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A NOTE ON THE SPECTRAL GAP FOR LOG-CONCAVE PROBABILITY MEASURES ON CONVEX BODIES

MICHEL BONNEFONT AND ALDÉRIC JOULIN

Abstract. Motivated by some statistical issues derived from Global Sensitivity Analysis, we provide explicit lower bounds with respect to some quantities of interest (parameters of the underlying distribution, dimension, geometrical characteristics of the domain, etc.) on the spectral gap of log-concave probability measures on convex bodies. Our results are illustrated by some classical and less classical examples.

1. Introduction

Many physical phenomena and industrial systems are now investigated by costly computer codes. To analyze these models, the Global Sensitivity Analysis (GSA) is a key ingredient that allows to rank the relative importance of the inputs of the system on the output variable. From a probabilistic point of view, inputs are encoded by i.i.d. random vectors X_i valued in $\mathbb{R}^d$, the computer code is some smooth unknown function $f : (\mathbb{R}^d)^K \to \mathbb{R}$ and $Y = f(X_1, \ldots, X_K)$ stands for the square-integrable scalar output of interest. Usually, the inputs responsible for such uncertainties are determined through some relevant quantities called sensitivity indices. Among those appearing in the literature, variance-based indices (or Sobol indices [24]) are popular because of their natural interpretability: Sobol indices are equal to fractions of the output variance explained by an input or a set of inputs. However, their estimation requires numerous computations. When the gradient of f is available, other sensitivity indices called DGSM (Derivative-based Global Sensitivity Measure) can be used. DGSM indices are defined as the integral of the square of the gradient of f over the domain of the inputs. As studied in [22] in the one-dimensional case $d = 1$, the independence assumption between the inputs X_i entails that Sobol indices and DGSM are connected by a Poincaré inequality related to the distribution of each X_i. In other words, when it is satisfied, such a functional inequality provides an upper bound on the variance-based index by using the derivative-based one, cheaper to compute. As a result, this allows to identify the unessential variables. Actually, controlling the mean square error by the L^2 norm of the gradient is at the basis of the so-called active subspace method introduced by Constantine and his collaborators

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in [10] for (at least) linear dimension reduction and is by-now the matter of a growing number of papers in the field, *cf.* for instance [29].

To apply these techniques, finding the Poincaré constant (the optimal constant in the devoted Poincaré inequality) is of crucial importance. Due to numerical issues, the law of each input X_i is often chosen to be classical (standard multidimensional Gaussian, product Laplace or more generally log-concave) but truncated on a domain Ω which may be convex or not, bounded or unbounded. However in theory it is quite hard to find explicitly the Poincaré constant beyond product spaces, a situation for which the problem is reduced to the one-dimensional case (the rare known examples of explicit constants are presented for instance in [22] through the Sturm-Liouville theory). To our knowledge, the Poincaré constant is known explicitly only for the uniform distribution on Euclidean balls [28] or on some specific triangles [20]. Therefore numerical methods based on finite elements discretizations of the dual Neumann eigenvalue problem are available to approximate those constants. Indeed, the Poincaré constant can be expressed as the inverse of the first eigenvalue $\lambda_1(\Omega,\mu)$ (called the spectral gap) of the weighted Laplacian of the type $L = e^V \mathrm{div}(e^{-V}\nabla) = \Delta - \langle \nabla V, \nabla\rangle$ associated to the probability measure μ with Lebesgue density proportional to e^{-V} on the domain Ω and endowed with Neumann conditions at the boundary $\partial\Omega$. As such, this numerical approach proved its efficiency in low dimension ($d = 1$ or 2, see for instance [22] and also the PhD Thesis of Steiner [26] in which the two dimensional inputs are modeled by copulae) but usually fails in large dimension. Hence, offering theoretical guarantees such as relevant upper bounds on the Poincaré constant (or equivalently lower bounds on the spectral gap) depending explicitly on all the quantities of interest of the problem (parameters of the underlying distribution, dimension, geometrical characteristics of Ω, etc.) is expected by practitioners and the present paper intends to make a first step in this direction.

In order to give an idea of the results we are able to obtain, let us already state one of our main contribution of the paper. Below $\mathcal{L}$ stands for the diagonal matrix operator acting on smooth vector fields F as $\mathcal{L}F = (LF_i)_{i=1,\dots,d}$ and $\rho(A)$ denotes the smallest eigenvalue of a given symmetric matrix A. We refer to Section 3.1 for the other missing definitions.

Theorem 1.1. *On a (connected) compact set $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) with smooth boundary $\partial\Omega$ and outer unit-normal η, we consider a probability measure μ whose Lebesgue density is proportional to e^{-V}, where $V : \Omega \to \mathbb{R}$ is some sufficiently smooth potential on Ω. Let W be some smooth invertible diagonal matrix mapping satisfying the two following assumptions:*

- (A_1) *The symmetric matrix mapping $\nabla^2 V - \mathcal{L}W\,W^{-1}$ is bounded from below (uniformly with respect to the space variable) by some positive constant on Ω.*
- (A_2) *At the boundary $\partial\Omega$ the symmetric matrix mapping $(\mathrm{Jac}\,\eta - W\,\langle \nabla W^{-1}, \eta\rangle)_{|\eta^\perp}$ is non-negative.*

Then the weighted Poincaré inequality holds: for all $g \in \mathcal{C}^\infty(\Omega)$,

$$\mathrm{Var}_\mu(g) \leq \int_\Omega \left\langle \nabla g,\, (\nabla^2 V - \mathcal{L}W\,W^{-1})^{-1}\nabla g \right\rangle d\mu.$$

In particular the spectral gap satisfies

$$\lambda_1(\Omega,\mu) \geq \inf_{x\in\Omega} \rho\left(\nabla^2 V(x) - \mathcal{L}W(x)\, W^{-1}(x)\right). \tag{1.1}$$

Theoretically, our result covers many various different situations as soon as we are able to find some W satisfying the announced assumptions. However in practice we are able to obtain convenient spectral gap estimates mainly for log-concave probability measures μ on a convex body Ω. On the one hand it includes in particular the uniform distribution on Ω so that our results can be compared with that of Payne and Weinberger [21] involving the diameter of Ω and also to Klartag's estimate [15] for unconditional convex bodies relying on a kind of monotonicity property. On the other hand, under the additional assumption that the potential V is uniformly convex on Ω, we reinforce the usual bound on the spectral gap provided by the Brascamp-Lieb inequality, *cf.* [9].

Let us describe the content of the paper. In Section 2 we recall some basic material on Poincaré inequalities on domains. Section 3 is then devoted to our main results of the paper contained in Theorem 1.1, Corollaries 3.3 and 3.5, in which we establish some convenient lower bounds on the spectral gap $\lambda_1(\Omega,\mu)$ in various situations of interest. Section 3.1 is concerned with two key ingredients, Lemmas 3.1 and 3.2, at the basis of the proof of Theorem 1.1 given in Section 3.2, whereas in Section 3.3 we apply the result to radial log-concave distributions on convex bodies, leading to Corollaries 3.3 and 3.5. In Section 4 we further study some log-concave examples which illustrate our main results and show that, at least in the standard Gaussian setting, our approach is sufficiently robust to offer a spectral gap estimate beyond the case of a convex domain, namely on the complement of an Euclidean ball. Finally in the Appendix, we provide some elements on the spectral gap of the uniform distribution on the Euclidean ball and discuss in this context a possible optimality of Theorem 1.1.

2. Basic material and notation

In this paper, we consider on the Euclidean space $(\mathbb{R}^d, |\cdot|)$ of dimension $d \geq 2$ a (connected) compact set Ω with sufficiently smooth boundary (say $\mathcal{C}^2$) $\partial\Omega$ and outer unit-normal η. Let $\mathcal{C}^\infty(\Omega)$ be the space of infinitely differentiable real-valued functions on Ω. We introduce a probability measure μ on Ω whose Lebesgue density is proportional to e^{-V}, where $V : \Omega \to \mathbb{R}$ is some sufficiently smooth potential on Ω, and consider on $\mathcal{C}^\infty(\Omega)$ the associated second-order differential operator

$$Lf = \Delta f - \langle \nabla V, \nabla f\rangle,$$

endowed with Neumann boundary conditions, *i.e.*,

$$\langle \nabla f, \eta\rangle = 0 \quad \text{on} \quad \partial\Omega.$$

In the sequel we denote $\mathcal{C}^\infty_N(\Omega)$ (N for Neumann) such a subspace of $\mathcal{C}^\infty(\Omega)$. Above Δ and ∇ stand respectively for the Euclidean Laplacian and gradient and $\langle\cdot,\cdot\rangle$ is the scalar product. By integration by parts, we have for all $f, g \in \mathcal{C}^\infty(\Omega)$,

$$\begin{aligned}\int_\Omega Lf\, g\, d\mu &= \int_{\partial\Omega} g\, \langle \nabla f, \eta\rangle\, d\mu - \int_\Omega \langle \nabla f, \nabla g\rangle\, d\mu \\ &= \int_{\partial\Omega} \left(g\, \langle \nabla f, \eta\rangle - f\, \langle \nabla g, \eta\rangle\right)\, d\mu + \int_\Omega f\, Lg\, d\mu,\end{aligned}$$

where by abuse of notation we still denote μ the measure on the boundary with density proportional to e^{-V} with respect to the normalized volume measure on $\partial\Omega$. Hence L is symmetric and non-positive on $\mathcal{C}_N^\infty(\Omega)$ and by completeness it admits a unique self-adjoint extension (still denoted L). In particular the (Neumann) spectrum $\sigma(-L)$ of the non-negative operator $-L$ is included in $[0,\infty)$, the zero eigenvalue corresponding to the constant eigenfunctions, and the first positive eigenvalue $\lambda_1(\Omega,\mu)$ (denoted as such in order to highlight the roles of the domain Ω and the dynamics related to the probability measure μ), called the spectral gap, is nothing but the optimal constant in the famous Poincaré inequality, that is, for all $g \in \mathcal{C}^\infty(\Omega)$,

$$\lambda_1(\Omega,\mu)\,\mathrm{Var}_\mu(g) \quad \leq \quad \int_\Omega |\nabla g|^2\, d\mu,$$

where $\mathrm{Var}_\mu(g)$ is the variance of function g under μ,

$$\mathrm{Var}_\mu(g) = \int_\Omega \left(g - \int_\Omega g\, d\mu \right)^2 d\mu.$$

In other words, the spectral gap is the inverse of the Poincaré constant mentioned in the Introduction and useful in the GSA methodology. Note that the Neumann boundary conditions do not appear directly in the Poincaré inequality.

Before turning to our main results, let us introduce some notation and definitions. By a matrix mapping (resp. an invertible matrix mapping, resp. a symmetric positive-definite matrix mapping) we mean a map defined on Ω and valued in $\mathcal{M}_d(\mathbb{R})$, the space of $d \times d$ matrices with real entries (resp. in the subset of invertible matrices, resp. in the subset of symmetric positive-definite matrices). Given a smooth matrix mapping M and a smooth vector field F defined on Ω, let ∇M and ∇F be respectively the matrix of gradients $(\nabla M_{i,j})_{i,j=1,\dots,d}$ and the column vector of gradients $(\nabla F_i)_{i=1,\dots,d}$. If $v \in \mathbb{R}^d$ then we define $\langle \nabla M, v\rangle$ and $\langle \nabla F, v\rangle$ to be respectively the matrix $(\langle \nabla M_{i,j}, v\rangle)_{i,j=1,\dots,d}$ and the vector $(\langle \nabla F_i, v\rangle)_{i=1,\dots,d}$. Moreover we define the vector field $\nabla M\, \nabla F$ by contraction as

$$(\nabla M\, \nabla F)_i = \sum_{j=1}^d \langle \nabla M_{i,j}, \nabla F_j\rangle.$$

For two column vectors of gradients ∇F and ∇G and a symmetric matrix $M \in \mathcal{M}_d(\mathbb{R})$, we define

$$[\nabla F]^T\, M\, \nabla G = \sum_{i,j=1}^d \langle \nabla F_i, M_{i,j} \nabla G_j\rangle.$$

Above the superscript T stands for the transpose of a vector or a matrix. Finally we denote $\rho(A)$ the smallest eigenvalue of a given symmetric matrix A and say that A is bounded from below by some constant $\kappa \in \mathbb{R}$ if $\rho(A) \geq \kappa$. If $\kappa = 0$ we say that A is non-negative.

3. Spectral gap estimates

3.1. **Preliminaries.** We start our analysis by stating an important lemma, which is more or less classical at least on the whole Euclidean space $\mathbb{R}^d$, and which might be seen as a dualized Poincaré-type inequality. Let us give the proof for completeness.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^d$ be a (connected) compact set with smooth boundary $\partial\Omega$ and outer unit-normal η. Assume that there exists some symmetric positive-definite matrix mapping K such that for every $f \in \mathcal{C}_N^\infty(\Omega)$,*

$$\int_\Omega (-Lf)^2 \, d\mu \geq \int_\Omega \langle \nabla f, K \nabla f \rangle \, d\mu. \tag{3.1}$$

Then for every $g \in \mathcal{C}^\infty(\Omega)$, we have the weighted Poincaré inequality

$$\mathrm{Var}_\mu(g) \leq \int_\Omega \left\langle \nabla g, K^{-1} \nabla g \right\rangle \, d\mu.$$

In particular if the mapping K is bounded from below (uniformly with respect to the space variable) by some $\kappa > 0$ then the spectral gap of the operator $-L$ is lower bounded as follows:

$$\lambda_1(\Omega, \mu) \geq \kappa.$$

Proof. Letting $g \in \mathcal{C}^\infty(\Omega)$ be centered, standard results for Neumann type Laplacians imply the existence of a unique solution $f \in \mathcal{C}_N^\infty(\Omega)$ to the Poisson equation $-Lf = g$. Then the trick is to write the variance as follows:

$$\begin{aligned}
\mathrm{Var}_\mu(g) &= 2 \int_\Omega g^2 \, d\mu - \int_\Omega g^2 \, d\mu \\
&= 2 \int_\Omega g \, (-Lf) \, d\mu - \int_\Omega (-Lf)^2 \, d\mu \\
&= 2 \int_\Omega \langle \nabla g, \nabla f \rangle \, d\mu - \int_\Omega (-Lf)^2 \, d\mu \\
&\leq 2 \int_\Omega \langle K^{-\frac{1}{2}} \nabla g, K^{\frac{1}{2}} \nabla f \rangle \, d\mu - \int_\Omega \langle \nabla f, K \nabla f \rangle \, d\mu \\
&= \int_\Omega \langle \nabla g, K^{-1} \nabla g \rangle \, d\mu - \int_\Omega |K^{\frac{1}{2}} \nabla f - K^{-\frac{1}{2}} \nabla g|^2 \, d\mu \\
&\leq \int_\Omega \langle \nabla g, K^{-1} \nabla g \rangle \, d\mu.
\end{aligned}$$

The proof of the spectral gap estimate is then straightforward. □

This approach, known to specialists as the L^2 method, is reminiscent of Hörmander's work [13] in the middle of the 60's for solving the Poisson equation associated to the operator $\bar{\partial}$ in complex analysis, and has been used then by several authors to establish Poincaré-type inequalities. For instance we have in mind the famous (integrated version of the) Γ_2 curvature dimension criterion of Bakry and Emery [3] and also the work of Helffer [12] for models arising in statistical mechanics. Moreover Klartag [15] used this method to prove, among other things, the variance conjecture in the case of log-concave unconditional distributions, that is, having log-concave density which is invariant under coordinate hyperplane reflections. The presence of a weight in the inequalities above through the matrix mapping K, which is just a refinement of this approach, already appeared for instance in [4, 17] and in our previous papers [2, 7], both works aiming at estimating conveniently the spectral gap or higher eigenvalues in various situations of interest. Actually, it takes its roots in the Brascamp-Lieb inequality [9] for strictly convex potentials V, the mapping K being $\nabla^2 V$, the Hessian matrix of V. Indeed it is known that we can decompose conveniently the term $\int_\Omega (-Lf)^2 \, d\mu$ for functions

$f \in \mathcal{C}_N^\infty(\Omega)$ by integrating the famous Bochner formula adapted to the measure μ. Since we work on the domain Ω, some extra boundary terms appear, cf. for instance [15]. As such, the formula is the following (we omit the proof since it is included in the one of Lemma 3.2 below): for all $f \in \mathcal{C}_N^\infty(\Omega)$,

$$\begin{aligned}\int_\Omega (-Lf)^2\, d\mu &= \int_\Omega \|\nabla^2 f\|_{HS}^2\, d\mu + \int_\Omega \langle \nabla f, \nabla^2 V \nabla f\rangle\, d\mu \\ &\quad + \int_{\partial\Omega} \langle \nabla f, \operatorname{Jac}\eta\, \nabla f\rangle\, d\mu, \end{aligned} \tag{3.2}$$

where $\|A\|_{HS} = \sqrt{\sum_{i,j=1,\dots,d} A_{i,j}^2}$ stands for the Hilbert-Schmidt norm of a given matrix $A \in \mathcal{M}_d(\mathbb{R})$ and $\operatorname{Jac}\eta = (\partial_j \eta_i)_{i,j=1,\dots,d}$ denotes the Jacobian matrix of η. Actually, the boundary term is related to the geometry of the domain Ω in the sense that it reveals to be non-negative when Ω is convex. Indeed, in most of the cases of interest the domain Ω is of the form

$$\Omega = \{x \in \mathbb{R}^d : F(x) \le 0\},$$

where F is some smooth function defined on $\mathbb{R}^d$. Function F is convex if and only if the domain Ω is. The boundary is described by the algebraic equation $F = 0$, the outer unit-normal is given by $\eta = \nabla F/|\nabla F|$ and for all vectors u, v in the hyperplane $\eta^\perp = \{m \in \mathbb{R}^d : \langle m, \eta\rangle = 0\}$ (depending on the space variable), we have

$$\begin{aligned}\langle u, \operatorname{Jac}\eta\, v\rangle &= \frac{1}{|\nabla F|}\langle u, \nabla^2 F\, v\rangle - \frac{1}{|\nabla F|^2}\underbrace{\langle u, \nabla F\rangle}_{=0}\langle \nabla|\nabla F|, v\rangle \\ &= \frac{1}{|\nabla F|}\langle u, \nabla^2 F\, v\rangle,\end{aligned}$$

i.e., the restriction $(\operatorname{Jac}\eta)_{|\eta^\perp}$ of the Jacobian matrix of η to the hyperplane $\eta^\perp$ is symmetric and coincides with $\nabla^2 F/|\nabla F|$.

Coming back to the consequences of the decomposition (3.2), the strict convexity of V (*i.e.*, $\nabla^2 V$ is a symmetric positive-definite matrix mapping) entails by Lemma 3.1 the famous Brascamp-Lieb inequality on convex domains: for every $g \in \mathcal{C}^\infty(\Omega)$,

$$\operatorname{Var}_\mu(g) \le \int_\Omega \langle \nabla g, \nabla^2 V^{-1} \nabla g\rangle\, d\mu. \tag{3.3}$$

Finally, if moreover V is uniformly convex on Ω, that is, the mapping $\nabla^2 V$ is uniformly bounded from below by some $\kappa > 0$, then the spectral gap of the operator $-L$ on the convex domain Ω satisfies

$$\lambda_1(\Omega, \mu) \ge \kappa. \tag{3.4}$$

In order to reinforce this spectral gap estimate or even to obtain a relevant bound beyond this convex situation, our idea is to introduce some matrix weight in the decomposition (3.2), freeing us from these strong convexity assumptions. This strategy is inspired by our previous works on the intertwinings, cf. [2, 7]. See also the approaches of Wang [27] and Kolesnikov and Milman [16] in a different context. Here is our key lemma. Below $\mathcal{L}$ stands for the diagonal matrix operator acting on smooth vector fields F as $\mathcal{L}F = (LF_i)_{i=1,\dots,d}$ and the notation W is used to remind us that it is interpreted as a weight, the unweighted version (*i.e.*, W is the identity) of our second identity (3.5) corresponding to the classical decomposition (3.2).

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^d$ be a (connected) compact set with smooth boundary $\partial\Omega$ and outer unit-normal η. Let W be some smooth invertible matrix mapping. Then for all $f \in \mathcal{C}^\infty(\Omega)$, it holds*

$$\begin{aligned}\int_\Omega (-Lf)^2\,d\mu &= \int_\Omega \Big[\nabla(W^{-1}\nabla f)\Big]^T W^T W\,\nabla(W^{-1}\nabla f)\,d\mu \\ &\quad + \int_\Omega \Big\langle W^{-1}\nabla f, (\nabla W^T\,W - W^T\,\nabla W)\,\nabla(W^{-1}\nabla f)\Big\rangle\,d\mu \\ &\quad + \int_\Omega \Big\langle \nabla f, (\nabla^2 V - \mathcal{L}W\,W^{-1})\,\nabla f\Big\rangle\,d\mu + \int_{\partial\Omega} Lf\,\langle\nabla f,\eta\rangle\,d\mu \\ &\quad - \int_{\partial\Omega}\langle \nabla f, \nabla^2 f\,\eta\rangle\,d\mu - \int_{\partial\Omega}\Big\langle \nabla f, W\,\langle\nabla W^{-1},\eta\rangle\,\nabla f\Big\rangle\,d\mu.\end{aligned}$$

Moreover, if f satisfies the Neumann boundary conditions $\langle\nabla f,\eta\rangle = 0$, then

$$\begin{aligned}\int_\Omega (-Lf)^2\,d\mu &= \int_\Omega \Big[\nabla(W^{-1}\nabla f)\Big]^T W^T W\,\nabla(W^{-1}\nabla f)\,d\mu \\ &\quad + \int_\Omega \Big\langle W^{-1}\nabla f, (\nabla W^T\,W - W^T\,\nabla W)\,\nabla(W^{-1}\nabla f)\Big\rangle\,d\mu \\ &\quad + \int_\Omega \Big\langle \nabla f, (\nabla^2 V - \mathcal{L}W\,W^{-1})\,\nabla f\Big\rangle\,d\mu \\ &\quad + \int_{\partial\Omega}\Big\langle \nabla f, (\operatorname{Jac}\eta - W\,\langle\nabla W^{-1},\eta\rangle)\,\nabla f\Big\rangle\,d\mu. \qquad (3.5)\end{aligned}$$

Proof. We use the notation $A = W^{-1}$ and $S = (AA^T)^{-1}$. Recall first the intertwining between operators and (weighted) gradients introduced and studied in [2]:

$$A\nabla Lf = (\mathcal{L}_A - \mathcal{M}_A)\,(A\nabla f),$$

where $\mathcal{L}_A$ denotes the matrix operator acting on smooth vector fields as

$$\mathcal{L}_A F = \mathcal{L}F + 2\,A\,\nabla A^{-1}\,\nabla F,$$

and $\mathcal{M}_A$ is the matrix corresponding to the multiplicative (or zero-order) operator

$$\mathcal{M}_A = A\,\nabla^2 V\,A^{-1} - A\,\mathcal{L}A^{-1}.$$

We have by integration by parts and the intertwining identity above,

$$\begin{aligned}\int_\Omega (-Lf)^2\,d\mu &= \int_\Omega \langle\nabla f, \nabla(-Lf)\rangle\,d\mu + \int_{\partial\Omega} Lf\,\langle\nabla f,\eta\rangle\,d\mu \\ &= \int_\Omega \langle A\nabla f, SA\nabla(-Lf)\rangle\,d\mu + \int_{\partial\Omega} Lf\,\langle\nabla f,\eta\rangle\,d\mu \\ &= \int_\Omega \langle A\nabla f, S(-\mathcal{L}_A + \mathcal{M}_A)(A\nabla f)\rangle\,d\mu + \int_{\partial\Omega} Lf\,\langle\nabla f,\eta\rangle\,d\mu \\ &= \int_\Omega \langle A\nabla f, S(-\mathcal{L})(A\nabla f)\rangle\,d\mu - \int_\Omega \langle A\nabla f, 2\,SA\,\nabla A^{-1}\,\nabla(A\nabla f)\rangle\,d\mu \\ &\quad + \int_\Omega \Big\langle \nabla f, (\nabla^2 V - \mathcal{L}A^{-1}\,A)\,\nabla f\Big\rangle\,d\mu + \int_{\partial\Omega} Lf\,\langle\nabla f,\eta\rangle\,d\mu,\end{aligned}$$

since

$$A^T\,S\,\mathcal{M}_A\,A = \nabla^2 V - \mathcal{L}A^{-1}\,A.$$

Dealing with the first term in the right-hand-side above, a second integration by parts gives

$$\begin{aligned}\int_\Omega \langle A\nabla f, S(-\mathcal{L})(A\nabla f)\rangle\, d\mu &= \int_\Omega [\nabla(A\nabla f)]^T\, S\, \nabla(A\nabla f)\, d\mu + \int_\Omega \langle A\nabla f, \nabla S\, \nabla(A\nabla f)\rangle\, d\mu \\ &\quad - \int_{\partial\Omega} \langle SA\nabla f, \langle \nabla(A\nabla f), \eta\rangle\rangle\, d\mu,\end{aligned}$$

so that reorganizing the terms in the initial computations lead to

$$\begin{aligned}\int_\Omega (-Lf)^2\, d\mu &= \int_\Omega [\nabla(A\nabla f)]^T\, S\, \nabla(A\nabla f)\, d\mu + \int_\Omega \Big\langle \nabla f, (\nabla^2 V - \mathcal{L}A^{-1}\, A)\, \nabla f\Big\rangle\, d\mu \\ &\quad + \int_\Omega \Big\langle A\nabla f, (\nabla S - 2\, SA\, \nabla A^{-1})\, \nabla(A\nabla f)\Big\rangle\, d\mu \\ &\quad - \int_{\partial\Omega} \langle SA\nabla f, \langle \nabla(A\nabla f), \eta\rangle\rangle\, d\mu + \int_{\partial\Omega} Lf\, \langle \nabla f, \eta\rangle\, d\mu \\ &= \int_\Omega [\nabla(A\nabla f)]^T\, S\, \nabla(A\nabla f)\, d\mu + + \int_\Omega \Big\langle \nabla f, (\nabla^2 V - \mathcal{L}A^{-1}\, A)\, \nabla f\Big\rangle\, d\mu \\ &\quad + \int_\Omega \Big\langle A\nabla f, \Big((\nabla A^{-1})^T\, A^{-1} - (A^{-1})^T\, \nabla A^{-1}\Big)\, \nabla(A\nabla f)\Big\rangle\, d\mu \\ &\quad - \int_{\partial\Omega} \langle \nabla f, \nabla^2 f\, \eta\rangle\, d\mu - \int_{\partial\Omega} \Big\langle \nabla f, A^{-1}\, \langle \nabla A, \eta\rangle\, \nabla f\Big\rangle\, d\mu \\ &\quad + \int_{\partial\Omega} Lf\, \langle \nabla f, \eta\rangle\, d\mu,\end{aligned}$$

since we have

$$\begin{aligned}\int_{\partial\Omega} \langle SA\nabla f, \langle \nabla(A\nabla f), \eta\rangle\rangle\, d\mu &= \int_{\partial\Omega} \Big\langle \nabla f, A^{-1}\, \langle \nabla(A\nabla f), \eta\rangle\Big\rangle\, d\mu \\ &= \int_{\partial\Omega} \langle \nabla f, \nabla^2 f\, \eta\rangle\, d\mu + \int_{\partial\Omega} \Big\langle \nabla f, A^{-1}\, \langle \nabla A, \eta\rangle\, \nabla f\Big\rangle\, d\mu.\end{aligned}$$

Hence the first desired identity is proved. Finally under the Neumann boundary conditions $\langle \nabla f, \eta\rangle = 0$ we have

$$0 = \nabla\langle \nabla f, \eta\rangle = \nabla^2 f\, \eta + (\mathrm{Jac}\, \eta)^T\, \nabla f,$$

from which the announced result follows. □

3.2. **Proof of Theorem 1.1.** Now we are able to prove the first main result of our paper stated in the Introduction, Theorem 1.1.

Proof of Theorem 1.1. In view of Lemma 3.1, our aim is to use Lemma 3.2 to find some convenient matrix mapping K such that the inequality (3.1) holds. To do so, we need to understand the four terms arising in the right-hand-side of (3.5). The first term is non-negative whereas the second one vanishes since $\nabla W^T\, W = W^T\, \nabla W$, the matrix weight W being diagonal. The most important terms are the two last ones, for which the assumptions of Theorem 1.1 directly apply: since assumption (A_2) means that the boundary term is non-negative, assumption (A_1) allows us to choose the matrix mapping K equal to $\nabla^2 V - \mathcal{L}W\, W^{-1}$. □

Note that Theorem 1.1 is written as such to optimize the presentation, but the weighted Poincaré inequality is valid under weaker assumptions, that is, when the boundedness from below by a positive constant in (A_1) is replaced by positive-definiteness only and still under (A_2).

As we will see on the examples, choosing a convenient matrix weight W which ensures simultaneously the conditions (A_1) and (A_2) leading to the spectral gap estimate (1.1) is not an easy task since these conditions are not of the same nature *a priori*. Indeed assumption (A_1), which already appeared in our previous study [2] on the whole space and provides the desired spectral gap estimate, strongly depends on the dynamics through the presence of the Hessian matrix of V and the matrix operator $\mathcal{L}$ whereas (A_2) does not depend on V but only on the geometry of the boundary of the domain. Therefore the strategy in the sequel is to find some convenient diagonal weight W balancing these two conditions (A_1) and (A_2).

3.3. **Convex bodies.** We concentrate in this part on convex bodies (*i.e.*, compact, convex sets of $\mathbb{R}^d$ with non-empty interior) with smooth boundary. The convexity of Ω means that the symmetric matrix $(\mathrm{Jac}\,\eta)_{|\eta^\perp}$ is non-negative, as we have seen previously. In other words, its smallest eigenvalue ρ (which depends on the space variable) is non-negative. Two interesting cases we have in mind are the following:

∘ The case of the uniform distribution on Ω, *i.e.*, the normalized volume measure with probability density function $1_\Omega/\mathrm{vol}(\Omega)$. The potential V is null and the underlying operator L is the Laplacian Δ on Ω. In this context we denote for simplicity the spectral gap $\lambda_1(\Omega)$.

∘ The case of a radial probability measure μ on Ω: the associated potential $V : \Omega \to \mathbb{R}$ only depends on the Euclidean norm, that is, $V(x) = V(r)$, $r = |x|$ (using an obvious abuse of notation). In this case μ is log-concave as soon as the one-dimensional function V is convex and non-decreasing on $\mathbb{R}^+$.

Recall first the log-concave version of the classical spectral gap estimate for convex bodies with smooth boundary, namely the famous Payne-Weinberger inequality [21] involving the diameter $\mathrm{diam}(\Omega) = \sup_{x,y\in\Omega} |x - y|$ of the convex domain Ω: if V is convex on Ω, then

$$\lambda_1(\Omega, \mu) \geq \frac{\pi^2}{\mathrm{diam}(\Omega)^2}. \tag{3.6}$$

The proof is based on hyperplane bisections (also known as the localization method, or the needle decomposition, such an approach has been formalized and further developed later by Gromov and Milman [11] and Kannan, Lovász and Simonovits [14]) allowing the authors to reduce the multidimensional case to a one-dimensional problem. Although optimal for the uniform distribution on one-dimensional intervals, such an estimate still leaves room for improvement in higher dimension since by tensorization it is known that in every dimension,

$$\lambda_1([-R, R]^d) = \frac{\pi^2}{4R^2},$$

whereas (3.6) applied to the hypercube gives only

$$\lambda_1([-R, R]^d) \geq \frac{\pi^2}{4R^2 d}.$$

In this part we improve the Payne-Weinberger estimate (3.6) under some additional assumptions provided by our Theorem 1.1. Let us start by stating the following corollary, which is presented for simplicity for convex bodies with the origin in their interior, but which can be adapted to the general situation up to some translation modifying only the potential V.

Corollary 3.3. *Consider a radial log-concave probability measure μ on a convex body $\Omega \subset \mathbb{R}^d$ with the origin in its interior. Assume that the following quantity is finite:*

$$C = \max\left\{\sup_{x\in\Omega} r^2 \left(1 + 2\,\max\left\{\frac{V'(r)}{rV''(r)}, 1\right\}\right), \sup_{x\in\partial\Omega} r^2 + \frac{2r}{\rho(x)}\right\}. \tag{3.7}$$

In the case of the uniform distribution, i.e., $V \equiv 0$, we set

$$C = \max\left\{\sup_{x\in\Omega} 3r^2, \sup_{x\in\partial\Omega} r^2 + \frac{2r}{\rho(x)}\right\}. \tag{3.8}$$

Then the spectral gap satisfies

$$\lambda_1(\Omega,\mu) \geq \frac{2d}{C}. \tag{3.9}$$

Proof. Let us choose some weight W which is a multiple of the identity and radial, say $W(x) = w(r)\,I$ for all $x \in \Omega$, where w is some smooth positive one-dimensional function. Then the quantities of interest appearing in the assumptions (A_1) and (A_2) of Theorem 1.1 rewrite as follows: first for all $x \in \Omega$,

$$\begin{aligned}\nabla^2 V(x) - \mathcal{L}W(x)\,W^{-1}(x) \;&=\; \frac{V'(r)}{r}\,I + \left(V''(r) - \frac{V'(r)}{r}\right)\frac{xx^T}{r^2}\\ &\quad -\frac{1}{w(r)}\left(w''(r) + \left(\frac{d-1}{r} - V'(r)\right)w'(r)\right)I,\end{aligned}$$

whereas at the boundary $x \in \partial\Omega$, we have on the hyperplane $\eta(x)^\perp$,

$$\operatorname{Jac}\eta(x) - W(x)\,\langle \nabla W^{-1}(x), \eta(x)\rangle = \operatorname{Jac}\eta(x) + \frac{w'(r)}{w(r)}\,\frac{\langle x,\eta(x)\rangle}{r}\,I.$$

Since we have $\langle x,\eta(x)\rangle/r \leq 1$, this matrix is non-negative on $\eta(x)^\perp$ if

$$\rho(x) \geq -\frac{w'(r)}{w(r)}, \quad x \in \partial\Omega,$$

provided w is non-increasing, and a good choice ensuring this latter inequality is then $w = C - |\cdot|^2$, where the constant C is defined by (3.7)-(3.8) (note that w is positive according to the definition of C). Now let us come back to assumption (A_1) and see if it is satisfied with this choice of function w. Since the matrix $\nabla^2 V(x)$ has only two eigenvalues $V''(r)$ and $V'(r)/r$ with respective eigenspaces $\mathbb{R}\,x$ and $x^\perp$, the smallest eigenvalue $\rho\left(\nabla^2 V(x) - \mathcal{L}W(x)\,W^{-1}(x)\right)$ is bounded from below by

$$\min\left\{\frac{V'(r)}{r}, V''(r)\right\} + \frac{2d}{C-r^2} - \frac{2V'(r)r}{C-r^2} \;\geq\; \frac{2d}{C-r^2},$$

the inequality being justified according to the definition of the constant C. Since the origin belongs to the domain Ω, we obtain the desired spectral gap estimate

$$\lambda_1(\Omega,\mu) \geq \inf_{x\in\Omega} \frac{2d}{C-r^2} = \frac{2d}{C}.$$

□

For instance, such a result will be applied in Section 4 to the so-called Subbotin distributions with potential $V = |\cdot|^\alpha/\alpha$, $\alpha > 1$.

Dealing with the uniform distribution, our result has to be interpreted as a spectral gap comparison between convex bodies and Euclidean balls. Denoting $\mathcal{B}(0,R)$ the (closed) Euclidean ball of radius $R > 0$, it is more or less known to specialists (see for instance our previous article [8] for a proof) that $\lambda_1(\mathcal{B}(0,R))$ is of order d/R^2 (recalled in the Appendix, its exact expression does not exhibit an explicit behaviour with respect to the dimension). Actually, Corollary 3.3 enables to recover easily this estimate. Indeed, note that on the hyperplane $x^\perp$ we have $\operatorname{Jac}\eta(x) = (1/R)\, I$, hence its smallest eigenvalue is $\rho(x) = 1/R$ and therefore $C = 3R^2$, so that we get

$$\lambda_1(\mathcal{B}(0,R)) \geq \frac{2d}{3R^2},$$

which is of the correct order up to the prefactor $2/3$. We mention that such a prefactor might be avoided in this particular case by rather considering in the proof of Corollary 3.3 the radial function $w(r) = \exp(-r^2/2R^2)$, leading to the slightly better estimate $\lambda_1(\mathcal{B}(0,R)) \geq (d-1)/R^2$. See also the discussion in the Appendix. Now the question is the following: since Weinberger [28] proved the following inequality:

$$\lambda_1(\Omega) \leq \left(\frac{\mathrm{vol}(\mathcal{B}(0,1))}{\mathrm{vol}(\Omega)}\right)^{2/d} \lambda_1(\mathcal{B}(0,1)),$$

i.e., the spectral gap $\lambda_1(\mathcal{B}(0,1))$ of the Euclidean unit ball $\mathcal{B}(0,1)$ maximizes all the spectral gaps $\lambda_1(\Omega)$ of bounded domains Ω with the same volume, does the reverse Weinberger inequality hold (up to some prefactor bounded with respect to the dimension) for convex bodies ? Actually, such an inequality reveals to be true by reformulating the conclusion of Corollary 3.3 at least under the additional assumption that the convex body is uniformly convex, that is, such that the smallest eigenvalue ρ of the symmetric matrix $(\operatorname{Jac}\eta)_{|\eta^\perp}$ satisfies $\inf_{\partial\Omega}\rho > 0$. This observation leads to the following result, which is the announced spectral gap comparison. Below we denote

$$\bar{r} = \max\{r : x \in \partial\Omega\} = \max\{r : x \in \Omega\} \quad \text{and} \quad \underline{r} = \min\{r : x \in \partial\Omega\},$$

the (finite) maximal (resp. (positive) minimal) distance of the boundary from the origin.

Corollary 3.4. *Let $\Omega \subset \mathbb{R}^d$ be a uniformly convex body with the origin in its interior. Then we have the spectral gap comparison*

$$\lambda_1(\Omega) \geq \frac{2d\,\underline{r}^2}{(d+2)\,\max\{3\bar{r}^2, \bar{r}^2 + 2\bar{r}/\rho\}} \left(\frac{\mathrm{vol}(\mathcal{B}(0,1))}{\mathrm{vol}(\Omega)}\right)^{2/d} \lambda_1(\mathcal{B}(0,1)),$$

where $\rho = \inf_{x\in\partial\Omega}\rho(x) > 0$.

Proof. The proof is straightforward from Corollary 3.3. Under the uniform convexity assumption, the constant C in (3.8) is bounded from above by $\max\{3\bar{r}^2, \bar{r}^2 + 2\bar{r}/\rho\}$ so that (3.9) yields

$$\lambda_1(\Omega) \geq \frac{2d}{\max\{3\bar{r}^2, \bar{r}^2 + 2\bar{r}/\rho\}}.$$

Now we have $\mathcal{B}(0, \underline{r}) \subset \Omega$ leading to the volume comparison

$$\underline{r}^d \operatorname{vol}(\mathcal{B}(0,1)) \leq \operatorname{vol}(\Omega).$$

Testing on linear functions, the spectral gap $\lambda_1(\mathcal{B}(0,1))$ is easily bounded from above as follows:

$$\lambda_1(\mathcal{B}(0,1)) \leq \frac{d}{\int_{\mathcal{B}(0,1)} |x|^2 \, d\mathcal{U}_{\mathcal{B}(0,1)}(x)} = \frac{d \int_0^1 r^{d-1} \, dr}{\int_0^1 r^{d+1} \, dr} = d + 2.$$

Finally combining all these inequalities entails the desired result. □

Now let us investigate a second interesting situation, that is, when the symmetric matrix mapping $(\operatorname{Jac} \eta)_{|\eta^\perp}$ is diagonal. This is the case when the function F describing the domain Ω has an additive form, that is for instance,

$$F(x) = \sum_{i=1}^d U_i(x_i) - 1, \quad x \in \mathbb{R}^d,$$

where the potentials $U_i : \mathbb{R} \to \mathbb{R}^+$ are smooth one-dimensional functions, since we have at the boundary $x \in \partial\Omega$,

$$\eta(x) = \frac{1}{\sqrt{\sum_{i=1}^d U_i'(x_i)^2}} \left(U_1'(x_1), \ldots, U_d'(x_d)\right)^T,$$

and

$$(\operatorname{Jac} \eta(x))_{|\eta(x)^\perp} = \frac{1}{\sqrt{\sum_{i=1}^d U_i'(x_i)^2}} \operatorname{diag} U_i''(x_i), \quad x \in \mathbb{R}^d.$$

Above the diagonal matrix $\operatorname{diag} U_i''(x_i)$ has the $U_i''(x_i)$ on the diagonal. In particular when the U_i are convex functions, the domain Ω is convex and called a generalized Orlicz ball, cf. [18] (note that similarly to [18] we do not assume any symmetry assumption on the U_i). Although the forthcoming result might be adapted to general probability measures, in particular product measures on Ω, let us provide a simplified version in the context of the uniform probability measure on the convex body Ω. Our second corollary derived from Theorem 1.1 exhibits a dimension free spectral gap estimate.

Corollary 3.5. *Let Ω be of the form*

$$\Omega = \left\{ x \in \mathbb{R}^d : \sum_{i=1}^d U_i(x_i) \leq 1 \right\},$$

where the smooth functions $U_i : \mathbb{R} \to \mathbb{R}^+$ are convex. We assume moreover the following properties: there exists some $R > 0$ such that

- $\Omega \subset [-R, R]^d$;
- *there exists some $q > 0$ such that for all $i \in \{1, \ldots, d\}$,*

$$q \, |U_i'(x_i)| \leq U_i''(x_i), \quad x_i \in [-R, R].$$

Then the spectral gap satisfies

$$\lambda_1(\Omega) \geq \frac{1}{R^2} \arctan\left(\frac{2Rq}{\pi}\right)^2.$$

Proof. Assume that for any $x \in \Omega$ the diagonal matrix weight $W(x)$ is of the form $\operatorname{diag} w_i(x_i)$. Then we have

$$\nabla^2 V(x) - \mathcal{L}W(x)\, W^{-1}(x) = \operatorname{diag} \frac{-w_i''(x_i)}{w_i(x_i)}, \quad x \in \Omega,$$

and at the boundary $x \in \partial\Omega$, we have on $\eta(x)^\perp$,

$$\operatorname{Jac} \eta(x) - W(x)\, \langle \nabla W^{-1}(x), \eta(x) \rangle = \frac{1}{\sqrt{\sum_{i=1}^d U_i'(x_i)^2}} \operatorname{diag}\left(U_i'' + \frac{w_i'}{w_i} U_i'\right)(x_i).$$

For any $i \in \{1, \ldots, d\}$ we set $w_i(x_i) = \cos(\beta x_i)$ for some $\beta > 0$ depending on the parameters q and R, so that assumption (A_2) in Theorem 1.1 is satisfied as soon as

$$U_i''(x_i) \geq \beta \tan(\beta x_i)\, U_i'(x_i), \quad x_i \in [-R, R].$$

Then the choice $\beta = \arctan(2Rq/\pi)/R \subset (0, \pi/2R)$ guarantees this inequality. Finally assumption (A_1) in Theorem 1.1 holds and we obtain the spectral gap estimate

$$\lambda_1(\Omega) \geq \inf_{x \in \Omega} \min_{i=1,\ldots,d} \frac{-w_i''(x_i)}{w_i(x_i)} = \beta^2,$$

which is the desired result. □

In contrast to Corollary 3.3, it is worth noticing that Corollary 3.5 is not really convenient when the spectral gap is expected to depend on the dimension. For instance in the case of the ℓ^p unit ball with $p \geq 1$, denoted $\mathcal{B}_p$, that is, the potentials U_i are of the form $U_i(x_i) = |x_i|^p$, $x_i \in [-1, 1]$, the devoted spectral gap $\lambda_1(\mathcal{B}_p)$ is of order $d^{2/p}$, cf. [25] for the case $p \in [1, 2]$ and [19] for $p \geq 2$. Thus it satisfies the famous KLS conjecture (introduced initially in an isoperimetric context by Kannan, Lovász and Simonovits [14], it states equivalently that the spectral gap of the operator $-\Delta$ on any convex domain Ω is of the order of the inverse of the operator norm of the covariance matrix of the uniform law on Ω, cf. [1]). However our estimate becomes relevant as p tends to infinity since Corollary 3.5 entails, when applied to the ℓ^p unit ball with $q = p-1$ and $R = 1$, the lower bound $\arctan\left(2(p-1)/\pi\right)^2$ which converges to $\pi^2/4$. This quantity is the expected value of the spectral gap obtained by tensorization, the ℓ^∞ unit ball being nothing but the hypercube $[-1, 1]^d$.

To go further into the analysis, it is known that there is no monotonicity properties of the spectral gap with respect to the inclusion of domains, even in the convex case. Indeed one could believe *a priori* that, similarly to the one-dimensional case, the spectral gap decreases when the domain increases since it is intimately related to the speed of convergence to equilibrium of the underlying Brownian motion. Nevertheless, considering some thin rectangle $\Omega \subset [-R, R]^d$ localized around the diagonal of the hypercube shows that this intuition is false: since its largest side is of order $R\sqrt{d}$, the spectral gap $\lambda_1(\Omega)$ is of order $1/dR^2$ whereas $\lambda_1([-R, R]^d) = \pi^2/4R^2$ as mentioned earlier. Note however

that Klartag [15] proved a kind of monotonicity property in the unconditional situation: if $\Omega \subset [-R,R]^d$ is unconditional then

$$\lambda_1(\Omega) \geq \lambda_1([-R,R]^d).$$

Hence Corollary 3.5 can be seen as a generalization of Klartag's result beyond the unconditional setting as soon as the parameter q does not depend on the dimension. For instance it should be applied to some domain Ω involving non symmetric one-dimensional potentials on a centered bounded interval of the type

$$U_i(x_i) = 1_{\{x_i \geq 0\}} |x_i|^{p_i} + 1_{\{x_i<0\}} |x_i|^{q_i},$$

for $p_i, q_i \geq 1$. See also the work of Kolesnikov and Milman [18] in which the authors show that the generalized Orlicz balls $\Omega_E = \left\{x \in \mathbb{R}^d : \sum_{i=1}^d U_i(x_i) \leq E\right\}$ (without the boundedness restriction $\Omega_E \subset [-R,R]^d$ for some $R>0$) satisfy the KLS conjecture for certain levels $E \in \mathbb{R}$ under an assumption on the rate of growth at infinity of the U_i.

4. The Subbotin and Gaussian settings

This part is devoted to apply the results of the previous section to simple radial examples on uniformly convex bodies and beyond.

4.1. **Subbotin distributions on uniformly convex bodies.** Let us start by the case of the Subbotin distribution μ with potential $V = |\cdot|^\alpha/\alpha$ on a convex body Ω containing the origin. Although we are able to obtain similar estimates in the case $\alpha > 2$, we concentrate our attention only on the case $\alpha \in (1,2]$ (including the standard Gaussian case $\alpha = 2$; the case $\alpha = 1$ could also be considered as well, but would require a slight modification of the argument below). Indeed, the potential V is uniformly convex on Ω and thus leads by (3.4) to the estimate

$$\begin{aligned} \lambda_1(\Omega,\mu) &\geq \inf_{x\in\Omega} \min\left\{V''(r), \frac{V'(r)}{r}\right\} \\ &= \inf_{x\in\Omega} \min\{1,\alpha-1\}\, r^{\alpha-2} \\ &= (\alpha-1)\,\bar{r}^{\,\alpha-2}, \end{aligned} \tag{4.1}$$

where we recall that $\bar{r} = \max\{r : x \in \partial\Omega\}$ denotes the maximal distance of the boundary of the convex body Ω from the origin. However we will see that this bound is not sufficient in general and has to be reinforced to reach the sharp order of the spectral gap with respect to the dimension. Indeed we expect a competition between two different regimes, depending at least when Ω is uniformly convex on the position of $\bar{r}$ with respect to the average value $\int_{\mathbb{R}^d} |x|\, d\mu(x)$, which is of order $d^{1/\alpha}$: when $\bar{r} \ll d^{1/\alpha}$ the spectral gap should be comparable to that of the uniform distribution on the ball $\mathcal{B}(0,\bar{r})$ since in this case the Subbotin distribution is close to the uniform law on $\mathcal{B}(0,\bar{r})$ (for instance in total variation distance), whereas for large $\bar{r} \gg d^{1/\alpha}$ the regime we expect should be similar to that of the Subbotin on the whole space $\mathbb{R}^d$, which is approximatively $d^{1-2/\alpha}$, cf. [8]. In both cases we observe that the estimate (4.1) obtained by using the uniform convexity is not sufficient to reach the expected results.

Recall that the notation ρ stands for the smallest eigenvalue of the symmetric matrix $(\mathrm{Jac}\,\eta)_{|\eta^\perp}$ and depends on the space variable.

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^d$ be a uniformly convex body with the origin in its interior and denote $\rho = \inf_{x\in\partial\Omega} \rho(x) > 0$. Let μ be the Subbotin distribution with parameter $\alpha \in (1,2]$ on Ω. Then the spectral gap satisfies*

$$\lambda_1(\Omega,\mu) \geq \max\left\{2C_{\alpha,\rho,\bar r}\, d, \frac{\alpha}{4}\left(\frac{2-\alpha}{\alpha-1}\right)^{1-2/\alpha} d^{1-2/\alpha}\right\},$$

where

$$C_{\alpha,\rho,\bar r} = \min\left\{\frac{\alpha-1}{(\alpha+1)\,\bar r^2}, \frac{\rho}{\bar r^2\rho + 2\bar r}\right\}.$$

Proof. Since Ω is uniformly convex, the constant C in (3.7) is bounded from above by

$$\max\left\{\left(1+\frac{2}{\min\{\alpha-1,1\}}\right)\bar r^2, \bar r^2 + \frac{2\bar r}{\rho}\right\} = \frac{1}{C_{\alpha,\rho,\bar r}},$$

so that (3.9) yields on the one hand the spectral gap estimate

$$\lambda_1(\Omega,\mu) \geq 2C_{\alpha,\rho,\bar r}\, d. \tag{4.2}$$

On the other hand a refinement is required to reach the sharp regime when $\bar r$ is large, as noticed above. To do so, we come back to Theorem 1.1 and set $W(x) = w(r)\, I$ with the non-decreasing radial function $w = \exp(\varepsilon|\cdot|^\alpha/\alpha)$, where $\varepsilon > 0$ has to be chosen conveniently. Then the boundary term in the assumption (A_2) rewrites as follows: for all $x \in \partial\Omega$ we have on the hyperplane $\eta(x)^\perp$,

$$\mathrm{Jac}\,\eta(x) - W(x)\,\langle \nabla W^{-1}(x), \eta(x)\rangle = \mathrm{Jac}\,\eta(x) + \frac{w'(r)}{w(r)}\,\frac{\langle x,\eta(x)\rangle}{r}\, I.$$

Since Ω is a convex body including the origin in its interior, we have $\langle x,\eta(x)\rangle \geq 0$ for all $x \in \partial\Omega$, see for instance Section 1.3 of [23]. Hence the smallest eigenvalue of this matrix is nonnegative since w is non-decreasing, thus (A_2) is satisfied. On the other hand, for all $x \in \Omega$ the smallest eigenvalue of the matrix $\nabla^2 V(x) - \mathcal{L}W(x)\, W^{-1}(x)$ is

$$\begin{aligned}\rho\left(\nabla^2 V(x) - \mathcal{L}W(x)\, W^{-1}(x)\right) &= (\alpha-1-\varepsilon\,(d+\alpha-2))\, r^{\alpha-2} + \varepsilon(1-\varepsilon) r^{2(\alpha-1)} \\ &\geq (\alpha-1-\varepsilon\,(d+\alpha-2))\, r^{\alpha-2} + \frac{\varepsilon}{2}\, r^{2(\alpha-1)},\end{aligned}$$

as soon as ε is chosen to belong to the interval $(0,(\alpha-1)/(d+\alpha-2)) \subset (0,1/2]$. Denoting φ the latter function of $r \in (0,\bar r]$, one observes that the minimum of φ on $\mathbb{R}^+$ is attained at point

$$r_0 = \left(\frac{(2-\alpha)(\alpha-1-\varepsilon(d+\alpha-2))}{\varepsilon(\alpha-1)}\right)^{1/\alpha},$$

so that for all $r \in (0,\bar r]$,

$$\varphi(r) \quad \geq \quad \frac{\alpha\,(\alpha-1-\varepsilon(d+\alpha-2))}{2(\alpha-1)}\left(\frac{(2-\alpha)(\alpha-1-\varepsilon(d+\alpha-2))}{\varepsilon(\alpha-1)}\right)^{1-2/\alpha}.$$

Choosing the parameter

$$\varepsilon = \frac{\alpha-1}{2(d+\alpha-2)} \in (0,1/2),$$

yields the inequality

$$\begin{aligned}\rho\left(\nabla^2 V(x) - \mathcal{L}W(x)\,W^{-1}(x)\right) &\geq \frac{\alpha}{4}\left(\frac{2-\alpha}{\alpha-1}\right)^{1-2/\alpha}(d+\alpha-2)^{1-2/\alpha}\\ &\geq \frac{\alpha}{4}\left(\frac{2-\alpha}{\alpha-1}\right)^{1-2/\alpha} d^{1-2/\alpha},\end{aligned}$$

so that the assumption (A_1) is satisfied and we obtain from (1.1),

$$\lambda_1(\Omega,\mu) \geq \frac{\alpha}{4}\left(\frac{2-\alpha}{\alpha-1}\right)^{1-2/\alpha} d^{1-2/\alpha}. \tag{4.3}$$

Finally combining (4.2) and (4.3) entails the desired spectral gap estimate. □

To see the relevance of our estimate, let us focus on a centered (closed) Euclidean ball $\mathcal{B}(0,R)$ of radius $R>0$ for which some results are already available in the literature. According to [5, 8], this radial situation can be reduced to a careful study of the one-dimensional radial part and in this case we have the two-sided estimates, cf. [8]:

$$\frac{(d-1)\,\mu(\mathcal{B}(0,R))}{\int_{\mathcal{B}(0,R)}|x|^2\,d\mu(x)} \leq \lambda_1(\mathcal{B}(0,R),\mu) \leq \frac{d\,\mu(\mathcal{B}(0,R))}{\int_{\mathcal{B}(0,R)}|x|^2\,d\mu(x)}.$$

Hence the following asymptotic result holds:

$$\lambda_1(\mathcal{B}(0,R),\mu) \underset{d\to\infty}{\sim} \frac{d\,\mu(\mathcal{B}(0,R))}{\int_{\mathcal{B}(0,R)}|x|^2\,d\mu(x)}. \tag{4.4}$$

Passing then in polar coordinates and using Laplace's method for the estimation of integrals leads to the exact asymptotics

$$\lambda_1(\mathcal{B}(0,R),\mu) \underset{d\to\infty}{\sim} \max\left\{\frac{d}{R^2}, d^{1-2/\alpha}\right\}.$$

Applying now Proposition 4.1 with $\rho = 1/R$ and $\bar{r} = R$, we obtain the spectral gap estimate

$$\lambda_1\left(\mathcal{B}(0,R),\mu\right) \geq \max\left\{\frac{2(\alpha-1)\,d}{(\alpha+1)\,R^2}, \frac{\alpha}{4}\left(\frac{2-\alpha}{\alpha-1}\right)^{1-2/\alpha} d^{1-2/\alpha}\right\},$$

which is therefore of the correct order up to prefactors depending only on α.

4.2. **Beyond convex bodies: the standard Gaussian case.** As usual, the Gaussian case has some advantages such as spherical symmetry and uniform convexity which allow to go further into the analysis. Hence one may wonder if our method enables to consider in this Gaussian situation some particular non-convex domains. This is the matter of this part. More precisely, we concentrate our attention on the standard Gaussian distribution μ on a non-convex domain given by the complement of a centered Euclidean ball, seen as an obstacle as in the paper [6]. Although we are able to extend the forthcoming result to more general obstacles, the result we obtain is not sufficiently convincing for the moment and this challenging extension could be the matter of a forthcoming research.

The asymptotic estimate (4.4) appearing in [8] has its counterpart on $\mathbb{R}^d \setminus \mathcal{B}(0,R)$:

$$\lambda_1(\mathbb{R}^d \setminus \mathcal{B}(0,R),\mu) \underset{d\to\infty}{\sim} \frac{d\,\mu(\mathbb{R}^d \setminus \mathcal{B}(0,R))}{\int_{\mathbb{R}^d\setminus\mathcal{B}(0,R)}|x|^2\,d\mu(x)}.$$

After a change in polar coordinates and using Laplace's method, it leads to the asymptotic estimate

$$\lambda_1(\mathbb{R}^d \setminus \mathcal{B}(0,R),\mu) \underset{d\to\infty}{\sim} \min\left\{\frac{d}{R^2}, 1\right\}, \tag{4.5}$$

which also reflects a competition between two regimes, depending on the position of the radius R with respect to the expected distance of the underlying random vector from the origin, which is of order $\sqrt{d}$. However since we work on the complement of the ball, the regimes are reversed compared to the previous section. Actually, such an estimate (4.5) has been quantified in the paper [6] in which the authors prove the following bound by exploiting the one-dimensional radial part,

$$\lambda_1\left(\mathbb{R}^d \setminus \mathcal{B}(0,R),\mu\right) \quad \geq \quad \frac{d}{2d+R^2}. \tag{4.6}$$

We would like to apply our method to this situation. Although $\mathbb{R}^d \setminus \mathcal{B}(0,R)$ is not compact (and even not bounded), the technology at the basis of our approach emphasized in Sections 3.1 and 3.2 and leading to Theorem 1.1 still applies at the price of working on the space $\mathcal{C}^\infty_{0,N}(\mathbb{R}^d \setminus \mathcal{B}(0,R))$ of compactly supported infinitely differentiable real-valued functions satisfying Neumann boundary conditions rather than on the space $\mathcal{C}^\infty_N(\mathbb{R}^d \setminus \mathcal{B}(0,R))$. It leads to the following non-asymptotic result, which is qualitatively comparable to the latter estimate (4.6). Below, the unessential restriction $d \geq 5$ on the dimension is due to the choice of function w we make in the proof and could certainly be relaxed if we modify conveniently w.

Proposition 4.2. *Let μ be the standard Gaussian distribution on $\mathbb{R}^d \setminus \mathcal{B}(0,R)$ ($d \geq 5$), the complement of the Euclidean ball of radius $R > 0$. Then the spectral gap satisfies*

$$\lambda_1\left(\mathbb{R}^d \setminus \mathcal{B}(0,R),\mu\right) \geq \min\left\{\frac{d-4}{R^2}, \frac{1}{3}\right\}.$$

Proof. First note that when considering the non-convex set $\mathbb{R}^d \setminus \mathcal{B}(0,R)$, then at the boundary $r = R$ the outer unit-normal is given by $\eta(x) = -x/r$. Therefore on the hyperplane $x^\perp$ we have $\operatorname{Jac}\eta(x) = (-1/R)\, I$ and thus its smallest eigenvalue $\rho(x) = -1/R$ is negative. Now we need to find some weight W such that the assumptions (A_1) and (A_2) of Theorem 1.1 are satisfied. Once again we choose $W(x) = w(r)\, I$ for all $x \in \mathbb{R}^d \setminus \mathcal{B}(0,R)$, for some smooth positive one-dimensional radial function w. Then we have for all $x \in \mathbb{R}^d \setminus \mathcal{B}(0,R)$,

$$\nabla^2 V(x) - \mathcal{L}W(x)\, W^{-1}(x) \quad = \quad \left(1 - \frac{w''(r)}{w(r)} - \left(\frac{d-1}{r} - r\right)\frac{w'(r)}{w(r)}\right) I,$$

and at the boundary, we have on the hyperplane $x^\perp$,

$$\operatorname{Jac}\eta(x) - W(x)\,\langle \nabla W^{-1}(x), \eta(x)\rangle = -\left(\frac{1}{R} + \frac{w'(R)}{w(R)}\right) I.$$

In this case an interesting choice of function w is $w = C + |\cdot|^{-2}$ for some $C > 0$, allowing to compensate the negativity of the smallest eigenvalue $\rho = -1/R$. Indeed, ensuring the assumption (A_2) requires in this case the condition $C \leq 1/R^2$. Choosing then $C = 1/R^2$,

the smallest eigenvalue of the matrix $\nabla^2 V(x) - \mathcal{L}W(x)\, W^{-1}(x)$ is

$$\rho\left(\nabla^2 V(x) - \mathcal{L}W(x)\, W^{-1}(x)\right) = 1 + \frac{2R^2(d-4-r^2)}{r^2(r^2+R^2)}, \quad r \geq R.$$

For a fixed $R > 0$ denote φ_R the latter function of r, defined on $\mathbb{R}^+$. We have $\varphi_R'(r) \geq 0$ if and only if $r^4 \geq (d-4)(2r^2+R^2)$, that is,

$$r^2 \geq (d-4)\left(1 + \sqrt{1 + \frac{R^2}{d-4}}\right) = R_0^2.$$

In particular φ_R is non-decreasing on $[R, \infty)$ if and only if $R^2 \geq R_0^2$, *i.e.*, $R^2 \geq 3(d-4)$. Hence in this case we have for all $r \geq R$,

$$\rho\left(\nabla^2 V(x) - \mathcal{L}W(x)\, W^{-1}(x)\right) \geq 1 + \frac{2R^2(d-4-R^2)}{R^2(R^2+R^2)} = \frac{d-4}{R^2}.$$

On the other hand for small radii, we expect a dimension-free spectral gap estimate similarly to that on the whole space $\mathbb{R}^d$ since the obstacle is small. For $R^2 \leq 3(d-4)$, that is, $R^2 \leq R_0^2$, the minimum of φ_R on $[R, \infty)$ is reached at point R_0 and thus

$$\begin{aligned}
\rho\left(\nabla^2 V(x) - \mathcal{L}W(x)\, W^{-1}(x)\right) &\geq 1 - \frac{2R^2(R_0^2-d+4)}{R_0^2(R_0^2+R^2)} \\
&= 1 - \frac{2\,(R/R_0)^2}{1+(R/R_0)^2}\left(1 - \frac{d-4}{R_0^2}\right) \\
&\geq \frac{d-4}{R_0^2} \\
&\geq \frac{1}{3},
\end{aligned}$$

where on the one hand we used the trivial inequality $2t/(1+t) \leq 1$ for all $t \in [0,1]$ and on the other hand we have $R_0^2 \leq 3(d-4)$ as soon as $R^2 \leq 3(d-4)$. Summarizing, we obtain for all $R > 0$ the inequality

$$\rho\left(\nabla^2 V(x) - \mathcal{L}W(x)\, W^{-1}(x)\right) \geq \min\left\{\frac{d-4}{R^2}, \frac{1}{3}\right\},$$

so that the assumption (A_1) of Theorem 1.1 is satisfied and we obtain from (1.1) the desired spectral gap estimate. □

5. Appendix

Considering the case of a product measure on a cube, we are able to reach optimality in Theorem 1.1. Indeed the idea is to take the matrix mapping W as the Jacobian matrix of the diffeomorphism whose coordinates are the eigenfunctions related to the spectral gap of the one-dimensional marginal distributions. As such, the weight W satifies the assumptions (A_1) and (A_2) and is thus diagonal (for assumption (A_2), an approximation procedure somewhat similar to that emphasized in Corollary 3.5 is required). In this appendix, we focus our attention on the spectral gap $\lambda_1(\mathcal{B}(0,R))$ of the (closed) Euclidean ball $\mathcal{B}(0,R)$ of radius $R > 0$ endowed with the uniform distribution and wonder if a possible optimality in Theorem 1.1 might be reached in this non-product context. We will

see that such an analysis leads to an interesting phenomenon and opens the door to a natural open question. Before going further into the details, let us recall how to identify the exact expression of the spectral gap $\lambda_1(\mathcal{B}(0,R))$, cf. for instance Weinberger [28]. Actually, the associated eigenspace is known to be of dimension d and the corresponding eigenfunctions are given (in a vector field notation) by

$$F(x) = \frac{g(r)}{r}\, x, \quad x \in \mathcal{B}(0,R),$$

where g is solution to the equation

$$g''(r) + (d-1)\,\frac{g'(r)}{r} + \left(\lambda_1(\mathcal{B}(0,R)) - \frac{d-1}{r^2}\right) g = 0,$$

which vanishes at 0. This is a generalized Bessel equation and classical computations provide the generic solution given by

$$g(r) = \left(\sqrt{\lambda_1(\mathcal{B}(0,R))}\, r\right)^{1-\frac{d}{2}} J_{\frac{d}{2}}\left(\sqrt{\lambda_1(\mathcal{B}(0,R))}\, r\right), \tag{5.1}$$

where $J_{d/2}$ stands for the Bessel function of the first kind J_ς, *i.e.*,

$$J_\varsigma(r) := \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{r}{2}\right)^{\varsigma+2k}}{k!\Gamma(\varsigma+k+1)},$$

with $\varsigma = d/2$. Moreover a standard analysis shows that the ratio g'/g is non-negative up to the first zero of g'. We have

$$\operatorname{Jac} F(x) = \frac{g(r)}{r}\, I + \left(g'(r) - \frac{g(r)}{r}\right) \frac{xx^T}{r^2}, \tag{5.2}$$

and since we have $\eta(x) = x/r$, we get at the boundary $\mathcal{S}(0,R)$ (the sphere centered at the origin and of radius R),

$$\operatorname{Jac} F(x)\, \eta(x) = \frac{g'(R)}{R}\, x,$$

so that each coordinate of the vector field F satisfies the Neumann boundary conditions if and only if $g'(R) = 0$. In other words, if we note the function $\tilde{J}_\varsigma : u \mapsto u^{1-d/2} J_{d/2}(u)$, it means that the derivative at point $u = \sqrt{\lambda_1(\mathcal{B}(0,R))}\, R$ of $\tilde{J}_\varsigma$ applied with $\varsigma = d/2$ vanishes. Thus if p_ς denotes the first positive zero of $\tilde{J}_\varsigma$, then the spectral gap $\lambda_1(\mathcal{B}(0,R))$, corresponding to the smallest positive eigenvalue, is given by

$$\lambda_1(\mathcal{B}(0,R)) = \frac{p_{\frac{d}{2}}^2}{R^2}. \tag{5.3}$$

Actually, an interesting question is the following: which radial function leads to the largest spectral gap estimate ? After some computations somewhat similar to the previous ones, such a radial function is given by

$$w(r) = \left(\sqrt{\lambda} r\right)^{1-\frac{d}{2}} J_{\frac{d}{2}-1}\left(\sqrt{\lambda} r\right),$$

for some convenient $\lambda > 0$ determined when saturating the boundary condition $w(R) + R\,w'(R) \geq 0$, that is, $\lambda = p^2_{d/2\,-1}/R^2$, so that we obtain

$$\lambda_1(\mathcal{B}(0,R)) \geq \frac{p^2_{\frac{d}{2}-1}}{R^2}.$$

However this lower bound is not really explicit in terms of the dimension, as the optimal one (5.3).

Let us return to our original questioning about a potential optimality in Theorem 1.1 in the non-product context of the uniform measure on $\mathcal{B}(0,R)$. Similarly to the product measure case, we intend to choose the weight W as the Jacobian matrix of the eigenfunctions associated to the spectral gap $\lambda_1(\mathcal{B}(0,R))$, cf. the formula (5.2). Although W is not diagonal (thus Theorem 1.1 cannot be used directly), let us observe however how the assumptions (A_1) and (A_2) are satisfied. Since W is not invertible on the boundary $\mathcal{S}(0,R)$, we still work on the ball $\mathcal{B}(0,R)$ but we consider on the extended ball $\mathcal{B}(0,R+\varepsilon)$ for some $\varepsilon > 0$ the smooth matrix mapping $W_\varepsilon = \mathrm{Jac}\,F_\varepsilon$. Here F_ε denotes the vector field whose coordinates are the eigenfunctions associated to the spectral gap $\lambda_1(\mathcal{B}(0,R+\varepsilon))$, with the corresponding function g_ε defined analogously to (5.1). From the computations below, we will see that W_ε is invertible on $\mathcal{B}(0,R)$. We simply denote W and g the respective quantities for $\varepsilon = 0$. Moreover we (still) denote Δ the diagonal matrix operator with the Laplacian acting on functions on the diagonal.
On the one hand the identity $\Delta F_\varepsilon = -\lambda_1(\mathcal{B}(0,R+\varepsilon))\,F_\varepsilon$ holds on the smaller ball $\mathcal{B}(0,R)$ and leads to

$$\nabla^2 V - \mathcal{L}W\,W^{-1} = -\Delta \mathrm{Jac}\,F_\varepsilon\,(\mathrm{Jac}\,F_\varepsilon)^{-1} = \lambda_1(B(0,R+\varepsilon))\,I,$$

since the Laplacian commutes with the Jacobian. Hence assumption (A_1) is satisfied. On the other hand we need some additional computations to verify assumption (A_2). Recall that by (5.2) we have

$$W_\varepsilon(x) = \frac{g_\varepsilon(r)}{r}\,I + \left(g'_\varepsilon(r) - \frac{g_\varepsilon(r)}{r}\right)\frac{xx^T}{r^2}, \quad x \in \mathcal{B}(0,R).$$

Above the matrix xx^T/r^2 is that of the orthogonal projection onto the line spanned by the vector x. In particular the matrix $W_\varepsilon(x)$ is diagonalizable and x is an eigenvector associated to the eigenvalue $g'_\varepsilon(r)$ whereas any vector orthogonal to x is an eigenvector associated to the eigenvalue $g_\varepsilon(r)/r$. Since g_ε and g'_ε do not vanish on $(0,R+\varepsilon)$, thus on $(0,R]$, W_ε is invertible on $\mathcal{B}(0,R)$ and

$$W_\varepsilon^{-1}(x) = \frac{r}{g_\varepsilon(r)}\,I + \left(\frac{1}{g'_\varepsilon(r)} - \frac{r}{g_\varepsilon(r)}\right)\frac{xx^T}{r^2}, \quad x \in \mathcal{B}(0,R).$$

Note that because $\eta(x) = x/r$ we have for all $i,j = 1,\ldots,d$,

$$\left\langle \nabla\left(\frac{x_i x_j}{r^2}\right), \eta(x)\right\rangle = \sum_{k=1}^{d}\left(\frac{x_j}{r^2}\,\delta_{i,k} + \frac{x_i}{r^2}\,\delta_{j,k} - \frac{2x_i x_j x_k}{r^4}\right)\frac{x_k}{r} = 0,$$

where δ is the usual Kronecker delta symbol. Since for any smooth radial function h we have $\nabla h(r) = h'(r)x/r$, we obtain

$$\left\langle \nabla W_\varepsilon^{-1}(x), \eta(x) \right\rangle = \frac{\partial}{\partial r}\left(\frac{r}{g_\varepsilon(r)}\right) I + \frac{\partial}{\partial r}\left(\frac{1}{g_\varepsilon'(r)} - \frac{r}{g_\varepsilon(r)}\right)\frac{xx^t}{r^2}.$$

Finally at the boundary $x \in \mathcal{S}(0,R)$, we have on the hyperplane $\eta(x)^\perp$,

$$\begin{aligned}\operatorname{Jac}\eta(x) - W_\varepsilon(x)\left\langle \nabla W_\varepsilon^{-1}(x), \eta(x)\right\rangle &= \left(\frac{1}{R} - \frac{g_\varepsilon(R)}{R}\frac{\partial}{\partial r}\left(\frac{r}{g_\varepsilon(r)}\right)\Big|_{r=R}\right) I \\ &= \frac{g_\varepsilon'(R)}{g_\varepsilon(R)} I,\end{aligned}$$

meaning that assumption (A_2) is satisfied for each fixed $\varepsilon > 0$ since $g_\varepsilon'/g_\varepsilon \geq 0$ on $(0, R+\varepsilon)$ and thus on $(0,R]$. Finally the desired optimality result in Theorem 1.1 is reached by passing to the limit $\varepsilon \to 0$ since $\lambda_1(\mathcal{B}(0,R+\varepsilon)) \to \lambda_1(\mathcal{B}(0,R))$ and also $g_\varepsilon'(R)/g_\varepsilon(R) \to g'(R)/g(R) = 0$, *i.e.*, the boundary term vanishes. The validity of these limits might be obtained either by a classical continuity argument or by using directly the exact expression (5.3) of the spectral gap of the Euclidean ball of any radius and plugging then into the formula (5.1) defining g_ε.

As announced earlier, the previous discussion does not allow us to use directly Theorem 1.1 because the matrix mapping W is not diagonal. However we can apply Lemma 3.2 with the above weight W_ε and let $\varepsilon \to 0$ to get the following identity: if μ stands for the uniform probability measure on $\mathcal{B}(0,R)$, then for all $f \in \mathcal{C}_N^\infty(\mathcal{B}(0,R))$,

$$\begin{aligned}\int_{\mathcal{B}(0,R)} (-Lf)^2\, d\mu &= \int_{\mathcal{B}(0,R)} \left[\nabla(W^{-1}\nabla f)\right]^T W^T W\, \nabla(W^{-1}\nabla f)\, d\mu \\ &\quad + \int_{\mathcal{B}(0,R)} \left\langle W^{-1}\nabla f, (\nabla W^T W - W^T \nabla W)\, \nabla(W^{-1}\nabla f)\right\rangle d\mu \\ &\quad + \lambda_1(\mathcal{B}(0,R)) \int_{\mathcal{B}(0,R)} |\nabla f|^2\, d\mu.\end{aligned}$$

Although the first term is always non-negative, some computations show that for all $i,j = 1,\dots,d$,

$$\left(\nabla W^T W - W^T \nabla W\right)_{i,j} = \frac{1}{r^2}\left(g'(r) - \frac{g(r)}{r}\right)^2 (x_j e_i - x_i e_j),$$

where $(e_k)_{k=1,\dots,d}$ is the usual canonical basis of $\mathbb{R}^d$. Therefore the matrix of vectors $\nabla W^T W - W^T \nabla W$ is not zero and thus it is not clear to us that the second integral above vanishes, as it is trivially the case when W is diagonal (or when f is one of the eigenfunctions associated to the spectral gap $\lambda_1(B(0,R))$ since in this case $W^{-1}\nabla f$ is constant). Nevertheless according to the (integrated version of) Bakry-Emery criterion, cf. [3], we know *a priori* that the sum of these two integrals is non-negative. Hence a challenging question would be to prove directly that this sum is non-negative for all $f \in \mathcal{C}_N^\infty(\mathcal{B}(0,R))$.

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