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► **To cite this version:**

| Spyridon Filippas. Whispering gallery modes for a transmission problem. 2023. ⟨hal-03979691⟩

HAL Id: hal-03979691

<https://hal.science/hal-03979691v1>

Preprint submitted on 9 Feb 2023

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Whispering gallery modes for a transmission problem

Spyridon Filippas *

Abstract

We construct a specific family of eigenfunctions for a Laplace operator with coefficients having a jump across an interface. These eigenfunctions have an exponential concentration arbitrarily close to the interface, and therefore could be considered as whispering gallery modes. The proof is based on an appropriate Agmon estimate. We deduce as a corollary that the quantitative unique continuation result for waves propagating in singular media proved by the author in [Fil22] is optimal.

Keywords

Agmon estimates, eigenfunctions, transmission problem, wave propagation

2010 Mathematics Subject Classification: 35B60, 47F05, 93B07, 35P20.

1 Introduction

Let \mathcal{M} be a compact, connected subset of \mathbb{R}^n with smooth boundary $\partial\mathcal{M}$ and S a smooth hypersurface such that we have the following partition: $\text{Int}(\mathcal{M}) \setminus S = \Omega_- \cup \Omega_+$ with $\Omega_- \cap \Omega_+ = \emptyset$. We consider the operator $-\Delta_c := -\text{div}(c\nabla\cdot)$ acting on \mathcal{M} with c being strictly positive and piecewise smooth but having a jump across the interface S . In this note we construct, for specific choices of Ω_- , Ω_+ , S and c eigenfunctions of $-\Delta_c$ which concentrate exponentially near the interface S . These maximally vanishing eigenfunctions are sometimes called *whispering gallery modes* (WGM). We show that (see Theorem 2.5 or Theorem 3.3 for a more precise statement):

Theorem 1.1. *There are sets \mathcal{M} , S and coefficients c such that there exist sequences $(\lambda_n)_{n \in \mathbb{N}}$ and $(u_n)_{n \in \mathbb{N}}$ with $\lambda_n \rightarrow +\infty$ and u_n satisfying the transmission conditions*

$$(u_n|_{\Omega_-})|_S = (u_n|_{\Omega_+})|_S \text{ and } c_+ \partial_\nu(u_n|_{\Omega_-})|_S = c_- \partial_\nu(u_n|_{\Omega_+})|_S, \quad (1.1)$$

such that for all $\omega \subset \mathcal{M}$ with $\text{dist}(\bar{\omega}, S) > 0$ there exist $C, d > 0$ with:

$$\|u_n\|_{L^2(\omega)} \leq C e^{-d\sqrt{\lambda_n}}, \quad -\Delta_c u_n = \lambda_n u_n, \quad \|u_n\|_{L^2(\mathcal{M})} = 1. \quad (1.2)$$

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The data \mathcal{M}, S, c considered in the proof of Theorem 1.1 have a rotational symmetry and possible sets for \mathcal{M} include a disk or an annulus. The proof of this is based on an *Agmon estimate* (see Section 2.2). Its main advantage is that it is quite simple and it allows to handle with minor modifications different geometries \mathcal{M}, S . Our basic toy model will be the case where \mathcal{M} is an annulus (see Figure 1 below) but in Section 3 we explain how one can deal with more general surfaces of revolution.

Our main motivation comes from tunneling estimates in control theory (see for instance [LL21a, Section 1.2]). Constructing such eigenfunctions allows to saturate certain observability estimates and therefore deduce their optimality. The idea of exhibiting such examples on surfaces of revolution can be traced back to [Leb96] and [All98].

Another motivation for studying WGM comes from optoelectronics. Indeed, the case where \mathcal{M} is a disk and S a smaller circle in its interior can be seen as a toy model for the orthogonal section of an optical fiber. Indeed, in this case Ω_- can be considered as the core of the optical fiber surrounded by a cladding Ω_+ . For the proof of Theorem 1.1 we shall assume that $c_- < c_+$ which means that the refractive index in Ω_- is higher than in Ω_+ and consequently light stays localized in the core by total internal reflection in the boundary between the core and the cladding. See as well the remarks after Theorem 1.3. Concerning the applications to optoelectronics, WGM have been studied numerically as well as from a theoretical point of view in [BDDM20, BDM21]. In these references the eigenvalue problem for $-\Delta_c$ is studied in an unbounded domain, thus becoming a resonance problem.

In the recent work [BBAD22] the authors obtain concentration and non-concentration properties for the eigenfunctions of $-\Delta_c$ depending on the regularity of the coefficient c . Discontinuity of the coefficient c corresponds to the case of *layered media*. The methods employed and the geometric context are however different than ours since the proofs are not based on Agmon estimates but rather on the explicit form of the Green kernel of the solutions.

1.1 Optimality of unique continuation results for operators with jumps

We consider the geometric context described in the beginning of the introduction.

The following spectral inequality for eigenfunctions of $-\Delta_c$ is proved in [LR10, Theorem 1.2] (and generalized in [LRL13]):

Theorem 1.2 (Theorem 1.2 in [LR10]). *Let $(u_j)_{j \in \mathbb{N}}$ be a Hilbert basis of eigenfunctions of the operator $-\Delta_c$ with Dirichlet boundary conditions, satisfying the transmission conditions (1.1). Denote by λ_j the associated eigenvalues, sorted in an increasing sequence. Then for any $\omega \subset \text{Int}(\mathcal{M})$ there exists $C > 0$ such that one has, for any $a_j \in \mathbb{C}$:*

$$\sum_{\lambda_j \leq \lambda} |a_j|^2 \leq C e^{C\sqrt{\lambda}} \int_{\omega} \left| \sum_{\lambda_j \leq \lambda} a_j u_j(x) \right|^2 dx.$$

The eigenfunctions exhibited in Theorem 1.1 prove that the spectral estimate of Theorem 1.2 above is sharp in general, even for a single eigenfunction.

The discontinuities of the operator $-\Delta_c$ can be used to describe waves propagating in non-homogeneous media. The coefficient c can be interpreted as the square of the speed of propagation. We consider the following system which describes the evolution of such a wave,

$$\begin{cases} (\partial_t^2 - \Delta_c)w = 0 & \text{in } (0, T) \times \Omega_- \cup \Omega_+ \\ w|_{S_-} = w|_{S_+} & \text{in } (0, T) \times S \\ (c\partial_\nu w)|_{S_-} = (c\partial_\nu w)|_{S_+} & \text{in } (0, T) \times S \\ w = 0 & \text{in } (0, T) \times \partial\mathcal{M} \\ (w, \partial_t w)|_{t=0} = (w_0, w_1) & \text{in } \mathcal{M}, \end{cases} \quad (1.3)$$

where we denote by ∂_ν the outward unit normal vector to S pointing into Ω_+ and by $w|_{S_\pm}$ the traces of $w|_{\Omega_\pm}$ on S . In unique continuation problems one tries to recover the whole wave from a partial observation. In [Fil22] the following quantitative unique continuation result is proved.

Theorem 1.3 (Theorem 1.3 in [Fil22]). *For any non empty subset $\omega \subset \mathcal{M}$ there exist $C, T > 0$ such that for all $(w_0, w_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ with $(w_0, w_1) \neq (0, 0)$ and w solution of (1.3) one has:*

$$\|(w_0, w_1)\|_{H^1 \times L^2} \leq C e^{C\Lambda} \|w\|_{L^2((0, T) \times \omega)},$$

where $\Lambda = \frac{\|(w_0, w_1)\|_{H^1 \times L^2}}{\|(w_0, w_1)\|_{L^2 \times H^{-1}}}$.

Theorem 1.3 above generalizes Theorem 1.1 in [LL19] where smooth coefficients are considered. An important aspect of Theorem 1.3 above is that there is no assumption on the sign of the jump of the coefficient c . Suppose, to fix ideas, that $c_- < c_+$ are two constants. We interpret c_- and c_+ as the square of the speed of propagation of a wave travelling through two isotropic media Ω_- and Ω_+ with different refractive indices, n_- and n_+ respectively (recall that $n_\pm = 1/\sqrt{c_\pm}$). Imagine that a wave starts travelling from a region that is inside Ω_- . One has $\sqrt{\frac{c_-}{c_+}} = \frac{n_+}{n_-}$ and therefore the assumption $c_- < c_+$ translates to $n_- > n_+$. Then Snell-Descartes law states that when a wave travels from a medium with a higher refractive index to one with a lower refractive index there is a critical angle from which there is *total internal reflection*, that is no refraction at all. At the level of geometric optics, that is to say, in the high frequency regime such a wave stays trapped inside Ω_- . Therefore one expects that, at least at high frequency, no information propagates from Ω_- to Ω_+ , following the laws of geometric optics. However, Theorem 1.3 states that a part of the wave can always be observed from Ω_+ with an intensity at least exponentially small in terms of the typical frequency Λ of the wave.

In this note we show that indeed, in situations where $\frac{c_-}{c_+}$ is sufficiently small depending on the geometric context one can find waves that are exponentially localized, in the high frequency limit, arbitrarily close to the interface S . As a consequence, we deduce that the estimate of Theorem 1.3 is, in general, optimal. For a solution w of (1.3) we define $\Lambda(w) := \frac{\|(w(0), \partial_t w(0))\|_{H^1 \times L^2}}{\|(w(0), \partial_t w(0))\|_{L^2 \times H^{-1}}}$. We have the following corollary of Theorem 1.1:

Theorem 1.4 (Whispering-gallery waves). *In the geometric setting of Theorem 1.1 there exist solutions $(w_n)_{n \in \mathbb{N}}$ of (1.3) with $\|w_n(0)\|_{L^2} = 1$ and such that for all $\omega \subset \mathcal{M}$ with $\text{dist}(\bar{\omega}, S) > 0$ there exist $C, d > 0$ with:*

$$\|w_n\|_{L^2((0, T) \times \omega)} \leq C e^{-d\Lambda(w_n)}. \quad (1.4)$$

Theorem 1.4 is an immediate consequence of Theorem 1.1. Indeed, take u_n as in Theorem 1.1, then

$$w_n(t, x) := \cos(\sqrt{\lambda_n t}) u_n(x).$$

satisfies (1.3) with $(w_n, \partial_t w_n)|_{t=0} = (u_n, 0)$, $\Lambda(w_n) = \sqrt{\lambda_n} + 1$ and (1.2) implies (1.4), up to changing the constant C .

The plan of the article is as follows. In Section 1.2 we deal with the domain of the operator $-\Delta_c$. Then in Section 2 we give a detailed proof of Theorem 1.1 in the case of an annulus based on an Agmon estimate for a 1D semiclassical Schrödinger operator. Finally, in Section 3 we explain how our arguments can be used to include more general surfaces of revolution.

1.2 Domain and self-adjointness of the operator

Let us recall first some basic facts concerning the operator $-\Delta_c$, its domain and some general spectral properties. Given a function $u = \mathbb{1}_{\Omega_-} u_- + \mathbb{1}_{\Omega_+} u_+$ with $u_{\pm} \in C^\infty(\mathcal{M})$ one has in the distributional sense

$$\nabla u = \mathbb{1}_{\Omega_-} \nabla u_- + \mathbb{1}_{\Omega_+} \nabla u_+ + (u_- - u_+) \delta_S \nu,$$

where δ_S is the surface measure on S and ν is the unit normal vector field pointing into Ω_+ . We impose then that

$$u_-|_S = u_+|_S, \tag{1.5}$$

and the singular term is removed. Similarly, calculating

$$\operatorname{div}(c(x) \nabla u),$$

we see that the condition

$$c_+ \partial_\nu u_+|_S = c_- \partial_\nu u_-|_S \tag{1.6}$$

combined with (1.5) gives the equality

$$\operatorname{div}(c(x) \nabla u) = \mathbb{1}_{\Omega_-} \operatorname{div}(c_- \nabla u_-) + \mathbb{1}_{\Omega_+} \operatorname{div}(c_+ \nabla u_+).$$

We define then \mathcal{W} as the space of functions of the form

$$u = \mathbb{1}_{\Omega_-} u_- + \mathbb{1}_{\Omega_+} u_+,$$

with $u_{\pm} \in C_0^\infty(\mathcal{M})$ and such that (1.5) and (1.6) hold. These conditions are called *transmission conditions* and for $u \in \mathcal{W}$ one has $-\Delta_c u \in L^2$. With \mathcal{W} as initial domain $-\Delta_c$ is symmetric and bounded from below. Indeed, writing (\cdot, \cdot) for the inner product in L^2 one has for $u \in \mathcal{W}$

$$(-\Delta_c u, u) = - \int_{\Omega_-} \operatorname{div}(c_- \nabla u_-) \bar{u} - \int_{\Omega_+} \operatorname{div}(c_+ \nabla u_+) \bar{u} = \int_{\Omega_-} c_- |\nabla u_-|^2 dx + \int_{\Omega_+} c_+ |\nabla u_+|^2 dx,$$

where we have used the transmission conditions and an integration by parts.

One can then consider the Friedrichs extension (see for instance [Lew18, Chapter 3.2.4]) of $-\Delta_c$, which is a self-adjoint extension of $-\Delta_c$ whose domain is given by:

$$\mathcal{A} = \{u \in H_0^1(\mathcal{M}) \mid -\Delta_c u \in L^2(\mathcal{M})\}.$$

Using elliptic regularity arguments (see eg [LRLR13, Appendix C.2]) one can see that in fact

$$\mathcal{A} = \{u \in H_0^1(\mathcal{M}) \mid u|_{\Omega_-} \in H^2(\Omega_-), u|_{\Omega_+} \in H^2(\Omega_+), c_+ \partial_\nu(u|_{\Omega_-})|_S = c_- \partial_\nu(u|_{\Omega_+})|_S\}. \tag{1.7}$$

In the sequel we shall denote by u_{\pm} the restriction $u|_{\Omega_{\pm}}$ of $u \in \mathcal{A}$ on Ω_- and Ω_+ .

The operator $-\Delta_c$ with domain \mathcal{A} is then positive, self-adjoint and has a compact resolvent. We deduce that its spectrum solely consists of positive eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \dots$ of finite multiplicity with $\lambda_j \rightarrow +\infty$.

Acknowledgements The author would like to thank C. Laurent and M. Léautaud for discussions, encouragements, and patient guidance.

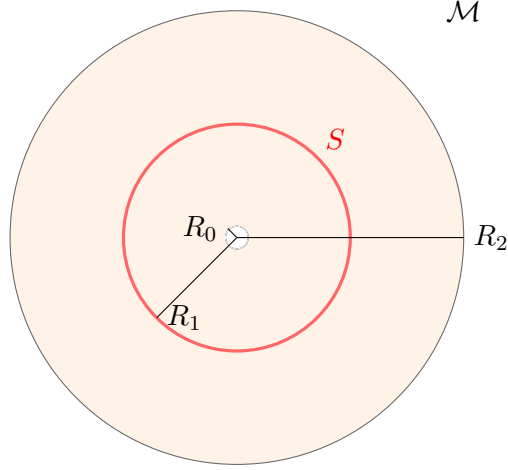


Figure 1: The annulus \mathcal{M} with the interface S where the coefficient c jumps.

2 The case of an annulus

We denote by B_R the open ball in \mathbb{R}^2 of radius R centered at 0 and study the eigenfunction problem for $(\mathcal{A}, -\Delta_c)$ in the following geometric context: Let $0 < R_0 < R_1 < R_2$ and set $\mathcal{M} = \overline{B_{R_2}} \setminus B_{R_0} \subset \mathbb{R}^2$, $\Omega_- = B_{R_1} \setminus \overline{B_{R_0}}$, $\Omega_+ = B_{R_2} \setminus \overline{B_{R_1}}$, $S = \overline{B_{R_1}} \setminus B_{R_1}$. For simplicity we assume c piecewise constant, that is $c = \mathbb{1}_{\Omega_-} c_- + \mathbb{1}_{\Omega_+} c_+$, with $0 < c_- < c_+$ to be chosen later on.

2.1 Reduction to a semiclassical Schrödinger operator

We work in polar coordinates, in which the operator $-\Delta_c$ takes the form

$$-\Delta_c = -\frac{1}{r} \partial_r (c r \partial_r) + \frac{c}{r^2} \partial_\theta^2.$$

For a function $u \in \mathcal{A}$ we have (see Section 1.2):

$$-\Delta_c u(r, \theta) = - \sum_{\pm} \mathbb{1}_{\Omega_{\pm}} \left(c_{\pm} \partial_r^2 u_{\pm} + \frac{c_{\pm}}{r} \partial_r u_{\pm} + \frac{c_{\pm}}{r^2} \partial_\theta^2 u_{\pm} \right), \quad u \in \mathcal{A},$$

where we recall that we write $u = \mathbb{1}_{\Omega_-} u_- + \mathbb{1}_{\Omega_+} u_+$. We look for solutions of the eigenvalue problem $-\Delta_c u = \lambda u$ under the form $u(r, \theta) = e^{in\theta} f(r)$ which yields

$$\sum_{\pm} \mathbb{1}_{\Omega_{\pm}} \left(-\frac{c_{\pm}}{n^2} \left(\partial_r^2 f_{\pm}(r) + \frac{1}{r} \partial_r f_{\pm}(r) \right) + \left(\frac{c_{\pm}}{r^2} - \frac{\lambda}{n^2} \right) f_{\pm}(r) \right) = 0.$$

We consider large angular momenta, $n \rightarrow +\infty$ and interpret $h := 1/n$ as a semiclassical parameter. Equation (2.1) writes then as $P_h f = E_h f$, where

$$P_h := -h^2 \frac{1}{r} \partial_r (c r \partial_r) + V_c(r), \quad V_c(r) := \frac{c}{r^2}, \quad E_h = h^2 \lambda, \quad (2.1)$$

and $V_c(r)$ can be seen as the effective potential with a jump discontinuity at $r = R_1$. We define then the continuous extension of V_c to $[R_0, R_1]$:

$$\overline{V}_c(r) = \begin{cases} V_c(r), & r \in [R_0, R_1) \\ \frac{c_-}{R_1^2}, & r = R_1, \end{cases} \quad (2.2)$$

and we write $E_0 := \frac{c_-}{R_1^2}$ for the infimum of the potential V_c .

We now study the eigenvalue problem for the one dimensional semiclassical Schrödinger operator P_h . We define the space

$$\mathcal{A}_r = \{f \in H_0^1((R_0, R_2)) \mid f|_{(R_0, R_1)} \in H^2((R_0, R_1)), f|_{(R_1, R_2)} \in H^2((R_1, R_2)), \\ c_+ \partial_r (f|_{(R_0, R_1)})(R_1) = c_- \partial_r (f|_{(R_1, R_2)})(R_1)\}. \quad (2.3)$$

We write as well $f_- := f|_{(R_0, R_1)}$ and $f_+ := f|_{(R_1, R_2)}$. The space \mathcal{A}_r is such that one has the following two properties:

$$f \in \mathcal{A}_r \Leftrightarrow f(r)e^{in\theta} \in \mathcal{A}, \quad n \in \mathbb{Z},$$

and P_h is self-adjoint with domain \mathcal{A}_r , for the same reason as in Section 1.2.

In the following proposition we show the existence of eigenfunctions for P_h close to some energy levels. For the proof we start by constructing a rough quasimode and then use the self-adjointness of the operator (see for instance [LL21a, Lemma 3.6], [Zwo12, Chapter 12.5]).

Proposition 2.1 (Existence of eigenfunctions for the 1D operator). *There exists h_0 such that for all $E \in [E_0, V_c(R_0))$ there exists $C > 0$ such that for all $h \in (0, h_0]$, there exist E_h and $\psi_h \in \mathcal{A}_r$ with*

$$P_h \psi_h = E_h \psi_h, \quad |E_h - E| \leq Ch^{2/3}.$$

Proof. The operator P_h with domain \mathcal{A}_r is self-adjoint in $L^2((R_0, R_2), r dr)$. Note that we can estimate indifferently with norms in $L^2((R_0, R_2), r dr)$ or $L^2((R_0, R_2), dr)$ since they are equivalent.

We may write $E = \overline{V}_c(\rho_E)$ for some $\rho_E \in (R_0, R_1]$ where we recall that \overline{V}_c is defined in (2.2). Consider $\chi \in C_0^\infty((-1, 0))$ such that $\chi = 1$ in a neighborhood of $-1/2$ and define $f_h(r) = h^{-1/3} \chi(h^{-2/3}(r - \rho_E))$. One then has that for $h \leq h_0$ with h_0 sufficiently small $f_h(r) \in C_0^\infty((R_0, R_1))$. This implies in particular that $f_h \in \mathcal{A}_r$. We estimate now:

$$\begin{aligned} & \left\| -ch^2 \left(\partial_r^2 f_h(r) + \frac{1}{r} \partial_r f_h(r) \right) \right\|_{L^2}^2 \\ &= \int ch^4 \left(h^{-5/3} \chi''(h^{-2/3}(r - \rho_E)) + \frac{1}{r} h^{-1} \chi'(h^{-2/3}(r - \rho_E)) \right)^2 dr \\ &\leq Ch^{4/3}. \end{aligned} \quad (2.4)$$

To estimate the next term we use that $\text{supp}(f_h) \subset (R_0, R_1)$. We use now the fact that V_c is uniformly Lipschitz in $[R_0, R_1]$ which implies that, in the support of f_h , one has

$$|V_c(r) - E| = |\overline{V}_c(r) - E| \leq C|r - \rho_E|,$$

and hence

$$\|(V_c(r) - E) f_h(r)\|_{L^2}^2 \leq C \int (r - \rho_E)^2 h^{-2/3} \chi^2(h^{-2/3}(r - \rho_E)) dr \leq Ch^{4/3}. \quad (2.5)$$

Putting together (2.4) and (2.5) yields

$$\|(P_h - E) f_h\|_{L^2} \leq Ch^{2/3}. \quad (2.6)$$

Noticing finally that

$$\|f_h\|_{L^2}^2 = \int h^{-2/3} \chi^2 \left(h^{-2/3} (r - \rho_E) \right) dr = c_0,$$

we can write (2.6) as $\|(P_h - E) f_h\|_{L^2} \leq Ch^{2/3} \|f_h\|_{L^2}$. If now $E \notin \text{Sp}(P_h)$ one can write

$$\begin{aligned} \|(P_h - E) f_h\|_{L^2} &\leq Ch^{2/3} \|f_h\|_{L^2} = Ch^{2/3} \left\| (P_h - E)^{-1} (P_h - E) f_h \right\| \\ &\leq Ch^{2/3} \left\| (P_h - E)^{-1} \right\|_{L^2 \rightarrow L^2} \|(P_h - E) f_h\|_{L^2}, \end{aligned}$$

which gives

$$\left\| (P_h - E)^{-1} \right\|_{L^2 \rightarrow L^2} \geq C^{-1} h^{-2/3}.$$

Finally, since P_h is self-adjoint one has $\left\| (P_h - E)^{-1} \right\|_{L^2 \rightarrow L^2} = \frac{1}{\text{dist}(\text{Sp}(P_h), E)}$ and consequently

$$\text{dist}(\text{Sp}(P_h), E) \leq Ch^{2/3}, \quad (2.7)$$

which is trivially true in the case $E \in \text{Sp}(P_h)$ as well. The existence of E_h, ψ_h is then a result of (2.7) and of the fact that the spectrum of P_h consists solely of eigenvalues. \square

2.2 The Agmon estimate

We follow [Hel88, Chapter 3] (see as well [DS99, Chapter 6.B], [LL22]). We start by defining the appropriate Agmon distance, which corresponds to a distance to the *classically allowed region* for the potential V_c at the energy level E . The classically allowed region K_E is defined as

$$K_E = \{r \in [R_0, R_2] \mid V_c(r) \leq E\}, \quad (2.8)$$

where we recall that $V_c(r) = \frac{c}{r^2}$ and $c = \mathbf{1}_{(R_0, R_1)} c_- + \mathbf{1}_{(R_1, R_2)} c_+$. The condition we impose on the coefficient c is then (see Figure 2):

$$0 < c_- < c_+, \quad \bar{V}_c(R_1) < V_c(R_2), \quad (2.9)$$

and as a consequence the function \tilde{V} defined as $\tilde{V} = V_c(r)$ for $r \in [R_0, R_2] \setminus \{R_1\}$ and $\tilde{V}(R_1) = E_0$ attains its minimum at $r = R_1$. We define the appropriate Agmon distance for $E \geq E_0$ as

$$d_{A,E}(r) = \inf_{y \in K_E} \left| \int_y^r \sqrt{\frac{(V_c(s) - E)_+}{c(s)}} ds \right|, \quad (2.10)$$

where $a_+ = \max(a, 0)$.

In the sequel we shall focus on energy levels situated close to the minimum E_0 . We remark that assumption (2.9) and continuity of \bar{V}_c in $[R_0, R_1]$ imply that there exists $\eta_0 > 0$ such that

$$K_{E_0+\eta} \subset [R_0, R_1], \quad \forall 0 < \eta < \eta_0. \quad (2.11)$$

For $E \in [E_0, E_0 + \eta]$, $\eta \leq \eta_0$ we can define $\rho_E := V_c^{-1}(E)$ (which is invertible in $[R_0, R_1]$) and obtain that $K_E = [\rho_E, R_1]$ as well as the following explicit expressions:

$$\begin{aligned} d_{A,E}(r) &= \int_r^{\rho_E} \sqrt{\frac{(V_c(s) - E)}{c_-(s)}} ds, \quad \text{for } r \in [R_0, \rho_E], \\ d_{A,E}(r) &= 0, \quad \text{for } r \in [\rho_E, R_1], \\ d_{A,E}(r) &= \int_{R_1}^r \sqrt{\frac{(V_c(s) - E)}{c_+(s)}} ds, \quad \text{for } r \in [R_1, R_2]. \end{aligned} \quad (2.12)$$

One has in particular that $d_{A,E}$ is C -Lipschitz with $C = \left(\frac{\max\{V_c(s)-E\}}{c_-}\right)^{1/2}$.

The following identity is the key ingredient of the Agmon estimate. Notice that all quantities appearing in the following lemma are well defined. Indeed, since $f \in \mathcal{A}_r$ one has $f \in H^1$ and the left hand side is well defined.

Lemma 2.2. *Let ϕ be real valued Lipschitz continuous on $[R_0, R_2]$ and $f \in \mathcal{A}_r$. Then one has:*

$$\begin{aligned} \int_{R_0}^{R_2} ch^2 \left| \partial_r (e^{\phi/h} f) \right|^2 r dr - \int_{R_0}^{R_2} c |\partial_r \phi|^2 e^{2\phi/h} |f|^2 r dr \\ = - \operatorname{Re} \int_{(R_0, R_1) \cup (R_1, R_2)} e^{2\phi/h} ch^2 \left(\partial_r^2 + \frac{1}{r} \partial_r \right) f \cdot \bar{f} r dr. \end{aligned}$$

Proof. We write $f = \mathbb{1}_{(R_0, R_1)} f_- + \mathbb{1}_{(R_1, R_2)} f_+$, split the integrals and integrate by parts. We have:

$$\begin{aligned} - \int_{R_0}^{R_1} c_- e^{2\phi/h} \left(\partial_r^2 + \frac{1}{r} \partial_r \right) f_- \cdot \bar{f}_- r dr &= - \int_{R_0}^{R_1} c_- \partial_r (r \partial_r f_-) e^{2\phi/h} \cdot \bar{f}_- dr \\ &= \int_{R_0}^{R_1} c_- \partial_r f_- \partial_r (e^{2\phi/h} \bar{f}_-) r dr - c_- R_1 \partial_r f_-(R_1) e^{2\phi/h} \bar{f}_-(R_1), \end{aligned} \tag{2.13}$$

and similarly

$$\begin{aligned} - \int_{R_1}^{R_2} c_+ e^{2\phi/h} \left(\partial_r^2 + \frac{1}{r} \partial_r \right) f_+ \cdot \bar{f}_+ r dr &= - \int_{R_1}^{R_2} c_+ \partial_r (r \partial_r f_+) e^{2\phi/h} \cdot \bar{f}_+ dr \\ &= \int_{R_1}^{R_2} c_+ \partial_r f_+ \partial_r (e^{2\phi/h} \bar{f}_+) r dr + c_+ R_1 \partial_r f_+(R_1) e^{2\phi/h} \bar{f}_+(R_1). \end{aligned} \tag{2.14}$$

Now by definition of the space \mathcal{A}_r in (2.3) we find that

$$c_- R_1 \partial_r f_-(R_1) e^{2\phi/h} \bar{f}_-(R_1) = c_+ R_1 \partial_r f_+(R_1) e^{2\phi/h} \bar{f}_+(R_1).$$

Adding (2.13) and (2.14) the boundary terms cancel out and we obtain

$$\begin{aligned} - \int_{(R_0, R_1) \cup (R_1, R_2)} ce^{2\phi/h} h^2 \left(\partial_r^2 + \frac{1}{r} \partial_r \right) f \cdot \bar{f} r dr \\ = \int_{R_0}^{R_2} c 2h \partial_r \phi e^{2\phi/h} \bar{f} \partial_r f r dr + \int_{R_0}^{R_2} ch^2 e^{2\phi/h} |\partial_r f|^2 r dr, \end{aligned}$$

which gives the sought identity after taking real parts. \square

We can now show the following Agmon estimate. We recall that η_0 is defined by (2.11).

Proposition 2.3. *Let $E \in [E_0, E_0 + \frac{\eta_0}{2}]$ and $\epsilon(h)$ with $\epsilon(h) \xrightarrow{h \rightarrow 0} 0$. Then for all $\delta > 0$, there exist C, h_0 such that for all ψ_h satisfying*

$$P_h \psi_h = (E + \epsilon(h)) \psi_h, \quad \|\psi_h\|_{L^2} = 1,$$

one has, for $h \leq h_0$:

$$\left\| h \partial_r \left(e^{\frac{d_{A,E}}{h}} \psi_h \right) \right\|_{L^2((R_0, R_2))} + \left\| e^{\frac{d_{A,E}}{h}} \psi_h \right\|_{L^2((R_0, R_2))} \leq C e^{\delta/h}.$$

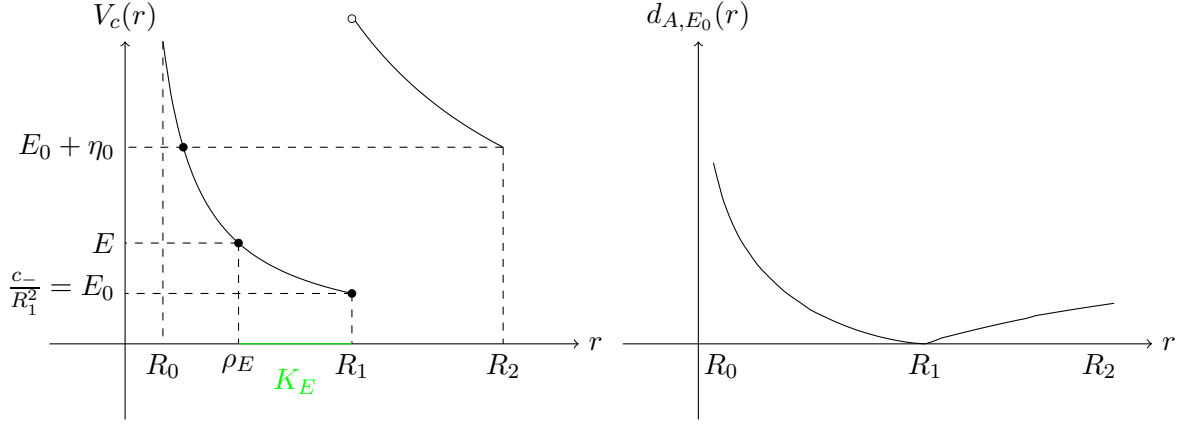


Figure 2: Left: The potential V_c with the points of interest. For $E < E_0 + \eta_0 = \frac{c_+}{R_2^2}$ the potential is continuous and injective in the classically allowed region K_E which is in green. Right: The graph of the Agmon distance related to the minimal energy level E_0 . Notice the C^1 singularity at $r = R_1$ due to the jump of the coefficient c .

Proof. We recall that $d_{A,E}$ is Lipschitz continuous and consider the weight $\phi = (1 - \delta)d_{A,E}$ with $0 < \delta < 1$. Let us write $E_h := E + \epsilon(h)$. We can then apply the identity of Lemma 2.2 with ψ_h which solves $P_h\psi_h = E_h\psi_h$, or equivalently

$$-\mathbf{1}_{(R_0, R_1)} ch^2 \left(\partial_r^2 + \frac{1}{r} \partial_r \right) \psi_{h,-} = \mathbf{1}_{(R_0, R_1)} (E_h - V_c(r)) \psi_{h,-},$$

and

$$-\mathbf{1}_{(R_1, R_2)} ch^2 \left(\partial_r^2 + \frac{1}{r} \partial_r \right) \psi_{h,+} = \mathbf{1}_{(R_1, R_2)} (E_h - V_c(r)) \psi_{h,+}.$$

We find:

$$\int_{R_0}^{R_2} ch^2 \left| \partial_r (e^{\phi/h} \psi_h) \right|^2 r dr - \int_{R_0}^{R_2} c |\partial_r \phi|^2 e^{2\phi/h} |\psi_h|^2 r dr = \int_{R_0}^{R_2} e^{2\phi/h} (E_h - V_c(r)) |\psi_h|^2 r dr.$$

Let us define $I_\alpha^+ := \{V - E > \alpha^2\}$, $I_\alpha^- = \{V - E \leq \alpha^2\}$ with $\alpha > 0$ small to be chosen. We split the integrals according to $(R_0, R_2) = I_\alpha^+ \cup I_\alpha^-$ and write the above equality as

$$\begin{aligned} \int_{R_0}^{R_2} ch^2 \left| \partial_r (e^{\phi/h} \psi_h) \right|^2 r dr + \int_{I_\alpha^+} e^{2\phi/h} (V_c(r) - E_h - c |\partial_r \phi|^2) |\psi_h|^2 r dr \\ = - \int_{I_\alpha^-} e^{2\phi/h} (V_c(r) - E_h - c |\partial_r \phi|^2) |\psi_h|^2 r dr. \end{aligned} \quad (2.15)$$

To control the integral on I_α^+ from below we notice that the Agmon distance satisfies the eikonal equation:

$$c |\partial_r d_{A,E}|^2 = (V_c(r) - E)_+, \quad \text{in } \mathcal{D}'((R_0, R_2)). \quad (2.16)$$

Hence, taking $h \leq h_0 = h_0(\alpha, \delta)$

$$\begin{aligned} \int_{I_\alpha^+} e^{2\phi/h} (V_c(r) - E_h - c |\partial_r \phi|^2) |\psi_h|^2 r dr &= \int_{I_\alpha^+} e^{2\phi/h} \left((V_c(r) - E)(1 - (1 - \delta)^2) - \epsilon(h) \right) |\psi_h|^2 r dr \\ &\geq \frac{\alpha^2 \delta}{2} \int_{I_\alpha^+} e^{2\phi/h} |\psi_h|^2 r dr, \end{aligned} \quad (2.17)$$

where h_0 is such that $\epsilon(h) \leq \frac{\alpha^2 \delta}{2}$ for all $h \leq h_0$. For the integral on I_α^- we have

$$\left| \int_{I_\alpha^-} e^{2\phi/h} (V_c(r) - E_h - c|\partial_r \phi|^2) |\psi_h|^2 r dr \right| \leq C \int_{I_\alpha^-} e^{2\phi/h} |\psi_h|^2 r dr, \quad (2.18)$$

where C depends on $\max V_c$. Putting (2.15), (2.17), (2.18) together we find

$$\int_{R_0}^{R_2} ch^2 \left| \partial_r (e^{\phi/h} \psi_h) \right|^2 r dr + \alpha^2 \delta \int_{I_\alpha^+} e^{2\phi/h} |\psi_h|^2 r dr \leq C \int_{I_\alpha^-} e^{2\phi/h} |\psi_h|^2 r dr,$$

which gives

$$\int_{R_0}^{R_2} ch^2 \left| \partial_r (e^{\phi/h} \psi_h) \right|^2 r dr + \alpha^2 \delta \int_{R_0}^{R_2} e^{2\phi/h} |\psi_h|^2 r dr \leq C \int_{I_\alpha^-} e^{2\phi/h} |\psi_h|^2 r dr. \quad (2.19)$$

We estimate now ϕ in I_α^- . Taking $\alpha^2 \leq \frac{\eta_0}{2}$ we have, with $\rho_E = \overline{V_c}^{-1}(E)$

$$d_{A,E}(r) = \int_r^{\rho_E} \sqrt{\frac{(V_c(s) - E)}{c(s)}} ds, \quad \text{for } r \in I_\alpha^-.$$

Using that $\overline{V_c}^{-1}$ is Lipschitz on $[R_0, \rho_E]$ implies that for $r \in I_\alpha^-$ one has

$$|r - \rho_E| \leq C |V_c(r) - V_c(\rho_E)| \leq C \alpha^2.$$

Choosing then $\alpha_0 = \alpha_0(\delta)$ sufficiently small we obtain for $r \in I_\alpha^-$ and $\alpha \leq \alpha_0$:

$$\phi(r) = (1 - \delta) d_{A,E} = (1 - \delta) \int_r^{\rho_E} \sqrt{\frac{(V_c(s) - E)}{c(s)}} ds \leq C(1 - \delta) |r - \rho_E| \leq \tilde{C} \alpha^2 \leq \delta.$$

which combined with (2.19) yields, using $R_0 > 0$,

$$\int_{R_0}^{R_2} h^2 \left| \partial_r (e^{\phi/h} \psi_h) \right|^2 dr + \alpha^2 \delta \int_{R_0}^{R_2} e^{2\phi/h} |\psi_h|^2 dr \leq C e^{2\delta/h}. \quad (2.20)$$

One needs finally to replace ϕ by $d_{A,E}$ in the above estimate. To do this we simply write, with $d_{A,E} \leq M$:

$$\begin{aligned} \int_{R_0}^{R_2} h^2 \left| \partial_r \left(e^{\frac{d_{A,E}}{h}} \psi_h \right) \right|^2 dr &= \int_{R_0}^{R_2} h^2 \left| \partial_r \left(e^{\frac{\delta d_{A,E}}{h}} e^{\frac{\phi}{h}} \psi_h \right) \right|^2 dr \\ &\leq C \int_{R_0}^{R_2} h^2 e^{\frac{2\delta d_{A,E}}{h}} \left| \partial_r (e^{\phi/h} \psi_h) \right|^2 dr + C \delta^2 \int_{R_0}^{R_2} |d'_{A,E}|^2 e^{\frac{2\delta d_{A,E}}{h}} e^{2\phi/h} |\psi_h|^2 dr \\ &\leq Ch^2 e^{\frac{2\delta M}{h}} \int_{R_0}^{R_2} \left| \partial_r (e^{\phi/h} \psi_h) \right|^2 dr + C \delta^2 \|V_c - E\|_{L^\infty} e^{\frac{2\delta M}{h}} \int_{R_0}^{R_2} e^{2\phi/h} |\psi_h|^2 dr. \end{aligned}$$

Combining this together with (2.20) concludes the proof of Proposition 2.3. \square

It follows from the Agmon estimate above that the mass of eigenfunctions close to the minimum energy level E_0 should be concentrated close to the point R_1 where the coefficient c exhibits a jump. This is the following corollary:

Corollary 2.4. *Let $\epsilon(h)$ with $\epsilon(h) \xrightarrow{h \rightarrow 0} 0$. There exist $h_0 > 0$ such that for all ψ_h satisfying*

$$P_h \psi_h = (E_0 + \epsilon(h)) \psi_h, \quad \|\psi_h\|_{L^2} = 1, \quad \psi_h \in \mathcal{A}_r,$$

and all $\varepsilon > 0$ there exist $C, d > 0$ such that

$$\|\psi_h\|_{L^2((R_0, R_2) \setminus [R_1 - \varepsilon, R_1 + \varepsilon])} \leq C e^{-\frac{d}{h}}$$

Proof. We recall that $E_0 = V_c(R_1^-)$ is the infimum of the potential V_c . Consider now a solution of $P_h \psi_h = (E_0 + \epsilon(h)) \psi_h$. The associated Agmon distance to the energy level E_0 satisfies

$$d_{A, E_0}(r) \geq m, \quad \text{for } r \in (R_0, R_2) \setminus [R_1 - \varepsilon, R_1 + \varepsilon].$$

The result then follows from the estimate of Proposition 2.3 by taking $\delta \leq m/2$. \square

2.3 Back to the two dimensional annulus

We now put all the pieces together to state our result for the initial operator $-\Delta_c = -\text{div}(c\nabla \cdot)$ defined on the annulus. Recall that the space \mathcal{A} has been defined in (1.7).

Theorem 2.5. *Consider \mathcal{M}, S as defined in the beginning of Section 2. Suppose that*

$$\frac{c_-}{R_1^2} < \frac{c_+}{R_2^2}.$$

Then there exist sequences $(\lambda_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}, (u_n)_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ and $E_0 > 0$ such that for all $\omega \subset \mathcal{M}$ with $\text{dist}(\bar{\omega}, S) > 0$ there exist $C, d > 0$ satisfying:

$$-\Delta_c u_n = \lambda_n u_n, \quad \lambda_n \underset{n \rightarrow +\infty}{\sim} E_0 n^2, \quad \|u_n\|_{L^2(\mathcal{M})} = 1, \quad \|u_n\|_{L^2(\omega)} \leq C e^{-dn},$$

for all $n \in \mathbb{N}$.

Proof. Let $h = 1/n$ and consider $\psi_h \in \mathcal{A}_r$ satisfying

$$P_h \psi_h = (E_0 + O(h^{2/3})) \psi_h, \quad \|\psi_h\|_{L^2((R_0, R_2), r dr)} = \frac{1}{\sqrt{2\pi}}.$$

the existence of such a family $(\psi_h)_h$ is given by Proposition 2.1. We define then

$$u_n(r, \theta) := e^{in\theta} \psi_h(r).$$

It follows (see the remark after the definition of \mathcal{A}_r in (2.3)) that $u_n \in \mathcal{A}$, $\|u_n\|_{L^2(\mathcal{M})} = 1$ and that (see (2.1))

$$-\Delta_c u_n = n^2 \left(E_0 + O(n^{-2/3}) \right) u_n.$$

Let finally $\omega \subset \mathcal{M}$ satisfy $\text{dist}(\bar{\omega}, S) > 0$. That means that there exists $\varepsilon > 0$ with

$$\begin{aligned} \|u_n\|_{L^2(\omega)} &\leq \|u_n\|_{L^2(\mathcal{M} \setminus (B(0, R_1 + \varepsilon) \setminus B(0, R_1 - \varepsilon)))} = \|\psi_h\|_{L^2((R_0, R_2) \setminus [R_1 - \varepsilon, R_1 + \varepsilon])}, \\ &\leq C e^{-dn}, \end{aligned}$$

thanks to Corollary 2.4. \square

3 The disk and other surfaces of revolution

We considered in Section 2 the case where \mathcal{M} is an annulus. The reason we presented the proof for an annulus is that the singularity coming from the change of variables to polar coordinates at 0 disappears, allowing us to work exclusively in one dimension. Of course, from an heuristic point of view this should not be a problem since we are interested in the behaviour of the eigenfunctions away from zero. However, in the case of the disk it is sometimes simpler to work with the initial operator $-\Delta_c$ and not with its 1-dimensional analogue (2.1) since in this case it becomes more intricate to describe its domain \mathcal{A}_r of self-adjointness when $r \in (0, R_2)$.

In this section we briefly explain how the method presented in Section 2 can be used to exhibit maximally vanishing eigenfunctions in the case of the disk or even for some surfaces of revolution embedded in \mathbb{R}^3 diffeomorphic to a disk. For the geometric description of such manifolds we follow [LL21a, Section 3] and [LL21b, Section 4].

Let (\mathcal{M}, g) be an embedded 2D submanifold of \mathbb{R}^3 having \mathbb{S}^1 as an effective isometry group. We denote by $\mathbb{S}^1 \times \mathcal{M} \ni (\theta, s) \rightarrow \mathcal{R}_\theta s$ the action of \mathbb{S}^1 on \mathcal{M} which satisfies $\mathcal{R}_\theta \mathcal{M} = \mathcal{M}$ and we suppose that it has exactly one fixed point denoted by $N \in \mathcal{M}$. We define $L = \text{dist}_g(N, \partial\mathcal{M})$. Then one can find coordinates

$$\mathcal{M} \ni m \rightarrow \zeta(m) = (s, \theta) \in (0, L] \times \mathbb{S}^1,$$

such that the metric becomes

$$(\zeta^{-1})^*g = ds^2 + R(s)d\theta^2,$$

and R is a smooth function $R : [0, L] \rightarrow \mathbb{R}_*^+$ which can be interpreted as the Euclidean distance in \mathbb{R}^3 of a point of \mathcal{M} to the symmetry axis. The disk of radius L centered at 0 corresponds to the case $R(s) = s$.

In the new coordinates, the Riemannian volume form is $R(s)dsd\theta$ and the Laplace-Beltrami operator is given by

$$\Delta_{s,\theta} = \frac{1}{R(s)}\partial_s(R(s)\partial_s) + \frac{1}{R^2(s)}\partial_\theta^2.$$

Choose now a point $s_0 \in (0, L)$ and define $\Omega_- = \{N\} \cup \zeta^{-1}((0, s_0) \times \mathbb{S}^1)$, $\Omega_+ = \zeta^{-1}((s_0, L] \times \mathbb{S}^1)$, $S = \zeta^{-1}(\{s_0\} \times \mathbb{S}^1)$. We consider as well the coefficient $c = c(s) = \mathbf{1}_{\Omega_-}c_- + \mathbf{1}_{\Omega_+}c_+$, and

$$\Delta_c := \frac{1}{R(s)}\partial_s(c(s)R(s)\partial_s) + \frac{c(s)}{R^2(s)}\partial_\theta^2,$$

which is well defined and self adjoint in the space \mathcal{A} , defined in (1.7) as explained in Section 1.2. More precisely, for $u \in \mathcal{A}$ one has

$$\partial_s(c(s)R(s)\partial_s u) = \mathbf{1}_{\Omega_-}c_- \partial_s(R(s)\partial_s u_-) + \mathbf{1}_{\Omega_+}c_+ \partial_s(R(s)\partial_s u_+), \quad \text{in } \mathcal{D}'(\mathcal{M}),$$

where $u_\pm = u|_{\Omega_\pm}$. The effective potential becomes now $V_c(s) = \frac{c(s)}{R^2(s)}$, and we define $V_c(s_0) = \frac{c_-}{R^2(s_0)}$ and $E_0 = \min V_c$. Consider now the operator \tilde{P}_h given by

$$\tilde{P}_h = -h^2 \left(\frac{1}{R(s)}\partial_s(c(s)R(s)\partial_s) \right) + \frac{c(s)}{R^2(s)}\partial_\theta^2.$$

It can be as shown as in [LL21a, Section 3.2] that if $\lambda \in \mathbb{R}$ is an eigenvalue of \tilde{P}_h then there is an eigenfunction of the form $e^{in\theta}\psi(s)$ with $n \in \mathbb{Z}$ and $\psi \in L^2((0, R_2), R(s)ds)$. This and the arguments used in the proof of Proposition 2.1 give us the following:

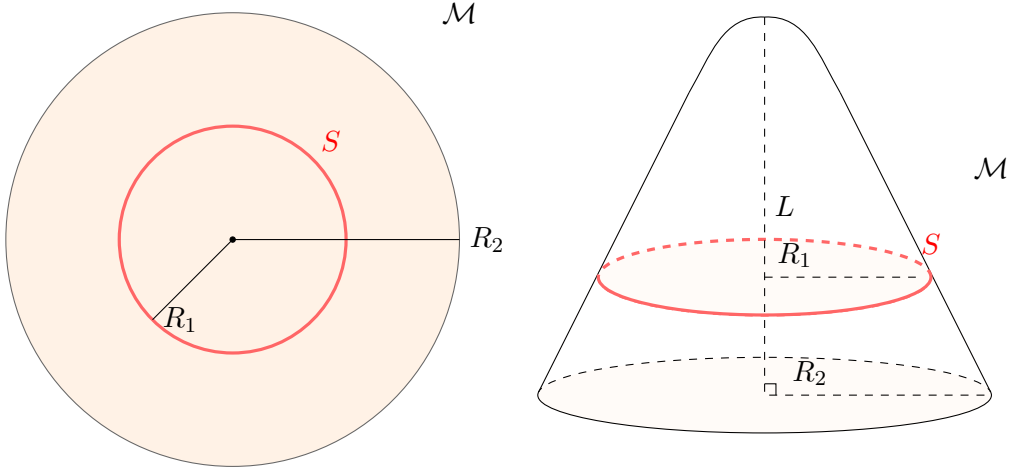


Figure 3: A disk and a surface of revolution diffeomorphic to the disk for which Theorem 3.3 applies under an assumption for the coefficient c similar to the one considered for the annulus.

Proposition 3.1. *For all $n \in \mathbb{N}$ there exists $u_n \in \mathcal{A}$ such that:*

$$-\Delta_c u_n = n^2 \left(E_0 + O\left(\frac{1}{n^{2/3}}\right) \right) u_n, \quad \|u_n\|_{L^2(\mathcal{M})} = 1, \quad u_n(s, \theta) = e^{in\theta} \psi_n(s).$$

We can then repeat the same steps as in Section 2. We define as in (2.10) the Agmon distance to the energy level E by

$$d_{A,E}(s) = \inf_{y \in K_E} \left| \int_y^r \sqrt{\frac{(V_c(x) - E)_+}{c(x)}} dx \right|,$$

which satisfies the equation $c|\partial_s d_{A,E}|^2 = (V_c(s) - E)_+$, in $\mathcal{D}'((0, L))$. Working directly on \mathcal{M} and using the fact that $u_n \in \mathcal{A}$ allows us to obtain the key identity of Lemma 2.2. We define the Agmon distance on \mathcal{M} as the pullback by ζ of the Agmon distance defined for $s \in (0, L]$.

We now assume that $V_c^{-1}(E_0) = \{s_0\}$. That is to say

$$\frac{1}{R^2(s)} > \frac{1}{R^2(s_0)}, \quad s \in (0, s_0) \quad \text{and} \quad \frac{c_-}{R^2(s_0)} < \min_{s \in [s_0, L]} \frac{c_+}{R^2(s)}.$$

With the key identity at hand we obtain then the Agmon estimate of Proposition 3.2. The proof is very similar to the proof of Proposition 2.3, but needs some care with respect to the degeneracy at the pole N where the Agmon distance tends to infinity (see Figure 2). This issue is treated in [LL22, Theorem 3.9] and we omit it.

Proposition 3.2. *For all $\delta > 0$ and $\epsilon(n)$ with $\epsilon(n) \xrightarrow{n \rightarrow \infty} 0$, there exist C, n_0 such that for all u_n satisfying*

$$-\Delta_c u_n = n^2(E_0 + \epsilon(n))u_n, \quad \|u_n\|_{L^2} = 1, \quad u_n \in \mathcal{A},$$

one has for $n \geq n_0$:

$$\left\| e^{nd_{A,E_0}} u_n \right\|_{L^2(\mathcal{M})} \leq C e^{n\delta}.$$

This gives finally the following result:

Theorem 3.3. Consider \mathcal{M}, S, c as defined in Section 3. Suppose that $V_c^{-1}(E_0) = \{s_0\}$.

Then there exist sequences $(\lambda_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}, (u_n)_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ such that for all $\omega \subset \mathcal{M}$ with $\text{dist}_g(\bar{\omega}, S) > 0$ there exist $C, d > 0$ satisfying:

$$-\Delta_c u_n = \lambda_n u_n, \quad \lambda_n \underset{n \rightarrow +\infty}{\sim} E_0 n^2, \quad \|u_n\|_{L^2(\mathcal{M})} = 1, \quad \|u_n\|_{L^2(\omega)} \leq C e^{-dn},$$

for all $n \in \mathbb{N}$.

Remark 3.4. If one supposes that the function $s \mapsto R(s)$ is increasing (this is the case if for instance \mathcal{M} is a disk or a cone-like surface) then the assumption $V_c^{-1}(E_0) = \{s_0\}$ is equivalent to

$$\frac{c_-}{R_1^2} < \frac{c_+}{R_2^2},$$

where $R_1 := R(s_0)$ and $R_2 := R(L)$. This is the same assumption as (2.9) for the annulus.

Remark 3.5. Theorem 3.3 and its proof are equally valid in the case where the coefficient c is rotationally invariant and piecewise smooth (not necessarily constant) satisfying $0 < c_{\min} \leq c \leq c_{\max}$. Notice that in the case of a disk or an annulus, in order to satisfy $V_c^{-1}(E_0) = \{s_0\}$ with $s_0 \in (0, L)$ the coefficient $c(s)$ has indeed to present a jump discontinuity at $s = s_0$ if one supposes that c is constant in Ω_+ (in terms of physical applications that means that the refractive index of the outer cladding of the optical fiber is constant).

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