourier transform with respect to  $t$ , from the system of  $y$ , we have

$$(2.48) \quad \begin{cases} iz\hat{y}(z, x) + \hat{y}_x(z, x) + \hat{y}_{xxx}(z, x) = 0 & \text{in } \mathbb{R} \times (0, L), \\ \hat{y}(z, 0) = \hat{y}(z, L) = 0 & \text{in } \mathbb{R}, \\ \hat{y}_{xx}(z, L) = \hat{h}_3(z) & \text{in } \mathbb{R}. \end{cases}$$

Taking into account the equation of  $\hat{y}$ , we search the solution of the form

$$(2.49) \quad \hat{y}(z, \cdot) = \sum_{j=1}^3 a_j e^{\lambda_j x},$$

where  $\lambda_j = \lambda_j(z)$  with  $j = 1, 2, 3$  are defined in Definition 4.1, and  $a_j = a_j(z)$  for  $j = 1, 2, 3$ .

We then have

$$\begin{aligned} \sum_{j=1}^3 a_j &= 0, \\ \sum_{j=1}^3 a_j e^{\lambda_j L} &= 0, \\ \sum_{j=1}^3 a_j \lambda_j^2 e^{\lambda_j L} &= \hat{h}_3, \end{aligned}$$

This implies, with the convention  $\lambda_{j+3} = \lambda_j$ ,

$$(2.50) \quad a_j = \frac{e^{\lambda_{j+2}L} - e^{\lambda_{j+1}L}}{\sum_{k=1}^3 e^{-\lambda_k L} (\lambda_{k+2}^2 - \lambda_{k+1}^2)} \hat{h}_3 \text{ for } j = 1, 2, 3.$$

Modifying the value of  $h_3$  in the interval  $(T, 2T)$  if necessary, as in the proof Proposition 2.1, one can assume that if  $z$  is a real solution of the equation  $(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1) \sum_{k=1}^3 e^{-\lambda_k L} (\lambda_{k+2}^2 - \lambda_{k+1}^2) = 0$  of order  $m$  then  $z$  is also a real solution of order at least  $m$  of  $\hat{h}_3(z)$ . Using the behavior of  $\lambda_j$  for large real  $z$ , see e.g. Lemma 2.2, one obtains, for  $m \geq 1$  and  $z \in \mathbb{R}$  with  $|z| \leq m$ ,

$$(2.51) \quad |a_j| \leq C_m \|h_3\|_{H^{-1/3}(\mathbb{R})},$$

and, for  $z \in \mathbb{R}$  with large  $|z|$ ,

$$(2.52) \quad |a_j| \leq \frac{C}{|z|^{2/3}} |\hat{h}_3(z)|$$

The conclusions now follows from (2.49)-(2.52).  $\square$

**2.4. Proof of Proposition 2.3.** We only derive the estimates. We have, by Proposition 2.1, we have

$$\|y\|_{X_T} \leq C_{T,L} \left( \|y_0\|_{L^2(0,L)} + \|u\|_{H^{1/3}(0,T)} + \|g\|_{L^1((0,T);L^2(0,L))} \right),$$

where

$$g(t, x) = f(t, x) - y(t, x)\partial_x y(t, x).$$

On the other hand, one has

$$\|yy_x\|_{L^1((0,T);L^2(0,L))} \leq C\|y\|_{X_T}^2.$$

By choosing  $\varepsilon_0$  sufficiently small, the conclusion follows.  $\square$

### 3. PROPERTIES OF THE UNREACHABLE SPACE

In this section, we first describe the unreachable space for the linearized KdV system:

$$(3.1) \quad \begin{cases} y_t + y_x + y_{xxx} = 0 & \text{in } (0, T) \times (0, L), \\ y(\cdot, 0) = y_x(\cdot, L) = 0 & \text{in } (0, T), \\ y(\cdot, L) = u & \text{in } (0, T). \end{cases}$$

This reveals more structures behind (1.4). The main result of this section is Proposition 3.1 which is based on an observability on the controllability with initial data in  $\mathcal{M}_D^\perp$ . In comparison with the right Neumann boundary control system, this part for the Neumann setting is much simpler. The proof of Proposition 3.1 involves the well-posedness and the estimates for solutions in  $X_T$  given in Proposition 2.1 and Proposition 2.2.

As in [35] (see also [14, 21]), we first describe the unreachable space  $\mathcal{M}_D$ . Let  $L \in \mathcal{N}_D$  be defined by (1.3). Denote

$$(3.2) \quad \mathcal{P}_D = \mathcal{P}_{L,D} := \left\{ iz_1 z_2 z_3 / L^3; z_1, z_2, z_3 \in \mathbb{C} : (1.4) \text{ holds, } z_3 = -(z_1 + z_2), \Im(z_1 z_2 z_3) \leq 0 \right\}.$$

Here and in what follows, for  $z \in \mathbb{C}$ ,  $\Re z$  and  $\Im z$  denote the real part and the imaginary part of  $z$ , respectively.

One can rewrite the definition of  $\mathcal{P}_D$  under the form

$$(3.3) \quad \mathcal{P}_D = \left\{ p \in \mathbb{C}; \Re(p) \geq 0 \text{ and (3.4) holds} \right\},$$

where, with  $z_1, z_2, z_3$  being three solutions of the equation  $z^3 + L^2 z + ipL^3 = 0$ ,

$$(3.4) \quad z_1 e^{z_1} = z_2 e^{z_2} = z_3 e^{z_3}.$$

Thus  $\mathcal{P}_D$  is bounded in  $\mathbb{C}$  since for large  $p$  the three solutions of the equations  $z^3 + L^2 z + ipL^3 = 0$  are close to the three solutions of the equation  $z^3 + ipL^3 = 0$ , and one can show that (3.4) does not hold then. Condition (3.4) also implies that

$$\chi(p) = 0,$$

where, with  $z_1, z_2, z_3$  being three solutions of the equation  $z^3 + L^2 z + ipL^3 = 0$ ,

$$\chi(p) = (z_1 - z_2)(z_1 - z_3)(z_2 - z_3)(z_1 e^{z_1} - z_2 e^{z_2})(z_1 e^{z_1} - z_3 e^{z_3})(z_2 e^{z_2} - z_3 e^{z_3}).$$

Since  $\chi$  is analytic in  $\mathbb{C}$ , the set of the solutions of the equation  $\chi(p) = 0$  has no accumulation point. We thus derive that  $\mathcal{P}_D$  is a finite set.

Set

$$(3.5) \quad n_D := \#\mathcal{P}_D.$$

Denote

$$(3.6) \quad \mathcal{P}_D = \left\{ p_\ell \in \mathbb{C}; 1 \leq \ell \leq n \right\}.$$

Thus, by (3.3),

$$\Re p_\ell \geq 0.$$

For  $1 \leq \ell \leq n_D$ , let  $z_{1,\ell}, z_{2,\ell}$  and  $z_{3,\ell}$  be three solutions of the equation  $z^3 + L^2 z + ip_\ell L^3 = 0$ . Denote

$$\eta_{j,\ell} = -z_{j,\ell}/L \quad \text{and} \quad \eta_{j+3,\ell} = \eta_{j,\ell} \text{ for } j \geq 1.$$

Then

$$(3.7) \quad \eta_{1,\ell} + \eta_{2,\ell} + \eta_{3,\ell} = 0, \quad \eta_{1,\ell}\eta_{2,\ell} + \eta_{1,\ell}\eta_{3,\ell} + \eta_{2,\ell}\eta_{3,\ell} = 1, \quad \text{and} \quad \eta_{1,\ell}\eta_{2,\ell}\eta_{3,\ell} = ip_\ell.$$

It follows that

$$(3.8) \quad \eta_{1,\ell}, \eta_{2,\ell}, \text{ and } \eta_{3,\ell} \text{ are the three solutions of the equation } \eta^3 + \eta - ip_\ell = 0.$$

From (3.4), one derives that

$$(3.9) \quad \eta_{1,\ell} e^{-\eta_{1,\ell} L} = \eta_{2,\ell} e^{-\eta_{2,\ell} L} = \eta_{3,\ell} e^{-\eta_{3,\ell} L}.$$

Define, for  $1 \leq \ell \leq n_D$ ,

$$(3.10) \quad \varphi_\ell(x) = \sum_{j=1}^3 (\eta_{j+2,\ell} - \eta_{j+1,\ell}) e^{\eta_{j,\ell} x} \text{ for } x \in [0, L],$$

and

$$(3.11) \quad \Psi_\ell(t, x) = e^{-ip_\ell t} \varphi_\ell(x) \text{ for } t \in \mathbb{R}, x \in [0, L].$$

Using (3.7), (3.8), and (3.9), one can check that

$$(3.12) \quad \Psi_\ell(t, 0) = \Psi_{\ell,x}(t, 0) = \Psi_\ell(t, L) = \Psi_{\ell,xx}(t, L) = 0 \text{ for } t \in \mathbb{R}$$

and

$$(3.13) \quad \Psi_{\ell,t} + \Psi_{\ell,x} + \Psi_{\ell,xxx} = 0 \text{ in } \mathbb{R} \times (0, L).$$

**Remark 3.1.** We later prove that  $\Im p_\ell \geq 0$  for  $1 \leq \ell \leq n_D$  (see Corollary B.1 in the appendix).

We are ready to state the main result of this section on the unreachable space for the linearized KdV system (3.1).

**Proposition 3.1.** *Let  $L \in \mathcal{N}_D$  be defined in (1.3) and set*

$$(3.14) \quad \mathcal{M}_D := \text{span} \left\{ \Re \varphi_\ell, \Im \varphi_\ell; \varphi_\ell \text{ is given in (3.10) with } 1 \leq \ell \leq n_D \right\}.$$

We have

i) for  $\varphi \in \mathcal{M}_D \setminus \{0\}$  and  $T > 0$ , there does not exist  $u \in H^{1/3}(0, T)$  such that  $y(T, \cdot) = \varphi$  where  $y \in X_T$  is the unique solution of the linearized KdV equation (3.1) with  $y(0, \cdot) = 0$ .

ii) there exists a linear continuous operator  $\mathcal{L} : \mathcal{M}_D^\perp \rightarrow H^{1/3}(0, T)$  such that  $y(T, \cdot) = \varphi$  where  $y \in X_T$  is the unique solution of (3.1) with  $y(0, \cdot) = 0$  and  $u = \mathcal{L}(\varphi)$ .

iii) There exists a linear continuous operator  $\hat{\mathcal{L}} : \mathcal{M}_D^\perp \rightarrow H^{1/3}(0, T)$  such that  $y(T, \cdot) = 0$  where  $y \in X_T$  is the unique solution of (3.1) with  $y(0, \cdot) = \varphi$  and  $u = \hat{\mathcal{L}}(\varphi)$ .

*Proof.* The proof of i) is standard as in [23]. For the convenience of the reader, we give the proof. Let  $\varphi \in \mathcal{M}_D \setminus \{0\}$ . By the definition of  $\mathcal{M}_D$  (see also (3.12) and (3.13)), there exists a smooth (real) solution  $\Psi$  of the system

$$\Psi_t + \Psi_x + \Psi_{xx} = 0 \text{ for } (t, x) \in \mathbb{R} \times (0, L),$$

and

$$\Psi(t, 0) = \Psi_x(t, 0) = \Psi(t, L) = \Psi_{xx}(t, L) = 0 \text{ for } t \in \mathbb{R},$$

with  $\Psi(0, \cdot) = \varphi$  in  $(0, L)$ .

Multiplying the equation of  $y$  by  $\Psi$  and integrating by parts, after using the boundary conditions of  $y$  and  $\Psi$ , one has

$$\frac{1}{2} \frac{d}{dt} \int_0^L y(t, x) \Psi(t, x) dx = 0.$$

This yields assertion i).

We next deal with assertion ii) and iii). Let  $\psi \in \mathcal{M}_D^\perp$ . Let  $y^* \in X_T$  be the unique solution of the backward linear KdV system

$$(3.15) \quad \begin{cases} y_t^* + y_x^* + y_{xxx}^* = 0 & \text{in } (0, T) \times (0, L), \\ y^*(\cdot, 0) = y_x^*(\cdot, 0) = y^*(\cdot, L) = 0 & \text{in } (0, T), \\ y^*(T, \cdot) = \psi & \text{in } (0, L). \end{cases}$$

Applying the observability inequality in Lemma 3.1 below to  $y^*(T - \cdot, L - \cdot)$ , we obtain

$$(3.16) \quad \lambda^{-1} \|\varphi\|_{L^2(0, L)} \leq \|y_{xx}^*(\cdot, L)\|_{[H^{1/3}(0, T)]^*} \leq \lambda \|\varphi\|_{L^2(0, L)}$$

for some constant  $\lambda \geq 1$ . Fix a continuous linear mapping

$$(3.17) \quad \mathcal{L}_1 : [H^{1/3}(0, T)]^* \rightarrow H^{1/3}(0, T)$$

such that

$$(3.18) \quad \langle \mathcal{L}_1(h), h \rangle_{H^{1/3}(0, T); [H^{1/3}(0, T)]^*} \geq C \|h\|_{[H^{1/3}(0, T)]^*} \text{ for all } h \in [H^{1/3}(0, T)]^*,$$

for some positive constant  $C$ . This can be done using the Fourier series or the Fourier transform appropriately.

We first prove assertion ii). Equipped  $\mathcal{M}_D^\perp$  with the  $L^2(0, L)$ -scalar product. Define

$$\mathcal{A} : \mathcal{M}_D^\perp \rightarrow \mathcal{M}_D^\perp$$

by

$$\mathcal{A}(\psi) = y(T, \cdot),$$

where  $y \in X_T$  is the unique solution of the following system

$$(3.19) \quad \begin{cases} y_t + y_x + y_{xxx} = 0 & \text{in } (0, T) \times (0, L), \\ y(\cdot, 0) = y_x(\cdot, L) = 0 & \text{in } (0, T), \\ y(\cdot, L) = h & \text{in } (0, T), \\ y(0, \cdot) = 0 & \text{in } (0, L), \end{cases}$$

with  $h = \mathcal{L}_1(y_{xx}^*(\cdot, L))$  where  $y^*$  is determined by (3.15) ( $y(T, \cdot) \in \mathcal{M}_D^\perp$  since  $y(0, \cdot) = 0$ ). An integration by part yields

$$(3.20) \quad \int_0^L y(T, x) z^*(T, x) dx = - \int_0^T y(t, L) z_{xx}^*(t, L) dt,$$

for all solutions  $z^* \in X_T$  of (3.15) with  $z^*(T, \cdot) \in \mathcal{M}_D^\perp$ .

Using (3.16) and applying the Lax-Milgram theory, we derive that  $\mathcal{A}$  is linear continuous and its inverse is also linear continuous. The conclusion of *ii*) follows by taking  $\mathcal{L}(\psi) = \mathcal{L}_1(h)$  with  $h = \mathcal{L}_1(y_{xx}^*(\cdot, L))$  and  $y^*$  is the solution of (3.15) with  $\psi$  being replaced by  $\mathcal{A}^{-1}(\psi)$ .

We now deal with assertion *iii*). Set

$$H = \left\{ y_{xx}^*(\cdot, L) \in [H^{1/3}(0, T)]^*; \text{ where } y^* \text{ is determined by (3.15) with } \psi \in \mathcal{M}_D^\perp \right\}.$$

It follows from (3.16) that  $H$  is a closed subspace of  $[H^{1/3}(0, T)]^*$  so is a Hilbert space. We next consider the following bilinear form on  $H$ :

$$\hat{a}(u, v) = \langle \mathcal{L}_1(u), v \rangle_{H^{1/3}(0, T); [H^{1/3}(0, T)]^*}.$$

Using (3.16), we derive from the Lax-Milgram theorem that there exists a continuous linear application  $\hat{\mathcal{A}}: \mathcal{M}_D^\perp \rightarrow H$  such that

$$(3.21) \quad \int_0^L \varphi(x) y^*(0, x) dx = \hat{a}(\hat{\mathcal{A}}\varphi, y_{xx}^*(\cdot, L))$$

for all solution  $y^* \in X_T$  of (3.15) with  $\psi \in \mathcal{M}_D^\perp$ .

Set

$$\hat{\mathcal{L}} = \mathcal{L}_1 \circ \hat{\mathcal{A}}.$$

The conclusion of *iii*) then follows after noting that if  $y(0, \cdot) \in \mathcal{M}_D^\perp$  then  $y(T, \cdot) \in \mathcal{M}_D^\perp$ , and

$$(3.22) \quad \int_0^L y(T, x) y^*(T, x) dx - \int_0^L y(0, x) y^*(0, x) dx = - \int_0^T y(t, L) y_{xx}^*(t, L) dt.$$

The proof is complete.  $\square$

Here is the observability inequality used in the proof of Proposition 3.1.

**Lemma 3.1.** *Let  $L \in \mathcal{N}_D$  and  $y \in X_T$  be a solution of the linearized KdV equation with  $y(0, \cdot) \in \mathcal{M}_D^\perp$  with  $y(\cdot, 0) = y(\cdot, L) = y_x(\cdot, L) = 0$ . Then there exists  $\Lambda \geq 1$  depending only on  $L$  and  $T$  such that*

$$(3.23) \quad \Lambda^{-1} \|y(0, \cdot)\|_{L^2(0, L)} \leq \|y_{xx}(\cdot, 0)\|_{[H^{1/3}(0, T)]^*} \leq \Lambda \|y(0, \cdot)\|_{L^2(0, L)}.$$

*Proof.* By Proposition 2.1, we have

$$(3.24) \quad \|y_{xx}(\cdot, 0)\|_{[H^{1/3}(0, T)]^*} \leq C \|y(0, \cdot)\|_{L^2(0, L)}.$$

We next prove

$$(3.25) \quad \|y(0, \cdot)\|_{L^2(0, L)} \leq C \|y_{xx}(\cdot, 0)\|_{[H^{1/3}(0, T)]^*}$$

by contradiction. Assume that there exists a sequence  $(\varphi_n) \subset \mathcal{M}_D^\perp$  such that

$$(3.26) \quad \|y_{n,xx}(\cdot, 0)\|_{[H^{1/3}(0, T)]^*} \leq \frac{1}{n} \|y_n(0, \cdot)\|_{L^2(0, L)} = \frac{1}{n},$$

where  $y_n \in X_T$  is the solution of the linearized KdV system with  $y_n(0, \cdot) = \varphi_n$  and  $y_n(\cdot, 0) = y_n(\cdot, L) = y_{n,x}(\cdot, L) = 0$ . Set

$$\mathbf{y}_n(t, x) = y_n(T - t, L - x).$$

Then  $\mathbf{y}_n \in X_T$  is a solution of the equation  $\mathbf{y}_{n,t} + \mathbf{y}_{n,x} + \mathbf{y}_{n,xxx} = 0$  in  $(0, T) \times (0, L)$ . By the regularizing effect of the linearized KdV equation, without loss of generality, one might assume that  $\mathbf{y}_n(0, \cdot) = y_n(T, L - \cdot)$  converges in  $L^2(0, L)$ . Applying Proposition 2.2 to  $\mathbf{y}_n$ , one derives that  $y_n(0, L - \cdot) = \mathbf{y}_n(T, \cdot)$  is a Cauchy sequence in  $L^2(0, L)$ . In other words,  $\varphi_n = y_n(0, \cdot)$  is a Cauchy sequence in  $L^2(0, L)$ . Let  $\varphi$  be the limit of  $\varphi_n$  in  $L^2(0, L)$  and denote  $y \in X_T$  be the corresponding solution of the linearized KdV system. Then  $\|\varphi\|_{L^2(0, L)} = 1$ .

Set  $\mathbf{y}(t, x) = y(T - t, L - x)$ . Then  $\mathbf{y} \in X_T$  is a solution of the system

$$(3.27) \quad \mathbf{y}_t + \mathbf{y}_x + \mathbf{y}_{xxx} = 0 \text{ in } (0, T) \times (0, L),$$

$$(3.28) \quad \mathbf{y}(\cdot, 0) = \mathbf{y}_x(\cdot, 0) = \mathbf{y}(\cdot, L) = \mathbf{y}_{xx}(\cdot, L) = 0 \text{ in } (0, T).$$

and

$$\mathbf{y}(0, \cdot) \in \mathcal{M}_D^\perp \quad \text{and} \quad \|\mathbf{y}(0, \cdot)\|_{L^2(0, L)} \stackrel{\text{Proposition 2.2}}{\geq} C \|\varphi\|_{L^2(0, L)} = C.$$

Moreover,  $\mathbf{y}(0, \cdot) \in C^\infty([0, L])$ .

Set

$$\mathcal{V} = \left\{ \varphi \in \mathcal{M}_D^\perp \cap C^\infty([0, L]); \exists \mathbf{y} \in X_T \text{ satisfying (3.27), (3.28), and } \mathbf{y}(0, \cdot) = \varphi \right\} \subset L^2(0, L).$$

Then  $\mathcal{V} \neq \{0\}$ . Using the same argument as above, one can show that any bounded sequence in  $\mathcal{V}$  (equipped  $L^2(0, L)$ -norm) has a subsequence converging in  $\mathcal{V}$ . Thus  $\mathcal{V}$  is of finite dimension.

We can now involve the arguments via spectral theory in the spirit of [23] or even simpler (see also [1, 38]) to show that there exists  $\varphi \in \mathcal{V} \setminus \{0\}$  and  $\lambda \in \mathbb{C}$  such that

$$(3.29) \quad \lambda\varphi + \varphi_x + \varphi_{xxx} = 0 \text{ in } (0, L),$$

$$(3.30) \quad \varphi(\cdot, 0) = \varphi_x(\cdot, 0) = \varphi(\cdot, L) = \varphi_{xx}(\cdot, L) = 0.$$

Indeed, this can be done by considering

$$(3.31) \quad \begin{array}{ccc} \mathcal{A} : \mathcal{V} & \rightarrow & \mathcal{V} \\ \psi & \mapsto & -(\psi_x + \psi_{xxx}) \end{array}$$

and taking  $\lambda \in \mathbb{C}$  and  $\varphi \in \mathcal{V} \setminus \{0\}$  such that  $\mathcal{A}\varphi = \lambda\varphi$ . The only point required to check is the fact that  $\psi_x + \psi_{xxx} \in \mathcal{V}$  for  $\psi \in \mathcal{V}$ . To this end, one just notes that  $-(\psi_x + \psi_{xxx}) = \mathbf{y}_t(0, \cdot)$  where  $\mathbf{y} \in X_T$  is the corresponding solution (thus  $\mathbf{y}(0, \cdot) = \psi$ ). One now can involve the same arguments in [23] (or apply [23, Propositions 1 and 2]), to derive that  $\varphi \in \mathcal{M}_D$ . Hence  $\varphi = 0$  since  $\varphi \in \mathcal{M}_D^\perp$ . We obtain a contradiction. The proof is complete.  $\square$

By the same arguments used in the proof of Lemma 3.1, we also have the following observability inequality in the case  $L \notin \mathcal{N}_D$ , which is the key point of the proof of Theorem 1.3.

**Lemma 3.2.** *Let  $L \notin \mathcal{N}_D$  and let  $y \in X_T$  be a solution of the linearized KdV equation with  $y(t = 0, \cdot) \in L^2(0, L)$  and  $y(\cdot, 0) = y(\cdot, L) = y_x(\cdot, 0) = 0$ . Then, for some  $\Lambda \geq 1$ ,*

$$(3.32) \quad \Lambda^{-1} \|y(0, \cdot)\|_{L^2(0, L)} \leq \|y_{xx}(\cdot, 0)\|_{[H^{1/3}(0, T)]^*} \leq \Lambda \|y(0, \cdot)\|_{L^2(0, L)}.$$

Here is a variant of Proposition 3.1 in the case  $L \notin \mathcal{N}_D$ .

**Proposition 3.2.** *Let  $L \notin \mathcal{N}_D$ . Then*

a) *There exists a linear continuous operator  $\mathcal{L} : L^2(0, L) \rightarrow H^{1/3}(0, T)$  such that  $y(T, \cdot) = \varphi$  where  $y \in X_T$  is the unique solution of (3.1) with  $y(0, \cdot) = 0$  and  $u = \mathcal{L}(\varphi)$ .*

b) *There exists a linear continuous operator  $\hat{\mathcal{L}} : L^2(0, L) \rightarrow H^{1/3}(0, T)$  such that  $y(T, \cdot) = 0$  where  $y \in X_T$  is the unique solution of (3.1) with  $y(0, \cdot) = \varphi$  and  $u = \hat{\mathcal{L}}(\varphi)$ .*

4. PROPERTIES OF CONTROLS WHICH STEER 0 AT TIME 0 TO 0 AT TIME  $T$ 

In this section, we study controls which steer 0 at time 0 to 0 at time  $T$  for the linearized KdV system of (1.1). To this end, it is convenient to introduce

**Definition 4.1.** For  $z \in \mathbb{C}$ , let  $(\lambda_j)_{1 \leq j \leq 3} = (\lambda_j(z))_{1 \leq j \leq 3}$  be the three solutions of

$$(4.1) \quad \lambda^3 + \lambda + iz = 0.$$

Set

$$(4.2) \quad Q = Q(z) := \begin{pmatrix} 1 & 1 & 1 \\ e^{\lambda_1 L} & e^{\lambda_2 L} & e^{\lambda_3 L} \\ \lambda_1 e^{\lambda_1 L} & \lambda_2 e^{\lambda_2 L} & \lambda_3 e^{\lambda_3 L} \end{pmatrix},$$

$$(4.3) \quad P_D = P_D(z) := \sum_{j=1}^3 \lambda_j^2 (\lambda_{j+1} e^{\lambda_{j+1} L} - \lambda_{j+2} e^{\lambda_{j+2} L}),$$

and

$$(4.4) \quad \Xi = \Xi(z) := (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_3),$$

with the convention  $\lambda_{j+3} = \lambda_j$  for  $j \geq 1$ .

Given  $u \in H^{1/3}(0, +\infty)$ , let  $y \in C([0, +\infty); L^2(0, L)) \cap L^2_{\text{loc}}([0, +\infty); H^1(0, L))$  be the unique solution of the system

$$(4.5) \quad \begin{cases} y_t + y_x + y_{xxx} = 0 & \text{in } (0, +\infty) \times (0, L), \\ y(\cdot, 0) = y_x(\cdot, L) = 0 & \text{in } (0, +\infty), \\ y(\cdot, L) = u & \text{in } (0, +\infty), \end{cases}$$

and

$$y(t = 0, \cdot) = 0 \text{ in } (0, L).$$

In what follows in this section, we extend  $y$  and  $u$  by 0 for  $t < 0$  and still denote these extensions by  $y$  and  $u$ , respectively. For an appropriate function  $v$  defined on  $\mathbb{R} \times (0, L)$ , let  $\hat{v}$  denote its Fourier transform with respect to  $t$ , i.e.,

$$\hat{v}(z, x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v(t, x) e^{-izt} dt.$$

From the system of  $y$ , we have

$$(4.6) \quad \begin{cases} iz\hat{y}(z, x) + \hat{y}_x(z, x) + \hat{y}_{xxx}(z, x) = 0 & \text{in } \mathbb{R} \times (0, L), \\ \hat{y}(z, 0) = \hat{y}_x(z, L) = 0 & \text{in } \mathbb{R}, \\ \hat{y}(z, L) = \hat{u}(z) & \text{in } \mathbb{R}. \end{cases}$$

This system has a unique solution outside a discrete set of  $z$  in  $\mathbb{C}$ , see [21, Lemma 2.1].

Taking into account the equation of  $\hat{y}$ , we search the solution of the form

$$\hat{y}(z, \cdot) = \sum_{j=1}^3 a_j e^{\lambda_j x},$$

where  $\lambda_1, \lambda_2, \lambda_3$  are the solutions of the equation

$$(4.7) \quad \lambda^3 + \lambda + iz = 0.$$

Using the boundary conditions for  $\hat{y}$ , we require that

$$\begin{cases} \sum_{j=1}^3 a_j = 0, \\ \sum_{j=1}^3 e^{\lambda_j L} a_j = \hat{u}, \\ \sum_{j=1}^3 \lambda_j e^{\lambda_j L} a_j = 0. \end{cases}$$

This implies

$$Q(a_1, a_2, a_3)^\top = (0, \hat{u}, 0)^\top,$$

where  $Q = Q(z)$  is given in Definition 4.1.

We have

$$(4.8) \quad \det Q = \sum_{j=1}^3 (\lambda_{j+1} - \lambda_j) e^{\lambda_j L + \lambda_{j+1} L} = \sum_{j=1}^3 (\lambda_{j+1} - \lambda_j) e^{-\lambda_{j+2} L},$$

since  $\sum_{j=1}^3 \lambda_j = 0$ . We then obtain

$$a_j = \frac{\hat{u}}{\det Q} (\lambda_{j+1} e^{\lambda_{j+1} L} - \lambda_{j+2} e^{\lambda_{j+2} L}).$$

This yields

$$(4.9) \quad \hat{y}(z, x) = \frac{\hat{u}}{\det Q} \sum_{j=1}^3 (\lambda_{j+1} e^{\lambda_{j+1} L} - \lambda_{j+2} e^{\lambda_{j+2} L}) e^{\lambda_j x}.$$

From (4.9), we derive that

$$(4.10) \quad \partial_{xx} \hat{y}(z, 0) = \frac{\hat{u}(z) P_D(z)}{\det Q(z)},$$

where  $P_D(z)$  is given in Definition 4.1.

Set

$$(4.11) \quad G(z) = P_D(z)/\Xi(z) \quad \text{and} \quad H(z) = \det Q(z)/\Xi(z),$$

where  $\Xi(z)$  is given in Definition 4.1. It is convenient to consider  $\partial_{xx} \hat{y}(z, 0)$  under the form

$$(4.12) \quad \partial_{xx} \hat{y}(z, 0) = \frac{\hat{u}(z) G(z)}{H(z)},$$

By [21, Lemmas A1 and B1],  $\partial_{xx} \hat{y}(z, 0)$  is a meromorphic function, and

$$(4.13) \quad G(z) \text{ and } H(z) \text{ are entire functions.}$$

We thus have just established the following result.

**Lemma 4.1.** *Let  $u \in H^{1/3}(\mathbb{R}_+)$  and let  $y \in C([0, +\infty); L^2(0, L)) \cap L^2_{\text{loc}}([0, +\infty); H^1(0, L))$  be the unique solution of*

$$(4.14) \quad \begin{cases} y_t + y_x + y_{xxx} = 0 & \text{in } (0, +\infty) \times (0, L), \\ y(\cdot, 0) = y_x(\cdot, L) = 0 & \text{in } (0, +\infty), \\ y(\cdot, L) = u & \text{in } (0, +\infty), \end{cases}$$

with

$$(4.15) \quad y(0, \cdot) = 0 \text{ in } (0, L).$$



Outside a discrete set  $z \in \mathbb{R}$ , we have

$$(4.16) \quad \hat{y}(z, x) = \frac{\hat{u}}{\det Q} \sum_{j=1}^3 (\lambda_{j+1} e^{\lambda_{j+1} L} - \lambda_{j+2} e^{\lambda_{j+2} L}) e^{\lambda_j x} \text{ for a.e. } x \in (0, L).$$

**Remark 4.1.** Assume that  $\hat{u}(z, \cdot)$  is well-defined for  $z \in \mathbb{C}$  (e.g. when  $u$  has a compact support). Then the conclusions of Lemma 4.1 hold outside of a discrete set  $z \in \mathbb{C}$ .

We end this section with the following result, which is the starting point of our approach.

**Proposition 4.1.** Let  $L > 0$ ,  $T > 0$ , and  $u \in H^{1/3}(\mathbb{R}_+)$ . Assume that  $u$  has a compact support in  $[0, T]$ , and  $u$  steers 0 at the time 0 to 0 at the time  $T$ , i.e., the unique solution  $y$  of (4.14) and (4.15) satisfies  $y(T, \cdot) = 0$  in  $(0, L)$ . Then  $\hat{u}$  and  $\hat{u}G/H$  satisfy the assumption of Paley-Wiener's theorem concerning the support in  $[-T, T]$ , i.e.,

$$(4.17) \quad \hat{u} \text{ and } \hat{u}G/H \text{ are entire functions,}$$

and

$$(4.18) \quad |\hat{u}(z)| + \left| \frac{\hat{u}G(z)}{H(z)} \right| \leq C e^{T|z|},$$

for some positive constant  $C$ .

**Remark 4.2.** The computations in this section are similar to the ones [21]. Nevertheless, in the conclusions of Proposition 4.1, we have/require that

$$\hat{y}(z, L) \text{ and } \partial_{xx} \hat{y}(z, 0) \text{ are entire functions satisfying (4.18).}$$

This is different with the one used in [21, Proposition 3.1] where one obtains that

$$\partial_x \hat{y}(z, L) \text{ and } \partial_x \hat{y}(z, 0) \text{ are entire functions satisfying a variant of (4.18).}$$

These differences are important to take into account different boundary conditions.

## 5. ATTAINABLE DIRECTIONS IN $\mathcal{M}_D$ IN SMALL TIME

In this section, we investigate directions in  $\mathcal{M}_D$ , defined in (3.14), which can be reached in small time. The starting point comes from the power series expansion approach. Let  $y_1$  and  $y_2$  be the solutions of

$$(5.1) \quad \begin{cases} y_{1,t} + y_{1,x} + y_{1,xxx} = 0 & \text{in } (0, T) \times (0, L), \\ y_1(\cdot, 0) = y_{1,x}(\cdot, L) = 0 & \text{in } (0, T), \\ y_1(\cdot, L) = u_1 & \text{in } (0, T), \\ y_1(0, \cdot) = 0 & \text{in } (0, L), \end{cases}$$

$$(5.2) \quad \begin{cases} y_{2,t} + y_{2,x} + y_{2,xxx} + y_1 y_{1,x} = 0 & \text{in } (0, T) \times (0, L), \\ y_2(\cdot, 0) = y_{2,x}(\cdot, L) = 0 & \text{in } (0, T), \\ y_2(\cdot, L) = 0 & \text{in } (0, T). \end{cases}$$

for some control  $u_1$ . The key point of this approach is to first understand how one can choose  $u_1$  so that

$$y_1(T, \cdot) = 0$$

and then analyse what the behavior of  $\text{proj}_{\mathcal{M}_D} y_2(T, \cdot)$  is. To this end, we compute the quantity

$$(5.3) \quad \int_0^L y_2(T, x) \Psi_\ell(t, x) dx$$

where  $\Psi_\ell$  is defined in (3.11). Multiplying the equation of  $y_2$  by  $\Psi_\ell$ , integrating by parts in  $[0, T] \times [0, L]$ , we obtain, after using the boundary conditions and the initial conditions,

$$(5.4) \quad \int_0^L y_2(T, x) \Psi_\ell(t, x) dx = \frac{1}{2} \int_0^T \int_0^L y_1^2(t, x) \Psi_{\ell, x}(t, x) dx.$$

The goal is then to understand the value of RHS of (5.4).

We will study the value of RHS of (5.4) in a more general setting. Motivated by the definition of  $y_1$ , we consider the unique solution  $y \in X_T$  of the system, for  $u \in H^{1/3}(0, T)$ ,

$$(5.5) \quad \begin{cases} y_t + y_x + y_{xxx} = 0 & \text{in } (0, T) \times (0, L), \\ y(\cdot, 0) = y_x(\cdot, L) = 0 & \text{in } (0, T), \\ y(\cdot, L) = u & \text{in } (0, T), \\ y(0, \cdot) = y(T, \cdot) = 0 & \text{in } (0, L). \end{cases}$$

Guided by the definition of  $\varphi_\ell$ , which appears in the definition of  $\Psi_\ell$  (see (3.10) and (3.11)), for  $\eta_1, \eta_2, \eta_3 \in \mathbb{C} \setminus \{0\}$ , we set

$$(5.6) \quad \varphi(x) = \sum_{j=1}^3 (\eta_{j+1} - \eta_j) e^{\eta_j + 2x} \text{ for } x \in [0, L],$$

with the convention  $\eta_{j+3} = \eta_j$  for  $j \geq 1$ . The following assumptions on  $\eta_j$  are used repeatedly throughout this section:

$$(5.7) \quad \eta_1 + \eta_2 + \eta_3 = 0, \quad \eta_1 \eta_2 + \eta_1 \eta_3 + \eta_2 \eta_3 = 1,$$

and

$$(5.8) \quad \eta_1 e^{-\eta_1 L} = \eta_2 e^{-\eta_2 L} = \eta_3 e^{-\eta_3 L}.$$

Extend  $y$  and  $u$  by 0 for  $t > T$  and still denote these extensions by  $y$  and  $u$ , respectively. Then, by Lemma A.1 in the appendix,

$$\|u\|_{H^{1/3}(\mathbb{R}_+)} \leq C \|u\|_{H^{1/3}(0, T)}.$$

Assume that  $y(T, \cdot) = 0$ . Then the extension  $y \in C([0, +\infty); L^2(0, L)) \cap L^2((0, +\infty); H^1(0, L))$  is also a solution of the linearized KdV system in  $[0, +\infty) \times (0, L)$  using the control which is the extension of  $u$  (by 0 outside  $(0, T)$ ), i.e.,

$$(5.9) \quad \begin{cases} y_t + y_x + y_{xxx} = 0 & \text{in } (0, +\infty) \times (0, L), \\ y(\cdot, 0) = y_x(\cdot, L) = 0 & \text{in } (0, +\infty), \\ y(\cdot, L) = u & \text{in } (0, +\infty), \\ y(0, \cdot) = 0 & \text{in } (0, L). \end{cases}$$

In what follows in this section, we study this quantity:

$$(5.10) \quad \int_0^T \int_0^L y^2(t, x) \varphi_x(x) e^{-ipt} dt dx \quad \left( = \int_0^{+\infty} \int_0^L y^2(t, x) \varphi_x(x) e^{-ipt} dt dx \right).$$

We have, by Lemma 4.1 (see also Remark 4.1), for  $z \in \mathbb{C}$  outside a discrete set,

$$(5.11) \quad \hat{y}(z, x) = \hat{u}(z) \frac{\sum_{j=1}^3 (\lambda_j e^{\lambda_j L} - \lambda_{j+1} e^{\lambda_{j+1} L}) e^{\lambda_j + 2x}}{\sum_{j=1}^3 (\lambda_{j+1} - \lambda_j) e^{-\lambda_j + 2L}}.$$

Recall that  $\lambda_j = \lambda_j(z)$  for  $j = 1, 2, 3$  are the three solutions of the equation

$$(5.12) \quad x^3 + x + iz = 0 \text{ for } z \in \mathbb{C}.$$

We begin with

**Lemma 5.1.** *Let  $p \in \mathbb{C}$ ,  $\eta_1, \eta_2, \eta_3 \in \mathbb{C} \setminus \{0\}$ , and let  $\varphi$  be defined by (5.6). Set, for  $(x, z) \in (0, L) \times \mathbb{R}$ ,*

$$B_D(x, z) = \frac{\sum_{j=1}^3 (\lambda_j e^{\lambda_j L} - \lambda_{j+1} e^{\lambda_{j+1} L}) e^{\lambda_{j+2} x}}{\sum_{j=1}^3 (\lambda_{j+1} - \lambda_j) e^{-\lambda_{j+2} L}} \cdot \frac{\sum_{j=1}^3 (\tilde{\lambda}_j e^{\tilde{\lambda}_j L} - \tilde{\lambda}_{j+1} e^{\tilde{\lambda}_{j+1} L}) e^{\tilde{\lambda}_{j+2} x}}{\sum_{j=1}^3 (\tilde{\lambda}_{j+1} - \tilde{\lambda}_j) e^{-\tilde{\lambda}_{j+2} L}} \cdot \varphi_x(x),$$

where  $\lambda_1 = \lambda_1(z)$ ,  $\lambda_2 = \lambda_2(z)$ ,  $\lambda_3 = \lambda_3(z)$  are three solutions of  $\lambda^3 + \lambda + iz = 0$ , and  $\tilde{\lambda}_j = \overline{\lambda_j(z)}$  denotes the conjugate of the roots corresponding to  $z - p$ . Let  $u \in H^{1/3}(0, +\infty)$  with compact support in  $[0, +\infty)$  and let  $y \in C([0, +\infty); L^2(0, L)) \cap L^2_{\text{loc}}([0, +\infty); H^1(0, L))$  be the unique solution of (5.9). Assume that  $y(t, \cdot) = 0$  for large  $t$ . We have

$$(5.13) \quad \int_0^L \int_0^{+\infty} |y(t, x)|^2 \varphi_x(x) e^{-ipt} dt dx = \int_{\mathbb{R}} \hat{u}(z) \overline{\hat{u}(z-p)} \int_0^L B_D(x, z) dx dz.$$

*Proof.* We have

$$\begin{aligned} \int_0^L \int_0^{\infty} |y(t, x)|^2 \varphi_x(x) e^{-ipt} dt dx &= \int_0^L \varphi_x(x) |\widehat{|y|^2}(p, x) dx = \int_0^L \varphi_x(x) \hat{y} * \widehat{\bar{y}}(p, x) dx \\ &= \int_0^L \varphi_x(x) \int_{\mathbb{R}} \hat{y}(z, x) \widehat{\bar{y}}(p-z, x) dz dx \\ &= \int_0^L \varphi_x(x) \int_{\mathbb{R}} \hat{y}(z, x) \widehat{\bar{y}}(z-p, x) dz dx. \end{aligned}$$

Using Fubini's theorem, we derive from (5.11) that

$$\int_0^L \int_0^{\infty} |y(t, x)|^2 \varphi_x(x) e^{-ipt} dt dx = \int_{\mathbb{R}} dz \hat{u}(z) \overline{\hat{u}(z-p)} \int_0^L B_D(x, z) dx,$$

which is (5.13). □

We next investigate the behavior of

$$\int_0^L B_D(x, z) dx$$

for  $z \in \mathbb{R}$  with large  $|z|$ . We have

**Lemma 5.2.** *Let  $p \in \mathbb{C}$  and  $\eta_1, \eta_2, \eta_3 \in \mathbb{C} \setminus \{0\}$ . Assume (5.8). We have*

$$(5.14) \quad \int_0^L B_D(x, z) dx = E_D |z|^{-1/3} + O(|z|^{-2/3}) \text{ for } z \in \mathbb{R} \text{ with large } |z|,$$

where  $E_D$  is defined by

$$(5.15) \quad E_D = \frac{1}{\sqrt{3}A} \sum_{j=1}^3 \eta_{j+2}^2 (\eta_{j+1} - \eta_j),$$

where <sup>5</sup>

$$A = A(\eta_1, \eta_2, \eta_3) := \eta_j e^{-\eta_j L}.$$

Here and in what follows, for  $s \in \mathbb{R}$ ,  $O(z^s)$  denotes a quantity bounded by  $Cz^s$  for large positive  $z$ . Similar convention is used for  $O(|z|^s)$  for  $z \in \mathbb{C}$ .

<sup>5</sup> $A$  does not depend on  $j$  by (5.8).

*Proof.* We first consider the case where  $z$  is positive and large. We use the following convention  $\Re(\lambda_1) < \Re(\lambda_2) < \Re(\lambda_3)$  and similarly for  $\tilde{\lambda}_j$  for large positive  $z$ .

We first look at the denominator of  $B_D(x, z)$ . We have, by Lemma 2.2,

$$(5.16) \quad \frac{1}{\sum_{j=1}^3 (\lambda_{j+1} - \lambda_j) e^{-\lambda_{j+2}L}} \cdot \frac{1}{\sum_{j=1}^3 (\tilde{\lambda}_{j+1} - \tilde{\lambda}_j) e^{-\tilde{\lambda}_{j+2}L}} = \frac{e^{\lambda_1 L} e^{\tilde{\lambda}_1 L}}{(\lambda_3 - \lambda_2)(\tilde{\lambda}_3 - \tilde{\lambda}_2)} \left(1 + O(e^{-C|z|^{1/3}})\right).$$

We next deal with the numerator of  $B_D(x, z)$ . Set, for  $(z, x) \in \mathbb{R} \times (0, L)$ ,

$$(5.17) \quad f(z, x) = \sum_{j=1}^3 (\lambda_j e^{\lambda_j L} - \lambda_{j+1} e^{\lambda_{j+1} L}) e^{\lambda_{j+2} x}, \quad g(z, x) = \sum_{j=1}^3 (\tilde{\lambda}_j e^{\tilde{\lambda}_j L} - \tilde{\lambda}_{j+1} e^{\tilde{\lambda}_{j+1} L}) e^{\tilde{\lambda}_{j+2} x},$$

and <sup>6</sup>

$$f_m(z, x) = \lambda_3 e^{\lambda_3 L} e^{\lambda_2 x} - \lambda_2 e^{\lambda_2 L} e^{\lambda_3 x} - \lambda_3 e^{\lambda_3 L} e^{\lambda_1 x}, \quad g_m(z, x) = \tilde{\lambda}_3 e^{\tilde{\lambda}_3 L} e^{\tilde{\lambda}_2 x} - \tilde{\lambda}_2 e^{\tilde{\lambda}_2 L} e^{\tilde{\lambda}_3 x} - \tilde{\lambda}_3 e^{\tilde{\lambda}_3 L} e^{\tilde{\lambda}_1 x}.$$

We have

$$\begin{aligned} \int_0^L f(z, x) g(z, x) \varphi_x(x) dx &= \int_0^L f_m(z, x) g_m(z, x) \varphi_x(x) dx + \int_0^L (f - f_m)(z, x) g_m(z, x) \varphi_x(x) dx \\ &\quad + \int_0^L f_m(z, x) (g - g_m)(z, x) \varphi_x(x) dx + \int_0^L (f - f_m)(z, x) (g - g_m)(z, x) \varphi_x(x) dx. \end{aligned}$$

By Lemma 2.2, we have

$$(5.18) \quad \int_0^L |(f - f_m)(z, x) g_m(z, x) \varphi_x(x)| dx + \int_0^L |(f - f_m)(z, x) (g - g_m)(z, x) \varphi_x(x)| dx \\ + \int_0^L |f_m(z, x) (g - g_m)(z, x) \varphi_x(x)| dx \leq C |e^{(\lambda_3 + \tilde{\lambda}_3)L}| e^{-C|z|^{1/3}}.$$

We next estimate

$$(5.19) \quad \int_0^L f_m(x, z) g_m(x, z) \varphi_x(x) = \int_0^L f_m(x, z) g_m(x, z) \left( \sum_{j=1}^3 \eta_{j+2} (\eta_{j+1} - \eta_j) e^{\eta_{j+2} x} \right) dx.$$

We first have, by Lemma 2.2,

$$(5.20) \quad \int_0^L \left( -\lambda_3 e^{\lambda_3 L} e^{\lambda_2 x} \tilde{\lambda}_2 e^{\tilde{\lambda}_2 L} e^{\tilde{\lambda}_3 x} - \lambda_2 e^{\lambda_2 L} e^{\lambda_3 x} \tilde{\lambda}_3 e^{\tilde{\lambda}_3 L} e^{\tilde{\lambda}_2 x} + \lambda_2 e^{\lambda_2 L} e^{\lambda_3 x} \tilde{\lambda}_2 e^{\tilde{\lambda}_2 L} e^{\tilde{\lambda}_3 x} \right) \\ \times \left( \sum_{j=1}^3 \eta_{j+2} (\eta_{j+1} - \eta_j) e^{\eta_{j+2} x} \right) dx \stackrel{(5.8)}{=} A^{-1} e^{(\lambda_3 + \tilde{\lambda}_3 + \lambda_2 + \tilde{\lambda}_2)L} \left( S_1(z) + O(e^{-C|z|^{1/3}}) \right),$$

where

$$(5.21) \quad S_1(z) := \sum_{j=1}^3 \eta_{j+2}^2 (\eta_{j+1} - \eta_j) \left( \frac{\lambda_2 \tilde{\lambda}_2}{\lambda_3 + \tilde{\lambda}_3 + \eta_{j+2}} - \frac{\lambda_2 \tilde{\lambda}_3}{\lambda_3 + \tilde{\lambda}_2 + \eta_{j+2}} - \frac{\lambda_3 \tilde{\lambda}_2}{\lambda_2 + \tilde{\lambda}_3 + \eta_{j+2}} \right).$$

<sup>6</sup>The index  $m$  stands the main part.

We next obtain, by Lemma 2.2,

$$(5.22) \quad \int_0^L \left( \lambda_3 e^{\lambda_3 L} e^{\lambda_1 x} \tilde{\lambda}_3 e^{\tilde{\lambda}_3 L} e^{\tilde{\lambda}_1 x} - \lambda_3 e^{\lambda_3 L} e^{\lambda_1 x} \tilde{\lambda}_3 e^{\tilde{\lambda}_3 L} e^{\tilde{\lambda}_2 x} - \lambda_3 e^{\lambda_3 L} e^{\lambda_2 x} \tilde{\lambda}_3 e^{\tilde{\lambda}_3 L} e^{\tilde{\lambda}_1 x} \right) \\ \times \left( \sum_{j=1}^3 \eta_{j+2} (\eta_{j+1} - \eta_j) e^{\eta_{j+2} x} \right) dx = e^{(\lambda_3 + \tilde{\lambda}_3)L} \left( S_2(z) + O(e^{-C|z|^{1/3}}) \right),$$

where

$$(5.23) \quad S_2(z) := \sum_{j=1}^3 \eta_{j+2} (\eta_{j+1} - \eta_j) \left( -\frac{\lambda_3 \tilde{\lambda}_3}{\lambda_1 + \tilde{\lambda}_1 + \eta_{j+2}} + \frac{\lambda_3 \tilde{\lambda}_3}{\lambda_1 + \tilde{\lambda}_2 + \eta_{j+2}} + \frac{\lambda_3 \tilde{\lambda}_3}{\lambda_2 + \tilde{\lambda}_1 + \eta_{j+2}} \right).$$

We have

$$(5.24) \quad \int_0^L \lambda_3 e^{\lambda_3 L} e^{\lambda_2 x} \tilde{\lambda}_3 e^{\tilde{\lambda}_3 L} e^{\tilde{\lambda}_2 x} \left( \sum_{j=1}^3 \eta_{j+2} (\eta_{j+1} - \eta_j) e^{\eta_{j+2} x} \right) dx = e^{(\lambda_3 + \tilde{\lambda}_3)L} S_3(z),$$

where

$$(5.25) \quad S_3(z) := \sum_{j=1}^3 \frac{\eta_{j+2} (\eta_{j+1} - \eta_j) \lambda_3 \tilde{\lambda}_3 \left( e^{\lambda_2 L + \tilde{\lambda}_2 L + \eta_{j+2} L} - 1 \right)}{\lambda_2 + \tilde{\lambda}_2 + \eta_{j+2}}.$$

We finally get, by Lemma 2.2,

$$(5.26) \quad \left| \int_0^L \left( \lambda_3 e^{\lambda_3 L} e^{\lambda_1 x} \tilde{\lambda}_2 e^{\tilde{\lambda}_2 L} e^{\tilde{\lambda}_3 x} + \lambda_2 e^{\lambda_2 L} e^{\lambda_3 x} \tilde{\lambda}_3 e^{\tilde{\lambda}_3 L} e^{\tilde{\lambda}_1 x} \right) \left( \sum_{j=1}^3 \eta_{j+2} (\eta_{j+1} - \eta_j) e^{\eta_{j+2} x} \right) dx \right| \\ = |e^{(\lambda_3 + \tilde{\lambda}_3)L}| O(e^{-Cz^{1/3}}).$$

By Lemma 2.2, we have

$$(5.27) \quad \begin{cases} \lambda_1 + \tilde{\lambda}_1 + \lambda_2 + \tilde{\lambda}_2 + \lambda_3 + \tilde{\lambda}_3 = O(z^{-1/3}), \\ \lambda_1 + \tilde{\lambda}_1 + \lambda_3 + \tilde{\lambda}_3 = O(z^{-1/3}), \\ (\lambda_3 - \lambda_2)(\tilde{\lambda}_3 - \tilde{\lambda}_2) = 3z^{2/3}(1 + O(z^{-1/3})). \end{cases}$$

We claim that

$$(5.28) \quad |z|^{-1/3} |S_1(z)| + |S_2(z)| + |S_3(z)| = O(1) \text{ for large positive } z.$$

Admitting this, by combining (5.16), (5.20), (5.22), (5.24), (5.26), and (5.27), and using (5.28), we obtain

$$(5.29) \quad \int_0^L B_D(x, z) dx = \frac{A^{-1} S_1(z)}{3z^{2/3}} + O(|z|^{-2/3}).$$

We first derive the the asymptotic behavior of  $S_1(z)$ . We have, by Lemma 2.2,

$$(5.30) \quad \lambda_2 \tilde{\lambda}_2 = z^{2/3} + O(1), \quad \lambda_2 \tilde{\lambda}_3 = z^{2/3} e^{i\pi/3} + O(1), \quad \lambda_3 \tilde{\lambda}_2 = z^{2/3} e^{-i\pi/3} + O(1),$$

and

$$(5.31) \quad \frac{1}{\lambda_3 + \tilde{\lambda}_3 + \eta_{j+2}} = \frac{1 + O(z^{-1/3})}{\sqrt{3}z^{1/3}}, \quad \frac{1}{\lambda_2 + \tilde{\lambda}_3 + \eta_{j+2}} = \frac{1 + O(z^{-1/3})}{(e^{i\pi/6} + i)z^{1/3}},$$

$$\frac{1}{\lambda_3 + \tilde{\lambda}_2 + \eta_{j+2}} = \frac{1 + O(z^{-1/3})}{(e^{-i\pi/6} - i)z^{1/3}},$$

It follows that

$$(5.32) \quad S_1(z) = \left( \frac{1}{\sqrt{3}} + 2\Re \frac{e^{i\pi/3}}{e^{i\pi/6} + i} \right) \sum_{j=1}^3 \eta_{j+2}^2 (\eta_{j+1} - \eta_j) |z|^{1/3} + O(1)$$

$$= \sqrt{3} \sum_{j=1}^3 \eta_{j+2}^2 (\eta_{j+1} - \eta_j) |z|^{1/3} + O(1).$$

We next deal with  $S_2(z)$ . Since

$$\sum_{j=1}^3 \eta_{j+2} (\eta_{j+1} - \eta_j) = 0,$$

it follows that

$$S_2(z) = \sum_{j=1}^3 \eta_{j+2} (\eta_{j+1} - \eta_j) \left( -\frac{\lambda_3 \tilde{\lambda}_3}{\lambda_1 + \tilde{\lambda}_1 + \eta_{j+2}} + \frac{\lambda_3 \tilde{\lambda}_3}{\lambda_1 + \tilde{\lambda}_1} \right)$$

$$+ \sum_{j=1}^3 \eta_{j+2} (\eta_{j+1} - \eta_j) \left( \frac{\lambda_3 \tilde{\lambda}_3}{\lambda_1 + \tilde{\lambda}_2 + \eta_{j+2}} - \frac{\lambda_3 \tilde{\lambda}_3}{\lambda_1 + \tilde{\lambda}_2} \right)$$

$$+ \sum_{j=1}^3 \eta_{j+2} (\eta_{j+1} - \eta_j) \left( \frac{\lambda_3 \tilde{\lambda}_3}{\lambda_2 + \tilde{\lambda}_1 + \eta_{j+2}} - \frac{\lambda_3 \tilde{\lambda}_3}{\lambda_2 + \tilde{\lambda}_1} \right).$$

Using Lemma 2.2, we derive that

$$(5.33) \quad S_2(z) = O(1).$$

We next derive the asymptotic behavior of  $S_3(z)$ . From Lemma 2.2, we have

$$(5.34) \quad \lambda_2 + \tilde{\lambda}_2 = O(z^{-2/3}).$$

Using (3.9), we derive from (5.25) that

$$S_3(z) = \lambda_3 \tilde{\lambda}_3 \sum_{j=1}^3 \frac{(\eta_{j+1} - \eta_j) \eta_{j+2}}{\eta_{j+2}} (A^{-1} \eta_{j+2} - 1) + O(1)$$

$$= \lambda_3 \tilde{\lambda}_3 \sum_{j=1}^3 (\eta_{j+1} - \eta_j) (A^{-1} \eta_{j+2} - 1) + O(1).$$

Since  $\sum_{j=1}^3 (\eta_{j+1} - \eta_j) = 0$  and  $\sum_{j=1}^3 (\eta_{j+1} - \eta_j) \eta_{j+2} = 0$ , it follows from (5.34) that

$$(5.35) \quad S_3 = O(1).$$

Combing (5.29) and (5.32) yields

$$\int_0^L B_D(x, z) dz = E_D |z|^{-1/3} + O(z^{-2/3}),$$

where

$$(5.36) \quad E_D = \frac{1}{\sqrt{3}A} \sum_{j=1}^3 \eta_{j+2}^2 (\eta_{j+1} - \eta_j).$$

The conclusion in the case where  $z$  is large and negative can be derived from the case where  $z$  is positive and large as follows. Define, for  $(z, x) \in \mathbb{R} \times (0, L)$ , with large  $|z|$ ,

$$M_D(z, x) = \frac{\sum_{j=1}^3 (\lambda_j e^{\lambda_j L} - \lambda_{j+1} e^{\lambda_{j+1} L}) e^{\lambda_{j+2} x}}{\sum_{j=1}^3 (\lambda_{j+1} - \lambda_j) e^{-\lambda_{j+2} L}}.$$

Then

$$B_D(z, x) = M_D(z, x) \overline{M_D(z - p, x)} \varphi_x(x).$$

It is clear from the definition of  $M_D$  that

$$M_D(-z, x) = \overline{M_D(z, x)}.$$

We then have

$$B_D(-z, x) = M_D(-z, x) \overline{M_D(-z - p, x)} \varphi_x(x) = \overline{M_D(z, x) M_D(z + p, x)} \overline{\varphi_x(x)}.$$

We thus obtain the result in the case where  $z$  is negative and large by taking the conjugate of the corresponding expression for large positive  $z$  in which  $\eta_j$  and  $p$  are replaced by  $\bar{\eta}_j$ , and  $-p$ . The conclusion follows.  $\square$

As a consequence of Lemmas 5.1 and 5.2, we obtain

**Lemma 5.3.** *Let  $p \in \mathbb{C}$ ,  $\eta_1, \eta_2, \eta_3 \in \mathbb{C} \setminus \{0\}$ , and let  $\varphi$  be defined by (5.6). Assume (5.8). Let  $u \in H^{1/3}(0, +\infty)$  with compact support in  $[0, +\infty)$ , and let  $y \in C([0, +\infty); L^2(0, L)) \cap L_{\text{loc}}^2([0, +\infty); H^1(0, L))$  be the unique solution of (5.9). Assume that  $y(t, \cdot) = 0$  for large  $t$ . We have*

$$(5.37) \quad \int_0^{+\infty} \int_0^L |y(t, x)|^2 \varphi_x(x) e^{-ipt} dx dt = \int_{\mathbb{R}} \hat{u}(z) \overline{\hat{u}(z - p)} \left( E_D |z|^{-1/3} + O(|z|^{-2/3}) \right) dz.$$

Recall that  $E_D$  is defined in (5.15).

In the next lemma, we show that  $E_D \neq 0$  under the assumptions (5.7) and (5.8).

**Lemma 5.4.** *Let  $\eta_1, \eta_2, \eta_3 \in \mathbb{C} \setminus \{0\}$  and  $L > 0$ . Assume (5.7) and (5.8). Then*

$$\sum_{j=1}^3 \eta_{j+2}^2 (\eta_{j+1} - \eta_j) \neq 0,$$

with the convention  $\eta_{j+3} = \eta_j$ .

*Proof.* We prove Lemma 5.4 by contradiction. Assume that

$$(5.38) \quad \sum_{j=1}^3 \eta_{j+2}^2 (\eta_{j+1} - \eta_j) = 0.$$

We have

$$0 = (\eta_3 - \eta_2) \eta_1^2 + (\eta_1 - \eta_3) \eta_2^2 + (\eta_2 - \eta_1) \eta_3^2 = \eta_3 (\eta_1^2 - \eta_2^2) + \eta_1 \eta_2 (\eta_2 - \eta_1) + (\eta_2 - \eta_1) \eta_3^2.$$

Since  $\eta_3 = -(\eta_1 + \eta_2)$  by (5.7), it follows that

$$(5.39) \quad 0 = -(\eta_1 + \eta_2)(\eta_1^2 - \eta_2^2) + \eta_1\eta_2(\eta_2 - \eta_1) + (\eta_2 - \eta_1)(\eta_1 + \eta_2)^2 = (\eta_2 - \eta_1)\left(2(\eta_1 + \eta_2)^2 + \eta_1\eta_2\right).$$

Set

$$c = \eta_1^2 + \eta_1\eta_2 + \eta_2^2 \quad \text{and} \quad d = \eta_1\eta_2.$$

Since, by (5.7)

$$1 = \eta_1\eta_2 + \eta_1\eta_3 + \eta_2\eta_3 = \eta_1\eta_2 - (\eta_1 + \eta_2)^2 = -(\eta_1^2 + \eta_1\eta_2 + \eta_2^2),$$

it follows that

$$(5.40) \quad c = -1$$

We first consider the case  $\eta_1 \neq \eta_2$ . From (5.39), we have

$$(5.41) \quad 2c + 3d = 2(\eta_1 + \eta_2)^2 + \eta_1\eta_2 = 0.$$

Combining (5.40) and (5.41) yields

$$d = -2c/3 = 2/3.$$

This yields

$$(\eta_1 + \eta_2)^2 = c + d = -1/3 \quad \text{and} \quad \eta_1\eta_2 = 2/3.$$

Thus either

$$\eta_1 = \frac{2i}{\sqrt{3}} \quad \text{and} \quad \eta_2 = -\frac{i}{\sqrt{3}}$$

or

$$\eta_1 = \frac{i}{\sqrt{3}} \quad \text{and} \quad \eta_2 = -\frac{2i}{\sqrt{3}}.$$

One can then check that, since  $L \in \mathbb{R}$ ,

$$\eta_1 e^{-\eta_1 L} \neq \eta_2 e^{-\eta_2 L}.$$

We have a contradiction with (5.8) in this case.

We next consider the case  $\eta_1 = \eta_2$ . Since  $c = -1$ , it follows that  $\eta_1 = \pm i/\sqrt{3}$ , which contradicts the fact that, by (5.8),

$$\eta_1 e^{-\eta_1 L} = \eta_3 e^{-\eta_3 L} = -2\eta_1 e^{2\eta_1 L}$$

since  $L \in \mathbb{R}$ .

The proof is complete.  $\square$

Using Lemma 5.4, we derive from Lemma 5.3 the following result which is the key ingredient for the analysis of the locally controllability of the KdV system (1.1).

**Proposition 5.1.** *Let  $0 < T < 1$ ,  $p \in \mathbb{C}$ ,  $\eta_1, \eta_2, \eta_3 \in \mathbb{C} \setminus \{0\}$ , and let  $\varphi$  be defined by (5.6). Assume (5.7) and (5.8). Let  $u \in H^{1/3}(0, +\infty)$  and let  $y \in C([0, +\infty); L^2(0, L)) \cap L^2_{\text{loc}}([0, +\infty); H^1(0, L))$  be the unique solution of (5.9). Assume that  $u \not\equiv 0$ ,  $u(t) = 0$  for  $t > T$ , and  $y(t, \cdot) = 0$  for large  $t$ . Then, there exists a real number  $N_D(u) \geq 0$  such that*

$$C^{-1}\|u\|_{H^{-1/6}(\mathbb{R}_+)} \leq N_D(u) \leq C\|u\|_{H^{-1/6}(\mathbb{R}_+)}$$

for some constant  $C \geq 1$  depending only on  $L$ , and

$$(5.42) \quad \int_0^\infty \int_0^L |y(t, x)|^2 e^{-ipt} \varphi_x(x) dx dt = N_D(u)(1 + O(1)T^{1/5}).$$



*Proof of Proposition 5.1.* After establishing Lemma 5.3 and Lemma 5.4, the proof of Proposition 5.1 is quite similar to the one of [21, Proposition 3.6] after applying several technical lemmas in the appendix.

By Proposition 4.1,

$$(5.43) \quad \hat{u}G/H \text{ is an entire function}$$

and by (4.13),  $G$  and  $H$  are entire functions. The same holds for  $\hat{u}$  since  $u(t) = 0$  for large  $t$ . One can show that the number of common roots of  $G$  and  $H$  in  $\mathbb{C}$  is finite, see Lemma B.1 in the appendix. Let  $z_1, \dots, z_k$  be the distinct common roots of  $G$  and  $H$  in  $\mathbb{C}$ . There exist  $m_1, \dots, m_k \in \mathbb{N}$  such that, with

$$\Gamma(z) = \prod_{j=1}^k (z - z_j)^{m_j} \quad \text{in } \mathbb{C},$$

the following two functions are entire

$$(5.44) \quad \mathcal{G}(z) := \frac{G(z)}{\Gamma(z)} \quad \text{and} \quad \mathcal{H}(z) := \frac{H(z)}{\Gamma(z)},$$

and  $\mathcal{G}$  and  $\mathcal{H}$  have no common roots. Since

$$\hat{u}\mathcal{G}/\mathcal{H} = \hat{u}G/H,$$

it follows from (5.43) that the function  $v$  defined by

$$(5.45) \quad v(z) = \hat{u}(z)/\mathcal{H}(z) \text{ in } \mathbb{C}$$

is also an entire function.

Using Lemma 2.2, one can show that

$$(5.46) \quad |v(z)| \leq C_\varepsilon e^{(T+\varepsilon)|z|} \text{ in } \mathbb{C}.$$

Since  $\mathcal{H}$  is a non-constant entire function, there exists  $\gamma > 0$  such that

$$(5.47) \quad \mathcal{H}'(z + i\gamma) \neq 0 \text{ for all } z \in \mathbb{R}.$$

Fix such an  $\gamma$  and denote  $\mathcal{H}_\gamma(z) = \mathcal{H}(z + i\gamma)$  for  $z \in \mathbb{C}$ . Set

$$(5.48) \quad \hat{w}(z) = v(z)\mathcal{H}'_\gamma(z) = \hat{u}(z)\mathcal{H}'_\gamma(z)\mathcal{H}(z)^{-1}.$$

Thus  $\hat{w}$  is an entire function and satisfies Paley-Wiener's conditions for the interval  $(-T - \varepsilon, T + \varepsilon)$  for all  $\varepsilon > 0$ , see e.g. [41, Theorem 19.3]. Indeed, this follows from the facts  $|\hat{w}(z)| \leq C_\varepsilon |v(z)| e^{\varepsilon|z|}$  for  $z \in \mathbb{C}$  by Lemma 2.2,  $|v(z)| \leq C_\varepsilon e^{(T+\varepsilon)|z|}$  for  $z \in \mathbb{C}$  by (5.46),  $|\mathcal{H}'_\gamma(z)v(z)| = |\mathcal{H}'_\gamma(z)\mathcal{H}(z)^{-1}\hat{u}(z)| \leq |\hat{u}(z)|$  for real  $z$  with large  $|z|$ , so that  $\int_{\mathbb{R}} |\hat{w}|^2 < +\infty$ .

Using Lemma 2.2, we have, for some  $\alpha \in \mathbb{C} \setminus \{0\}$ , it holds, with  $\chi(z) = z$  or  $\chi(z) = z - p$

$$(5.49) \quad \lim_{z \in \mathbb{R}, z \rightarrow +\infty} \mathcal{H}(\chi(z))|z|^{-2/3}/\mathcal{H}'_\gamma(\chi(z)) = \alpha,$$

$$(5.50) \quad \lim_{z \in \mathbb{R}, z \rightarrow -\infty} \mathcal{H}(\chi(z))|z|^{-2/3}/\mathcal{H}'_\gamma(\chi(z)) = -\bar{\alpha}.$$

and, for large positive  $z$ ,

$$(5.51) \quad |\mathcal{H}(\chi(z))|z|^{-2/3} - \alpha\mathcal{H}'_\gamma(\chi(z))| \leq C|\mathcal{H}(\chi(z))||z|^{-1} \leq C|\mathcal{H}'_\gamma(\chi(z))||z|^{-1/3},$$

and, for large negative  $z$ ,

$$(5.52) \quad |\mathcal{H}(\chi(z))|z|^{-2/3} + \bar{\alpha}\mathcal{H}'_\gamma(\chi(z))| \leq C|\mathcal{H}(\chi(z))||z|^{-1} \leq C|\mathcal{H}'_\gamma(\chi(z))||z|^{-1/3}.$$

From the definition of  $w$  by (5.48), we derive from (5.51) and (5.52) and the fact  $v$  being an entire function that

$$(5.53) \quad \int_{\mathbb{R}} (1 + |z|) |\hat{w}(z)|^2 dz \sim \int_{\mathbb{R}} \frac{|\hat{u}(z)|^2}{1 + |z|^{1/3}} dz,$$

and

$$(5.54) \quad \int_{\mathbb{R}} |\hat{w}(z)|^2 dz \sim \int_{\mathbb{R}} \frac{|\hat{u}(z)|^2}{1 + |z|^{4/3}} dz.$$

We have

$$\left| \hat{u}(z) \overline{\hat{u}(z-p)} B_D(z, x) \right| = |\varphi_x(x) \hat{y}(z, x) \overline{\hat{y}(z-p, x)}| \leq C |\hat{y}(z, x)| |\hat{y}(z-p, x)|.$$

Since  $\text{supp } y \subset [0, T] \times [0, L]$ , applying Lemma 2.3, we have, for  $(z, x) \in \mathbb{R} \times (0, L)$ ,

$$|\hat{y}(z, x)| \leq C(1 + |z|) \|u\|_{H^{-2/3}(\mathbb{R})} \quad \text{and} \quad |\hat{y}(z-p, x)| \leq C(1 + |z|) \|u\|_{H^{-2/3}(\mathbb{R})}.$$

The real zeros of  $H(z)$  and  $H(z-p)$  are simple by Lemma B.3 in the appendix, it follows from Lemma 5.3 that

$$(5.55) \quad \left| \hat{u}(z) \overline{\hat{u}(z-p)} \int_0^L B_D(z, x) dx \right| \leq C(1 + |z|)^2 \|u\|_{H^{-2/3}(0, +\infty)}^2.$$

Combining (5.54) and (5.55) yields

$$(5.56) \quad \left| \hat{u}(z) \overline{\hat{u}(z-p)} \int_0^L B_D(z, x) dx \right| \leq C(1 + |z|)^2 \|w\|_{L^2(0, +\infty)}^2,$$

Applying Lemmas 5.1 and 5.2 and using (5.51), (5.52), (5.56) and the fact  $\int_{\mathbb{R}} dz = \int_{z \in \mathbb{R}; |z| < m} dz + \int_{z \in \mathbb{R}; |z| \geq m} dz$ , we derive that

$$(5.57) \quad \left| \int_{\mathbb{R}} \hat{u}(z) \overline{\hat{u}(z-p)} \int_0^L B_D(z, x) dx dz - E_D |\alpha|^2 \int_{\mathbb{R}} |z| \hat{w}(z) \overline{\hat{w}(z-p)} dz \right| \\ \leq Cm^3 \|w\|_{L^2(\mathbb{R})}^2 + Cm^{-1/3} \int_{|z| > m} |z| |\hat{w}(z)| |\hat{w}(z-p)| dz.$$

Using the fact

$$\int_{\mathbb{R}} |z| |\hat{w}(z)| |\hat{w}(z-p) - \hat{w}(z)| dz \leq \|w\|_{\dot{H}^{1/2}(\mathbb{R})} \|w(\cdot) - w(\cdot)e^{-ip}\|_{\dot{H}^{1/2}(\mathbb{R})}$$

and, by Lemma A.2 in the appendix <sup>7</sup>,

$$\|w(\cdot) - w(\cdot)e^{-ip}\|_{\dot{H}^{1/2}(\mathbb{R})}^2 \leq C \left( T^2 \|w\|_{\dot{H}^{1/2}} + T \|w\|_{L^2}^2 \right),$$

and, since  $\text{supp } w \subset (-T, T)$ , <sup>8</sup>

$$(5.58) \quad \|w\|_{L^2(\mathbb{R})} \leq CT^{1/2} \|w\|_{\dot{H}^{1/2}(\mathbb{R})},$$

we derive that

$$\left| \int_{\mathbb{R}} |z| \hat{w}(z) \overline{\hat{w}(z-p) - \hat{w}(z)} dz \right| \leq CT^2 \|w\|_{\dot{H}^{1/2}(\mathbb{R})}^2.$$

<sup>7</sup>Here we apply with  $u = w$  and  $\varphi = \chi(1 - e^{-ip\cdot})$  where  $\chi$  is a cutoff function which is 1 on  $(-T, T)$  and 0 outside  $(-2T, 2T)$  such that  $|\nabla \chi| \leq C/T$  for some universal constant  $C$ .

<sup>8</sup>The inequality can be proved first for  $T = 1$  and then for  $0 < T < 1$  by scaling.

It then follows from (5.57) that

$$(5.59) \quad \left| \int_{\mathbb{R}} \hat{u}(z) \overline{\hat{u}(z-p)} \int_0^L B_D(z, x) dx dz - E_D |\alpha|^2 \int_{\mathbb{R}} |z| |\hat{w}(z)|^2 dz \right| \\ \leq C m^3 T^2 \|w\|_{\dot{H}^{1/2}(\mathbb{R})}^2 + C m^{-1/3} \|w\|_{\dot{H}^{1/2}(\mathbb{R})}^2 + C T^2 \|w\|_{\dot{H}^{1/2}(\mathbb{R})}^2.$$

By choosing  $m = 1/T^{3/5}$  and using again (5.58), from (5.59), we obtain

$$\int_{\mathbb{R}} \hat{u}(z) \overline{\hat{u}(z-p)} \int_0^L B_D(z, x) dx dz = E_D |\alpha|^2 \int_{\mathbb{R}} (1 + |z|) |\hat{w}(z)|^2 dz \left(1 + O(1)T^{1/5}\right).$$

The conclusion now follows from (5.53). The proof is complete.  $\square$

We present now one of the direct consequences of Proposition 5.1. Denote  $\xi_1(t, x) = \Re\{\varphi(x)e^{-ipt}\}$  and  $\xi_2(t, x) = \Im\{\varphi(x)e^{-ipt}\}$ . Then

$$(5.60) \quad \xi_1(t, x) + i\xi_2(t, x) = \varphi(x)e^{-ipt}.$$

Denote  $E_{1,D} = \Re(E_D)$  and  $E_{2,D} = \Im(E_D)$ , and set

$$(5.61) \quad \Psi(t, x) = E_{1,D}\xi_1(t, x) + E_{2,D}\xi_2(t, x).$$

Multiplying (5.42) by  $\overline{E}_D$  and normalizing appropriately, we have

**Corollary 5.1.** *Let  $p \in \mathbb{C}$ ,  $\eta_1, \eta_2, \eta_3 \in \mathbb{C} \setminus \{0\}$  and let  $\varphi$  be defined by (5.6). Assume (5.7) and (5.8). There exists  $T_* > 0$  such that, for any (real)  $u \in H^{1/3}(\mathbb{R}_+)$  with  $u(t) = 0$  for  $t > T_*$  and  $y(t, \cdot) = 0$  for large  $t$  where  $y \in C([0, +\infty); L^2(0, L)) \cap L_{\text{loc}}^2([0, +\infty); H^1(0, L))$  is the unique solution of (5.9), we have*

$$(5.62) \quad \int_0^\infty \int_0^{+\infty} y^2(t, x) \Psi_x(t, x) dx dt \geq C \|u\|_{H^{-1/6}(\mathbb{R})}^2.$$

## 6. FINITE TIME LOCAL EXACT CONTROLLABILITY OF THE KdV EQUATION FOR ALL CRITICAL LENGTH - PROOF OF THEOREM 1.1

This section is devoted to the proof of Theorem 1.1. We begin with

**Lemma 6.1.** *Let  $L \in \mathcal{N}_D$  and  $p_\ell \in \mathcal{P}_D$ . We have*

$$\Re p_\ell > 0 \text{ for } 1 \leq \ell \leq n_D.$$

*Proof.* We prove by contradiction that  $\Re p_\ell \leq 0$ . Then

$$\Re p_\ell = 0,$$

since  $\Re p_\ell \geq 0$ . It follows that  $ip_\ell$  is real.

The equation  $\eta^3 + \eta - ip_\ell = 0$  then has one real solution and two (possibly complex) solutions which are conjugate each other. Assume that  $\eta_3$  is real. With  $\eta_1 = a + bi$  and  $\eta_2 = a - bi$  ( $a, b \in \mathbb{R}$ ), we have

$$-(\eta_1\eta_2 + \eta_1\eta_3 + \eta_2\eta_3) \stackrel{\eta_3 = -(\eta_1 + \eta_2)}{=} \eta_1^2 + \eta_2^2 + \eta_1\eta_2 = (a + bi)^2 + (a - bi)^2 + a^2 + b^2 = 3a^2 - b^2 = -1.$$

This implies

$$(6.1) \quad b^2 = 3a^2 + 1.$$

We also have

$$\eta_1\eta_2\eta_3 = -2a(a^2 + b^2) \quad \text{and} \quad \eta_1\eta_2\eta_3 = ip_\ell \stackrel{\text{Corollary B.1}}{\leq} 0.$$

This yields

$$(6.2) \quad a \geq 0,$$

since  $a^2 + b^2 \neq 0$  by (6.1).

On the other hand, since  $\eta_1 e^{\eta_1 L} = \eta_2 e^{\eta_2 L} = \eta_3 e^{\eta_3 L}$ , it follows that

$$\eta_3^2 e^{2\eta_3 L} = \eta_1 \eta_2 e^{(\eta_1 + \eta_2)L}.$$

We thus obtain

$$4a^2 e^{-4aL} = (a^2 + b^2) e^{2aL}.$$

This yields

$$4a^2 e^{-6aL} = a^2 + b^2 \stackrel{(6.1)}{=} 4a^2 + 1.$$

This contradicts (6.2). The conclusion follows.  $\square$

We introduce, for  $1 \leq \ell \leq n_D$ ,

$$(6.3) \quad \mathcal{M}_\ell := \text{span} \left\{ \Re \varphi_\ell, \Im \varphi_\ell \right\}.$$

From Lemma 6.1, we know that  $\dim \mathcal{M}_\ell = 2$ . It follows that

$$(6.4) \quad \dim \mathcal{M}_D = 2n_D.$$

In what follows, for  $\xi \in \mathcal{M}_\ell$  with

$$\xi = a \Re \varphi_\ell + b \Im \varphi_\ell,$$

we denote

$$(6.5) \quad \|\xi\|_{\mathcal{M}_\ell} = (a^2 + b^2)^{1/2}.$$

For  $c > 0$  and  $\varphi \in L^2(0, L)$ , let  $B_c(\varphi)$  denote the open ball in  $L^2(0, L)$  centered at  $\varphi$  and with radius  $c$ , i.e.,

$$(6.6) \quad B_c(\varphi) = \left\{ \phi \in L^2(0, L); \|\phi - \varphi\|_{L^2(0, L)} < c \right\}.$$

The following result is important for the proof of Theorem 1.1.

**Lemma 6.2.** *For  $L \in \mathcal{N}_D$ . There exists  $T = T_L$  such that for all  $0 < c_1 < c_2$ , there exists  $0 < c_3 < c_1$  such that for  $\varphi \in \mathcal{M}_D$  with  $c_1 \leq \|\text{proj}_{\mathcal{M}_\ell} \varphi\|_{\mathcal{M}_\ell} \leq c_2$  for all  $1 \leq \ell \leq n_D$ , there exist two mappings  $U_1 : B_{c_3}(\varphi) \rightarrow H^{1/3}(0, T)$  and  $U_2 : B_{c_3}(\varphi) \rightarrow H^{1/3}(0, T)$  such that for  $\psi \in B_{c_3}(\varphi)$ , the unique solutions  $y_1 \in X_T$  and  $y_2 \in X_T$  of the following two systems, with  $u_1 = U_1(\psi)$  and  $u_2 = U_2(\psi)$ ,*

$$(6.7) \quad \begin{cases} y_{1,t} + y_{1,x} + y_{1,xxx} = 0 & \text{in } (0, T) \times (0, L), \\ y_1(\cdot, 0) = y_1(\cdot, L) = 0 & \text{in } (0, T), \\ y_{1,x}(\cdot, L) = u_1 & \text{in } (0, T), \\ y_1(0, \cdot) = 0 & \text{in } (0, L), \end{cases}$$

$$(6.8) \quad \begin{cases} y_{2,t} + y_{2,x} + y_{2,xxx} + y_1 y_{1,x} = 0 & \text{in } (0, T) \times (0, L), \\ y_2(\cdot, 0) = y_2(\cdot, L) = 0 & \text{in } (0, T), \\ y_{2,x}(\cdot, L) = u_2 & \text{in } (0, T), \\ y_2(0, \cdot) = 0 & \text{in } (0, T), \end{cases}$$

satisfy

$$y_1(T, \cdot) = 0 \quad \text{and} \quad y_2(T, \cdot) = \psi.$$

Moreover, for  $\psi, \tilde{\psi} \in B_{c_3}(\varphi)$ ,

$$(6.9) \quad \|U_1(\psi) - U_1(\tilde{\psi})\|_{H^{1/3}(0,T)} \leq C\|\psi - \tilde{\psi}\|_{L^2(0,T)}$$

and

$$(6.10) \quad \|U_2(\psi) - U_2(\tilde{\psi})\|_{H^{1/3}(0,T)} \leq C\|\psi - \tilde{\psi}\|_{L^2(0,T)},$$

for some positive constant  $C$  depending only on  $L, T, c_1$ , and  $c_2$ .

*Proof.* In this proof, for notational ease, we denote  $n_D$  by  $n$ . We first claim that for

$$(6.11) \quad 1 \leq k \leq n \text{ and } 1 \leq i_1, \dots, i_k \leq n \text{ distinct,}$$

there exists  $T$ , depending only on  $L$ , and  $u_1 \in H^{1/3}(0, T)$  such that

$$(6.12) \quad y_1(T, \cdot) = 0, \quad \text{proj}_{\mathcal{M}_{i_1}} y_2(T, \cdot) \neq 0 \text{ and } \text{proj}_{\mathcal{M}_{i_j}} y_2(T, \cdot) = 0 \text{ for } 2 \leq j \leq k,$$

where  $y_1 \in X_T$  is the unique solution of (6.7) and  $y_2 \in X_T$  is the unique solution of (6.8) with  $u_2 = 0$  in  $(0, T)$ .

We prove this claim by recurrence in  $k$ . We first note that this claim holds for  $k = 1$  by Corollary 5.1. Assume that the claim holds for  $1 \leq k \leq n - 1$  and for all  $1 \leq i_1, \dots, i_k \leq n$  distinct, we prove it for  $k + 1$  and for all  $1 \leq i_1, \dots, i_{k+1} \leq n$  distinct.

Let  $1 \leq i_1, \dots, i_{k+1} \leq n$  be distinct. For notational ease, we assume that

$$i_j = j \text{ for } j \leq 1 \leq k + 1.$$

By the recurrence we know that there exist  $T > 0$  and  $u_1 \in H^{1/3}(0, T)$  and  $\hat{T} > 0$  and  $\hat{u}_1 \in H^{1/3}(0, \hat{T})$  such that the corresponding solutions  $y_1$  and  $y_2$  in the time interval  $(0, T)$  with  $y_{1,x}(\cdot, L) = u_1$  in  $(0, T)$  of (6.7) and  $y_{2,x}(\cdot, L) = 0$  in  $(0, T)$  of (6.8), and the corresponding solutions  $\hat{y}_1$  and  $\hat{y}_2$  in the time interval  $(0, \hat{T})$  with  $\hat{y}_{1,x}(\cdot, L) = \hat{u}_1$  in  $(0, \hat{T})$  and  $\hat{y}_{2,x}(\cdot, L) = 0$  in  $(0, \hat{T})$  satisfy

$$(6.13) \quad y_1(T, \cdot) = 0, \quad \text{proj}_{\mathcal{M}_1} y_2(T, \cdot) \neq 0, \quad \text{and} \quad \text{proj}_{\mathcal{M}_j} y_2(T, \cdot) = 0 \text{ for } 2 \leq j \leq k,$$

$$(6.14) \quad \hat{y}_1(\hat{T}, \cdot) = 0, \quad \text{proj}_{\mathcal{M}_{k+1}} \hat{y}_2(\hat{T}, \cdot) \neq 0, \quad \text{and} \quad \text{proj}_{\mathcal{M}_j} \hat{y}_2(\hat{T}, \cdot) = 0 \text{ for } 2 \leq j \leq k.$$

We still denote  $y_1, y_2$  the solution of (6.7) and (6.8), respectively, with  $t \in \mathbb{R}_+$  where we extend  $u_1$  by 0 for  $t > T$  ( $y_1(t, \cdot) = 0$  for  $t > T$  then), and do similarly for  $\hat{y}_1$  and  $\hat{y}_2$ .

Without loss of generality, we can assume that

$$(6.15) \quad \text{proj}_{\mathcal{M}_{k+1}} y_2(T, \cdot) \neq 0$$

since the claim is proved in this case otherwise.

For  $\tau_1$  and  $\tau_2$  being two non-negative constants and  $\alpha$  being a positive constant all determined later, we define  $U_1$  in  $(0, T + \hat{T} + \tau_1 + \tau_2)$  by

$$(6.16) \quad U_1(t) = \begin{cases} u_1(t) & \text{in } (0, T), \\ 0 & \text{in } (T, T + \tau_1), \\ \alpha \hat{u}_1(t - T - \tau_1) & \text{in } (T + \tau_1, T + \tau_1 + \hat{T}), \\ 0 & \text{in } (T + \hat{T} + \tau_1, T + \hat{T} + \tau_1 + \tau_2), \end{cases}$$

Then  $U_1 \in H^{1/3}(0, T + \hat{T} + \tau_1 + \tau_2)$  and

$$(6.17) \quad \|U\|_{H^{1/3}((0, T + \hat{T} + \tau_1 + \tau_2))} \leq C \left( \|u_1\|_{H^{1/3}(0, T)} + |\alpha| \|\hat{u}_1\|_{H^{1/3}(0, \hat{T})} \right)$$

by Lemma A.1 in the appendix.

Let  $Y_1$  and  $Y_2$  be the corresponding solution of (6.7) and (6.8) in the time interval  $(0, T + \hat{T} + \tau_1 + \tau_2)$  with  $Y_{1,x}(\cdot, L) = U_1$  and  $Y_{2,x}(\cdot, L) = 0$ . Since  $y_1(T, \cdot) = 0$  and  $\hat{y}_1(\hat{T}, \cdot) = 0$ , it follows that

$$Y_1(T + \hat{T} + \tau_1 + \tau_2, \cdot) = 0.$$

Using (6.15), by choosing appropriately  $\tau_1$ , one has, by Lemma 6.1,

$$\text{proj}_{\mathcal{M}_{k+1}} y_2(T + \hat{T} + \tau_1, \cdot) = -c \text{proj}_{\mathcal{M}_{k+1}} \hat{y}_2(\hat{T}, \cdot),$$

for some  $c > 0$ . We now choose  $\alpha = c^{1/2}$ . Then

$$\begin{aligned} (6.18) \quad \text{proj}_{\mathcal{M}_{k+1}} Y_2(T + \hat{T} + \tau_1, \cdot) \\ &= \text{proj}_{\mathcal{M}_{k+1}} y_2(T + \hat{T} + \tau_1, \cdot) + \alpha^2 \text{proj}_{\mathcal{M}_{k+1}} \hat{y}_2(\hat{T}, \cdot) \\ &= -c \text{proj}_{\mathcal{M}_{k+1}} y_2(\hat{T}, \cdot) + c \text{proj}_{\mathcal{M}_{k+1}} y_2(\hat{T}, \cdot) = 0, \end{aligned}$$

which yields

$$(6.19) \quad \text{proj}_{\mathcal{M}_{k+1}} Y_2(T + \hat{T} + \tau_1 + \tau_2, \cdot) = 0.$$

We have

$$(6.20) \quad \text{proj}_{\mathcal{M}_1} Y_2(T + \hat{T} + \tau_1, \cdot) = \text{proj}_{\mathcal{M}_1} y_2(T + \hat{T} + \tau_1, \cdot) + c \text{proj}_{\mathcal{M}_1} \hat{y}_2(\hat{T}, \cdot).$$

If  $\text{proj}_{\mathcal{M}_1} \hat{y}_2(\hat{T}, \cdot) = 0$ , we choose  $\tau_2 = 0$ . Then

$$(6.21) \quad \text{proj}_{\mathcal{M}_1} Y_2(T + \hat{T} + \tau_1 + \tau_2, \cdot) = \text{proj}_{\mathcal{M}_1} y_2(T + \hat{T} + \tau_1, \cdot) \neq 0$$

since  $\text{proj}_{\mathcal{M}_1} y_2(T, \cdot) \neq 0$ . The claim for  $k+1$  follows with the control  $U_1$  (with  $\tau_2 = 0$ ). Otherwise,  $\text{proj}_{\mathcal{M}_1} \hat{y}_2(\hat{T}, \cdot) \neq 0$ . Then if  $\mathfrak{R}p_1 \neq \mathfrak{R}p_{k+1}$ , one can choose  $\tau_2 > 0$  such that  $\text{proj}_{\mathcal{M}_1} y_2(T + \hat{T} + \tau_1 + \tau_2)$  and  $\text{proj}_{\mathcal{M}_1} y_2(\hat{T} + \tau_2)$  are independent. The claim for  $k+1$  follows with the control  $U_1$ . Otherwise,  $\mathfrak{S}p_1 \neq \mathfrak{S}p_{k+1}$  since  $\mathfrak{R}p_1 = \mathfrak{R}p_{k+1}$  and  $p_1 \neq p_{k+1}$ . Since, after recalling (6.5),

$$\|\text{proj}_{\mathcal{M}_1} y_2(T + \hat{T} + \tau_1 + \tau_2)\|_{\mathcal{M}_1} = \|\text{proj}_{\mathcal{M}_1} y_2(T + \hat{T} + \tau_1)\|_{\mathcal{M}_1} e^{-\mathfrak{S}p_1 \tau_2}$$

and

$$c \|\text{proj}_{\mathcal{M}_1} \hat{y}_2(T + \hat{T} + \tau_1 + \tau_2)\|_{\mathcal{M}_1} = c \|\text{proj}_{\mathcal{M}_1} \hat{y}_2(T + \hat{T} + \tau_1)\|_{\mathcal{M}_1} e^{-\mathfrak{S}p_{k+1} \tau_2}.$$

It follows from (6.20) that

$$\text{proj}_{\mathcal{M}_1} Y_2(T + \hat{T} + \tau_1 + \tau_2, \cdot) \neq 0$$

for some  $\tau_2 > 0$ .

Therefore the claim is proved.

From the claim with  $k = n+1$ , one can derive that there exist a positive  $T$  and a Lipschitz map from

$$V_1 : B_{c_3}(\varphi) \cap \mathcal{M}_D \rightarrow H^{1/3}(0, T)$$

such that

$$y_1(T, \cdot) = 0 \quad \text{and} \quad \text{proj}_{\mathcal{M}_D} y_2 = \psi,$$

for  $\psi \in B_{c_3}(\varphi) \cap \mathcal{M}_D$ . The conclusion now follows by considering

$$U_1(\psi) = V_1 \circ \text{proj}_{\mathcal{M}_D}(\psi) \quad \text{and} \quad U_2 = \mathcal{L}_T(y_2(T) - \text{proj}_{\mathcal{M}_D} \psi)$$

where  $\mathcal{L}_T = \mathcal{L}$  is given by Proposition 3.1 and  $y_2$  is the solution corresponding to  $U_1(\psi)$ .

The proof is complete.  $\square$

**Remark 6.1.** When  $\dim \mathcal{M}_D = 2$ , the proof of Lemma 6.2 is similar to the analysis in [13] where the rotation idea is involved. In the case,  $\dim \mathcal{M}_D > 2$ , we cannot extend the analysis in [15]. Our situation is more complicated than the one in [15] since  $\mathfrak{S}p_\ell$  might be not zero and therefore, it might happen that  $\mathfrak{S}p_{\ell_1} = \mathfrak{S}p_{\ell_2}$  for some  $\ell_1 \neq \ell_2$ .

We are ready to give the proof of Theorem 1.1

*Proof of Theorem 1.1.* Fix  $y_0, y_T \in L^2(0, L)$  with small  $L^2$ -norms. For simplicity of the presentation (and by Proposition 3.1), we will assume that  $\|y_0\|_{L^2(0, L)} \leq \|y_T\|_{L^2(0, L)}$ . Set  $\rho = \|y_T\|_{L^2(0, L)}$  and assume that  $\rho > 0$  otherwise, one just takes the zero control and the conclusion follows.

Let  $w_0$  be the state at the time  $T$  of the solution of the linearized KdV system with the zero control starting from  $P_{\mathcal{M}_D} y_0$  at the time 0. We first consider the case where

$$(6.22) \quad \|\text{proj}_{\mathcal{M}_\ell} y_T\|_{\mathcal{M}_\ell} \geq c_1 \rho \text{ for } 1 \leq \ell \leq n = n_D,$$

for some small constant  $c_1$  independent of  $\rho$  and defined later.

Denote  $c$  the constant  $c_3$  in Lemma 6.2 corresponding to two constants  $c_1$  and  $1 (= c_2)$ . Set

$$\begin{aligned} \mathbb{G}: \overline{B_{c\rho}(y_T)} &\rightarrow H^{1/3}(0, T) \\ \varphi &\mapsto \rho \mathbf{u}_0 + \rho^{1/2} u_1 + \rho u_2. \end{aligned}$$

Recall that  $B_c(y_T)$  is defined in (6.6) and the closure here is considered with the  $L^2(0, L)$ -norm. Here we decompose  $\varphi$  as

$$\varphi = \text{proj}_{\mathcal{M}_D^\perp} \varphi + \text{proj}_{\mathcal{M}_D} \varphi,$$

$\mathbf{u}_0 \in H^{1/3}(0, T)$  is a control for which the corresponding solution  $\mathbf{y}_0$  of the linearized KdV system (3.1) starting from  $P_{\mathcal{M}_D^\perp} y_0 / \rho$  at 0 and arriving  $P_{\mathcal{M}_D^\perp} \varphi / \rho$  at the time  $T$ , and  $u_1$  and  $u_2$  are controls for which the solutions  $y_1$  and  $y_2$  of the system (6.7) and ((6.8) with the initial data  $P_{\mathcal{M}_D} y_0 / \rho$  instead of 0) satisfy  $y_1(T, \cdot) = 0$  and  $y_2(T, \cdot) = P_{\mathcal{M}_D} \varphi / \rho$ . Moreover, by Proposition 3.1, one can choose  $\mathbf{u}_0$  in such a way that  $\mathbf{u}_0 = \mathbf{u}_0(\varphi)$  is a Lipschitz function of  $\varphi$  with the Lipschitz constant bounded by a positive constant independent of  $\rho$ , and by Lemma 6.2 one can choose  $u_1 = u_1(\varphi)$  and  $u_2 = u_2(\varphi)$  as Lipschitz functions of  $P_{\mathcal{M}_D} \varphi / \rho$  with the Lipschitz constants bounded by positive constants independent of  $\rho$ .

Let  $\varepsilon_0$  be a small positive constant such that the conclusion of Proposition 2.3 holds if  $\|u\|_{H^{1/3}(0, T)} \leq \varepsilon_0$  and  $\|y_0\|_{L^2(0, L)} < \varepsilon_0$ . Set

$$\begin{aligned} \mathcal{P}: \left\{ w \in H^{1/3}(0, T); \|w\|_{H^{1/3}(0, T)} \leq \varepsilon_0 \right\} &\rightarrow L^2(0, L) \\ w &\mapsto y(T, \cdot), \end{aligned}$$

where  $y$  is the unique solution of the KdV system (1.1) with  $u = w$  starting from  $y_0$  at time 0. Consider the map<sup>9</sup>

$$\begin{aligned} \Lambda: \overline{B_{c\rho}(y_T)} &\rightarrow L^2(0, L) \\ \varphi &\mapsto \varphi - \mathcal{P} \circ \mathbb{G}(\varphi) + y_T. \end{aligned}$$

We will prove that

$$(6.23) \quad \Lambda(\varphi) \in \overline{B_{c\rho}(y_T)},$$

and

$$(6.24) \quad \|\Lambda(\varphi) - \Lambda(\phi)\|_{L^2(0, L)} \leq \lambda \|\varphi - \phi\|_{L^2(0, L)},$$

for some  $\lambda \in (0, 1)$ . Assuming this, one derives from the contraction mapping theorem that there exists a unique  $\varphi_0 \in \overline{B_{c\rho}(y_T)}$  such that  $\Lambda(\varphi_0) = \varphi_0$ . As a consequence,

$$y_T = \mathcal{P} \circ \mathbb{G}(\varphi_0),$$

and  $\mathbb{G}(\varphi_0)$  is hence a required control.

<sup>9</sup>We thus implicitly require that  $\rho$  is much smaller than  $\varepsilon_0$ .

We next establish (6.23) and (6.24) using similar ideas as in [21]. Indeed, assertion (6.23) follows from the fact

$$\|\varphi - \mathcal{P} \circ \mathbb{G}(\varphi)\|_{L^2(0,L)} \leq C \|\varphi\|_{L^2(0,L)}^{3/2} \text{ for } \varphi \in \overline{B_{c\rho}(y_T)}.$$

This can be proved using the approximation via the power series method as follows. Set <sup>10</sup>

$$(6.25) \quad u = \rho \mathbf{u}_0 + \rho^{1/2} u_1 + \rho u_2 \quad \text{and} \quad y_a = \rho \mathbf{y}_0 + \rho^{1/2} y_1 + \rho y_2.$$

Let  $y$  be the solution of the KdV system (1.1) with  $y(t=0, \cdot) = y_0$  and with  $u$  defined above. Considering the system of  $y - y_a$ , we derive that, by e.g. Proposition 2.1 with  $(h_1, h_2, h_3) = (0, 0, 0)$  and with the zero initial datum, for small  $\rho$ ,

$$(6.26) \quad \|y - y_a\|_{X_T} \leq C \|(yy_x, \rho^{3/2}(y_1 y_{2,x} + y_2 y_{1,x}), \rho^2 y_2 y_{2,x})\|_{L^1((0,T);L^2(0,L))} \leq C \rho^{3/2}.$$

Assertion (6.23) follows since  $y(T, \cdot) = \mathcal{P} \circ \mathbb{G}(\varphi)$  and  $y_a(T, \cdot) = \varphi$ .

We next establish (6.24). To this end, we estimate

$$\left( \varphi - \mathcal{P} \circ \mathbb{G}(\varphi) \right) - \left( \tilde{\varphi} - \mathcal{P} \circ \mathbb{G}(\tilde{\varphi}) \right).$$

Denote  $\tilde{\mathbf{u}}_0, \tilde{u}_1, \tilde{u}_2, \tilde{u}$  and  $\tilde{\mathbf{y}}_0, \tilde{y}_1, \tilde{y}_2, \tilde{y}_a, \tilde{y}$  the functions corresponding to  $\tilde{\varphi}$  which are defined in the same way as the functions  $\mathbf{u}_0, u_1, u_2, u$  and  $\mathbf{y}_0, y_1, y_2, y_a, y$  defined for  $\varphi$ .

We have

$$(6.27) \quad (y - \tilde{y})_t + (y - \tilde{y})_x + (y - \tilde{y})_{xxx} + yy_x - \tilde{y}\tilde{y}_x = 0,$$

$$(6.28) \quad (y_a - \tilde{y}_a)_t + (y_a - \tilde{y}_a)_x + (y_a - \tilde{y}_a)_{xxx} + y_a y_{a,x} - \tilde{y}_a \tilde{y}_{a,x} = g(t, x),$$

where

$$(6.29) \quad g(t, x) = \rho^{3/2} \left( (y_1 y_2)_x - (\tilde{y}_1 \tilde{y}_2)_x \right) + \rho^2 \left( y_2 y_{2,x} - \tilde{y}_2 \tilde{y}_{2,x} \right) + \rho^2 \left( \mathbf{y}_0 \mathbf{y}_{0,x} - \tilde{\mathbf{y}}_0 \tilde{\mathbf{y}}_{0,x} \right) \\ + \rho^{3/2} \left( \mathbf{y}_0 (y_1 + \rho^{1/2} y_2) - \tilde{\mathbf{y}}_0 (\tilde{y}_1 + \rho^{1/2} \tilde{y}_2) \right)_x.$$

By Proposition 2.1, we have

$$(6.30) \quad \|y - \tilde{y}\|_{X_T} \leq C \rho^{-1/2} \|\varphi - \tilde{\varphi}\|_{L^2(0,L)} \leq C \rho^{1/2}.$$

By (6.26) for  $y - y_a$  and similar fact for  $\tilde{y} - \tilde{y}_a$ , we obtain

$$(6.31) \quad \|(y - y_a, \tilde{y} - \tilde{y}_a)\|_{X_T} \leq C \rho^{3/2}.$$

Using (6.25) for  $y_a$  and similar one for  $\tilde{y}_a$ , and applying Proposition 2.1, one gets

$$(6.32) \quad \|y_a - \tilde{y}_a\|_{X_T} \leq C \rho^{-1/2} \|\varphi - \tilde{\varphi}\|_{L^2(0,L)} \leq C \rho^{1/2}.$$

From the definition of  $g$  (6.29), we deduce that

$$(6.33) \quad \|g\|_{L^1((0,T);L^2(0,L))} \leq C \rho^{1/2} \|\varphi - \tilde{\varphi}\|_{L^2(0,L)}.$$

Set

$$Y = y - y_a - \tilde{y} + \tilde{y}_a \text{ in } (0, T) \times (0, L).$$

Using (6.27) and (6.28), we have

$$(6.34) \quad \partial_t Y + \partial_x Y + \partial_{xxx} Y = f(t, x) \text{ in } (0, T) \times (0, L),$$

where

$$f(t, x) = -g(t, x) - \left( yy_x - \tilde{y}\tilde{y}_x - (y_a y_{a,x} - \tilde{y}_a \tilde{y}_{a,x}) \right)$$

<sup>10</sup>The index  $a$  stands the approximation.



From (6.30) and (6.32), we obtain

$$(6.35) \quad \|yy_x - \widetilde{y}\widetilde{y}_x - (y_a y_{a,x} - \widetilde{y}_a \widetilde{y}_{a,x})\|_{L^1((0,T);L^2(0,L))} \leq C\rho^{1/2}\|\varphi - \widetilde{\varphi}\|_{L^2(0,L)}.$$

Using (6.33) and (6.35) and applying Proposition 2.1 to  $Y$ , we derive from (6.34) that

$$\|(y - y_a - \widetilde{y} + \widetilde{y}_a)(T, \cdot)\|_{L^2(0,L)} \leq C\rho^{1/2}\|\varphi - \phi\|_{L^2(0,L)}.$$

Assertion (6.24) follows.

We next consider the case  $\|P_{\mathcal{M}_\ell} y_T\|_{\mathcal{M}_\ell} \leq c_1 \|y_T\|_{L^2(0,L)}$  for some  $1 \leq \ell \leq n_D$ . One can bring this case to the previous case as follows. Fix  $\varepsilon > 0$  small. By Lemma 5.4 and Corollary 5.1, there exists  $v_1, v_2 \in H^{1/3}(0, \varepsilon)$  such that if  $y_1 \in X_\varepsilon$  (with  $T = \varepsilon$ ) is the solution of (6.7) with  $u_1 = v_1$  and  $y_2 \in X$  is the solution of (6.8) with  $u_2 = v_2$  then

$$y_1(\varepsilon, \cdot) = 0, \quad \text{proj}_{\mathcal{M}_D^\perp} y_2 = 0, \quad \text{and} \quad \|\text{proj}_{\mathcal{M}_\ell} y_2(\varepsilon, \cdot)\|_{L^2(0,L)} \sim 1 \text{ for all } 1 \leq \ell \leq n_D.$$

Let  $u_{0,T}, u_{1,T}, u_{2,T}$  defined in  $(0, \varepsilon)$  be such that  $u_{0,T}$  is a control for which the corresponding solution in  $X_\varepsilon$  of the linearized KdV system starting from  $\text{proj}_{\mathcal{M}_D^\perp} y_T(L - \cdot)/\rho$  at 0 and arriving 0 at the time  $\varepsilon$ ,  $u_{1,T} = \gamma v_1$ ,  $u_{2,T} = \gamma^2 v_2$  for some  $\gamma > 0$  defined later. Let  $\mathbf{y}$  be the unique solution of the (nonlinear) KdV system in the time interval  $[T, T + \varepsilon]$  using the control

$$\rho u_0(\cdot - T) + \rho^{1/2} u_1(\cdot - T) + \rho u_2(\cdot - T),$$

with  $\mathbf{y}(T, \cdot) = y_T(L - \cdot)$ . By choosing  $\gamma$  large enough,  $y_0$  and  $\mathbf{y}(T + \varepsilon, L - \cdot)$  satisfy the setting of the previous case for the time interval  $[0, T + \varepsilon]$  (instead of  $[0, T]$ ). One now considers the control (for the nonlinear KdV system) in the time interval  $[0, T + 2\varepsilon]$  which is equal to the one which brings  $y_0$  at the time 0 to  $\mathbf{y}(T + \varepsilon, L - \cdot)$  at the time  $T + \varepsilon$  obtained in the previous case in the time interval  $[0, T + \varepsilon]$ , and is equal to  $-\mathbf{y}_x(2(T + \varepsilon) - t, 0)$  for  $t \in [T + \varepsilon, T + 2\varepsilon]$ . It is clear that the solution of the nonlinear KdV system at the time  $T + 2\varepsilon$  is  $y_T$ . The proof is complete by changing  $T + 2\varepsilon$  to  $T$ .  $\square$

## 7. SMALL TIME LOCAL CONTROLLABILITY PROPERTIES OF THE KdV SYSTEM - PROOF OF THEOREM 1.2

The main result of this section is the following, which implies in particular Theorem 1.2.

**Theorem 7.1.** *Let  $L \in \mathcal{N}_D$  and  $1 \leq \ell \leq n_D$ . Let  $\Psi$  be defined in (5.61) with  $\varphi = \varphi_\ell$  given in (3.10) and  $E_D$  is determined by (5.15) with  $(\eta_1, \eta_2, \eta_3) = (\eta_{1,\ell}, \eta_{2,\ell}, \eta_{3,\ell})$ . For all  $0 < T < T_*/2$ <sup>11</sup>, there exists  $\varepsilon_0 > 0$  such that for all  $0 < \varepsilon < \varepsilon_0$ , and for all solutions  $y \in X_T$  of*

$$(7.1) \quad \begin{cases} y_t + y_x + y_{xxx} + yy_x = 0 & \text{in } (0, T) \times (0, L), \\ y(\cdot, 0) = y(\cdot, L) = 0 & \text{in } (0, T), \\ y_x(\cdot, L) = u & \text{in } (0, T), \\ y(0, \cdot) = y_0 := \varepsilon \Psi(0, \cdot) & \text{in } (0, L), \end{cases}$$

with  $\|u\|_{H^{1/3}(0,T)} < \varepsilon_0$ , we have

$$y(T, \cdot) \neq 0.$$

**Remark 7.1.** Recall that, by (3.12) and (3.13), the function  $\Psi(t, x)$  satisfies

$$(7.2) \quad \begin{cases} \Psi_t + \Psi_x + \Psi_{xxx} = 0 & \text{in } \mathbb{R}_+ \times (0, L), \\ \Psi(\cdot, 0) = \Psi(\cdot, L) = \Psi_x(\cdot, 0) = \Psi_{xx}(\cdot, L) = 0 & \text{in } \mathbb{R}_+. \end{cases}$$

<sup>11</sup> $T_*$  is the constant in Corollary 5.1.

*Proof of Theorem 7.1.* Fix  $0 < T < T_*/2$ . Let  $\varepsilon_0$  be a small positive constant, which depends only on  $L$  and  $T$  and is determined later. We prove Theorem 7.1 by contradiction. Assume that there exists a solution  $y \in C([0, +\infty); L^2(0, L)) \cap L^2_{\text{loc}}([0, +\infty); H^1(0, L))$  of (7.1) with  $y(t, \cdot) = 0$  for  $t \geq T$ , for some  $u \in H^{1/3}(0, +\infty)$ , for some  $0 < \varepsilon < \varepsilon_0$ , and for some  $0 < T < T_*/2$  with  $\|u\|_{H^{1/3}(\mathbb{R}_+)} < \varepsilon_0$ , and  $\text{supp } u \subset [0, T]$ .

Using the fact  $y(t, \cdot) = 0$  for  $t \geq T_*/2$ , from Proposition 2.1, we have, for  $\varepsilon_0$  small,

$$(7.3) \quad \|y\|_{L^2(\mathbb{R}_+; H^1(0, L))} \leq C \left( \|y_0\|_{L^2(0, L)} + \|u\|_{H^{1/3}(\mathbb{R}_+)} \right).$$

and [21, Lemma 4.6]

$$(7.4) \quad \|y\|_{L^2(\mathbb{R}_+ \times (0, L))} \leq C \left( \|y_0\|_{L^2(0, L)} + \|u\|_{H^{-1/3}(\mathbb{R}_+)} \right).$$

Here and in what follows,  $C$  denotes a positive constant depending on  $T_*$  but not on  $T$ .

Let  $y_1$  be the solution of the linearized system in the time interval  $(0, T)$  with the control  $u$  starting at 0 at time 0, i.e.,

$$\begin{cases} y_{1,t} + y_{1,x} + y_{1,xxx} = 0 & \text{in } \mathbb{R}_+ \times (0, L), \\ y_1(\cdot, 0) = y_1(\cdot, L) = 0 & \text{in } \mathbb{R}_+, \\ y_{1,x}(\cdot, L) = u & \text{in } \mathbb{R}_+, \\ y_1(0, \cdot) = 0 & \text{in } \mathbb{R}_+, \end{cases}$$

By [21, Lemma 4.6], we also have

$$(7.5) \quad \|y_1\|_{L^2((0, T_*) \times (0, L))} \leq C \|u\|_{H^{-1/3}(\mathbb{R}_+)}.$$

Using (7.2), we obtain

$$\begin{cases} (y - y_1 - \varepsilon\Psi)_t + (y - y_1 - \varepsilon\Psi)_x + (y - y_1 - \varepsilon\Psi)_{xxx} + yy_x = 0 & \text{in } \mathbb{R}_+ \times (0, L), \\ (y - y_1 - \varepsilon\Psi)(\cdot, 0) = (y - y_1 - \varepsilon\Psi)(\cdot, L) = 0 & \text{in } \mathbb{R}_+, \\ (y - y_1 - \varepsilon\Psi)_x(\cdot, L) = 0 & \text{in } \mathbb{R}_+, \\ (y - y_1 - \varepsilon\Psi)(0, \cdot) = 0 & \text{in } (0, L), \end{cases}$$

Applying [21, Lemma 5.4], we derive that

$$(7.6) \quad \begin{aligned} & \|y - y_1 - \varepsilon\Psi\|_{L^2((0, T_*) \times (0, L))} \\ & \leq C \|yy_x\|_{L^1((0, T_*) \times (0, L))} \leq C \|y\|_{L^2((0, T_*) \times (0, L))} \|y_x\|_{L^2((0, T_*) \times (0, L))} \\ & \stackrel{(7.3), (7.4)}{\leq} C \left( \|y_0\|_{L^2(0, L)} + \|u\|_{H^{1/3}(\mathbb{R}_+)} \right) \left( \|y_0\|_{L^2(0, L)} + \|u\|_{H^{-1/3}(\mathbb{R}_+)} \right). \end{aligned}$$

Since  $y = 0$  for  $t \geq T_*/2$  and  $u = 0$  for  $t \geq T_*/2$ , after considering the projection into  $\mathcal{M}_D^\perp$ , we derive that

$$(7.7) \quad \|y_1(T_*, \cdot)\|_{L^2(0, L)} \leq C \left( \|y_0\|_{L^2(0, L)} + \|u\|_{H^{1/3}(\mathbb{R}_+)} \right) \left( \|y_0\|_{L^2(0, L)} + \|u\|_{H^{-1/3}(\mathbb{R}_+)} \right).$$

By Proposition 3.1, there exists  $u_1 \in H^{1/3}(0, T_*)$  such that

$$(7.8) \quad \|u_1\|_{H^{1/3}(0, T_*)} \leq C \|y_1(T_*, \cdot)\|_{L^2(0, L)}$$

and the solution  $\tilde{y}_1 \in X_{T_*}$  of the system

$$\begin{cases} \tilde{y}_{1,t} + \tilde{y}_{1,x} + \tilde{y}_{1,xxx} = 0 & \text{in } (0, T_*) \times (0, L), \\ \tilde{y}_1(\cdot, 0) = \tilde{y}_1(\cdot, L) = 0 & \text{in } (0, T_*), \\ \tilde{y}_{1,x}(\cdot, L) = u_1 & \text{in } (0, T_*), \\ \tilde{y}_1(0, \cdot) = 0 & \text{in } (0, T_*), \end{cases}$$

satisfies

$$\tilde{y}_1(T_*, \cdot) = -y_1(T_*, \cdot)$$

Using (7.7), we derive from (7.8) that

$$(7.9) \quad \|u_1\|_{H^{1/3}(0, T_*)} \leq C \left( \|y_0\|_{L^2(0, L)} + \|u\|_{H^{1/3}(\mathbb{R}_+)} \right) \left( \|y_0\|_{L^2(0, L)} + \|u\|_{H^{-1/3}(\mathbb{R}_+)} \right),$$

Set

$$\hat{y} = y_1 + \tilde{y}_1 \text{ in } (0, T_*) \times (0, L).$$

We have

$$\begin{cases} (y - \hat{y} - \varepsilon\Psi)_t + (y - \hat{y} - \varepsilon\Psi)_x + (y - \hat{y} - \varepsilon\Psi)_{xxx} + yy_x = 0 & \text{in } (0, T_*) \times (0, L), \\ (y - \hat{y} - \varepsilon\Psi)(\cdot, 0) = (y - \hat{y} - \varepsilon\Psi)(\cdot, L) = 0 & \text{in } (0, T_*), \\ (y - \hat{y} - \varepsilon\Psi)_x(\cdot, L) = -u_1 & \text{in } (0, T_*), \\ (y - \hat{y} - \varepsilon\Psi)(0, \cdot) = 0 & \text{in } (0, L). \end{cases}$$

We then have, by [21, Lemma 4.6] and Proposition 2.1,

$$(7.10) \quad \|y - \hat{y} - \varepsilon\Psi\|_{L^2((0, T_*) \times (0, L))} \leq C \|yy_x\|_{L^1((0, T_*) \times (0, L))} + C \|u_1\|_{H^{1/3}(0, T_*)} \\ \stackrel{(7.3), (7.4), (7.9)}{\leq} C \left( \|y_0\|_{L^2(0, L)} + \|u\|_{H^{1/3}(\mathbb{R}_+)} \right) \left( \|y_0\|_{L^2(0, L)} + \|u\|_{H^{-1/3}(\mathbb{R}_+)} \right).$$

Multiplying the equation of  $y$  with  $\Psi(t, x)$ , integrating by parts on  $[0, L]$ , and using (7.2), we have

$$(7.11) \quad \frac{d}{dt} \int_0^L y(t, x) \Psi(t, x) dx - \frac{1}{2} \int_0^L y^2(t, x) \Psi_x(t, x) dx = 0.$$

Integrating (7.11) from 0 to  $T_*$  and using the fact  $y(T_*, \cdot) = 0$  yield

$$(7.12) \quad \int_0^L y_0(x) \Psi(0, x) dx + \frac{1}{2} \int_0^{T_*} \int_0^L y^2(t, x) \Psi_x(t, x) dx dt = 0.$$

We have

$$(7.13) \quad \int_0^{T_*} \int_0^L |y^2 - (\hat{y} + \varepsilon\Psi)^2| dx dt \\ = \int_0^{T_*} \int_0^L |y - (\hat{y} + \varepsilon\Psi)| |y + (\hat{y} + \varepsilon\Psi)| dx dt \\ \stackrel{(7.4), (7.10)}{\leq} C \left( \|y_0\|_{L^2(0, L)} + \|u\|_{H^{1/3}(\mathbb{R}_+)} \right) \left( \|y_0\|_{L^2(0, L)} + \|u\|_{H^{-1/3}(\mathbb{R}_+)} \right)^2$$

$$(7.14) \quad \int_0^{T_*} \int_0^L |y - (\hat{y} + \varepsilon\Psi)| |\hat{y} + \varepsilon\Psi| dx dt$$

$$\stackrel{(7.4),(7.5),(7.10)}{\leq} C \left( \|y_0\|_{L^2(0,L)} + \|u\|_{H^{-1/3}(\mathbb{R}_+)} \right) \left( \|y_0\|_{L^2(0,L)} + \|u\|_{H^{-1/3}(\mathbb{R}_+)} \right)^2$$

and

$$(7.15) \quad \int_0^{T_*} \int_0^L (\hat{y} + \varepsilon\Psi)^2 \Psi_x dx dt = \int_0^{T_*} \int_0^L \left( \hat{y}^2 + 2\varepsilon\hat{y}\Psi + \varepsilon^2\Psi^2 \right) \Psi_x dx dt$$

$$= \int_0^{T_*} \int_0^L \left( \hat{y}^2 + 2\varepsilon\hat{y}\Psi \right) \Psi_x dx dt$$

since  $\Psi(\cdot, 0) = \Psi(\cdot, L) = 0$ .

Using the fact

$$y^2 = (y - (\hat{y} + \varepsilon\Psi))^2 - 2(y - (\hat{y} + \varepsilon\Psi))(\hat{y} + \varepsilon\Psi) + (\hat{y} + \varepsilon\Psi)^2,$$

we derive from (7.12), (7.13), (7.14), and (7.15) that

$$\int_0^L y_0(x)\Psi(0,x) dx + \frac{1}{2} \int_0^{T_*} \int_0^L \hat{y}^2(t,x)\Psi_x(t,x) dx dt$$

$$\leq C \left( \varepsilon + \|u\|_{H^{1/3}(\mathbb{R}_+)} \right) \left( \varepsilon + \|u\|_{H^{-1/3}(\mathbb{R}_+)} \right)^2 + C\varepsilon \|\hat{y}\|_{L^2((0,T_*) \times (0,L))}.$$

Applying Corollary 5.1 to  $\hat{y}$  yield, for  $\varepsilon_0$  sufficiently small,

$$\varepsilon + C\|u + u_1\|_{H^{-1/6}(\mathbb{R}_+)}^2 \leq C \left( \varepsilon + \|u\|_{H^{1/3}(\mathbb{R}_+)} \right) \left( \varepsilon + \|u\|_{H^{-1/3}(\mathbb{R}_+)} \right)^2.$$

Using (7.9), it follows that, for sufficiently small  $\varepsilon_0$ ,

$$\|u\|_{H^{-1/6}(\mathbb{R}_+)}^2 \leq C\|u\|_{H^{1/3}(\mathbb{R}_+)}\|u\|_{H^{-1/3}(\mathbb{R}_+)}^2.$$

So, for fixed sufficiently small  $\varepsilon_0$ ,

$$u = 0.$$

Hence  $y(t, \cdot) = \varepsilon\Psi(t, \cdot) \not\equiv 0$  for all  $t > 0$ . We have a contradiction.

The proof is complete.  $\square$

**Remark 7.2.** The constant  $\varepsilon_0$  can be chosen independently of  $T$  if instead of assuming  $\|u\|_{H^{1/3}(0,T)} < \varepsilon_0$ , one requires that

$$\|U\|_{H^{1/3}(\mathbb{R}_+)} < \varepsilon_0,$$

where  $U$  is the extension of  $u$  by 0 in  $(T, +\infty)$ .

## 8. PROOF OF THEOREM 1.3

Theorem 1.3 follows from Proposition 2.3 and Lemma 3.2 as usual. The details are omitted.

## APPENDIX A. HARDY TYPE INEQUALITIES

In this section, we prove two results related to the Hardy inequality. The first one is the following.

**Lemma A.1.** *Let  $-\infty < a < b < c \leq +\infty$ , and  $0 < s < 1/2$  and let  $u \in H^{1/3}(a, b)$ . Set*

$$v = \begin{cases} u & \text{in } (a, b), \\ 0 & \text{in } (b, c). \end{cases}$$

Then  $v \in H^s(a, c)$  and

$$\|v\|_{H^s(a,c)} \leq C\|u\|_{H^s(a,b)},$$

for some positive constant  $C$  depending only on  $a, b, c$ , and  $s$ .

*Proof.* For notational ease, we assume that  $a = -1$  and  $b = 0$ . Without loss of generality, we then can assume that  $c = +\infty$ . Let  $V \in H^s(\mathbb{R})$  be an extension of  $u$  such that

$$\|V\|_{H^s(\mathbb{R})} \leq C\|u\|_{H^s(-1,0)}.$$

Applying [36, Theorem 1.1] with  $\gamma = -s$ ,  $\tau = 2$ ,  $p = 2$  to  $V$ , one obtains

$$\| |x|^{-s} V \|_{L^2(\mathbb{R})} \leq C \|V\|_{H^s(\mathbb{R})}.$$

The condition  $s < 1/2$  is required here. This yields

$$(A.1) \quad \| |x|^{-s} u \|_{L^2(-1,0)} \leq C \|u\|_{H^s(-1,0)}.$$

Using the equivalent Gagliardo-Nirenberg definition of the semi-norm  $H^s$ , we have

$$\|v\|_{H^s(-1,+\infty)}^2 \sim \int_{-1}^{\infty} \int_{-1}^{\infty} \frac{|v(x) - v(y)|^2}{|x - y|^{1+2s}} dx dy + \int_{-1}^{\infty} |v|^2 dx.$$

Since

$$\int_{-1}^{\infty} \int_{-1}^{\infty} \frac{|v(x) - v(y)|^2}{|x - y|^{1+2s}} dx dy \leq \int_{-1}^0 \int_{-1}^0 \frac{|u(x) - u(y)|^2}{|x - y|^{1+2s}} dx dy + C_s \int_{-1}^0 \frac{|u(x)|^2}{|x|^{2s}} dx,$$

and

$$\int_{-1}^{\infty} |v|^2 dx = \int_{-1}^0 |u|^2 dx,$$

we derive from (A.1) that

$$\|v\|_{H^s(-1,+\infty)} \leq C\|u\|_{H^s(-1,0)}.$$

The proof is complete.  $\square$

We next prove the following result.

**Lemma A.2.** *Let  $T > 0$ ,  $u \in H^{1/2}(\mathbb{R})$ , and  $\varphi \in C^1(\mathbb{R})$  be such that  $\text{supp } u \subset (-T, T)$ . There exists a positive constant  $C$ , independent of  $T$ ,  $u$ , and  $\varphi$ , such that*

$$(A.2) \quad \|\varphi u\|_{\dot{H}^{1/2}(\mathbb{R})} \leq C\|\varphi\|_{L^\infty(-2T,2T)}\|u\|_{\dot{H}^{1/2}(\mathbb{R})} \\ + C\left(T^{-1/2}\|\varphi\|_{L^\infty(-T,T)} + T^{1/2}\|\nabla\varphi\|_{L^\infty(-2T,2T)}\right)\|u\|_{L^2(\mathbb{R})}.$$

*Proof.* Using the equivalent Gagliardo-Nirenberg definition of the semi-norm  $H^{1/2}$ , we have, since  $\text{supp } u \subset [-T, T]$ ,

$$(A.3) \quad \|\varphi u\|_{\dot{H}^{1/2}(\mathbb{R})}^2 \sim \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\varphi(s)u(s) - \varphi(t)u(t)|^2}{|s-t|^2} ds dt \\ \leq \int_{-2T}^{2T} \int_{-2T}^{2T} \frac{|\varphi(s)u(s) - \varphi(t)u(t)|^2}{|s-t|^2} ds dt + 2 \int_{\mathbb{R} \setminus [-2T, 2T]} \int_{-2T}^{2T} \frac{|\varphi(s)u(s)|^2}{|s-t|^2} ds dt.$$

We next estimate the RHS of (A.3). We first deal with the first term. We have

$$\int_{-2T}^{2T} \int_{-2T}^{2T} \frac{|\varphi(s)u(s) - \varphi(t)u(t)|^2}{|s-t|^2} ds dt \\ \leq 2 \int_{-2T}^{2T} \int_{-2T}^{2T} \frac{|\varphi(s)|^2 |u(s) - u(t)|^2}{|s-t|^2} ds dt + 2 \int_{-2T}^{2T} \int_{-2T}^{2T} \frac{|\varphi(s) - \varphi(t)|^2 |u(t)|^2}{|s-t|^2} ds dt.$$

This implies

$$(A.4) \quad \int_{-2T}^{2T} \int_{-2T}^{2T} \frac{|\varphi(s)u(s) - \varphi(t)u(t)|^2}{|s-t|^2} ds dt \\ \leq C \|\varphi\|_{L^\infty(-2T, 2T)}^2 \|u\|_{\dot{H}^{1/2}(\mathbb{R})}^2 + CT \|\nabla \varphi\|_{L^\infty(-2T, 2T)}^2 \|u\|_{L^2(\mathbb{R})}^2.$$

Concerning the second term of the RHS of (A.3), we have

$$(A.5) \quad \int_{\mathbb{R} \setminus [-2T, 2T]} \int_{-2T}^{2T} \frac{|\varphi(s)u(s)|^2}{|s-t|^2} ds dt \\ = \int_{\mathbb{R} \setminus [-2T, 2T]} \int_{-T}^T \frac{|\varphi(s)u(s)|^2}{|s-t|^2} ds dt \leq C \|\varphi\|_{L^\infty(-T, T)}^2 \frac{1}{T} \int_{-T}^T |u(s)|^2 ds.$$

The conclusion now follows from (A.4) and (A.5).  $\square$

## APPENDIX B. ON THE ZEROS OF $G$ AND $H$

In this section, we establish several results concerning the zeros of  $G$  and  $H$  defined in (4.11). We begin with

**Lemma B.1.** *Let  $L > 0$ . Assume that  $G(z) = H(z) = 0$  for some  $z \in \mathbb{C} \setminus \{\pm 2/(3\sqrt{3})\}$ . Then*

$$L \in \mathcal{N}_D \quad \text{and} \quad (-z \in \mathcal{P}_D \text{ or } \bar{z} \in \mathcal{P}_D).$$

Recall that  $\mathcal{N}_D$  and  $\mathcal{P}_D$  are defined in (1.3) and (3.2), respectively.

*Proof.* Set

$$y(t, x) = e^{izt} \sum_{j=1}^3 (\lambda_{j+1} e^{\lambda_{j+1}L} - \lambda_{j+2} e^{\lambda_{j+2}L}) e^{\lambda_j x},$$

where  $\lambda_j = \lambda_j(z)$  with  $j = 1, 2, 3$  are the solution of the equation  $\lambda^3 + \lambda + iz = 0$ . Then

$$y_t(t, x) + y_x(t, x) + y_{xxx}(t, x) = 0 \text{ in } \mathbb{R} \times (0, L).$$

Note that  $\Xi(z) \neq 0$  for  $z \neq \pm 2/(3\sqrt{3})$  (recall that  $\Xi$  is defined in Definition 4.1). Hence  $P(z) = \sum_{j=1}^3 \lambda_j^2 (\lambda_{j+1} e^{\lambda_{j+1}L} - \lambda_{j+2} e^{\lambda_{j+2}L}) = 0$  and  $\det Q(z) = \sum_{j=1}^3 (\lambda_{j+1} - \lambda_j) e^{-\lambda_{j+2}L} = 0$ . Using this, one can check from the definition of  $y$  that

$$y(t, 0) = y_{xx}(t, 0) = y(t, L) = y_x(t, L) = 0 \text{ for } t \in \mathbb{R}.$$

This implies that  $L \in \mathcal{N}_D$  and  $(-z \in \mathcal{P}_D$  or  $\bar{z} \in \mathcal{P}_D)$ . □

Set

$$D(\mathcal{A}) = \left\{ v \in H^3(0, L), v(0) = v(L) = v'(L) = 0 \right\}$$

and let  $\mathcal{A}$  be the unbounded operator on  $L^2(0, L)$  with domain  $D(\mathcal{A})$  and defined by  $\mathcal{A}v = v''' + v'$  for  $v \in D(\mathcal{A})$ . It is known from [21, Lemma 2.1] that the spectrum of  $\mathcal{A}$  is discrete. In the following result, we prove that all eigenvalues of  $\mathcal{A}$  has its real part non-negative.

**Lemma B.2.** *Let  $L > 0$ ,  $z \in \mathbb{C}$ , and  $\varphi \in C^\infty([0, L])$  be such that*

$$\varphi_{xxx} + \varphi_x + iz\varphi = 0 \text{ in } [0, L],$$

and

$$\varphi(0) = \varphi(L) = \varphi_x(L) = 0.$$

Then

$$\Im(z) \geq 0.$$

*Proof.* Set

$$\Psi(t, x) = \Re \left\{ e^{izt} \varphi(x) \right\} \text{ in } \mathbb{R}_+ \times [0, L].$$

Then  $\Psi$  is a solution of the linearized KdV system

$$(B.1) \quad \begin{cases} \Psi_t + \Psi_x + \Psi_{xxx} = 0 & \text{in } \mathbb{R}_+ \times [0, L], \\ \Psi(\cdot, 0) = \Psi(\cdot, L) = \Psi_x(\cdot, L) = 0 & \text{in } \mathbb{R}_+. \end{cases}$$

We then derive that

$$\frac{d}{dt} \int_0^L |\Psi(t, x)|^2 dx \leq 0.$$

This implies

$$\Im(z) \geq 0.$$

The proof is complete. □

As a consequence of Lemma B.2, we have the following result.

**Corollary B.1.** *Let  $L \in \mathcal{N}_D$  and  $1 \leq \ell \leq n_D$ . Then*

$$\Im p_\ell \geq 0 \text{ for } 1 \leq \ell \leq n_D.$$

The following result is useful.

**Lemma B.3.** *Let  $L \in \mathcal{N}_D$  and  $p \in \mathbb{C}$  be such that  $\Im p \geq 0$ . Then the zero of  $H(z - p)$  is simple for  $z \in \mathbb{R}$ .*

*Proof.* By [21, Lemma B1], we can assume that  $z - p \neq \pm 2/(3/\sqrt{3})$ . We now prove the assertion by contradiction. Assume that  $H(z - p) = 0$  for some  $z \in \mathbb{R}$  with  $z - p \neq \pm 2/(3/\sqrt{3})$ . Thus  $\det Q(z - p) = 0$ . Then there exists  $\varphi \in C^\infty[0, L]$  such that

$$\varphi_{xxx} + \varphi_x + i(z - p)\varphi = 0 \text{ in } [0, L],$$

and

$$\varphi(0) = \varphi(L) = \varphi_x(L) = 0.$$

Applying Lemma B.2, we have

$$\Im p \leq 0.$$

Since  $\Im p \geq 0$ , we derive that  $\Im p = 0$ , which is equivalent to the fact  $z - p \in \mathbb{R}$ . The conclusion now follows from [21, Lemma B1]. □

## APPENDIX C. A LEMMA RELATED TO THE MOMENT METHOD

The following lemma from [21] is used in the proof of Proposition 2.1.

**Lemma C.1.** *Let  $\varphi$  be an analytic function in  $\mathbb{C}$  such that  $\varphi$  has a finite number of zeros on the real line, and*

$$(C.1) \quad |\varphi(z)| \leq c_1 e^{c_2 |z|^\alpha} \text{ in } \mathbb{C},$$

for some  $0 < \alpha < 1$ , and  $c_1, c_2 > 0$ . Let  $T_1, T_2 > 0$ , and  $h \in H^s(\mathbb{R})$  for some  $s \leq 0$  with support in  $(0, T_1)$ . There exists  $g \in C^\infty(\mathbb{R})$  with support in  $[T_1, T_1 + T_2]$  such that if  $z$  is a real solution of order  $m$  of the equation  $\varphi(z) = 0$ , then  $z$  is also a real solution of order  $m$  of the equation  $\hat{h} - \hat{g} = 0$ , and

$$(C.2) \quad \|g\|_{H^k(\mathbb{R})} \leq C_k \|h\|_{H^s(\mathbb{R})} \text{ for } k \in \mathbb{N},$$

for some positive constant  $C_k$  depending only on  $k, T_1, T_2, s$ , and real zeros and their multiplicity of  $\varphi$ .

*Proof.* The proof of Lemma C.1 is as in [21], where a special case is considered. The construction of  $g$ , inspired by the moment method, see e.g. [44], can be done as follows. Set  $\eta(t) = e^{-1/(t^2 - (T_2/2)^2)} \mathbb{1}_{|t| < T_2}$  for  $t \in \mathbb{R}$ . Assume that  $z_1, \dots, z_k$  are real, distinct solutions of the equation  $\varphi(z) = 0$ , and  $m_1, \dots, m_k$  are the corresponding orders. Set, for  $z \in \mathbb{C}$ ,

$$\zeta(z) = \sum_{i=1}^k \left( \hat{\eta}(z - z_i) \prod_{\substack{j=1 \\ j \neq i}}^k (z - z_j)^{m_j} \left( \sum_{l=0}^{m_i} c_{i,l} (z - z_i)^l \right) \right),$$

where  $c_{i,l} \in \mathbb{C}$  is chosen such that

$$\frac{d^l}{dz^l} \left( e^{i(T_1 + T_2/2)z} \zeta(z) \right)_{z=z_i} = \frac{d^l}{dz^l} \hat{h}_3(z_i) \text{ for } 0 \leq l \leq m_i, 1 \leq i \leq k.$$

This can be done since  $\hat{\eta}(0) \neq 0$ . Since

$$|\hat{\eta}(z)| \leq C e^{T_2 |\Im(z)|/2},$$

and, by [44, Lemma 4.3],

$$|\hat{\eta}(z)| \leq C_1 e^{-C_2 |z|^{1/2}} \text{ for } z \in \mathbb{R},$$

using (C.1), and applying Paley-Wiener's theorem, one can prove that  $\zeta$  is the Fourier transform of a function  $\psi$  of class  $C^1$ ; moreover,  $\psi$  has the support in  $[-T_2/2, T_2/2]$ . Set, for  $z \in \mathbb{C}$ ,

$$g(t) = \psi(t + T_1 + T_2/2).$$

Using the fact  $\hat{g}(z) = e^{i(T_1 + T_2/2)z} \zeta(z)$ , one can check that  $\hat{g} - \hat{h}$  has zeros  $z_1, \dots, z_k$  with the corresponding orders  $m_1, \dots, m_k$ . One can check that

$$\|\psi\|_{H^k(\mathbb{R})} \leq C_{T,L,k} \sum_{i=1}^k \sum_{l=0}^{m_i} \left| \frac{d^l}{dz^l} \hat{h}(z_i) \right|,$$

which yields

$$\|\psi\|_{H^k(\mathbb{R})} \leq C_{T,L,k} \|h\|_{H^s(\mathbb{R})}.$$

The required properties of  $g$  follow.  $\square$

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