



HAL
open science

Eulerian description for anisotropic hyper-élastic materials

Emmanuelle Rouhaud, Benoît Panicaud

► **To cite this version:**

Emmanuelle Rouhaud, Benoît Panicaud. Eulerian description for anisotropic hyper-élastic materials. Colloque Mécamat Aussois 2023, Jan 2023, Aussois (FR), France. hal-03979833

HAL Id: hal-03979833

<https://hal.science/hal-03979833v1>

Submitted on 9 Feb 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Description eulérienne pour les matériaux anisotropes hyper-élastiques

Author Emmanuelle Rouhaud, Benoît Panicaud

Université de technologie de Troyes



Introduction

Several approaches are possible to construct a model for elastic materials undergoing finite transformations:

- Rate type, hypo-elastic models with the use of objective transports: does not guarantee a reversible behavior.
- Thermodynamics of irreversible processes with an intrinsic dissipation rate equal to zero. For adiabatic or isothermal reversible transformations the first principle of thermodynamics leads to:

$$\boldsymbol{\sigma} : \mathbf{d} - \rho \dot{\psi} = 0 \quad (1)$$

where ψ is the specific internal energy (an energy per unit mass), $\boldsymbol{\sigma}$ is the Cauchy stress tensor, \mathbf{d} is the rate of deformation and ρ is the mass density. The dot over the quantity indicates a total derivative with respect to time.

Small perturbations

Under the hypothesis of small perturbations, one can make the following approximation:

$$\mathbf{d} \approx \dot{\boldsymbol{\varepsilon}} \quad (2)$$

where $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor. The mass density ρ is considered as constant.

Define a strain energy density function $W(\boldsymbol{\varepsilon})$ (energy per unit volume) that depends on the strain tensor alone. Equation 1 becomes:

$$\left(\boldsymbol{\sigma} - \frac{\partial W}{\partial \boldsymbol{\varepsilon}} \right) : \dot{\boldsymbol{\varepsilon}} = 0. \quad (3)$$

If there is no constraint on $\dot{\boldsymbol{\varepsilon}}$, it comes:

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}} \quad (4)$$

and an adequate choice for the expression of W as a function of $\boldsymbol{\varepsilon}$ eventually leads to the classical Hooke model.

Finite transformations

In the context of finite transformations, the density can no more be considered as a constant, an equivalent of $\mathbf{d} \approx \dot{\boldsymbol{\varepsilon}}$ has to be proposed, the invariance with respect to superposition of rigid body rotation has to be enforced. To derive the model, one uses objective quantities and a derivation on the reference configuration.

Define $W_0 = \rho_0 \psi$ a strain energy per unit undeformed volume (ρ_0 is the density in the reference configuration considered constant in space and time) and choose $W_0(\mathbf{E})$ where \mathbf{E} is the Lagrangian strain tensor, then:

$$\boldsymbol{\Sigma} : \mathbf{D} - \dot{W}_0(\mathbf{E}) = 0 \quad \text{with} \quad \mathbf{D} = \dot{\mathbf{E}} \quad \text{and} \quad \boldsymbol{\Sigma} = \frac{\partial W_0}{\partial \mathbf{E}} \quad (5)$$

where $\boldsymbol{\Sigma}$ is the second Piola-Kirchhoff tensor and \mathbf{D} is the lagrangian rate of deformation. Eulerian models limited to the isotropic case may then be derived :

$$\boldsymbol{\sigma} = 2 \frac{\rho}{\rho_0} \mathbf{B} \frac{\partial W_0}{\partial \mathbf{B}} \quad \boldsymbol{\sigma} = 2 \frac{\rho}{\rho_0} \frac{\partial W_0}{\partial \mathbf{g}} \quad (6)$$

where \mathbf{B} is the left Cauchy-Green deformation tensor and is the right equation known as Doyle-Ericksen formula; \mathbf{g} is the ambient metric.

Questions

Why is it necessary to derive the model in the reference configuration? One could wonder how to treat the derivation with respect to time in the reference configuration: is it possible to take into account the variation of the density ρ_0 in space? Is it possible to write such an equation with an energy that is not per unit of undeformed volume? Equation $\mathbf{D} = \dot{\mathbf{E}}$ is written for the reference configuration, does it have a counterpart in an Eulerian description? This raises the question of the geometric nature of $\dot{\mathbf{E}}$ and thus \mathbf{D} . Is it then possible to derive the equivalent of Equation 5 without resorting to the reference configuration?

The Lie derivative

The Lie derivative in the velocity field $\mathcal{L}_v(\cdot)$ of a scalar a , a scalar density α , a second order contravariant tensor \mathbf{A} or covariant tensor \mathbf{B} is:

$$\mathcal{L}_v a = \dot{a} \quad \mathcal{L}_v \alpha = \dot{\alpha} + \alpha \nabla \cdot \mathbf{v} \quad \mathcal{L}_v \mathbf{A} = \dot{\mathbf{A}} - (\nabla \mathbf{v}) \mathbf{A} - \mathbf{A} (\nabla \mathbf{v})^T \quad \mathcal{L}_v \mathbf{B} = \dot{\mathbf{B}} + (\nabla \mathbf{v}) \mathbf{B} + \mathbf{B} (\nabla \mathbf{v})^T \quad (7)$$

Interesting relations:

$$\mathcal{L}_v \rho = 0, \quad \mathcal{L}_v \mathbf{g} = 2\mathbf{d}, \quad \mathcal{L}_v \mathbf{e} = \mathbf{d} \quad \mathcal{L}_v \mathbf{b} = 0 \quad \text{with} \quad \mathbf{b} = (\mathbf{F}^T)^{-1} \mathbf{F}^{-1} \quad (8)$$

where \mathbf{F} is the deformation gradient and $\mathbf{e} = \frac{1}{2}(\mathbf{g} - \mathbf{b})$ is the Euler-Almansi strain tensor. The first equation is the mass conservation.

Eulerian anisotropic hyper elasticity

Rewrite Equation 1 ($\boldsymbol{\sigma} : \mathbf{d} - \rho \dot{\psi} = 0$) as:

$$\frac{1}{2} \boldsymbol{\sigma} : \mathbf{d} - \rho \mathcal{L}_v \psi = 0 \quad \rightarrow \quad \frac{1}{2} \boldsymbol{\sigma} : \mathbf{d} - \mathcal{L}_v(\rho \psi) = 0 \quad \frac{1}{2} \boldsymbol{\sigma} : \mathbf{d} - \mathcal{L}_v W = 0 \quad \text{with} \quad W = \rho \psi \quad (9)$$

Choose the energy density such that $W(\rho, \mathbf{C}, \mathbf{e})$ where \mathbf{C} is a fourth order tensor containing material parameters and verifying $\mathcal{L}_v \mathbf{C} = 0$ (or for the isotropic case, $W(\rho, C_i, I_i(\mathbf{e}))$ where $I_i(\mathbf{e})$ corresponds to a number i of invariants of the strain tensor \mathbf{e} and C_i are material parameters), then:

$$\frac{\partial W}{\partial \rho} \overbrace{\mathcal{L}_v \rho}^{=0} + \frac{\partial W}{\partial \mathbf{C}} \overbrace{\mathcal{L}_v \mathbf{C}}^{=0} + \left(2 \frac{\partial W}{\partial \mathbf{g}} - \boldsymbol{\sigma} \right) : \mathbf{d} = 0 \quad \text{then} \quad \boldsymbol{\sigma} = 2 \frac{\partial W}{\partial \mathbf{g}} \quad (10)$$

Conclusion

It is possible to derive an elastic model in the Eulerian configuration for anisotropic materials in the context of finite deformations.

- The Lie derivative is used.
- The mass conservation is used to treat properly the variation of the mass density.
- The relation $\mathcal{L}_v \mathbf{e} = \mathbf{d}$ is geometrically sound and is the Eulerian counterpart of $\mathbf{D} = \dot{\mathbf{E}}$.