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Questions in linear recurrence: From the $T \oplus T$ -problem to lineability

by

Sophie Grivaux, Antoni López-Martínez & Alfred Peris

Abstract

We study, for a continuous linear operator T on an F-space X , when the direct sum operator $T \oplus T$ is recurrent on $X \oplus X$. In particular: we establish the analogous notion, for recurrence, to that of (topological) *weak-mixing* for transitivity/hypercyclicity, namely *quasi-rigidity*; and we construct a recurrent but not quasi-rigid operator on each separable infinite-dimensional Banach space, solving the $T \oplus T$ -recurrence problem in the negative way. The quasi-rigidity notion is closely related to the *dense lineability* of the set of recurrent vectors, and using similar conditions we study the lineability and dense lineability properties for the set of \mathcal{F} -recurrent vectors, under very weak assumptions on the Furstenberg family \mathcal{F} .

1 Introduction

This paper focuses on some aspects of *Linear Dynamics* and the general setting is the following: a (*real or complex*) *linear dynamical system* (X, T) is a pair formed by a continuous linear operator $T : X \rightarrow X$ on a (real or complex) separable infinite-dimensional F-space (i.e. X is a completely metrizable topological vector space). We will denote by $\mathcal{L}(X)$ the *set of continuous linear operators* acting on such a space X , and \mathbb{K} will stand for either the real or complex field, \mathbb{R} or \mathbb{C} .

Given a linear dynamical system (X, T) , the T -orbit of a vector $x \in X$ is the set

$$\text{Orb}(x, T) := \{x, Tx, T^2x, T^3x, \dots\} = \{T^n x : n \in \mathbb{N}_0\},$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We say that a vector $x \in X$ is *hypercyclic* for T if $\text{Orb}(x, T)$ is a dense set in X ; and that the operator T is *hypercyclic* if it admits a hypercyclic vector. The so-called *Birkhoff transitivity theorem* (see for instance [7, Theorem 1.2]) shows the equivalence between hypercyclicity and the notion of topological transitivity: a system (X, T) is called *topologically transitive* if for each pair of non-empty open subsets $U, V \subset X$ one can find some (and hence infinitely many) $n \in \mathbb{N}_0$ such that $T^n(U) \cap V \neq \emptyset$. It follows that the set $\text{HC}(T)$ of hypercyclic vectors for T is a dense G_δ subset of X as soon as T is hypercyclic.

Hypercyclicity has been, historically, the main and most studied notion in Linear Dynamics. For instance, the following long-standing problem, which we call the $T \oplus T$ -*hypercyclicity problem*, was posed in 1992 by D. Herrero [33]:

Question 1.1 (The $T \oplus T$ -hypercyclicity problem). Let T be a hypercyclic operator on the F-space X . Is the operator $T \oplus T$ acting on the direct sum $X \oplus X$ hypercyclic?

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Recall that an operator T is said to be (topologically) *weakly-mixing* if and only if $T \oplus T$ is transitive, so the question above asks if there exists any transitive but not weakly-mixing operator. The active research on hypercyclicity yielded many equivalent reformulations of Question 1.1, and the approach which will interest us most is the one taken in 1999 by Bès and the third author of this paper. Here is the main result they obtained in [10]:

- *A continuous linear operator $T \in \mathcal{L}(X)$ is weakly-mixing if and only if it satisfies the so-called Hypercyclicity Criterion.*

In other words, $T \oplus T \in \mathcal{L}(X \oplus X)$ is a hypercyclic operator if and only if the following holds: *there exist two dense subsets $X_0, Y_0 \subset X$, an increasing sequence of positive integers $(n_k)_{k \in \mathbb{N}}$, and a family of (not necessarily continuous) mappings $S_{n_k} : Y_0 \rightarrow X$ such that*

- (i) $T^{n_k} x \rightarrow 0$ for each $x \in X_0$;
- (ii) $S_{n_k} y \rightarrow 0$ for each $y \in Y_0$;
- (iii) $T^{n_k} S_{n_k} y \rightarrow y$ for each $y \in Y_0$.

Question 1.1 was finally answered negatively in 2006 by De La Rosa and Read, when they exhibited a hypercyclic operator on a particular Banach space whose direct sum with itself is not hypercyclic (see [21]). The techniques used there were later refined by Bayart and Matheron in order to construct hypercyclic but not weakly-mixing operators, on each Banach space admitting a normalized unconditional basis whose associated forward shift is continuous, as for instance on the classical $c_0(\mathbb{N})$ and $\ell^p(\mathbb{N})$ spaces, $1 \leq p < \infty$ (see [5, 6]).

The aim of this paper is to study the previous problem, and to develop the corresponding theory, for the notion of *recurrence*: we say that a vector $x \in X$ is *recurrent* for T if x belongs to the closure of its forward orbit $\overline{\text{Orb}(Tx, T)}$; and that the operator T is *recurrent* if its set $\text{Rec}(T)$ of recurrent vectors is *dense* in X . Note that, in order to say that an operator T has a *recurrent behaviour*, it is not enough to assume that $\text{Rec}(T) \neq \emptyset$ since the zero-vector is recurrent for every linear map. It is shown in [19, Proposition 2.1] that the notion of recurrence coincides with that of *topological recurrence*, i.e. the property that for each non-empty open subset $U \subset X$ one can find some (and hence infinitely many) $n \in \mathbb{N}$ such that $T^n(U) \cap U \neq \emptyset$. In addition, and as in the hypercyclicity case, when T is recurrent its set $\text{Rec}(T)$ of recurrent vectors is a dense G_δ subset of X .

Recurrence is one of the oldest and most studied concepts in the *Topological Dynamics* area of knowledge (see [24, 3, 36]), but in Linear Dynamics its systematic study began with the 2014 paper of Costakis, Manoussos and Parissis [19]. As one can naturally expect, many questions in *linear recurrence* remain open and here we will be particularly interested in the so-called *$T \oplus T$ -recurrence problem*, which was stated in [19, Question 9.6]:

Question 1.2 (The $T \oplus T$ -recurrence problem). Let T be a recurrent operator on the F-space X . Is the operator $T \oplus T$ acting on the direct sum $X \oplus X$ recurrent?

This question has been recently restated as an open problem in [20] for strongly continuous semigroups of operators. In the first part of this work we identify the systems (X, T) for which the recurrence of T implies that of $T \oplus T$. This characterization is given in terms of *quasi-rigidity* (see Definition 2.2), which will be for recurrence, the analogous property to that of *weak-mixing* for hypercyclicity (see Theorem 2.5). We then answer Question 1.2 negatively, by constructing some recurrent but not quasi-rigid operators (see Section 3).

The second part of this paper is dedicated to the study of some properties related to the notion of lineability: recall that a subset Y of an F-space X is called (*dense*) *lineable* if $Y \cup \{0\}$ contains a (dense) infinite-dimensional vector space. A well-known result (due to Herrero and Bourdon) states that the set $\text{HC}(T)$ is *dense lineable* for every hypercyclic operator T (see [31, Theorem 2.55]); moreover, one easily observes that if T is a quasi-rigid operator, then $\text{Rec}(T)$ is also *dense lineable* (see Proposition 2.7). Here we generalize these results to a rather general class of Furstenberg families \mathcal{F} : first we show that it makes sense to study quasi-rigidity from the \mathcal{F} -recurrence point of view (see Proposition 4.5); and then we check that, for an \mathcal{F} -recurrent operator T , the set $\mathcal{F}\text{Rec}(T)$ of \mathcal{F} -recurrent vectors is always *lineable* and usually even *dense lineable* (see Theorems 5.5 and 5.8). As a consequence we obtain the *Herrero-Bourdon theorem* for \mathcal{F} -hypercyclicity (see Subsection 5.3).

The paper is organized as follows. Section 2 is devoted to define and study *quasi-rigidity*, which is the analogous property, for recurrence, to that of *weak-mixing* for hypercyclicity. We show in Section 3 that there exist recurrent but not quasi-rigid operators, answering Question 1.2 in a negative way and solving the *$T \oplus T$ -recurrence problem*. In Section 4 we recall the definition of \mathcal{F} -recurrence, showing that quasi-rigidity is a particular case of such a concept, and studying both the weakest and strongest possible \mathcal{F} -recurrence notions. The *lineability* and *dense lineability* properties, for the set $\mathcal{F}\text{Rec}(T)$ of \mathcal{F} -recurrent vectors, are studied in Section 5. We finally gather, in Section 6, some left open problems.

We refer the reader to the textbooks [7, 31] for any unexplained but standard notion about hypercyclicity, or more generally, about Linear Dynamics. Along the paper we use the environment “**Question**” to state the different problems that we solve, and we use “**Problem(s)**” to state the left open problems.

2 Quasi-rigid dynamical systems

In this section we introduce and study the concept of *quasi-rigidity*, which is naturally defined in the broader framework of *Topological Dynamics* (i.e. without assuming linearity): a pair (X, T) is called a *dynamical system* if T is a continuous self-map on a Hausdorff topological space X . An important class of dynamical systems, which we will repeatedly use, is the family of Polish systems: a pair (X, T) is said to be a *Polish dynamical system* whenever T is a continuous self-map of a separable completely metrizable space X .

Definition 2.1. Given a dynamical system (X, T) and any $N \in \mathbb{N}$ we will denote by $T_N : X^N \rightarrow X^N$ the *N -fold direct product* of T with itself, i.e. (X^N, T_N) is the dynamical system

$$T_N := \underbrace{T \times \cdots \times T}_N : \underbrace{X \times \cdots \times X}_N \longrightarrow \underbrace{X \times \cdots \times X}_N.$$

where $X^N := \underbrace{X \times \cdots \times X}_N$ is the *N -fold product* of X and $T_N(x_1, \dots, x_N) := (Tx_1, \dots, Tx_N)$.

If (X, T) is linear, then (X^N, T_N) will refer to the *N -fold direct sum* linear dynamical system

$$T_N := \underbrace{T \oplus \cdots \oplus T}_N : \underbrace{X \oplus \cdots \oplus X}_N \longrightarrow \underbrace{X \oplus \cdots \oplus X}_N,$$

and in this case $X^N := \underbrace{X \oplus \cdots \oplus X}_N$ is the *N -fold direct sum* of X .

2.1 Quasi-rigidity: definition and equivalences

By a well-known theorem of Furstenberg, once a dynamical system (X, T) is weakly-mixing (i.e. if $T \times T$ is topologically transitive) then so is every N -fold direct product system (X^N, T_N) . See [31, Theorem 1.51]. Hence, in order to completely answer Question 1.2 we should first study the recurrent dynamical systems (X, T) for which every N -fold direct product is again recurrent. A first attempt would be to rely on the notion of *rigidity*: a dynamical system (X, T) is said to be *rigid* if there exists an increasing sequence of natural numbers $(n_k)_{k \in \mathbb{N}}$ such that $T^{n_k}x \rightarrow x$ for every $x \in X$.

Rigidity has been studied in different contexts such as measure theoretic recurrence [25], dynamical systems on topological spaces [27] and also for linear systems [22, 19]. It is a really strong form of recurrence, as it implies that $X^N = \text{Rec}(T_N)$ for all $N \in \mathbb{N}$, so that is not the exact notion we are looking for. Nonetheless, this concept motivates the definition:

Definition 2.2 (Quasi-Rigidity). Let (X, T) be a dynamical system and let $(n_k)_{k \in \mathbb{N}}$ be a strictly increasing sequence of positive integers. We say that T is:

- *quasi-rigid* with respect to the sequence $(n_k)_{k \in \mathbb{N}}$, if there exists a dense subset Y of X such that $T^{n_k}x \rightarrow x$ for every $x \in Y$.
- *topologically quasi-rigid* with respect to $(n_k)_{k \in \mathbb{N}}$, if for every non-empty open subset $U \subset X$ there exists some $k_U \in \mathbb{N}$ such that $T^{n_k}(U) \cap U \neq \emptyset$ for every $k \geq k_U$.

Our aim is to show that, given a dynamical system (X, T) , the previous notions characterize the recurrent behaviour of every N -fold direct product system (X^N, T_N) under very weak assumptions (dynamically speaking) on the underlying space X . The following result shows that, when the underlying space X is second-countable, *topological quasi-rigidity* is for recurrence the analogous property to that of *weak-mixing* for transitivity:

Lemma 2.3. *Let (X, T) be a dynamical system on a second-countable space X . The following are equivalent:*

- (i) T is topologically quasi-rigid;
- (ii) T_N is topologically recurrent for every $N \in \mathbb{N}$.

Proof. (i) \Rightarrow (ii): suppose that T is topologically quasi-rigid with respect to $(n_k)_{k \in \mathbb{N}}$, let $N \in \mathbb{N}$ and consider any finite sequence of non-empty open sets $U_1, \dots, U_N \subset X$. By topological quasi-rigidity one can find $k_0 \in \mathbb{N}$ such that

$$T^{n_k}(U_j) \cap U_j \neq \emptyset \quad \text{for every } 1 \leq j \leq N \text{ and all } k \geq k_0.$$

Then, for the open set $U = U_1 \times \dots \times U_N \subset X^N$ we have that $T_N^{n_k}(U) \cap U \neq \emptyset$ for all $k \geq k_0$, so (X^N, T_N) is topologically recurrent.

(ii) \Rightarrow (i): assume that T_N is topologically recurrent for every $N \in \mathbb{N}$ and let $(U_s)_{s \in \mathbb{N}}$ be a countable base of (non-empty) open sets for X . We can construct recursively an increasing sequence of positive integers $(n_k)_{k \in \mathbb{N}}$ such that $T^{n_k}(U_s) \cap U_s \neq \emptyset$ for every $s, k \in \mathbb{N}$ with $1 \leq s \leq k$: given any $k \in \mathbb{N}$, the set $U := U_1 \times \dots \times U_k$ is a non-empty open subset of X^k and by the topological recurrence of the map T_k we can pick $n_k \in \mathbb{N}$ sufficiently large with

$$T_k^{n_k}(U) \cap U \neq \emptyset \quad \text{and hence} \quad T^{n_k}(U_s) \cap U_s \neq \emptyset \quad \text{for all } 1 \leq s \leq k.$$

It is easy to check that T is topologically quasi-rigid with respect to $(n_k)_{k \in \mathbb{N}}$. □

As one would expect, if the underlying space is completely metrizable, we can identify the “pointwise” *quasi-rigidity* notion with that of *topological quasi-rigidity*:

Proposition 2.4. *Let (X, T) be a Polish dynamical system. The following are equivalent:*

- (i) T is quasi-rigid;
- (ii) T is topologically quasi-rigid.

Proof. The implication (i) \Rightarrow (ii) is straightforward: assume that there is a dense subset $Y \subset X$ and an increasing sequence $(n_k)_{k \in \mathbb{N}}$ such that $T^{n_k}x \rightarrow x$ for every $x \in Y$. For each non-empty open subset $U \subset X$ we can select $x \in U \cap Y$, and then

$$T^{n_k}x \in T^{n_k}(U) \cap U \quad \text{for every sufficiently large } k \in \mathbb{N}.$$

Hence T is topologically quasi-rigid with respect to $(n_k)_{k \in \mathbb{N}}$. Let us prove (ii) \Rightarrow (i): assume that $d(\cdot, \cdot)$ is a metric defining the complete topology of X , let $(U_s)_{s \in \mathbb{N}}$ be a countable base of open sets for X and set $V_{s,s} := U_s$ for each $s \in \mathbb{N}$. By (ii) we can find a positive integer $n_1 \in \mathbb{N}$ such that

$$T^{n_1}(U_1) \cap U_1 = T^{n_1}(V_{1,1}) \cap V_{1,1} \neq \emptyset.$$

The continuity of T ensures the existence of a non-empty open subset $V_{1,2} \subset X$ of diameter (with respect to d) less than $\frac{1}{2^2}$ for which $\overline{V_{1,2}} \subset V_{1,1}$ and $T^{n_1}(\overline{V_{1,2}}) \subset V_{1,1}$. Suppose now that for some $k \in \mathbb{N}$ we have already constructed:

- finite sequences $(V_{s,j})_{s \leq j \leq k}$ of open subsets of X , for each $1 \leq s \leq k-1$;
- and a finite increasing sequence of positive integers $(n_j)_{1 \leq j \leq k-1}$;

with the properties that each set $V_{s,j}$ has d -diameter less than $\frac{1}{2^j}$, and also that

$$\overline{V_{s,j}} \subset V_{s,j-1} \quad \text{and} \quad T^{n_{j-1}}(\overline{V_{s,j}}) \subset V_{s,j-1} \quad \text{for all } 1 \leq s \leq k-1 \text{ and } s < j \leq k.$$

Then, considering the open subsets $V_{1,k}, V_{2,k}, \dots, V_{k,k} \subset X$ and using again (ii) we can select a positive integer $n_k \in \mathbb{N}$ with $n_k > n_{k-1}$ for which $T^{n_k}(V_{s,k}) \cap V_{s,k} \neq \emptyset$ for all $1 \leq s \leq k$. Again the continuity of T ensures the existence, for each $1 \leq s \leq k$, of a non-empty open subset $V_{s,k+1} \subset X$ of d -diameter less than $\frac{1}{2^{k+1}}$ and such that

$$\overline{V_{s,k+1}} \subset V_{s,k} \quad \text{and} \quad T^{n_k}(\overline{V_{s,k+1}}) \subset V_{s,k} \quad \text{for all } 1 \leq s \leq k.$$

A recursive argument gives us an increasing sequence of positive integers $(n_k)_{k \in \mathbb{N}}$ and, for each $s \in \mathbb{N}$, a sequence of non-empty open sets $(V_{s,j})_{j=s}^\infty$ such that each set $V_{s,j}$ has d -diameter less than $\frac{1}{2^j}$, but also satisfying

$$\overline{V_{s,k+1}} \subset V_{s,k} \quad \text{and} \quad T^{n_k}(\overline{V_{s,k+1}}) \subset V_{s,k} \quad \text{for all } s, k \in \mathbb{N} \text{ with } s \leq k.$$

By the Baire Category theorem, for each $s \in \mathbb{N}$ there exists a unique vector $y_s \in X$ such that

$$\{y_s\} = \bigcap_{k \geq s} \overline{V_{s,k}} \subset V_{s,s} = U_s.$$

We deduce that $Y := \{y_s : s \in \mathbb{N}\}$ is a dense subset of X and for each $s, k \in \mathbb{N}$ we have that $T^{n_k}y_s \in T^{n_k}(\overline{V_{s,k+1}}) \subset V_{s,k}$ whenever $s \leq k$, so

$$\limsup_{k \rightarrow \infty} d(T^{n_k}y_s, y_s) \leq \limsup_{k \rightarrow \infty} \text{diam}_d(V_{s,k}) \leq \limsup_{k \rightarrow \infty} \frac{1}{2^k} = 0,$$

and hence T is quasi-rigid with respect to the sequence $(n_k)_{k \in \mathbb{N}}$. \square

The proof of the implication (i) \Rightarrow (ii) in Proposition 2.4 shows that every quasi-rigid map is topologically quasi-rigid even if X is not a Polish space. The converse fails in general (consider for instance [31, Example 12.9]). Combining the previous results we get:

Theorem 2.5. *Let (X, T) be a Polish dynamical system. The following are equivalent:*

- (i) T is quasi-rigid;
- (ii) T is topologically quasi-rigid;
- (iii) T_N is topologically recurrent for every $N \in \mathbb{N}$;
- (iv) T_N is recurrent for every $N \in \mathbb{N}$.

Proof. The implications (i) \Rightarrow (ii) \Rightarrow (iii) (and also (i) \Rightarrow (iv) \Rightarrow (iii)) are trivial even if the space X is not second-countable neither complete. When X is second-countable we get the equivalence (ii) \Leftrightarrow (iii) by Lemma 2.3. If we just assume completeness, then we have the equivalence (iii) \Leftrightarrow (iv) by [19, Proposition 2.1]. Finally, using the Polish hypothesis we get the equivalence (i) \Leftrightarrow (ii) by Proposition 2.4. \square

Remark 2.6. Even though we have worked in the general setting of *Polish dynamical systems*, it is worth mentioning that Theorem 2.5 holds true for every:

- *compact dynamical system*, i.e. when T is a continuous self-map of a compact metrizable space X (many of references just study this class of systems, see for instance [3, 24, 36]);
- continuous linear operator T acting on any separable F-space X . In other words, the previous result holds true for *linear dynamical systems* as defined in the Introduction.

In the linear setting we have got the following relations between the already exposed concepts:

$$\begin{array}{ccc}
 \textit{satisfying the HC} & \xrightarrow{\text{Theorem 2.5}} & \textit{quasi-rigidity} \\
 \downarrow & & \downarrow \\
 \textit{hypercyclicity} & \xrightarrow{\text{already known}} & \textit{recurrence}
 \end{array}$$

i.e. the *quasi-rigidity* notion is for recurrence, the analogous property to that of *satisfying the Hypercyclicity Criterion* (and hence to that of *weak-mixing*) for hypercyclicity.

2.2 Quasi-rigid operators

It is now time to discuss how the quasi-rigidity notion influences the structure of the set of recurrent vectors for a linear dynamical system (X, T) . We start by showing that quasi-rigidity implies the dense lineability of the set of recurrent vectors:

Proposition 2.7. *If $T : X \rightarrow X$ is a quasi-rigid operator, then $\text{Rec}(T)$ is dense lineable.*

Proof. Assume that $Y \subset X$ is a dense subset such that there exists a sequence $(n_k)_{k \in \mathbb{N}}$ of integers with $T^{n_k}x \rightarrow x$ for all $x \in Y$. Then any $z \in Z := \text{span}(Y)$ satisfies that $T^{n_k}z \rightarrow z$, so $Z \subset X$ is an infinite-dimensional dense vector space contained in $\text{Rec}(T)$. \square

Remark 2.8. Let $T : X \rightarrow X$ be a quasi-rigid operator with respect to a sequence $(n_k)_{k \in \mathbb{N}}$ and consider $Y := \{x \in X : T^{n_k}x \rightarrow x \text{ as } k \rightarrow \infty\}$. This is a dense subset of X which can be either of first or second Baire category, depending on the operator T :

- (a) If T is a *rigid* operator, then $Y = X$. This is the case of the identity operator, but there also exist examples of rigid systems which are even (Devaney) chaotic (see [22]).
- (b) If X is a Banach space and Y is a second category set, then $\sup_{k \in \mathbb{N}} \|T^{n_k}\| < \infty$ by the Banach-Steinhaus theorem. Consider the operator $T := \lambda B$ with $|\lambda| > 1$, where B is the (unilateral) *backward shift* on $c_0(\mathbb{N})$ or any $\ell^p(\mathbb{N})$ ($1 \leq p < \infty$). This operator is weakly-mixing and hence quasi-rigid, but the set Y as defined before has to be of first category since $x = (\frac{1}{n^2})_{n \in \mathbb{N}} \in X$ has the property that $\|T^n x\| \rightarrow \infty$ as $n \rightarrow \infty$.

Now we give some (usually fulfilled) sufficient conditions for a dynamical system to be quasi-rigid. Recall that a vector $x \in X$ is called *cyclic* for an operator $T : X \rightarrow X$ if

$$\text{span}(\text{Orb}(x, T)) = \text{span}\{T^n x : n \in \mathbb{N}_0\} = \{p(T)x : p \text{ polynomial}\},$$

is a dense set in X . Moreover, a point $x \in X$ is called *periodic* for a map T whenever $T^p x = x$ for some positive integer $p \in \mathbb{N}$. See [7, 31] for more on cyclicity and periodicity.

Proposition 2.9. *Let (X, T) be a dynamical system for which any of the following hold:*

- (a) *the set $\text{Per}(T)$ of periodic points for T is dense in X ;*
- (b) *T admits a recurrent vector $x \in \text{Rec}(T)$ for which the following set is dense*

$$\{Sx : ST = TS, S : X \rightarrow X \text{ continuous mapping}\};$$

then T is quasi-rigid. In particular, a continuous linear operator T acting on a Hausdorff topological vector space X is quasi-rigid whenever any of the following conditions hold:

- *T has dense periodic vectors;*
- *T admits a recurrent and cyclic vector;*
- *T admits a hypercyclic vector.*

Proof. To check (a) consider $Y = \text{Per}(T)$ and $(n_k)_{k \in \mathbb{N}} = (k!)_{k \in \mathbb{N}}$. For (b) consider

$$Y = \{Sx : ST = TS, S : X \rightarrow X \text{ continuous mapping}\}.$$

Indeed, since x is recurrent there exists $(n_k)_{k \in \mathbb{N}}$ such that $T^{n_k} x \rightarrow x$, but then, given any point $y = Sx \in Y$ we get that $T^{n_k}(Sx) = S(T^{n_k} x) \rightarrow Sx$. Finally, recall that every recurrent and cyclic (and hence every hypercyclic) vector fulfills condition (b). \square

Proposition 2.9 together with [19, Corollary 9.2] show that recurrence plus cyclicity imply quasi-rigidity when X is a complex Banach space. This result extends to (real and complex) linear dynamical systems as defined in the Introduction via the following lemma, in which the concept of *complexification* is used (see Remark 5.2). We write $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Lemma 2.10. *Let T be a topologically recurrent continuous linear operator acting on a (real or complex) Hausdorff topological vector space X . The following statements hold:*

- (a) *For every (real or complex) number $\lambda \in \mathbb{K} \setminus \mathbb{T}$ the operator $(T - \lambda)$ has dense range.*
- (b) *If X is a real space, then for every $\lambda \in \mathbb{C} \setminus \mathbb{T}$ the complexified operator $(\tilde{T} - \lambda)$, acting on the complexification \tilde{X} of X , has dense range.*

As a consequence, for any (real or complex) polynomial p without unimodular roots, the operator $p(T)$ has dense range on the (real or complex) space X .

The proof of this lemma is based on [31, Lemma 12.13], a well-known result of Wengenroth for topologically transitive operators acting on Hausdorff topological vector spaces. We just show statement (a), since (b) follows exactly as in [31, Exercise 12.2.6] (*Hint*: when X is a real space define the *complexification* $\tilde{T} : \tilde{X} \rightarrow \tilde{X}$ as it is done in Remark 5.2 for operators on F-spaces, and use the fact that for each non-empty open subset $U \subset X$ there are infinitely many $n \in \mathbb{N}$ with $\tilde{T}^n(U + iU) \cap (U + iU) \neq \emptyset$).

Proof of Lemma 2.10. Given $\lambda \in \mathbb{K} \setminus \mathbb{T}$ let $E := \overline{(T - \lambda)(X)}$, which is a closed subspace of X , and suppose that $E \neq X$. The quotient space X/E is then a (real or complex) Hausdorff topological vector space (since E is closed). If $q : X \rightarrow X/E$ is the quotient map we have that $q((T - \lambda)x) = 0$ and hence $q(Tx) = \lambda q(x)$ for every $x \in X$. The operator S on X/E given by $S[x] = \lambda[x]$ is easily seen to be topologically recurrent since so is T .

However, if $|\lambda| < 1$ we can choose $[x] \in X/E \setminus \{0\}$ and a balanced 0-neighbourhood W such that $[x] \notin \overline{W}$. Since $\lambda^n[x] \rightarrow 0$ and $\lambda W \subset W$ there exists a neighbourhood U of $[x]$ and a natural number $N \in \mathbb{N}$ for which $U \cap W = \emptyset$ and $\lambda^n U \subset W$ for every $n \geq N$. Hence $S^n(U) \cap U = \emptyset$ for all $n \geq N$ so S is not topologically recurrent, which is a contradiction. If $|\lambda| > 1$ we can consider the map $S^{-1} : X/E \rightarrow X/E$ for which $S^{-1}[x] = \lambda^{-1}[x]$. By the previous argument S^{-1} is not topologically recurrent, but then S is not topologically recurrent (see [29, Section 6]), which is again a contradiction.

Let now p be any (real or complex) polynomial without unimodular roots: if X is a complex space then $p(T)$ has dense range since, splitting the polynomial, $p(T)$ can be written as a composition of dense range operators by (a); and if X is a real space then $p(\tilde{T})$ also has dense range in \tilde{X} via the same argument as above but using (b). Finally, if p has real coefficients, then $p(\tilde{T}) = p(T) + ip(T)$ and $p(T)$ has dense range in the real space X . \square

The proof of [19, Corollary 9.2] is highly based on the following result of D. Herrero: *given an operator T on a complex Banach space X , if $\sigma_p(T^*)$ has empty interior then the set of cyclic vectors for T is dense, and hence it is a dense G_δ set* (see [32]). By using Lemma 2.10 we obtain this density result in the more general setting of operators on F-spaces:

Corollary 2.11 (Extension of [19, Corollary 9.2]). *Let (X, T) be a (real or complex) linear dynamical system. If T is recurrent and cyclic then the set of cyclic vectors for T is a dense G_δ set. In particular, if T is recurrent and cyclic, then it is quasi-rigid.*

Proof. The set of cyclic vectors is always a G_δ set. To check its density let $x \in X$ be a cyclic vector for T and pick a non-empty open set $U \subset X$. Then there is a (real or complex) polynomial p such that $p(T)x \in U$. Splitting p we have that

$$p(T) = \prod_{j=1}^N (T - \lambda_j) \quad \text{for some } (\lambda_j)_{j=1}^N \subset \mathbb{C}.$$

Choose a small perturbation $(\mu_j)_{j=1}^N \subset \mathbb{C} \setminus \mathbb{T}$ of $(\lambda_j)_{j=1}^N \subset \mathbb{C}$ with the property that the polynomial

$$\hat{p}(T) := \prod_{j=1}^N (T - \mu_j) \quad \text{is still such that } \hat{p}(T)x \in U.$$

Note that if p has real coefficients, then $(\mu_i)_{i=1}^N \subset \mathbb{C} \setminus \mathbb{T}$ can be chosen such that \hat{p} has still real coefficients. Finally, $\hat{p}(T)x$ is a cyclic vector for T since

$$\hat{p}(T) (\{q(T)x : q \text{ polynomial}\}) = \{q(T)(\hat{p}(T)x) : q \text{ polynomial}\}$$

is a dense set by Lemma 2.10. Then $\text{Rec}(T)$ and the set of cyclic vectors for T are both residual, so T admits a recurrent and cyclic vector. Proposition 2.9 ends the proof. \square

3 Existence of recurrent but not quasi-rigid operators

After having studied in Section 2 the linear dynamical systems (X, T) for which every of its N -fold direct sums (X^N, T_N) is recurrent, we are now ready to answer negatively Question 1.2 showing that there exist recurrent but not quasi-rigid operators. If for the moment we continue in the (not necessarily linear) *dynamical systems* setting, the following are natural questions in view of the theory already developed:

Question 3.1. Let (X, T) be a recurrent dynamical system:

- (a) Is $(X \times X, T \times T)$ again a recurrent dynamical system?
- (b) Assume that $(X \times X, T \times X)$ is recurrent. Is then (X, T) quasi-rigid?

If we change the “recurrence” assumption into that of “topological transitivity” the respective answers are:

- (a) **No**, and in particular:
 - for non-linear systems consider any irrational rotation, see [31, Example 1.43];
 - for linear maps we already mentioned in the Introduction the references [21, 5, 6].
- (b) **Yes**, in both linear and non-linear cases (X^N, T_N) is transitive for every $N \in \mathbb{N}$ if and only if $T \times T$ is transitive (see [31, Theorem 1.51]).

However, the answer to both questions is **no** in the recurrence setting, and indeed: in [3, Lemma 9 and Example 4] it is shown that for each $N \in \mathbb{N}$ there exists a **non-linear dynamical system** $f : X \rightarrow X$ on a **compact metric space** X for which

$$(X^N, f_N) \text{ is recurrent} \quad \text{while} \quad (X^{N+1}, f_{N+1}) \text{ is not recurrent.}$$

This construction is **highly non-linear** since each compact space X is the disjoint union of $N + 1$ *sub-shifts* of the well-known *shift on two symbols*. It turns out that the answer to Question 3.1 in the framework of Linear Dynamics is also negative. Let us state the main result of this section:

Theorem 3.2. *Let X be any separable infinite-dimensional Banach space. For each $N \in \mathbb{N}$ there exists an operator $T \in \mathcal{L}(X)$ such that*

$$T_N : X^N \rightarrow X^N \text{ is recurrent, and even } \text{Rec}(T_N) = X^N,$$

but for which $T_{N+1} : X^{N+1} \rightarrow X^{N+1}$ is not recurrent any more.

Considering $N = 1$ we obtain examples of recurrent operators $T \in \mathcal{L}(X)$ for which $T \oplus T$ is not recurrent. Thus we get a negative answer to the $T \oplus T$ -recurrence problem, i.e. we answer [19, Question 9.6] and hence Question 1.2 in the negative.

Our proof of Theorem 3.2 **relies heavily** on a construction of Augé in [2] of operators T on (infinite-dimensional, separable) Banach spaces X with **wild dynamics**: the two sets

$$A_T = \left\{ x \in X : \lim_{n \rightarrow \infty} \|T^n x\| = \infty \right\} \quad \text{and} \quad B_T = \left\{ x \in X : \liminf_{n \rightarrow \infty} \|T^n x - x\| = 0 \right\}$$

have non-empty interior and form a partition of X . Since $B_T = \text{Rec}(T)$, these operators have plenty (but not a dense set) of recurrent vectors. The remainder of this section is devoted to prove the **complex** version of Theorem 3.2 by modifying the construction of [2]. The **real** case follows in a really similar way by using the same arguments as in [2].

3.1 Necessary prerequisites

Assume that X is a complex separable infinite-dimensional Banach space, denote by X^* its *topological dual space* and by B_{X^*} the *unit ball* of such a space X^* . Moreover, given $(x, x^*) \in X \times X^*$ we will denote by $x^*(x) = \langle x^*, x \rangle$ the dual evaluation. As in [2], our operator will be built using a bounded biorthogonal system of X . The following result is well-known (see [37, Vol I, Section 1.f] or [41]):

- Given a separable Banach space X one can find $(e_n, e_n^*)_{n \in \mathbb{N}} \subset X \times X^*$ such that:

- $\text{span}\{e_n : n \in \mathbb{N}\}$ is dense in X ;
- $\langle e_n^*, e_m \rangle = \delta_{n,m}$ where $\delta_{n,m} = 0$ if $n \neq m$ and 1 if $n = m$;
- for each $n \in \mathbb{N}$ we have that $\|e_n\| = 1$, and $K := \sup_{n \in \mathbb{N}} \|e_n^*\| < \infty$.

Once we have fixed a sequence $(e_n, e_n^*)_{n \in \mathbb{N}} \subset X \times X^*$ with the previous properties we set

$$c_{00} := \text{span}\{e_n : n \in \mathbb{N}\}.$$

Given $x \in X$ we will write $x_k := \langle e_k^*, x \rangle$ for each $k \in \mathbb{N}$, and we will repeatedly use that

$$\|x_k e_k\| \leq K \|x\| \quad \text{for each } k \in \mathbb{N}, \quad (1)$$

since

$$\|x\| := \sup_{x^* \in B_{X^*}} |\langle x^*, x \rangle| \geq \frac{|\langle e_k^*, x \rangle|}{\|e_k^*\|} = \frac{|x_k|}{\|e_k^*\|} = \frac{\|x_k e_k\|}{\|e_k^*\|} \geq \frac{\|x_k e_k\|}{K},$$

for each $k \in \mathbb{N}$, and hence that

$$\|x - y\| < \varepsilon \quad \text{implies} \quad |x_k - y_k| < K\varepsilon \quad \text{for all } x, y \in X \text{ and } k \in \mathbb{N}. \quad (2)$$

Remark 3.3. If $(e_n)_{n \in \mathbb{N}}$ is not a Schauder basis of X , the above inequalities are still true but a vector $x \in X$ cannot in general be written as a convergent series $x = \sum_{k \in \mathbb{N}} x_k e_k$. However, for the vectors in $c_{00} = \text{span}\{e_n : n \in \mathbb{N}\}$ this equality will be true: for each $x \in c_{00}$ there is some $n \in \mathbb{N}$ such that

$$x = \sum_{k=1}^n \langle e_k^*, x \rangle e_k = \sum_{k=1}^n x_k e_k.$$

From now on we fix a natural number $N \in \mathbb{N}$. We are going to construct

- a *projection* P from X into itself;
- a *sequence of functionals* $(w_k^*)_{k \geq N+2} \subset X^*$;
- two *sequences* $(m_k)_{k \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and $(\lambda_k)_{k \in \mathbb{N}} \in \mathbb{T}^{\mathbb{N}}$;
- and two *operators* R and T on X ;

in such a way that T_N is recurrent while T_{N+1} is not.

3.2 The projection P and the sequences $(w_k^*)_{k \geq N+2}$, $(m_k)_{k \in \mathbb{N}}$ and $(\lambda_k)_{k \in \mathbb{N}}$

Denote by $P : X \rightarrow \text{span}\{e_1, e_2, \dots, e_{N+1}\}$ the projection of X onto the linear span of the first $N + 1$ basis vectors defined by

$$Px := \sum_{j=1}^{N+1} \langle e_j^*, x \rangle e_j \quad \text{for every } x \in X.$$

Note that P is continuous. In fact $\|P\| \leq (N + 1)K$. Let $E := \text{span}\{e_1^*, e_2^*, \dots, e_{N+1}^*\}$ endowed with the norm $\|\cdot\|^*$ of X^* , and consider on E the equivalent norm $\|\cdot\|_\infty$ defined in the following way for each $\alpha_1, \alpha_2, \dots, \alpha_{N+1} \in \mathbb{C}$,

$$\left\| \sum_{j=1}^{N+1} \alpha_j e_j^* \right\|_\infty := \max\{|\alpha_1|, |\alpha_2|, \dots, |\alpha_{N+1}|\}.$$

This makes sense because the vectors e_1^*, \dots, e_{N+1}^* are linearly independent. There exist constants $M \geq m > 0$ such that

$$m\|w^*\|_\infty \leq \|w^*\|^* \leq M\|w^*\|_\infty \quad \text{for every } w^* \in E.$$

Moreover, since the unit sphere $S_{E_\infty} := \{w^* \in E : \|w^*\|_\infty = 1\}$ is a metrizable compact space there exists a sequence $(w_k^*)_{k \geq N+2} \subset S_{E_\infty}$, which is dense in S_{E_∞} . We include here a fact which will be used in the sequel:

Fact 3.2.1. *For each $w^* \in S_{E_\infty}$ there exists an index $i \in \{1, 2, \dots, N + 1\}$ such that the following holds: for every $\varepsilon > 0$ and every $x \in X$ with $\|x - e_i\| < \varepsilon$ we have that*

$$|\langle w^*, Px \rangle| > 1 - (N + 1)K\varepsilon.$$

Proof. Since $w^* \in S_{E_\infty}$ there are coefficients $(\alpha_j)_{j=1}^{N+1} \in \mathbb{C}^{N+1}$ such that $w^* = \sum_{j=1}^{N+1} \alpha_j e_j^*$, and there exists at least one index $i \in \{1, 2, \dots, N + 1\}$ with $|\alpha_i| = 1$. Given $\varepsilon > 0$ and $x \in X$ with $\|x - e_i\| < \varepsilon$, the inequality (2) implies that $|x_i - 1| < K\varepsilon$ while $|x_j| < K\varepsilon$ for every $1 \leq j \leq N + 1$ with $j \neq i$. Since $0 \leq |\alpha_j| \leq 1$ for every $1 \leq j \leq N + 1$ we get that

$$|\langle w^*, Px \rangle| = \left| \sum_{j=1}^{N+1} \alpha_j x_j \right| \geq |x_i| - \sum_{j=1, j \neq i}^{N+1} |\alpha_j| |x_j| > (1 - K\varepsilon) - NK\varepsilon = 1 - (N + 1)K\varepsilon. \quad \square$$

Let $(m_k)_{k \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ be a sequence of positive integers with the following properties:

- (a) $m_k \mid m_{k+1}$ for each $k \geq 1$;
- (b) $m_1 = m_2 = \dots = m_{N+1} = 1$;

and starting from $k = N + 2$, the sequence $(m_k)_{k \geq N+2}$ grows fast enough to satisfy:

- (c) $\lim_{j \rightarrow \infty} \left(m_j \cdot \sum_{k=j+1}^{\infty} \frac{1}{m_k} \right) = 0$.

The sequence $(\lambda_k)_{k \in \mathbb{N}} \in \mathbb{T}^{\mathbb{N}}$ is then defined by setting $\lambda_k := e^{2\pi i \frac{1}{m_k}}$ for each $k \in \mathbb{N}$. Using the inequality $|e^{i\theta} - 1| \leq |\theta|$, which is true for every $\theta \in \mathbb{R}$, and also the fact that $\lambda_1 = \lambda_2 = \dots = \lambda_{N+1} = 1$, we deduce the inequality

$$\sum_{k=1}^{\infty} |\lambda_k - 1| \leq 2\pi \sum_{k=N+2}^{\infty} \frac{1}{m_k} < \infty, \quad (3)$$

since condition (c) on the sequence $(m_k)_{k \in \mathbb{N}}$ implies that the series $\sum_{k=1}^{\infty} \frac{1}{m_k}$ is convergent.

3.3 The operators R and T

For each $x \in c_{00} = \text{span}\{e_n : n \in \mathbb{N}\}$, which we write as $x = \sum_{k=1}^n x_k e_k$, we set

$$Rx := \sum_{k=1}^n \lambda_k x_k e_k.$$

This defines a bounded operator R on c_{00} . Indeed, for every $x = \sum_{k=1}^n x_k e_k \in c_{00}$,

$$\|Rx\| \leq \|Rx - x\| + \|x\| \leq \sum_{k=1}^n |\lambda_k - 1| \cdot \|x_k e_k\| + \|x\| \leq \left(K \sum_{k=1}^{\infty} |\lambda_k - 1| + 1 \right) \|x\|$$

by (1). Using the inequality (3) we see that the map R extends to a bounded operator on X still denoted by R . We now define the operator T on X by setting

$$Tx := Rx + \sum_{k=N+2}^{\infty} \frac{1}{m_{k-1}} \langle w_k^*, Px \rangle e_k \quad \text{for every } x \in X. \quad (4)$$

The second sum in the expression (4) defines a bounded operator by the assumption (c) on the sequence $(m_k)_{k \in \mathbb{N}}$, the fact that $\|P\| \leq (N+1)K$ and also that $\|w_k^*\|^* \leq M$ for each $k \geq N+2$. Indeed,

$$\left\| \sum_{k=N+2}^{\infty} \frac{1}{m_{k-1}} \langle w_k^*, Px \rangle e_k \right\| \leq \sum_{k=N+2}^{\infty} \frac{\|w_k^*\|^* \|Px\|}{m_{k-1}} \leq \left(M(N+1)K \sum_{k=N+2}^{\infty} \frac{1}{m_{k-1}} \right) \|x\|,$$

for every $x \in X$, where the last parenthesis has finite value by assumption (c) on $(m_k)_{k \in \mathbb{N}}$. It follows that T is a continuous operator on X . Let us now compute its n -th power:

Fact 3.3.1 (Modification of [2, Lemma 3.5]). *For every $x \in X$ and $n > 1$ we have that*

$$T^n x = R^n x + \sum_{k=N+2}^{\infty} \frac{\lambda_{k,n}}{m_{k-1}} \langle w_k^*, Px \rangle e_k,$$

where $\lambda_{k,n} := \sum_{l=0}^{n-1} \lambda_k^l = \frac{\lambda_k^n - 1}{\lambda_k - 1}$ for each $k \geq N+2$.

Proof. Suppose that the formula holds for some $n > 1$. Then

$$\begin{aligned} T^{n+1}x &= TR^n x + \sum_{k=N+2}^{\infty} \frac{\lambda_{k,n}}{m_{k-1}} \langle w_k^*, Px \rangle T e_k = R^{n+1}x + \sum_{k=N+2}^{\infty} \frac{1}{m_{k-1}} \langle w_k^*, PR^n x \rangle e_k \\ &+ \sum_{k=N+2}^{\infty} \frac{\lambda_{k,n}}{m_{k-1}} \langle w_k^*, Px \rangle \left(R e_k + \sum_{j=N+2}^{\infty} \frac{1}{m_{j-1}} \langle w_j^*, P e_k \rangle e_j \right) \\ &= R^{n+1}x + \sum_{k=N+2}^{\infty} \frac{1 + \lambda_{k,n} \cdot \lambda_k}{m_{k-1}} \langle w_k^*, Px \rangle e_k = R^{n+1}x + \sum_{k=N+2}^{\infty} \frac{\lambda_{k,n+1}}{m_{k-1}} \langle w_k^*, Px \rangle e_k. \end{aligned}$$

In order to obtain these equalities we have used the fact that $PR^n x = Px$, because of $\lambda_1 = \lambda_2 = \dots = \lambda_{N+1} = 1$, and also that $P e_k = 0$ for $k \geq N+2$. \square

We will also need the following properties regarding the numbers $\lambda_{k,n}$:

Fact 3.3.2 (Modification of [2, Fact 3.6]). *Let $n \geq 1$. Then:*

- (i) $|\lambda_{k,n}| \leq n$ for all $k \geq N + 2$;
- (ii) $\lambda_{k,m_n} = 0$ whenever $n \geq k \geq N + 2$;
- (iii) $|\lambda_{k,n}| \geq \frac{2}{\pi}n > \frac{m_{k-1}}{\pi}$ whenever $k = \min\{j \geq N + 2 : 2n \leq m_j\}$.

Proof. Each $\lambda_{k,n}$, for $k \geq N + 2$, is the geometric sum

$$\lambda_{k,n} = \sum_{l=0}^{n-1} \lambda_k^l = \frac{\lambda_k^n - 1}{\lambda_k - 1} \quad \text{with } \lambda_k = e^{2\pi i \frac{1}{m_k}};$$

then (i) follows from the triangle inequality; (ii) since $n \geq k$ implies that $\lambda_k^{m_n} = e^{2\pi i \frac{m_n}{m_k}} = 1$ by condition (a) on the sequence $(m_k)_{k \in \mathbb{N}}$; and (iii) because

$$|\lambda_{k,n}| = \left| \frac{e^{2\pi i \frac{n}{m_k}} - 1}{e^{2\pi i \frac{1}{m_k}} - 1} \right| = \frac{|\sin(\pi \frac{n}{m_k})|}{|\sin(\pi \frac{1}{m_k})|} > \frac{2}{\pi}n > \frac{m_{k-1}}{\pi},$$

using that $\sin(\theta) > \frac{2}{\pi}\theta$ for each $\theta \in [0, \frac{\pi}{2}]$ and that $|\sin(\theta)| \leq |\theta|$ for every $\theta \in \mathbb{R}$. \square

Up to now, we have defined the operator T and computed its powers T^n . It remains to check that T_N is recurrent while T_{N+1} is not. We start by showing that $R : X \rightarrow X$ is a rigid operator (see Subsection 2.1 or [19, Definition 1.2]) with respect to the (eventually) increasing sequence of positive integers $(m_k)_{k \in \mathbb{N}}$:

Fact 3.3.3 (Modification of [2, Lemma 3.4]). *We have $\lim_{k \rightarrow \infty} R^{m_k} x = x$ for all $x \in X$.*

Proof. Following [2, Lemma 3.4] we first show that the sequence $(\|R^{m_j}\|)_{j \in \mathbb{N}}$ is bounded. In fact, if we fix $x = \sum_{k=1}^n x_k e_k \in c_{00}$ such that $\|x\| = 1$, for all $j \geq 1$ we have that

$$\begin{aligned} \|R^{m_j} x - x\| &= \left\| \sum_{k=1}^n (\lambda_k^{m_j} - 1) x_k e_k \right\| \\ &= \left\| \sum_{k=j+1}^n (\lambda_k^{m_j} - 1) x_k e_k \right\| \quad \text{since } \lambda_k^{m_j} = 1 \text{ for all } k \leq j, \\ &\leq K \|x\| \sum_{k=j+1}^n |\lambda_k^{m_j} - 1| = K \sum_{k=j+1}^n \left| e^{2\pi i \frac{m_j}{m_k}} - 1 \right| \quad \text{by (1),} \\ &\leq 2\pi K \sum_{k=j+1}^{\infty} \frac{m_j}{m_k} \quad \text{since } |e^{i\theta} - 1| \leq |\theta| \text{ for every } \theta \in \mathbb{R}, \end{aligned}$$

which is less than a constant independent of j by condition (c) on $(m_k)_{k \in \mathbb{N}}$. The density of c_{00} in X implies that $(\|R^{m_j}\|)_{j \in \mathbb{N}}$ is a bounded sequence. Now if given $x \in X$ and $\varepsilon > 0$ we find $y \in c_{00}$ with $\|y - x\| < \varepsilon$, then

$$\|R^{m_j} x - x\| \leq \|R^{m_j}(x - y)\| + \|R^{m_j} y - y\| + \|y - x\| < \sup_{j \in \mathbb{N}} \|R^{m_j}\| \varepsilon + \|R^{m_j} y - y\| + \varepsilon.$$

The claim follows since $R^{m_j} y = y$ for every $j \in \mathbb{N}$ large enough, and ε was arbitrary. \square

Now we can prove the *recurrence* property of T_N :

Proposition 3.4. *The operator $T_N : X^N \rightarrow X^N$ is recurrent. Moreover, $\text{Rec}(T_N) = X^N$.*

Proof. Fix $x^{(1)}, x^{(2)}, \dots, x^{(N)} \in X$ and set $x_j^{(i)} := \langle e_j^*, x^{(i)} \rangle$ for each $1 \leq i \leq N$ and $j \geq 1$. Choose $(\alpha_j)_{j=1}^{N+1} \in \mathbb{C}^{N+1}$ such that

$$\begin{pmatrix} x_1^{(1)} & x_2^{(1)} & \cdots & x_{N+1}^{(1)} \\ x_1^{(2)} & x_2^{(2)} & \cdots & x_{N+1}^{(2)} \\ \vdots & \vdots & \cdots & \vdots \\ x_1^{(N)} & x_2^{(N)} & \cdots & x_{N+1}^{(N)} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{N+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad \max_{1 \leq j \leq N+1} |\alpha_j| = 1.$$

This is indeed possible since the system considered has N equations and $N+1$ unknowns. Set $w^* := \sum_{j=1}^{N+1} \alpha_j e_j^* \in E$. Then $w^* \in S_{E_\infty}$ and clearly $\langle w^*, Px^{(i)} \rangle = 0$ for all $1 \leq i \leq N$. Let $(k_n)_{n \in \mathbb{N}}$ be an increasing sequence of positive integers (all greater than $N+2$) such that $\lim_{n \rightarrow \infty} w_{k_n}^* = w^*$ in E . By Fact 3.3.1, for each $1 \leq i \leq N$ we have that

$$\begin{aligned} (T^{m_{k_n}-1} - R^{m_{k_n}-1})x^{(i)} &= \sum_{k=N+2}^{\infty} \frac{\lambda_{k, m_{k_n}-1}}{m_{k-1}} \langle w_k^*, Px^{(i)} \rangle e_k = \underbrace{\sum_{N+2 \leq k < k_n} \frac{\lambda_{k, m_{k_n}-1}}{m_{k-1}} \langle w_k^*, Px^{(i)} \rangle e_k}_{(*)} \\ &+ \underbrace{\frac{\lambda_{k_n, m_{k_n}-1}}{m_{k_n-1}} \langle w_{k_n}^*, Px^{(i)} \rangle e_{k_n}}_{(**)} + \underbrace{\sum_{k_n < k} \frac{\lambda_{k, m_{k_n}-1}}{m_{k-1}} \langle w_k^*, Px^{(i)} \rangle e_k}_{(***)}. \end{aligned}$$

Note that for each $1 \leq i \leq N$,

$$(*) = \sum_{N+2 \leq k < k_n} \frac{\lambda_{k, m_{k_n}-1}}{m_{k-1}} \langle w_k^*, Px^{(i)} \rangle e_k = 0,$$

since by (ii) of Fact 3.3.2 we have that $\lambda_{k, m_{k_n}-1} = 0$ for $k_n - 1 \geq k$. By (i) of Fact 3.3.2 we have that $|\lambda_{k_n, m_{k_n}-1}| \leq m_{k_n-1}$ and hence

$$|(**)| = \left| \frac{\lambda_{k_n, m_{k_n}-1}}{m_{k_n-1}} \langle w_{k_n}^*, Px^{(i)} \rangle \right| \leq |\langle w_{k_n}^*, Px^{(i)} \rangle| \xrightarrow{n \rightarrow \infty} 0,$$

since $\lim_{n \rightarrow \infty} w_{k_n}^* = w^*$ and $\langle w^*, Px^{(i)} \rangle = 0$ for every $1 \leq i \leq N$. Moreover, by condition (c) on the sequence $(m_k)_{k \in \mathbb{N}}$ we get that, for each $1 \leq i \leq N$,

$$|(***)| \leq \sum_{k_n < k} \frac{|\lambda_{k, m_{k_n}-1}|}{m_{k-1}} \|w_k^*\|^* \cdot \|P\| \cdot \|x^{(i)}\| \leq M(N+1) \|x^{(i)}\| \cdot \sum_{k_n < k} \frac{m_{k_n-1}}{m_{k-1}} \xrightarrow{n \rightarrow \infty} 0.$$

Finally, for each $1 \leq i \leq N$ we have that

$$\begin{aligned} \|T^{m_{k_n}-1}x^{(i)} - x^{(i)}\| &\leq \|(T^{m_{k_n}-1} - R^{m_{k_n}-1})x^{(i)}\| + \|R^{m_{k_n}-1}x^{(i)} - x^{(i)}\| \\ &\leq |(*)| + |(**)| + |(***)| + \|R^{m_{k_n}-1}x^{(i)} - x^{(i)}\| \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

where $\|R^{m_{k_n}-1}x^{(i)} - x^{(i)}\| \rightarrow 0$ by Fact 3.3.3. That is, $(x^{(1)}, x^{(2)}, \dots, x^{(N)}) \in \text{Rec}(T_N)$ and the arbitrariness of the vectors $x^{(1)}, x^{(2)}, \dots, x^{(N)}$ implies that $\text{Rec}(T_N) = X^N$. \square

To end the proof of Theorem 3.2 we just have to show the following:

Proposition 3.5. *The operator $T_{N+1} : X^{N+1} \rightarrow X^{N+1}$ is not recurrent.*

Proof. By contradiction suppose that T_{N+1} is recurrent. Fix any positive value

$$0 < \varepsilon < \frac{1}{K(N+1+2\pi)}, \quad (5)$$

and consider the open balls $B(e_1, \varepsilon), B(e_2, \varepsilon), \dots, B(e_{N+1}, \varepsilon) \subset X$ centred at each vector e_i , $1 \leq i \leq N+1$, with radius ε . Then there exists some $n \in \mathbb{N}$ such that

$$T^n(B(e_i, \varepsilon)) \cap B(e_i, \varepsilon) \neq \emptyset \quad \text{for every } i = 1, 2, \dots, N+1.$$

Let $k_n := \min\{j \geq N+2 : 2n \leq m_j\} \geq N+2$. By (iii) of Fact 3.3.2 we have that

$$|\lambda_{k_n, n}| \geq \frac{m_{k_n-1}}{\pi}. \quad (6)$$

Moreover, Fact 3.2.1 applied to the functional $w_{k_n}^*$ yields the existence of (at least) one index $i_n \in \{1, 2, \dots, N+1\}$ for which

$$|\langle w_{k_n}^*, Px \rangle| > 1 - (N+1)K\varepsilon \quad \text{whenever } \|x - e_{i_n}\| < \varepsilon. \quad (7)$$

Picking any $x \in B(e_{i_n}, \varepsilon) \cap T^{-n}(B(e_{i_n}, \varepsilon)) \cap c_{00}$ we get the following inequalities

$$\begin{aligned} \varepsilon &> \|T^n x - e_{i_n}\| \geq \frac{1}{K} |\langle e_{k_n}^*, T^n x - e_{i_n} \rangle| = \frac{1}{K} |\langle e_{k_n}^*, T^n x \rangle| \quad \text{by (1),} \\ &= \frac{1}{K} \left| \lambda_{k_n}^n x_{k_n} + \frac{\lambda_{k_n, n}}{m_{k_n-1}} \langle w_{k_n}^*, Px \rangle \right| \quad \text{by Fact 3.3.1.} \end{aligned}$$

Hence

$$\begin{aligned} \varepsilon &> \frac{1}{K} \left(\left| \frac{\lambda_{k_n, n}}{m_{k_n-1}} \right| \cdot |\langle w_{k_n}^*, Px \rangle| - K\varepsilon \right) \geq \frac{1}{K\pi} \cdot |\langle w_{k_n}^*, Px \rangle| - \varepsilon \quad \text{by (2) and (6),} \\ &> \frac{(1 - (N+1)K\varepsilon)}{K\pi} - \varepsilon > \frac{1}{K(N+1+2\pi)} \quad \text{by (7) and (5).} \end{aligned}$$

This is a contradiction with (5). □

The **complex** version of Theorem 3.2 is now proved. The construction can be adapted to the **real** case using the same arguments as in [2], so every separable infinite-dimensional Banach space supports a recurrent operator which is not quasi-rigid. Moreover, there are operators whose N -fold direct sum is recurrent while its $N+1$ -fold direct sum is not, and taking $N=1$ we solve negatively the $T \oplus T$ -recurrence problem. We recall here that the respective problem for C_0 -continuous semigroups of operators, stated in [20], remains open.

4 Furstenberg families for pointwise recurrence

In the last two decades hypercyclicity has been studied from the *frequency of visits* point of view: one investigates “how often” the orbit of a vector returns to every open subset of the space. The recent notions of *frequent*, \mathcal{U} -*frequent* or *reiterative hypercyclicity* have emerged following this line of thought (see [4, 42, 8, 9]). As shown in [8, 13], under some natural assumptions this behaviour can be decomposed in two necessary ingredients:

- 1) *usual hypercyclicity* (we have to require that the operator under study has a dense orbit);
- 2) some kind of *strong recurrence* property (we have to require that each non-empty open subset of the space contains an orbit returning to it with some frequency).

Even though the study of *linear recurrence* began in 2014 with [19], *stronger recurrence* notions were considered only recently in the 2020 work [13], and then in the following 2022 papers [29, 17], where the concepts of \mathcal{F} -*recurrence* and \mathcal{F} -*hypercyclicity* with respect to a Furstenberg family \mathcal{F} are deeply studied. In this section we show that quasi-rigidity can be studied from an \mathcal{F} -recurrence perspective for a very particular kind of Furstenberg families called *free filters* (see Proposition 4.5). Then we study both the weakest and strongest possible \mathcal{F} -recurrence notions together with the Furstenberg families associated to them (see Subsection 4.2), and we finish the section recalling that \mathcal{F} -recurrence behaves well under homomorphisms, quasi-conjugacies and commutants (see Subsection 4.3).

Let us start by recalling the precise definitions, as well as some known examples.

4.1 Definitions, examples and $\mathcal{F}(A)$ -recurrence

A collection of sets $\mathcal{F} \subset \mathcal{P}(\mathbb{N}_0)$ is said to be a *Furstenberg family* (a *family* for short) provided that: each set $A \in \mathcal{F}$ is infinite; \mathcal{F} is hereditarily upward (i.e. $B \in \mathcal{F}$ whenever $A \in \mathcal{F}$ and $A \subset B$); and also $A \cap [n, \infty[\in \mathcal{F}$ for all $A \in \mathcal{F}$ and $n \in \mathbb{N}$. The *dual family* \mathcal{F}^* for a given family \mathcal{F} is the collection of sets $A \subset \mathbb{N}_0$ such that $A \cap B \neq \emptyset$ for every $B \in \mathcal{F}$. The rather vague notion of *frequency* mentioned above will be defined in terms of families.

Given a dynamical system (X, T) , a point $x \in X$ and a non-empty (open) subset $U \subset X$ we denote by

$$N_T(x, U) := \{n \in \mathbb{N}_0 : T^n x \in U\},$$

the *return set from x to U* . It will be denoted by $N(x, U)$ if no confusion with the map arises. Following [8, 9, 12, 13, 15, 16, 17, 29] we can now define:

Definition 4.1. Let (X, T) be a dynamical system and let \mathcal{F} be a Furstenberg family. A point $x \in X$ is said to be:

- \mathcal{F} -*recurrent* if $N(x, U) \in \mathcal{F}$ for every neighbourhood U of x ;
- \mathcal{F} -*hypercyclic* if $N(x, U) \in \mathcal{F}$ for every non-empty open subset $U \subset X$.

Moreover, we will denote by:

- $\mathcal{F}\text{Rec}(T)$ the *set of \mathcal{F} -recurrent points*, and we will say that T is \mathcal{F} -*recurrent* whenever the set $\mathcal{F}\text{Rec}(T)$ is dense in X ;
- $\mathcal{F}\text{HC}(T)$ the *set of \mathcal{F} -hypercyclic points*, and we will say that T is \mathcal{F} -*hypercyclic* whenever the set $\mathcal{F}\text{HC}(T)$ is non-empty.

Example 4.2. The previous Linear Dynamics works [7, 8, 9, 12, 13, 15, 16, 17, 19, 29, 31] have considered the Furstenberg families formed by:

- (a) the *infinite sets*

$$\mathcal{I} = \{A \subset \mathbb{N}_0 : A \text{ is infinite}\}.$$

The notion of \mathcal{I} -recurrence (resp. \mathcal{I} -hypercyclicity) coincides with the concept of recurrence (resp. hypercyclicity) as defined in the Introduction. We denote by $\text{Rec}(T)$ (resp. $\text{HC}(T)$) the set of recurrent (resp. hypercyclic) vectors associated to this family.

- (b) the sets containing arbitrarily long arithmetic progressions

$$\mathcal{AP} := \{A \subset \mathbb{N}_0 : A \text{ contains arbitrarily long arithmetic progressions}\}.$$

In other words, $A \in \mathcal{AP}$ if $\forall l \in \mathbb{N}$, $\exists n, a \in \mathbb{N}$ such that $\{a + kn : 0 \leq k \leq l\} \subset A$, or equivalently A contains the *arithmetic progression* of length $l + 1$, common difference $n \in \mathbb{N}$ and initial term $a \in \mathbb{N}_0$. The concept of \mathcal{AP} -recurrence is equivalent to the notion of (topological) *multiple recurrence*, which requires that for each non-empty open subset $U \subset X$ and each length $l \in \mathbb{N}$ there exists some $n \in \mathbb{N}$ fulfilling that

$$U \cap T^{-n}(U) \cap \dots \cap T^{-ln}(U) \neq \emptyset.$$

This concept was introduced for linear dynamical systems in [18] and further developed in [36] and [16], where \mathcal{AP} -hypercyclicity is also considered. We will denote by $\mathcal{AP}\text{Rec}(T)$ (resp. $\mathcal{AP}\text{HC}(T)$) the set of \mathcal{AP} -recurrent (resp. \mathcal{AP} -hypercyclic) vectors.

- (c) the sets with positive upper Banach density $\overline{\mathcal{BD}} := \{A \subset \mathbb{N}_0 : \overline{\text{Bd}}(A) > 0\}$, where for each $A \subset \mathbb{N}_0$ its upper Banach density is defined as

$$\overline{\text{Bd}}(A) := \lim_{N \rightarrow \infty} \left(\max_{n \geq 0} \frac{\# A \cap [n+1, n+N]}{N} \right). \quad \text{See [28] for alternative definitions.}$$

The notion of $\overline{\mathcal{BD}}$ -recurrence (resp. $\overline{\mathcal{BD}}$ -hypercyclicity) has been called *reiterative recurrence* (resp. *reiterative hypercyclicity*), and $\text{RRec}(T)$ (resp. $\text{RHC}(T)$) is the set of reiteratively recurrent (resp. reiteratively hypercyclic) vectors, see [8, 9, 13, 29].

- (d) the sets with positive upper density $\overline{\mathcal{D}} := \{A \subset \mathbb{N}_0 : \overline{\text{dens}}(A) > 0\}$, where for each $A \subset \mathbb{N}_0$ its upper density is defined as

$$\overline{\text{dens}}(A) := \limsup_{N \rightarrow \infty} \frac{\# A \cap [1, N]}{N}.$$

The $\overline{\mathcal{D}}$ -recurrence (resp. $\overline{\mathcal{D}}$ -hypercyclicity) notion is commonly called *\mathcal{U} -frequent recurrence* (resp. *\mathcal{U} -frequent hypercyclicity*) and $\text{UFRec}(T)$ (resp. $\text{UFHC}(T)$) is the set of \mathcal{U} -frequent recurrent (resp. \mathcal{U} -frequent hypercyclic) vectors, see [42, 8, 9, 13, 29].

- (e) the sets with positive lower density $\underline{\mathcal{D}} := \{A \subset \mathbb{N}_0 : \underline{\text{dens}}(A) > 0\}$, where for each $A \subset \mathbb{N}_0$ its lower density is defined as

$$\underline{\text{dens}}(A) := \limsup_{N \rightarrow \infty} \frac{\# A \cap [1, N]}{N}.$$

The notion of $\underline{\mathcal{D}}$ -recurrence (resp. $\underline{\mathcal{D}}$ -hypercyclicity) is that of *frequent recurrence* (resp. *frequent hypercyclicity*), and $\text{FRec}(T)$ (resp. $\text{FHC}(T)$) is the set of frequently recurrent (resp. frequently hypercyclic) vectors, see [4, 8, 9, 13, 29].

- (f) the syndetic sets

$$\mathcal{S} := \{A \subset \mathbb{N}_0 : A \text{ is syndetic}\},$$

i.e. $A \in \mathcal{S}$ whenever $\exists m \in \mathbb{N}$ such that $\forall a \in \mathbb{N}_0$, $[a, a+m] \cap A \neq \emptyset$. The notion of \mathcal{S} -recurrence has been called *uniform recurrence* and we denote by $\text{URec}(T)$ the set of uniformly recurrent vectors, see [24, 13, 29]. The family \mathcal{S} coincides with the family of sets with positive lower Banach density $\underline{\mathcal{BD}} := \{A \subset \mathbb{N}_0 : \underline{\text{Bd}}(A) > 0\}$, where for each $A \subset \mathbb{N}_0$ its lower Banach density is defined as

$$\underline{\text{Bd}}(A) := \lim_{N \rightarrow \infty} \left(\inf_{n \geq 0} \frac{\# A \cap [n+1, n+N]}{N} \right). \quad \text{See again [28] for alternative definitions.}$$

(g) the *IP-sets*

$$\mathcal{IP} := \{A \subset \mathbb{N}_0 : A \text{ is an IP-set}\},$$

i.e. $A \in \mathcal{IP}$ whenever $\{\sum_{k \in F} n_k : F \subset \mathbb{N} \text{ finite}\} \subset A$ for some sequence of natural numbers $(n_k)_{k \in \mathbb{N}}$. In this case we use the dual family \mathcal{IP}^* , and for the \mathcal{IP}^* -recurrence notion we denote by $\mathcal{IP}^*\text{Rec}(T)$ the set \mathcal{IP}^* -recurrent vectors, see [24, 26, 13, 29].

(h) the Δ -sets

$$\Delta := \{A \subset \mathbb{N}_0 : A \text{ is an } \Delta\text{-set}\},$$

i.e. $A \in \Delta$ whenever $\exists B \in \mathcal{I}$ with $(B - B) \cap \mathbb{N} \subset A$. In this case we again use the dual family Δ^* , and for the Δ^* -recurrence notion we denote by $\Delta^*\text{Rec}(T)$ the set of Δ^* -recurrent vectors, see [24, 29].

We have the following relations between the families listed above:

$$\Delta^* \subset \mathcal{IP}^* \subset \mathcal{S} = \underline{\mathcal{BD}} \subset \underline{\mathcal{D}} \subset \overline{\mathcal{D}} \subset \overline{\mathcal{BD}} \subset \mathcal{AP} \subset \mathcal{I}.$$

Among these inclusions, some of them follow easily from the definitions while others depend on deep theorems like the celebrated *Szemerédi theorem*, see [11, 16, 34, 43]. They imply the respective inclusions between the introduced sets of recurrent vectors

$$\begin{aligned} \Delta^*\text{Rec}(T) \subset \mathcal{IP}^*\text{Rec}(T) \subset \text{URec}(T) \subset \text{FRec}(T) \subset \dots \\ \dots \subset \text{UFRec}(T) \subset \text{RRec}(T) \subset \mathcal{AP}\text{Rec}(T) \subset \text{Rec}(T), \end{aligned}$$

and also between the sets of hypercyclic vectors

$$\text{FHC}(T) \subset \text{UFHC}(T) \subset \text{RHC}(T) \subset \mathcal{AP}\text{HC}(T) \subset \text{HC}(T).$$

It is worth mentioning that the families \mathcal{F} for which there exist \mathcal{F} -hypercyclic operators are by far less common than those for which \mathcal{F} -recurrence exists - which is not surprising since having an orbit distributed around the whole space is much more complicated than having it just coming back around the initial point of the orbit (it is known that there does not exist any Δ^* , \mathcal{IP}^* neither \mathcal{S} -hypercyclic operator, see [8, 13]). Other families have been considered from the \mathcal{F} -hypercyclicity perspective in the works [12, 23]. Our aim now is to show that a dynamical system is *quasi-rigid* if and only if it is \mathcal{F} -recurrent with respect to a *free filter* \mathcal{F} . Let us recall the following classical definitions (see [14]):

Definition 4.3. Let $\mathcal{F} \subset \mathcal{P}(\mathbb{N}_0)$ be a collection of sets of natural numbers (note that \mathcal{F} is not necessarily a Furstenberg family here). We say that \mathcal{F} :

- (a) has the *finite intersection property*, if $\bigcap_{A \in \mathcal{A}} A \neq \emptyset$ for every $\mathcal{A} \subset \mathcal{F}$ with $\#\mathcal{A} < \infty$;
- (b) is a *filter*, if \mathcal{F} is hereditarily upward and for every $A, B \in \mathcal{F}$ we have that $A \cap B \in \mathcal{F}$;
- (c) is a *free filter*, if it is a filter and $\bigcap_{A \in \mathcal{F}} A = \emptyset$.

Finally, given any infinite subset $A \subset \mathbb{N}_0$ we denote by $\mathcal{F}(A)$ the *free filter generated by A*, which is the Furstenberg family

$$\mathcal{F}(A) := \{B \subset \mathbb{N}_0 : \# A \setminus B < \infty\}.$$

In the literature, $\mathcal{F}(A)$ has been called the *Fréchet filter* on A , or also the *eventuality filter* generated by the increasing sequence of integers $(n_k)_{k \in \mathbb{N}}$ forming the set $A \subset \mathbb{N}_0$ (see [14]).

Remark 4.4. It is clear that every *free filter* is a *filter*, and that every *filter* has the *finite intersection property*. However, when \mathcal{F} is a *Furstenberg family* as defined in this paper the previous definitions have some extra (and immediate) **consequences**:

- (a) A *Furstenberg family* \mathcal{F} has the *finite intersection property* if and only if $\bigcap_{A \in \mathcal{A}} A \in \mathcal{I}$ for every $\mathcal{A} \subset \mathcal{F}$ with $\#\mathcal{A} < \infty$: indeed, if \mathcal{F} has the finite intersection property but for some $\mathcal{A} \subset \mathcal{F}$ with $\#\mathcal{A} < \infty$ we had that $\bigcap_{A \in \mathcal{A}} A$ is finite, then for any fixed $n > \max(\bigcap_{A \in \mathcal{A}} A)$ we would arrive to the contradiction:

$$\mathcal{A}' := \{A \cap [n, \infty[: A \in \mathcal{A}\} \subset \mathcal{F} \quad \text{with} \quad \#\mathcal{A}' < \infty \quad \text{but} \quad \bigcap_{A \in \mathcal{A}'} A = \emptyset.$$

- (b) Let \mathcal{F} be a *Furstenberg family*. Then, \mathcal{F} is a *filter* if and only if it is a *free filter*: note that given any B from a filter \mathcal{F} which is also a Furstenberg family we have that

$$\bigcap_{A \in \mathcal{F}} A \subset \bigcap_{n \in \mathbb{N}} B \cap [n, \infty[= \emptyset,$$

by the definition of Furstenberg family used in this paper.

We are now ready to characterize quasi-rigidity in terms of \mathcal{F} -recurrence and free filters:

Proposition 4.5. *Let (X, T) be a dynamical system. The following are equivalent:*

- (i) T is *quasi-rigid*;
- (ii) T is $\mathcal{F}(A)$ -*recurrent* for some infinite subset $A \subset \mathbb{N}_0$;
- (iii) T is \mathcal{F} -*recurrent* with respect to a free filter \mathcal{F} with a countable base.

Moreover, if X is a second-countable space, the previous statements are equivalent to:

- (iv) T is \mathcal{F} -*recurrent* for a family \mathcal{F} with the finite intersection property.

Proof. For (i) \Rightarrow (ii), if T is quasi-rigid with respect to $(n_k)_{k \in \mathbb{N}}$ it is enough to take $A := \{n_k : k \in \mathbb{N}\}$. The implication (ii) \Rightarrow (iii) is obvious since $\mathcal{F}(A)$ has a countable base. To see (iii) \Rightarrow (i) assume that the map T is \mathcal{F} -recurrent and let $(A_k)_{k \in \mathbb{N}}$ be a decreasing base of the free filter \mathcal{F} . Setting $n_k := \min(A_k)$ for each $k \in \mathbb{N}$, and taking a subsequence if necessary, we get an increasing sequence $(n_k)_{k \in \mathbb{N}}$ with respect to which T is quasi-rigid.

We always have that (i), (ii) and (iii) \Rightarrow (iv) even if the space X is not second-countable. Assume now that X is second-countable and that T is \mathcal{F} -recurrent for a family \mathcal{F} with the finite intersection property. Let $\{x_s : s \in \mathbb{N}\} \subset \mathcal{F}\text{Rec}(T)$ be a dense set in X and, for each $s \in \mathbb{N}$, let $(U_{s,k})_{k \in \mathbb{N}}$ be a decreasing neighbourhood basis of x_s . We can now recursively construct an increasing sequence of positive integers $(n_k)_{k \in \mathbb{N}}$ such that

$$T^{n_k} x_s \in U_{s,k} \quad \text{for every } s, k \in \mathbb{N} \text{ with } 1 \leq s \leq k,$$

by taking a sufficiently large integer $n_k \in \bigcap_{s=1}^k N_T(x_s, U_{s,k}) \in \mathcal{I}$. It is easily seen that T is quasi-rigid with respect to $(n_k)_{k \in \mathbb{N}}$, which shows that (iv) \Rightarrow (i), (ii) and (iii). \square

4.2 Appropriate Furstenberg families for \mathcal{F} -recurrence

We have just shown that quasi-rigidity is indeed a particular case of \mathcal{F} -recurrence for a Furstenberg family not listed in Example 4.2. It is then natural to ask the following:

Question 4.6. Are the families of Example 4.2, i.e. $\mathcal{F} = \mathcal{I}, \mathcal{AP}, \overline{\mathcal{BD}}, \overline{\mathcal{D}}, \underline{\mathcal{D}}, \mathcal{S}, \mathcal{IP}^*, \Delta^*$, and the free filters $\mathcal{F}(A)$ the only ones for which \mathcal{F} -recurrence should be considered?

This may seem a very open query. Our objective in this part of the paper is look in detail into two very different classes of Furstenberg families: those for which \mathcal{F} -recurrence is the *weakest* possible recurrence notion, i.e. families $\mathcal{F} \subsetneq \mathcal{I}$ such that $\mathcal{F}\text{Rec}(T) = \text{Rec}(T)$; and those for which \mathcal{F} -recurrence is the *strongest* possible recurrence notion, i.e. periodicity.

Starting with the first of these two classes: *for any dynamical system (X, T) every recurrent point $x \in \text{Rec}(T)$ is an \mathcal{IP} -recurrent point* (this is shown in [24, Theorem 2.17] when X is a *metric space*, and that proof easily extends for arbitrary dynamical systems). Hence $\text{Rec}(T) \subset \mathcal{IP}\text{Rec}(T) \subset \Delta\text{Rec}(T) \subset \mathcal{I}\text{Rec}(T)$. Since $\mathcal{I}\text{Rec}(T) = \text{Rec}(T)$ we always have the equalities

$$\mathcal{IP}\text{Rec}(T) = \Delta\text{Rec}(T) = \text{Rec}(T).$$

This simple relation allow us to give an alternative (and much simpler) proof of the so-called *Ansari and León-Müller recurrence theorems*, which assert that powers and unimodular multiples of an operator share the same set of recurrent vectors (see [19, Proposition 2.3] for the original, different for each of the T^p and λT cases, and rather long proof):

Proposition 4.7 ([19, Proposition 2.3]). *Let (X, T) be a dynamical system. Then:*

- (a) *For every positive integer $p \in \mathbb{N}$ we have that $\text{Rec}(T) = \text{Rec}(T^p)$.*
- (b) *If (X, T) is linear, then for every $\lambda \in \mathbb{T}$ we have that $\text{Rec}(T) = \text{Rec}(\lambda T)$.*

Proof. Suppose that $x \in \text{Rec}(T)$, i.e. that $N_T(x, V)$ belongs to Δ for every neighbourhood V of x . Fixed any $p \in \mathbb{N}$ and any neighbourhood U of x , since the set $(p \cdot \mathbb{N}_0)$ belongs to Δ^* we have that

$$\emptyset \neq N_T(x, U) \cap (p \cdot \mathbb{N}_0) \subset N_{T^p}(x, U),$$

which implies that $x \in \text{Rec}(T^p)$, proving (a). To check (b) assume moreover that (X, T) is a linear dynamical system and fix any $\lambda \in \mathbb{T}$. Let $\varepsilon > 0$ and let $U_0 \subset U$ be a neighbourhood of x such that $\mu \cdot U_0 \subset U$ for all $|\mu - 1| < \varepsilon$. Since the set $\{n \in \mathbb{N} : |\lambda^n - 1| < \varepsilon\}$ belongs to Δ^* (see for instance [29, Proposition 4.1]) we have that

$$\emptyset \neq N_T(x, U_0) \cap \{n \in \mathbb{N} : |\lambda^n - 1| < \varepsilon\} \subset N_{\lambda T}(x, U),$$

which implies that $x \in \text{Rec}(\lambda T)$. □

Let us now look at the *strongest* possible recurrence notion by using families of big subsets of natural numbers, which have been also historically considered in (linear) dynamics:

Example 4.8. Consider the Furstenberg families formed by:

- (a) the *cofinite sets* $\mathcal{I}^* := \{A \subset \mathbb{N}_0 : A \text{ is cofinite}\}$, which is the dual family of that formed by the infinite sets \mathcal{I} , and which is used to define the notion of (topological) *mixing*: a dynamical system (X, T) is called *mixing* if the set $\{n \in \mathbb{N}_0 : T^n(U) \cap V \neq \emptyset\}$ belongs to \mathcal{I}^* for every pair of non-empty open subset $U, V \subset X$ (see [31, Definition 1.38]).

(b) the *thick sets*

$$\mathcal{T} := \{A \subset \mathbb{N}_0 : A \text{ is thick}\},$$

i.e. $A \in \mathcal{T}$ if $\forall m \in \mathbb{N}$, $\exists a_m \in A$ with $[a_m, a_m + m] \subset A$. This family is known to characterize the *weak-mixing* property (see [30, Theorem 3] or [31, Theorem 1.54]): *for every dynamical system (X, T) the following statements are equivalent:*

- (i) (X, T) is *weakly mixing*;
- (ii) the set $\{n \in \mathbb{N}_0 : T^n(U) \cap V \neq \emptyset\}$ belongs to \mathcal{T} for every pair of non-empty open subsets $U, V \subset X$.

(c) the *thickly syndetic sets*

$$\mathcal{TS} := \{A \subset \mathbb{N}_0 : A \text{ is thickly syndetic}\},$$

i.e. $A \in \mathcal{TS}$ if $\forall m \in \mathbb{N}$, $\exists A_m$ syndetic such that $A_m + [0, m] \subset A$. This family characterizes the *topological ergodicity* (i.e. the property that $\{n \in \mathbb{N}_0 : T^n(U) \cap V \neq \emptyset\}$ is syndetic for every pair of non-empty open subsets $U, V \subset X$) for **linear** dynamical systems (see [31, Exercise 2.5.4] and [9]): *for every linear dynamical system (X, T) the following statements are equivalent:*

- (i) (X, T) is *topologically ergodic*;
- (ii) the set $\{n \in \mathbb{N}_0 : T^n(U) \cap V \neq \emptyset\}$ belongs to \mathcal{TS} for every pair of non-empty open subsets $U, V \subset X$.

(d) the *sets of density greater than $\delta > 0$* , which for each $0 < \delta \leq 1$ are the families

$$\begin{aligned} \underline{\mathcal{BD}}_\delta &:= \{A \subset \mathbb{N}_0 : \underline{\text{Bd}}(A) \geq \delta\}, & \overline{\mathcal{BD}}_\delta &:= \{A \subset \mathbb{N}_0 : \overline{\text{Bd}}(A) \geq \delta\}, \\ \underline{\mathcal{D}}_\delta &:= \{A \subset \mathbb{N}_0 : \underline{\text{dens}}(A) \geq \delta\}, & \overline{\mathcal{D}}_\delta &:= \{A \subset \mathbb{N}_0 : \overline{\text{dens}}(A) \geq \delta\}. \end{aligned}$$

The density families have also been studied in Linear Dynamics, and we refer the reader to [8, 9] for more about them.

We are about to show that the recurrence notions associated to the Furstenberg families introduced in Example 4.8 above imply different periodicity notions (see Proposition 4.11). We replicate the arguments from [8, Proposition 3], which have been used in an independent way in [17, Lemma 2.14 and Corollary 2.15]. Our contribution here is to rewrite these results in their “*pointwise \mathcal{F} -recurrence*” version. We start by proving two key lemmas:

Lemma 4.9. *Let (X, T) be a dynamical system. Given $x \in X$ and $N \in \mathbb{N}$ the following statements are equivalent:*

- (i) $x \in \text{Per}(T)$ and its period is strictly lower than N ;
- (ii) for every neighbourhood U of x there exists $n_U \in \mathbb{N}_0$ such that

$$\# N(x, U) \cap [n_U + 1, n_U + N] \geq 2;$$

- (iii) for every neighbourhood U of x we have that $T^p(U) \cap U \neq \emptyset$ for some $1 \leq p < N$.

Proof. For (i) \Rightarrow (ii) let n_U be the period of x minus one. For (ii) \Rightarrow (iii) recall that

$$\text{given } n_1 < n_2 \in N_T(x, U) \quad \text{we have that} \quad T^{n_2 - n_1}(U) \cap U \neq \emptyset.$$

Conversely, if we suppose that $T^p x \neq x$ for $1 \leq p < N$, by continuity we can find an open neighbourhood U of x such that $T^p(U) \cap U = \emptyset$ for $1 \leq p < N$, so (iii) \Rightarrow (i). \square

Lemma 4.10. *Let $0 < \delta \leq 1$ and $N \in \mathbb{N}$ with $\frac{1}{N} < \delta$. Then, for every $A \in \overline{\mathcal{BD}}_\delta$ there is $n_A \in \mathbb{N}_0$ such that*

$$\# A \cap [n_A + 1, n_A + N] \geq 2.$$

Proof. Otherwise we would have that $\# A \cap [n + 1, n + N] \leq 1$ for every $n \in \mathbb{N}_0$ obtaining the contradiction

$$\overline{\text{Bd}}(A) = \lim_{K \rightarrow \infty} \left(\max_{n \geq 0} \frac{\# A \cap [n + 1, n + (N \cdot K)]}{N \cdot K} \right) \leq \lim_{K \rightarrow \infty} \frac{K}{N \cdot K} = \frac{1}{N} < \delta. \quad \square$$

Finally we get the desired result, in which we classify the periodic points of a dynamical system in terms of the density of the return sets:

Proposition 4.11. *Let (X, T) be a dynamical system and let $x \in X$. Then:*

(a) *Given $0 < \delta \leq 1$, the following statements are equivalent:*

- (i) $x \in \text{Per}(T)$ and its period is lower or equal to $\lfloor \frac{1}{\delta} \rfloor$;
- (ii) $x \in \underline{\mathcal{BD}}_\delta \text{Rec}(T)$;
- (iii) $x \in \underline{\mathcal{D}}_\delta \text{Rec}(T)$;
- (iv) $x \in \overline{\mathcal{D}}_\delta \text{Rec}(T)$;
- (v) $x \in \overline{\mathcal{BD}}_\delta \text{Rec}(T)$.

In particular, if T is $\overline{\mathcal{BD}}_\delta$ -recurrent then $T^N = I$ for $N = 1 \cdot 2 \cdot \dots \cdot (\lfloor \frac{1}{\delta} \rfloor - 1) \cdot \lfloor \frac{1}{\delta} \rfloor$.

(b) *The following statements are equivalent:*

- (i) x is a fixed point, i.e. $Tx = x$;
- (ii) $x \in \mathcal{I}^* \text{Rec}(T)$;
- (iii) $x \in \mathcal{TS} \text{Rec}(T)$;
- (iv) $x \in \mathcal{T} \text{Rec}(T)$;
- (v) $x \in \overline{\mathcal{BD}}_\delta \text{Rec}(T)$ for some $\delta > \frac{1}{2}$;
- (vi) for every neighbourhood U of x the set $N(x, U)$ contains two consecutive integers.

Proof. (a): Let us first show that (i) \Rightarrow (ii): let $p \in \{1, 2, \dots, \lfloor \frac{1}{\delta} \rfloor\}$ be such that $T^p x = x$. Then

$$\underline{\text{Bd}}(N(x, U)) \geq \underline{\text{Bd}}(p \cdot \mathbb{N}_0) = \frac{1}{p} \geq \delta$$

for every neighbourhood U of x , so $N(x, U) \in \overline{\mathcal{BD}}_\delta$. For (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) recall that $\underline{\mathcal{BD}}_\delta \subset \underline{\mathcal{D}}_\delta \subset \overline{\mathcal{D}}_\delta \subset \overline{\mathcal{BD}}_\delta$. For (v) \Rightarrow (i) let $N := \lfloor \frac{1}{\delta} \rfloor + 1$. By Lemma 4.10, for every neighbourhood U of x there is $n_U \in \mathbb{N}$ such that

$$\# N(x, U) \cap [n_U + 1, n_U + N] \geq 2,$$

so Lemma 4.9 finishes the work.

(b): The implications (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) follow from the following well-known relations $\mathbb{N}_0 \in \mathcal{I}^* \subset \mathcal{TS} \subset \mathcal{T} = \overline{\mathcal{BD}}_1 \subset \overline{\mathcal{BD}}_\delta$ for every $0 < \delta \leq 1$ (see [11, 34]). In order to prove (v) \Rightarrow (vi) use Lemma 4.10 applied to $N = 2$. Finally, the equivalence (vi) \Leftrightarrow (i) follows from Lemma 4.9. \square

Statement (a) of Proposition 4.11 shows that we should restrict ourselves to study the \mathcal{F} -recurrence notions not implying periodicity (as those of Example 4.2): recall that $p \cdot \mathbb{N}_0$ belongs to the family Δ^* for each $p \in \mathbb{N}$, so $\text{Per}(T) \subset \Delta^*\text{Rec}(T)$ for every system (X, T) and Δ^* -recurrence is weaker than periodicity. We have reproved [8, Proposition 3]:

Corollary 4.12. *There is no $\overline{\mathcal{BD}}_\delta$ -hypercyclic operator for any $0 < \delta \leq 1$.*

4.3 Quasi-conjugacies and commutants in \mathcal{F} -recurrence

We end Section 4 by presenting some basic but useful tools regarding *quasi-conjugacies* and (non-linear) *commutants* which will be used in Section 5 to study the *lineability* and *dense lineability* properties of the set of \mathcal{F} -recurrent vectors. Following [24, 31] we define:

Definition 4.13. Given two dynamical systems (X, T) and (Y, S) , we say that a map $\phi : Y \rightarrow X$ is an *homomorphism of dynamical systems* if ϕ is continuous and the diagram

$$\begin{array}{ccc} Y & \xrightarrow{S} & Y \\ \phi \downarrow & & \downarrow \phi \\ X & \xrightarrow{T} & X \end{array}$$

commutes, i.e. $\phi \circ S = T \circ \phi$. Moreover, we say that the map ϕ is a:

- *quasi-conjugacy*, if ϕ has dense range, and hence that (X, T) is *quasi-conjugate* to (Y, S) ;
- *conjugacy*, if ϕ is an homeomorphism between X and Y (see Remark 5.2).

Many dynamical properties are preserved by quasi-conjugacy (see [31, Chapter 1]): if (Y, S) admits a dense orbit, is topologically transitive, weakly-mixing or even (Devaney) chaotic, then so is (X, T) . On the other hand, to preserve the *recurrence* of a point it is enough to have an homomorphism. This is a really well-known fact (see [24, Proposition 1.3]), and indeed similar arguments were already used in [24, Proposition 9.9] for \mathcal{F} -recurrence with respect to the Furstenberg families Δ^* and \mathcal{IP}^* , or in [17, Proposition 2.7] for general Furstenberg families. We include here the general argument regarding return sets:

Lemma 4.14. *Let (X, T) and (Y, S) be dynamical systems and let $\phi : Y \rightarrow X$ be an homomorphism between them. Given $y \in Y$ and any neighbourhood $U \subset X$ of $\phi(y)$, the set $V := \phi^{-1}(U) \subset Y$ is a neighbourhood of y for which $N_S(y, V) = N_T(\phi(y), U)$. In particular, for any Furstenberg family \mathcal{F} we have that:*

- (a) *If $y \in Y$ is \mathcal{F} -recurrent for S then $\phi(y)$ is \mathcal{F} -recurrent for T .*
- (b) *If ϕ is a quasi-conjugacy and S is \mathcal{F} -recurrent then so is T .*

Proof. By continuity of ϕ the set $V = \phi^{-1}(U)$ is a neighbourhood of y . Since $\phi \circ S = T \circ \phi$ we have that $\phi(S^n y) = T^n \phi(y)$ for every $n \in \mathbb{N}_0$. It follows that

$$N_T(\phi(y), U) = \{n \in \mathbb{N}_0 : T^n \phi(y) \in U\} = \{n \in \mathbb{N}_0 : S^n y \in \phi^{-1}(U)\} = N_S(y, V).$$

In particular, if y is \mathcal{F} -recurrent we have that $N_T(\phi(y), U) = N_S(y, V) \in \mathcal{F}$ for any neighbourhood U of $\phi(y)$. Statement (b) follows immediately from (a). \square

Considering $Y := X$, $S := T$ and $\phi := S$ we obtain the following:

Corollary 4.15. *Let (X, T) be a dynamical system and let $S : X \rightarrow X$ be a continuous map commuting with T , that is $S \circ T = T \circ S$. Given $x \in X$ and any neighbourhood U of Sx , the set $V := S^{-1}(U)$ is a neighbourhood of x for which $N_T(x, V) = N_T(Sx, U)$. In particular, for any Furstenberg family \mathcal{F} we have that*

$$S(\mathcal{F}\text{Rec}(T)) \subset \mathcal{F}\text{Rec}(T).$$

Corollary 4.15 unifies some of the well-known arguments already used in [1, Proposition 2.5], [19, Proof of Theorem 9.1] and [24, Propositions 1.3 and 9.9], generalizing them to arbitrary Furstenberg families and (not necessarily linear) maps acting on arbitrary topological spaces. From now on we will write

$$\mathcal{C}(X) := \{S : X \rightarrow X \text{ continuous map}\},$$

i.e. the *set of continuous* (and not necessarily linear) *self-maps* of a Hausdorff topological space X , and following the notation of [35] we define:

Definition 4.16. Let (X, T) be a dynamical system. The (non-linear) *commutant* of T is the set

$$\mathcal{C}_T := \{S \in \mathcal{C}(X) : S \circ T = T \circ S\}.$$

Given $x \in X$ we define the \mathcal{C}_T -orbit of x as $\mathcal{C}_T(x) := \{Sx : S \in \mathcal{C}_T\}$. We say that a subset $Y \subset X$ is \mathcal{C}_T -invariant if $S(Y) \subset Y$ for every $S \in \mathcal{C}_T$.

Remark 4.17. Since $T \in \mathcal{C}_T$ we always have that:

- every \mathcal{C}_T -invariant set is also T -invariant;
- if X is a linear space, then $\{p(T) : p \text{ polynomial}\} \subset \mathcal{C}_T$ and for every $x \in X$ we get that

$$\text{span}(\text{Orb}(x, T)) = \text{span}\{T^n x : n \in \mathbb{N}_0\} = \{p(T)x : p \text{ polynomial}\} \subset \mathcal{C}_T(x).$$

Moreover, in this case every \mathcal{C}_T -orbit is a vector subspace of X :

Lemma 4.18. *Let (X, T) be a linear dynamical system. Then $(\mathcal{C}_T, +, \circ)$ is a subring of $(\mathcal{C}(X), +, \circ)$. In particular, given any $x \in X$ the set $\mathcal{C}_T(x)$ is a vector subspace of X , and it is the smallest \mathcal{C}_T -invariant subset of X containing the point x .*

Proof. Given $S, R \in \mathcal{C}_T$ and $\alpha, \beta \in \mathbb{K}$ it is easily checked that $(\alpha S + \beta R) \in \mathcal{C}_T$ and also $(S \circ R) \in \mathcal{C}_T$. In particular, given $x \in X$ the set $\mathcal{C}_T(x)$ is a vector subspace of X . Moreover, if $Y \subset X$ is a \mathcal{C}_T -invariant subset of X and $x \in Y$ then

$$\mathcal{C}_T(x) = \{Sx : S \in \mathcal{C}_T\} \subset \bigcup_{S \in \mathcal{C}_T} S(Y) \subset Y. \quad \square$$

When (X, T) is a non-linear dynamical system, it remains true that $\mathcal{C}_T(x)$ is the *smallest \mathcal{C}_T -invariant subset of X containing the point $x \in X$* . We now generalize Corollary 4.15 to direct products. We will use the following notation: given a dynamical system (X, T) and a subset $Y \subset X$ we denote by

$$\mathcal{N}_{pr}(Y) := \{A \subset \mathbb{N}_0 : N(x, U) \subset A \text{ for some } x \in Y \text{ and some neighbourhood } U \text{ of } x\},$$

the *family of pointwise-recurrent return sets* of the points of Y . Note that $\mathcal{N}_{pr}(\{x\})$ is a filter, for a point $x \in X$, if and only if $x \in \text{Rec}(T)$; and that a dynamical system (X, T) is \mathcal{F} -recurrent whenever there is a dense set $Y \subset X$ such that $\mathcal{N}_{pr}(Y) \subset \mathcal{F}$.

Theorem 4.19. *Let (X, T) be a (linear) dynamical system and let \mathcal{F} be a Furstenberg family. Given $x \in \mathcal{F}\text{Rec}(T)$ we have that*

$$\mathcal{N}_{pr}(\mathcal{C}_T(x)) = \mathcal{N}_{pr}(\{x\}),$$

and hence $\mathcal{C}_T(x)$ is a \mathcal{C}_T -invariant (vector) subspace of X with the property that

$$\mathcal{C}_T(x)^N \subset \mathcal{F}\text{Rec}(T_N) \quad \text{for every } N \in \mathbb{N}.$$

Proof. Obviously $\mathcal{N}_{pr}(\{x\}) \subset \mathcal{N}_{pr}(\mathcal{C}_T(x))$. The inclusion $\mathcal{N}_{pr}(\mathcal{C}_T(x)) \subset \mathcal{N}_{pr}(\{x\})$ follows from Corollary 4.15: given $y \in \mathcal{C}_T(x)$ and any neighbourhood U of y there is a neighbourhood V of x such that

$$N_T(y, U) = N_T(x, V) \in \mathcal{N}_{pr}(\{x\}).$$

The rest of the result follows from the filter condition satisfied by the family $\mathcal{N}_{pr}(\{x\})$: indeed, given any $N \in \mathbb{N}$ and a set $\{x_1, x_2, \dots, x_N\} \subset \mathcal{C}_T(x)$ write

$$z := (x_1, x_2, \dots, x_N) \in X^N.$$

Then, given any neighbourhood $U \subset X^N$ of z we can find neighbourhoods $U_i \subset X$ of x_i , for $1 \leq i \leq N$, such that $U_1 \times \dots \times U_N \subset U$, and hence

$$N_{T_N}(z, U) \supset \bigcap_{i=1}^N N_T(x_i, U_i) \in \mathcal{N}_{pr}(\{x\}) \subset \mathcal{F}.$$

The arbitrariness of U implies that $z \in \mathcal{F}\text{Rec}(T_N)$. Finally, if (X, T) is a linear dynamical system, then the set $\mathcal{C}_T(x)$ is a \mathcal{C}_T -invariant vector subspace of X by Lemma 4.18. \square

Remark 4.20. Theorem 4.19 is an extension of Corollary 4.15 to every N -fold direct product of a given dynamical system. As we have already mentioned, these kind of arguments were already used in [19, Theorem 9.1] for usual recurrence and operators on Banach spaces, and in [17, Proposition 2.7] for arbitrary Furstenberg families.

5 Infinite-dimensional vector spaces in $\mathcal{F}\text{Rec}(T)$

The objective of this section is to study, for a linear dynamical system (X, T) and arbitrary Furstenberg families \mathcal{F} , when the set of \mathcal{F} -recurrent vectors is *lineable* or *dense lineable*, i.e. to establish whether it contains a (possibly dense) infinite-dimensional vector subspace. Theorem 4.19 will be our main tool for this study, which motivation stems from the following two facts already shown in previous sections:

- quasi-rigidity coincides with $\mathcal{F}(A)$ -recurrence (see Proposition 4.5);
- quasi-rigidity of an operator T implies that $\text{Rec}(T)$ is dense lineable (see Proposition 2.7).

It is then natural to ask whether the set $\mathcal{F}\text{Rec}(T)$ contains an infinite-dimensional vector space for other Furstenberg families \mathcal{F} . We show that $\mathcal{F}\text{Rec}(T)$ is lineable as soon as T is \mathcal{F} -recurrent (see Theorem 5.5 and Corollary 5.6 below), and we obtain some (natural) sufficient conditions implying that $\mathcal{F}\text{Rec}(T)$ is dense lineable (see Theorem 5.8). As a consequence we obtain the *Herrero-Bourdon theorem* for \mathcal{F} -hypercyclicity (see Subsection 5.3).

5.1 Lineability

Given a vector space X and a subset of vectors $Y \subset X$ with some property, we say that Y is *lineable* if there exists an infinite-dimensional vector subspace $Z \subset X$ such that $Z \setminus \{0\} \subset Y$. In our case, given a linear dynamical system (X, T) we always have that $0 \in \mathcal{F}\text{Rec}(T)$ for every Furstenberg family \mathcal{F} , so the set of \mathcal{F} -recurrent vectors will be lineable if it admits an infinite-dimensional vector space (which includes the zero-vector).

In order to prove that $\mathcal{F}\text{Rec}(T)$ is lineable as soon as (X, T) is \mathcal{F} -recurrent, we will observe that the (span of the) *unimodular eigenvectors* are the only recurrent vectors whose orbit has a finite-dimensional linear span (see Lemma 5.3 below). Recall that:

Definition 5.1. Given a **complex-linear** dynamical system $T : X \rightarrow X$, a vector $x \in X$ is called a *unimodular eigenvector* for T if $x \neq 0$ and $Tx = \lambda x$ for some unimodular complex number $\lambda \in \mathbb{T}$. We denote by $\mathcal{E}(T)$ the *set of unimodular eigenvectors* for T , i.e.

$$\mathcal{E}(T) = \{x \in X \setminus \{0\} : Tx = \lambda x \text{ for some } \lambda \in \mathbb{T}\}.$$

Every finite linear combination of unimodular eigenvectors is a Δ^* -recurrent vector (see for instance [29, Proposition 4.1]) and the following holds (see [31, Proposition 2.33]):

$$\text{Per}(T) = \text{span}\{x \in X : Tx = e^{\alpha\pi i}x \text{ for some } \alpha \in \mathbb{Q}\} \subset \text{span}(\mathcal{E}(T)) \subset \Delta^*\text{Rec}(T).$$

In order to treat both real and complex cases at the same time, we need a set of (**real**) vectors having an analogous recurrent-behaviour to that of unimodular eigenvectors:

Remark 5.2 (A well-known conjugacy). The *complexification* (\tilde{X}, \tilde{T}) of a **real-linear** system $T : X \rightarrow X$ is defined in the following way (see [40, 39] and [31, Exercise 2.2.7]):

- the space $\tilde{X} := \{x + iy : x, y \in X\}$, which is topologically identified with $X \oplus X$, and becomes a **complex** F-space when endowed with the multiplication by complex scalars $(\alpha + i\beta)(x + iy) = (\alpha x - \beta y) + i(\alpha y + \beta x)$ for every $\alpha, \beta \in \mathbb{R}$ and every $x, y \in X$;
- and the operator $\tilde{T} : \tilde{X} \rightarrow \tilde{X}$ is defined as $\tilde{T}(x + iy) = Tx + iTy$ for every $x, y \in X$. It is a continuous **complex-linear** operator acting on \tilde{X} .

Defining the map $J : X \oplus X \rightarrow \tilde{X}$ as $J(x, y) := x + iy \in \tilde{X}$ for every $(x, y) \in X \oplus X$, the diagram

$$\begin{array}{ccc} X \oplus X & \xrightarrow{T \oplus T} & X \oplus X \\ J \downarrow & & \downarrow J \\ \tilde{X} & \xrightarrow{\tilde{T}} & \tilde{X} \end{array}$$

commutes so J is a conjugacy (see Definition 4.13). In this setting we define the (**real**) *set of unimodular eigenvectors* for T as

$$\mathcal{E}(T) := \left\{ x \in X : \text{there exists } y \in X \text{ such that } x + iy \in \mathcal{E}(\tilde{T}) \right\}.$$

These **real** unimodular eigenvectors have the same recurrent-behaviour than the **complex** unimodular eigenvectors. In fact, the following properties are easily checked:

$$\text{span}(\mathcal{E}(T)) = \left\{ x \in X : \text{there exists } y \in X \text{ such that } x + iy \in \text{span}(\mathcal{E}(\tilde{T})) \right\},$$

$$\text{Per}(T) \subset \text{span}(\mathcal{E}(T)) \subset \Delta^*\text{Rec}(T).$$

Lemma 5.3. *Let (X, T) be a (real or complex) linear dynamical system. For each recurrent vector $x \in \text{Rec}(T)$ the following statements are equivalent:*

- (i) $x \in \text{span}(\mathcal{E}(T))$;
- (ii) $\dim(\text{span}\{T^n x : n \in \mathbb{N}_0\}) < \infty$.

In particular, given any Furstenberg family \mathcal{F} for which $\mathcal{F}\text{Rec}(T) \setminus \text{span}(\mathcal{E}(T)) \neq \emptyset$, there exists an infinite-dimensional T -invariant vector subspace $Z \subset X$ with the property that $Z^N \subset \mathcal{F}\text{Rec}(T_N)$ for every $N \in \mathbb{N}$.

Proof. For (i) \Rightarrow (ii) compute the orbit of a vector from $\text{span}(\mathcal{E}(T))$ in both real and complex cases, and observe that its span is finite-dimensional. For (ii) \Rightarrow (i) let $x \in \text{Rec}(T)$ and suppose that the T -invariant subspace $E = \text{span}\{T^n x : n \in \mathbb{N}_0\}$ is finite-dimensional:

- (1) If (X, T) is complex, then $T|_E : E \rightarrow E$ is a recurrent complex-linear operator on a finite-dimensional space. By [19, Theorem 4.1] there exists a basis of E formed by unimodular eigenvectors for $T|_E$ (and hence for T) so $x \in E \subset \text{span}(\mathcal{E}(T))$.
- (2) If (X, T) is real, then we can identify the complexification $\widetilde{T|_E} : \widetilde{E} \rightarrow \widetilde{E}$ with the direct sum $T|_E \oplus T|_E$ (see Remark 5.2). Theorem 4.19 implies that

$$E \oplus E \subset \text{Rec}(T|_E \oplus T|_E),$$

so $\widetilde{T|_E}$ is a recurrent complex-linear operator on a finite-dimensional space. As in case (1) above, by [19, Theorem 4.1] there exists a basis of \widetilde{E} formed by unimodular eigenvectors for \widetilde{T} , so $x + i0 \in \widetilde{E} \subset \text{span}(\mathcal{E}(\widetilde{T}))$ and hence $x \in \text{span}(\mathcal{E}(T))$.

Finally, if \mathcal{F} is a Furstenberg family for which there exists some $x \in \mathcal{F}\text{Rec}(T) \setminus \text{span}(\mathcal{E}(T))$, the equivalence (i) \Leftrightarrow (ii) implies that $Z := \text{span}\{T^n x : n \in \mathbb{N}_0\}$ is an infinite-dimensional T -invariant vector subspace of X . Theorem 4.19 then shows

$$Z^N \subset \mathcal{C}_T(x)^N \subset \mathcal{F}\text{Rec}(T_N) \quad \text{for every } N \in \mathbb{N}. \quad \square$$

The strongest \mathcal{F} -recurrence notion considered in this paper and fulfilling the condition that $\text{span}(\mathcal{E}(T)) \subset \mathcal{F}\text{Rec}(T)$ for every $T \in \mathcal{L}(X)$ is that of Δ^* -recurrence. The properties of *having a spanning set of unimodular eigenvectors* and that of *being Δ^* -recurrent* have been deeply related in the recent work [29] and they are specially near when one considers power-bounded operators (see [29, Theorem 1.9]). It is then natural to ask if the equality $\text{span}(\mathcal{E}(T)) = \Delta^*\text{Rec}(T)$ holds for some general class of linear dynamical systems (X, T) , a natural candidate being the class of power-bounded operators. The answer is negative as the following easy example shows by using Lemma 5.3:

Example 5.4. *There exists a linear dynamical system admitting a Δ^* -recurrent vector such that the linear span of its orbit is infinite-dimensional: consider $X = c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$ for some $1 \leq p < \infty$ with their usual norms. Define a linear map $T : c_{00}(\mathbb{N}) \rightarrow c_{00}(\mathbb{N})$ in the following way: $Te_1 = e_1$ and for every $k \geq 2$*

$$Te_k = \begin{cases} e_{k+1}, & \text{if } 2^m < k < 2^{m+1}, \\ e_{2^{m+1}}, & \text{if } k = 2^{m+1}, \end{cases}$$

for each $m \geq 0$, where $e_k = \{\delta_{k,n}\}_{n=1}^\infty$ is the k -th vector of the canonical basis of X . Since $\|Tx\| = \|x\|$ for each $x \in c_{00}(\mathbb{N})$, T extends to a *linear isometry* on the whole space X . We now show that the vector $x = \sum_{m \geq 0} \frac{1}{2^m} e_{2^{m+1}} \in X$ has the required properties:

- (1) $x \in \Delta^*\text{Rec}(T)$: given any $\varepsilon > 0$ there is $m_\varepsilon \in \mathbb{N}$ such that $\left\| \sum_{m > m_\varepsilon} \frac{1}{2^m} e_{2^m+1} \right\| < \frac{\varepsilon}{2}$. Then, given $n \in 2^{m_\varepsilon} \cdot \mathbb{N}_0$ we have that

$$T^n \left(\sum_{m \leq m_\varepsilon} \frac{1}{2^m} e_{2^m+1} \right) = \sum_{m \leq m_\varepsilon} \frac{1}{2^m} e_{2^m+1} \quad \text{so} \quad \|T^n x - x\| \leq 2 \left\| \sum_{m > m_\varepsilon} \frac{1}{2^m} e_{2^m+1} \right\| < \varepsilon.$$

Hence $2^{m_\varepsilon} \cdot \mathbb{N}_0 \subset N_T(x, B(x, \varepsilon))$, so $x \in \Delta^*\text{Rec}(T)$ since $2^m \cdot \mathbb{N}_0 \in \Delta^*$ for all $m \in \mathbb{N}$.

- (2) $\text{span}\{T^n x : n \in \mathbb{N}_0\}$ is infinite-dimensional: otherwise there would exist a polynomial

$$p(z) = \sum_{i=1}^N a_i z^i \quad \text{with } a_i \in \mathbb{K} \text{ and } a_N \neq 0 \quad \text{such that} \quad p(T)x = 0,$$

and taking $m_0 \in \mathbb{N}$ such that $N < 2^{m_0}$ we would arrive to the contradiction

$$0 = [p(T)x]_{2^{m_0+1}+N} = [a_N \cdot T^N x]_{2^{m_0+1}+N} = \frac{a_N}{2^{m_0}} \neq 0.$$

We finally state the desired *lineability* results:

Theorem 5.5 (Lineability). *Let \mathcal{F} be a Furstenberg family such that the inclusion $\text{span}(\mathcal{E}(S)) \subset \mathcal{F}\text{Rec}(S)$ holds for every linear dynamical system (Y, S) . Then, for each \mathcal{F} -recurrent linear system (X, T) there exists an infinite-dimensional T -invariant vector subspace $Z \subset X$ with the property that $Z^N \subset \mathcal{F}\text{Rec}(T_N)$ for every $N \in \mathbb{N}$.*

Proof. By assumption $\mathcal{F}\text{Rec}(T)$ is dense. We distinguish two cases:

- (1) If $\text{span}(\mathcal{E}(T))$ is finite-dimensional (or simply if it is not dense in X), then we have that $\mathcal{F}\text{Rec}(T) \setminus \text{span}(\mathcal{E}(T)) \neq \emptyset$ and Lemma 5.3 yields the desired vector subspace.
- (2) If $\text{span}(\mathcal{E}(T))$ is infinite-dimensional, then $(\text{span}(\mathcal{E}(T)))^N = \text{span}(\mathcal{E}(T_N)) \subset \mathcal{F}\text{Rec}(T_N)$ for every $N \in \mathbb{N}$, and $Z := \text{span}(\mathcal{E}(T))$ is the required vector subspace. \square

The previous theorem is “optimal” for the families considered in this paper. Indeed, we always have the inclusions

$$\text{Per}(T) \subset \text{span}(\mathcal{E}(T)) \subset \Delta^*\text{Rec}(T),$$

and we observed in Section 4 that the families \mathcal{F} considered in this paper for which $\mathcal{F}\text{Rec}(T) \subset \Delta^*\text{Rec}(T)$ are such that $\mathcal{F}\text{Rec}(T) \subset \text{Per}(T)$, for every system (X, T) . We deduce (at least) the following result:

Corollary 5.6. *Let (X, T) be a linear dynamical system. If the operator T is recurrent (resp. \mathcal{AP} , reiterative, \mathcal{U} -frequent, frequent, uniformly, \mathcal{IP}^* or Δ^* -recurrent), then $\text{Rec}(T)$ (resp. $\mathcal{AP}\text{Rec}(T)$, $\text{RRec}(T)$, $\text{UFRec}(T)$, $\text{FRec}(T)$, $\text{URec}(T)$, $\mathcal{IP}^*\text{Rec}(T)$ or $\Delta^*\text{Rec}(T)$) contains an infinite-dimensional T -invariant vector space, and in particular it is lineable.*

Theorem 5.5 (and hence Corollary 5.6) is still true if we replace the original assumption that “ T is \mathcal{F} -recurrent” by the much less restrictive hypothesis that “the set $\mathcal{F}\text{Rec}(T)$ is dense in some infinite-dimensional closed subspace $Y \subset X$ ”, which is also a necessary condition for the lineability property. If this is not satisfied the following could happen:

Example 5.7. Let $X = c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$ for some $1 \leq p < \infty$ with their usual norm. Consider a bounded sequence $(\lambda_n)_{n \in \mathbb{N}} \subset \mathbb{C}$ and the respective multiplication operator defined as

$$T((x_n)_{n \in \mathbb{N}}) := (\lambda_n \cdot x_n)_{n \in \mathbb{N}} \quad \text{for each } (x_n)_{n \in \mathbb{N}} \in X.$$

For any fixed any $N \in \mathbb{N}$, consider a sequence $(\lambda_n)_{n \in \mathbb{N}}$ with the properties that

$$\lambda_n = 1 \text{ for } n \leq N \quad \text{and} \quad \lambda_n \notin \mathbb{T} \text{ for } n > N.$$

Then it is trivial to check that, for any Furstenberg family \mathcal{F} , the set $\mathcal{F}\text{Rec}(T)$ is exactly a finite-dimensional vector space of dimension N .

5.2 Dense Lineability

Given a topological vector space X and a subset of vectors $Y \subset X$ with some property, we say that Y is *dense lineable* if there exists a dense infinite-dimensional vector subspace $Z \subset X$ such that $Z \setminus \{0\} \subset Y$. As it happens for the *lineability* condition, given a linear dynamical system (X, T) we always have that $0 \in \mathcal{F}\text{Rec}(T)$ for every Furstenberg family \mathcal{F} , so the set of \mathcal{F} -recurrent vectors will be dense lineable if it admits a dense infinite-dimensional vector space (which includes the zero-vector).

Given a linear dynamical system (X, T) we provide some sufficient conditions for the set $\mathcal{F}\text{Rec}(T)$ to be dense lineable. Filters will play a fundamental role in our study. The motivation for our main result (see Theorem 5.8 below) is again the notion of quasi-rigidity:

- quasi-rigidity implies that $\text{Rec}(T) = \mathcal{I}\text{Rec}(T)$ is dense lineable (see Proposition 2.7);
- quasi-rigidity coincides with $\mathcal{F}(A)$ -recurrence, and $\mathcal{F}(A)$ is a filter (see Proposition 4.5);
- i.e. *there is a dense set $Y \subset X$ and a filter $\mathcal{F}' = \mathcal{F}(A)$ such that $\mathcal{N}_{pr}(Y) \subset \mathcal{F}' \subset \mathcal{I}$.*

We can generalize these ideas to arbitrary Furstenberg families \mathcal{F} , finding some filter \mathcal{F}' contained in \mathcal{F} . In particular, we extend [13, Theorem 6.1] where the set of $\mathcal{I}\mathcal{P}^*$ -recurrent vectors is shown to be dense lineable:

Theorem 5.8 (Dense Lineability). *Let (X, T) be a linear dynamical system and let \mathcal{F} be a Furstenberg family. If there exists a dense subset $Y \subset X$ and a filter \mathcal{F}' such that*

$$\mathcal{N}_{pr}(Y) \subset \mathcal{F}' \subset \mathcal{F},$$

then $Z := \text{span}\left(\bigcup_{S \in \mathcal{C}_T} S(Y)\right) \subset X$ is a dense infinite-dimensional T -invariant vector subspace with the property that $Z^N \subset \mathcal{F}\text{Rec}(T_N)$ for every $N \in \mathbb{N}$. In particular, if \mathcal{F} is a filter itself, then for every $N \in \mathbb{N}$ we have the equality

$$\mathcal{F}\text{Rec}(T)^N = \mathcal{F}\text{Rec}(T_N),$$

which is a T -invariant vector subspace of X^N , and the following are equivalent:

- (i) T is \mathcal{F} -recurrent;
- (ii) $\mathcal{F}\text{Rec}(T_N)$ is a dense infinite-dimensional vector subspace of X^N for every $N \in \mathbb{N}$.

Proof. It is clear that $Z \subset X$ is a dense infinite-dimensional T -invariant vector subspace. Let us show that $\mathcal{N}_{pr}(Z) \subset \mathcal{F}'$: given $y_1, y_2, \dots, y_N \in \bigcup_{S \in \mathcal{C}_T} S(Y)$ and any neighbourhood $U \subset X$ of $x := \sum_{i=1}^N y_i \in Z$ we can find neighbourhoods U_i of y_i , for $1 \leq i \leq N$, such that

$$\sum_{i=1}^N U_i \subset U \subset X \quad \text{and hence} \quad N_T(x, U) \supset \bigcap_{i=1}^N N_T(y_i, U_i) \in \mathcal{F}' \subset \mathcal{F},$$

since $\mathcal{N}_{pr}(\{y_i\}) \subset \mathcal{F}'$ for all $1 \leq i \leq N$ by Theorem 4.19. Finally, given any $N \in \mathbb{N}$, $x_1, x_2, \dots, x_N \in Z$ and any neighbourhood $V \subset X^N$ of the N -tuple $z := (x_1, \dots, x_N) \in X^N$, we can find neighbourhoods V_i of z_i , for $1 \leq i \leq N$, such that

$$V_1 \oplus \dots \oplus V_N \subset V \subset X^N \quad \text{and hence} \quad N_{T_N}(z, V) \supset \bigcap_{i=1}^N N_T(z_i, U_i) \in \mathcal{F}' \subset \mathcal{F}.$$

The arbitrariness of $V \subset X^N$ implies that $z \in \mathcal{F}\text{Rec}(T_N)$. □

As mentioned in the comments before Theorem 5.8, this result extends Proposition 2.7 to every N -fold direct sum operator. Moreover, in view of Theorems 4.19 and 5.8 we can generalize Proposition 2.9 establishing the following (natural) sufficient conditions:

Proposition 5.9. *Let (X, T) be a linear dynamical system and let \mathcal{F} be a Furstenberg family. If T admits an \mathcal{F} -recurrent vector $x \in \mathcal{F}\text{Rec}(T)$ with a dense \mathcal{C}_T -orbit*

$$\mathcal{C}_T(x) = \{Sx : S \in \mathcal{C}_T\},$$

then $\mathcal{F}\text{Rec}(T_N)$ is dense lineable in X^N for every $N \in \mathbb{N}$. In particular, the later is true whenever any of the following conditions hold:

- T admits an \mathcal{F} -recurrent and cyclic vector;
- T admits an \mathcal{F} -hypercyclic vector.

Proof. Given a vector $x \in \mathcal{F}\text{Rec}(T)$ for which $\mathcal{C}_T(x)$ is dense in X , Theorem 4.19 implies that $\mathcal{N}_{pr}(\mathcal{C}_T(x)) \subset \mathcal{N}_{pr}(\{x\}) \subset \mathcal{F}$ and the result follows from Theorem 5.8. □

The dense lineability of the set $\mathcal{F}\text{Rec}(T_N)$ implies that T_N is \mathcal{F} -recurrent. Under a slightly weaker hypothesis, we loose the dense lineability but we keep the \mathcal{F} -recurrence of every N -fold direct product of T with itself:

Proposition 5.10 (\mathcal{F} -recurrence of every (X^N, T_N)). *Let (X, T) be a linear dynamical system and let \mathcal{F} be a Furstenberg family. If T is \mathcal{F} -recurrent and if there exists a vector $x \in X$ with a dense \mathcal{C}_T -orbit*

$$\mathcal{C}_T(x) = \{Sx : S \in \mathcal{C}_T\},$$

then $T_N : X^N \longrightarrow X^N$ is \mathcal{F} -recurrent for every $N \in \mathbb{N}$. In particular, the later is true whenever any of the following conditions hold:

- T is \mathcal{F} -recurrent and cyclic;
- T is \mathcal{F} -recurrent and hypercyclic.

This has been recently proved in [17, Proposition 2.7], but we repeat here the argument for the sake of completeness:

Proof. Fix $N \in \mathbb{N}$. By Theorem 4.19 we have the inclusion

$$Y := \bigcup_{y \in \mathcal{F}\text{Rec}(T)} \mathcal{C}_T(y)^N \subset \mathcal{F}\text{Rec}(T_N),$$

so it is enough to show that Y is dense in X^N . By assumption the point $x \in X$ has a dense \mathcal{C}_T -orbit, so given any non-empty open subset $U \subset X^N$ there are N continuous maps $S_{(1)}, S_{(2)}, \dots, S_{(N)} \in \mathcal{C}_T$ such that $(S_{(1)}x, S_{(2)}x, \dots, S_{(N)}x) \in U$. By continuity and since T is \mathcal{F} -recurrent we can find $y \in \mathcal{F}\text{Rec}(T)$ near enough x such that

$$(S_{(1)}y, S_{(2)}y, \dots, S_{(N)}y) \in \mathcal{C}_T(y)^N \cap U \subset Y \cap U. \quad \square$$

Remark 5.11. We include here some comments about the previous results:

(a) Proposition 5.10 remains valid for non-linear dynamical systems. It is a slight improvement of [19, Theorem 9.1] in terms of Furstenberg families and non-linear commuting maps, and it has been independently proved in the recent work [17, Proposition 2.7].

(b) Given a Furstenberg family \mathcal{F} , the following is a natural open question:

Problem 5.12. Suppose that a linear dynamical system (X, T) does not admit any dense \mathcal{C}_T -orbit. Can every N -fold direct sum (X^N, T_N) be \mathcal{F} -recurrent?

(c) The assumptions of Proposition 5.9 imply those of Proposition 5.10. Conversely:

Problem 5.13. Let \mathcal{F} be a Furstenberg family. Can an \mathcal{F} -recurrent linear system (X, T) admit dense \mathcal{C}_T -orbits but fulfill that $\mathcal{F}\text{Rec}(T) \cap \{x \in X : \overline{\mathcal{C}_T(x)} = X\} = \emptyset$?

This seems to be a tough question since the commutator of an operator is usually difficult to describe. If instead of dense \mathcal{C}_T -orbits we just consider the set of *hypercyclic* vectors, then the answer is yes: Menet constructed a *chaotic* operator T (and hence with dense periodic vectors) which is not \mathcal{U} -frequently hypercyclic (see [38]), i.e.

- T fulfills the assumptions of Proposition 5.10 for every Furstenberg family \mathcal{F} fulfilling that $\Delta^* \subset \mathcal{F} \subset \overline{\mathcal{D}}$, but it has not any \mathcal{F} -recurrent and hypercyclic vector.

Considering now the set of *cyclic* vectors (instead of dense \mathcal{C}_T -orbits or hypercyclic vectors), Corollary 2.11 and Proposition 5.9 imply that:

- If (X, T) is a recurrent and cyclic linear dynamical system, then the set $\text{Rec}(T_N)$ is dense lineable for every $N \in \mathbb{N}$.

In general we cannot change $\text{Rec}(T_N)$ into $\mathcal{F}\text{Rec}(T_N)$ unless we could ensure that $\mathcal{F}\text{Rec}(T)$ is a co-meager set. This is the case for \mathcal{AP} -recurrence (see [36, 16]):

- If (X, T) is an \mathcal{AP} -recurrent (i.e. multiple recurrent) and cyclic linear dynamical system, then the set $\mathcal{AP}\text{Rec}(T_N)$ is dense lineable for every $N \in \mathbb{N}$.

In [13, Example 2.4] it is exhibited a $\overline{\mathcal{BD}}$ -recurrent (i.e. reiteratively recurrent) operator T for which the set $\overline{\mathcal{BD}}\text{Rec}(T) = \text{RRec}(T)$ is meager. Such a operator T is not cyclic, and in view of [13, Theorem 2.1] (result which states that: *the set of $\overline{\mathcal{BD}}$ -recurrent vectors is co-meager for every $\overline{\mathcal{BD}}$ -recurrent and hypercyclic operator*) the following is a natural open question:

Problem 5.14. Let (X, T) be a $\overline{\mathcal{BD}}$ -recurrent (i.e. reiteratively recurrent) and cyclic linear dynamical system. Is $\text{RRec}(T)$ a co-meager set?

5.3 Dense Lineability for \mathcal{F} -hypercyclicity

Under some natural conditions on the Furstenberg family \mathcal{F} we can apply the \mathcal{F} -recurrence theory developed above in order to obtain some results regarding \mathcal{F} -hypercyclicity. We will consider *right-invariant* and *upper* Furstenberg families. Let us recall the definitions:

A family \mathcal{F} is said to be *right-invariant*, if for every $A \in \mathcal{F}$ and every $n \in \mathbb{N}_0$ the set $A+n = \{k+n : k \in A\}$ also belongs to \mathcal{F} . For example, the families $\mathcal{I}, \mathcal{AP}, \mathcal{S}, \mathcal{T}, \mathcal{TS}, \mathcal{I}^*$ and the density ones are easily seen to be right-invariant. However, the families \mathcal{IP} and Δ together with their dual families \mathcal{IP}^* and Δ^* are not right-invariant (see [9, Proposition 5.6]). In [13] it is shown that, for a right-invariant family \mathcal{F} , every \mathcal{F} -recurrent and hypercyclic vector is \mathcal{F} -hypercyclic. As a consequence we obtain a kind of *Herrero-Bourdon theorem* (see [31, Theorem 2.55]) for \mathcal{F} -hypercyclicity:

Corollary 5.15. *Let (X, T) be a linear dynamical system and assume that \mathcal{F} is a right-invariant Furstenberg family. If x is an \mathcal{F} -hypercyclic vector for T , then*

$$\{p(T)x : p \text{ polynomial}\} \setminus \{0\},$$

is a dense set of \mathcal{F} -hypercyclic vectors. In particular, any \mathcal{F} -hypercyclic operator admits a dense invariant vector space consisting, except for zero, of \mathcal{F} -hypercyclic vectors.

Proof. Follows from the original Herrero-Bourdon theorem, together with Theorem 4.19 and the equality

$$\mathcal{FHC}(T) = \mathcal{FRec}(T) \cap \text{HC}(T),$$

shown in [13] for every right-invariant Furstenberg family \mathcal{F} . □

If apart from *right-invariance* we also assume the *upper* condition (defined in [12]), we obtain even stronger results: a family \mathcal{F} is said to be *upper* if it can be written as

$$\mathcal{F} = \bigcup_{\delta \in D} \mathcal{F}_\delta \quad \text{with} \quad \mathcal{F}_\delta = \bigcap_{m \in M} \mathcal{F}_{\delta, m}$$

for sets $\mathcal{F}_{\delta, m} \subset \mathcal{P}(\mathbb{N}_0)$ such that $\delta \in D$ and $m \in M$, where D is arbitrary, M is countable and they have the following properties:

- (i) for any $\mathcal{F}_{\delta, m}$ and any $A \in \mathcal{F}_{\delta, m}$ there exists a finite set $F \subset \mathbb{N}_0$ such that

$$A \cap F \subset B \text{ implies } B \in \mathcal{F}_{\delta, m};$$

- (ii) for any $A \in \mathcal{F}$ there exists some $\delta \in D$ such that for all $n \in \mathbb{N}_0$ we have

$$(A - n) \cap \mathbb{N}_0 \in \mathcal{F}_\delta.$$

The families $\mathcal{I}, \mathcal{AP}, \overline{\mathcal{BD}}$ and $\overline{\mathcal{D}}$ are easily checked to be upper, while $\underline{\mathcal{D}}$ is not (see [12, 16]). As it is shown in [12], there exists a kind of *Birkhoff transitivity theorem* for these *upper* families, and in particular, the following holds:

Corollary 5.16. *Let (X, T) be an \mathcal{F} -recurrent linear dynamical system for an upper and right-invariant Furstenberg family \mathcal{F} . If there is a dense set $X_0 \subset X$ such that $T^n x \rightarrow 0$ for each $x \in X_0$, then $\mathcal{FRec}(T_N)$ is dense lineable for every $N \in \mathbb{N}$. In particular:*

- (a) If T is recurrent then $\text{Rec}(T_N)$ is dense lineable for every $N \in \mathbb{N}$;
- (b) If T is \mathcal{AP} -recurrent then $\mathcal{AP}\text{Rec}(T_N)$ is dense lineable for every $N \in \mathbb{N}$;
- (c) If T is reiteratively recurrent then $\text{RRec}(T_N)$ is dense lineable for every $N \in \mathbb{N}$;
- (d) If T is \mathcal{U} -frequently recurrent then $\text{UFRec}(T_N)$ is dense lineable for every $N \in \mathbb{N}$.

Proof. If T is \mathcal{F} -recurrent for an upper and right-invariant Furstenberg family \mathcal{F} and if there exists a dense set of vectors $x \in X$ such that $T^n x \rightarrow 0$ as $n \rightarrow \infty$, then T is an \mathcal{F} -hypercyclic operator by [13, Theorem 8.5]. By Proposition 5.9 the set $\mathcal{F}\text{Rec}(T_N)$ is dense lineable for every $N \in \mathbb{N}$. The particular cases follow from the fact that the families \mathcal{I} , \mathcal{AP} , $\overline{\mathcal{BD}}$ and $\overline{\mathcal{D}}$ are upper and right-invariant. \square

Moreover, if we consider the Furstenberg family $\overline{\mathcal{BD}}$ of positive upper Banach density sets we can prove a particular case of [23, Theorem 2.5 and Corollary 2.8] in a much simpler way, which is also a particular case of [17, Proposition 2.8]:

Corollary 5.17 ([23, Theorem 2.5 and Corollary 2.8] and [17, Proposition 2.8]). *Let (X, T) be a reiteratively hypercyclic linear dynamical system. Then $T_N : X^N \rightarrow X^N$ is reiteratively hypercyclic for every $N \in \mathbb{N}$.*

Proof. By [8, Proposition 4] we have that T_N is hypercyclic. By Corollary 5.10 we have that T_N is reiteratively recurrent. By [13, Theorem 2.1] every hypercyclic vector of T_N is also reiteratively hypercyclic so T_N is reiteratively hypercyclic. \square

6 Open problems

In this section we include three open problems and a few comments related to them. They essentially ask “what happens if we cannot apply Theorem 5.8”:

Problems 6.1. Let \mathcal{F} be a Furstenberg family and assume that it is not a filter (i.e. the second part of Theorem 5.8 does not apply in the following situations):

- (a) Is there any \mathcal{F} -recurrent operator T for which some $\mathcal{F}\text{Rec}(T_N)$ are not dense lineable?
- (b) Is there any \mathcal{F} -recurrent operator T for which $\mathcal{F}\text{Rec}(T)$ is not dense lineable?
- (c) Is there any \mathcal{F} -recurrent operator T for which $T \oplus T$ is not \mathcal{F} -recurrent?

Note that **Problems (b)** and **(c)** are weaker questions than **Problem (a)**, i.e. a negative answer to **(a)** would imply a negative answer to both **(b)** and **(c)**.

Moreover, for usual recurrence (i.e. for \mathcal{I} -recurrence) we already have a positive answer to **Problem (c)**: in Section 3 we have constructed a recurrent operator T such that $T \oplus T$ is not recurrent. Nevertheless, that example is not solving **Problem (b)** since this operator is such that $\text{Rec}(T) = X$ (its set of recurrent vectors is dense lineable). Thus, finding a possible recurrent operator T for which $\text{Rec}(T)$ is not dense lineable demands to solve again **Problem (c)**, for usual recurrence, but with an example fulfilling that $\text{Rec}(T) \neq X$.

These questions seem to be difficult for Furstenberg families implying stronger recurrence notions than usual recurrence (see [29, Theorems 5.1 and 5.8]). In addition, following the ideas of [8, Proposition 4] we can show that every $\overline{\mathcal{BD}}$ -recurrent (i.e. reiteratively recurrent) linear dynamical system is quasi-rigid:

Proposition 6.2 (Modification of [8, Proposition 4] and [9, Proposition 5.6]). *Let (X, T) be a linear dynamical system. If T is reiteratively recurrent, then T is quasi-rigid, and in particular $\text{Rec}(T_N)$ is dense lineable for every $N \in \mathbb{N}$.*

Proof. Given a non-empty subset $U \subset X$ we can pick a vector $x \in \text{RRec}(T) \cap U$ and hence $\overline{\text{Bd}}(N_T(x, U)) > 0$. Recall that

$$(N_T(x, U) - N_T(x, U)) := \{s_2 - s_1 : s_1 \leq s_2 \in N_T(x, U)\} \subset \{n \geq 0 : T^n(U) \cap U \neq \emptyset\}.$$

Now we use [24, Theorem 3.18], which implies that $A - A \in \Delta^*$ whenever $A \in \overline{\mathcal{BD}}$, obtaining that: for every non-empty open subset $U \subset X$ we have that

$$\{n \geq 0 : T^n(U) \cap U \neq \emptyset\} \in \Delta^*.$$

Since Δ^* is a filter (see [11, 34]) it is easily checked that every N -fold direct sum (X^N, T_N) is topologically recurrent. By Theorem 2.5 we have that T is quasi-rigid and Proposition 2.7 (or Theorem 5.8) implies that $\text{Rec}(T_N)$ is dense lineable for every $N \in \mathbb{N}$. \square

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