Extremal shot noise processes and random cutout sets

Clément Foucart, Linglong Yuan

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Clément Foucart[1] and Linglong Yuan[2]

[1]Université Sorbonne Paris Nord, [2]University of Liverpool

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Abstract

We study some fundamental properties, such as the transience, the recurrence, the first passage times and the zero-set of a certain type of sawtooth Markov processes, called extremal shot noise processes. The sets of zeros of the latter are Mandelbrot's random cutout sets, i.e. the sets obtained after placing Poisson random covering intervals on the positive half-line. Based on this connection, we provide a new proof of Fitzsimmons-Fristedt-Shepp Theorem [FFS85] which characterizes the random cutout sets.

Key words. Extremal process, Shot noise process, Subordinator, First passage times, Random covering, Sawtooth process, Invariant function.

MSC (2010): primary 60K30, 60G70; secondary 60J35, 60J50, 60J55

1 Introduction

Extremal shot noise processes (ESNs) first appeared in the eighties in the framework of applied stochastic geometry and of random sets for modelling extremes in a spatial setting, see Serra [Ser82, page 470] and Heinrich and Molchanov [HM94]. They have been then reintroduced in a more general setting by Dombry [Dom12] who has studied some of their properties and shed light on their connection with max-stable random fields.

We work here in the setting of one-dimensional Markov processes. Let $\mathcal{N} := \sum_{s\geq 0} \delta_{(s,\xi_s)}$ be a Poisson point process (PPP) on $[0,\infty)\times(0,\infty)$ with intensity $\lambda\times\mu$ with μ a Borel measure on $(0,\infty)$ and λ the Lebesgue measure. We denote the tail of μ by $\bar{\mu}(x) = \mu([x,\infty))$ and suppose $\bar{\mu}(x) < \infty$ for all $x > 0$. Denote by $(a)_+ := \max(a,0)$, the positive part of any real number a. Let $b \in \mathbb{R}$.

Definition 1. *We call standard* $\mathrm{ESN}(b,\mu)$ *the process* $(M(s), s \geq 0)$ *valued in* $[0,\infty)$ *and obtained from* $\mathcal{N}$ *as follows:*

$$M(t) := \sup_{0\leq s\leq t} \big(\xi_s - b(t-s)\big)_+.$$

When $b = 0$, the process M is a classical extremal process. We refer to Resnick's book [Res08, Chapter 4.3], see also the references therein. When $b \neq 0$, the contribution of any atom to the process is affected by its age, which is the so-called shot noise structure.

Extremal shot noise processes as defined above form a certain class of sawtooth processes, in the sense that they evolve linearly or stay constant between their jumps. Such processes are known to play an important role in the theory of Markov processes, see for instance Blumenthal [Blu92, page 49]. We refer also to Boxma et al. [BPSZ06] and Löpker and Stadje [LS11] for works on a general class of sawtooth processes close to ESNs. It turns out that many natural problems can be solved with closed-form solutions for ESNs. Their finite dimensional laws, their semigroup and their long-term behavior are for instance obtained in Theorem 1. The generator is studied in Theorem 2, and last but not least, the Laplace transform of their first-passage times, also available explicitly in terms of b and μ, is given in Theorem 4.

Random cutout sets were introduced by Mandelbrot [Man72]. They are defined as the sets of real numbers left uncovered by Poisson random covering intervals on the positive half line. Namely, the random cutout set based on the PPP $\mathcal{N}$ is given by

$$\mathcal{R} := [0, \infty) - \bigcup_{s \geq 0} (s, s + \xi_s). \tag{1.1}$$

Those random sets are at the core of the theory of random coverings, see Kahane [Kah00] and [Kah90]. Barral and Fan have studied in [BF04] and [BF05] some of their multifractal properties. They also appear in many other contexts. We refer the reader for instance to Bertoin [Ber91] and [Ber94] where they are used for studying the existence of increase times in Lévy processes and the differentiability of their sample paths. They also play a crucial role in the study of zero-sets of certain processes, see e.g. Bi and Delmas [BD14], Evans and Ralph [ER10] and Foucart and Uribe Bravo [FUB14]. Some random sets with closely related constructions are studied in Marchal [Mar15] and Rivero [Riv03].

The question of when the random set $\mathcal{R}$ is almost surely reduced to the singleton $\{0\}$, that is to say when the whole open half-line is covered, was asked in [Man72] and the necessary and sufficient condition was found by Shepp [She72]. A definitive answer on how to characterize the law of the random cutout set is given by Fitzsimmons-Fristedt-Shepp's Theorem [FFS85], with the potential measure of $\mathcal{R}$ given explicitly. Arguments in [FFS85] were based on approximations of $\mathcal{R}$ by intersections of regenerative sets. We refer also to Bertoin [Ber99, Chapter 7].

A striking feature of the ESN process lies in its simple connection with the random cutout set associated to $\mathcal{N}$. We shall see that the random set $\mathcal{R}$ in (1.1) coincides with the closure of the zero-set of a standard ESN$(1, \mu)$ process. The main contribution of the paper is a new proof of Fitzsimmons-Fristedt-Shepp's Theorem, see Corollary 1, based on this connection and on classical arguments of potential theory of Markov processes. Most important properties of $\mathcal{R}$, such as the regenerative property and the fact that it is a perfect set (i.e. it has no isolated point), will also directly follow from this representation.

Notation. Let $\mathbb{R}_+ := [0, \infty)$. For any subset $A \subset \mathbb{R}_+$, we denote its closure by $\bar{A}$. For any $x, y \in \mathbb{R}$, we denote by $x \wedge y$ and $x \vee y$ the minimum and the maximum of x and y. In any integral $\int_a^b$, we adhere to the convention that the lower delimiter a is excluded from the integration, while the upper delimiter b is included (except for $b = \infty$). We denote by $C_0([0, \infty))$ the space of continuous functions vanishing at ∞, and by $\|f\|_\infty$ the supremum norm of f. We set $C_b^1([0, \infty))$ the space of continuously differentiable bounded functions and $C_c^1([0, \infty))$ the space of those with compact support included in $[0, \infty)$. For any function f, we denote by $f_{|[a,b)}$ the restriction of f on the interval $[a, b)$. The limit inferior and superior of a function f are denoted respectively by $\underline{\lim}\, f$ and $\overline{\lim}\, f$. For any event A, we denote by A^c the complementary event. Lastly, $X \overset{\text{law}}{=} Y$ means that the random variables X and Y have the same law.

2 Extremal shot noise processes as Markov processes

We first collect some basic observations from Definition 1. Let M be a standard ESN(b,μ) process. The process M takes only positive values, starts from 0 and has clearly càdlàg paths.

In the case $b \le 0$ (i.e. the slopes are nonnegative) the process has almost-surely non-decreasing sample paths which go towards ∞. When $b < 0$, paths are increasing, and they are not monotonic when $b > 0$, see Figure 1 below. Note also that by construction, for any $b \in \mathbb{R}$, we have $M(t) \ge (-bt)_+ \ge 0$ for all $t \ge 0$, $\mathbb{P}$-a.s.. We shall mainly focus on the case $b \neq 0$ in the article.

The Markov property of the Poisson point process $\mathcal{N}$ and the fact that the "response function" $s \mapsto -b(t-s)$ in the shot noise structure is linear in time will imply that the standard ESN process $(M(t), t \ge 0)$ is a time-homogeneous Markov process. In particular there exists a family of probability distributions $(\mathbb{P}_x, x \in \mathbb{R}_+)$ on the space of non-negative càdlàg paths such that $\mathbb{P}_x$ is the law of the process M started with initial value $M_0 = x$. The probability law $\mathbb{P}_x$ can be constructed on the same probability space as $\mathcal{N}$ by adjoining a point $(0,x)$ to the PPP $\mathcal{N}$. Namely set $\mathcal{N}^x := \delta_{(0,x)} + \mathcal{N}$ and define

$$M^x(t) = \sup_{0\le s\le t} \left\{ \left(\xi_s - b(t-s)\right)_+ : (s,\xi_s) \text{ is an atom of } \mathcal{N}^x \right\} = (x-bt)_+ \vee M^0(t), \tag{2.2}$$

where M^0 is the standard ESN(b,μ) process.

We see from (2.2) that the process will leave x along $(x-bt)_+$ until it encounters the first atom of $\mathcal{N}$ satisfying $\xi > (x-bt)_+$ and jumps there. If the process is able to reach the boundary 0, then it stays at 0 until the next atom of $\mathcal{N}$.

In particular, any point $x > 0$ is instantaneous (it is left immediately) and is irregular for itself (the process does not return to it immediately). Indeed, when $b < 0$, the paths being increasing, the point x will not be reached again. When $b > 0$, the process will return to x by firstly getting back above it and secondly reaching it by linear decay. Since by assumption $\bar{\mu}(x) < \infty$, it can occur only after a strictly positive time a.s..

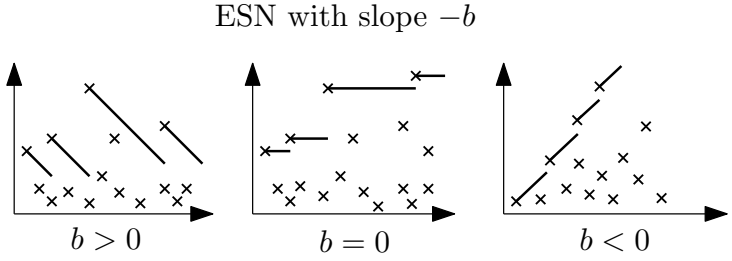

Figure 1: Sample paths of an ESN(b,μ).

When $b > 0$ and $\bar{\mu}(0) < \infty$, it is clear that the point 0 will be reached with positive probability. This is furthermore an holding point. Indeed the process started at 0 stays at 0 for an exponential time with parameter $\bar{\mu}(0)$, the rate at which a new atom of $\mathcal{N}$ is encountered. When $\bar{\mu}(0) = \infty$, the set $\{0 \le s \le t : \xi_s \in (0,1)\}$ is a dense subset of $[0,t]$. Hence, when starting with initial value 0, the process M will make immediately a positive jump a.s..

The only point at which the process may have a non trivial behavior is thus 0. A natural question in the case $b > 0$ is whether or not the negative slopes are strong enough for the paths to be able to reach 0 when $\bar{\mu}(0) = \infty$. We may also wonder if the process can be transient when $b > 0$. Before tackling this problem, see Theorem 4, we gather in our first theorem fundamental properties of ESNs, as for instance their Markov property.

Theorem 1 (Finite dimensional laws, semigroup and stationary distribution)**.** *Let $b \in \mathbb{R}$, μ be a measure on $(0,\infty)$ and M be a* ESN(b,μ).

1. *Let $n \geq 1$ and $0 = s_0 < s_1 < s_2 < \cdots < s_n$. For any $u_1, \cdots, u_n \in \mathbb{R}_+$.*

$$\begin{aligned}&\mathbb{P}_0(M(s_1) \leq u_1, \cdots, M(s_n) \leq u_n)\\ &= \exp\left(-\sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} \bar{\mu}\left(\bigwedge_{j=i}^{n}\Big(u_j + b(s_j - t)\Big)\right)\mathrm{d}t\right)\prod_{i=1}^{n} \mathbb{1}_{\{u_i \geq (-bs_i)_+\}}. \end{aligned} \tag{2.3}$$

   In particular the one-dimensional law at time s has the following cumulative distribution function: for any $u \in \mathbb{R}$,

$$F_s^0(u) := \mathbb{P}_0(M(s) \leq u) = \exp\left(-\int_0^s \bar{\mu}\left(u + b(s-t)\right)\mathrm{d}t\right)\mathbb{1}_{\{u \geq (-bs)_+\}} \tag{2.4}$$

$$= \exp\left(-\frac{1}{b}\int_u^{u+bs} \bar{\mu}(y)\mathrm{d}y\right)\mathbb{1}_{\{u \geq (-bs)_+\}} \text{ if } b \neq 0. \tag{2.5}$$

2. *Let $x \in [0,\infty)$ and $t \geq 0$. For any $u \in \mathbb{R}$,*

$$F_t^x(u) := \mathbb{P}_x(M(t) \leq u) = F_t^0(u)\mathbb{1}_{\{u \geq (x-bt)_+\}}.$$

   Namely the random variable $M(t)$ under $\mathbb{P}_x$ takes the value $(x-bt)_+$ with probability $F_t^0\big(x-bt)_+\big)$ and conditionally given that $M(t) > (x-bt)_+$, it is distributed as $M^0(t)$.

   The process M is a Markov process with the Feller property, i.e. its semigroup (P_t) satisfies

   (a) $P_t C_0([0,\infty)) \subset C_0([0,\infty))$,

   (b) For any $f \in C_0([0,\infty))$, $P_t f \underset{t\to 0}{\longrightarrow} f$ uniformly.

3. *Assume $b > 0$. The following equivalence holds*

$$\forall s > 0, \mathbb{P}_0(M(s) = 0) > 0 \text{ if and only if } \int_0^1 \bar{\mu}(v)\mathrm{d}v < \infty.$$

4. *Assume $b > 0$. The process M admits a stationary distribution if and only if $\int_1^\infty \bar{\mu}(u)\mathrm{d}u < \infty$. In this case, it is unique and it is the limiting distribution of M: namely the stationary distribution π satisfies for any $u \geq 0$,*

$$\pi([0,u]) = \mathbb{P}_0(M(\infty) \leq u) := \lim_{t\to\infty} \mathbb{P}_0(M(t) \leq u) = \exp\left(-\frac{1}{b}\int_u^\infty \bar{\mu}(v)\mathrm{d}v\right).$$

   If $\int_1^\infty \bar{\mu}(u)\mathrm{d}u = \infty$ then $M(s) \to \infty$ as s goes to ∞ in probability.

We now study the infinitesimal generator of M.

Theorem 2 (Generator of ESN)**.** - *Let $\mathcal{A}$ be the generator of the* ESN(b,μ) *process $(M(t), t \geq 0)$ and set*

$$\mathcal{D}^{b>0} := \{f \in C_b^1([0,\infty)) : \exists \varepsilon > 0, \text{such that } f_{|[0,\varepsilon]} \text{ is constant}\} \text{ and } \mathcal{D}^{b\leq 0} := C_c^1([0,\infty)). \tag{2.6}$$

Then according to $b > 0$ (negative slopes) or $b \leq 0$ (non negative slopes) the generator $\mathcal{A}$ acts on any $f \in \mathcal{D}^{b>0}$ or $f \in \mathcal{D}^{b\leq 0}$ as follows :

$$\mathcal{A}f(x) = \begin{cases} \int_x^\infty \Big(f(y) - f(x)\Big)\mu(\mathrm{d}y) - bf'(x) \text{ for any } x > 0, \\ \\ \int_0^\infty \Big(f(y) - f(0)\Big)\mu(\mathrm{d}y) \neq 0 \text{ for } x = 0. \end{cases} \tag{2.7}$$

Moreover the set $\mathcal{D}^{b>0}$ and $\mathcal{D}^{b\leq 0}$ are cores.

Remark 3. 1. A simple use of Fubini-Lebesgue theorem gives the following alternative expression for the generator $\mathcal{A}$. Let $f \in C_b^1([0,\infty))$, for any $x > 0$:

$$\mathcal{A}f(x) = \int_x^\infty \bar{\mu}(v) f'(v) \mathrm{d}v - b f'(x). \tag{2.8}$$

2. Although the generator takes this simple form, the $\mathrm{ESN}(b,\mu)$ process is of course not the superposition of a linear drift with an extremal process.

In the next theorem, the first passage time below any level a of M is studied and the questions whether the process is recurrent or transient and if 0 is accessible are addressed.

For any $a \in (0,\infty)$, we set $\sigma_a := \inf\{s \geq 0 : M(s) \leq a\}$.

Theorem 4 (First passage times, transience/recurrence and zero-set)**.** *Let M be an $\mathrm{ESN}(b,\mu)$ with $b > 0$.*

1. *Let $\theta > 0$. Define for any $x > 0$,*

$$f_\theta(x) := \int_x^\infty e^{-\frac{\theta}{b}s} \exp\left(\frac{1}{b}\int_s^1 \bar{\mu}(u)\mathrm{d}u\right)\mathrm{d}s. \tag{2.9}$$

One has $f_\theta(x) < \infty$ for all $x > 0$. For any $x > a > 0$, the Laplace transform of σ_a is given by

$$\mathbb{E}_x[e^{-\theta\sigma_a}] = \frac{f_\theta(x)}{f_\theta(a)}.$$

2. *Set*

$$\mathcal{I} := \int_1^\infty \exp\left(\frac{1}{b}\int_s^1 \bar{\mu}(v)\mathrm{d}v\right)\mathrm{d}s. \tag{2.10}$$

We have the following dichotomy:

- *If $\mathcal{I} = \infty$ then M is recurrent (i.e. it returns almost surely to any point $a > 0$). Moreover,*
  - *in the case $\int_1^\infty \bar{\mu}(v)\mathrm{d}v = \infty$, M is null recurrent,*
  - *in the case $\int_1^\infty \bar{\mu}(v)\mathrm{d}v < \infty$, M is positive recurrent.*
- *If $\mathcal{I} < \infty$ then M is transient (i.e. $M(s) \to \infty$ a.s.)*

3. *Set*

$$\mathcal{J} := \int_0^1 \exp\left(\frac{1}{b}\int_s^1 \bar{\mu}(v)\mathrm{d}v\right)\mathrm{d}s. \tag{2.11}$$

We have the following dichotomy:

- *If $\mathcal{J} = \infty$ then 0 is inaccessible (i.e. $M(s) > 0$ for all $s > 0$ almost surely).*
- *If $\mathcal{J} < \infty$ then 0 is accessible (i.e. $M(s) = 0$ for some $s > 0$ with positive probability). Moreover, the inverse of the local time at 0 of M is a subordinator $(\tau_x, 0 \leq x < \zeta)$ with lifetime ζ and whose Laplace exponent is $\varphi : \theta \mapsto c/f_\theta(0)$ for some renormalisation constant $c > 0$.*

Remark 5. A strong parallel can be drawn between the first passage times of ESNs and those of branching processes with immigration, see Duhalde et al. [DFM14].

The next lemma establishes the connection between extremal shot noise processes and random cutout sets.

Lemma 1 (Form of the zero-set of ESN). *Let $b > 0$. The closure of the zero-set of the* ESN(b, μ) *process, $(M(t), t \geq 0)$, is of the following form*

$$\bar{\mathcal{Z}} = \overline{\{t \geq 0 : M(t) = 0\}} = [0, \infty) - \bigcup_{s \geq 0} (s, s + \xi_s / b) \text{ a.s..}$$

Lemma 1 and Theorem 4-(3) have for direct corollary Fitzsimmons-Fristedt-Shepp Theorem, see [FFS85], which characterizes the random cutout set.

Corollary 1 (Theorem 1 in [FFS85]). *Let μ be a measure on $(0, \infty)$ and $\mathcal{N} := \sum_{s \geq 0} \delta_{(s, \xi_s)}$ be a PPP on $[0, \infty) \times (0, \infty)$ with intensity $\lambda \times \mu$. Consider the random cutout set*

$$\mathcal{R} := [0, \infty) - \bigcup_{s \geq 0} (s, s + \xi_s),$$

we have that

1. *$\mathcal{R} = \{0\}$ a.s. if and only if $\int_0^1 \exp\left(\int_s^1 \bar{\mu}(v) \mathrm{d}v\right) \mathrm{d}s = \infty$ (Shepp's criterion, see [She72]).*
2. *When $\int_0^1 \exp\left(\int_s^1 \bar{\mu}(v) \mathrm{d}v\right) \mathrm{d}s < \infty$,*

$$\mathcal{R} \overset{\text{law}}{=} \overline{\{\tau_x : 0 \leq x < \zeta\}},$$

where $(\tau_x, 0 \leq x < \zeta)$ is a subordinator with lifetime ζ and Laplace exponent $\theta \mapsto c / f_\theta(0)$ for some positive constant $c > 0$. In particular $\mathcal{R}$ is a regenerative set which is almost surely perfect, i.e. $\mathcal{R}$ has no isolated point. It is furthermore bounded almost surely if and only if $\int_1^\infty \exp\left(\int_s^1 \bar{\mu}(s) \mathrm{d}s\right) \mathrm{d}s < \infty$ and has positive Lebesgue measure a.s. if and only if $\int_0^1 \bar{\mu}(s) \mathrm{d}s < \infty$.

Remark 6. ESN processes satisfy the property of max-infinite divisibility, see [Dom12, Proposition 2.3], that is to say,

$$(M(t), t \geq 0) \overset{\text{law}}{=} \left(\vee_{i=1}^n M_i(t), t \geq 0\right), \tag{2.12}$$

where M is a standard ESN(b, μ) process and the processes $(M_i, i = 1, \cdots, n)$ are i.i.d. standard ESN$(b, \frac{1}{n}\mu)$ processes. The identity (2.12) is a direct consequence of the superposition theorem of Poisson point processes. In terms of the random cutout sets, this entails

$$\mathcal{R} := \overline{\{t \geq 0 : M(t) = 0\}} \overset{\text{law}}{=} \overline{\{t \geq 0 : \vee_{i=1}^n M_i(t) = 0\}} = \cap_{i=1}^n \mathcal{R}_i,$$

with $\mathcal{R}_i := \overline{\{t \geq 0 : M_i(t) = 0\}}$ for $i = 1, \cdots, n$. We recover here the fact that the random cutout sets are infinitely divisible for the intersection, see [FFS85, Section 5] and Fristedt [Fri96].

Many explicit examples can be designed by choosing specific tails for the measure μ. We refer the reader to the examples in [FFS85]. We only shed light on the following important example in which the inverse local time at 0 of the standard ESN is a stable subordinator.

Example 1 (Selfsimilar ESN with negative slopes). Let $b \in \mathbb{R}_+$ and $c > 0$. Assume $\bar{\mu}(x) = \frac{c}{x}$ for all $x > 0$ and let M be an ESN(b, μ). Then

1. For any $x \geq 0$, $t \geq 0$,

$$F_t^x(u) := \mathbb{P}_x(M(t) \leq u) = \left(\frac{1}{1 + \frac{bt}{u}}\right)^{c/b} \mathbb{1}_{\{u \geq (x - bt)_+\}}.$$

Moreover, for any $k > 0$ and $u \geq 0$, $F_{t/k}^x(u/k) = F_t^{kx}(u)$ so that $(kM^x(t/k), t \geq 0)$ has same law as $(M^{kx}(t), t \geq 0)$, i.e. M is selfsimilar with index 1.

2. The finite-dimensional law of M satisfies for any $u_1, \cdots, u_n \in \mathbb{R}$,

$$\mathbb{P}_0(M(s_1) \le u_1, \cdots M(s_n) \le u_n) = \prod_{i=1}^{n} \bigwedge_{j=i}^{n} \left(\frac{u_j + b(s_j - s_{i-1})}{u_j + b(s_j - s_i)} \right)^{\frac{c}{b}} \mathbb{1}_{\{u_i \ge (-bs_i)_+\}}, \tag{2.13}$$

where $0 = s_0 < s_1 < \cdots < s_n$.

3. When $b > 0$, Theorem 4-(2) ensures that when $c/b > 1$, M is transient, otherwise it is null recurrent. Furthermore, by applying Theorem 4-(3), we see that 0 is accessible for M if and only if $c/b < 1$. In this case, $f_\theta(0) = \theta^{c/b-1}$ for all $\theta > 0$ (up to a multiplicative positive constant) and by Theorem 4-(3), the zero-set of M is the closure of the range of a stable subordinator with index $1 - c/b$.

Remark 7. The selfsimilar ESN(b, μ) process studied in Example 1 appears as the functional limit of certain Galton-Watson processes with immigration, see Iksanov and Kabluchko in [IK18]. They have shown that if $(Y_n, n \in \mathbb{N})$ is a Galton-Watson process with immigration (GWI) whose offspring distribution has finite mean m and whose immigration distribution, say ν, is such that $\bar{\nu}(n) \underset{n\to\infty}{\sim} \frac{c}{\log n}$ for some $c > 0$ then

$$\left(\frac{1}{n} \big(\log Y_{[ns]} \big)_+, s \ge 0 \right) \underset{n\to\infty}{\Longrightarrow} (M(s), s \ge 0)$$

where M is a selfsimilar standard ESN with $b = -\log m \in \mathbb{R}$ and with $\bar{\mu}(x) = c/x$ for all $x > 0$. The process M describes therefore the evolution of the growth rate of the GWI process. In the subcritical case ($b > 0$), a phase transition occurs in the growth rate: it is null-recurrent when $c/b \le 1$ and transient otherwise. It does not return to 0 almost surely if and only if $c/b \ge 1$. In the supercritical case ($b < 0$), the process M is transient and becomes less and less "jumping" as time goes to ∞, this is simply due to the fact that fewer and fewer atoms of $\mathcal{N}$ can be found above the line $t \mapsto -bt$, see Figure 1. At an intuitive level, this is saying that the growth rate is mostly governed by the first immigration arrivals.

3 Study of ESN processes

We establish here the results of Section 2.

3.1 Finite-dimensional laws, semigroup and stationary law of ESNs: proof of Theorem 1

Proof of Theorem 1-(1). Recall $\mathcal{N}$ the Poisson point process with intensity $\lambda \times \mu$ and the Poisson construction of M in Definition 1. Recall that almost surely for all $s \ge 0$, $M(s) \ge (-bs)_+$. Let $s_1 > 0$ and $u_1 \in [0, \infty)$. The event $\{M(s_1) \le u_1\}$ coincides almost surely with the event that all atoms (t, ξ_t) of $\mathcal{N}$ on $[0, s_1]$ are such that $(\xi_t - b(s_1 - t))_+ \le u_1$. Note that since $u_1 \ge 0$, the inequality is equivalent to $\xi_t - b(s_1 - t) \le u_1$, and since any atom ξ_t is positive, it is also equivalent to $\xi_t \le (u_1 + b(s_1 - t))_+$, a.s.. More generally, for any $s_1 < s_2 < \cdots < s_n$ and $u_1, \ldots, u_n \ge 0$,

$$\begin{aligned}
&\{M(s_1) \le u_1, M(s_2) \le u_2, \cdots, M(s_n) \le u_n\} \\
&= \Big\{ \forall t \in [0, s_1], \big(\xi_t - b(s_1 - t)\big)_+ \le u_1, \forall t \in [0, s_2], \big(\xi_t - b(s_2 - t)\big)_+ \le u_2, \cdots, \\
&\qquad\qquad \forall t \in [0, s_n], \big(\xi_t - b(s_n - t)\big)_+ \le u_n \Big\} \cap \{u_1 \ge (-bs_1)_+\} \cap \cdots \cap \{u_n \ge (-bs_n)_+\} \\
&= \big\{ \mathcal{N}(A^c) = 0 \big\} \cap \{u_1 \ge (-bs_1)_+\} \cap \cdots \cap \{u_n \ge (-bs_n)_+\},
\end{aligned}$$

with A, obtained by gathering all conditions on each disjoint intervals $(s_i, s_{i+1}]$, given by

$$A := \Big\{ 0 \le t < s_1, \xi \le \bigwedge_{i=1}^{n} \big(u_i + b(s_i - t)\big)_+, s_1 \le t < s_2, \xi \le \bigwedge_{i=2}^{n} \big(u_i + b(s_i - t)\big)_+, \cdots,$$
$$s_{n-1} \le t \le s_n, \xi \le \big(u_n + b(s_n - t)\big)_+ \Big\}.$$

Finally, since $\mathcal{N}(A^c)$ is a Poisson random variable with parameter

$$(\lambda \times \mu)(A^c) = \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} \bar{\mu}\left(\bigwedge_{j=i}^{n} \big(u_j + b(s_j - t)\big)_+ \right) \mathrm{d}t,$$

we get

$$\mathbb{P}_0(M(s_1) \le u_1, \cdots M(s_n) \le u_n) = \mathbb{P}(\mathcal{N}(A^c) = 0) \mathbb{1}_{\{u_i \ge (-bs_i)_+, \forall 1 \le i \le n\}}$$
$$= \exp\left(- \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} \bar{\mu}\Big(\bigwedge_{j=i}^{n} \big(u_j + b(s_j - t)\big)\Big) \mathrm{d}t \right) \mathbb{1}_{\{u_i \ge (-bs_i)_+, \forall 1 \le i \le n\}}.$$

The case $n = 1, s_1 = s, u_1 = u$ gives (2.4), namely

$$F_s^0(u) := \mathbb{P}_0(M(s) \le u) = \exp\left(- \int_0^s \bar{\mu}\left(u + b(s - t)\right) \mathrm{d}t \right) \mathbb{1}_{\{u \ge (-bs)_+\}},$$

and (2.5) is obtained by change of variable. □

Proof of Theorem 1: (2). Recall that the ESN process started from x is defined at any time t by (2.2), namely $M^x(t) = M^0(t) \vee (x - bt)_+$. One has

$$\mathbb{P}(M^x(t) \le y) = F_t^0(y) \mathbb{1}_{\{y \ge (x - bt)_+\}} = F_t^0(y) \mathbb{1}_{\{y \ge x - bt\}}. \tag{3.14}$$

Since the process M takes only non-negative values, $F_t^0(y) = 0$ if $y < 0$ and one can replace in (3.14) in the indicator function $(x - bt)_+$ by $x - bt$. We shall use both writings. The expression in (3.14) is simpler to handle in some calculations.

The fact that M satisfies the Markov property is checked as follows. Let $s, t \ge 0$ and $x \ge 0$, then

$$\begin{aligned} M^x(t + s) &= \sup_{\substack{0 \le u \le t+s \\ \xi_0 = x}} (\xi_u - b(s + t - u))_+ \\ &= \sup_{\substack{0 \le u \le t \\ \xi_0 = x}} (\xi_u - b(t - u) - bs)_+ \vee \sup_{\substack{t \le u \le t+s \\ \xi_0 = x}} (\xi_u - b(s + t - u))_+ \\ &= (M^x(t) - bs)_+ \vee M(t, t + s), \end{aligned}$$

with $M(t, t + s) := \sup_{t \le u \le t+s} (\xi_u - b(t + s - u))_+ = \sup_{0 \le u \le s} (\xi_{u+t} - b(s - u))_+$ which is independent from $M^x(t)$. Note that $(M(t, t + s), s \ge 0)$ is a standard ENS(b, μ) constructed from the PPP $\mathcal{N}$ shifted by time t.

We now check the Feller property.

a) Let $f \in C_0([0, \infty))$. We plainly see from (2.2) that almost surely for any $x_0 \ge 0$ and any $t \ge 0$, $M^x(t) \to M^{x_0}(t)$ as x goes to x_0. For the case $x_0 = 0$, recall that $M^0(t) \ge -bt$ a.s. so that $(-bt) \vee M^0(t) = M^0(t)$. Therefore by continuity under expectation, the map $P_t f : x \mapsto \mathbb{E}[f(M^x(t)]$ is continuous on $[0, \infty)$.

b) For any $x \in [0,\infty)$, and all $t \geq 0$, by (3.14)

$$\mathbb{P}_x(M(t) \leq u) = \mathbb{P}_0(M(t) \leq u)\mathbb{1}_{\{x-bt\leq u\}}.$$

By (2.4), $\mathbb{P}_0(M(t) \leq u) \to 1$ as t goes to 0. Therefore, as t goes to 0, $\mathbb{P}_x(M(t) \leq u) \to \mathbb{1}_{\{x\leq u\}}$ and $M(t)$ converges in law towards x under $\mathbb{P}_x$. This implies the pointwise continuity of the semigroup for given $x \geq 0$ as $t \to 0$, which is equivalent to the uniform one since $P_t C_0([0,\infty)) \subset C_0([0,\infty))$, see e.g. Rogers and Williams book [RW00, Lemma 6.7 Chapter III].

Proof of Theorem 1: (3) and (4). Recall $b > 0$. Since we assume (2.5), for any $u \geq 0$,

$$F_s^0(u) := \mathbb{P}_0(M(s) \leq u) = \exp\left(-\frac{1}{b}\int_u^{u+bs} \bar{\mu}(y)\mathrm{d}y\right).$$

1. By letting u go to 0 in the expression above, we see that for all $s > 0$, $\mathbb{P}_0(M(s) = 0) = e^{-\frac{1}{b}\int_0^{bs}\bar{\mu}(v)\mathrm{d}v}$. The latter is strictly positive if and only if $\int_0^1 \bar{\mu}(v)\mathrm{d}v < \infty$.

2. By letting s go to ∞, we see that

$$\lim_{s\to\infty} \mathbb{P}_0(M(s) \leq u) = \exp\left(-\frac{1}{b}\int_u^{\infty} \bar{\mu}(y)\mathrm{d}y\right) = \begin{cases} 0 & \text{if } \int_1^\infty \bar{\mu}(y)\mathrm{d}y = \infty \\ > 0 & \text{if } \int_1^\infty \bar{\mu}(y)\mathrm{d}y < \infty. \end{cases}$$

Hence, the process converges towards ∞ in probability if and only if $\int_1^\infty \bar{\mu}(y)\mathrm{d}y = \infty$. Assume now $\int_1^\infty \bar{\mu}(y)\mathrm{d}y < \infty$ and let π be a stationary distribution. Then for any $y \geq 0$,

$$\begin{aligned}\pi([0,y]) = \int_0^\infty \mathbb{P}_x(M(t) \leq y)\pi(\mathrm{d}x) &= \int_0^\infty \mathbb{P}_0(M(t) \leq y)\mathbb{1}_{\{y\geq x-bt\}}\pi(\mathrm{d}x) \\ &= \mathbb{P}_0(M(t) \leq y)\pi([0,y+bt]).\end{aligned}$$

By letting t to ∞, we see that π exists if and only if M admits a limiting distribution and that in this case π and the latter coincide.

□

3.2 Infinitesimal generator of ESNs: proof of Theorem 2.

Proof. Note that since the process is Feller, the generator $\mathcal{A}$ (obtained as a strong derivative of the semigroup) matches with the weak generator, see e.g. Böttcher et al. [BSW13, Theorem 1.33].

Let f be a $C_b^1([0,\infty))$ function. We see from (3.14) that the semigroup of M takes the form

$$\mathbb{E}_x[f(M(t))] = f\big((x-bt)_+\big)F_t^0\big((x-bt)_+\big) + \int_{((x-bt)_+,\infty)} f(m)\mathrm{d}F_t^0(m) \tag{3.15}$$

where $\mathrm{d}F_t^0$ denotes the Stieltjes measure associated to F_t^0 restricted on $(0,\infty)$.

One has for all $x > 0$

$$\begin{aligned}\lim_{t\to 0+}\frac{\mathbb{E}_x[f(M(t))] - f(x)}{t} &= \lim_{t\to 0+}\frac{1}{t}\big(f\big((x-bt)_+\big) - f(x)\big)F_t^0\big((x-bt)_+\big) &(3.16)\\ &\quad + \lim_{t\to 0+}\frac{1}{t}\int_{(x-bt)_+}^\infty (f(y)-f(x))\mathrm{d}F_t^0(y). &(3.17)\end{aligned}$$

For (3.16), since for any $x > 0$, $\mathbb{P}_0(M(t) \le (x - bt)_+) = F_t^0((x - bt)_+) \to 1$ as t goes to 0, and $f \in C^1$, we have that the middle term below can be deleted

$$\lim_{t\to 0+} \frac{1}{t}(f\big((x-bt)_+\big) - f(x))F_t^0\big((x-bt)_+\big) = \lim_{t\to 0+} \frac{1}{t}(f(x-bt) - f(x))F_t^0(x-bt) = -bf'(x).$$

We deal now with (3.17). By (2.4), for any $u > 0$:

$$\frac{1}{t}\mathbb{P}_0(M(t) \ge u) = \frac{1}{t}\left(1 - \exp\big(-\int_0^t \bar{\mu}(u + b(t-v))\mathrm{d}v\big)\right) \underset{t\to 0}{\to} \bar{\mu}(u). \tag{3.18}$$

By assumption f is continuous and bounded, hence for all $x > 0$, and for any small $\varepsilon > 0$,

$$\overline{\lim_{t\to 0+}} \left|\frac{1}{t}\int_{(x-bt)_+}^{x} (f(y) - f(x))\mathrm{d}F_t^0(y)\right| \le \sup_{y\in[x-\varepsilon,x]} |f(y) - f(x)|\mu([x-\varepsilon, x]),$$

which converges towards $0 \times \mu(\{x\}) = 0$ as $\varepsilon \to 0$. Thus $\lim_{t\to 0+} \frac{1}{t}\int_{(x-bt)_+}^{x} (f(y) - f(x))\mathrm{d}F_t^0(y) = 0$ for all $x > 0$. Combining this with (3.18) we have

$$\lim_{t\to 0+} \frac{1}{t}\int_{(x-bt)_+}^{\infty} (f(y) - f(x))\mathrm{d}F_t^0(y) = \int_x^{\infty} (f(y) - f(x))\mu(\mathrm{d}y).$$

Hence for all $x > 0$:

$$\mathcal{A}f(x) = \int_x^{\infty} (f(y) - f(x))\mu(\mathrm{d}y) - bf'(x). \tag{3.19}$$

Recall that no assumption is made on the measure μ near 0. The generator at $x = 0$ is thus more involved to study because the right-hand side above might not be well-defined when $x = 0$ even if $f \in C_b^1([0,\infty))$. We thus restrict ourselves to the function space $\mathcal{D}^{b>0}$ in the case $b > 0$ and $\mathcal{D}^{b\le 0}$ in the case $b \le 0$, see (2.6). When $b > 0$, for any $f \in \mathcal{D}^{b>0}$, the term in (3.16) vanishes at $x = 0$ for all $t \ge 0$, and a similar study of (3.17) provides

$$\mathcal{A}f(0) = \int_0^{\infty} (f(y) - f(0))\mu(\mathrm{d}y).$$

We now verify that $\mathcal{D}^{b>0}$ is a core for the ESN(b, μ). Recall (3.15). For fixed time t, with $x < bt$, we have that

$$P_t f(x) = f(0)F_t^0(0) + \int_0^{\infty} f(y)\mathrm{d}F_t^0(y) = P_t f(0),$$

so that the function $P_t f$ is also constant near the boundary 0. By assumption $f \in C^1$ and we see that $P_t f$ is differentiable on $[0,\infty)$. Since $P_t\mathcal{D}^{b>0} \subset \mathcal{D}^{b>0}$, and $\mathcal{D}^{b>0}$ is dense in $C_0([0,\infty))$, $\mathcal{D}^{b>0}$ is a core, see e.g. Kallenberg [Kal02, Proposition 19.9]. Everything works similarly when $b \le 0$. Note that the term in (3.16) vanishes at $x = 0$ for t small enough. One readily checks that $P_t\mathcal{D}^{b\le 0} \subset \mathcal{D}^{b\le 0}$ with by definition $\mathcal{D}^{b\le 0} := C_c^1([0,\infty))$.

3.3 First passage times, transience, recurrence and zero-set : proofs of Theorem 4 and Corollary 1

In all this section, we consider an ESN(b, μ) with negative slopes, i.e. $b > 0$. Notice that in the case $b \le 0$, the process has almost-surely non-decreasing sample paths and questions to be addressed in this section are pointless.

Proof of Theorem 4-(1). Let $\theta > 0$ and set for any $x > 0$:

$$f_\theta(x) := \int_x^{\infty} e^{-\frac{\theta}{b}s} \exp\left(\frac{1}{b}\int_s^1 \bar{\mu}(u)\mathrm{d}u\right)\mathrm{d}s. \tag{3.20}$$

For any $x > 0$, there is a constant $C(x) > 0$ such that $f_\theta(x) \le C(x) \int_x^\infty e^{-\frac{\theta+\bar\mu(x)}{b}s} \mathrm{d}s < \infty$. Moreover, for all $x > 0$, $f'_\theta(x) = -e^{-\frac{\theta}{b}x} \exp\left(\frac{1}{b}\int_x^1 \bar\mu(u)\mathrm{d}u\right)$. Recall the generator of M, $\mathcal{A}$ in (3.19), and the form (2.8). We verify now that f_θ is θ-invariant for $\mathcal{A}$, i.e. $\mathcal{A}f_\theta = \theta f_\theta$. One has

$$\begin{aligned}
\mathcal{A}f_\theta(x) &= \int_x^\infty \bar\mu(v) f'_\theta(v)\mathrm{d}v - b f'_\theta(x) \\
&= \int_x^\infty \big(-\bar\mu(v)\big) e^{-\frac{\theta}{b}v} \exp\left(\frac{1}{b}\int_v^1 \bar\mu(u)\mathrm{d}u\right)\mathrm{d}v + b e^{-\frac{\theta}{b}x} \exp\left(\frac{1}{b}\int_x^1 \bar\mu(u)\mathrm{d}u\right) \\
&= \left[b e^{-\frac{\theta}{b}v} \exp\left(\frac{1}{b}\int_v^1 \bar\mu(u)\mathrm{d}u\right)\right]_{v=x}^\infty + \int_x^\infty \theta e^{-\frac{\theta}{b}v} \exp\left(\frac{1}{b}\int_v^1 \bar\mu(u)\mathrm{d}u\right)\mathrm{d}v \\
&\qquad\qquad + b e^{-\frac{\theta}{b}x} \exp\left(\frac{1}{b}\int_x^1 \bar\mu(u)\mathrm{d}u\right) \\
&= \theta \int_x^\infty e^{-\frac{\theta v}{b}} \exp\left(\frac{1}{b}\int_v^1 \bar\mu(u)\mathrm{d}u\right) = \theta f_\theta(x),
\end{aligned}$$

where in the third equality we have performed an integration by parts, together with the fact that $e^{-\frac{\theta}{b}v} \exp\left(-\frac{1}{b}\int_1^v \bar\mu(u)\mathrm{d}u\right) \underset{v\to\infty}{\to} 0$.

Let $\tilde f_\theta$ be a $C_b^1([0,\infty))$ function such that $\tilde f_\theta(v) = f_\theta(v)$ for any $v \in [a/2, \infty)$ and $\tilde f_\theta$ is a constant on $[0, a/3]$. Then $\tilde f_\theta \in \mathcal{D}^{b>0}$, see (2.6), and in particular by Theorem 2 is in the domain of $\mathcal{A}$. Then by applying Dynkin's formula, see [RW00, (10.11), Chapter III.10, page 254] we have, for any $x > a$.

$$\begin{aligned}
\mathbb{E}_x[e^{-\theta\sigma_a\wedge t} f_\theta(M(\sigma_a \wedge t))] - f_\theta(x) &= \mathbb{E}_x[e^{-\theta\sigma_a\wedge t} \tilde f_\theta(M(\sigma_a \wedge t))] - \tilde f_\theta(x) \\
&= \mathbb{E}_x\left[\int_0^{\sigma_a\wedge t} e^{-\theta s}(\mathcal{A}\tilde f_\theta - \theta \tilde f_\theta)(M(s))\mathrm{d}s\right] \\
&= \mathbb{E}_x\left[\int_0^{\sigma_a\wedge t} e^{-\theta s}(\mathcal{A} f_\theta - \theta f_\theta)(M(s))\mathrm{d}s\right] = 0.
\end{aligned}$$

Since the process M has no negative jumps, one has almost surely $M(\sigma_a) = a$ on the event $\{\sigma_a < \infty\}$. The function f_θ being continuous, we get by letting t go to ∞:

$$\mathbb{E}_x[e^{-\theta\sigma_a} \mathbb{1}_{\{\sigma_a<\infty\}}] = \mathbb{E}_x[e^{-\theta\sigma_a}] = f_\theta(x)/f_\theta(a). \tag{3.21}$$

□

Proof of Theorem 4-2). We now study the recurrence and the transience of the process. Recall $\mathcal{I} := \int_1^\infty \exp\left(\frac{1}{b}\int_s^1 \bar\mu(v)\mathrm{d}v\right)\mathrm{d}s$ and define the function $g(s) := \exp\left(\frac{1}{b}\int_s^1 \bar\mu(v)\mathrm{d}v\right)$ for all $s > 0$ so that $\mathcal{I} = \int_1^\infty g(s)\mathrm{d}s$ and $f_\theta(x) = \int_x^\infty e^{-\frac{\theta}{b}s} g(s)\mathrm{d}s$ for all $x > 0$. Moreover $\int_a^1 g(s)\mathrm{d}s \le e^{\frac{\theta}{b}} f_\theta(a) < \infty$ for all $a > 0$ and by Lemma 3.21:

$$\begin{aligned}
\mathbb{E}_x[e^{-\theta\sigma_a}] = \frac{f_\theta(x)}{f_\theta(a)} &= \frac{\int_x^1 e^{-\frac{\theta}{b}s} g(s)\mathrm{d}s + \int_1^\infty e^{-\frac{\theta}{b}s} g(s)\mathrm{d}s}{\int_a^1 e^{-\frac{\theta}{b}s} g(s)\mathrm{d}s + \int_1^\infty e^{-\frac{\theta}{b}s} g(s)\mathrm{d}s} \\
&= \frac{\frac{\int_x^1 e^{-\frac{\theta}{b}s} g(s)\mathrm{d}s}{\int_1^\infty e^{-\frac{\theta}{b}s} g(s)\mathrm{d}s} + 1}{\frac{\int_a^1 e^{-\frac{\theta}{b}s} g(s)\mathrm{d}s}{\int_1^\infty e^{-\frac{\theta}{b}s} g(s)\mathrm{d}s} + 1}.
\end{aligned} \tag{3.22}$$

1. Recurrence: assume $\mathcal{I} = \infty$. One has $\int_1^\infty g(s)\mathrm{d}s = \infty$ and by monotone convergence, we have
$$f_\theta(1) = \int_1^\infty e^{-\frac{\theta}{b}s} g(s)\mathrm{d}s \underset{\theta\to 0^+}{\to} \mathcal{I} = \infty,$$

and $\int_a^1 e^{-\frac{\theta}{b}s} g(s)\mathrm{d}s \underset{\theta\to 0^+}{\to} \int_a^1 g(s)\mathrm{d}s < \infty$. Hence, by letting θ go to 0 in (3.22), we get

$$\mathbb{P}_x(\sigma_a < \infty) = \lim_{\theta\to 0^+} \mathbb{E}_x[e^{-\theta\sigma_a}] = \lim_{\theta\to 0^+} \frac{f_\theta(x)}{f_\theta(a)} = 1.$$

2. Transience: assume $\mathcal{I} < \infty$. One has $\int_1^\infty g(s)\mathrm{d}s < \infty$. By letting θ go to 0 in (3.22), we see that

$$\mathbb{P}_x(\sigma_a < \infty) = \lim_{\theta\to 0+} \frac{f_\theta(x)}{f_\theta(a)} < 1. \tag{3.23}$$

It remains to show that the process M goes to ∞ a.s.. Denote by θ_t the time shift operator, i.e. $\theta_t(M(\cdot)) = M(t+\cdot)$. For any $a > 0$, one has

$$\begin{aligned}\mathbb{P}_x(\varliminf_{t\to\infty} M(t) < a) &\le \varliminf_{t\to\infty} \mathbb{P}_x(\sigma_a \circ \theta_t < \infty) \\ &= \varliminf_{t\to\infty} \mathbb{E}_x\left[\mathbb{P}_{M(t)}(\sigma_a < \infty)\right].\end{aligned} \tag{3.24}$$

Moreover

$$\begin{aligned}\mathbb{E}_x\left[\mathbb{P}_{M(t)}(\sigma_a < \infty)\right] &\le \mathbb{P}_x(M(t) \le a) + \mathbb{E}_x\left[\mathbb{1}_{\{M(t)>a\}}\mathbb{P}_{M(t)}(\sigma_a < \infty)\right] \\ &=: \mathrm{I} + \mathrm{II}.\end{aligned}$$

Note that the condition $\mathcal{I} < \infty$ implies that $\int^\infty \bar{\mu}(u)\mathrm{d}u = \infty$ and by Theorem 1-(3), in this case the process goes to ∞ in probability. The first term I at the right-hand side in (3.24) goes towards 0 as t goes to ∞. We study now the second term II. By the assumption $\mathcal{I} < \infty$, we see that for any $a > 0$, $\theta \mapsto f_\theta(a)$ is well-defined and continuous at $\theta = 0$. Recall $f_0(x) = \int_x^\infty g(s)\mathrm{d}s$ for all $x \ge 0$. One has by (3.23) and by applying Fubini-Tonelli theorem in the last inequality,

$$\begin{aligned}\mathbb{E}_x[\mathbb{1}_{\{M(t)>a\}}\mathbb{P}_{M(t)}(\sigma_a < \infty)] &= \mathbb{E}_x\left[\mathbb{1}_{\{M(t)>a\}}\frac{f_0(M(t))}{f_0(a)}\right] \\ &= \frac{1}{f_0(a)}\mathbb{E}_x\left[\int_0^\infty \mathbb{1}_{\{a<M(t)\le s\}} g(s)\mathrm{d}s\right] \\ &\le \frac{1}{f_0(a)}\int_a^\infty \mathbb{P}_x(M(t) \le s) g(s)\mathrm{d}s.\end{aligned}$$

By dominated convergence, since $\mathbb{P}_x(M(t) \le s) \underset{t\to\infty}{\longrightarrow} 0$, we have that

$$\mathrm{II} := \mathbb{E}_x[\mathbb{1}_{\{M(t)>a\}}\mathbb{P}_{M(t)}(\sigma_a < \infty)] \underset{t\to\infty}{\longrightarrow} 0.$$

Therefore $\mathbb{P}_x(\varliminf_{t\to\infty} M(t) < a) = 0$ and since a can be arbitrarily large, $\varliminf_{t\to\infty} M(t) = \infty$ a.s.

□

Proof of Theorem 4-3). Recall $\mathcal{J} = \int_0^1 \exp\left(\frac{1}{b}\int_s^1 \bar{\mu}(v)\mathrm{d}v\right)\mathrm{d}s$. Given an initial value $x > 0$, for any $x > a_1 > a_2 > 0$, we have that $\sigma_{a_1} \le \sigma_{a_2} \le \sigma_0$ a.s.. Therefore $\sigma_{0^+} := \lim_{a\to 0^+} \uparrow \sigma_a \le \sigma_0$ a.s.. Since the process M is Feller, it is quasi-continuous to the left and one has by the absence of negative jumps: on the event $\{\sigma_{0^+} < \infty\}$:

$$M(\sigma_{0^+}) = \lim_{a\to 0^+} M(\sigma_a) = \lim_{a\to 0^+} a = 0.$$

Thus, since by definition σ_0 is the first hitting time of 0, $\sigma_{0^+} \geq \sigma_0$ a.s. This entails $\sigma_{0^+} = \sigma_0$. On the event $\sigma_{0^+} = \infty$, trivially $\sigma_0 = \infty$. To sum up, we have $\sigma_{0+} = \sigma_0$ a.s. By Lemma 3.21, for any $\theta > 0$, $\mathbb{E}_x[e^{-\theta\sigma_a}] = \frac{f_\theta(x)}{f_\theta(a)}$. In this equality, by letting a go to 0, we see that

$$\mathbb{E}_x[e^{-\theta\sigma_0}] = \frac{f_\theta(x)}{f_\theta(0)},$$

with

$$\begin{aligned} f_\theta(0) := f_\theta(0^+) &= \int_0^\infty e^{-\frac{\theta}{b}s} \exp\left(\int_s^1 \bar{\mu}(u)\mathrm{d}u\right)\mathrm{d}s \\ &= \int_0^1 e^{-\frac{\theta}{b}s} \exp\left(\int_s^1 \bar{\mu}(u)\mathrm{d}u\right)\mathrm{d}s + \int_1^\infty e^{-\frac{\theta}{b}s} \exp\left(\int_s^1 \bar{\mu}(u)\mathrm{d}u\right)\mathrm{d}s. \end{aligned}$$

The second term in the right-hand side is nothing but $f_\theta(1)$ which is always finite. The first term is finite if and only if $\mathcal{J} < \infty$. In the case $\mathcal{J} = \infty$, one therefore has $\mathbb{E}_x[e^{-\theta\sigma_0}] = 0$ and $\sigma_0 = \infty$ a.s. Otherwise, when $\mathcal{J} < \infty$, $f_\theta(0) < \infty$ and $\sigma_0 < \infty$ with positive probability.

It remains to identify the law of the inverse local time of M when 0 is accessible, i.e. when $\mathcal{J} < \infty$. Recall that this entails $f_\theta(0) < \infty$ for all $\theta > 0$. The fact that the inverse of the local time is a subordinator with Laplace exponent $\varphi(\theta) = 1/f_\theta(0)$ (up to a multiplicative constant) follows from standard arguments that we briefly explain here. We refer for the following facts to [RW00, Chapter III-16, p. 266]. The map $x \mapsto f_\theta(x) = f_\theta(0)\mathbb{E}_x[e^{-\theta\sigma_0}]$ is a θ-excessive function and is the θ-potential of the local time [1] $(L_t, t \geq 0)$ at 0 of M, namely for all $x \geq 0$ and $\theta \geq 0$, $f_\theta(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\theta t}\mathrm{d}L_t\right]$. The inverse of the local time defined at any $x \geq 0$, by $\tau_x := \inf\{t \geq 0 : L_t > x\}$, is a subordinator $(\tau_x, 0 \leq x < \zeta)$ with lifetime $\zeta = L_\infty$. We denote by φ its Laplace exponent i.e. $\mathbb{E}_0[e^{-\theta\tau_x}] = e^{-x\varphi(\theta)}$ and one has,

$$f_\theta(0) = \mathbb{E}_0\left(\int_0^\infty e^{-\theta t}\mathrm{d}L_t\right) = \mathbb{E}_0\left(\int_0^{L_\infty} e^{-\theta\tau_x}\mathrm{d}x\right) = 1/\varphi(\theta),$$

hence $\varphi(\theta) = 1/f_\theta(0)$. We recall also that the closure of the zero set of M and the closure of the range of $(\tau_x, 0 \leq x < \zeta)$ coincide. For all those latter facts on the inverse local time, we refer to Bertoin's book [Ber96, Chapter IV, page 120 and Theorem 4-(iii)]. □

The proof of Theorem 4 is achieved. We now explain the connection between ESNs and random cutout sets by establishing Lemma 1.

Proof of Lemma 1. Recall $b > 0$. Recall Definition 1. The ESN(b, μ) started at 0 is given for all $t \geq 0$ by $M(t) = \sup_{0\leq s\leq t}(\xi_s - b(t-s))_+$. We see that $M(t) > 0$ if and only if there exists an atom (s, ξ_s) such that $s \leq t$ and $\xi_s - b(t-s) > 0$, namely $t \in \bigcup_{s\geq 0}[s, s+\xi_s/b)$. Recall that $M(t) \geq 0$ a.s. Therefore $\{M(t) > 0\}^c = \{M(t) = 0\}$ for all t a.s. and

$$\mathcal{Z} = \{t > 0 : M(t) = 0\} = [0, \infty[- \bigcup_{s\geq 0}[s, s + \xi_s/b).$$

Plainly, $\bar{\mathcal{Z}} := \overline{\{t > 0 : M(t) = 0\}} \subset [0, \infty[- \bigcup_{s\geq 0}(s, s + \xi_s/b)$. Moreover, the only points which belong to $\bar{\mathcal{Z}}$ and not $\mathcal{Z}$ are those atoms of times s which are accumulation points of zeros from the left, so that finally $\bar{\mathcal{Z}} = [0, \infty[- \bigcup_{s\geq 0}(s, s + \xi_s/b) = \mathcal{R}$. □

It only remains to explain how Corollary 1, which restates Fitzsimmons-Fristedt-Shepp Theorem, is deduced.

[1] we work here up to a positive multiplicative constant

Proof of Corollary 1. It will follow directly by applying Lemma 1, Theorem 4 and Theorem 1-3. in the case $b = 1$. Consider M the ESN$(1, \mu)$ started at 0, constructed from the same Poisson point process $\mathcal{N}$ as the random cutout set $\mathcal{R}$. By Lemma 1, one has the identity

$$\mathcal{R} = \overline{\{t \geq 0 : M(t) = 0\}}.$$

In particular, by Theorem 4-3, 0 is inaccessible for M, equivalently $\mathcal{R} = \{0\}$ a.s. if and only if $\mathcal{J} = \infty$ (with $b = 1$). When $\mathcal{J} < \infty$, $\mathcal{R}$ coincides with $\bar{\mathcal{Z}} = \overline{\{\tau_x : 0 \leq x < \zeta\}}$ with $(\tau_x, 0 \leq x < \zeta)$ the subordinator defined as the inverse local time of M at 0 and whose Laplace exponent is $\varphi : \theta \mapsto c/f_\theta(0)$ (for some constant $c > 0$). The fact that $\mathcal{R}$ is a regenerative set is immediate since $\bar{\mathcal{Z}}$ is the closure of the range of the subordinator $(\tau_x, x \geq 0)$, see [Ber99, Chapter 2.1]. It is perfect a.s. since $1/f_\theta(0) \to \infty$ as θ goes to ∞ which ensures that the Lévy measure of $(\tau_x, 0 \leq x < \zeta)$ is infinite. The set $\mathcal{R}$ is bounded a.s. if and only if M is transient, namely $\mathcal{I} < \infty$.

We have seen in Theorem 1-3, that $\mathbb{P}(t \in \mathcal{Z}) > 0$ for all $t > 0$ if and only if $\int_0^1 \bar{\mu}(s)\mathrm{d}s < \infty$. Fix $a > 0$, an application of Fubini-Tonelli theorem yields $\int_0^a \mathbb{P}_0(M(s) = 0)\mathrm{d}s = \mathbb{E}[\lambda(\mathcal{Z} \cap [0, a])] > 0$ (which is equivalent to $\lambda(\mathcal{Z} \cap [0, a]) > 0$ with positive probability) if and only if $\int_0^1 \bar{\mu}(v)\mathrm{d}v < \infty$. Recall that the $\lambda(\mathcal{Z})$ is either zero almost surely or positive almost surely, see e.g. [Ber99, Proposition 1.8]. Then we can conclude that $\lambda(\mathcal{Z})$ is positive almost surely. □

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Clément Foucart
LAGA, Institut Galilée
Université Sorbonne Paris Nord
E-mail: foucart@math.univ-paris13.fr

Linglong Yuan
Department of Mathematical Sciences
University of Liverpool
E-mail: linglong.yuan@liverpool.ac.uk