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Abstract

We develop a mathematical formalism that allows to study decoherence with a great level generality, so as to make it appear as a geometrical phenomenon between reservoirs of dimensions. It enables us to give quantitative estimates of the level of decoherence induced by a purely random environment on a system according to their respectives sizes, and to exhibit some links with entanglement entropy.

Introduction

The theory of decoherence is arguably one of the greatest advances in fundamental physics of the past forty years. Without adding anything new to the quantum mechanical framework, and by assuming that the Schrödinger equation is universally valid, it explains why quantum interferences virtually disappear at macroscopic scales. Since the pioneering papers [13] [14], a wide variety of models have been designed to understand decoherence in different specific contexts (see the review [16] or [3] and the numerous references therein). In this paper, we would like to embrace a more general point of view and understand why decoherence is so ubiquitous between quantum mechanical systems.

We start by introducing general notations to present as concisely as possible the idea underlying the theory of decoherence (§1). We then build two simple but very general models to reveal the mathematical mechanisms that make decoherence so universal, thereby justifying why quantum interferences disappear due to the Schrödinger dynamics only (§2 and §3). The most important result is Theorem 2.3, proved in §2.3, giving estimates for the level of decoherence induced by a random environment on a system of given sizes. We conclude in §2.4 that even very small environments (of typical size at least $N_{\mathcal{E}} = \ln(N_{\mathcal{S}})$ with $N_{\mathcal{S}}$ the size of the system) suffice, under assumptions discussed in §2.5. We also give a general formula estimating the level of classicality of a quantum system in terms of the entropy of entanglement with its environment (§3.2, proved in the annex A).

1 The basics of decoherence

The theory of decoherence sheds light on the reason why quantum interferences disappear when a system gets entangled with a macroscopic one, for example an electron in a double-slit experiment that doesn't interfere anymore when entangled with a detector. According to Di Biagio and Rovelli [1], the deep difference between classical and quantum is the way probabilities behave: all classical phenomena satisfy the total probability formula

$$\mathbb{P}(B = y) = \sum_{x \in \mathrm{Im}(A)} \mathbb{P}(A = x)\mathbb{P}(B = y \mid A = x)$$

relying on the fact that, *even though the actual value of the variable A is not known, one can still assume that it has a definite value among the possible ones.* This, however, is not correct for quantum systems. It is well-known that the diagonal elements of the density matrix account for the classical behavior of a

system (they correspond to the terms of the total probability formula) while the non-diagonal terms are the additional interference terms[1].

Consider a system $\mathcal{S}$, described by a Hilbert space $\mathcal{H}_\mathcal{S}$ of dimension d, that interacts with an environment $\mathcal{E}$ described by a space $\mathcal{H}_\mathcal{E}$ of dimension n, and let $\mathcal{B} = (|i\rangle)_{1\leqslant i\leqslant d}$ be an orthonormal basis of $\mathcal{H}_\mathcal{S}$. In the sequel, we will say that each $|i\rangle$ corresponds to a **possible history** of the system in this basis (this expression will be given its full meaning in an ongoing article dedicated to the measurement problem [11]). Let's also assume that $\mathcal{B}$ is a **conserved basis** during the interaction with $\mathcal{E}$ (in some contexts called a pointer basis). When $\mathcal{E}$ is a measurement apparatus for the observable A, the eigenbasis of $\hat{A}$ is clearly a conserved basis; in general, any observable that commute with the interaction Hamiltonian is suitable.

Inspired by [1], let's denote $|\Psi\rangle = \left(\sum_{i=1}^d c_i\,|i\rangle\right)\otimes|\mathcal{E}_0\rangle$ the initial state of $\mathcal{S}+\mathcal{E}$ before interaction. After a time t, the total state evolves to $|\Psi(t)\rangle = \sum_{i=1}^d c_i\,|i\rangle\otimes|\mathcal{E}_i(t)\rangle$. Let also $\eta(t) = \max\limits_{i\neq j}|\langle\mathcal{E}_i(t)|\mathcal{E}_j(t)\rangle|$. If $(|e_k\rangle)_{1\leqslant k\leqslant n}$ denotes an orthonormal basis of $\mathcal{H}_\mathcal{E}$, the state of $\mathcal{S}$, obtained by tracing out the environment, is:

$$\begin{aligned}
\rho_\mathcal{S}(t) &= \mathrm{tr}_\mathcal{E}\,|\Psi(t)\rangle\,\langle\Psi(t)| \\
&= \sum_{k=1}^n\left(\sum_{i=1}^d|c_i|^2|\langle e_k|\mathcal{E}_i(t)\rangle|^2\,|i\rangle\,\langle i| + \sum_{1\leqslant i\neq j\leqslant d}c_i\overline{c_j}\,\langle e_k|\mathcal{E}_i(t)\rangle\,\langle\mathcal{E}_j(t)|e_k\rangle\,|i\rangle\,\langle j|\right) \\
&= \sum_{i=1}^d|c_i|^2\underbrace{\sum_{k=1}^n|\langle e_k|\mathcal{E}_i(t)\rangle|^2}_{=1}\,|i\rangle\,\langle i| + \sum_{1\leqslant i\neq j\leqslant d}c_i\overline{c_j}\,\langle\mathcal{E}_j(t)|\Big(\underbrace{\sum_{k=1}^n|e_k\rangle\,\langle e_k|}_{=\mathbb{1}}\Big)|\mathcal{E}_i(t)\rangle\;|i\rangle\,\langle j| \\
&= \sum_{i=1}^d|c_i|^2\,|i\rangle\,\langle i| + \sum_{1\leqslant i\neq j\leqslant d}c_i\overline{c_j}\,\langle\mathcal{E}_j(t)|\mathcal{E}_i(t)\rangle\,|i\rangle\,\langle j| \\
&\equiv \rho_\mathcal{S}^{(d)} + \rho_\mathcal{S}^{(q)}(t)
\end{aligned}$$

where $\rho_\mathcal{S}^{(d)}$ stands for the (time independent) diagonal part of $\rho_\mathcal{S}(t)$ (which corresponds to the total probability formula), and $\rho_\mathcal{S}^{(q)}(t)$ for the remaining non diagonal terms responsible for the interferences between the possible histories. It is not difficult to show (see the Annex A) that $|||\rho_\mathcal{S}^{(q)}(t)||| \leqslant \eta(t)$. Therefore η *measures how close the system is from being classical* because, as shown in A, we have for all subspaces[2] $F\subset\mathcal{H}_\mathcal{S}$:

$$|\underbrace{\mathrm{tr}(\rho_\mathcal{S}(t)\Pi_F)}_{\text{quantum probability}} - \underbrace{\mathrm{tr}(\rho_\mathcal{S}^{(d)}\Pi_F)}_{\text{classical probability}}| \leqslant \dim(F)\,\eta(t). \tag{1}$$

In other words, $\eta(t)$ estimates how decohered the system is. Notice well that it is only during an interaction between $\mathcal{S}$ and $\mathcal{E}$ that decoherence can occur; any future internal evolution U of $\mathcal{E}$ lets η unchanged since $\langle U\mathcal{E}_j|U\mathcal{E}_i\rangle = \langle\mathcal{E}_j|\mathcal{E}_i\rangle$.

The aim of the theory of decoherence is to explain why $\eta(t)$ rapidly goes to zero when n is large, so that the state of the system almost immediately[3] evolves from $\rho_\mathcal{S}$ to $\rho_\mathcal{S}^{(d)}$ in the conserved basis. As recalled in the introduction, a lot of different models already explain this phenomenon in specific contexts. In this paper, we shall build two (excessively) simple but quite universal models that highlight the fundamental reason why $\eta(t)\to 0$ so quickly, and that will allow us to determine the typical size of an environment needed to entail proper decoherence on a system.

[1] As a reminder, this is because the probability to obtain an outcome x is:
$\mathrm{tr}(\rho\,|x\rangle\,\langle x|) = \sum_{i,j=1}^n\rho_{ij}\,\langle j|x\rangle\,\langle x|i\rangle = \sum_{i=1}^n\underbrace{\rho_{ii}|\,\langle x|i\rangle\,|^2}_{\mathbb{P}(i)\mathbb{P}(x|i)} + \sum_{1\leqslant i<j\leqslant n}\underbrace{2\mathrm{Re}(\rho_{ij}\,\langle j|x\rangle\,\langle x|i\rangle)}_{\text{interferences}}$

[2] Recall that, in the quantum formalism, probabilistic events correspond to subspaces.

[3] It is actually very important that the decoherence process (in particular a measurement) is not instantaneous. Otherwise, it would be impossible to explain why an unstable nucleus continuously measured by a Geiger counter is not frozen due to the quantum Zeno effect. See the wonderful model of [2, §8.3 and §8.4] that quantifies the effect of continuous measurement on the decay rate.

2 First model: purely random environment

When no particular assumption is made to specify the type of environment under study, the only reasonable behaviour to assume for $|\mathcal{E}_i(t)\rangle$ is that of a Brownian motion on the sphere $\mathbb{S}^n = \{|\Psi\rangle \in \mathcal{H}_\mathcal{E} \mid \||\Psi\| = 1\} \subset \mathcal{H}_\mathcal{E} \simeq \mathbb{C}^n \simeq \mathbb{R}^{2n}$. It boils down to representing the environment as a purely random system with no preferred direction of evolution. This choice will be discussed in §2.5. Another bold assumption would be the independence of the $(|\mathcal{E}_i(t)\rangle)_{1\leqslant i\leqslant d}$; we will dare to make this assumption anyway.

2.1 Convergence to the uniform measure

We will first show that the probabilistic law of each $|\mathcal{E}_i(t)\rangle$ converges exponentially fast to the uniform probability measure on $\mathbb{S}^n$. To make things precise, endow $\mathbb{S}^n$ with its Borel σ-algebra $\mathcal{B}$ and with the canonical Riemannian metric g which induces the uniform measure μ that we suppose normalized to a probability measure. Denote $\Delta f = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j f)$ the Laplacian operator on $\mathcal{C}^\infty(\mathbb{S}^n)$ which can be extended to $L^2(\mathbb{S}^n)$, the completion of $\mathcal{C}^\infty(\mathbb{S}^n)$ for the scalar product $(f,h) = \int_{\mathbb{S}^n} f(x)h(x)\mathrm{d}\mu$. Finally, recall that the total variation norm of a measure defined on $\mathcal{B}$ is given by $\|\sigma\|_{TV} = \sup_{B\in\mathcal{B}}|\sigma(B)|$.

Proposition 2.1. *Let ν_t be the law of the random variable $|\mathcal{E}_i(t)\rangle$, that is $\nu_t(B) = \mathbb{P}\big(|\mathcal{E}_i(t)\rangle \in B\big)$ for all $B \in \mathcal{B}$. Then $\|\nu_t - \mu\|_{TV} \underset{t\to+\infty}{\longrightarrow} 0$ exponentially fast.*

Proof. The overall idea is to decompose the density of the measure ν_t in an eigenbasis of the Laplacian so that the Brownian motion (which is generated by Δ) will exponentially kill all modes but the one associated with the eigenvalue 0, that is the constant one.

It is recalled in [9] that the eigenvalues of Δ take the form $\lambda_k = -k(k+2n-2)$ for $k \in \mathbb{N}$ with multiplicity $d_k = \frac{(k+2n-3)!}{(2n-2)!k!}(2k+2n-2)$. Denote $(f_{k,l})_{\substack{k\in\mathbb{N}\\1\leqslant l\leqslant d_k}}$ an orthonormal Hilbert basis in $L^2(\mathbb{S}^n)$ of eigenfunctions of Δ, where $f_{k,l}$ is associated with the eigenvalue λ_k. Note that $d_0 = 1$ and that $f_{0,1}$ is constant (as any harmonic function on a manifold without boundary, due to the maximum principle) so it is the density of a uniform measure. The law ν_0 of the deterministic variable $|\mathcal{E}_i(0)\rangle = |\mathcal{E}_0\rangle$ corresponds to a Dirac distribution, which is not strictly speaking in $L^2(\mathbb{S}^n)$, so we rather consider it as given by a sharply peaked density (with respect to μ) denoted $p_0 \in L^2(\mathbb{S}^n)$; the latter can be decomposed in the Hilbert basis $(f_{k,l})$ as $p_0 = \sum_{k=0}^{+\infty}\sum_{l=1}^{d_k} a_{k,l}f_{k,kl}$. The fact that $\|p_0\|_{L^2}^2 = \sum_{k=0}^{+\infty}\sum_{l=1}^{d_k}|a_{k,l}|^2$ yields $|a_{k,l}| \leqslant \|p_0\|_{L^2}$. Denote also p_t the density after a time t, *i.e.* $\nu_t(\mathrm{d}x) = p_t(x)\mu(\mathrm{d}x)$. The Hille-Yosida theory allows to define the Brownian motion on the sphere as the Markov semigroup of stochastic kernels generated by Δ; in particular, this implies $p_t = e^{t\Delta}p_0 = \sum_{k=0}^{+\infty} e^{-k(k+2n-2)t}\sum_{l=1}^{d_k} a_{k,l}f_{k,l}$. Note that $a_{0,1}f_{0,1}$ is a probability density because for all $k \geqslant 1$, $\int_{\mathbb{S}^n} f_{k,l} = \int_{\mathbb{S}^n}\frac{\Delta f_{k,l}}{\lambda_k} = 0$ due to Stokes' theorem, thus $\int_{\mathbb{S}^n} a_{0,1}f_{0,1} = \int_{\mathbb{S}^n} a_{0,1}f_{0,1} + \sum_{k=1}^{+\infty}\sum_{l=1}^{d_k} a_{k,l}\int_{\mathbb{S}^n} f_{k,l} = \int_{\mathbb{S}^n} p_0 = 1$. Hence $a_{0,1}f_{0,1} = 1$, and therefore:

$$\begin{aligned}
\|\nu_t - \mu\|_{TV} &= \frac{1}{2}\int_{\mathbb{S}^n}|p_t(x) - 1|\mathrm{d}\mu \qquad \text{(classical result on the total variation norm)}\\
&\leqslant \frac{1}{2}\sum_{k=1}^{+\infty} e^{-k(k+2n-2)t}\sum_{l=1}^{d_k}|a_{k,l}|\underbrace{\|f_{k,l}\|_{L^1}}_{\leqslant 1\ (*)}\\
&\leqslant \frac{1}{2}\|p_0\|_{L^2}\sum_{k=1}^{+\infty} e^{-k(k+2n-2)t}d_k
\end{aligned}$$

where $(*)$ relies on Hölder's inequality $\|f_{k,l}\|_{L^1} \leqslant \mu(\mathbb{S}^n)^{1/2}\|f_{k,l}\|_{L^2} = 1$. It remains to find a characteristic time after which the above series is efficiently bounded. We reproduce the argument of [9], setting $u_k = e^{-k(k+2n-2)t}d_k$ so that $u_1 = 2ne^{-(2n-1)t}$ and :

$$\frac{u_{k+1}}{u_k} = \frac{k+2n-2}{k+1}\frac{2k+2n}{2k+2n-2}e^{-[(2n-1)(k+1)+k]t}$$

If $n \geqslant 2$ and $t \geqslant t_n = \frac{\ln(2n-1)}{2n-1}$, then $\frac{u_{k+1}}{u_k} \leqslant \frac{2n-1}{2}\frac{2n+2}{2n}\frac{1}{2n-1} \leqslant \frac{3}{4}$, which implies :

$$\|\nu_t - \mu\|_{TV} \leqslant \frac{1}{2}\|p_0\|_{L^2} u_1 \sum_{k=1}^{+\infty} (\frac{3}{4})^k \leqslant 3\|p_0\|_{L^2} n e^{-(2n-1)t}$$

Interestingly enough, the convergence is faster as n increases since the characteristic time to equilibrium satisfies $t_n \underset{n\to\infty}{\longrightarrow} 0$ and the exponential is sharper. □

2.2 Most vectors are almost orthogonal

Consequently, we are now interested in the behavior of the scalar products between random vectors uniformly distributed on the complex n-sphere $\mathbb{S}^n$. The first thing to understand is that, *in high dimension, most pairs of unit vectors are almost orthogonal.*

Proposition 2.2. *Denote by $S = \langle \mathcal{E}_1 | \mathcal{E}_2 \rangle \in \mathbb{C}$ the random variable where $|\mathcal{E}_1\rangle$ and $|\mathcal{E}_2\rangle$ are two independent uniform random variables on $\mathbb{S}^n$. Then $\mathbb{E}(S) = 0$ and $\mathbb{V}(S) = \mathbb{E}(|S|^2) = \frac{1}{n}$.*

Proof. Clearly, $|\mathcal{E}_1\rangle$ and $-|\mathcal{E}_1\rangle$ have the same law, hence $\mathbb{E}(S) = \mathbb{E}(-S) = 0$. What about its variance ? One can rotate the sphere to impose for example $|\mathcal{E}_1\rangle = (1, 0, \dots, 0)$, and by independence $|\mathcal{E}_2\rangle$ still follows a uniform law. Such a uniform law can be achieved by generating $2n$ independent normal random variables $(X_i)_{1\leqslant i\leqslant 2n}$ following $\mathcal{N}(0,1)$ and by considering the random vector $|\mathcal{E}_2\rangle = \left(\frac{X_1+iX_2}{\sqrt{X_1^2+\cdots+X_{2n}^2}}, \dots, \frac{X_{2n-1}+iX_{2n}}{\sqrt{X_1^2+\cdots+X_{2n}^2}}\right)$. Indeed, for any continuous function $f : \mathbb{S}^n \to \mathbb{R}$ (with $\mathrm{d}\sigma^n$ denoting the measure induced by Lebesgue's on $\mathbb{S}^n$):

$$\begin{aligned}
\mathbb{E}[f(|\mathcal{E}_2\rangle)] &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} f\left(\frac{x_1+ix_2}{\sqrt{x_1^2+\cdots+x_{2n}^2}}, \dots, \frac{x_{2n-1}+ix_{2n}}{\sqrt{x_1^2+\cdots+x_{2n}^2}}\right) e^{-(x_1^2+\cdots+x_{2n}^2)/2}\mathrm{d}x_1\dots\mathrm{d}x_{2n} \\
&= \frac{1}{(2\pi)^n}\int_0^\infty \left[\int_{\mathbb{S}^n} f(u)\mathrm{d}\sigma^n(u)\right] e^{-\frac{r^2}{2}} r^{2n-1}\mathrm{d}r \\
&= \omega_n \int_{\mathbb{S}^n} f(u)\mathrm{d}\sigma^n(u)
\end{aligned}$$

which means that $|\mathcal{E}_2\rangle$ defined this way follows indeed the uniform law. In these notations, $S = \frac{X_1+iX_2}{\sqrt{X_1^2+\cdots+X_{2n}^2}}$. Note that, up to relabelling the variables, we have $\forall k \in [\![1, n]\!]$, $\mathbb{E}\left(\frac{X_1^2+X_2^2}{X_1^2+\cdots+X_{2n}^2}\right) = \mathbb{E}\left(\frac{X_{2k-1}^2+X_{2k}^2}{X_1^2+\cdots+X_{2n}^2}\right)$ and so:

$$\mathbb{V}(S) = \mathbb{E}\left(\frac{X_1^2+X_2^2}{X_1^2+\cdots+X_{2n}^2}\right) = \frac{1}{n}\sum_{k=1}^n \mathbb{E}\left(\frac{X_{2k-1}^2+X_{2k}^2}{X_1^2+\cdots+X_{2n}^2}\right) = \frac{1}{n}\mathbb{E}(1) = \frac{1}{n}.$$

Another way to recover this result would have been to define the unitary evolution operators $(U^{(i)}(t))_{1\leqslant i\leqslant d}$ such that $|\mathcal{E}_i(t)\rangle = U^{(i)}(t)\,|\mathcal{E}_0\rangle$, resulting from the interaction Hamiltonian. Again, if no direction of evolution is preferred, it is reasonable to consider the law of each $U^{(i)}(t)$ to be given by the Haar measure $\mathrm{d}U$ on the unitary group $\mathcal{U}_n$. If moreover they are independent, then $U^{(i)}(t)^\dagger U^{(j)}(t)$ also follows the Haar measure for all i, j so that, using [12, (112)]:

$$\mathbb{V}\big(\,\langle \mathcal{E}_i(t)|\mathcal{E}_j(t)\rangle\,\big) = \int_{\mathcal{U}_n} |\langle \mathcal{E}_0|U\mathcal{E}_0\rangle|^2 dU = \prod_{i=2}^n \frac{i-1}{i} = \frac{1}{n}.$$

□

Therefore, $|\langle \mathcal{E}_i(t)|\mathcal{E}_j(t)\rangle|$ is, after a very short time, of order $\sqrt{\mathbb{V}(S)} = \frac{1}{\sqrt{\dim(\mathcal{H}_\mathcal{E})}}$. When $d = 2$, if $\mathcal{E}$ is composed of $N_\mathcal{E}$ particles and each of them is described by a p-dimensional Hilbert space, then very rapidly:

$$\eta \sim p^{-N_\mathcal{E}/2} \tag{2}$$

which is virtually zero for macroscopic environments, therefore decoherence is guaranteed. Of course, this is not true anymore if d is large, because there will be so many pairs that some of them will inevitably

become non-negligible, and so will η. We would like to determine a condition between n and d under which proper decoherence is to be expected. In other words, what is the minimal size of an environment needed to decohere a given system?

2.3 Direct study of η

To answer this question, we should be more precise and consider directly the random variable $\eta_{n,d} = \max_{i\neq j} |\langle \mathcal{E}_i | \mathcal{E}_j \rangle|$ where the $(|\mathcal{E}_i\rangle)_{1\leqslant i\leqslant d}$ are d random vectors uniformly distributed on the complex n-sphere $\mathbb{S}^n$. In the following, we fix $\varepsilon \in]0,1[$ as well as a threshold $s \in [0,1[$ close to 1, and define $d^{\varepsilon,s}_{max}(n) = \min\{d \in \mathbb{N} \mid \mathbb{P}(\eta_{n,d} \geqslant \varepsilon) \geqslant s\}$, so that if $d^{\varepsilon,s}_{max}(n)$ points or more are placed randomly on $\mathbb{S}^n$, it is very likely (with probability $\geqslant s$) that at least one of the scalar products will be greater that ε.

Theorem 2.3. *The following estimates hold :*

1. $d^{\varepsilon,s}_{max}(n) \sim \sqrt{-2\ln(1-s)} \left(\dfrac{1}{1-\varepsilon^2}\right)^{\frac{n-1}{2}}$
2. $\mathbb{V}(\eta_{n,d}) \simeq 0$ *and* $\boxed{\eta_{n,d} \simeq \mathbb{E}(\eta_{n,d}) \sim \sqrt{1-d^{-\frac{2}{n}}}}$

To derive these formulas, we first need the following geometrical lemma.

Lemma 2.4. *Let $A_n = |\mathbb{S}^n|$ be the area of the complex n-sphere for $\mathrm{d}\sigma^n$ (induced by Lebesgue's measure), $C^\varepsilon_n(x) = \{u \in \mathbb{S}^n \mid |\langle u|x\rangle| \geqslant \varepsilon\}$ the 'spherical cap'[4] centered in x of parameter ε, and $A^\varepsilon_n = |C^\varepsilon_n|$ the area of any spherical cap of parameter ε. Then for all $n \geqslant 1$:*

$$\frac{A^\varepsilon_n}{A_n} = (1-\varepsilon^2)^{n-1}$$

Proof of Lemma. Recall that $\mathbb{S}^n \subset \mathbb{C}^n \simeq \mathbb{R}^{2n}$ can be seen as a real manifold of dimension $2n-1$. Consider the set of coordinates $(r, \theta, \varphi_1, \ldots, \varphi_{2n-3})$ on $\mathbb{S}^n$ defined by the chart

$$\begin{array}{rccl} F: & [0,1]\times[0,2\pi[\times[0,\pi]^{2n-4}\times[0,2\pi[& \longrightarrow & \mathbb{S}^n \\ & (r,\theta,\varphi_1,\ldots,\varphi_{2n-3}) & \longmapsto & (x_1+ix_2,\ldots,x_{2n-1}+ix_{2n}) \simeq (x_1,\ldots,x_{2n}) = \\ & & & (r\cos(\theta), r\sin(\theta), \sqrt{1-r^2}\cos(\varphi_1), \sqrt{1-r^2}\sin(\varphi_1)\cos(\varphi_2),\ldots, \\ & & & \sqrt{1-r^2}\sin(\varphi_1)\ldots\cos(\varphi_{2n-3}), \sqrt{1-r^2}\sin(\varphi_1)\ldots\sin(\varphi_{2n-3})) \end{array}$$

This amounts to choose the modulus r and the argument θ of x_1+ix_2, and then describe the remaining parameters using the standard spherical coordinates on $\mathbb{S}^{n-1}$, seen as a sphere of real dimension $2n-3$, including a radius factor $\sqrt{1-r^2}$. The advantage of these coordinates is that $C^\varepsilon_n(1,0,\ldots,0)$ simply corresponds to the set of points for which $r \geqslant \varepsilon$.

Let's determine the metric g.

- $\vec{e_r} = \partial_r F(r,\theta,\varphi_1,\ldots,\varphi_{2n-3}) = (\cos(\theta), \sin(\theta), \frac{-r}{\sqrt{1-r^2}}[\vec{u}])$, where $[\vec{u}]$ stands for the standard expression of the spherical coordinates on $\mathbb{S}^{n-1}$
- $\vec{e_\theta} = \partial_\theta F(r,\theta,\varphi_1,\ldots,\varphi_{2n-3}) = (-r\sin(\theta), r\cos(\theta), 0, \ldots, 0)$
- $\vec{e_{\varphi_i}} = \partial_{\varphi_i} F(r,\theta,\varphi_1,\ldots,\varphi_{2n-3}) = (0, 0, \sqrt{1-r^2}[\vec{e_{\varphi_i}}])$ where $[\vec{e_{\varphi_i}}]$ stands for the tangent vector on $\mathbb{S}^{n-1}$

Obviously, $\vec{e_r} \perp \vec{e_\theta}$ and $\vec{e_{\varphi_i}} \perp \vec{e_\theta}$, as well as $\vec{e_{\varphi_i}} \perp \vec{e_{\varphi_j}}$ for $i \neq j$ as is the case in the standard spherical coordinates. Moreover, since $[\vec{u}]$ is radial and $[\vec{e_{\varphi_i}}]$ tangent to $\mathbb{S}^{n-1}$, we also have $\vec{e_r} \perp \vec{e_{\varphi_i}}$, therefore g is diagonal in these coordinates. Its components are given by :

- $g_{rr} = \langle \vec{e_r} | \vec{e_r} \rangle = 1 + \frac{r^2}{1-r^2}$
- $g_{rr} = \langle \vec{e_\theta} | \vec{e_\theta} \rangle = r^2$

[4] We use the quotation marks, because on $\mathbb{S}^n$ equipped with its complex scalar product, this set doesn't look like a cap as it does in the real case. QM is nothing but a strange way of calculating probabilities (in which all possible histories interfere) based on a geometrical structure, but the geometry in use is also quite different from the intuitive one given by the familiar real scalar product...

- $g_{\varphi_i\varphi_i} = (1-r^2)[g_{\varphi_i\varphi_i}]$ with $[g]$ the metric corresponding to the spherical coordinates on $\mathbb{S}^{n-1}$

It is now easy to compute the desired quantity :

$$\begin{aligned}
A_n^\varepsilon &= \int_\varepsilon^1 \sqrt{1+\frac{r^2}{1-r^2}}\mathrm{d}r \int_{[0,2\pi[\times[0,\pi[^{2n-4}\times[0,2\pi[} r\mathrm{d}\theta\sqrt{1-r^2}^{2n-3}\sqrt{[g]}\mathrm{d}\varphi_1\dots\mathrm{d}\varphi_{2n-3} \\
&= 2\pi A_{n-1}\int_\varepsilon^1 r(1-r^2)^{n-2}\mathrm{d}r \\
&= \frac{\pi A_{n-1}}{n-1}\int_\varepsilon^1 2(n-1)r(1-r^2)^{n-2}\mathrm{d}r \\
&= \frac{\pi A_{n-1}}{n-1}(1-\varepsilon^2)^{n-1}
\end{aligned}$$

and, finally,

$$\frac{A_n^\varepsilon}{A_n} = \frac{A_n^\varepsilon}{A_n^0} = (1-\varepsilon^2)^{n-1}.$$

□

We are now ready to prove the theorem.

Proof of Theorem 2.3. For this proof, we find some inspiration in [6], but eventually find sharper bounds with simpler arguments. We say that a set of vectors on a sphere are ε-separated if all scalar products between any pairs among them are not greater than ε in modulus. Consider the following events :

- $A : \forall k \in [\![1, d-1]\!], |\langle\mathcal{E}_d|\mathcal{E}_k\rangle| \leqslant \varepsilon$
- $B : (|\mathcal{E}_k\rangle)_{1\leqslant k\leqslant d-1}$ are $\varepsilon-$separated

and write $\mathbb{P}(\eta_{n,d}\leqslant\varepsilon) = \mathbb{P}(A \mid B)\mathbb{P}(B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}\mathbb{P}(\eta_{n,d-1}\leqslant\varepsilon)$, with :

$$\begin{aligned}
\frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} &= \frac{\frac{1}{A_n^d}\displaystyle\int_{(\mathbb{S}^n)^{d-1}} \mathrm{d}\sigma^n(x_1)\dots\mathrm{d}\sigma^n(x_{d-1})\mathbb{1}_{\{x_1,\dots,x_{d-1}\text{ are }\varepsilon\text{-separated}\}}\left(A_n - \left|\bigcup_{k=1}^{d-1} C_n^\varepsilon(x_k)\right|\right)}{\frac{1}{A_n^{d-1}}\displaystyle\int_{(\mathbb{S}^n)^{d-1}} \mathrm{d}\sigma^n(x_1)\dots\mathrm{d}\sigma^n(x_{d-1})\mathbb{1}_{\{x_1,\dots,x_{d-1}\text{ are }\varepsilon\text{-separated}\}}} \\
&= 1 - \mathbb{E}\left(\frac{\left|\bigcup_{k=1}^{d-1} C_n^\varepsilon(|\mathcal{E}_k\rangle)\right|}{A_n}\middle| B\right)
\end{aligned}$$

We need to find bounds on the latter quantity. Obviously, $\mathbb{E}\left(\frac{\left|\bigcup_{k=1}^{d-1} C_n^\varepsilon(|\mathcal{E}_k\rangle)\right|}{A_n}\middle| B\right) \leqslant (d-1)\frac{A_n^\varepsilon}{A_n}$, corresponding to the case when all the caps are disjoint. For the lower bound, define the sequence $u_d = \mathbb{E}\left(\frac{\left|\bigcup_{k=1}^{d} C_n^\varepsilon(|\mathcal{E}_k\rangle)\right|}{A_n}\right)$, which clearly satisfies $u_d \leqslant \mathbb{E}\left(\frac{\left|\bigcup_{k=1}^{d} C_n^\varepsilon(|\mathcal{E}_k\rangle)\right|}{A_n}\middle| B\right)$, because conditioning on the vectors being separated can only decrease the overlap between the different caps. First observe that $u_1 = \frac{A_n^\varepsilon}{A_n} \equiv \alpha$, and compute :

$$
\begin{aligned}
u_d &= u_{d-1} + \frac{1}{A_n}\mathbb{E}\left(\left|C_n^\varepsilon(|\mathcal{E}_d\rangle) \setminus \bigcup_{k=1}^{d-1} C_n^\varepsilon(|\mathcal{E}_k\rangle)\right|\right) \\
&= u_{d-1} + \frac{1}{A_n}\frac{1}{A_n^d}\int_{(\mathbb{S}^n)^d} \mathrm{d}\sigma^n(x_1)\dots\mathrm{d}\sigma^n(x_d) \int_{C_n^\varepsilon(x_d)} \mathbb{1}_{\{y\notin\bigcup_{k=1}^{d-1} C_n^\varepsilon(x_k)\}}\mathrm{d}\sigma^n(y) \\
&= u_{d-1} + \frac{1}{A_n}\frac{1}{A_n^{d-1}}\int_{(\mathbb{S}^n)^{d-1}} \mathrm{d}\sigma^n(x_1)\dots\mathrm{d}\sigma^n(x_{d-1}) \int_{\mathbb{S}^n} |C_n^\varepsilon(y)|\,\mathbb{1}_{\{y\notin\bigcup_{k=1}^{d-1} C_n^\varepsilon(x_k)\}}\frac{\mathrm{d}\sigma^n(y)}{A_n} \\
&= u_{d-1} + \frac{A_n^\varepsilon}{A_n}\frac{1}{A_n^{d-1}}\int_{(\mathbb{S}^n)^{d-1}} \mathrm{d}\sigma^n(x_1)\dots\mathrm{d}\sigma^n(x_{d-1})\left(1 - \frac{\left|\bigcup_{k=1}^{d-1} C_n^\varepsilon(x_k)\right|}{A_n}\right) \\
&= u_{d-1} + \frac{A_n^\varepsilon}{A_n}(1-u_{d-1}) \\
&= (1-\alpha)u_{d-1} + \alpha
\end{aligned}
$$

where the main trick was to invert the integrals on x_d and on y. This result is actually quite intuitive: it states that when adding a new cap, only a fraction $1-u_{d-1}$ of it on average will be outside the previous caps and contribute to the new total area covered by the caps. Hence $u_d = 1-(1-\alpha)^d$, and the recurrence relation becomes:

$$
\left(1-(d-1)\frac{A_n^\varepsilon}{A_n}\right)\mathbb{P}(\eta_{n,d-1}\leqslant\varepsilon) \leqslant \mathbb{P}(\eta_{n,d}\leqslant\varepsilon) \leqslant \left(1-\frac{A_n^\varepsilon}{A_n}\right)^{d-1}\mathbb{P}(\eta_{n,d-1}\leqslant\varepsilon).
$$

Applying the lemma, we get by induction:

$$
\prod_{k=1}^{d-1}(1-k(1-\varepsilon^2)^{n-1}) \leqslant \mathbb{P}(\eta_{n,d}\leqslant\varepsilon) \leqslant (1-(1-\varepsilon^2)^{n-1})^{\frac{d(d-1)}{2}}
$$

Note that the left inequality is valid only as long as $d \leqslant \left(\frac{1}{1-\varepsilon^2}\right)^{n-1}$, but when d is larger than this critical value, the right hand side becomes very small (of order $e^{-1/2(1-\varepsilon^2)^{n-1}}$), so taking 0 as a lower bound in this case is actually very precise. The two bounds are in fact extremely close to each other, and get closer as n goes larger, as exemplified by numerical simulations (a precise argument could be given but it would be quite lengthy and would not bring any additional understanding to the problem). Consequently, we can make the following approximation for the distribution function of $\eta_{n,d}$: $\mathbb{P}(\eta_{n,d}\leqslant\varepsilon) \simeq (1-(1-\varepsilon^2)^{n-1})^{\frac{d(d-1)}{2}}$. It is then straightforward to obtain the estimation $d_{max}^{\varepsilon,s}(n) \simeq \sqrt{-2\ln(1-s)}\left(\frac{1}{1-\varepsilon^2}\right)^{\frac{n-1}{2}}$, which is the first part of the theorem.

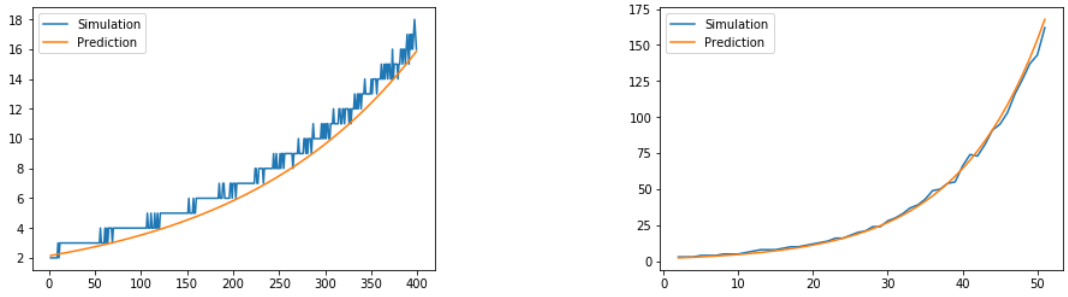
The function $\varepsilon \mapsto (1-(1-\varepsilon^2)^{n-1})^{\frac{d(d-1)}{2}}$ happens to be almost constant equal to 0 in the vicinity of $\varepsilon = 0$, almost 1 in the vicinity of $\varepsilon = 1$, and to have a very sharp step between the two; this step sharpens as n and d grow larger. This explains why the mass of probability is highly peaked around a critical value ε_c, so that $\mathbb{V}(\eta_{n,d}) \simeq 0$ and $\eta_{n,d} \simeq \mathbb{E}(\eta_{n,d})$ is almost a deterministic variable. This is certainly due to the averaging effect of considering the maximum of a set of $\frac{d(d+1)}{2}$ scalar products. The critical value ε_c satisfies:

$$
(1-(1-\varepsilon_c^2)^{n-1})^{\frac{d(d-1)}{2}} = \frac{1}{2} \Leftrightarrow \varepsilon_c = \sqrt{1-(1-2^{-2/d(d-1)})^{1/n-1}} \underset{n,d\to\infty}{\sim} \sqrt{1-d^{-2/n}}
$$

It simply remains to use the well-known formula: $\mathbb{E}(\eta_{n,d}) = \int_0^1 \mathbb{P}(\eta_{n,d}\leqslant\varepsilon)\mathrm{d}\varepsilon \simeq \varepsilon_c \simeq \sqrt{1-d^{-2/n}}$, which completes the proof. □

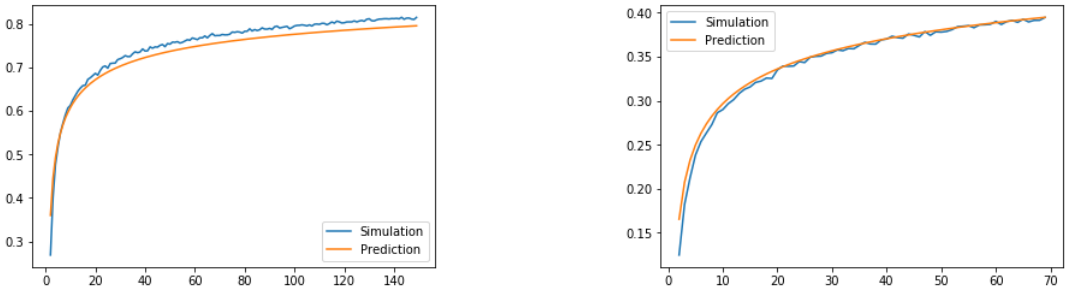
2.4 Comparison with simulation and consequences

The above expressions actually give incredibly good estimations for $d_{max}^{\varepsilon,s}(n)$ and $\eta_{n,d}$, as shown in Figures 1 and 2.



(a) $\varepsilon = 0.1$ et $s = 0.9$ (b) $\varepsilon = 0.4$ et $s = 0.9$

Figure 1: Simulation *vs* prediction for $d^{\varepsilon,s}_{max}(n)$



(a) $n = 10$ (b) $n = 50$

Figure 2: Simulation *vs* prediction for $d \mapsto \mathbb{E}(\eta_{n,d})$ at fixed n

This theorem has a strong physical consequence. Indeed, $\mathcal{E}$ induces proper decoherence on $\mathcal{S}$ as long as $\eta_{n,d} \ll 1$, that is when $d^{-2/n}$ is very close to 1, *i.e.* when $d \ll e^{n/2}$. Going back to physically meaningful quantities, we write as previously $n = p^{N_\mathcal{E}}$ and $d = p^{N_\mathcal{S}}$ where $N_\mathcal{E}$ and $N_\mathcal{S}$ stand for the number of particles composing $\mathcal{E}$ and $\mathcal{S}$. The condition becomes: $2\ln(p)N_\mathcal{S} \ll p^{N_\mathcal{E}}$ or simply :

$$\boxed{\frac{\ln(N_\mathcal{S})}{\ln(p)} \ll N_\mathcal{E}}$$

A more precise condition can be obtained using d_{max}, because $\mathcal{E}$ induces proper decoherence on $\mathcal{S}$ as long as $d \leqslant d^{\varepsilon,s}_{max}(n)$ for an arbitrary choice of ε close to 0 and s close to 1. This rewrites: $2\ln(p)N_\mathcal{S} \leqslant \ln(\sqrt{-2\ln(1-s)}) + \ln\left(\frac{1}{1-\varepsilon^2}\right) p^{N_\mathcal{E}} \simeq \varepsilon^2 p^{N_\mathcal{E}}$ or simply: $\ln(N_\mathcal{S}) \leqslant 2\ln(\varepsilon) + \ln(p)N_\mathcal{E}$. Thus, for instance, a gas composed of thousands of particles will lose most of its coherence if it interacts with only a few external particles. It is rather surprising that so many points can be placed randomly on a n-sphere before having the maximum of the scalar products becoming non-negligible. *It is this property that makes decoherence an extremely efficient high-dimensional geometrical phenomenon.*

2.5 Discussing the hypotheses

On the one hand, this result could be seen as a worst case scenario for decoherence, since realistic Hamiltonians are far from random and actually discriminate even better the different possible histories. This is especially true if $\mathcal{E}$ is a measurement apparatus for example, whose Hamiltonian is by construction such that the $(|\mathcal{E}_i(t)\rangle)_{1\leqslant i\leqslant d}$ evolve quickly and deterministically towards orthogonal points of the sphere.

On the other hand, pursuing such a high level of generality led us to abstract and unphysical assumptions. First, realistic dynamics are not isotropic on the n-sphere (some transitions are more probable

than others). Then, the assumption that each $|\mathcal{E}_i(t)\rangle$ can explore indistinctly all the states of $\mathcal{H}_\mathcal{E}$ is very criticizable. As explained in [8]:

> '... the set of quantum states that can be reached from a product state with a polynomial-time evolution of an arbitrary time-dependent quantum Hamiltonian is an exponentially small fraction of the Hilbert space. This means that the vast majority of quantum states in a many-body system are unphysical, as they cannot be reached in any reasonable time. As a consequence, all physical states live on a tiny submanifold'

It would then be more accurate in our model to replace $\mathbb{S}^n$ by this submanifold. By how does it look like geometrically and what is its dimension? If it were a subsphere of $\mathbb{S}^n$ of exponentially smaller dimension, then n should be replaced everywhere by something like $\ln(n)$ in what precedes, so the condition would rather be $N_\mathcal{S} \ll N_\mathcal{E}$ which is a completely different conclusion. Some clues to better grasp the submanifold are found in [7, §3.4]:

> '... one can prove that low-energy eigenstates of gapped Hamiltonians with local interactions obey the so-called area-law for the entanglement entropy. This means that the entanglement entropy of a region of space tends to scale, for large enough regions, as the size of the boundary of the region and not as the volume. (...) In other words, low-energy states of realistic Hamiltonians are not just "any" state in the Hilbert space: they are heavily constrained by locality so that they must obey the entanglement area-law.'

More work is needed in order to draw precise conclusions taking this physical remarks into account...

3 Second model: interacting particles

3.1 The environment feels the system

At present, let's better specify the nature of the environment. Suppose that the energy of interaction dominates the evolution of the whole system $\mathcal{S}+\mathcal{E}$ and can be expressed in terms of the positions $x_1, \dots, x_N$ of the N particles composing the environment, together with the state of $\mathcal{S}$ (this is the typical regime for macroscopic systems which decohere in the position basis [10, §III.E.2.]). If the latter is $|i\rangle$, denote $H(i, x_1 \dots x_N)$ this energy. The initial state $|\Psi\rangle = \left(\sum_{i=1}^d c_i |i\rangle\right) \otimes \int f(x_1 \dots x_N) |x_1 \dots x_N\rangle \, \mathrm{d}x_1 \dots \mathrm{d}x_N$ evolves into:

$$\sum_{i=1}^d c_i |i\rangle \otimes \underbrace{\int f(x_1 \dots x_N) e^{\frac{i}{\hbar} H(i,x_1 \dots x_N)t} |x_1 \dots x_N\rangle \, \mathrm{d}x_1 \dots \mathrm{d}x_N}_{=|\mathcal{E}_i(t)\rangle}.$$

Therefore:

$$\langle \mathcal{E}_i(t) | \mathcal{E}_j(t) \rangle = \int |f(x_1 \dots x_N)|^2 e^{\frac{i}{\hbar}\Delta(i,j,x_1 \dots x_N)t} \mathrm{d}x_1 \dots \mathrm{d}x_N$$

where $\Delta(i, j, x_1 \dots x_N) = H(j, x_1 \dots x_N) - H(i, x_1 \dots x_N)$ is a spectral gap between eigenvalues of the Hamiltonian, measuring how much the environment feels the transition of $\mathcal{S}$ from $|i\rangle$ to $|j\rangle$ in a given configuration of the environment. In a time interval $[-T, T]$, the mean value $\frac{1}{2T}\int_{-T}^T \langle \mathcal{E}_i(t) | \mathcal{E}_j(t) \rangle \, \mathrm{d}t$ yields $\int |f(x_1 \dots x_N)|^2 \mathrm{sinc}(\frac{\Delta(i,j,x_1 \dots x_N)}{\hbar} T)$ which is close to zero for all i and j as soon as $T > \dfrac{\pi\hbar}{\min\limits_{i,j,x_1 \dots x_N} \Delta(i, j, x_1 \dots x_N)}$, which is likely to be small if $\mathcal{E}$ is a macroscopic system, for the energies involved will be much greater than $\hbar$. Similarly, the empirical variance is:

$$\mathbb{V} = \frac{1}{2T} \int_{-T}^T |\langle \mathcal{E}_i(t) | \mathcal{E}_j(t) \rangle|^2 \mathrm{d}t \sim \int |f(x_1 \dots x_N)|^4 \mathrm{d}x_1 \dots \mathrm{d}x_N$$

plus terms that go to zero after a short time. Note that the variables $x_1 \dots x_N$ could be discretized to take p possible values, in which case $n = \dim(\mathcal{H}_\mathcal{E}) = p^N$, and the integral becomes a finite sum. For a delocalized initial state with constant f, this sum is equal to p^{-N}, and we recover the previous estimate (2) if $d = 2$: $\eta \sim p^{-N/2}$. This model teaches us that *the more the environment feels the difference between the possible histories, the more they decohere.*

3.2 Entanglement entropy as a measure of decoherence

What precedes suggests the following intuition: the smaller η is, the more information the environment has stored about the system because the more distinguishable (*i.e.* orthogonal) the $(|\mathcal{E}_i(t)\rangle)_{1\leqslant i\leqslant d}$ are; on the other hand, the smaller η is, the fewer quantum interferences occur. It motivates the search for a general relationship between entanglement entropy[5] (how much $\mathcal{E}$ knows about $\mathcal{S}$) and the level of classicality of a system. Such results have already been derived for specific environments [3, (3.76)] [4] [5] but not, to our knowledge, in the general case. The following formula is proved in the annex A when S stands for the linear entropy (or purity defect) $1-\mathrm{tr}(\rho^2)$, and some justifications are given when S denotes the entanglement entropy:

$$\forall F\subset\mathcal{H}_\mathcal{S},\quad |\mathrm{tr}(\rho_\mathcal{S}(t)\Pi_F)-\mathrm{tr}(\rho_\mathcal{S}^{(d)}\Pi_F)|\leqslant \dim(F)\sqrt{1-\inf_{|\Psi_\mathcal{S}(0)\rangle}\frac{S(\rho_\mathcal{S}(t))}{S(\rho_\mathcal{S}^{(d)})}}. \tag{3}$$

Conclusion

We introduced general mathematical notations that can be relevant for any study on decoherence, in particular the parameter $\eta(t)$ that quantifies the level of decoherence at a given instant. Two simple models were then presented, designed to feel more intuitively the general process of decoherence. Most importantly, our study revealed the mathematical reason why it is so fast and universal, namely because surprisingly many points can be placed randomly on a n-sphere before having the maximum of the scalar products becoming non-negligible. We also learned that decoherence is neither perfect nor everlasting, since η is not expected to be exactly 0 and will eventually become large again (according to Borel-Cantelli's lemma for the first model, and finding particular times such that all the exponentials are almost real in the second) pretty much like the ink drop in the glass of water will re-form again due to Poincaré's recurrence theorem, even though the recurrence time can easily exceed the lifetime of the universe for realistic systems [15]. Finally, decoherence can be estimated by entanglement entropy because η is linked to what the environment knows about the system.

Further works could include the search for a description of the submanifold of reachable states mentioned in §2.5, and the study of the infinite dimensional case, where the very definition of η is not obvious anymore (since the scalar products vary continuously, their supremum is necessarily 1). Another interesting question would be to investigate how η depends on the basis in which decoherence is considered: quantum interferences are indeed suppressed in the conserved basis, but how strong are they in the other bases?

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A Annex: decoherence estimated by the entanglement entropy with the environment

We establish here the formula (3): we first derive the inequality (1), and then look for a relation between η and the linear entropy or the entanglement entropy. Inserting the second into the first directly yields (3).

[5] Recall that for a bipartite quantum system, the entanglement entropy is defined as the von Neumann entropy of the reduced density matrix for any of the subsystems.

A.1 Relation between η and the level of classicality

Let's keep the notations of §1, where we defined $\rho_{\mathcal{S}}^{(q)}(t) = \sum_{i\neq j} c_i\overline{c_j}\,\langle\mathcal{E}_j(t)|\mathcal{E}_i(t)\rangle\,|i\rangle\,\langle j|$. We have $|||\rho_{\mathcal{S}}^{(q)}(t)||| \leqslant \eta(t)$ because for all vectors $|\Psi\rangle = \sum_k \alpha_k\,|k\rangle \in \mathcal{H}_{\mathcal{S}}$ of norm 1,

$$\rho_{\mathcal{S}}^{(q)}(t)\,|\Psi\rangle = \sum_{1\leqslant i\neq j\leqslant d} c_i\overline{c_j}\,\langle\mathcal{E}_j(t)|\mathcal{E}_i(t)\rangle\,\alpha_j\,|i\rangle$$

$$\Rightarrow \quad \|\rho_{\mathcal{S}}^{(q)}(t)\,|\Psi\rangle\|^2 = \sum_{i=1}^{d}|c_i|^2\Big|\sum_{\substack{j=1\\ j\neq i}}^{d}\overline{c_j}\,\langle\mathcal{E}_j(t)|\mathcal{E}_i(t)\rangle\,\alpha_j\Big|^2 \leqslant \eta(t)^2\sum_{i=1}^{d}|c_i|^2\sum_{j=1}^{d}|c_j|^2 \leqslant \eta(t)^2.$$

Now, if F is a subspace of $\mathcal{H}_{\mathcal{S}}$ (*i.e.* a probabilistic event), let $(\varphi_k)_k$ be an orthonormal basis of F. We have:

$$\mathrm{tr}(\rho_{\mathcal{S}}(t)\Pi_F) - \mathrm{tr}(\rho_{\mathcal{S}}^{(d)}\Pi_F) = \mathrm{tr}(\rho_{\mathcal{S}}^{(q)}(t)\Pi_F) = \sum_{k=1}^{\dim(F)}\langle\varphi_k|\rho_{\mathcal{S}}^{(q)}(t)\varphi_k\rangle$$

$$\Rightarrow \quad |\mathrm{tr}(\rho_{\mathcal{S}}(t)\Pi_F) - \mathrm{tr}(\rho_{\mathcal{S}}^{(d)}\Pi_F)| \leqslant \sum_{k=1}^{\dim(F)}|||\rho_{\mathcal{S}}^{(q)}(t)||| \leqslant \dim(F)\eta(t).$$

In a nutshell: $\mathbb{P}_{quantum} = \mathbb{P}_{classical} + \mathcal{O}(\eta)$.

A.2 Relation between η and the linear entropy

We define the linear entropy (or purity defect) of a state ρ to be $S_{lin}(\rho) = 1 - \mathrm{tr}(\rho^2)$. Since $\mathcal{S}$ is initially in a pure state, the quantity $\frac{S_{lin}(\rho_{\mathcal{S}}(t))}{S_{lin}(\rho_{\mathcal{S}}^{(d)})}$ goes from 0 at $t = 0$ to almost 1 when $t \to +\infty$. It measures the ratio of purity that has already been lost compared to its final ideal value. Recall that $\rho_{\mathcal{S}}(t) = \sum_{i=1}^{d}|c_i|^2\,|i\rangle\,\langle i| + \sum_{1\leqslant i\neq j\leqslant d} c_i\overline{c_j}\,\langle\mathcal{E}_j(t)|\mathcal{E}_i(t)\rangle\,|i\rangle\,\langle j|$, so that:

$$\begin{aligned}\frac{S_{lin}(\rho_{\mathcal{S}}(t))}{S_{lin}(\rho_{\mathcal{S}}^{(d)})} &= \frac{1-\sum_i|c_i|^4 - 2\sum_{i\neq j}|c_i|^2|c_j|^2|\langle\mathcal{E}_i(t)|\mathcal{E}_j(t)\rangle|^2}{1-\sum_i|c_i|^4}\\ &\geqslant 1-\eta^2(t)\frac{2\sum_{i\neq j}|c_i|^2|c_j|^2}{1-\sum_i|c_i|^4}\\ &\geqslant 1-\eta^2(t)\end{aligned}$$

since the last fraction always equals 1 because $1 = (\sum_i|c_i|^2)(\sum_i|c_i|^2) = \sum_i|c_i|^4 + 2\sum_{i\neq j}|c_i|^2|c_j|^2$. Note that, for any given time t, this inequality is actually an equality for the initial state $|\Psi_{\mathcal{S}}(0)\rangle = c_{i_0}\,|i_0\rangle + c_{j_0}\,|j_0\rangle$ where i_0 and j_0 denote two indices such that $\eta(t) = |\langle\mathcal{E}_{i_0}(t)|\mathcal{E}_{j_0}(t)\rangle|$. Thus:

$$\eta(t) = \sqrt{1 - \inf_{|\Psi_{\mathcal{S}}(0)\rangle}\frac{S_{lin}(\rho_{\mathcal{S}}(t))}{S_{lin}(\rho_{\mathcal{S}}^{(d)})}}.$$

A.3 Relation between η and the entanglement entropy

The entanglement entropy is always much harder to manipulate. We were not able to prove in the general case a similar result when the linear entropy S_{lin} is replaced by the entanglement entropy S, but numerical simulations tend to indicate that the same formula is still (almost) true and that there exists a deep link between the quantity $1-\eta^2(t)$ and the ratio $\frac{S(\rho_{\mathcal{S}}(t))}{S(\rho_{\mathcal{S}}^{(d)})}$. Here are some considerations to get convinced.

In dimension $d = 2$, if one denotes $f(t) = \langle\mathcal{E}_2(t)|\mathcal{E}_1(t)\rangle$, one can write $\rho_{\mathcal{S}}(t) = \begin{pmatrix}|c_1|^2 & c_1\overline{c_2}f(t)\\ \overline{c_1}c_2\overline{f}(t) & |c_1|^2\end{pmatrix}$, whose eigenvalues are $ev_\pm = \frac{1}{2}(1 \pm \sqrt{(|c_1|^2-|c_2|^2)^2 + 4f^2(t)|c_1|^2|c_2|^2})$. At large times, $f \ll 1$ and after some calculations we get at leading order:

$$\frac{S(\rho_{\mathcal{S}}(t))}{S(\rho_{\mathcal{S}}^{(d)})} = \frac{ev_+\ln(ev_+) + ev_-\ln(ev_-)}{|c_1|^2\ln(|c_1|^2) + |c_2|^2\ln(|c_2|^2)} \simeq 1 + \frac{\frac{|c_1|^2|c_2|^2}{|c_1|^2-|c_2|^2}(\ln(|c_1|^2) - \ln(|c_2|^2))}{|c_1|^2\ln(|c_1|^2) + |c_2|^2\ln(|c_2|^2)}\eta^2(t)$$

The ratio preceding η^2 is a one-parameter real function in $|c_1|^2$ (since $|c_2|^2 = 1 - |c_1|^2$) defined on $[0,1]$; it turns out that is takes only values in $[-1,-0.7]$ and tends to -1 only when $|c_1|^2$ tends to 0 or 1. Therefore, in dimension 2, we still have (at least at leading order):

$$\eta(t) = \sqrt{1 - \inf_{|\Psi_{\mathcal{S}}(0)\rangle} \frac{S(\rho_{\mathcal{S}}(t))}{S(\rho_{\mathcal{S}}^{(d)})}}.$$

In higher dimension, if we suppose that one of the $\langle \mathcal{E}_j(t)|\mathcal{E}_i(t)\rangle$ decreases much slower than the others (assume without loss of generality that it is $\langle \mathcal{E}_2(t)|\mathcal{E}_1(t)\rangle$, still denoted $f(t)$), then after some time $\rho_{\mathcal{S}}(t)$ is not very different from:

$$\begin{pmatrix} |c_1|^2 & c_1\overline{c_2}f(t) & & & & \\ \overline{c_1}c_2\overline{f}(t) & |c_1|^2 & & & & \\ & & |c_3|^2 & & & \\ & & & \ddots & & \\ & & & & |c_d|^2 \end{pmatrix}$$

Using the previous inequality in dimension 2:

$$\frac{S(\rho_{\mathcal{S}}(t))}{S(\rho_{\mathcal{S}}^{(d)})} \geqslant \frac{(1-\eta^2(t))(|c_1|^2+|c_2|^2)+|c_3|^2+\cdots+|c_d|^2}{|c_1|^2+\cdots+|c_d|^2} \geqslant 1-\eta^2(t)$$

and, once again, this bound is attained for an appropriate choice of the $(c_i)_{1\leqslant i\leqslant d}$.

References

[1] A. Di Biagio and C. Rovelli, *Stable facts, relative facts*, Foundations of Physics, 51 (2021), pp. 1–13.

[2] L. Fonda, G. Ghirardi, and A. Rimini, *Decay theory of unstable quantum systems*, Reports on Progress in Physics, 41 (1978), p. 587.

[3] E. Joos, *Decoherence through interaction with the environment*, in Decoherence and the appearance of a classical world in quantum theory, Springer, 1996, pp. 35–136.

[4] K. Le Hur, *Entanglement entropy, decoherence, and quantum phase transitions of a dissipative two-level system*, Annals of Physics, 323 (2008), pp. 2208–2240.

[5] M. Merkli, G. Berman, R. Sayre, X. Wang, and A. I. Nesterov, *Production of entanglement entropy by decoherence*, Open Systems & Information Dynamics, 25 (2018), p. 1850001.

[6] P. Moran, *The closest pair of n random points on the surface of a sphere*, Biometrika, 66 (1979), pp. 158–162.

[7] R. Orús, *A practical introduction to tensor networks: Matrix product states and projected entangled pair states*, Annals of physics, 349 (2014), pp. 117–158.

[8] D. Poulin, A. Qarry, R. Somma, and F. Verstraete, *Quantum simulation of time-dependent hamiltonians and the convenient illusion of hilbert space*, Physical review letters, 106 (2011), p. 170501.

[9] L. Saloff-Coste, *Precise estimates on the rate at which certain diffusions tend to equilibrium*, Mathematische Zeitschrift, 217 (1994), pp. 641–677.

[10] M. Schlosshauer, *Decoherence, the measurement problem, and interpretations of quantum mechanics*, Reviews of Modern physics, 76 (2005), p. 1267.

[11] A. Soulas, *On the measurement problems in (relativistic) quantum mechanics*. Work in progress.

[12] C. Spengler, M. Huber, and B. C. Hiesmayr, *Composite parameterization and Haar measure for all unitary and special unitary groups*, Journal of mathematical physics, 53 (2012), p. 013501.

[13] H. D. Zeh, *On the interpretation of measurement in quantum theory*, Foundations of Physics, 1 (1970), pp. 69–76.

[14] W. H. Zurek, *Pointer basis of quantum apparatus: Into what mixture does the wave packet collapse?*, Physical review D, 24 (1981), p. 1516.

[15] W. H. Zurek, *Environment-induced superselection rules*, Physical review D, 26 (1982), p. 1862.

[16] ———, *Decoherence, einselection, and the quantum origins of the classical*, Reviews of modern physics, 75 (2003), p. 715.