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# An inductive model structure for strict $\infty$ -categories

Simon Henry\*and Felix Loubaton†

## Abstract

We construct a left semi-model category of “marked strict  $\infty$ -categories” for which the fibrant objects are those whose marked arrows satisfy natural closure properties and are weakly invertible. The canonical model structure on strict  $\infty$ -categories can be recovered as a left Bousfield localization of this model structure. We show that an appropriate extension of the Street nerve to the marked setting produces a Quillen adjunction between our model category and the Verity model structure for complicial sets, generalizing previous results by the second named author. Finally, we use this model structure to study, in the setting of strict  $\infty$ -categories, the idea that there are several non-equivalent notions of weak  $(\infty, \infty)$ -categories - depending on what tower of  $(\infty, n)$ -categories is used. We show that there ought to be at least three different notions of  $(\infty, \infty)$ -categories.

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# 1 Introduction

In the present paper, we introduce (in Section 2.2) a category  $\infty\text{-Cat}^m$  of “ $m$ -marked (strict)  $\infty$ -categories” for  $m \in \mathbb{N} \cup \{\infty\}$ . The objects of  $\infty\text{-Cat}^m$  are strict  $\infty$ -categories, with, similarly to stratified simplicial sets, some arrows being “marked”. The marked arrows are required to be closed under composition, and all identities arrows as well as all arrows of dimension  $> m$  are marked. This category  $\infty\text{-Cat}^m$  is equipped with two monoidal closed structures denoted  $\otimes$  and  $\odot$  that are both the Gray-Crans tensor product on the underlying strict  $\infty$ -categories but act differently on markings. These two monoidal structures are meant to respectively be models for the “lax-Gray tensor product” and the “pseudo-Gray tensor product”.

Our main result is the construction of a model structure<sup>1</sup> on  $\infty\text{-Cat}^m$  similar to the canonical (or “Folk”) model structure on strict  $\infty$ -category from [19]:

**1.1 Theorem.** *There is a combinatorial left semi-model structure on the category  $\infty\text{-Cat}^m$  of  $m$ -marked  $\infty$ -categories such that:*

- *This model structure is monoidal for both tensor products  $\otimes$  and  $\odot$  (from Section 2.3).*
- *The cofibrations are the map that are cofibrations of the canonical model structure between the underlying  $\infty$ -categories.*
- *The fibrant objects are the marked  $\infty$ -categories in which all marked arrows admit marked weak inverses, and in which if there is a marked arrow  $a \rightarrow b$  then  $a$  is marked if and only if  $b$  is marked.*
- *Fibrations between fibrant objects are the “isofibrations” (as defined in Section 3.3).*
- *Weak equivalences between fibrant objects are “equivalence of marked  $\infty$ -categories” (as defined in Section 3.4).*

This model structure is a model for strict “ $(\infty, m)$ -categories” where “invertibility” or arrows of dimension  $> m$  is taken in a weak sense. The existence of this model structure is established in Section 2.4, but some of its properties, in particular, the characterization of fibrant objects and fibrations between fibrant objects will only be established in Section 3.

We also consider two left Bousfield localizations of this model structure:

- The *saturated inductive model structure*, studied in Section 3.5, whose fibrant objects are the  $\infty$ -categories in which every arrow which is weakly invertible up to marked arrows is also marked.
- The *coinductive model structure*, studied in Section 4.2, whose fibrant objects are the  $\infty$ -categories in which every coinductively invertible (see Definition 4.16) arrow is marked.

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<sup>1</sup>We use the term “model category” as a generic name for all sorts of model categories (Quillen model categories, semi-model categories, weak model categories, etc...)

This second localization is equivalent<sup>2</sup> to the canonical model structure on  $\infty$ -categories from [19].

The motivations to introduce this model structure come from two different lines of investigations that we will explain separately below:

## 1.1 The street nerve as a right Quillen functor

In [20], the second named author has shown that the Street nerve of a strict  $\infty$ -category can be made into a complicial set by defining the “thin” simplexes as being those whose top dimensional arrows are weakly invertible.

From there, it is natural to ask whether this stratified version of the Street nerve, also preserves fibrations, and hence is a morphism of categories of fibrant objects (and this will be shown in the present paper as Proposition 4.51).

In fact, more generally, one could ask if it is possible to make this version of the Street nerve into a right Quillen functor (for the Verity model structure on complicial sets from [30]). This is not directly possible simply because this stratified Street nerve is not a right adjoint functor. The solution to this problem is to work with marking on both sides: The usual Street nerve from strict  $\infty$ -categories to simplicial sets is a right adjoint functor, and one can extend it to a right adjoint functor from marked  $\infty$ -categories to “marked” simplicial sets (or rather *stratified* simplicial sets to follow the terminology of [30]). In Section 4.4 we show that this functor is indeed a right Quillen functor.

This right Quillen functor from marked  $\infty$ -categories to stratified simplicial sets is meant to be a model for the forgetful functor from strict  $(\infty, \infty)$ -categories to weak  $(\infty, \infty)$ -categories. In particular, the corresponding left Quillen functor from stratified simplicial sets to marked  $\infty$ -categories is a model for the more mysterious “strictification functor”, sending weak  $\infty$ -categories to strict  $\infty$ -categories.

At the level of  $\infty$ -groupoids, this strictification functor corresponds essentially to (non-abelian) homology, through the equivalence between strict  $\infty$ -groupoids and crossed chain complexes ([7]) which is well-known to be a conservative functor by Whitehead’s theorem for homology. The first named author has conjectured [17] that more generally this strictification functor should be conservative on weak  $(\infty, m)$ -categories for all  $m$ . This allows us to state a concrete version of this conjecture here:

**1.2 Conjecture.** *The left Quillen functor  $|-|: \mathbf{sSet}^m \rightarrow \infty\text{-Cat}^m$  from Section 4.4 reflects weak equivalence between cofibrant objects.*

## 1.2 The two (?) notions of $(\infty, \infty)$ -categories

C.Schommer-Pries and C.Rezk have independently argued ([16]) that there should be more than one notion of weak  $(\infty, \infty)$ -categories. More precisely, they both arrive at the conclusion that even if one accepts (which seems to be a clear consensus nowadays) that there is only one notion of weak  $(\infty, n)$ -categories for finite  $n$ , there are at least two different ways to build a notion of  $(\infty, \infty)$ -categories out of it.

Before we go into further details, we should say that the following discussion is mostly informal and speculative and most of it has not been formalized in

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<sup>2</sup>Though not through a Quillen equivalence.

any models - in fact, one motivation for the present paper is to formalize some of it in the context of strict  $\infty$ -categories.

First, let us go over the argument put forward by Rezk and Schommer-Pries, or at least how we understand it: The forgetful (or inclusion) functor from  $(\infty, n)$ -categories to  $(\infty, n+1)$ -categories is supposed to have both a left adjoint  $\pi_n$ , which freely adds inverses to all  $(n+1)$ -arrows and a right adjoint  $\tau_n$  which remove all non-invertible  $(n+1)$ -arrows.

This allows to produce two different towers:

$$(\infty, 0)\text{-Cat} \xleftarrow{\pi_0} (\infty, 1)\text{-Cat} \xleftarrow{\pi_1} (\infty, 2)\text{-Cat} \xleftarrow{\pi_2} \dots \xleftarrow{\pi_{n-1}} (\infty, n)\text{-Cat} \xleftarrow{\pi_n} \dots$$

$$(\infty, 0)\text{-Cat} \xleftarrow{\tau_0} (\infty, 1)\text{-Cat} \xleftarrow{\tau_1} (\infty, 2)\text{-Cat} \xleftarrow{\tau_2} \dots \xleftarrow{\tau_{n-1}} (\infty, n)\text{-Cat} \xleftarrow{\tau_n} \dots$$

and one can take the projective limit of either of these two towers to give a definition of what is an  $(\infty, \infty)$ -category.

If one takes the limits of the  $\pi$ -tower then one can see that an arrow that is “coinductively” invertible (see Definition 4.16) has to be considered invertible. To be precise, we mean that if  $F: X \rightarrow Y$  is a morphism in the limit of the  $\pi$ -tower which admits an inverse up to a coinductively invertible natural transformation then  $F$  is an equivalence.

The situation in the limit of the  $\tau$ -tower however is fairly different: Given an  $(\infty, \infty)$ -category in this sense, it corresponds to a collection of  $(\infty, n)$ -categories  $X_n$  such that  $X_n \simeq \tau_n X_{n+1}$ , and an  $n$ -arrow corresponds to an  $n$ -arrow of  $X_n$  (or of  $X_k$  for  $k > n$ ). In this setting one has an intrinsic notion of equivalence: an  $n$ -arrow is said to be an equivalence if it belongs to  $X_{n-1}$  (equivalently if it is invertible in the  $(\infty, n)$ -category  $X_n$ ). In this setting, coinductively invertible arrows do not have to be invertible if none of the higher cells witnessing the coinductive invertibility are not themselves invertible.

To clearly show that the two are different, one can for example consider the  $(\infty, \infty)$ -category of cobordisms. In the limit of the  $\tau$ -tower one can define it by taking  $X_n$  to be the  $(\infty, n)$ -categories of cobordisms. In this  $(\infty, \infty)$ -category, every arrow has a dual, so it follows from a result of E.Cheng (see [9]) that every arrow in the cobordisms  $(\infty, \infty)$ -category is coinductively invertible, although there are many non-invertible  $n$ -arrows in  $X_n$  for all  $n$ . Hence, if one were trying to define  $X_n$  in the limit of the  $\pi$ -tower, it would be equivalent to an  $\infty$ -groupoid.

Using our model structure of marked strict  $\infty$ -category, we will make these two constructions formal in the context of strict  $\infty$ -category. This is of course only meant to be a toy model for the case of weak  $\infty$ -categories, but it is already interesting, and it will show that the picture above while correct, needs to be refined a little.

First, we will show in Section 4.1 that our model structure on  $\infty\text{-Cat}^m$  for  $m = \infty$  corresponds to the limit of  $\tau$ -tower as above. More precisely, we will show that it is Quillen equivalent to an appropriate homotopy limit of the  $\infty\text{-Cat}^m$  for  $m < \infty$  using the  $\tau_n$  functor as transition functors.

The notion of homotopy limit of a tower of model structure we are using has been introduced in [6], and we will use their construction of the homotopy limit. Here there is a small gap we should disclaim: [6] only develops the theory of such limits for Quillen model categories and not semi-model categories, and we will

apply their construction to our left semi-model categories directly. In order for our argument to be complete despite this, we will prove that the construction from [6] does yield to a left semi-model category, but we will not reprove that it corresponds to a homotopy limit as in [6, Theorem 5.1]. However, it should be noted that in practice, the argument of [6] seems to carry over to our setting with almost no changes, so this gap is not really a concern.

In Section 4.2 we will show that the folk model structure is equivalent to the left Bousfield localization of our model structure which corresponds to turning all coinductively invertible arrows into equivalences.

However, we will also show in Section 4.3, that the folk model structure is not equivalent to the limit of the  $\pi$ -tower. It is unclear if the limit of the tower of  $\pi_n$  corresponds to further localization of our model structure, or is something entirely different, but we find that the argument we will give in Section 4.3 to distinguish between the folk model structure and the limit of  $\pi$ -tower shows that this limit is exhibiting behaviors that are not really expected from a notion of  $(\infty, \infty)$ -categories, or at least are not typical of any known model of  $\infty$ -categories.

Coming back to the world of weak  $(\infty, \infty)$ -categories, this suggests that the two most interesting notions of weak  $(\infty, \infty)$ -categories are the limit of  $\tau_n$  tower, which corresponds to an “inductive” notion of equivalences, and its localization that turn the coinductive equivalence into equivalences, but this localization should be different from the limit of the  $\pi_n$ -tower which might not be an interesting notion of  $(\infty, \infty)$ -categories. What we mean here is that we are not aware of any attempt of giving a concrete definition of  $(\infty, \infty)$ -categories that seems to produce something that could be equivalent to this limit. All definitions we have seen can be reasonably conjectured to be equivalent to either the limit of the  $\tau_n$  tower or to its “coinductive” localization.

## 2 $\infty$ -categories and marked $\infty$ -categories

### 2.1 $\infty$ -categories

A globular set is a presheaf on the globular category  $\mathbb{G}$ :

$$\mathbb{D}_0 \begin{array}{c} \xrightarrow{i_0^+} \\ \xrightarrow{i_0^-} \end{array} \mathbb{D}_1 \begin{array}{c} \xrightarrow{i_1^+} \\ \xrightarrow{i_1^-} \end{array} \mathbb{D}_2 \begin{array}{c} \xrightarrow{i_2^+} \\ \xrightarrow{i_2^-} \end{array} \mathbb{D}_3 \begin{array}{c} \xrightarrow{i_3^+} \\ \xrightarrow{i_3^-} \end{array} \mathbb{D}_4 \dots$$

With the relations  $i_n^\epsilon i_{n-1}^+ = i_n^\epsilon i_{n-1}^-$  for all  $n > 0$  and  $\epsilon \in \{+, -\}$ . We also denote by  $i_k^\epsilon$  the map  $\mathbb{D}_k \rightarrow \mathbb{D}_n$  for  $k < n$  obtained by composing any string of arrow ending with  $i_k^\epsilon$ . These and the identity arrows are the only maps in the category  $\mathbb{G}$ .

If  $X$  is a globular set, one denotes by  $X_n$  the set  $X(\mathbb{D}_n)$  whose elements are called  $n$ -arrows. The map  $X_n \rightarrow X_k$  induced by  $i_k^\epsilon: \mathbb{D}_k \rightarrow \mathbb{D}_n$  is denoted by  $\pi_k^\epsilon$ .

**2.1 Definition.** An  $\infty$ -category is a globular set  $X$  together with operations of *compositions*

$$X_n \times_{X_k} X_n \rightarrow X_n \quad (0 \leq k < n)$$

which associates to two  $n$ -arrows  $(x, y)$  verifying  $\pi_k^+(x) = \pi_k^-(y)$ , one  $n$ -arrow  $x \#_k y$ , as well as *identities*

$$X_n \rightarrow X_{n+1}$$

associating to an  $n$ -arrow  $x$ , an  $(n+1)$ -arrow  $\mathbb{I}_x$ , and satisfying the following axioms:

- (1)  $\forall x \in X_n, \pi_n^\epsilon(\mathbb{I}_x) = x$ .
- (2)  $\pi_k^-(x \#_n y) = \pi_k^-(x)$  and  $\pi_k^+(x \#_n y) = \pi_k^+(y)$  whenever the composition is defined and  $k \leq n$ .
- (3)  $\pi_k^\epsilon(x \#_n y) = \pi_k^\epsilon(x) \#_n \pi_k^\epsilon(y)$  whenever the composition is defined and  $k > n$ .
- (4)  $x \#_n \mathbb{I}_{\pi_n^+ x} = x$  and  $\mathbb{I}_{\pi_n^- x} \#_n x = x$ .
- (5)  $(x \#_n y) \#_n z = x \#_n (y \#_n z)$  as soon as one of these is defined.
- (6) If  $k < n$

$$(x \#_n y) \#_k (z \#_n w) = (x \#_k z) \#_n (y \#_k w)$$

when the left-hand side is defined.

A morphism of  $\infty$ -categories is a map of globular sets commuting with both operations. The category of  $\infty$ -categories is denoted  $\infty\text{-Cat}$ .

**2.2 Definition.** An  $(n+1)$ -arrow  $c$  in an  $\infty$ -category is said to be *trivial*, or an *identity arrow*, if there exists an  $n$ -cell  $d$  such that  $c = \mathbb{I}_d$ .

**2.3 Example.** By abuse of notation, we also denote  $\mathbb{D}_n$  the  $\infty$ -category that admits for any  $k < n$  only two  $k$ -non-trivial arrows, denoted  $e_k^-$  and  $e_k^+$ , and a single non-trivial  $n$ -arrow, denoted  $e_n$  verifying :

$$\begin{array}{lll} \pi_l^-(e_k^\epsilon) = e_l^- & \pi_l^+(e_k^\epsilon) = e_l^+ & \text{for } l \leq k < n \\ \pi_l^-(e_n) = e_l^- & \pi_l^+(e_n) = e_l^+ & \text{for } l \leq n \end{array}$$

The  $\infty$ -category  $\partial\mathbb{D}_n$  is obtained from  $\mathbb{D}_n$  by removing the  $n$ -arrow  $e_n$ . We thus have a morphism

$$i_n: \partial\mathbb{D}_n \rightarrow \mathbb{D}_n.$$

Note that  $\partial\mathbb{D}_0 = \emptyset$

**2.4 Definition.** If  $X$  is an  $\infty$ -category, we define the globular set  $\Sigma X$ , called the *suspension of  $X$* , by the formula

$$(\Sigma X)_0 = \{a, b\} \quad (\Sigma X)_{n+1} := X_n \cup \{\mathbb{I}^n a, \mathbb{I}^n b\}$$

where  $\mathbb{I}_a^n$  (resp.  $\mathbb{I}_b^n$ ) is the  $n$ -times iterated unity of  $a$  (resp. of  $b$ ). Moreover,  $\Sigma X$  inherits from  $X$  a structure of  $\infty$ -category.

Eventually, for an integer  $n$ , we define the  $\infty$ -category  $\Sigma^n X$ , called the  *$n$ -suspension of  $X$* , as the  $n$ -times iterated suspension of  $X$ .

Next, we define the notion of polygraphs, first introduced under the name “computads” by R. Street in [28] for 2-categories, with the general notion being hinted at in [29]. As far as we know the first formal introduction of polygraphs in the literature is in [25] and independently in [8], where the name “polygraphs” was introduced. Here we will exploit that the category of polygraphs identifies with a (non-full) subcategory of  $\infty\text{-Cat}$  to give a shorter definition. We refer to the references above for a more complete introduction.

**2.5 Definition.**

- We say that an  $\infty$ -category  $X$  is a polygraph if it can be constructed from the empty  $\infty$ -category by freely adding arrows with specified source and target. That is if  $X$  can be obtained as a transfinite composition  $\emptyset = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_i \rightarrow \text{Colim } X_i = X$  where for each  $i$ , the map  $X_i \rightarrow X_{i+1}$  is a pushout of  $Y \times \partial\mathbb{D}_n \rightarrow Y \times \mathbb{D}_{n+1}$ .
- An arrow of a polygraph is said to be a *generator* if it is one of the arrows that has been freely added at some stage.
- A morphism of  $\infty$ -categories between two polygraphs is said to be a *morphism of polygraphs* or a *polygraphic* morphism if it sends each generator to a generator.
- An *n-polygraph* is a polygraph whose generators are all of dimension  $\leq n$ .

**2.6 Remark.** Generators of a polygraph can be shown to be exactly the arrows that cannot be written as a composite in a non-trivial way, so the notion of generator does not depend on the choice of the presentation of  $X$ , and any isomorphism between polygraphs is automatically polygraphic.

**2.7 Example.** The only  $n$ -polygraphs for  $n < 0$  is the empty  $\infty$ -category, the category of 0-polygraphs is equivalent to the category of sets and corresponds to discrete  $\infty$ -categories, the category of 1-polygraphs (and polygraphic morphisms between them) is equivalent to the category of directed graphs, and they corresponds to categories that are free on a graph.

We will sometimes distinguish between a polygraph seen as an object of the category of polygraphs and polygraphic morphisms, and the corresponding  $\infty$ -category, which we call the free  $\infty$ -category on the polygraph.

**2.8 Remark.** Each arrow in a polygraph can be written as an iterated composite of the generators (not necessarily in a unique way). For an  $n$ -arrow  $f$ , the set of generators of dimension  $n$  that appear in such an expression (and even the number of times they appear) is the same for all such expressions. We will say that an  $n$ -generator *appears* in an  $n$ -arrow if it appears in any such expression.

**2.9 Construction.** The category  $\infty\text{-Cat}$  admits a closed monoidal structure, called the Gray tensor product or Crans-Gray tensor product, which we denote as

$$\begin{array}{ccc} \infty\text{-Cat} \times \infty\text{-Cat} & \rightarrow & \infty\text{-Cat} \\ X, Y & \mapsto & X \otimes Y \end{array}$$

Its explicit construction is very involved and we will assume the reader is already familiar with it. It was first introduced by S. Crans in his Ph.D. thesis [10]. We refer to [1] for an introduction to this tensor product close to its original definition, and to [27] for a more modern account. The proof of the existence of this monoidal structure in [27] contains some gaps that have been fixed in [4].

It is easy to see from either of these definitions that  $\mathbb{D}_n \otimes \mathbb{D}_m$  has a unique non-trivial arrow of dimension  $n + m$ . If  $f$  and  $g$  are respectively an  $n$ -arrow of  $X$  and an  $m$ -arrow of  $Y$ , which corresponds to morphisms  $f: \mathbb{D}_n \rightarrow X$  and  $g: \mathbb{D}_m \rightarrow Y$ , we denote by  $f \otimes g$  the  $m + n$  arrow of  $X \otimes Y$  obtained as the image of this non-trivial  $(n + m)$ -arrow by the functor  $f \otimes g: \mathbb{D}_n \otimes \mathbb{D}_m \rightarrow X \otimes Y$ .

We recall from [3]:

**2.10 Proposition.** *If  $X$  and  $Y$  are polygraphs then  $X \otimes Y$  is also a polygraph. The generators of  $X \otimes Y$  are the arrow of the form  $x \otimes y$  where  $x$  and  $y$  are respectively generators of  $X$  and  $Y$ .*

Finally, we recall from [19] that  $\infty\text{-Cat}$  carries a model structure, called the folk model structure in which every object is fibrant and where the generating cofibrations are the  $\partial\mathbb{D}_n \rightarrow \mathbb{D}_n$ . Its weak equivalences are a natural class of equivalence of  $\infty$ -categories that generalizes the equivalences of ordinary categories. It was shown in [23] that the cofibrant objects are exactly the polygraphs and it also follows from this that the cofibrations between cofibrant objects are the polygraphic inclusions. It was shown in [3] that this model structure is a monoidal model structure for the Gray tensor product.

## 2.2 Marked $\infty$ -categories

For the rest of the article, we fix an  $m \in \mathbb{N} \cup \{\infty\}$

**2.11 Definition.** An  $m$ -marked  $\infty$ -category is an  $\infty$ -category  $X$ , together with a set  $M \subset \coprod_{k>0} X(k)$  of arrows of positive dimension called *marked* arrows such that:

- All identity arrows  $\mathbb{I}_x$  are marked.
- All arrows of dimension strictly superior to  $m$  are marked.
- If  $x$  and  $y$  are marked  $n$ -arrows and  $x\#_k y$  is defined, then  $x\#_k y$  is marked.

A morphism of  $m$ -marked  $\infty$ -categories is a morphism between the underlying  $\infty$ -categories that sends marked arrows to marked arrows. The category of  $m$ -marked  $\infty$ -categories is denoted  $\infty\text{-Cat}^m$ .

Note that if  $m = \infty$ , then the second condition of the definition simply disappears, this is the main case we are interested in.

**2.12 Example.** If  $X$  is an  $\infty$ -category we denote by  $X^\#$  the  $m$ -marked  $\infty$ -category  $(X, X_{>0})$  where all arrows of positive dimension are marked. We denote by  $X^\flat$  the  $m$ -marked  $\infty$ -category where only identity arrows and  $k$ -arrows for  $k > m$  are marked.

**2.13 Construction.** If  $X$  is an  $\infty$ -category and  $M \subset \coprod_{k>0} X_k$  is a set of arrows of  $X$ , we denote by  $\overline{M}$  the smallest set of arrows such that  $M \subset \overline{M}$  and  $(X, \overline{M})$  is an  $m$ -marked  $\infty$ -category. That is  $\overline{M}$  is the reunion of the set of arrows of dimension strictly superior to  $m$  and the set of all  $n$ -arrows that can be written as iterated composites of  $n$ -arrows in  $M$  and arrows of the form  $\mathbb{I}_x$  for  $x$  an  $(n - 1)$ -arrow. For example  $X^\flat = (X, \overline{\emptyset})$ .

**2.14 Construction.** The category of  $m$ -marked  $\infty$ -categories has all colimits, and they are easily described in terms of colimits of  $\infty$ -category and of Construction 2.13: if  $(X_i, M_i)_{i \in I}$  is a diagram of  $m$ -marked  $\infty$ -category indexed by a category  $I$  then:

$$\operatorname{Colim}_{i \in I}(X_i, M_i) = \left( \operatorname{Colim}_{i \in I} X_i, \overline{\cup_i f_i(M_i)} \right)$$

where  $f_i$  denotes the canonical map  $f_i: X_i \rightarrow \operatorname{Colim}_{i \in I} X_i$  and  $f_i(M_i)$  is simply the set of arrows of the form  $f_i(x)$  for  $x \in M_i$ .

This is easily shown by checking that the right-hand side has the universal property of the colimit.

**2.15 Definition.** A *special  $m$ -marked polygraph* is an  $m$ -marked  $\infty$ -category of the form  $(X, \overline{M})$  where  $X$  is free on a polygraph and  $M$  only contains generators of  $X$ .

**2.16 Proposition.** *If  $(X, \overline{M})$  is a special  $m$ -marked polygraph, then an  $n$ -arrow  $f$  is in  $\overline{M}$  if and only if  $n > m$  or if all the generating  $n$ -arrows that appear in  $f$  are in  $M$ .*

*Proof.* An arrow satisfying this condition is a composite of marked  $n$ -arrows and identities of lower dimensional arrows, so it has to be in  $\overline{M}$ . Conversely, this set of arrows contains  $M$  and all identities (as no  $n$ -dimensional arrows appear in their expression) and is closed under composition.  $\square$

### 2.3 Tensor product of $m$ -marked $\infty$ -categories

In this section we construct two monoidal closed structures on the category of  $m$ -marked  $\infty$ -categories, respectively called the *pseudo-Gray* tensor product  $\odot$  and the *lax-Gray* tensor product  $\ominus$ . Both are obtained by putting different markings on the Gray tensor product from Construction 2.9. For example, the lax-Gray tensor product  $\mathbb{D}_1 \ominus \mathbb{D}_1$  is  $C_1^?$  where  $C_1$  is the polygraph

$$C_1 = \left( \begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & \swarrow & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array} \right)$$

while  $\mathbb{D}_1 \odot \mathbb{D}_1$  is the special  $m$ -marked polygraph  $(C_1, \overline{D})$  where  $D$  only contains the unique 2 dimension generator of  $C_1$ . So, unless  $m = 0$  or  $m = 1$ , the two tensor products are distinct. At the derived or homotopy theoretic level, the pseudo-Gray tensor product should correspond to the cartesian product.

The formal definition goes as follows

**2.17 Construction.** Given two  $m$ -marked  $\infty$ -categories  $(X, M)$  and  $(Y, N)$  we define two sets of arrows in  $X \otimes Y$ :

- $M \ominus N$  is the set of arrows of the form  $x \otimes y \in X \otimes Y$  where either  $x \in M$  or  $y \in N$ .
- $M \odot N$  contains all arrows in  $M \ominus N$  together with all arrows of the form  $x \otimes y$  with  $x$  and  $y$  both of dimension  $> 0$ .

Note that  $M \ominus N$  and  $M \odot N$  are not marking on  $X \otimes Y$ : they are not stable under composition. So we define:

$$(X, M) \ominus (Y, N) = (X \otimes Y, \overline{M \ominus N})$$

$$(X, M) \odot (Y, N) = (X \otimes Y, \overline{M \odot N})$$

We will show in Lemma 2.37 that both make the category of  $m$ -marked  $\infty$ -categories into a monoidal closed category.

In order to show this, it is convenient to introduce the following notations:

**2.18 Notation.** For  $A$  and  $B$  subsets of arrows in  $\infty$ -categories, we denote by  $A \otimes B$  the set of arrows of the form  $a \otimes b \in X \otimes Y$  for  $a \in A$  and  $b \in B$ . For  $X$  and  $\infty$ -category, we denote by  $X_{\geq 0}$  the set of all arrows of  $X$  and by  $X_{> 0}$  the set of all arrows of dimension  $> 0$ . We can hence, for  $(X, M)$  and  $(Y, N)$  to  $m$ -marked  $\infty$ -category rewrite the definitions above as:

$$\begin{aligned} M \ominus N &= (M \otimes Y_{\geq 0}) \cup (X_{\geq 0} \otimes N) \\ M \odot N &= (M \ominus N) \cup (X_{> 0} \otimes Y_{> 0}) \\ &= (M \otimes Y_{\geq 0}) \cup (X_{\geq 0} \otimes N) \cup (X_{> 0} \otimes Y_{> 0}) \end{aligned}$$

By definition of the Gray tensor product, we have the following result:

**2.19 Lemma.** *Let  $X$  and  $Y$  be two  $\infty$ -categories, then*

$$\begin{aligned} \overline{X_{\geq 0} \otimes Y_{\geq 0}} &= (X \otimes Y)_{\geq 0} \\ \overline{X_{> 0} \otimes Y_{\geq 0} \cup X_{\geq 0} \otimes Y_{> 0}} &= (X \otimes Y)_{> 0}. \end{aligned}$$

That is  $X \otimes Y$  is generated under composition by arrows of the form  $x \otimes y$ , and the arrows of dimension  $> 0$  of  $X \otimes Y$  are generated under compositions by arrows of the form  $x \otimes y$  with  $x$  or  $y$  of dimension  $> 0$

**2.20 Lemma.** *Let  $X$  be an  $\infty$ -category and  $M, N$  two subsets of arrows of  $X$  then:*

$$\overline{M \cup N} = \overline{\overline{M} \cup N} = \overline{M \cup \overline{N}} = \overline{\overline{M} \cup \overline{N}}$$

*Proof.* This is straightforward.  $\square$

**2.21 Lemma.** *Let  $X, Y$  be two  $\infty$ -categories and  $M \subset X_{\geq 0}$  and  $N \subset Y_{\geq 0}$ . Then:*

$$\overline{M \otimes N} = \overline{\overline{M} \otimes N} = \overline{M \otimes \overline{N}} = \overline{\overline{M} \otimes \overline{N}}$$

*Proof.* We will only show the equality  $\overline{M \otimes N} = \overline{\overline{M} \otimes N}$ . The equality  $\overline{\overline{M} \otimes N} = \overline{M \otimes \overline{N}}$  is proved in the exact same way and the last equality follows immediately by applying the result to  $M$  and  $\overline{N}$ . We will also only prove the results for  $m = \infty$ , the case of a general  $m$  follows immediately as it marks all arrow of dimension  $> m$  on each side of these equalities. The evident inclusion  $M \subset \overline{M}$  implies  $\overline{M \otimes N} \subset \overline{\overline{M} \otimes N}$ , so it is then enough to show that  $\overline{\overline{M} \otimes N} \subset \overline{M \otimes N}$ . Let  $K$  be the set of arrows  $k$  in  $X$  such that  $k \otimes n \in \overline{M \otimes N}$  for all  $n \in N$ . We need to show that  $K$  is closed by identity and composition to finish the proof. If  $k = \mathbb{1}_x$ , then  $k \otimes n = \mathbb{1}_{x \otimes n} \in \overline{M \otimes N}$ . Let now  $k, k' \in K$  of dimension  $n$  such

that  $k\#_i k'$  is defined. They are encoded by a map  $\mathbb{D}_n \coprod_{\mathbb{D}_i} \mathbb{D}_n \rightarrow X$  and let  $y \in N$  be an arrow of dimension  $m$  of  $Y$ , encoded by a map  $\mathbb{D}_m \rightarrow Y$ .

Together these induced a map  $e: (\mathbb{D}_n \coprod_{\mathbb{D}_i} \mathbb{D}_n) \otimes \mathbb{D}_m \rightarrow X \otimes Y$ .  $(\mathbb{D}_n \coprod_{\mathbb{D}_i} \mathbb{D}_n) \otimes \mathbb{D}_m$  is a polygraph of dimension  $m+n$  with only two generating arrows of maximal dimensions that are sent to  $k \otimes y$  and  $k' \otimes y$ , which are by hypothesis in  $\overline{M \otimes N}$ .

Now the arrow corresponding to  $(k\#_i k') \otimes y$  in  $(\mathbb{D}_n \coprod_{\mathbb{D}_i} \mathbb{D}_n) \otimes \mathbb{D}_m$  is in  $\overline{M \otimes N}$  as all the top dimensional generators that appear in it are in  $\overline{M \otimes N}$ . We have proved that  $k\#_i k' \otimes y \in \overline{M \otimes N}$  for all  $y \in N$ , hence  $k\#_i k' \in K$  and this concludes the proof.  $\square$

**2.22 Lemma.** *Let  $X, Y$  be two  $\infty$ -categories,  $M \subset X_{\geq 0}$  and  $N \subset Y_{\geq 0}$ . Then we have*

$$\begin{aligned} \overline{M \ominus N} &= \overline{\overline{M} \ominus \overline{N}} \\ \overline{M \odot N} &= \overline{\overline{M} \odot \overline{N}}. \end{aligned}$$

*Proof.* Given the formula for  $M \ominus N$  and  $M \odot N$  from Notation 2.18, this is a direct consequence of Lemma 2.20 and Lemma 2.21.  $\square$

**2.23 Lemma.** *Let  $X, Y, Z$  be three  $\infty$ -categories,  $M \subset X_{> 0}$ ,  $N \subset Y_{> 0}$  and  $P \subset Z_{> 0}$ . Then we have*

$$\begin{aligned} \overline{(M \ominus N) \ominus P} &= \overline{\overline{M \ominus (N \ominus P)}} \\ \overline{(M \odot N) \odot P} &= \overline{\overline{M \odot (N \odot P)}} \end{aligned}$$

*Proof.* We begin with the first equality. Let

$$E := (M \otimes Y_{\geq 0} \otimes Z_{\geq 0}) \cup (X_{\geq 0} \otimes N \otimes Z_{\geq 0}) \cup (X_{\geq 0} \otimes Y_{\geq 0} \otimes P).$$

The lemmas 2.19, 2.20 and 2.21 implies the following equalities:

$$\begin{aligned} \overline{E} &= \overline{M \otimes \overline{Y_{\geq 0} \otimes Z_{\geq 0}} \cup X_{\geq 0} \otimes (N \otimes \overline{Z_{\geq 0} \cup Y_{\geq 0} \otimes P})} \\ &= \overline{M \otimes (\overline{Y \otimes Z})_{\geq 0} \cup X_{\geq 0} \otimes (N \ominus P)} \\ &= \overline{M \ominus (N \ominus P)} \end{aligned}$$

A very similar computation also shows that  $\overline{E} = \overline{(M \ominus N) \ominus P}$ , which concludes the proof of the first equality.

For the second equality, we define

$$F := (X_{\geq 0} \otimes Y_{> 0} \otimes Z_{> 0}) \cup (X_{> 0} \otimes Y_{\geq 0} \otimes Z_{> 0}) \cup (X_{> 0} \otimes Y_{> 0} \otimes Z_{\geq 0})$$

The second equality of Lemma 2.19 implies that:

$$\overline{F} = \overline{X_{k \geq 0} \otimes Y_{> 0} \otimes Z_{> 0} \cup X_{> 0} \otimes (Y \otimes Z)_{> 0}}$$

and then that

$$\begin{aligned} \overline{E \cup F} &= \overline{M \otimes (\overline{Y \otimes Z})_{\geq 0} \cup X_{\geq 0} \otimes (N \odot P) \cup X_{> 0} \otimes (Y \otimes Z)_{> 0}} \\ &= \overline{M \odot (N \odot P)} \end{aligned}$$

and here again, a similar computation shows  $\overline{E \cup F} = \overline{(M \odot N) \odot P}$ , which concludes the proof.  $\square$

**2.24 Lemma.** *Let  $X$  be an  $\infty$ -category,  $M \subset X_{>0}$ . Then the empty set, considered as a subset of the  $\infty$ -category  $\mathbb{D}_0$ , verifies (up to the identifications  $\mathbb{D}_0 \otimes X \simeq X \otimes \mathbb{D}_0 \simeq X$ ):*

$$\begin{aligned}\emptyset \ominus M &= M \ominus \emptyset = M \\ \overline{\emptyset \ominus M} &= \overline{M \ominus \emptyset} = \overline{M}\end{aligned}$$

*Proof.* The first equality is a straightforward application of the definition of  $\ominus$ . For the second case, we also use that all arrows of  $(\mathbb{D}_0)_{>0} \otimes X_{>0}$  are identities and so all belong to  $\overline{M}$ .  $\square$

**2.25 Proposition.** *Both the lax-Gray tensor product  $\ominus$  and the pseudo-Gray tensor product  $\odot$  as defined above are monoidal structures on the category of  $m$ -marked  $\infty$ -categories. In both cases the forgetful functor to  $\infty$ -categories is monoidal and their unit is  $\mathbb{D}_0^{\flat} = \mathbb{D}_0^{\sharp}$ .*

*Proof.* Note that  $\mathbb{D}_0^{\flat} = \mathbb{D}_0^{\sharp} = (\mathbb{D}_0, \overline{\emptyset})$  as all arrows of  $\mathbb{D}_0$  of dimension  $> 0$  are identities.

The proposition exactly says that the structural map (associativity and unit isomorphism) of the Gray tensor product of  $\infty$ -categories preserves the marking we specified on the tensor product.

For the unit, let  $(X, M)$  be an  $m$ -marked  $\infty$ -category. The Lemmas 2.21 and 2.24 imply that

$$\begin{aligned}(X, M) \ominus (\mathbb{D}_0, \overline{\emptyset}) &= (X \otimes \mathbb{D}_0, \overline{M \ominus \emptyset}) = (X, M) \\ (X, M) \odot (\mathbb{D}_0, \overline{\emptyset}) &= (X \otimes \mathbb{D}_0, \overline{M \odot \emptyset}) = (X, M)\end{aligned}$$

and

$$\begin{aligned}(\mathbb{D}_0, \overline{\emptyset}) \ominus (X, M) &= (\mathbb{D}_0 \otimes X, \overline{\emptyset \ominus M}) = (X, M) \\ (\mathbb{D}_0, \overline{\emptyset}) \odot (X, M) &= (\mathbb{D}_0 \otimes X, \overline{\emptyset \odot M}) = (X, M).\end{aligned}$$

For the associativity isomorphism, let  $(X, M)$ ,  $(Y, N)$  and  $(Z, P)$  be three  $\infty$ -categories. Lemma 2.21 implies that

$$\begin{aligned}((X, M) \ominus (Y, N)) \ominus (Z, P) &= (X \otimes Y \otimes Z, \overline{(M \ominus N) \ominus P}) \\ ((X, M) \odot (Y, N)) \odot (Z, P) &= (X \otimes Y \otimes Z, \overline{(M \odot N) \odot P})\end{aligned}$$

and

$$\begin{aligned}(X, M) \ominus ((Y, N) \ominus (Z, P)) &= (X \otimes Y \otimes Z, \overline{M \ominus (N \ominus P)}) \\ (X, M) \odot ((Y, N) \odot (Z, P)) &= (X \otimes Y \otimes Z, \overline{M \odot (N \odot P)}).\end{aligned}$$

Lemma 2.23 shows that these two marking on  $X \otimes Y \otimes Z$ , in the lax and the pseudo case, coincide.  $\square$

**2.26 Proposition.** *The pseudo and lax-Gray tensor product  $\ominus$  and  $\odot$  preserves colimits in each variable.*

In particular, as  $\infty\text{-Cat}^m$  is locally presentable, this immediately implies that both tensor products are closed monoidal structures.

*Proof.* It follows from the fact that the Gray tensor product  $\otimes$  preserves colimits in each variables, the description of colimits of  $m$ -marked  $\infty$ -category given in Construction 2.14 and Lemma 2.21.  $\square$

## 2.4 The semi-model structure

In this section, we will construct a left semi-model structure on the category  $\infty\text{-Cat}^m$ .

**2.27 Definition.** We define the set  $I = I^m \cup I^a$  to be our set of generating cofibrations in  $\infty\text{-Cat}^m$  where:

$$I^a = \{i_n: \partial\mathbb{D}_n \rightarrow \mathbb{D}_n, |n \geq 0\}$$

$$I^m = \{\mathbb{D}_n \rightarrow (\mathbb{D}_n, \overline{\{e_n\}}) \quad , n \geq 0\}$$

An arrow in  $\infty\text{-Cat}^m$  is said to be a trivial fibration if it has the right lifting property against all arrows in  $I$ . An arrow in  $\infty\text{-Cat}^m$  is said to be a cofibration if it has the left lifting property against all trivial fibration.

**2.28 Remark.** It immediately follows from the small object argument that every arrow can be factored into a cofibration followed by a trivial fibration and that all cofibrations are retracts of transfinite compositions of pushouts of arrows in  $I$ .

**2.29 Remark.** An arrow  $\pi: X \rightarrow Y$  has the right lifting property against all arrows in  $I^a$  if its image by the forgetful functor to  $\infty\text{-Cat}$  is a trivial fibration, that is if for every pair of parallel  $n$ -arrows  $u, v$  in  $X$ , the map  $\text{Hom}_X(u, v) \rightarrow \text{Hom}_Y(\pi(u), \pi(v))$  is surjective.

$\pi$  has the right lifting property against all arrows in  $I^m$  if and only if for every arrow  $f \in X$  such that  $\pi(f)$  is marked in  $Y$ ,  $f$  is marked in  $X$ . A trivial fibration is a map that has both these properties.

**2.30 Remark.** The cofibrant objects of  $\infty\text{-Cat}^m$  are exactly the  $m$ -marked  $\infty$ -categories whose underlying  $\infty$ -category is free on a polygraph, with any possible marking on them (not just the special markings of 2.15). Indeed, transfinite compositions of pushouts by arrows in  $I^a$  only starting from the empty  $\infty$ -category exactly give all polygraphs with no markings on them. Pushouts by  $I^m$  are simply changing the marking and can make any arrow marked, so by also taking pushouts by arrows in  $I^m$  one obtains all polygraphs with any possible marking on them. Finally, it was shown in [23] that polygraphs are closed under retract in  $\infty\text{-Cat}$ , so they constitute all cofibrant objects.

The pushout-product, or corner-product (sometimes also called Leibniz product)  $f \hat{\otimes} g$  and  $f \hat{\circ} g$  is defined as usual: if  $f: X \rightarrow Y$  and  $g: A \rightarrow B$  are two arrows in  $\infty\text{-Cat}^m$ , then  $f \hat{\otimes} g$  is the canonical arrow:

$$X \hat{\otimes} B \coprod_{X \hat{\circ} A} Y \hat{\otimes} A \rightarrow Y \hat{\otimes} B$$

and  $f \hat{\circ} g$  is the canonical arrow

$$X \hat{\circ} B \coprod_{X \hat{\otimes} A} Y \hat{\circ} A \rightarrow Y \hat{\circ} B$$

We refer to the appendix of [18] for the general theory of pushout products and their formal properties.

**2.31 Proposition.** *If  $f$  and  $g$  are two cofibrations in  $\infty\text{-Cat}^m$  then  $f \hat{\odot} g$  and  $f \hat{\circ} g$  are both cofibrations.*

*Proof.* By the usual properties of the corner-product, it is enough to check this when  $f$  and  $g$  are generating cofibrations. If  $f$  and  $g$  are both in  $I^a$ , then  $f \odot g$  has no marked arrows in either its domain or codomain and coincides with the corner-product  $f \hat{\odot} g$  in  $\infty\text{-Cat}$ , which has been shown to be a cofibration in [3].  $f \circ g$  is the same except that some arrows are marked, but we can always add these marking by taking additional pushouts by arrows in  $I^m$ , so it is again a cofibration.

The forgetful functor  $\infty\text{-Cat}^m \rightarrow \infty\text{-Cat}$  is monoidal for both tensor product and preserves colimits, so it preserves the corner-product. In particular, if either  $f$  or  $g$  is in  $I^m$  then it is sent to isomorphisms by this forgetful functor and hence  $f \hat{\odot} g$  and  $f \hat{\circ} g$  induces isomorphisms between their underlying  $\infty$ -categories. Now, if  $f: (X, N) \rightarrow (X, M)$  is a morphism in  $\infty\text{-Cat}^m$  that induces an isomorphism on underlying  $\infty$ -categories, then it is a pushout of arrows in  $I^m$ : one simply needs to take such pushout to make all arrows in  $M$  marked.  $\square$

**2.32 Construction.** We define  $I := \mathbb{D}_1^\sharp = (\mathbb{D}_1, \{e_1\})$ . It is the  $\infty$ -category with two objects,  $e_0^-$  and  $e_0^+$  and a marked arrow  $e_1: e_0^- \rightarrow e_0^+$ . We denote by  $j_-$  and  $j_+$  the two maps  $\mathbb{D}_0 \rightarrow I$  corresponding respectively to the two objects  $e_0^-$  and  $e_0^+$ . This gives a diagram:

$$\mathbb{D}_0 \coprod \mathbb{D}_0 \hookrightarrow I \rightarrow \mathbb{D}_0$$

Which will play the role of the interval object for our semi-model structure on  $\infty\text{-Cat}^m$ .

We will take as a set of “generating anodyne cofibrations” (also called a “pseudo-generating set of trivial cofibrations”) the set of maps of the form  $j_+ \hat{\circ} i$  where  $i$  is a generating cofibration, more precisely:

**2.33 Definition.**

- We say that an arrow in  $\infty\text{-Cat}^m$  is a *naive fibration* if it has the right lifting property against all arrows of the form  $j_+ \hat{\circ} i$ , where  $j_+: \mathbb{D}_0 \rightarrow I$  is as in Construction 2.32, and  $i$  is one of the generating cofibrations as in Definition 2.27.
- We say that an arrow in  $\infty\text{-Cat}^m$  is an *anodyne cofibrations* if it has the right lifting property against all naive fibrations.
- We say that a cofibration in  $\infty\text{-Cat}^m$  is *acyclic* if it has the lifting property against all naive fibrations between (naively) fibrant objects.
- We say that a map in  $\infty\text{-Cat}^m$  is a *fibration* if it has the right lifting property against all acyclic cofibrations.

As before, it immediately follows from the small object argument that every arrow factors as an anodyne cofibration followed by a naive fibration, and all anodyne cofibrations are retracts of transfinite compositions of pushouts of the “generating anodyne cofibrations”.

**2.34 Remark.** It immediately follows from Proposition 2.31 that, as  $j_+$  is a cofibration, all maps of the form  $j_+ \hat{\circ} i$  are cofibrations. In particular, all trivial fibrations are also naive fibrations and all anodyne cofibrations are cofibrations.

**2.35 Proposition.** *Acyclic cofibrations and fibrations form a cofibrantly generated weak factorization system on  $\infty\text{-Cat}^m$ . An object is “naively fibrant” if and only if it is fibrant and more generally an arrow between fibrant objects is a fibration if and only if it is a naive fibration.*

*Proof.* This is a direct application of the results of Section 4 of [15]. Starting from the premodel structure on  $\infty\text{-Cat}^m$  whose weak factorization systems are (cofibrations, trivial fibrations) and (anodyne cofibrations, naive fibrations), we obtain the one with (cofibrations, trivial fibrations) and (acyclic cofibrations, fibrations) as its “left saturation”  $\mathbb{L}(\infty\text{-Cat}^m)$  in the sense of Theorem 4.1 of [15]. All the claim in the proposition follows from this Theorem 4.1.  $\square$

**2.36 Remark.** Note that replacing  $\hat{\odot}$  by  $\hat{\ominus}$  in 2.33 would not change the definition. Indeed, if  $X = Y^\sharp$  is an  $m$ -marked  $\infty$ -category whose arrows of dimension  $> 0$  are all marked then for any  $m$ -marked  $\infty$ -category  $Z$  one has  $X \hat{\odot} Z = X \hat{\ominus} Z$ . As this applies to both the domain and the co-domain of  $j_+$  it follows that  $j_+ \hat{\odot} i = j_+ \hat{\ominus} i$ .

Also, the reader should not be worried about the use of  $j_+$  in Definition 2.33 rather than  $j_-$  or both  $j_-$  and  $j_+$ . While putting  $j_-$  or both  $j_-$  and  $j_+$  instead of  $j_+$  would change the definition of naive fibrations and anodyne cofibrations, this does not affect the definition of (naive) fibrations between fibrant objects, hence the acyclic cofibrations and fibrations would not be changed. Indeed, once the existence of a (monoidal) model structure is established, it follows that  $j_-$  is acyclic by 2-out-of-3, and hence all the maps  $j_- \hat{\odot} i = j_- \hat{\ominus} i$  are also acyclic cofibrations.

**2.37 Lemma.** *If  $f$  is an anodyne (resp. acyclic) cofibration and  $g$  is a cofibration then  $f \hat{\odot} g$  and  $f \hat{\ominus} g$  are anodyne (resp. acyclic).*

*Proof.* To get the result for “anodyne cofibrations” it is enough to prove it for the generating anodyne cofibrations. Let  $i$  be one of the generating cofibrations and  $f = j_+ \hat{\odot} i'$  be one of the generating anodyne cofibrations. We have  $f \hat{\odot} i = j_+ \hat{\odot} (i \hat{\odot} i')$ . As  $i' \hat{\odot} i$  is a pushout of generating cofibrations  $i_1, \dots, i_k$  by Proposition 2.31 it follows that  $j_+ \hat{\odot} (i \hat{\odot} i')$  is a pushout of the  $j_+ \hat{\odot} i_k$  and hence is an anodyne cofibration.

The result for acyclic cofibrations follows from formal properties of the pushout product: it follows that if  $i$  is a cofibration and  $p$  is a naive fibration then the (right) pullback exponential  $\langle p/i \rangle$  is a naive fibration. If  $p$  is a (naive) fibration between fibrant objects then  $\langle p/i \rangle$  is a naive fibration between fibrant objects hence a fibration. It follows that if  $i$  is an acyclic cofibration and  $j$  is a cofibration then  $i \hat{\odot} j$  is an acyclic cofibration as it is a cofibration by Definition 2.27 and if  $p$  is a fibration between fibrant objects then  $i \hat{\odot} j$  has the right lifting property against  $p$  because  $j$  has the left lifting property against  $\langle p/i \rangle$ .

The case of  $\hat{\ominus}$  works exactly the same considering the first half of Remark 2.36.  $\square$

**2.38 Theorem.** *The category  $\infty\text{-Cat}^m$  of  $m$ -marked  $\infty$ -category admits a left semi-model structure, called the inductive model structure, in which the cofibrations and trivial fibrations are as in Definition 2.27 and the fibrations are as in Definition 2.33.*

*Proof.* This immediately follows from Theorem 6.12 of [15]. Because of Proposition 2.31 and Lemma 2.37, tensoring by the interval object  $I$  of Construction 2.32 is a “strong Quillen functor” in the sense of section 6 of [15]. Note that to apply Theorem 6.12 one needs to observe that  $\infty\text{-Cat}^m$ , with the (cofibrations, trivial fibrations) and (acyclic cofibrations, fibrations) weak factorization systems, is both “right saturated” and “left saturated” that is, that a fibration that has the right lifting property against all cofibrations between cofibrant objects is a trivial fibration and that a cofibration that has the left lifting property against all fibrations between fibrant objects is a trivial cofibration. The first ones hold because the generating cofibrations are cofibrations between cofibrant objects and the second because that is how we defined acyclic fibrations.  $\square$

**2.39 Remark.** The proof of Theorem 2.38 above also shows that  $\infty\text{-Cat}^m$  also admits a right semi-model category structure whose fibrations and trivial cofibrations are the fibrations and acyclic cofibrations of Definition 2.33 and whose cofibrations are as in Definition 2.27.

This however does not clearly make  $\infty\text{-Cat}^m$  into a Quillen model structure but rather into a “two-sided model category” as in Section 5 of [15]. We refer to Section 5 of [15] for what this means more precisely, but in short, the problem is that the left and right semi-model categories have different classes of weak equivalences. The two classes of equivalence however coincide for arrows that are between fibrant or cofibrant objects. Another way to talk about this difference is that left and the right semi-model categories are Quillen equivalent and have the same homotopy category, but define different functors  $\infty\text{-Cat}^m \rightarrow \text{Ho}(\infty\text{-Cat}^m)$ . The two functors agree on objects that are either fibrant or cofibrant but differ on general objects: one sends an object  $X$  to its cofibrant replacement while the other sends it to a fibrant replacement, and we do not know if these are always homotopy equivalent when  $X$  is neither fibrant nor cofibrant itself.

**2.40 Remark.** We do not know if  $\infty\text{-Cat}^m$  is actually a Quillen model category or not. In the unmarked case, this follows from the fact that all objects are fibrant. But that is no longer the case in this situation. In terms of the “two-sided model structure” mentioned in the previous remark, the question is whether  $\infty\text{-Cat}^m$  satisfies one of the equivalent conditions of Proposition 5.3 of [15].

We conclude this section with the following lemma that will be useful later:

**2.41 Lemma.** *The map*

$$i_n^+ : \mathbb{D}_n^b \rightarrow (\mathbb{D}_{n+1}, \overline{\{e_{n+1}\}})$$

where  $e_{n+1}$  is the unique non-identity arrow of  $\mathbb{D}_{n+1}$ , is an anodyne cofibration.

*Proof.* We will show it is a retract of the map  $j_+ \hat{\circ} i_n$  where  $i_n$  is the map  $\partial\mathbb{D}_n \rightarrow \mathbb{D}_n$ .

In order to achieve this, we will compute  $j_+ \hat{\circ} i_n$  more explicitly using the description of  $\mathbb{D}_1 \otimes \mathbb{D}_n$  given in appendix B.1 of [4] (see proposition B.1.4): As a polygraph, the generating arrows of  $\mathbb{D}_1 \otimes \mathbb{D}_n$  are the:

$$a_0^- \otimes e_k^\epsilon \quad a_0^+ \otimes e_k^\epsilon \quad a \otimes e_k^\epsilon$$

where the arrows of  $\mathbb{D}_1$  have been denoted “ $a$ ” instead of “ $e$ ” to distinguish them, and  $\epsilon$  is either  $+$  or  $-$ ,  $k \leq n$  and  $e_n^+ = e_n^-$ . Their source and target are given as follows:

$$\begin{aligned}\pi^-(a_0^- \otimes e_k^\epsilon) &= a_0^- \otimes e_{k-1}^- & \pi^+(a_0^- \otimes e_k^\epsilon) &= a_0^- \otimes e_{k-1}^+ \\ \pi^-(a_0^+ \otimes e_k^\epsilon) &= a_0^+ \otimes e_{k-1}^- & \pi^+(a_0^+ \otimes e_k^\epsilon) &= a_0^+ \otimes e_{k-1}^+ \\ \pi^-(a \otimes e_k^\epsilon) &= (a_0^- \otimes e_k^\epsilon) \#_0 (a \otimes e_0^+) \#_1 \dots \#_{k-1} (a \otimes e_{k-1}^+) \\ \pi^+(a \otimes e_k^\epsilon) &= (a \otimes e_{k-1}^-) \#_{k-1} \dots \#_1 (a \otimes e_0^-) \#_0 (a_0^+ \otimes e_k^\epsilon)\end{aligned}$$

We did not put parenthesis in the expression above, to keep them shorter, the default convention is to do the composition  $\#_i$  in order of increasing values of  $i$ . The last two equations are given by proposition B.1.4 of [4], though note that this reference is using a different convention than ours regarding the composition order. Note that the object we are interested in is  $I \odot \mathbb{D}_n^b$  which is the same polygraph endowed with the special marking where all the arrows  $a \otimes e_k^\epsilon$  are marked.

We then realize  $(\mathbb{D}_{n+1}, \overline{\{e_{n+1}\}})$  as a retract of  $I \odot \mathbb{D}_{n+1}^b$  as follows: We call  $i: (\mathbb{D}_{n+1}, \overline{\{e_{n+1}\}}) \rightarrow I \odot \mathbb{D}_n^b$  the unique morphism sending  $e_{n+1}$  to  $a \otimes e_n$ . This is well defined because  $a \otimes e_n$  is a marked arrow. Next, we define a map  $p: I \odot \mathbb{D}_n^b \rightarrow (\mathbb{D}_{n+1}, \overline{\{e_{n+1}\}})$  by:

$$\begin{aligned}p(a_0^\epsilon \otimes e_k^\mu) &= e_k^\mu \text{ if } k < n. \\ p(a_0^\epsilon \otimes e_n) &= e_n^\epsilon \\ p(a \otimes e_k^\epsilon) &= \mathbb{I}_{e_k^\epsilon} \text{ if } k < n. \\ p(a \otimes e_n) &= e_{n+1}\end{aligned}$$

In order to check that this is well defined, we first need to check that this definition is compatible with the source and target given above, which follow from an immediate calculation. Then we need to show that this is compatible with the marking, which is the case as both  $\mathbb{I}_{e_k^\epsilon}$  and  $e_{n+1}$  are marked.

Finally, the composite  $p \circ i$  send the arrow  $e_{n+1}$  to  $p(a \otimes e_n) = e_{n+1}$  and hence is the identity of  $\mathbb{D}_{n+1}$ .

To conclude the proof, we just have to observe that the maps  $f$  and  $i$  defined above send the domain of  $i_n^+$  and of  $j_+ \hat{\odot} i_n$  to each other.

The domain of  $j_+ \hat{\odot} i_n$  is the sub-polygraph of  $I \odot \mathbb{D}_n^b$  which contains all the generators except  $a_0^- \otimes e_n$  and  $a \otimes e_n$ , while the domain of  $i_n^+$  contains all generators of  $\mathbb{D}_{n+1}$  except  $e_{n+1}$  and  $e_n^-$ .

In order to check that the map  $i$  is compatible with these sub-polygraphs, it is enough to check that  $i(e_n^+)$  is in the domain of  $j_+ \hat{\odot} i_n$ , to see this, we compute:

$$i(e_n^+) = \pi^+ i(e_{n+1}) = \pi^+(a \otimes e_n) = (a \otimes e_{n-1}^-) \#_{n-1} \dots \#_1 (a \otimes e_0^-) \#_0 (a_0^+ \otimes e_n)$$

and we observe that this expression involves neither  $a_0^- \otimes e_n$  nor  $a \otimes e_n$ , hence it does belong to the domain of  $j_+ \hat{\odot} i_n$ .

In order to check that the map  $p$  is compatible with these sub-polygraphs, we need to check the image by  $p$  of all the generators of  $I \odot \mathbb{D}_n^b$  except  $a_0^- \otimes e_n$  and  $a \otimes e_n$ . These are given by the formulas  $p(a_0^\epsilon \otimes e_k^\mu) = e_k^\mu$  if  $k < n$ ,  $p(a_0^+ \otimes e_n) = e_n^+$  and  $p(a \otimes e_k^\epsilon) = \mathbb{I}_{e_k^\epsilon}$ , which all indeed belong to the image of  $i_n^+$ .  $\square$

### 3 Equations and saturations in an $m$ -marked $\infty$ -category.

The general goal of this section is to arrive at a better description of the fibrant objects and fibrations between fibrant objects of the model structure of Theorem 2.38. This is achieved using the notion of “equations” in  $\infty$ -categories introduced by the second named author in [20]. We will recall the basic theory of equations, in a slightly different language and introduce an analog of equations to deal with the markings, which we call saturations.

#### 3.1 Definitions of equations and saturations

**3.1 Definition.** A *left equation* is a special  $m$ -marked polygraph  $(P, \overline{M})$  with two arrows  $x, y \in P$  such that:

- (1)  $y$  is the unique arrow of dimension  $n + 1$  and  $P$  contains no arrows of dimension  $> n + 1$ .
- (2)  $y$  is a marked arrow.
- (3) if  $n \leq m$ ,  $x$  is an unmarked arrow of  $P$ .
- (4) The source of  $y$  admits a decomposition:

$$\pi_n^- y = l_n \#_{n-1} (l_{n-1} \#_{n-2} \dots \#_1 (l_1 \#_0 x \#_0 r_1) \#_1 \dots \#_{n-2} r_{n-1}) \#_{n-1} r_n$$

where for each  $i$ ,  $l_i$  and  $r_i$  are marked  $i$ -arrow in  $P$ , with  $l_n$  and  $r_n$  not containing  $x$ . In particular,  $x$  appears only once in  $\pi_n^- y$ .

- (5)  $x$  does not appear in the target of  $y$ .

*Right equations* are defined in the exact same way except the source and target of  $y$  are exchanged in the last two conditions.

We say that  $(P, \overline{M})$  is an equation to mean that it is either a left or right equation. If  $P$ , with its arrows  $x$  and  $y$  as in the definition, is an equation one denotes by  $\Lambda P$  the sub-polygraphs of  $P$  that contains all arrow except  $x$  and  $y$ .

**3.2 Remark.** Note that specifying the arrows  $x, y \in P$  is exactly the same as specifying the subpolygraphs  $\Lambda P \subset P$ . For this reason, we will often also call “equation” the map  $\Lambda P \rightarrow P$ .

We say that an equation  $\Lambda P \rightarrow P$  has solutions in  $C \in \infty\text{-Cat}^m$  if  $C$  has the right lifting property against  $\Lambda P \rightarrow P$  and we say that a morphism  $f: C \rightarrow D$  lifts solutions of the equation if it has the right lifting property against the map  $\Lambda P \rightarrow P$ .

**3.3 Remark.** The name “equation” comes from the idea that we are looking for an element  $x$  such that a certain composite of  $x$  with other arrows is isomorphic to another given arrow. From this point of view, a map  $\Lambda P \rightarrow X$  corresponds to such an equation in  $X$ , and an extension  $P \rightarrow X$  corresponds to a solution of the equation, or rather the image of  $x$  is the solution and  $y$  represents the isomorphism witnessing that  $x$  is a solution.

**3.4 Definition.** A *left saturation* is a special marked polygraph  $(P, \overline{M})$  with arrows  $x$  and  $y$  satisfying the conditions of Definition 3.1 except that  $x$  is a marked arrow and the target of  $y$  is a marked arrow. *Right saturations* are defined in the same way. If  $P$  is a saturation, one denotes  $\Omega P$  the special  $m$ -marked polygraph  $(P, \overline{M - \{x\}})$ .

**3.5 Definition.** If  $P$  is an equation, we define the  $m$ -marked  $\infty$ -category  $Uni(P)$  which is the colimit of the following diagram:

$$\begin{array}{ccc} \partial\mathbb{D}_n & \longrightarrow & \mathbb{D}_n^\# \\ x \amalg x' \downarrow & \ulcorner & \downarrow z \\ P \amalg_{\Lambda P} P & \longrightarrow & Uni(P) \end{array}$$

A map  $Uni(P) \rightarrow X$  corresponds to a map  $\Lambda P \rightarrow X$ , which is an equation in  $X$ , together with two solutions  $P \rightarrow X$ , given by pairs  $(x, y)$  and  $(x', y')$ , and a marked arrow  $z: x \rightarrow x'$  which express that the two solutions are isomorphic.

**3.6 Definition.** Let  $C$  be an  $m$ -marked  $\infty$ -category  $C$  and  $P$  a left equation (resp. right equation).

The equation  $P$  has solutions in  $C$  if for all morphisms  $\Lambda P \rightarrow C$ , there exists a lifting  $(x, y): P \rightarrow C$  such that  $x$  is sent on a marked arrow whenever the target of  $y$  is (resp. the source of  $y$  is).

Solutions to an equation  $P$  are *weakly unique* if  $C$  has the right lifting property against  $P \amalg_{\Lambda P} P \rightarrow Uni(P)$ .

The equation  $P$  has *unique solutions* in  $C$  if the equation  $P$  has solutions in  $C$  and they are weakly unique.

It will be useful to have a ‘‘coherent’’ version of  $Uni(P)$ , noted  $Uni^{coh}(P)$ . If  $P$  is a left equation,  $P \amalg_{\Lambda P} P \rightarrow Uni^{coh}(P)$  is obtained as the following sequence of pushout:

$$\begin{array}{ccccc} \partial\mathbb{D}_n & \longrightarrow & \mathbb{D}_n & & \\ x \amalg x' \downarrow & & \downarrow z & & \\ P \amalg_{\Lambda P} P & \xrightarrow{\ulcorner} & \bullet & \longrightarrow & Uni^{coh}(P) \\ s[x/z] \#_n y' \amalg y \uparrow & & \uparrow & \lrcorner & \uparrow \\ \partial\mathbb{D}_{n+1} & \longrightarrow & \mathbb{D}_{n+1} & & \end{array}$$

where  $s$  is the source of  $y$ . Conversely, if  $P$  is a right equation,  $P \amalg_{\Lambda P} P \rightarrow Uni^{coh}(P)$  is obtained as the following sequence of pushout:

$$\begin{array}{ccccc} \partial\mathbb{D}_n & \longrightarrow & \mathbb{D}_n & & \\ x \amalg x' \downarrow & & \downarrow z & & \\ P \amalg_{\Lambda P} P & \xrightarrow{\ulcorner} & \bullet & \longrightarrow & Uni^{coh}(P) \\ y \#_n t[x/z] \amalg y' \uparrow & & \uparrow & \lrcorner & \uparrow \\ \partial\mathbb{D}_{n+1} & \longrightarrow & \mathbb{D}_{n+1} & & \end{array}$$

where  $t$  is the source of  $y$ . Remarks that in both cases,  $P \coprod_{\Delta P} P \rightarrow \text{Uni}^{\text{coh}}(P)$  is an equation. By definition, if  $C$  is an  $m$ -marked  $\infty$ -category such that  $\text{Uni}^{\text{coh}}(P)$  has a solution in  $C$ , then  $C$  as the right lifting property against  $P \coprod_{\Delta P} P \rightarrow \text{Uni}(P)$ .

**3.7 Example.** Let  $n$  be a non-negative integer. The morphism

$$j_+ \hat{\circ} i_n := I \odot \partial \mathbb{D}_n \coprod \{1\} \odot \mathbb{D}_n \rightarrow I \odot \mathbb{D}_n$$

is a left equation. Indeed, let  $y$  be the top dimensional generator of  $I \odot \mathbb{D}_n$ . If we denote by  $x$  the top dimensional arrow of  $\{0\} \odot \mathbb{D}_n$ , and for  $0 < k \leq n$ , by  $a_k$  the image of the top dimensional  $k$ -generator of  $I \odot \mathbb{D}_{k-1}$  by the morphism

$$I \odot \delta_{k-1}^- : I \odot \mathbb{D}_{k-1} \rightarrow I \odot \mathbb{D}_n,$$

Section B.1. of [4] allows to give an explicit description of  $I \odot \mathbb{D}_n$ , which we recalled in the proof of Lemma 2.41. Using this description, we see that if we name  $y = a \otimes e_n$  and  $x = a_0^- \otimes e_n$  the two arrows of  $I \odot \mathbb{D}_n$  that are not in the image of  $j_+ \hat{\circ} i_n$ , then we have a decomposition of the source of  $y$  of the form:

$$(((x \#_0 a_0) \#_1 a_2) \dots) \#_{n-1} a_n$$

and all the  $a_k$  are marked. We denote it

$$\mathbf{eq}_n^{\circledast} : \Lambda \mathbf{Eq}_n^{\circledast} \rightarrow \mathbf{Eq}_n^{\circledast}.$$

**3.8 Example.** Similarly, the morphism

$$j_+ \hat{\circ} s_n : I \odot \mathbb{D}_n \coprod \{1\} \odot (\mathbb{D}_n, \overline{\{e_n\}}) \rightarrow I \odot (\mathbb{D}_n, \overline{\{e_n\}})$$

where  $s_n$  is the ‘‘identity’’ map  $\mathbb{D}_n \rightarrow (\mathbb{D}_n, \overline{\{e_n\}})$  is a left saturation. which we denote

$$\mathbf{sat}_n^{\circledast} : \Omega \mathbf{Sat}_n^{\circledast} \rightarrow \mathbf{Sat}_n^{\circledast}$$

**3.9 Definition.** We define some left equations which play an important role. In each case,  $k$  and  $n$  are integers with  $k \leq n$ .

- $\mathbf{eq}_{k,n}^{\circlearrowleft} : \Lambda \mathbf{Eq}_{k,n}^{\circlearrowleft} \rightarrow \mathbf{Eq}_{k,n}^{\circlearrowleft}$ , whose target is generated by  $x$  and  $b$  of dimension  $n$ ,  $a$  a marked arrow of dimension  $k$  and  $y : (a \#_{k-1} x) \Rightarrow b$ .
- $\mathbf{eq}_{k,n}^{\circlearrowright} : \Lambda \mathbf{Eq}_{k,n}^{\circlearrowright} \rightarrow \mathbf{Eq}_{k,n}^{\circlearrowright}$ , whose target is generated by  $x$  and  $b$  of dimension  $n$ ,  $a$  a marked arrow of dimension  $k$  and  $y : (x \#_{k-1} a) \Rightarrow b$ .

In all equations above, the domain of the arrow is obtained by removing  $x$  and  $y$ . Also, in each case, we have not listed all the constraints of the source and target that are necessary to make sense of the definition of  $y$ . For example, in  $\mathbf{Eq}_{k,n}^{\circlearrowleft}$ , we have the relation  $\pi_{k-1}^+(a) = \pi_{k-1}^-(x)$  for the composition  $a \#_{k-1} x$  to exists, and the relations  $\pi_{n-1}^c(b) = \pi_{n-1}^c(a \#_{k-1} x)$ , as  $b$  needs to be parallel to  $a \#_{k-1} x$  for  $y$  to exists.

## 3.2 Characterization of fibrant objects

In this section, we will give a simple characterization of the fibrant objects of the model structure introduced in Theorem 2.38. We will temporarily call the objects satisfying this characterization “prefibrant” (Definition 3.12) and then show in Proposition 3.19 that these are exactly the fibrant objects.

**3.10 Notation.** Suppose given an equation  $P$  and a lifting problem of the form:

$$\begin{array}{ccc} \Lambda P & \longrightarrow & C \\ \downarrow & \nearrow \text{dotted} & \downarrow p \\ P & \longrightarrow & D \end{array}$$

Given  $a$  a generator of  $P$ , we will denote its image in  $D$  also by  $a$ . If  $a \in \Lambda P$ , we denote by  $\bar{a}$  its image in  $C$ . So in general  $p(\bar{a}) = a$ . If the dotted diagonal lift exists, or in the process of constructing such a lift, the image of  $x, y \in P$  in  $C$  are also denoted  $\bar{x}$  and  $\bar{y}$ , and we hence also have  $p(\bar{x}) = x$  and  $p(\bar{y}) = y$ .

**3.11 Definition.** Let  $a$  be a  $(n+1)$ -arrow. An *inverse* for  $a$  is an arrow  $a^{-1}$  such that there exist two marked arrows:

$$\epsilon: a \#_n a^{-1} \rightarrow \mathbb{I} \quad \nu: a^{-1} \#_n a \rightarrow \mathbb{I}.$$

An arrow is *invertible* if it has an inverse.

**3.12 Definition.** An  $m$ -marked  $\infty$ -category  $C$  is *prefibrant* if

- (1) marked arrows are invertible and their inverses are marked,
- (2) whenever  $a$  and  $c: a \rightarrow b$  are marked, so is  $b$ .

This directly implies that if  $b$  and  $c: a \rightarrow b$  are marked, so is  $a$ .

This notion is purely temporary: we will show in Proposition 3.19 that an object is fibrant for the model structure of Theorem 2.38 if and only if it is prefibrant.

**3.13 Proposition.** *If  $C$  is prefibrant, then equations  $\mathbf{Eq}_{k,n}^{\circlearrowleft}$  and  $\mathbf{Eq}_{k,n}^{\circlearrowright}$  have weakly unique solutions in  $C$ .*

*Proof.* We show the result by a decreasing induction on  $k \leq n$ . The initialization corresponds to  $k = n$ . In this case, the data of a morphism  $\Lambda \mathbf{Eq}_{n,n}^{\circlearrowright} \rightarrow C$  corresponds to two  $n$ -arrows  $a$  and  $b$  sharing the same source and such that  $a$  is marked. Let  $\nu: a^{-1} \#_n a \rightarrow \mathbb{I}$ . If we define  $x := a^{-1} \#_n b$  and  $y: \psi \#_n b: a \#_n x \rightarrow b$ , the couple  $(x, y)$  is a solution of  $\mathbf{Eq}_{n,n}^{\circlearrowright}$ . If  $b$  is marked so is  $x$ . We now show the weak unicity of the solution. Let  $(\bar{x}, \bar{y})$  be another solution. We then have a marked arrow:

$$z: \bar{x} \xrightarrow{\nu^{-1}} a^{-1} \#_n a \#_n \bar{x} \xrightarrow{\bar{y}} a^{-1} * b.$$

The assertion for  $\mathbf{Eq}_{n,n}^{\circlearrowleft}$  is similar.

Suppose now the result is true for all  $k+1$ . We start by showing that solutions of  $\mathbf{Eq}_{k,n}^{\circlearrowright}$  and  $\mathbf{Eq}_{k,n}^{\circlearrowleft}$  are weakly unique in  $C$ . The data of a morphism

$\Lambda \mathbf{Eq}_{k,n}^{\circ\leftarrow} \rightarrow C$  corresponds to an  $n$ -arrow  $x: s \rightarrow t$ , a  $k$ -invertible arrow  $a$ , and an arrow  $b: a\#_{k-1}s \rightarrow a\#_k t$ . Let  $(x, y: a\#_{k-1}x \rightarrow b)$  be a solution of this equation. Let  $\nu: a^{-1}\#_n a \rightarrow \mathbb{I}$ . The arrow  $x$  is then also a solution of  $\mathbf{Eq}_{k+1,n}^{\circ\leftarrow}$ :

$$(\nu\#_0 s)\#_k x = (a^{-1}\#_{k-1} a\#_{k-1} x)\#_k (\nu\#_{k-1} t) \xrightarrow{(a^{-1}\#_{k-1} y)\#_k (\nu\#_{k-1} t)_1} (a^{-1}\#_{k-1} b)\#_k (\nu\#_{k-1} t)$$

and so is weakly unique. The unicity of solution of  $\mathbf{Eq}_{k,n}^{\circ\leftarrow}$  is proved similarly.

We show now that  $\mathbf{Eq}_{k,n}^{\circ\leftarrow}$  and  $\mathbf{Eq}_{k,n}^{\circ\rightarrow}$  have solutions in  $C$ . Let  $(x, y)$  be a solution of the equation

$$(\nu\#_0 s)\#_k x \xrightarrow{y} (a^{-1}\#_{k-1} b)\#_k (\nu\#_{k-1} t).$$

Moreover, we can find such  $x$  marked whenever  $b$  is. We then have

$$(\nu\#_0 s)\#_k x = (a^{-1}\#_{k-1} a\#_{k-1} x)\#_k (\nu\#_{k-1} t).$$

By weakly unicity of solution of  $\mathbf{Eq}_{k+1,n}^{\circ\leftarrow}$ , we then have a marked arrow

$$z: a^{-1}\#_{k-1} a\#_{k-1} x \rightarrow a^{-1}\#_{k-1} b.$$

But  $a\#_{k-1} x$  and  $b$  are solutions of an equation  $\mathbf{Eq}_{k,n}^{\circ\leftarrow}$ , and so there exist a marked arrow

$$\tilde{y}: a\#_{k-1} x \rightarrow b.$$

If  $b$  is marked, the arrow  $x$  that we produce is also marked. The existence of solution of  $\mathbf{Eq}_{k,n}^{\circ\leftarrow}$  is proved similarly.  $\square$

**3.14 Lemma.** *If equations  $\mathbf{Eq}_{k,n}^{\circ\leftarrow}$  and  $\mathbf{Eq}_{k,n}^{\circ\rightarrow}$  have solutions in  $C$ , then all equations have solutions in  $C$ .*

*Proof.* Let  $P$  be a left equation. There is a decomposition of the source of  $y$  of the shape

$$\pi_n^- y = l_n\#_{n-1}(l_{n-1}\#_{n-2}\dots\#_1(l_1\#_0 x\#_0 r_1)\#_1\dots\#_{n-2}r_{n-1})\#_{n-1}r_n$$

where for each  $i$ ,  $l_i$  and  $r_i$  are marked  $i$ -arrow in  $P$ . We can then use the existence of solutions to  $\mathbf{Eq}_{k,n}^{\circ\leftarrow}$  and  $\mathbf{Eq}_{k,n}^{\circ\rightarrow}$  to get two sequences of arrows  $(x_k)_{0 < k < 2n}$  and  $(y_k)_{0 < k < 2n}$  such that:

- (1)  $y_{2n-1}: x_{2n-1}\#_{n-1}r_n \rightarrow \pi_n^+ y$ ;
- (2)  $y_{2k-1}: x_{2k-1}\#_{k-1}r_k \rightarrow x_{2k}$ ;
- (3)  $y_{2k-2}: l_k\#_{k-1}x_{2k-2} \rightarrow x_{2k-1}$ .

Moreover, arrows  $x_k$  are marked whenever  $\pi_n^+ y$  is. The couple  $(x_0, \bar{y})$  is then a solution to  $P$  where  $\bar{y}$  is the composite:

$$\begin{aligned} \bar{y}: = & \quad (l_n\#_{n-1}(l_{n-1}\#_{n-2}\dots\#_1((y_0\#_0 r_1)\#_n y_1)\#_1\dots\#_{n-2}r_{n-1})\#_{n-1}r_n) \\ & \#_n (l_n\#_{n-1}(l_{n-1}\#_{n-2}\dots\#_2((y_2\#_1 r_2)\#_n y_3)\#_2\dots\#_{n-2}r_{n-1})\#_{n-1}r_n) \\ & \#_n \dots \\ & \#_n (y_{2n-2}\#_{n-1}r_n)\#_n y_{2n-1} \end{aligned}$$

If  $P$  is a right equation, we define  $P^{op}$  to be the left equation obtained by inverting the direction of the arrow of maximum dimension. A solution of  $P$  is given by  $(x, y^{-1})$  where  $(x, y)$  is a solution of  $P^{op}$ . Moreover, one can find an arrow  $x$  marked whenever the source of  $y^{-1}$  is.  $\square$

**3.15 Lemma.** *Let  $C$  be an  $m$ -marked  $\infty$ -category such that all equations have solutions in  $C$  and whenever  $a$  and  $c: a \rightarrow b$  are marked, so is  $b$ . Then  $C$  has the right lifting property against all equations and saturations.*

*Proof.* It is obvious that  $C$  has the right lifting property against all equations. Suppose now that we have a morphism  $f: \Omega Q \rightarrow C$  where  $Q$  is a saturation. This corresponds to a solution  $(x, y)$  of the equation  $P := (Q, \overline{M - \{x, t(y)\}})$ . We then know that there exists another solution  $(\bar{x}, \bar{y})$  of the equation where  $\bar{x}$  is marked. Furthermore, as  $P \coprod_{\Lambda P} Q \rightarrow \text{Uni}^{coh}(P)$  has solutions in  $C$ , there exists a marked arrow  $z': \bar{x} \rightarrow x$  and  $x$  is then marked. That shows that we can lift the morphism  $f$  to  $Q$ .  $\square$

**3.16 Lemma.** *Fibrant objects have the right lifting property against the equations  $\mathbf{eq}_{n,n}^{\hat{\circ}}$  and saturations  $\mathbf{sat}_{n,n}^{\hat{\circ}}$*

*Proof.* Consider a lifting problem of  $\mathbf{eq}_{n,n}^{\hat{\circ}}$  against  $C$ . This means that we have in  $C$  an  $n$ -arrow  $b$  and a marked  $n$ -arrow  $a$  that share the same source.

Since  $C$  is fibrant, it has, by definition, the right lifting property against  $\mathbf{eq}_n^{\hat{\circ}}$  as in Example 3.7. Using the same notations as in Example 3.7 for the generators of  $\Lambda \mathbf{Eq}_n^{\hat{\circ}}$ , we chose the image of  $a_l$  in  $C$  to be an identity for all  $l < n$ , and  $a_n = a$ . This gives us a commutative square:

$$\begin{array}{ccc} \Lambda \mathbf{Eq}_n^{\hat{\circ}} & \longrightarrow & C \\ \mathbf{eq}_n^{\hat{\circ}} \downarrow & & \\ \mathbf{Eq}_n^{\hat{\circ}} & & \end{array}$$

which has a dotted diagonal filling  $(x, y)$ . But this pair verifies  $y: x \#_{k-1} a \rightarrow b$ , and is then a solution of the lifting problem above.

The proof for the saturation  $\mathbf{sat}_{n,n}^{\hat{\circ}}$  is similar.  $\square$

**3.17 Lemma.** *In a fibrant  $m$ -marked  $\infty$ -category, all marked arrows are invertible. Moreover, their inverses are marked.*

*Proof.* First, the right lifting property against  $\mathbf{eq}_{n,n}^{\hat{\circ}}$  shows that for any marked arrow  $a$ , there exists a pair  $(a^{-1}, \nu)$  where  $\nu$  is marked and  $\nu: a^{-1} \#_n a \rightarrow \mathbb{I}$ . The fact that  $a^{-1}$  is marked, come from the right lifting property against  $\mathbf{sat}_{n,n}^{\hat{\circ}}$ .

Using again the right lifting property against  $\mathbf{eq}_{n,n}^{\hat{\circ}}$ , we deduce that there is two marked arrows  $(a^{-1})^{-1}$  and  $\beta$  such that:

$$\beta: (a^{-1})^{-1} \#_n a^{-1} \rightarrow \mathbb{I}.$$

Finally, in the same way, we obtain a marked arrow:

$$\beta^{-1}: \mathbb{I} \rightarrow (a^{-1})^{-1} \#_n a^{-1}.$$

We then define  $\epsilon: a\#_n a^{-1} \rightarrow \mathbb{I}$  as the composite:

$$\begin{array}{ccc} a\#_n a^{-1} & & \mathbb{I} \\ \beta^{-1}\#_n a\#_n a^{-1} \downarrow & & \uparrow \beta \\ (a^{-1})^{-1}\#_n a^{-1}\#_n a\#_n a^{-1} & \xrightarrow{(a^{-1})^{-1}\#_n \nu\#_n a^{-1}} & (a^{-1})^{-1}\#_n a^{-1} \end{array}$$

As it is a composite of marked arrows,  $\epsilon$  is also marked. This then shows that  $a^{-1}$  is an inverse of  $a$ .  $\square$

**3.18 Lemma.** *Fibrant objects are prefibrant.*

*Proof.* Lemma 3.17 implies the first condition. For the second one, let  $y: x \rightarrow b$  be a marked arrow where  $b$  is marked. The right lifting property against  $\mathbf{sat}_{n,n}^{\circlearrowright}$ , choosing  $a$  to be an identity, implies that  $x$  is marked. Now suppose given a marked arrow  $y: b \rightarrow x$  where  $x$  is marked. We have a marked arrow  $y^{-1}: x \rightarrow b$  and  $b$  is then marked.  $\square$

**3.19 Proposition.** *For an  $m$ -marked  $\infty$ -category  $C$ , the following assertions are equivalent:*

- (1)  $C$  is prefibrant in the sense of Definition 3.12.
- (2) All equations have solutions in  $C$  and whenever  $a$  and  $c: a \rightarrow b$  are marked, so is  $b$ .
- (3)  $C$  has the right lifting property against all equations and saturations.
- (4)  $C$  is fibrant for the model structure of Theorem 2.38.

*Proof.* The implication (1)  $\Rightarrow$  (2) is a consequence of Proposition 3.13 and Lemma 3.14. Lemma 3.15 states (2)  $\Rightarrow$  (3). As generating anodyne extensions are either equations or saturations, (3)  $\Rightarrow$  (4). Eventually, the implication (4)  $\Rightarrow$  (1) is the content of Lemma 3.18.  $\square$

### 3.3 Isofibrations

In this section we will give as Proposition 3.23 a simpler characterization of fibrations between fibrant objects, as the ‘‘isofibrations’’ in the following sense:

**3.20 Definition.** A morphism between  $m$ -marked  $\infty$ -categories is said to be an *isofibration* if it has the lifting property against the maps:

$$i_n^+: \mathbb{D}_n^b \rightarrow (\mathbb{D}_{n+1}, \{\overline{e_{n+1}}\})$$

where  $e_{n+1}$  is the unique non-identity arrow of  $\mathbb{D}_{n+1}$ .

Explicitly, a morphism  $\pi: X \rightarrow Y$  between fibrant  $m$ -marked  $\infty$ -categories is an isofibration if for every  $n$ -dimensional arrow  $f: a \rightarrow b$  in  $X$ , such that in  $Y$  there is a parallel arrow  $g: \pi(a) \rightarrow \pi(b)$  with a marked arrow  $h: g \rightarrow \pi(f)$ , then  $g$  and  $h$  can be lifted to arrows  $\bar{g}: a \rightarrow b$  and  $\bar{h}: \bar{g} \rightarrow f$  of  $X$ , with  $\bar{h}$  marked such that  $\pi(\bar{g}) = g$  and  $\pi(\bar{h}) = h$ .

Note that it follows from Lemma 2.41 that fibrations are isofibrations. We insist on the fact that we will only consider the notion of isofibration between *fibrant*  $m$ -marked  $\infty$ -categories. We do not expect the definition given above to be very interesting outside this context.

**3.21 Lemma.** *Any isofibrations between fibrant  $m$ -marked  $\infty$ -category also has the lifting property against*

$$i_n^-: \mathbb{D}_n^b \rightarrow (\mathbb{D}_{n+1}, \overline{\{e_{n+1}\}})$$

*Proof.* Let  $\pi: X \rightarrow Y$  be an isofibration between fibrant  $m$ -marked  $\infty$ -categories,  $f: a \rightarrow b$  an  $n$ -arrow in  $X$ , with  $g: \pi(a) \rightarrow \pi(b)$  and  $h: \pi(f) \rightarrow g$  two arrows in  $Y$ ,  $h$  being marked.

As  $Y$  is fibrant, accorded to proposition 3.17 the arrow  $h$  admits an inverse, i.e. there is a marked arrow  $h^{-1}: g \rightarrow \pi(f)$  and another marked arrow  $t: h^{-1} \#_n h \rightarrow \mathbb{I}_g$  witnessing the inverse relation. One can then apply the isofibration property to lift  $g$  and  $h^{-1}$  to two arrows  $\bar{g}: a \rightarrow b$  and  $\bar{h}^{-1}: \bar{g} \rightarrow f$ .

As  $X$  is also fibrant, One can then consider an inverse  $\bar{h}'$  of  $\bar{h}^{-1}$  in  $X$ , whose image by  $\pi$  is going to be a second inverse of  $h^{-1}$  in  $Y$ , and again because  $Y$  is fibrant, one can hence construct a marked arrow  $h \rightarrow \pi(\bar{h}')$ , applying the isofibration property one more time then gives us a lift of  $h$  and concludes the proof.  $\square$

**3.22 Lemma.** *An isofibration between fibrant  $m$ -marked  $\infty$ -categories has the right lifting property against all equations and saturations*

*Proof.* We will show that such a morphism has the lifting property against all left equations, the exact same argument shows that it also has the lifting property against all right equations.

We consider an isofibration  $\pi: X \rightarrow Y$  between two fibrant  $m$ -marked  $\infty$ -categories and a lifting problem of  $\pi$  against  $\Lambda P \rightarrow P$ :

$$\begin{array}{ccc} \Lambda P & \longrightarrow & X \\ \downarrow & & \downarrow \pi \\ P & \xrightarrow{(x,y)} & Y \end{array}$$

we want to show that  $x$  and  $y$  can be lifted to  $X$ .

One first remark is that as  $X$  is fibrant, the equation  $P$  has solutions in  $X$  according to Proposition 3.19. This implies that one can find arrows  $x'$  and  $y'$  in  $X$  that have the correct shape (although they might not be lifts of  $x$  and  $y$ ).

Now, in  $Y$ ,  $(\pi(x'), \pi(y'))$  also is a solution to the equation  $\Lambda P$ . As  $Y$  is fibrant,  $Uni^{coh}(P)$  has solutions in  $Y$ , and there exist marked arrows:

$$\begin{aligned} z: x &\rightarrow \pi(x') \\ t: s[x/z] \#_n \pi(y') &\rightarrow y \end{aligned}$$

where  $s$  is the source of  $y$ .

One can then use the isofibration property, as well as its dual version from Lemma 3.21 to gradually lifts all these arrows to  $X$ : one lifts first  $x$  and  $z$ , then one can lift  $y$  and  $t$ .

Now, to show that isofibrations have the right lifting property against saturation, one simply remarks that lifts against saturations are unique when they exist (saturations are epimorphisms), so as fibrant objects have the right lifting property against these maps, any map between fibrant objects also has the lifting property against all saturation.  $\square$

**3.23 Proposition.** *A morphism between fibrant  $m$ -marked  $\infty$ -categories is a fibration if and only if it is an isofibration.*

*Proof.* According to Lemma 2.41, the morphism  $i_n^+$  is an anodyne cofibration, so that all fibrations (between fibrant objects) are isofibrations.

For the converse, as a morphism between fibrant objects is a fibration if and only if it has the right lifting property against generating acyclic cofibrations, which are either equations or saturations, the last lemma implies that isofibrations between fibrant objects are fibrations.  $\square$

As a consequence we have:

**3.24 Corollary.** *Equations and saturations are acyclic cofibrations.*

Indeed, as these maps have the lifting property against all isofibrations between fibrant objects, they have the lifting property against all fibrations between fibrant objects.

### 3.4 Equivalences

We now turn to the characterization of weak equivalences between fibrant objects.

**3.25 Definition.** A morphism  $p: X \rightarrow Y$  between fibrant  $m$ -marked  $\infty$ -categories is a *equivalence of  $m$ -marked  $\infty$ -categories* if:

- (1) For any arrow  $x \in X$ , if  $p(x)$  is a marked in  $Y$ , then  $x$  is marked in  $X$ .
- (2) For any object  $c \in Y$ , there exists an object  $\tilde{c} \in X$  and a marked arrow  $e: p(\tilde{c}) \rightarrow c$ .
- (3) For any pair of parallel arrows  $(a, b)$  in  $X$ , and any arrow  $c: p(a) \rightarrow p(b)$  in  $Y$ , there exist an arrow  $\tilde{c}: a \rightarrow b$  in  $X$  and a marked arrow  $e: p(\tilde{c}) \rightarrow c$  in  $X$ .

So informally, a functor is an equivalence if it is conservative, essentially surjective, and “essentially surjective on each Hom  $\infty$ -category”.

**3.26 Proposition.** *An arrow  $f: X \rightarrow Y$  between fibrant objects in  $\infty\text{-Cat}^m$  is a weak equivalence of the left semi-model structure of Theorem 2.38 if and only if it is an equivalence in the sense of Definition 3.25.*

*Proof.* We will use the general characterization of weak equivalence between fibrant objects as the morphisms having the “weak left lifting properties” against the generating cofibrations, that is if given a lifting problem against one of the generating cofibration, there exists a diagonal lift in which is the upper triangle commutes and the lower triangle commutes up to a relative homotopy. This is established for weak model categories (in particular left semi-model categories) in [14] as theorem A.2.6 (see also remark A.2.7). We recall that in our model structure, the generating cofibrations are given by

$$I^a = \{i_n: \partial\mathbb{D}_n \rightarrow \mathbb{D}_n, |n \geq 0\} \quad I^m = \{\mathbb{D}_n \rightarrow (\mathbb{D}_n, \overline{\{e_n\}}) \quad , n \geq 0\}$$

To express what the “weak left lifting property” against these morphisms means, we need a relative cylinder object for each of these cofibrations.

For a map of the form,  $\mathbb{D}_n \rightarrow (\mathbb{D}_n, \overline{\{e_n\}})$ , we have that the canonical map

$$(\mathbb{D}_n, \overline{\{e_n\}}) \coprod_{\mathbb{D}_n} (\mathbb{D}_n, \overline{\{e_n\}}) \rightarrow (\mathbb{D}_n, \overline{\{e_n\}})$$

is an isomorphism, so that  $(\mathbb{D}_n, \overline{\{e_n\}})$  is already a cylinder object. In particular, the weak left lifting property against these maps is exactly the same as the ordinary left lifting property and it corresponds exactly to the first condition of Definition 3.25.

For the map  $i_n: \partial\mathbb{D}_n \rightarrow \mathbb{D}_n$  one obtains a relative cylinder object by considering the factorization:

$$\mathbb{D}_n \coprod_{\partial\mathbb{D}_n} \mathbb{D}_n \xrightarrow{\quad} (\mathbb{D}_{n+1}, \overline{\{e_{n+1}\}}) \rightarrow \mathbb{D}_n$$

the first map freely adds a (marked)  $(n+1)$ -arrow between the two non-trivial arrows of the domain, so it is a cofibration. And one of the two maps  $\mathbb{D}_n \rightarrow (\mathbb{D}_{n+1}, \overline{\{e_{n+1}\}})$  was shown to be an anodyne cofibration in Lemma 2.41, hence proving that this is a relative cylinder object for this cofibration. Using this cylinder to express the weak lifting property against  $i_n$ , one obtains exactly the second condition (for  $n=0$ ) and the third condition (for  $n>0$ ) of Definition 3.25. Indeed, given a weak lifting diagram:

$$\begin{array}{ccc}
 \partial\mathbb{D}_n & \xrightarrow{\quad} & X \\
 \downarrow & \searrow & \downarrow p \\
 & \mathbb{D}_n & \\
 & \downarrow & \\
 & (\mathbb{D}_{n+1}, \overline{\{e_n\}}) & \\
 \downarrow & \nearrow & \downarrow \\
 \mathbb{D}_n & \xrightarrow{\quad} & Y
 \end{array}$$

The solid part of the diagram corresponds to a pair of parallel  $(n-1)$ -arrows  $(a, b)$  in  $X$ , together with an  $n$ -arrow  $c: p(a) \rightarrow p(b)$  in  $Y$ , the top dotted morphism gives us an arrow  $\tilde{c}: a \rightarrow b$ , while the bottom dotted morphism corresponds to a marked  $(n+1)$ -arrow  $e: p(\tilde{c}) \rightarrow c$ , so this lifting condition corresponds exactly to the third point of Definition 3.25 (with the second point corresponding to the case  $n=0$ ).

□

### 3.5 The saturated localization.

Proposition 3.19 produces a characterization of fibrant objects of the model structure of Theorem 2.38: a marked  $\infty$ -categories is fibrant if the marked arrows have inverses and if an arrow isomorphic to a marked arrow is marked.

A careful reader might have noticed however that this is not sufficient to show that the marked arrows are exactly the arrows that have inverses in the sense of Definition 3.11.

**3.27 Example.** Let  $C$  be a category, seen as an  $\infty$ -category with no non-identity arrows of dimension  $> 1$ . We endow  $C$  with the marking  $C^b$ , where only the identity arrows are marked.

With this marking  $C$  is fibrant, indeed, it satisfies all the conditions of Proposition 3.19. But if the category  $C$  has non-identity invertible arrows, these would be arrows that have inverses in the sense of Definition 3.11 without being marked.

In this section, we “fix” this problem by introducing a Bousfield localization in which the fibrant objects have these properties.

**3.28 Definition.** A marked  $\infty$ -category  $C$  is said to satisfy the *2-out-of-6 property* if given three composable  $n$ -arrow  $f, g$  and  $h$  such that  $f\#_{n-1}g$  and  $g\#_{n-1}h$  are marked, then  $f, g$  and  $h$  are marked.

**3.29 Remark.** If  $C$  is a fibrant  $m$ -marked  $\infty$ -category. Then the relation  $f \sim g$  defined by  $\exists c: f \rightarrow g$  a marked  $(n+1)$ -arrow, is an equivalence relation on  $n$ -arrow. Indeed it is reflexive and transitive as identities are marked and composites of marked arrows are marked, and it is symmetric as marked arrows have inverses.

This equivalence relation is moreover compatible with all composition operations, so that one can define a “homotopy  $n$ -category”  $h_n C$ , which is an  $n$ -category whose  $k$  arrows for  $k < n$  are these of  $C$  and its  $n$ -arrows are equivalence classes for this relations. We will use in particular that given two parallel  $n-2$  arrows  $u, v$  in  $C$  we have a category  $h_n C(u, v)$  whose objects are  $n-1$  arrows  $u \rightarrow v$  and whose morphisms are equivalence classes of  $n$  arrows between them.

**3.30 Lemma.** *For an  $m$ -marked  $\infty$ -category  $C$  the following conditions are equivalent:*

- (1) *An arrow in  $C$  is marked if and only if it has an inverse in the sense of Definition 3.11.*
- (2)  *$C$  is fibrant in the model structure of Theorem 2.38 and satisfies the 2-out-of-6 property.*

*Proof.* We first consider  $C$  an  $m$ -marked  $\infty$ -category which satisfies (1), and we check it is fibrant using Proposition 3.19. The first condition of Proposition 3.19 is immediate, we check the second condition : if  $c: a \rightarrow b$  is marked and  $b$  is marked, then considering  $b^{-1}$  an inverse of  $b$ , the marked arrow connecting  $\epsilon: b^{-1}\#_n b \rightarrow \mathbb{I}$  and  $\nu: b\#_n b^{-1} \rightarrow \mathbb{I}$ , we can simply compose:

$$\begin{aligned} b^{-1}\#_n a &\xrightarrow{b^{-1}\#_n c} b^{-1}\#_n b \xrightarrow{\epsilon} \mathbb{I} \\ a\#_n b^{-1} &\xrightarrow{c\#_n b^{-1}} b\#_n b^{-1} \xrightarrow{\epsilon} \mathbb{I} \end{aligned}$$

This shows that  $b^{-1}$  is also an inverse for  $a$ , and hence if all arrows with an inverse are marked  $a$  is marked as well. Note that if it is  $a$  which is marked in the first place then one can consider an inverse  $c^{-1}: b \rightarrow a$  and apply the same argument.

Next, we show that  $C$  satisfies 2-out-of-6. For this, we can rely on Remark 3.29. An  $n$  arrow has an inverse in the sense of Definition 3.11 if and

only if it is an isomorphism in the category  $h_n C(u, v)$  where  $u$  and  $v$  are its  $(n - 2)$ -dimensional source and target. Our assumption is then that an  $n$ -arrow is marked if and only if its equivalence class is invertible in the category  $h_n C(u, v)$ . The fact that marked arrows satisfy 2-out-of-6 then follows from the fact that isomorphism in a category satisfies the 2-out-of-6 condition.

Conversely, assuming that  $C$  satisfies condition (2) we have that marked arrows have inverses because  $C$  is fibrant and Proposition 3.19. If an arrow  $a$  has an inverse  $a^{-1}$  then both  $a\#_{n-1}a^{-1}$  and  $a^{-1}\#_{n-1}a$  are marked because they are equivalence to identities, and it follows from the 2-out-of-6 condition that  $a$  (and  $a^{-1}$ ) is marked. □

**3.31 Theorem.** *The model structure of Theorem 2.38 admits a Bousfield localization (as a left semi-model structure) in which the fibrant objects are the marked  $\infty$ -categories which satisfy the equivalent conditions of Lemma 3.30.*

*We call this model structure the saturated inductive model structure.*

As a Bousfield localization, this model structure has the same cofibrations and the same fibration between fibrant objects as the model structure from Theorem 2.38.

*Proof.* The key point here is that the 2-out-of-6 condition for a marked  $\infty$ -category corresponds to the lifting property against certain cofibrations.

For each  $n$  we consider the polygraphs  $X_n$  generated by three composable  $n$  arrow

$$\mathbb{D}_n \coprod_{\mathbb{D}_{n-1}} \mathbb{D}_n \coprod_{\mathbb{D}_{n-1}} \mathbb{D}_n$$

Where each pushout uses the target maps on the left and the source map on the right. We call  $f$ ,  $g$  and  $h$  the three  $n$ -dimensional generators of  $X_n$ . We consider the map  $s_n$ :

$$s_n: \left( X_n, \overline{\{f\#_{n-1}g, g\#_{n-1}h\}} \right) \rightarrow \left( X_n, \overline{\{f, g, h\}} \right)$$

which is the identity of  $X_n$  (with two different markings).  $s_n$  is a cofibration, and a marked  $m$ -category has the right lifting property against all the  $s_n$  if and only if it satisfies the 2-out-of-6 property.

We hence take the left Bousfield localization of the model structure of Theorem 2.38 at the set  $s_n$ . The theory of left Bousfield localization for left semi-model structure can be found in [5] or [15].

Given that the maps in  $s_n$  are already cofibration between cofibrant objects, the fibrant objects of the left Bousfield localization are the fibrant objects that have the right lifting property against the maps  $s_n$  and all their iterated cylinder maps  $\nabla^k s_n$ , where if  $i: A \rightarrow B$  is a cofibration,  $\nabla i$  is the cofibration  $B \coprod_A B \rightarrow I_A B$  for some relative cylinder object. However, in the case of the map  $s_n$ , given that it only changes the marking, the pushout  $B \coprod_A B$  is just  $B$ , hence it is already a cylinder object, and the map  $\nabla s_n$  is an isomorphism. It hence follows that an object is fibrant in the localization if it is fibrant and has the lifting property against all the  $s_n$ , i.e. has the 2-out-of-6 property. This concludes the proof. □

## 4 Comparison with other model structures

### 4.1 Truncation functors

**4.1 Definition.** Let  $m < p \leq \infty$ . There is a functor:

$$\begin{array}{ccc} \pi_m: \infty\text{-}\mathbf{Cat}^{m+1} & \rightarrow & \infty\text{-}\mathbf{Cat}^m \\ (X, M) & \mapsto & (X, \overline{M}). \end{array}$$

that marks every arrow of dimension  $m + 1$ , an obvious inclusion functor:

$$\begin{array}{ccc} \iota_{m+1}: \infty\text{-}\mathbf{Cat}^m & \rightarrow & \infty\text{-}\mathbf{Cat}^{m+1} \\ (X, M) & \mapsto & (X, M) \end{array}$$

and eventually, a functor:

$$\begin{array}{ccc} \tau_m: \infty\text{-}\mathbf{Cat}^{m+1} & \rightarrow & \infty\text{-}\mathbf{Cat}^m \\ (X, M) & \mapsto & (\tau(X), M) \end{array}$$

where  $\tau(X)$  is the sub  $\infty$ -category of  $X$  whose arrows of dimension strictly superior to  $m$  are the ones in  $M$ . As  $M$  is assumed to be closed under composition and contains the identities,  $\tau(X)$  is indeed an  $\infty$ -category.

These functors fit in following adjunctions:

$$\pi_m \dashv \iota_{m+1} \dashv \tau_m.$$

**4.2 Notation.** For  $m < p$ , we also denote by  $\iota_p: \infty\text{-}\mathbf{Cat}^p \rightarrow \infty\text{-}\mathbf{Cat}^m$  (resp.  $\pi_p: \infty\text{-}\mathbf{Cat}^m \rightarrow \infty\text{-}\mathbf{Cat}^p$ , resp.  $\tau_p: \infty\text{-}\mathbf{Cat}^m \rightarrow \infty\text{-}\mathbf{Cat}^p$ ) the iterate composite of  $\iota_k$  (resp.  $\pi_k$ , resp.  $\tau_k$ ) when  $k$  range over  $[p, m]$ .

Moreover, because  $\iota_p$  is the inclusion of a full subcategory, we will often identify  $X$  and  $\iota_p X$  in our notation. In the same way, for a morphism  $f \in \text{Hom}(X, \tau_m(Y))$ , the corresponding morphism in  $\text{Hom}(\iota_p X, Y)$  will also be denoted  $f$ .

**4.3 Proposition.** For  $m < p$ , the adjoint pairs  $(\pi_m \dashv \iota_p)$  and  $(\iota_p \dashv \tau_m)$  are Quillen pairs.

*Proof.* The functors  $\pi_m$  and  $\iota_p$  obviously preserve cofibrations and anodyne cofibrations.  $\square$

As mentioned in the introduction, we can consider the two towers of model structures:

$$\begin{array}{c} \infty\text{-}\mathbf{Cat}^0 \xleftarrow{\pi_0} \infty\text{-}\mathbf{Cat}^1 \xleftarrow{\pi_1} \infty\text{-}\mathbf{Cat}^2 \xleftarrow{\pi_2} \dots \xleftarrow{\pi_{n-1}} \infty\text{-}\mathbf{Cat}^n \xleftarrow{\pi_n} \dots \\ \infty\text{-}\mathbf{Cat}^0 \xleftarrow{\tau_0} \infty\text{-}\mathbf{Cat}^1 \xleftarrow{\tau_1} \infty\text{-}\mathbf{Cat}^2 \xleftarrow{\tau_2} \dots \xleftarrow{\tau_{n-1}} \infty\text{-}\mathbf{Cat}^n \xleftarrow{\tau_n} \dots \end{array}$$

and take the projective limit of either tower to get a definition of strict  $(\infty, \infty)$ -categories.

Our goal in this section is to show that the inductive model structure on  $\infty\text{-}\mathbf{Cat}^\infty$  is equivalent to the limit of the second tower (with  $\tau$  functors). Here by projective limit we mean a homotopy theoretic limit of these towers, that is a homotopy limit of the corresponding tower of  $(\infty, 1)$ -categories. Such projective limits of model structures have been studied in [6] and [12] and we will use the construction from these papers.

**4.4 Remark.** It should be noted that the results from [6] and [12] are only proved for Quillen model structures, so they do not immediately apply to the left semi-model structures that we are using here. The proof from these two papers easily adapts to the setting of left semi-model structures with very few modifications, so it should be safe to assume these results can be applied here as well. Though to avoid relying on this, we will give an independent proof that the model structure we use as a model of this projective limits exists and state our main theorem as an equivalence with this model structure. The only aspect that still relies on applying the results of [6] or [12] to semi-model structures is in order to interpret our results as saying something about homotopy limits of towers.

**4.5 Definition.** A *category with weak equivalences* is a couple  $(C, W)$  where  $C$  is a category and  $W$  is a class of map  $C$  satisfying the two-out-of-three property. We define the *homotopy category* of  $(C, W)$  as  $ho(C, W) := C[W^{-1}]$ .

**4.6 Definition.** We define the category with weak equivalences  $\text{LimLax}_{n \in \mathbb{N}} \infty\text{-Cat}^n$ , whose object are sequences  $X_\bullet = \{(X_n, f_n)\}_{n \in \mathbb{N}}$  where  $X_n \in \infty\text{-Cat}^n$  and  $f_n: X_n \rightarrow \tau_n X_{n+1}$ , and whose weak equivalences are pointwise equivalences. By adjunction, objects are in bijection with sequences

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \dots$$

where each  $X_n \in \infty\text{-Cat}^n$ .

The category with weak equivalences  $\lim_{n \in \mathbb{N}} \infty\text{-Cat}^n$ , is the full sub-category of  $\text{LimLax}_{n \in \mathbb{N}} \infty\text{-Cat}^n$  composed of objects  $\{(X_n, f_n)\}_{i \in \mathbb{N}}$  where for all  $n$ ,  $f_n: X_n \rightarrow \tau_n X_{n+1}$  is a weak equivalence of the model structure on  $\infty\text{-Cat}^n$ . Weak equivalences are pointwise equivalences.

**4.7 Proposition.** *There exist a model structure on  $\text{LimLax}_{n \in \mathbb{N}} \infty\text{-Cat}^n$ , called the lax-limit structure, where fibrations and weak equivalences are pointwise, and cofibrations are morphisms  $h: X_\bullet \rightarrow Y_\bullet$  such that  $h_0: X_0 \rightarrow Y_0$  is a cofibration in  $\infty\text{-Cat}^0$ , and for all  $n$ , the dotted morphism in the following diagrams is a cofibration in  $\infty\text{-Cat}^{i+1}$ :*

$$\begin{array}{ccc}
 X_n & \longrightarrow & X_{n+1} \\
 \downarrow & & \downarrow \\
 Y_n & \longrightarrow & Y_n \amalg_{X_n} X_{n+1} \\
 & \searrow & \downarrow \text{dotted} \\
 & & Y_{n+1}
 \end{array}$$

*Proof.* First let us notice that  $\text{LimLax}_{n \in \mathbb{N}} \infty\text{-Cat}^n$ , can be identified with the full subcategory of the functors  $X: \mathbb{N} \rightarrow \infty\text{-Cat}^\infty$  such that  $X_n \in \infty\text{-Cat}^n$ .

There is a model structure on such functors, where fibrations and weak equivalences are pointwise: the projective (or Reedy) model structure. The cofibrations of this model structure are as described in the proposition and this model structure restricts to  $\text{LimLax}_{n \in \mathbb{N}} \infty\text{-Cat}^n$ .  $\square$

**4.8 Definition.** We have an adjunction

$$\begin{array}{ccc} & \xrightarrow{c} & \\ \text{LimLax}_{n \in \mathbb{N}} \infty\text{-Cat}^n & \perp & \infty\text{-Cat}^\infty \\ & \xleftarrow{\tau} & \end{array}$$

where the left adjoint sends a sequence  $X_\bullet$  to its colimit:

$$c(X_\bullet) := \text{Colim}_{n \in \mathbb{N}} X_n,$$

and the right adjoint sends a  $\infty$ -marked  $\infty$ -category  $X$  on the sequence

$$\tau_0(X) \rightarrow \cdots \rightarrow \tau_n(X) \rightarrow \cdots$$

**4.9 Proposition.** *This adjunction induces a Quillen adjunction between the lax-limit model structure and the inductive model structure where the left adjoint preserves weak equivalences and fibrant objects.*

*Proof.* The left adjoint  $c$  clearly preserves cofibrations. Secondly, because the model structure on  $\infty\text{-Cat}^\infty$  is  $\omega$ -combinatorial, its fibrant objects are closed under  $\omega$ -filtered colimits, and because its factorization systems can be obtained as  $\omega$ -accessible functors its weak equivalences are closed under  $\omega$ -filtered colimit (this is shown for Quillen model structure as Proposition 7.3 of [11] and for left semi-model structure as Proposition 7.7 of [13]). This implies that the functor  $c$  preserves cofibrations and weak equivalences (as it is a filtered colimit) and hence it preserves acyclic cofibrations.  $\square$

**4.10 Proposition.** *There is a left Bousfield localization of the model structure on  $\text{LimLax}_{n \in \mathbb{N}} \infty\text{-Cat}^n$ , called the limit structure, where  $X_\bullet$  is fibrant if and only if it is fibrant in the lax-limit model structure and if for all integer  $n$ ,  $f_n: X_n \rightarrow \tau_n X_{n+1}$  is a weak equivalence. Moreover, weak equivalences between fibrant objects are pointwise equivalences.*

According to our claim (see Remark 4.4) that the results of [6] or [12] can be applied to left semi-model structures, the  $\infty$ -category obtained as the localization of this Bousfield localization is equivalent to the limit of the  $\infty$ -categories obtained as the localization of the  $\infty\text{-Cat}^n$  (with the  $\tau_n$  functors as transitions).

We need to introduce certain constructions before proving the theorem:

**4.11 Construction.** Let  $k$  be any integer. We define  $\text{LimLax}_{i \in k, k+1}(\infty\text{-Cat}^i, \tau_i)$  to be the category whose objects are triplet  $(X, X', f: X \rightarrow \tau_k(X'))$  where  $X$  and  $X'$  are respectively  $k$ -marked and  $(k+1)$ -marked  $\infty$ -categories. By adjunction, these objects are in bijection with sequences:

$$X \xrightarrow{f} X'$$

where  $X$  and  $X'$  are respectively  $k$ -marked and  $(k+1)$ -marked  $\infty$ -categories.

There is an adjunction

$$\begin{array}{ccc} & \xrightarrow{U} & \\ \text{LimLax}_{i \in k, k+1}(\infty\text{-Cat}^i, \tau_i) & \perp & \text{LimLax}_{i \in \mathbb{N}}(\infty\text{-Cat}^i, \tau_i) \\ & \xleftarrow{r} & \end{array}$$

where the left adjoint  $U$  sends  $X \rightarrow Y$  to the sequence

$$\emptyset \rightarrow \dots \rightarrow \emptyset \rightarrow X \xrightarrow{f} Y \rightarrow Y \rightarrow \dots \rightarrow Y \rightarrow \dots$$

while the right adjoint sends  $X_\bullet$  to

$$X_k \xrightarrow{f} X_{k+1}.$$

**4.12 Remark.** Given  $i: A \rightarrow B$  a cofibration between cofibrant objects in a (possibly left semi-) model category, we call the (or a) homotopy codiagonal of  $i$  the cofibration  $B \amalg_A B \rightarrow I_A B$  where  $I_A B$  is some choice of a relative cylinder object for this cofibration. Given that this homotopy codiagonal is itself a cofibration between cofibrant objects this construction can be iterated. When constructing a left Bousfield localization at a set  $S$  of cofibration between cofibrant objects, the fibrant objects of the localization are exactly the objects that have the right lifting property against all arrows in  $S$  as well as all their iterated homotopy codiagonal. This is a fairly standard result on Bousfield localization, which is proved for weak model structure (in particular for left semi-model structure) in [15] (See the proof of Theorem 7.3 and Remark 7.6).

**4.13 Construction.** Given  $A \rightarrow B$  a cofibration between cofibrant objects in  $\infty\text{-Cat}^k$ , we can see it as a cofibrant object of  $\text{LimLax}_{i \in k, k+1}(\infty\text{-Cat}^i, \tau_i)$ . Given a choice of a relative cylinder object  $I_A B$  for  $A \rightarrow B$ , we have a cofibration in  $\text{LimLax}_{i \in k, k+1}(\infty\text{-Cat}^i, \tau_i)$  given by the square:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ B & \longrightarrow & I_A B \end{array}$$

A key observation for the proof below is that there is a way to choose a homotopy codiagonal for this map that is also of this form.

Indeed to construct such a codiagonal map, one needs to construct a (cofibration, weak equivalence) of a map (in the vertical direction):

$$\begin{array}{ccc} B \amalg_A B & \longrightarrow & I_A B \amalg_B I_A B \\ \downarrow & & \downarrow \\ B & \longrightarrow & I_A B \end{array}$$

One can observe that the horizontal map  $B \amalg_A B \rightarrow I_A B \amalg_B I_A B$  is already a relative cylinder object for  $A \rightarrow B$ , so that one can first factorize the leftmost map

$$\begin{array}{ccc} B \amalg_A B & \longrightarrow & I_A B \amalg_B I_A B \\ \downarrow & & \downarrow \\ I_A B \amalg_B I_A B & & \\ \downarrow \sim & & \\ B & \longrightarrow & I_A B \end{array}$$

One then forms the pushout  $P$ :

$$\begin{array}{ccc}
B \amalg_A B & \longrightarrow & I_A B \amalg_B I_A B \\
\downarrow & & \downarrow \\
I_A B \amalg_B I_A B & \longrightarrow & P \\
\downarrow & & \downarrow \\
B & \longrightarrow & I_A B
\end{array}$$

And the map  $P \rightarrow I_A B$  can then be factored in a cofibration followed by a weak equivalence.

$$\begin{array}{ccc}
B \amalg_A B & \longrightarrow & I_A B \amalg_B I_A B \\
\downarrow & & \downarrow \\
I_A B \amalg_B I_A B & \longrightarrow & P \\
\downarrow & & \downarrow \\
B & \longrightarrow & I_A B
\end{array}$$

$$\begin{array}{ccc}
& & W \\
& & \downarrow \\
& & \sim \\
& & I_A B
\end{array}$$

Which gives a relative cylinder object, and hence a homotopy codiagonal for our map of the form:

$$\begin{array}{ccc}
B \amalg_A B & \longrightarrow & I_A B \amalg_B I_A B \\
\downarrow & & \downarrow \\
I_A B \amalg_B I_A B & \longrightarrow & W
\end{array}$$

But one can see that the object  $W$  we constructed above is itself a relative cylinder object for the map  $B \amalg_A B \rightarrow I_A B \amalg_B I_A B$  and hence this homotopy codiagonal is again of the desired form.

*Proof of Proposition 4.10.* As in Construction 4.13, given a cofibration  $A \twoheadrightarrow B$  in  $\infty\text{-Cat}^k$  we consider the cofibration  $\{U(A \rightarrow B) \rightarrow U(B \rightarrow I_A B)\}$  in  $\text{LimLax}_{i \in \mathbb{N}}(\infty\text{-Cat}^i, \tau_i)$ . We call  $I_k$  the set of cofibrations obtained for  $A \twoheadrightarrow B$  a generating cofibration of  $\infty\text{-Cat}^k$ . We claim that a fibrant object  $(X_i, f_i)$  of  $\text{LimLax}_{i \in \mathbb{N}}(\infty\text{-Cat}^i, \tau_i)$  has the right lifting property against all maps in  $I_k$  if and only if the map  $f_k: X_k \rightarrow \tau_k X_{k+1}$  is a weak equivalence, and that this also implies that  $(X_i, f_i)$  has the right lifting property against all maps of the form  $\{U(A \rightarrow B) \rightarrow U(B \rightarrow I_A B)\}$  when  $A \twoheadrightarrow B$  is an arbitrary cofibration in  $\infty\text{-Cat}^k$ .

For this, we will use the criterion that in any model category a morphism between fibrant objects  $f: X \rightarrow Y$  is a weak equivalence if and only if for every generating cofibration  $A \rightarrow B$ , there is, in the category of arrows, a lifting in all

diagram of shape:

$$\begin{array}{ccc}
 (A \rightarrow B) & \longrightarrow & (X \xrightarrow{f} Y) \\
 \downarrow & \nearrow \text{dotted} & \\
 (B \rightarrow I_A B) & & 
 \end{array}$$

where  $I_A B$  is a relative cylinder object for the cofibration  $A \rightarrow B$ . This is proved for weak model categories in Appendix A.2 of [14], see Theorem A.2.6 and Remark A.2.7.

Now, an object  $(X_i, f_i)$  of the lax-limit has the right lifting property against morphisms of  $I_k$  if and only if  $(X_k \rightarrow X_{k+1})$  has the right lifting property against  $(A \rightarrow B) \rightarrow (B \rightarrow I_A B)$ . This last condition is, by adjunction, equivalent to asking that  $f_k: X_k \rightarrow \tau_k X_{k+1}$  has the right lifting property against  $(A \rightarrow B) \rightarrow (B \rightarrow I_A B)$ , which is, accorded to the criterium, equivalent to ask that  $f_k: X_k \rightarrow \tau_k X_{k+1}$  is a weak equivalence. And conversely, if  $f_k: X_k \rightarrow \tau_k X_{k+1}$  is an equivalence, then it has the right lifting property against  $(A \rightarrow B) \rightarrow (B \rightarrow I_A B)$  for any cofibration  $A \rightarrow B$  and any relative cylinder object.

We then defined the limit model structure as the left Bousfield localization of the lax-limit model structure by all set  $I_k$  (for all values of  $k$ ). The existence of this localization is asserted by theorem 7.3 of [15].

It remains to show that the fibrant object of this localization are the fibrant object satisfying this condition that  $f_k: X_k \rightarrow \tau_k X_{k+1}$  is a weak equivalence. As discussed in Remark 4.12, the fibrant object of this localization are the objects that are fibrant in the lax-limit model structure on which have the left lifting property against all maps in  $I_k$  and all their iterated homotopy codiagonal, but by Construction 4.13, all these iterated homotopy codiagonals are of the form  $U(A \rightarrow B) \rightarrow U(B \rightarrow I_A B)$  and hence, we the discussion above show that their fibrant objects are exactly the objects such that the map  $f_k: X_k \rightarrow \tau_k X_{k+1}$  is an equivalence as claimed in the proposition.  $\square$

**4.14 Proposition.** *The adjunction of Definition 4.8 is a Quillen adjunction between the limit model structure of Proposition 4.10 and the inductive model structure.*

*Proof.* We need to show that the adjunction of Proposition 4.14 passes to the localization, which means that all morphisms of  $I$  are sent to trivial cofibrations in  $\infty\text{-Cat}^\infty$ . This is immediate because

$$U(A \rightarrow B) \rightarrow U(B \rightarrow I_A B) \xrightarrow{c} B \rightarrow I_A B.$$

$\square$

**4.15 Theorem.** *The Quillen adjunction between the limit model structure of Proposition 4.10 and the inductive model structure is a Quillen equivalence. This induces an equivalence of categories:*

$$ho \lim_{n \in \mathbb{N}} \infty\text{-Cat}^n \cong ho \infty\text{-Cat}^\infty.$$

*Proof.* Because the left adjoint preserves all weak equivalences and fibrant objects, we have to show that for every fibrant  $\infty$ -marked  $\infty$ -category  $X$ , and for every cofibrant and fibrant sequence  $X_\bullet$  we have two weak equivalences:

$$c\tau X \rightarrow X \quad \text{and} \quad X_\bullet \rightarrow \tau c X_\bullet.$$

The first one is immediate because

$$X \cong \operatorname{Colim}_{n \in \mathbb{N}} \tau_n X.$$

Let  $X_\bullet$  be a cofibrant and fibrant object of the limit model structure. Because  $X_\bullet$  and  $\tau c X_\bullet$  are fibrant, the second comparison morphism is a weak equivalence, if and only if for all of  $k$ ,  $X_k \rightarrow \operatorname{Colim}_{n \in \mathbb{N}} \tau_k(X_n)$  is a weak equivalence. In order to show this, consider the following diagram:

$$\begin{array}{ccccccccccc} X_0 & \longrightarrow & \dots & \longrightarrow & X_k & \xrightarrow{\sim} & X_k & \xrightarrow{\sim} & \dots & \xrightarrow{\sim} & X_k & \xrightarrow{\sim} & \dots \\ \downarrow \sim & & & & \downarrow \sim & & \downarrow \sim & & & & \downarrow \sim & & \\ X_0 & \longrightarrow & \dots & \longrightarrow & X_k & \xrightarrow{\sim} & \tau_k(X_{k+1}) & \xrightarrow{\sim} & \dots & \xrightarrow{\sim} & \tau_k(X_n) & \xrightarrow{\sim} & \dots \end{array}$$

where all the vertical morphisms are weak equivalence. Because the left adjoint  $c$  preserves weak equivalence, this induces a weak equivalence:

$$X_k \xrightarrow{\sim} \operatorname{Colim}_{n \in \mathbb{N}} \tau_k(X_n) \cong \operatorname{Colim}_{n \in \mathbb{N}} \tau_k(X_n).$$

□

## 4.2 Comparison with the folk model structure on $\infty$ -Cat

Following [19, Definition 6], we can also give a coinductive definition of invertibility of arrows in an  $\infty$ -category. The notion is called “weakly invertible” in [19]. Explicitly, we define by coinduction:

**4.16 Definition.** We say that an  $n$ -arrow  $f: a \rightarrow b$  in a (marked)  $\infty$ -category is *coinductively invertible* if there exist  $g: b \rightarrow a$  and two coinductively invertible  $(n+1)$ -arrows  $\alpha: f \#_{n-1} g \rightarrow 1_a$  and  $\beta: g \#_{n-1} f \rightarrow 1_b$ .

Of course, this implies that there are also  $(n+1)$ -arrows  $\alpha': 1_a \rightarrow f \#_{n-1} g$  and  $\beta': 1_b \rightarrow g \#_{n-1} f$ , and then  $(n+2)$ -arrows  $\alpha \#_n \alpha' \rightarrow 1_{1_a}$ ,  $1_{f \#_{n-1} g} \rightarrow \alpha' \#_n \alpha$ , ... then followed by several  $(n+3)$ -arrows and so on up to infinity.

**4.17 Lemma.** *Let  $X$  be a  $\infty$ -category, and  $M$  the set of coinductively invertible arrows. The set  $M$  satisfies the two following properties:*

- (1)  $M = \overline{M}$ .
- (2) For all  $c: a \rightarrow b$  in  $M$ ,  $a \in M \Leftrightarrow b \in M$ .

*Proof.* The first point is the third and the fourth point of example 1.1.9 of [20], and the second one is a consequence of proposition 1.1.10 of *loc cit*. □

**4.18 Proposition.** *If  $X$  is a fibrant  $m$ -marked  $\infty$ -category, all marked arrows in  $X$  are coinductively invertible in the underlying  $\infty$ -category.*

*Proof.* This is a direct consequence of Lemma 3.17. □

**4.19 Proposition.** *Let  $X$  be an  $\infty$ -category and  $M$  the set of coinductively invertible arrows. The marked  $\infty$ -category  $(X, M)$  is then fibrant in the inductive model structure.*

*Proof.* We show that  $(X, M)$  verifies the conditions of Proposition 3.19. By definition, marked arrows of  $(X, M)$  have inverses in the sense of definition 3.11, and the first condition is fulfilled. The second condition is implied by Lemma 4.17.  $\square$

**4.20 Definition.** Let  $G_1$  be the  $\infty$ -category obtained in the factorization of  $\mathbb{D}_1 \rightarrow \mathbb{D}_0$  in a cofibration  $k_1: \mathbb{D}_1 \rightarrow G_1$  followed by a trivial fibration  $t_1: G_1 \rightarrow \mathbb{D}_1$  of the folk model structure. We then define  $G_n := \Sigma^{n-1}G_1$  and  $k_n := \Sigma^{n-1}k_1: \mathbb{D}_n \rightarrow G_n$ ,  $t_n := \Sigma^{n-1}t_1: G_n \rightarrow \mathbb{D}_{n-1}$ . Let us recall that the definition of the functor  $\Sigma^{n-1}$  is given in Definition 2.4. As the suspension preserves trivial fibrations and cofibrations, the pair  $(k_n, t_n)$  is a factorization of  $\mathbb{D}_n \rightarrow \mathbb{D}_{n-1}$  into a cofibration followed by a trivial fibration.

**4.21 Proposition.** *Let  $X$  be a  $\infty$ -category, and  $f$  an  $n$ -arrow of  $X$ . There exist a lifting in the following diagram :*

$$\begin{array}{ccc} \mathbb{D}_n & \xrightarrow{f} & X \\ \downarrow & \nearrow \text{dotted} & \uparrow \\ G_n & & \end{array}$$

*if and only if  $f$  is weakly invertible.*

*Proof.* This is a reformulation of lemma 18 of [19].  $\square$

**4.22 Definition.** The *coinductive model structure* is the left Bousfield localization of the model structure on  $\infty\text{-Cat}^\infty$  by the set of morphisms:

$$\{(G_n, \bar{\emptyset}) \rightarrow \mathbb{D}_{n-1}, n \in \mathbb{N}^*\}$$

**4.23 Remark.** Remark that if we define  $\tilde{G}_n := \pi_{n-1}(G_n, \bar{\emptyset})$ , the sequence

$$(G_n, \bar{\emptyset}) \xrightarrow{p_n} \tilde{G}_n \xrightarrow{\tilde{k}_n} \mathbb{D}_{n-1}$$

is a factorization in a cofibration followed by a trivial fibration in the inductive model structure. Using the terminology of [15], we will say that the cofibration  $p_n$  represents the morphism  $(G_n, \bar{\emptyset}) \rightarrow \mathbb{D}_{n-1}$ . As we can see in the construction of the left Bousfield localization provides in the proof of the theorem 7.3 of *op cit*, a marked  $\infty$ -category  $X$  is fibrant in the coinductive model structure if and only if  $X$  is fibrant in the inductive model structure and has the right lifting property against morphisms  $k_n$  and iterated homotopy codiagonal of  $k_n$  for all  $n > 0$ .

**4.24 Proposition.** *Let  $X$  be a fibrant  $\infty$ -marked  $\infty$ -category in the inductive model structure. Then  $X$  is fibrant in the coinductive model structure if and only if marked arrows are exactly the coinductively invertible arrows of the underlying  $\infty$ -category.*

*Proof.* Suppose first that  $X$  is fibrant in the coinductive model structure and let  $f$  be a coinductively invertible arrow of the underlying  $\infty$ -category. By proposition 4.21, this corresponds to a morphism  $f: (G_n, \bar{\emptyset}) \rightarrow X$ . As remarked in 4.23,  $X$  as the right lifting property against  $k_n$ , which implies that  $f$  can be lifted by  $\pi_{n-1}G_n$ . That shows that  $f$  is marked. Moreover, the proposition

4.18 states that all marked arrows are coinductively invertible. This shows that marked arrows exactly correspond to coinductively invertible ones.

For the other direction, suppose that  $X$  is a marked  $\infty$ -category, fibrant in the inductive model structure, whose marked arrows are the coinductively invertible ones. We want to show that  $X$  is fibrant in the coinductive model structure. Accorded to Proposition 4.19,  $X$  is fibrant in the nonlocalized model structure. We then have to show for all integers  $n > 0$ ,  $X$  has the left lifting property against  $k_n$  and iterated homotopy codiagonal of  $k_n$ . Remarks now that, as  $\tilde{G}_n \coprod_{(G_n, \bar{0})} \tilde{G}_n = \tilde{G}_n$ , all the iterated homotopy codiagonals are identities. To conclude, it is enough to show that  $X$  has the left lifting property against morphisms  $k_n$  for  $n > 0$ , which is obvious by assumption.  $\square$

**4.25 Theorem.** *The subcategory of fibrant objects of the coinductive model structure on  $\infty\text{-Cat}^\infty$  is isomorphic to  $\infty\text{-Cat}$ . Moreover, a morphism between fibrant  $\infty$ -marked  $\infty$ -categories is a weak equivalence if and only if the corresponding morphism in  $\infty\text{-Cat}$  is a weak equivalence of the folk model structure.*

We then have

$$ho^{coind}(\infty\text{-Cat}^\infty) \cong ho^{folk}(\infty\text{-Cat}).$$

*Proof.* Let  $Fib(\infty\text{-Cat}^\infty)$  be the subcategory of fibrant objects of the coinductive model structure on  $\infty\text{-Cat}^\infty$ . We define  $\phi: Fib(\infty\text{-Cat}^\infty) \rightarrow \infty\text{-Cat}$  to be the functor that forgets the marking. Proposition 4.24 implies that this functor is an equivalence of category.

Eventually, a morphism  $f: X \rightarrow Y$  between fibrant  $\infty$ -marked  $\infty$ -categories is a weak equivalence if and only if, every diagram in the category of arrows of shape:

$$\begin{array}{ccc} (\partial\mathbb{D}_n \rightarrow \mathbb{D}_n) & \longrightarrow & (X \xrightarrow{f} Y) \\ \downarrow & \nearrow \text{dotted} & \\ (\mathbb{D}_n \rightarrow \tilde{G}_{n+1}) & & \end{array}$$

admit a lifting. This is equivalent to asking that every diagram in the category of arrows of  $\infty\text{-Cat}$  of shape

$$\begin{array}{ccc} (\partial\mathbb{D}_n \rightarrow \mathbb{D}_n) & \longrightarrow & (X \xrightarrow{\phi(f)} Y) \\ \downarrow & \nearrow \text{dotted} & \\ (\mathbb{D}_n \rightarrow G_{n+1}) & & \end{array}$$

admit a lifting, which is equivalent to ask that  $\phi(f)$  is a weak equivalence.  $\square$

Note that if  $m < \infty$ , then every  $m$ -marked  $\infty$ -category which is fibrant for the saturated inductive model structure is also fibrant for the coinductive model structure, hence when restricting the previous theorem to  $m$ -marked objects for  $m < \infty$ , we no longer need to move to the coinductive model structure and we directly obtain the following:



Informally, taking a pushout along  $\mathbb{D}_{n+1} \rightarrow E_n$  means freely adding a left and a right inverse to an arrow  $f$  (except there is no marking yet) and so  $P_{n+1}$  is constructed by freely adding left and right inverses to all  $(n+1)$ -arrows of  $P_n$ .

When writing  $\mathbb{D}_1 \rightarrow P_n$ , we will always consider the morphisms representing the 1-arrow  $P_0 \rightarrow P_n$ . Finally, for  $n \in \mathbb{N} \cup \{\infty\}$  we define  $C_n$  and  $D_n$  as the following pushouts:

$$\begin{array}{ccccc} \coprod_{k < n} \mathbb{D}_1 & \longrightarrow & \mathbb{D}_1 & \longrightarrow & \mathbb{D}_0 \\ \downarrow & & \downarrow & & \downarrow \\ \coprod_{k < n} P_k & \longrightarrow & C_n & \longrightarrow & D_n \end{array}$$

The morphism  $C_\infty \rightarrow D_\infty$  will be the map  $f$  of Proposition 4.27. The informal idea is that in  $C_\infty$  the 1-arrow corresponding to the vertical map  $\mathbb{D}_1 \rightarrow C_\infty$  has ‘‘coinductive inverse up to height  $n$ ’’ for all  $n$ , but is not coinductively invertible. So when  $C_\infty$  is seen as an object of the folk (or coinductive) model structure this 1-arrow is not invertible, but as soon as we localize to make all the  $n$ -arrows, for large enough, invertible, then this 1-arrow will become invertible. In contrast in  $D_\infty$  this arrow becomes an identity, so it is invertible from the start. In the rest of the section, we will justify this rigorously.

We begin by showing the first point of Proposition 4.27, namely that  $C_\infty \rightarrow D_\infty$  is not a weak equivalence of the coinductive model structure.

**4.30 Lemma.** *Let  $P$  be a polygraph and  $f$  a coinductively invertible  $k$ -arrow in  $P$ . For every  $k$ -generator  $g$  appearing in the decomposition of  $f$ , there exists a sequence of generating arrows  $(g_n)_{n \in \mathbb{N}}$  such that*

- (1) for  $n > 0$ ,  $g_n$  is a  $(n+k)$ -generator and  $g_0 = g$ ,
- (2) for  $n > 0$ ,  $g_n$  appears in the decomposition of the source of  $g_{n+1}$ .

*Proof.* We show this result by coinduction on  $k$ . Suppose the result is true for all  $(k+1)$ -arrows, and let  $f: a \rightarrow b$  be a coinductively invertible  $k$ -arrow, and  $g$  a  $k$ -generator appearing in the decomposition of  $f$ . There exists a  $k$ -arrow  $f': b \rightarrow a$  and a coinductively invertible  $(n+1)$ -arrow  $\alpha: f \#_{k-1} f' \rightarrow \mathbb{I}_a$ . As  $g$  is a  $k$ -generator appearing in the decomposition  $f \#_{k-1} f'$  (which is the source of  $\alpha$ ), we can find a  $(k+1)$ -generator  $\beta$  appearing in the decomposition of  $\alpha$  and such that  $g$  is in the decomposition of the source of  $\beta$ . As  $\alpha$  is coinductively invertible, one can continue this process coinductively starting from  $\beta$  to build a sequence of generators  $(\beta_n)_{n \in \mathbb{N}}$  satisfying the desired property. We then set  $g_0 := g$ , and  $g_n := \beta_{n-1}$ . This sequence also satisfies the desired property.  $\square$

**4.31 Corollary.** *The  $\infty$ -categories  $C_\infty$  and  $D_\infty$  have no coinductively invertible arrow except identities.*

*Proof.* We will show this assertion for  $C_\infty$ , the proof for  $D_\infty$  is essentially the same. We proceed by contradiction: let  $f$  be a non-identity coinductively invertible  $k$ -arrow of  $C_\infty$ . As  $f$  is not an identity there should be at least one  $k$ -generator  $g$  appearing in its decomposition. As  $C_\infty$  is a polygraph one can apply the previous lemma and obtain a sequence  $(g_m)_{m \in \mathbb{N}}$  of generators of  $C_\infty$ .

Eventually shifting the sequence one can freely assume that  $g_0$  is of dimension  $> 1$ . The generators of  $C_\infty$  are obtained by gluing the generators of  $P_n$  for all  $n$  at the unique generator of  $\mathbb{D}_1$ , so this  $g_0$  will have to be in one of the  $P_n$ , it then follows by induction that all the  $g_m$  are in the same  $P_n$ , but this leads to a contradiction as the dimension of the generator of  $P_n$  is bounded above.  $\square$

**4.32 Corollary.** *The marked  $\infty$ -categories  $C_\infty^b$  and  $D_\infty^b$  are fibrant in the coinductive model structure.*

*Proof.* It is immediate that  $C_\infty^b$  and  $D_\infty^b$  are prefibrant (as in Definition 3.12) and hence they are fibrant in the inductive model structure. Hence by Proposition 4.24 we only need to check that all their coinductively invertible arrows are marked, but by the previous corollary, only their identity arrows are coinductively invertible, which concludes the proof.  $\square$

**4.33 Lemma.** *The morphism  $C_\infty \rightarrow D_\infty$  is not a weak equivalence of the coinductive model structure.*

*Proof.* As both  $C_\infty$  and  $D_\infty$  are fibrant in the coinductive model structure, which is a Bousfield localization of the inductive model structure, this map is a coinductive equivalence if and only if it is an inductive equivalence. Hence one can test whether it is an equivalence using Definition 3.25 and Proposition 3.26, but this map fails to satisfy condition (1) of Definition 3.25, as the 1-arrow of  $C_\infty$  corresponding to the vertical map  $\mathbb{D}_1 \rightarrow C_\infty$  is not marked and send to an identity arrow (hence marked) in  $D_\infty$ .  $\square$

Let us now show the second point, namely that for any integer  $n$ ,  $\pi_n C_\infty \rightarrow \pi_n D_\infty$  is a weak equivalence.

**4.34 Lemma.** *For all  $n > 0$ , the map  $\pi_{n+1} E_{n+1} \rightarrow \pi_n E_{n+1}$  is a weak equivalence in saturated inductive model structure for  $n$ -marked  $\infty$ -category (Theorem 3.31).*

*Proof.* One should first note that this map is an isomorphism of the underlying  $\infty$ -categories and only corresponds to marking all the  $n$ -arrows. In particular, it is a cofibration. Moreover,  $\pi_{n+1} E_{n+1}$  is cofibrant as its underlying  $\infty$ -category is a polygraph. Using the characterization of fibrant objects in the saturated inductive model structure (see Lemma 3.30 and Theorem 3.31), one easily sees that fibrant objects have the left lifting property against  $\pi_{n+1} E_{n+1} \rightarrow \pi_n E_{n+1}$ . As lifts against these morphisms are unique if they exist, we deduce that any fibration between fibrant objects has the left lifting property against them. It follows that this map is an acyclic cofibration and hence a weak equivalence.  $\square$

**4.35 Lemma.** *For all  $n$ ,  $\pi_n P_n \rightarrow \mathbb{D}_0$  is a weak equivalence of the saturated inductive model structure.*

*Proof.* We define  $\tilde{P}_{n+1}$  as the pushouts:

$$\begin{array}{ccc} \coprod_{(P_n)_{n+1}} \mathbb{D}_{n+1} & \longrightarrow & P_n \\ \downarrow & & \downarrow \\ \coprod_{(P_n)_{n+1}} \pi_n E_{n+1} & \longrightarrow & \tilde{P}_{n+1} \end{array}$$

As weak equivalences between cofibrant objects are stable by pushouts, the previous lemma and the fact that  $(\mathbb{D}_{n+1}, \{e_{n+1}\}) \rightarrow \pi_n E_{n+1}$  is an acyclic cofibration, imply that all arrows labeled by  $\sim$  in the following diagrams are weak equivalences:

$$\begin{array}{ccc}
\coprod_{(P_n)_{n+1}} \mathbb{D}_{n+1} & \longrightarrow & P_n \\
\downarrow & \lrcorner & \downarrow \\
\coprod_{(P_n)_{n+1}} \pi_{n+1} E_{n+1} & \longrightarrow & \pi_{n+1} P_{n+1} \\
\sim \downarrow & & \downarrow \sim \\
\coprod_{(P_n)_{n+1}} \pi_n E_{n+1} & \longrightarrow & \tilde{P}_n
\end{array}
\qquad
\begin{array}{ccc}
\coprod_{(P_n)_{n+1}} \mathbb{D}_{n+1} & \longrightarrow & P_{n+1} \\
\downarrow & \lrcorner & \downarrow \\
\coprod_{(P_n)_{n+1}} (\mathbb{D}_{n+1}, \overline{\{e_n\}}) & \longrightarrow & \pi_n P_n \\
\sim \downarrow & & \downarrow \sim \\
\coprod_{(P_n)_{n+1}} \pi_n E_{n+1} & \longrightarrow & \tilde{P}_n
\end{array}$$

By two out of three,  $\pi_{n+1} P_{n+1} \rightarrow \mathbb{D}_0$  is a weak equivalence if and only if  $\tilde{P}_{n+1} \rightarrow \mathbb{D}_0$  is, and so if and only if  $\pi_n P_n \rightarrow \mathbb{D}_0$  is. It remains to show the case  $n = 0$  which is obvious.  $\square$

**4.36 Lemma.** *For all  $n$ , the induced morphism  $\pi_n C_\infty \rightarrow \pi_n D_\infty$  is a weak equivalence of the saturated inductive model structure on  $n$ -marked  $\infty$ -categories.*

*Proof.* Using the last lemma and as weak equivalences between cofibrant objects are stable by pushout, we have a diagram where all arrows labeled by  $\sim$  are weak equivalences

$$\begin{array}{ccccc}
\coprod_{k \in \mathbb{N}} \pi_n \mathbb{D}_1 & \longrightarrow & \coprod_{k \in \mathbb{N}} \pi_n P_k & \xrightarrow{\sim} & (\coprod_{k < n} P_k) \amalg (\coprod_{k \geq n} \mathbb{D}_0) \\
\downarrow & \lrcorner & \downarrow & & \downarrow \\
\pi_n \mathbb{D}_1 & \longrightarrow & \pi_n C_\infty & \xrightarrow{\sim} & D_n \\
\downarrow & \lrcorner & \downarrow & & \parallel \\
\mathbb{D}_0 & \longrightarrow & \pi_n D_\infty & \xrightarrow{\sim} & D_n
\end{array}$$

By two out of three, this shows the result.  $\square$

*Proof of Proposition 4.27.* We choose  $f$  to be the morphism  $C_\infty^b \rightarrow D_\infty^b$ . The first point is Lemma 4.33 and the second is Lemma 4.36.  $\square$

## 4.4 Complicial sets and stratified Street nerve

In this section we show that the Street nerve can be made into a right Quillen functor from the saturated inductive model structure on  $\infty\text{-Cat}^\infty$  to the Ozornova-Rovelli-Verity model structure for complicial sets. We refer to [30] and [26] for a detailed introduction to complicial sets, we will simply recall the important definition below.

**4.37 Definition.** An  $m$ -stratified simplicial set is a simplicial set  $X$ , together with a set  $M \subset \coprod_{k > 0} X_n$  of simplex of positive dimension called *thin* simplexes that includes all degenerate simplexes.

A morphism of  $m$ -stratified simplicial sets is a morphism between the underlying simplicial sets that sends thin simplexes to thin simplexes. The category of  $m$ -stratified simplicial sets is denoted **Strat**.

The *join* is an important operation for simplicial sets, which is defined on representable by the formula

$$\Delta[n] \star \Delta[m] := \Delta[n + m + 1]$$

and extended by colimits to any pair of simplicial set

$$X \star Y := \operatorname{Colim}_{([n],[m]) \in \Delta \times \Delta} \coprod_{X_n \times Y_m} \Delta[n] \star \Delta[m].$$

See for example [22] Definition 1.2.8.1 and below. We now defined it for stratified simplicial sets as follows:

**4.38 Definition.** If  $(X, M)$  and  $(Y, N)$  are two stratified simplicial sets, we define  $M \star N$  as the set of simplices of  $X \star Y$  of the form  $x \star y$  where either  $x$  or  $y$  is thin. We then define

$$(X, M) \star (Y, N) := (X \star Y, M \star N).$$

**4.39 Definition.** We define several marking on  $\Delta[n]$ :

- (1)  $\Delta[n]_t$ . The top  $n$ -simplex is thin.
- (2)  $\Delta^k[n]$ . All simplices that include  $\{k-1, k, k+1\} \cap [n]$  are thin.
- (3)  $(\Delta^k[n])'$ . All simplices that include  $\{k-1, k, k+1\} \cap [n]$ , together with the  $(k-1)$ -face and the  $(k+1)$  face are thin.
- (4)  $(\Delta^k[n])''$ . All simplices that include  $\{k-1, k, k+1\} \cap [n]$ , together with the  $(k-1)$ -face, the  $k$ -face and the  $(k+1)$  face are thin.
- (5)  $\Delta[3]^{eq}$ . All simplices of dimension strictly higher than 2, together with  $[0, 2]$  and  $[1, 3]$  are thin.
- (6)  $\Delta[n]^\sharp$ . All simplices are thin.

**4.40 Definition** ([24, Definition 1.19]). An *elementary anodyne extension* is one of the following:

- (1) The *compcial horn inclusions* are the regular extensions

$$\Lambda^k[n] \rightarrow \Delta^k[n], \quad n \geq 1, \quad n \geq k \geq 0.$$

- (2) The *compcial thinness extensions*:

$$(\Delta^k[n])' \rightarrow (\Delta^k[n])'', \quad n \geq 2, \quad n \geq k \geq 0.$$

- (3) The *saturation extensions*:

$$\Delta[n] \star \Delta[3]^{eq} \rightarrow \Delta[n] \star \Delta[3]^\sharp, \quad n \geq -1.$$

- (4) The *m-triviality extensions*:

$$\Delta[n] \rightarrow \Delta[n]_t, \quad n > m$$

**4.41 Remark.** In the case where  $m = \infty$ , there is no  $m$ -triviality extension.

**4.42 Definition.** A (saturated)  $m$ -complicial set is a marked simplicial set having the right lifting property against all elementary anodyne extensions.

As demonstrated in [21],  $m$ -complicial sets are a model for  $(\infty, m)$ -categories. For example, 0-complicial sets and 1-complicial sets are essentially the same as Kan complexes and quasicategories respectively. The word *saturated* refers to the fact that (as in [24]) we have included the “saturation extension” as part of our elementary anodyne extensions. These are not always included and play a role similar to the saturated localization of the inductive model structure considered in Section 3.5. See also [26] for a more general discussion of saturation for complicial sets.

**4.43 Theorem** (Verity [30], Riehl [26], Ozornova-Rovelli [24]). *There is a model structure on  $\mathbf{Strat}$  where cofibrations are all monomorphisms, and acyclic cofibrations are generated by elementary anodyne extension. Fibrant objects of this structure are the (saturated)  $m$ -complicial sets. We denote  $\mathbf{Strat}^m$  the category  $\mathbf{Strat}$  endowed with this model structure.*

We will use the join to define the adjunction between stratified simplicial sets and marked  $\infty$ -categories.

**4.44 Definition.** Let  $C$  and  $D$  be two marked  $\infty$ -categories. The *joint* of  $C$  and  $D$ , noted  $C \star D$ , is the colimit of the following diagram:

$$\begin{array}{ccc} A \odot \{0\} \odot B \amalg A \odot \{1\} \odot B & \longrightarrow & A \odot \mathbb{D}_1 \odot B \\ \downarrow & & \downarrow \\ A \amalg B & \longrightarrow & A \star B \end{array}$$

As noted in proposition 3.3.11 of [2] at the level of  $\infty$ -categories, this is the usual join of  $\omega$ -categories, as defined in paragraph 6.30 of [4]. This operation is then associative.

**4.45 Proposition.** *Let  $X \rightarrow Y$  be a cofibration and  $K \rightarrow L$  an acyclic cofibration. Morphisms*

$$K \star Y \amalg_{X \star K} L \star X \rightarrow L \star Y \quad \text{and} \quad Y \star K \amalg_{K \star X} X \star L \rightarrow Y \star L$$

*are acyclic cofibrations.*

*Proof.* Consider the following diagram:

$$\begin{array}{ccccc} L \amalg Y & \longleftarrow & K \odot \partial \mathbb{D}_1 \odot Y \cup L \odot \partial \mathbb{D}_1 \odot X & \longrightarrow & K \odot \mathbb{D}_1 \odot Y \cup L \odot \mathbb{D}_1 \odot X \\ \downarrow & & \downarrow & & \downarrow \\ L \amalg Y & \longleftarrow & L \odot \partial \mathbb{D}_1 \odot Y & \longrightarrow & L \odot \mathbb{D}_1 \odot Y \end{array}$$

Taking colimit of the lines, this induces a comparison morphism:

$$K \star Y \amalg_{X \star K} L \star X \rightarrow L \star Y.$$

Proposition 7.5 of [3] implies that this morphism is a cofibration. Lemma 2.37 implies that vertical morphisms of the previous diagram are weak equivalence. Furthermore, these colimits are homotopy colimits, the comparison morphism is then a weak equivalence, and then an acyclic cofibration. We proceed analogously for the second morphism.  $\square$

**4.46 Definition.** The terminal category  $1$  has a monoid structure for this operation. The multiplication  $1 \star 1 \rightarrow 1$  is the unique morphism to the terminal  $\infty$ -category.

By the universal property of the category  $\Delta$ , this induces a cosimplicial object  $|-|: \Delta \rightarrow \infty\text{-Cat}^\infty$  where

$$|\Delta[n]| := 1 \star 1 \star \dots \star 1.$$

The  $\omega$ -category  $|\Delta[n]|$  is traditionally called the  $n^{\text{th}}$  oriental. We denote  $|-|: \mathbf{Sset} \rightarrow \infty\text{-Cat}^\infty$  the extension by colimits of this cosimplicial object. For all  $n$ ,  $|\Delta[n]|$  is an  $n$ -polygraph that admits only one  $n$ -generator. If  $M$  is a marking for  $K$ , we denote  $|M|$  the set of arrows obtained as composition:

$$\mathbb{D}_n \rightarrow \Delta[n] \xrightarrow{|v|} K$$

where the left morphism corresponds to the top cell of the  $n^{\text{th}}$  orientals, and the right morphism is in  $M$ . We can now extend the realization to stratified simplicial sets:

$$\begin{aligned} |-|: \mathbf{Strat} &\rightarrow \infty\text{-Cat}^m \\ (K, M) &\mapsto (|K|, \overline{|M|}) \end{aligned}$$

This functor is cocontinuous, and induces an adjunction:

$$\begin{array}{ccc} \mathbf{Strat} & \begin{array}{c} \xrightarrow{|-|} \\ \perp \\ \xleftarrow{N} \end{array} & \infty\text{-Cat}^m \end{array}$$

The right adjoint is called the *stratified Street nerve*. By construction, if  $K$  and  $L$  are two stratified simplicial sets, we have  $|K \star L| = |K| \star |L|$ .

**4.47 Remark.** In the case  $m = \infty$ , this adjunction model the forgetful functor from strict  $\infty$ -categories to weak  $\infty$ -categories (given by the stratified Street nerve  $N$ ). The left adjoint corresponds to the “strictification functor” that sends a weak  $\infty$ -category to a strict  $\infty$ -category in a universal way.

**4.48 Proposition.** *The stratified nerve preserves fibrant objects.*

*Proof.* Suppose first that  $m < \infty$  and let  $(X, M)$  be a fibrant  $m$ -marked  $\infty$ -category for the saturated inductive model structure. According to Corollary 4.26,  $M$  consist of coinductively invertible arrows of  $X$ , and  $N((X, M))$  is equal to the stratified simplicial set associated to the Street nerve of  $X$  defines in [20, Définition 5.2.1]. Theorem 5.2.12 of *op.cit.* then imply that the stratified Street nerve sends fibrant objects of the saturated inductive model structure on  $\infty\text{-Cat}^m$  to an  $m$ -complicial sets.

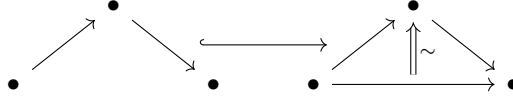
Now, let  $C$  be a fibrant  $\infty$ -marked  $\infty$ -category for the saturated inductive model structure. As the stratified nerve preserves directed colimits, there is an isomorphism

$$N(C) \cong \operatorname{Colim}_{n \in \mathbb{N}} N(\tau_n C)$$

For all  $n$ ,  $\tau_n C$  is fibrant for the saturated inductive model structure for  $n$ -marked  $\infty$ -categories, and  $N(\tau_n C)$  is then a fibrant of the model structure for  $n$ -complicial sets. As the model structure for  $\infty$ -complicial sets is  $\omega$ -combinatorial, fibrant objects are stable by directed colimits, and  $N(C)$  is fibrant.  $\square$

**4.49 Lemma.** *The realization functor sends complicial horn inclusion to acyclic cofibration of the saturated inductive model structure for  $m$ -marked  $\infty$ -categories.*

*Proof.* The complicial horn inclusion  $\Lambda^1[2] \rightarrow \Delta[2]^1$  corresponds to the following inclusion of marked  $\infty$ -categories:



which obviously is an equation. The two complicial horn inclusions  $\Lambda^0[2] \rightarrow \Delta^0[2]$  and  $\Lambda^2[2] \rightarrow \Delta^2[2]$  are respectively equal to  $\mathbf{eq}_{1,1}^{\circlearrowleft}$  and  $\mathbf{eq}_{1,1}^{\circlearrowright}$ . The realization functor commutes with the join. Furthermore, we can see that for all  $0 < k < n$ , we have:

$$\Delta^k[n] = \Delta[k-2] \star \Delta^1[2] \star \Delta[n-k-2]$$

and  $\Lambda^k[n]$  is the sub-object:

$$\begin{aligned} & \partial\Delta[k-2] \star \Delta^1[2] \star \Delta[n-k-2] \\ \cup & \Delta[k-2] \star \Lambda^1[2] \star \Delta[n-k-2] \\ \cup & \Delta[k-2] \star \Delta^1[2] \star \partial\Delta[n-k-2]. \end{aligned}$$

Proposition 4.45 then implies that  $\Lambda^k[n] \rightarrow \Delta^k[n]$  is an equation. We proceed analogously for the case  $k = 0$  and  $k = n$ .  $\square$

**4.50 Theorem.** *The strictification functor and the stratified Street nerve form a Quillen adjunction between the model structure for  $m$ -complicial sets and the inductive model structure on  $\infty$ -Cat <sup>$m$</sup> .*

*Proof.* Because of Lemma 4.49, it just remains to show that complicial thinness extensions, saturation extensions, and  $m$ -triviality extensions are sent to acyclic cofibrations. Let  $i$  be such a morphism. According to Proposition 4.48, any fibrant object of the saturated inductive model structure has the right lifting property against  $|i|$ . As  $|i|$  is an identity on the underlying  $\infty$ -category, lifts against it are unique if there exist. This implies that any morphism between fibrant objects has the right lifting property against  $|i|$ , and this morphism is then an acyclic cofibration. This concludes the proof.  $\square$

Finally, we can use this to generalize the results from [20]: The stratified Street nerve:

$$\mathcal{N}: \infty\text{-Cat} \rightarrow \mathbf{sSet}^m$$

introduced in [20], is exactly the stratified Street nerve  $N$  of the present paper combined with the fully faithful inclusion  $\infty\text{-Cat} \subset \infty\text{-Cat}^m$  constructed in Section 4.2, which makes all coinductively invertible arrow marked. Hence:

**4.51 Proposition.** *Let  $f: X \rightarrow Y$  a fibration (resp. a trivial fibration, resp. weak equivalence) of the canonical model structure on  $\infty\text{-Cat}$ , then its stratified Street nerve  $\mathcal{N}(f): \mathcal{N}(X) \rightarrow \mathcal{N}(Y)$  is a fibration (resp. a trivial fibration, resp. a weak equivalence) in the Verity model structure on  $\mathbf{sSet}^m$ .*

The main result of [20] corresponds to the special case of preservation of fibrant objects.

Note, that in particular the proposition shows that the stratified Street nerve from [20], while not being a right Quillen functor, is still a morphism of Brown categories of fibrant objects, and so it does defines a limit preserving functor on the corresponding associated  $\infty$ -categories.

*Proof.* As the stratified Street nerve  $N: \infty\text{-Cat}^m \rightarrow \mathbf{sSet}^m$  is a right Quillen functor, it preserves fibrations and trivial fibrations, as well as weak equivalences between fibrant objects. Moreover, we have shown in Section 4.2 that the functor sending a strict  $\infty$ -category to the marked one where the marked arrows are the coinductively invertible ones, preserves fibrations, trivial fibrations and weak equivalences.  $\square$

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