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# Description eulérienne pour les matériaux anisotropes hyper-élastiques

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## Introduction

Several approaches are possible to construct a model for elastic materials undergoing finite transformations:

- Rate type, hypo-elastic models with the use of objective transports: does not guarantee a reversible behavior.
- Thermodynamics of irreversible processes with an intrinsic dissipation rate equal to zero. For adiabatic or isothermal reversible transformations the first principle of thermodynamics leads to:

$$\boldsymbol{\sigma} : \mathbf{d} - \rho \dot{\psi} = 0 \quad (1)$$

where  $\psi$  is the specific internal energy (an energy per unit mass),  $\boldsymbol{\sigma}$  is the Cauchy stress tensor,  $\mathbf{d}$  is the rate of deformation and  $\rho$  is the mass density. The dot over the quantity indicates a total derivative with respect to time.

## Small perturbations

Under the hypothesis of small perturbations, one can make the following approximation:

$$\mathbf{d} \approx \dot{\boldsymbol{\varepsilon}} \quad (2)$$

where  $\boldsymbol{\varepsilon}$  is the infinitesimal strain tensor. The mass density  $\rho$  is considered as constant.

Define a strain energy density function  $W(\boldsymbol{\varepsilon})$  (energy per unit volume) that depends on the strain tensor alone. Equation 1 becomes:

$$\left( \boldsymbol{\sigma} - \frac{\partial W}{\partial \boldsymbol{\varepsilon}} \right) : \dot{\boldsymbol{\varepsilon}} = 0. \quad (3)$$

If there is no constraint on  $\dot{\boldsymbol{\varepsilon}}$ , it comes:

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}} \quad (4)$$

and an adequate choice for the expression of  $W$  as a function of  $\boldsymbol{\varepsilon}$  eventually leads to the classical Hooke model.

## Finite transformations

In the context of finite transformations, the density can no more be considered as a constant, an equivalent of  $\mathbf{d} \approx \dot{\boldsymbol{\varepsilon}}$  has to be proposed, the invariance with respect to superposition of rigid body rotation has to be enforced. To derive the model, one uses objective quantities and a derivation on the reference configuration.

Define  $W_0 = \rho_0 \psi$  a strain energy per unit undeformed volume ( $\rho_0$  is the density in the reference configuration considered constant in space and time) and choose  $W_0(\mathbf{E})$  where  $\mathbf{E}$  is the Lagrangian strain tensor, then:

$$\boldsymbol{\Sigma} : \mathbf{D} - \dot{W}_0(\mathbf{E}) = 0 \quad \text{with} \quad \mathbf{D} = \dot{\mathbf{E}} \quad \text{and} \quad \boldsymbol{\Sigma} = \frac{\partial W_0}{\partial \mathbf{E}} \quad (5)$$

where  $\boldsymbol{\Sigma}$  is the second Piola-Kirchhoff tensor and  $\mathbf{D}$  is the lagrangian rate of deformation. Eulerian models limited to the isotropic case may then be derived :

$$\boldsymbol{\sigma} = 2 \frac{\rho}{\rho_0} \mathbf{B} \frac{\partial W_0}{\partial \mathbf{B}} \quad \boldsymbol{\sigma} = 2 \frac{\rho}{\rho_0} \frac{\partial W_0}{\partial \mathbf{g}} \quad (6)$$

where  $\mathbf{B}$  is the left Cauchy-Green deformation tensor and is the right equation known as Doyle-Ericksen formula;  $\mathbf{g}$  is the ambient metric.

## Questions

Why is it necessary to derive the model in the reference configuration? One could wonder how to treat the derivation with respect to time in the reference configuration: is it possible to take into account the variation of the density  $\rho_0$  in space? Is it possible to write such an equation with an energy that is not per unit of undeformed volume? Equation  $\mathbf{D} = \dot{\mathbf{E}}$  is written for the reference configuration, does it have a counterpart in an Eulerian description? This raises the question of the geometric nature of  $\dot{\mathbf{E}}$  and thus  $\mathbf{D}$ . Is it then possible to derive the equivalent of Equation 5 without resorting to the reference configuration?

## The Lie derivative

The Lie derivative in the velocity field  $\mathcal{L}_v(\cdot)$  of a scalar  $a$ , a scalar density  $\alpha$ , a second order contravariant tensor  $\mathbf{A}$  or covariant tensor  $\mathbf{B}$  is:

$$\mathcal{L}_v a = \dot{a} \quad \mathcal{L}_v \alpha = \dot{\alpha} + \alpha \nabla \cdot \mathbf{v} \quad \mathcal{L}_v \mathbf{A} = \dot{\mathbf{A}} - (\nabla \mathbf{v}) \mathbf{A} - \mathbf{A} (\nabla \mathbf{v})^T \quad \mathcal{L}_v \mathbf{B} = \dot{\mathbf{B}} + (\nabla \mathbf{v}) \mathbf{B} + \mathbf{B} (\nabla \mathbf{v})^T \quad (7)$$

Interesting relations:

$$\mathcal{L}_v \rho = 0, \quad \mathcal{L}_v \mathbf{g} = 2\mathbf{d}, \quad \mathcal{L}_v \mathbf{e} = \mathbf{d} \quad \mathcal{L}_v \mathbf{b} = 0 \quad \text{with} \quad \mathbf{b} = (\mathbf{F}^T)^{-1} \mathbf{F}^{-1} \quad (8)$$

where  $\mathbf{F}$  is the deformation gradient and  $\mathbf{e} = \frac{1}{2}(\mathbf{g} - \mathbf{b})$  is the Euler-Almansi strain tensor. The first equation is the mass conservation.

## Eulerian anisotropic hyper elasticity

Rewrite Equation 1 ( $\boldsymbol{\sigma} : \mathbf{d} - \rho \dot{\psi} = 0$ ) as:

$$\frac{1}{2} \boldsymbol{\sigma} : \mathbf{d} - \rho \mathcal{L}_v \psi = 0 \quad \rightarrow \quad \frac{1}{2} \boldsymbol{\sigma} : \mathbf{d} - \mathcal{L}_v(\rho \psi) = 0 \quad \frac{1}{2} \boldsymbol{\sigma} : \mathbf{d} - \mathcal{L}_v W = 0 \quad \text{with} \quad W = \rho \psi \quad (9)$$

Choose the energy density such that  $W(\rho, \mathbf{C}, \mathbf{e})$  where  $\mathbf{C}$  is a fourth order tensor containing material parameters and verifying  $\mathcal{L}_v \mathbf{C} = 0$  (or for the isotropic case,  $W(\rho, C_i, I_i(\mathbf{e}))$  where  $I_i(\mathbf{e})$  corresponds to a number  $i$  of invariants of the strain tensor  $\mathbf{e}$  and  $C_i$  are material parameters), then:

$$\frac{\partial W}{\partial \rho} \overbrace{\mathcal{L}_v \rho}^{=0} + \frac{\partial W}{\partial \mathbf{C}} \overbrace{\mathcal{L}_v \mathbf{C}}^{=0} + \left( 2 \frac{\partial W}{\partial \mathbf{g}} - \boldsymbol{\sigma} \right) : \mathbf{d} = 0 \quad \text{then} \quad \boldsymbol{\sigma} = 2 \frac{\partial W}{\partial \mathbf{g}} \quad (10)$$

## Conclusion

It is possible to derive an elastic model in the Eulerian configuration for anisotropic materials in the context of finite deformations.

- The Lie derivative is used.
- The mass conservation is used to treat properly the variation of the mass density.
- The relation  $\mathcal{L}_v \mathbf{e} = \mathbf{d}$  is geometrically sound and is the Eulerian counterpart of  $\mathbf{D} = \dot{\mathbf{E}}$ .