A Strict Constrained Superposition Calculus for Graphs

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Abstract. We propose a superposition-based proof procedure to reason on equational first order formulas defined over graphs. First, we introduce the considered graphs that are directed labeled graphs with lists of roots standing for pins or interfaces for replacements. Then the syntax and semantics of the considered logic are defined. The formulas at hand are clause sets built on equations and disequations on graphs. Afterwards, a sound and complete proof procedure is provided, and redundancy criteria are introduced to dismiss useless clauses and improve the efficiency of the procedure. In a first step, a set of inferences rules is provided in the case of uninterpreted labels. In a second step, the proposed rules are lifted to take into account labels defined as terms interpreted in some arbitrary theory. Particular formulas of interest are Horn clauses, for which stronger redundancy criteria can be devised. Essential differences with the usual term superposition calculus are emphasized.

1 Introduction

Graphs are ubiquitous structures in computer science. They are used to model several notions such as data, program runs (transition systems), networks, software and hardware architectures. They are also often used as foundational structures to model knowledge or data bases, cognitive or intelligent systems as well as physical, chemical or biological phenomena. They constitute, in addition, the basis of operational research or combinatorics. Graphs are, definitely, fundamental structures for modelling, computing and reasoning. Graph transformations have been studied since the early 70's [30]. Some of their applications can be found in [17, 19]. In the literature, one can distinguish two main streams of approaches for graph transformation, namely the algebraic approaches [16, 13] where category theory is used to define structure transformations in a very abstract and elegant way and the algorithmic approaches where graph transformations are defined by means of the actual algorithms involved in the transformations [21, 14].

During the last decade, a very interesting application of graph transformations has emerged in the area of quantum models of computation, see e.g., the calculi ZX [12], ZH [4], ZW [25] or PBS [11]. In these calculi, one can specify quantum algorithms using particular graphs and can make some equational reasoning on them to verify correctness of quantum algorithms, see e.g. the

Quantomatic tool [26]. In such situations, making automated equational reasoning over graphs is very desirable even though equational theories over graphs are not recursively enumerable in general (see e.g. [8]).

The *superposition calculus* [2] is one of the most successful automated proof procedures which handles equational theories (on terms) which is being actually implemented in various theorem provers such as Vampire [29], Spass [33], or E [31]. The calculus operates on finite sets of equational clauses. It is defined as a set of *inference rules*, which deduce new clauses from previous ones. To prune the search space, strong restrictions (based on term orderings and literal selection functions) are imposed on the inferences, and redundancy criteria are provided to detect and dismiss useless clauses. The rules are applied until a contradiction (i.e., the empty clause) is derived or until the set is *saturated*, i.e., no further non-redundant clause may be deduced. The calculus is *refutationally complete*, in the sense that it is able to derive a contradiction from any unsatisfiable clause set. In a recent work [15], we proposed a superposition calculus for testing the unsatisfiability of sets of equations and disequations between graphs whose shapes are inspired by those used in the ZX calculus, where nodes are labeled by first-order (uninterpreted) terms. In the present paper we extend this work in several directions: (i) We tackle full clauses, i.e., disjunctions of equations and disequations. This extension turned out to be much more difficult than we initially expected, due to the fact that no reduction order exists on the considered graphs (see Examples 20 and 23), which complicates the completeness proof. We introduce redundancy criteria that cover some usual deletion and simplification rules. (ii) We lift the obtained calculus into a constrained calculus operating on graphs labeled by terms interpreted in some base theory. The procedure is a semi-decision procedure for unsatisfiability if the underlying theory is (semi) decidable and compact. (iii) We consider a slightly different class of graphs, where multi-edges are allowed. The new framework has the advantage of being both more general and simpler, and it also improves the efficiency of the calculus (more precisely for the computation of "merges" between graphs, see Remark 10).

Why defining a graph superposition calculus is difficult. We wish to emphasize some important differences between term and graph superposition. (i) It is well-known that term rewrite systems that are terminating and in which all critical pairs are joinable are confluent. This property plays a key rôle in the completeness proof of the superposition calculus. However, such a property does *not* hold for graph rewrite systems, and, worse, confluence is undecidable for terminating graph rewrite rules (if confluence is meant modulo isomorphism). As it is done in [15] we overcome this issue by considering a special class of graphs, for which the above property holds. This class is obtained by restricting the way graphs can be composed and replaced, using a sequence of distinguished nodes in the graphs, called roots. (ii) The usual superposition calculus is based on the use of a *reduction order*, i.e., a well-founded order on terms that is total on ground terms and closed under instantiation and embedding. Unfortunately

no such order exists for graphs in general (see Example 20). Thus the model construction algorithm used to establish refutational completeness must cope with non terminating systems (indeed, since a ground equation $\mathfrak{g} \approx \mathfrak{h}$ cannot always be oriented, one must consider both rules: $\mathfrak{g} \to \mathfrak{h}$ and $\mathfrak{h} \to \mathfrak{g}$, which entails that the system does not terminate). Confluence is harder to establish for non terminating systems and we need to devise a new confluence criterion. (iii) The usual redundancy criterion of [2] (where a clause is considered redundant if it is implied by smaller clauses) does not apply to graphs. For instance the conclusion of an inference may be strictly bigger than all the premises (see Example 22). This is due to the fact that two graphs may overlap without one of them being included in the other. Such a behavior cannot be avoided, since, as proven in [15, Theorem 45], satisfiability is undecidable for sets of ground equational clauses defined on graphs (whereas it is well known to be decidable for standard ground clauses based on terms), thus superposition cannot terminate on ground graphs. Furthermore, we show (see Example 23) that the calculus is – rather surprisingly – not compatible with tautology deletion in general (tautology deletion is possible for Horn clauses).

Related work. The graphs we are considering are intended to capture (possibly cyclic) circuit shaped structures such as those used in the ZX or related calculi. They are close to hypergraphs with interfaces as used in some papers (see, e.g. [6]) where the roots or interfaces are used in the gluing process while transforming a graph. We follow an algorithmic approach when transforming the graphs. This approach eases the completeness proofs of the proposed superposition calculus. However, the performed graph transformations used in the present paper can be encoded as simple double pushout (DPO) [20] steps of the form $L \longleftarrow \mathit{Roots} \longrightarrow R$ with some additional constraints on matched subgraphs. It is also a particular case of DPOI steps (DPO with interfaces) where the roots play the rôle of the interfaces [6]. Automated reasoning in presence of graph structures is not an easy task in general. Several authors did tackle this problem and one can distinguish different approaches in the literature. Variants of Hoare-like calculi have been proposed for the verification of graph transformation systems see, e.g., [24, 27, 7, 9]. Likewise, model checking procedures have also been devised in presence of graph structures see, e.g. [28, 32]. In these works, a dynamic logic underlying program execution is assumed. In addition, a dedicated logic is used to express graph properties to be proven. Other techniques have been used to prove graph equivalences such as bisimulation [18] or normalization using terminating and confuent graph rewriting systems [10]. In the paper at hand, we are rather concerned by a refutational proof technique based on superposition dedicated to a class of graphs. Thus our proof procedure departs from all the aforementioned works. To our knowledge, only the report [23] presents a refutational procedure dedicated to ZX diagrams which is close to ours. However, the authors use the classical superposition calculus [2] over first-order terms and provide a translation from the considered graphs to first-order terms. Such translation needs the use of additional axioms encoding some graph properties such as associativ-

ity and commutativity of graph constructor operations. Such additional axioms are useless in our framework. The class of graph rewriting systems handled in our proof procedure are not necessarily terminating and thus we had to devise new criteria to ensure their (ground) confluence instead of using joinability of pre-critical pairs as done in [5].

The paper is organized as follows. Section 2 introduces some basic notations and defines the considered graphs and the operations used over them. In Section 3 the syntax and semantics of the formulas are introduced. In Section 4, a first set of inference rules is defined to test the satisfiability of sets of clauses where graphs are endowed with uninterpreted labels and its completeness is established modulo a redundancy criterion that captures usual deletion or simplification rules (such as subsumption). In Section 5 the obtained calculus is lifted to graphs labeled with terms that can be interpreted in some arbitrary theory and possibly containing variables. Completeness is guaranteed if the theory is semi-decidable and compact. This last calculus is proven complete and an enhanced redundancy test is proposed. Concluding remarks are given in Section 6. Proofs can be found in the appendix.

2 Graphs and Graph Operations

We briefly review some usual definitions and notations. For any partial function f, we denote by $dom(f)$ the domain of f. If f and g are partial functions, we write $f(x) = g(x)$ to state that either $x \notin dom(f) \cup dom(g)$ or that $x \in dom(f) \cap dom(g)$ and the images of x by f and g are identical. Given a multiset $\mathfrak{m}$ and an element e, $\mathfrak{m}(e)$ denotes the multiplicity of e in $\mathfrak{m}$. For all multisets $\mathfrak{m}_1$ and $\mathfrak{m}_2$, we denote by $\mathfrak{m}_1 + \mathfrak{m}_2$ and $\mathfrak{m}_1 - \mathfrak{m}_2$ the sum and difference of $\mathfrak{m}_1$ and $\mathfrak{m}_2$, respectively. We write $\mathfrak{m}_1 \sqsubseteq \mathfrak{m}_2$ to state that $\mathfrak{m}_1$ is included in $\mathfrak{m}_2$. A multiset containing exactly the elements $e_1, \ldots, e_n$ is written $\{e_1, \ldots, e_n\}$. We denote by $\mathfrak{m}_1 \sqcup \mathfrak{m}_2$ the union of $\mathfrak{m}_1$ and $\mathfrak{m}_2$ (i.e., the minimal multiset containing $\mathfrak{m}_1$ and $\mathfrak{m}_2$) defined as follows: for all elements e, $(\mathfrak{m}_1 \sqcup \mathfrak{m}_2)(e) = \max(\mathfrak{m}_1(e), \mathfrak{m}_2(e))$. Finite sequences may sometimes be identified with sets if the order is not important, e.g., if $\boldsymbol{y} = (y_1, \ldots, y_n)$, we may write $x \in \boldsymbol{y}$ to state that $x = y_i$, for some $i = 1, \ldots, n$. We recall that a preorder is a binary relation that is reflexive and transitive. Any preorder $\leq$ may be associated with a strict order $<$ defined as follows: $x < y \iff (x \leq y \wedge y \not\leq x)$.

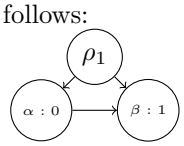
The graphs we consider are directed, labeled graphs enriched with a sequence of distinguished nodes, called *roots*:

Definition 1. *Let $\mathcal{N}$ be a countably infinite set of* nodes *and let $\mathcal{L}$ be a set of* labels*, disjoint from $\mathcal{N}$. An $\mathcal{L}$-*graph $\mathfrak{g}$ *is a tuple $\langle N, E, R, L\rangle$, where:*

- *$N \subseteq \mathcal{N}$ is a finite set of nodes in $\mathcal{N}$, called* vertices *or* nodes*;*
- *E is a finite multiset of pairs in $N \times N$, called* edges*;*
- *R is a sequence of nodes in N, with no repetition, called the* roots *of $\mathfrak{g}$;*
- *L is a function mapping every node in $N \setminus R$ to a* label *in $\mathcal{L}$.*

The components N, E, R and L of a graph $\mathfrak{g}$ are denoted by $N_\mathfrak{g}$, $E_\mathfrak{g}$, $R_\mathfrak{g}$ and $L_\mathfrak{g}$, respectively. We denote by $\widehat{N}_\mathfrak{g}$ the set of nodes $\alpha \in N_\mathfrak{g}$ that do not occur in $R_\mathfrak{g}$. The profile *of a graph $\mathfrak{g}$, written $pr(\mathfrak{g})$, is the length of $R_\mathfrak{g}$.*

Example 2. The $\mathcal{L}$-graph $\mathfrak{g}$ with $N_\mathfrak{g} = \{\rho_1, \alpha, \beta\}$, $E_\mathfrak{g} = \{(\rho_1, \alpha), (\rho_1, \beta), (\alpha, \beta)\}$, $R_\mathfrak{g} = (\rho_1)$, $dom(L_\mathfrak{g}) = \{\alpha, \beta\}$, $L_\mathfrak{g}(\alpha) = 0$ and $L_\mathfrak{g}(\beta) = 1$ is depicted graphically as follows:



We write $\alpha : \ell$ to state that a node named α is labeled by ℓ. In many cases, the names of the non-root nodes will be irrelevant, and will thus be omitted. When possible, root nodes will be named $\rho_1, \rho_2, \rho_3, \ldots$ in this order.

In the following, $\mathcal{L}$-graphs will be considered up to a renaming of nodes. More precisely, the isomorphism relation on $\mathcal{L}$-graphs is defined as follows.

Definition 3. *An $\mathcal{N}$-*renaming *μ is an injective mapping from $\mathcal{N}$ to $\mathcal{N}$. It is extended to any $\mathcal{L}$-graph $\mathfrak{g}$ by replacing every occurrence of a node α by $\mu(\alpha)$. In particular, the function $L_{\mu(\mathfrak{g})}$ is defined as follows: $L_{\mu(\mathfrak{g})}(\alpha) = \ell$ iff $L_\mathfrak{g}(\beta) = \ell$ for some $\beta \in N_\mathfrak{g}$ such that $\mu(\beta) = \alpha$ ($L_{\mu(\mathfrak{g})}$ is well-defined since μ is injective). We write $\mathfrak{g} \equiv \mathfrak{h}$ if $\mathfrak{h} = \mu(\mathfrak{g})$, for some $\mathcal{N}$-renaming μ. It is easy to check that $\equiv$ is an equivalence relation. Two $\mathcal{L}$-graphs $\mathfrak{g}, \mathfrak{h}$ such that $\mathfrak{g} \equiv \mathfrak{h}$ are* isomorphic*.*

The following result is straightforward to prove:

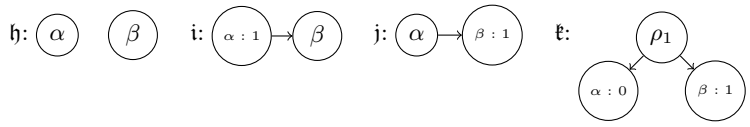
Proposition 4. *If $\mathfrak{g}$ is an $\mathcal{L}$-graph and μ is an $\mathcal{N}$-renaming then $\mu(\mathfrak{g})$ is an $\mathcal{L}$-graph such that $F_{\mu(\mathfrak{g})} = \mu(F_\mathfrak{g})$, for $F \in \{N, E, R, L\}$, and $pr(\mu(\mathfrak{g})) = pr(\mathfrak{g})$.*

2.1 Subgraphs and Replacement

We define the notion of a subgraph. The definition is slightly stronger than the usual one in graph theory because it imposes that only nodes that are roots in the subgraph can be connected to a node outside the subgraph. These roots can be viewed as an "interface" which restricts the way graphs may be connected and composed.

Definition 5 (Subgraph). *A graph $\mathfrak{h}$ is a* subgraph *of $\mathfrak{g}$ (written $\mathfrak{h} \leq^g \mathfrak{g}$) if $N_\mathfrak{h} \subseteq N_\mathfrak{g}$, $E_\mathfrak{h} \sqsubseteq E_\mathfrak{g}$, $\widehat{N}_\mathfrak{h} \subseteq \widehat{N}_\mathfrak{g}$, $L_\mathfrak{h}(\alpha) = L_\mathfrak{g}(\alpha)$ for all $\alpha \in \widehat{N}_\mathfrak{h}$ and if a node α occurs in an edge in $E_\mathfrak{g} - E_\mathfrak{h}$ then $\alpha \notin \widehat{N}_\mathfrak{h}$.*

Example 6. Consider the $\mathcal{L}$-graphs $\mathfrak{h}$, $\mathfrak{i}$, $\mathfrak{j}$ and $\mathfrak{k}$ with respective roots (α, β), (β), (α) and (ρ_1), defined as follows:



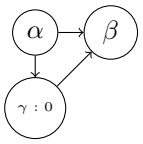
The $\mathcal{L}$-graph $\mathfrak{h}$ is a subgraph of the $\mathcal{L}$-graph $\mathfrak{g}$ from Example 2, but $\mathfrak{i}$, $\mathfrak{j}$ and $\mathfrak{k}$ are not. Indeed, α has different labels in $\mathfrak{g}$ and $\mathfrak{i}$; $\mathfrak{g}$ contains an edge between ρ_1 and β that does not occur in $\mathfrak{j}$ and β is not a root node in $\mathfrak{j}$; and $E_\mathfrak{g} - E_\mathfrak{k}$ contains the edge (α, β) between nodes that are not roots in $\mathfrak{k}$.

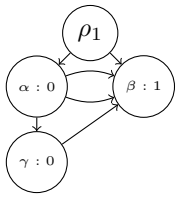
The replacement operation is defined in a natural way: all vertices and edges occurring from the replaced subgraph are deleted and replaced by those in the replacing graph (we assume that the considered graphs share the same roots).

Definition 7 (Subgraph replacement). *Let $\mathfrak{g}$ be an $\mathcal{L}$-graph and let $\mathfrak{h}$ be a subgraph of $\mathfrak{g}$. An $\mathcal{L}$-graph $\mathfrak{i}$ is* substitutable for $\mathfrak{h}$ in $\mathfrak{g}$ *if $R_\mathfrak{i} = R_\mathfrak{h}$ and $N_\mathfrak{g} \cap \widehat{N}_\mathfrak{i} = \emptyset$. If $\mathfrak{i}$ is substitutable for $\mathfrak{h}$ in $\mathfrak{g}$, then we denote by $\mathfrak{g}\{\mathfrak{h} \leftarrow \mathfrak{i}\}$ (the $\mathcal{L}$-graph obtained by replacing $\mathfrak{h}$ by $\mathfrak{i}$ in $\mathfrak{g}$) the tuple $\langle N', E', R', L' \rangle$, where:*

- $N' \stackrel{def}{=} (N_\mathfrak{g} \setminus N_\mathfrak{h}) \cup N_\mathfrak{i}$. *Note that since $R_\mathfrak{i} = R_\mathfrak{h}$ we have $N' = (N_\mathfrak{g} \setminus \widehat{N}_\mathfrak{h}) \cup \widehat{N}_\mathfrak{i}$.*
- $E' \stackrel{def}{=} (E_\mathfrak{g} - E_\mathfrak{h}) + E_\mathfrak{i}$.
- $R' \stackrel{def}{=} R_\mathfrak{g}$.
- $L'(\alpha) \stackrel{def}{=} \begin{cases} L_\mathfrak{g}(\alpha) & \text{if } \alpha \in N_\mathfrak{g} \setminus \widehat{N}_\mathfrak{i} \\ L_\mathfrak{i}(\alpha) & \text{if } \alpha \in \widehat{N}_\mathfrak{i} \end{cases}$ *for all $\alpha \in N' \setminus R'$.*

Example 8. Let $\mathfrak{i}'$ be the $\mathcal{L}$-graph with root (α, β) defined below. Using the $\mathcal{L}$-graphs $\mathfrak{g}$ and $\mathfrak{h}$ from Examples 2 and 6, we get the following $\mathcal{L}$-graph $\mathfrak{g}\{\mathfrak{h} \leftarrow \mathfrak{i}'\}$ (the edge (α, β) occurs twice because it occurs both in $E_{\mathfrak{i}'}$ and in $E_\mathfrak{g} - E_\mathfrak{h}$):

$\mathfrak{i}' =$ 

$\mathfrak{g}\{\mathfrak{h} \leftarrow \mathfrak{i}'\} =$ 

The notation $\mathfrak{g}\{\mathfrak{h} \leftarrow \mathfrak{i}\}$ is extended to the case where $pr(\mathfrak{i}) = pr(\mathfrak{h})$ as follows: $\mathfrak{g}\{\mathfrak{h} \leftarrow \mathfrak{i}\} \stackrel{def}{=} \mathfrak{g}\{\mathfrak{h} \leftarrow \mathfrak{i}'\}$, where $\mathfrak{i}'$ is any $\mathcal{L}$-graph substitutable for $\mathfrak{h}$ in $\mathfrak{g}$ such that $\mathfrak{i} \equiv \mathfrak{i}'$. Thus the replacement operation possibly involves a renaming step, to avoid conflicts on the names of the nodes. (Proposition 46 in Appendix B ensures that the result does not depend on the renaming, up to isomorphism). The next proposition states a straightforward property of subgraph replacement:

Proposition 9. *Let $\mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{j}$ be $\mathcal{L}$-graphs, where $\mathfrak{i} \leq^g \mathfrak{h} \leq^g \mathfrak{g}$ and $pr(\mathfrak{i}) = pr(\mathfrak{j})$. Then $\mathfrak{g}\{\mathfrak{h} \leftarrow \mathfrak{h}\{\mathfrak{i} \leftarrow \mathfrak{j}\}\} \equiv \mathfrak{g}\{\mathfrak{i} \leftarrow \mathfrak{j}\}$.*

Remark 10. Note that Proposition 9 would not hold if edges were defined as sets and not as multisets. For instance, consider $\mathcal{L}$-graphs $\mathfrak{g}, \mathfrak{h}$ with two root nodes ρ_1, ρ_2, where $\mathfrak{g}$ contains an edge (ρ_1, ρ_2) and $\mathfrak{h}$ contains no edges. If edges are taken as sets then we get $\mathfrak{g}\{\mathfrak{h} \leftarrow \mathfrak{g}\} = \mathfrak{g}$ and $\mathfrak{g}\{\mathfrak{g} \leftarrow \mathfrak{h}\} = \mathfrak{h}$, whereas $\mathfrak{g}\{\mathfrak{h} \leftarrow \mathfrak{h}\} = \mathfrak{g}$. In our previous work [15], this problem was overcome by restricting ourselves to induced subgraphs (which prevents the replacement of $\mathfrak{h}$ by $\mathfrak{g}$ in $\mathfrak{g}$), but this causes a combinatorial explosion in the definition of the calculus: when one "merges" two subgraphs, it is necessary to add every possible combination of edges connecting a root of the first $\mathcal{L}$-graph to a root of the second one, yielding exponentially many solutions w.r.t. the number of roots (see [15, Definition 30]). Such a behavior is avoided in the new framework.

We now introduce a notion of orthogonality between graphs. The intuition is that two $\mathcal{L}$-graphs will be considered orthogonal if they share no edges and no nodes other than roots.

Definition 11 (Orthogonal graphs). *Let $\mathfrak{g}$ be an $\mathcal{L}$-graph. Two subgraphs $\mathfrak{h}$ and $\mathfrak{i}$ of $\mathfrak{g}$ are* orthogonal in $\mathfrak{g}$*, or simply* orthogonal*, if $\widehat{N}_\mathfrak{h} \cap \widehat{N}_\mathfrak{i} = \emptyset$ and $E_\mathfrak{h} + E_\mathfrak{i} \sqsubseteq E_\mathfrak{g}$.*

Note that $\mathfrak{h}$ and $\mathfrak{i}$ may share root nodes. Proposition 12 states that the result of the replacement of two orthogonal subgraphs does not depend on the order in which the $\mathcal{L}$-graphs are considered.

Proposition 12. *Let $\mathfrak{g}$ be an $\mathcal{L}$-graph, and let $\mathfrak{h}_1$, $\mathfrak{h}_2$ be orthogonal subgraphs of $\mathfrak{g}$. For all $\mathcal{L}$-graphs $\mathfrak{i}_1, \mathfrak{i}_2$ of respective profiles $pr(\mathfrak{h}_1)$ and $pr(\mathfrak{h}_2)$, $\mathfrak{h}_2$ and $\mathfrak{h}_1$ are subgraphs of $\mathfrak{g}\{\mathfrak{h}_1 \leftarrow \mathfrak{i}_1\}$ and $\mathfrak{g}\{\mathfrak{h}_2 \leftarrow \mathfrak{i}_2\}$, respectively, and $\mathfrak{g}\{\mathfrak{h}_1 \leftarrow \mathfrak{i}_1\}\{\mathfrak{h}_2 \leftarrow \mathfrak{i}_2\} \equiv \mathfrak{g}\{\mathfrak{h}_2 \leftarrow \mathfrak{i}_2\}\{\mathfrak{h}_1 \leftarrow \mathfrak{i}_1\}$.*

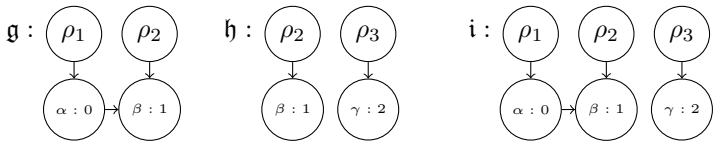
2.2 Graph Merging

Intuitively, a merge of two $\mathcal{L}$-graphs $\mathfrak{g}_1$ and $\mathfrak{g}_2$ denotes any minimal $\mathcal{L}$-graph containing all vertices, labels and edges in $\mathfrak{g}_1$ and $\mathfrak{g}_2$. More formally:

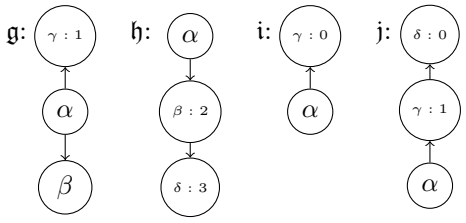
Definition 13. *A* merge *of two $\mathcal{L}$-graphs $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is an $\mathcal{L}$-graph $\mathfrak{h}$ such that: (i) $\mathfrak{g}_i \leq^g \mathfrak{h}$, for all $i = 1, 2$; (ii) $N_\mathfrak{h} = N_{\mathfrak{g}_1} \cup N_{\mathfrak{g}_2}$, $E_\mathfrak{h} = E_{\mathfrak{g}_1} \sqcup E_{\mathfrak{g}_2}$ and $\widehat{N}_\mathfrak{h} = \widehat{N}_{\mathfrak{g}_1} \cup \widehat{N}_{\mathfrak{g}_2}$; (iii) for all $i = 1, 2$ and for all $\alpha \in \widehat{N}_{\mathfrak{g}_i}$, $L_\mathfrak{h}(\alpha) = L_{\mathfrak{g}_i}(\alpha)$.*

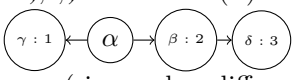
Note that in contrast to [15, Definition 30], the merge contains no node and edge other than those occurring in $\mathfrak{g}_1$ or $\mathfrak{g}_2$. Moreover, the multiplicity of edges is minimal ($E_\mathfrak{h}$ is defined as $E_{\mathfrak{g}_1} \sqcup E_{\mathfrak{g}_2}$ instead of $E_{\mathfrak{g}_1} + E_{\mathfrak{g}_2}$). It is easy to check that a merge of $\mathfrak{g}_1$, $\mathfrak{g}_2$ exists iff $L_{\mathfrak{g}_1}(\alpha) = L_{\mathfrak{g}_2}(\alpha)$ holds for all $\alpha \in \widehat{N}_{\mathfrak{g}_1} \cap \widehat{N}_{\mathfrak{g}_2}$. Moreover, all the merges are equal up to a permutation of their roots.

Example 14. Consider the following $\mathcal{L}$-graphs $\mathfrak{g}$ and $\mathfrak{h}$ below of respective roots (ρ_1, ρ_2) and (ρ_2, ρ_3), where the nodes α, β, γ are labeled by 0, 1 and 2, respectively. These $\mathcal{L}$-graphs admit the following merge $\mathfrak{i}$, of root (ρ_1, ρ_2, ρ_3):



Example 15. Let $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{i}$ and $\mathfrak{j}$ be the $\mathcal{L}$-graphs, defined as follows:



The $\mathcal{L}$-graph $\mathfrak{g}$ has roots (α, β) and $\mathfrak{h}, \mathfrak{i}, \mathfrak{j}$ have roots (α). Then $\mathfrak{g}$ and $\mathfrak{h}$ admit the following merge, of root (α): 

In contrast, $\mathfrak{g}$ and $\mathfrak{i}$ admit no merge (since γ has different labels in the two graphs), and neither do $\mathfrak{g}$ and $\mathfrak{j}$ (due to the edge connecting the non-root node γ to δ, that is outside of $\mathfrak{g}$).

Lemma 16. *Let $\mathfrak{g}$ be an $\mathcal{L}$-graph and let $\mathfrak{h}, \mathfrak{i}$ be subgraphs of $\mathfrak{g}$. Then $\mathfrak{h}$ and $\mathfrak{i}$ admit a merge $\mathfrak{j}$, and for all merges $\mathfrak{j}$ of $\mathfrak{h}$ and $\mathfrak{i}$ we have $\mathfrak{j} \leq^g \mathfrak{g}$.*

3 An Equational Logic on Graphs

We now define equational clauses built on $\mathcal{L}$-graphs and their semantics.

Definition 17. *An* equation *is an unordered pair written $\mathfrak{g} \approx \mathfrak{h}$, where $\mathfrak{g}, \mathfrak{h}$ are $\mathcal{L}$-graphs such that $R_\mathfrak{g} = R_\mathfrak{h}$. A* literal *is either an equation (positive literal) or the negation of an equation, written $\mathfrak{g} \not\approx \mathfrak{h}$ (negative literal). A* clause *is a disjunction of literals. The disjunction may be empty, in which case the clause is written $\Box$. A clause is* Horn *if it contains at most one positive literal. A set of clauses is* Horn *if it contains only Horn clauses.*

Note that we assume for technical convenience that the two members of an equation share the same roots. $\mathcal{N}$-renamings μ are extended to equations, literals and clauses in a straightforward way: $\mu(\mathfrak{g} \approx \mathfrak{h}) \stackrel{def}{=} \mu(\mathfrak{g}) \approx \mu(\mathfrak{h})$, $\mu(\mathfrak{g} \not\approx \mathfrak{h}) \stackrel{def}{=} \mu(\mathfrak{g}) \not\approx \mu(\mathfrak{h})$ and $\mu(C \vee D) \stackrel{def}{=} \mu(C) \vee \mu(D)$. The relation $\equiv$ is extended accordingly.

Sets of clauses built on $\mathcal{L}$-graphs will be interpreted w.r.t. a congruence on $\mathcal{L}$-graphs. Graph congruences are defined in same way as for terms, except that we also assume that they are closed under isomorphism.

Definition 18 (Graph Congruence). *A binary relation $\bowtie$ on $\mathcal{L}$-graphs is* closed under isomorphisms *if $\mathfrak{i} \bowtie \mathfrak{h}$ when $\mathfrak{g} \bowtie \mathfrak{h}$ and $\mathfrak{g} \equiv \mathfrak{i}$. It is* closed under embeddings *if $\mathfrak{h} \bowtie \mathfrak{i}$ entails $\mathfrak{g}\{\mathfrak{h} \leftarrow \mathfrak{i}\} \bowtie \mathfrak{g}$. A* congruence *is an equivalence relation on $\mathcal{L}$-graphs that is closed under isomorphisms and embeddings.*

Definition 19. *A congruence $\sim$* validates *an expression E (written $\sim \models E$) iff one of the following conditions holds: (i) E is an equation $\mathfrak{g} \approx \mathfrak{h}$ and $\mathfrak{g} \sim \mathfrak{h}$; (ii) E is a literal $\mathfrak{g} \not\approx \mathfrak{h}$ and $\mathfrak{g} \not\sim \mathfrak{h}$; (iii) E is a clause C and $\sim$ validates at least one literal in C; (iv) E is a set of clauses Γ and $\sim$ validates all the clauses in Γ. A congruence $\sim$ is a* model *of E if $\sim \models E$. An expression is* satisfiable *if it admits a model and* unsatisfiable *otherwise. A* tautology *is a clause that is true in all congruences.*

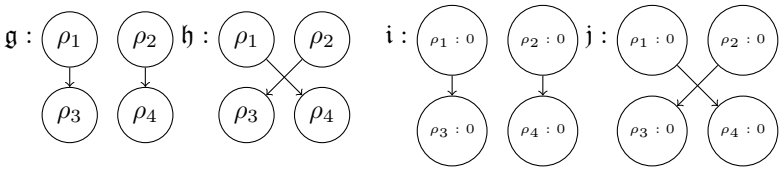
4 Superposition Calculus with Uninterpreted Labels

We define a superposition calculus for testing the satisfiability of sets of clauses. This calculus is *strict* (see, e.g., [3]) in the sense that it does not use the equational

factorization rule (as defined in [2]), but uses instead the standard factorization rule that unifies both members of two equations. This choice is motivated by the fact that, as shown in Example 23, graph superposition is not compatible with tautology deletion (except when the clauses are Horn). Since tautology deletion is disabled for non-Horn clauses, equational factorization is not needed anyway. Selection functions are not considered, since they are not compatible with the redundancy criterion.

The usual superposition calculus [2] is parameterized by a *reduction order*, i.e., an order on terms that is well-founded, total on ground terms, and closed under substitutions and embeddings. In the case of $\mathcal{L}$-graphs, no such order possibly exists, if we also add the natural requirement that the order must be closed under renamings, as evidenced by the following example:

Example 20. Assume that an order $<$ exists, satisfying the following properties: $<$ is well-founded, closed under isomorphisms and embeddings, and total up to isomorphism (i.e., if $\mathfrak{g} \not\equiv \mathfrak{h}$ then either $\mathfrak{g} < \mathfrak{h}$ or $\mathfrak{h} < \mathfrak{g}$). Consider the $\mathcal{L}$-graphs $\mathfrak{g}$ and $\mathfrak{h}$ with roots $(\rho_1, \rho_2, \rho_3, \rho_4)$ and containing no labels, as well as the $\mathcal{L}$-graphs $\mathfrak{i}, \mathfrak{j}$ with an empty sequence of roots, where all nodes are labeled by 0:



It is clear that $\mathfrak{g} \not\equiv \mathfrak{h}$. Indeed, if $\mu(\mathfrak{g}) = \mathfrak{h}$ holds for some $\mathcal{N}$-renaming μ, then $\mu(R_\mathfrak{g}) = R_\mathfrak{h}$, i.e., $\mu((\rho_1, \rho_2, \rho_3, \rho_4)) = (\rho_1, \rho_2, \rho_3, \rho_4)$, which entails that μ is the identity on these nodes. Thus we cannot have $\mu(E_\mathfrak{g}) = E_\mathfrak{h}$, as the first root (ρ_1) is connected to the third root (ρ_3) in $\mathfrak{g}$ and to the fourth one (ρ_4) in $\mathfrak{h}$. Consequently, we have either $\mathfrak{g} < \mathfrak{h}$ or $\mathfrak{h} < \mathfrak{g}$. Now we also have $\mathfrak{g} \leq^g \mathfrak{i}$ and $\mathfrak{h} \leq^g \mathfrak{j}$, and it is easy to check that $\mathfrak{i}\{\mathfrak{g} \leftarrow \mathfrak{h}\} = \mathfrak{j}$ and $\mathfrak{j}\{\mathfrak{h} \leftarrow \mathfrak{g}\} = \mathfrak{i}$. Thus we have either $\mathfrak{i} < \mathfrak{j}$ or $\mathfrak{j} < \mathfrak{i}$. But since $R_\mathfrak{i} = R_\mathfrak{j} = ()$ we have $\mathfrak{i} \equiv \mathfrak{j}$: indeed, if $\mu(\rho_1) = \rho_1$, $\mu(\rho_2) = \rho_2$, $\mu(\rho_3) = \rho_4$ and $\mu(\rho_4) = \rho_3$, then $\mu(\mathfrak{i}) = \mathfrak{j}$.

We thus slightly relax the requirement of having a reduction order, and consider instead a pre-order $<$ on $\mathcal{L}$-graphs, that is well-founded, closed under isomorphisms and embeddings, and contains $\leq^g$. We write $\mathfrak{g} < \mathfrak{h}$ if $\mathfrak{g} \leq \mathfrak{h}$ and $\mathfrak{h} \not\leq \mathfrak{g}$, and we write $\mathfrak{g} \simeq \mathfrak{h}$ if $\mathfrak{g} \leq \mathfrak{h}$ and $\mathfrak{h} \leq \mathfrak{g}$. We also assume that the equivalence classes of $\simeq$ are finite, up to isomorphism. It is clear that such pre-orders exist, for instance, the pre-order: $\mathfrak{g} \leq \mathfrak{h} \iff \mathit{card}(N_\mathfrak{g}) \leq \mathit{card}(N_\mathfrak{h})$ fulfills the above properties.

Similarly to the usual superposition calculus, we associate every literal L with a multiset defined as follows: $\mathit{mset}(\mathfrak{g} \not\approx \mathfrak{h}) \stackrel{\mathit{def}}{=} \{\!\{\mathfrak{g}, \mathfrak{h}\}\!\}$ and $\mathit{mset}(\mathfrak{g} \approx \mathfrak{h}) \stackrel{\mathit{def}}{=} \{\!\{\mathfrak{g}\}\!\}, \{\!\{\mathfrak{h}\}\!\}\}$. For every clause $C = L_1 \vee \cdots \vee L_n$, we define: $\mathit{mset}(C) \stackrel{\mathit{def}}{=} \{\!\{\mathit{mset}(L_i) \mid i = 1, \ldots, n\}\!\}$. Any order or preorder $\triangleright$ on $\mathcal{L}$-graphs may then be extended into an order on clauses as follows: $C \triangleright D \iff \mathit{mset}(C) \triangleright_m \mathit{mset}(D)$, where $\triangleright_m$ denotes the multiset extension of $\triangleright$ (note that $\triangleright_m$ is also a (pre)order). A literal

L is $<$-*maximal* in a clause C if there is no literal $L' \in C$ such that $L' > L$. An $\mathcal{L}$-graph $\mathfrak{g}$ is $<$-*maximal* in a literal L if L contains no $\mathcal{L}$-graph $\mathfrak{g}'$ such that $\mathfrak{g}' > \mathfrak{g}$. A literal L is *eligible* in a clause C if L is a $<$-maximal literal in C. Intuitively, eligible literals are those that may be considered for performing inferences. For instance, given a clause $(\mathfrak{g} \approx \mathfrak{h}) \vee (\mathfrak{i} \approx \mathfrak{j})$, if $(\mathfrak{g} \approx \mathfrak{h}) > (\mathfrak{i} \approx \mathfrak{j})$, then $\mathfrak{g} \approx \mathfrak{h}$ is eligible but not $\mathfrak{i} \approx \mathfrak{j}$. Consequently the inference rules (as defined in Section 4.1) will be allowed to replace $\mathfrak{g}$ by $\mathfrak{h}$ using the equation $\mathfrak{g} \approx \mathfrak{h}$ (provided $\mathfrak{g} \not< \mathfrak{h}$) but not, e.g., $\mathfrak{i}$ by $\mathfrak{j}$ (this restricts the number of inferences and prune the search space). Non eligible literals are simply attached to the conclusion of the inference but they play no active role until they (eventually) become eligible.

4.1 Inference Rules

The Superposition calculus $\mathtt{SC}$ is defined by the following rules: $\mathtt{Sp}^+$ (positive superposition), $\mathtt{Sp}^-$ (negative superposition), $\mathtt{R}$ (Reflection) and $\mathtt{F}$ (Factoring). The rules and their side conditions are very similar to those of the usual (ground) superposition calculus, except for the use of the merging operation for positive superposition. To simplify notations, the rules are defined modulo isomorphims, which means that one has to find a renaming of the premises such that the considered rule applies (this can be done using standard algorithms for finding graph homomorphisms). For instance, with this convention, the Reflection rule $\mathtt{R}$ actually removes all equations of the form $\mathfrak{g} \not\approx \mathfrak{h}$, with $\mathfrak{g} \equiv \mathfrak{h}$.

$$\mathtt{Sp}^+ : \frac{\mathfrak{g}_1 \approx \mathfrak{h}_1 \vee C_1 \quad \mathfrak{g}_2 \approx \mathfrak{h}_2 \vee C_2}{\mathfrak{i}\{\mathfrak{g}_1 \leftarrow \mathfrak{h}_1\} \approx \mathfrak{i}\{\mathfrak{g}_2 \leftarrow \mathfrak{h}_2\} \vee C_1 \vee C_2}$$

where:

1. $\mathfrak{i}$ is a merge of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, and $\mathfrak{g}_1, \mathfrak{g}_2$ are not orthogonal;
2. $\mathfrak{g}_i \approx \mathfrak{h}_i$ is eligible in $\mathfrak{g}_i \approx \mathfrak{h}_i \vee C_i$ for $i = 1, 2$.
3. $\mathfrak{g}_i \not< \mathfrak{h}_i$ for $i = 1, 2$.

The non-orthogonality condition is the analogous of the non-variable condition of the usual calculus, it dismisses trivial replacements.

$$\mathtt{Sp}^- : \frac{\mathfrak{g} \approx \mathfrak{h} \vee C \quad \mathfrak{i} \not\approx \mathfrak{j} \vee D}{\mathfrak{i}\{\mathfrak{g} \leftarrow \mathfrak{h}\} \not\approx \mathfrak{j} \vee C \vee D}$$

where:

1. $\mathfrak{g} \leq^g \mathfrak{i}$;
2. $\mathfrak{g} \approx \mathfrak{h}$ and $\mathfrak{i} \not\approx \mathfrak{j}$ are eligible in $\mathfrak{g} \approx \mathfrak{h} \vee C$ and $\mathfrak{i} \not\approx \mathfrak{j} \vee D$, respectively.
3. $\mathfrak{g} \not< \mathfrak{h}$ and $\mathfrak{i} \not< \mathfrak{j}$.

$$\mathtt{F} : \frac{\mathfrak{g} \approx \mathfrak{h} \vee \mathfrak{g} \approx \mathfrak{h} \vee C}{\mathfrak{g} \approx \mathfrak{h} \vee C} \qquad \text{if } \mathfrak{g} \approx \mathfrak{h} \text{ is eligible in } \mathfrak{g} \approx \mathfrak{h} \vee \mathfrak{g} \approx \mathfrak{h} \vee C.$$

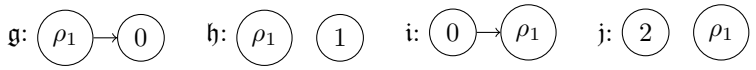
$$\mathtt{R} : \frac{\mathfrak{g} \not\approx \mathfrak{g} \vee C}{C} \qquad \text{if } \mathfrak{g} \not\approx \mathfrak{g} \text{ is eligible in } \mathfrak{g} \not\approx \mathfrak{g} \vee C.$$

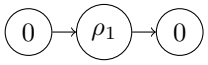
Lemma 21. *The rules* $\mathtt{Sp}^+$, $\mathtt{Sp}^-$, $\mathtt{F}$ *and* $\mathtt{R}$ *are sound, i.e., for all congruences* $\sim$ *and for all clauses* C *deducible from a set of premises* Γ*, we have* $\sim\models \Gamma \implies \sim\models C$.

4.2 Redundancy

In the usual superposition calculus [2], a clause is redundant if all its ground instances are entailed by smaller clauses (w.r.t. the considered order). Such clauses can be deleted without threatening refutational completeness, which reduces the search space. In our context, such a definition cannot be used, because one of the inference rules –namely $\mathtt{Sp}^+$– may generate clauses that are strictly larger than the premises (hence such clauses would be considered as redundant if the usual criterion were to be used).

Example 22. Consider the clauses: $\mathfrak{g} \approx \mathfrak{h}$ and $\mathfrak{i} \approx \mathfrak{j}$, where $\mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{j}$ are $\mathcal{L}$-graphs with root (ρ_1) that are defined as follows:



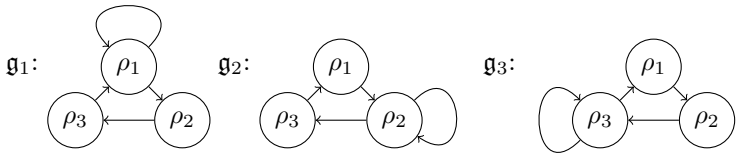
The $\mathcal{L}$-graphs $\mathfrak{g}$ and $\mathfrak{i}$ admit the following merge (of root (ρ_1)): 

Therefore, rule $\mathtt{Sp}^+$ applies, yielding $\mathfrak{g}' \approx \mathfrak{g}''$, where:

If $\mathcal{L}$-graphs are ordered according to their number of nodes, then we have $(\mathfrak{g}' \approx \mathfrak{g}'') > (\mathfrak{g} \approx \mathfrak{h})$ and $(\mathfrak{g}' \approx \mathfrak{g}'') > (\mathfrak{i} \approx \mathfrak{j})$.

Worse, the calculus is actually incomplete if tautologies are deleted, as shown in the following example.

Example 23. Consider the $\mathcal{L}$-graphs $\mathfrak{g}_1, \mathfrak{g}_2$ and $\mathfrak{g}_3$ with roots (ρ_1, ρ_2, ρ_3):



Let $\dot{\mathfrak{g}}_i$ denote the graph obtained from $\mathfrak{g}_i$ by adding one additional non root node α distinct from ρ_1, ρ_2, ρ_3, with some arbitrary (but fixed) label, e.g., 0. Assume that the graphs are ordered by the number of nodes, so that $\dot{\mathfrak{g}}_i > \mathfrak{g}_j$, $\dot{\mathfrak{g}}_i \simeq \dot{\mathfrak{g}}_j$ and $\mathfrak{g}_i \simeq \mathfrak{g}_j$ (for all $i, j \in \{1, 2, 3\}$). Let $\Gamma = \{\dot{\mathfrak{g}}_1 \approx \mathfrak{g}_2 \vee \dot{\mathfrak{g}}_2 \approx \mathfrak{g}_3 \vee \dot{\mathfrak{g}}_3 \approx \mathfrak{g}_1, \dot{\mathfrak{g}}_1 \not\approx \mathfrak{g}_2 \vee \dot{\mathfrak{g}}_2 \not\approx \mathfrak{g}_3 \vee \dot{\mathfrak{g}}_3 \not\approx \mathfrak{g}_1\}$. Intuitively, every equation $\dot{\mathfrak{g}}_i \approx \mathfrak{g}_j$ where $(i, j) \in \{(1, 2), (2, 3), (3, 1)\}$ states that the semantics of the graph is

preserved when the isolated node is deleted and the graph is rotated by 90 degrees clockwise, for each possible position of the loop. Since the graphs are invariant by rotation, all these transformations are actually equivalent. It is easy to check that every clause that can be generated from Γ by applying the negative superposition rule from the first clause into the second clause contains two complementary literals (i.e. two literals of the form $\dot{\mathfrak{g}}_i \approx \mathfrak{g}_j$ and $\dot{\mathfrak{g}}_i \not\approx \mathfrak{g}_j$) hence is a tautology. Moreover, the clauses obtained by superposition using the first clause only either are subsumed by the first clause (if the superposition rule is applied on two different literals) or contains a literal $\mathfrak{g}_i \approx \mathfrak{g}_i$ (hence is a tautology). The equational factorization rule (as defined in [2]) does not apply since $\dot{\mathfrak{g}}_i$ and $\dot{\mathfrak{g}}_j$ are not isomorphic if $i \neq j$. However, consider the $\mathcal{L}$-graphs $\mathfrak{g}'_i, \dot{\mathfrak{g}}'_i$ which contain the same nodes and edges as $\mathfrak{g}_i$ and $\dot{\mathfrak{g}}_i$ respectively, but with roots (ρ_2, ρ_3, ρ_1). It is clear that $\mathfrak{g}'_2 \equiv \mathfrak{g}_1$ and $\mathfrak{g}'_3 \equiv \mathfrak{g}_2$, so that $\dot{\mathfrak{g}}_1 \approx \mathfrak{g}_2 \models \dot{\mathfrak{g}}'_2 \approx \mathfrak{g}'_3$. However, $\dot{\mathfrak{g}}'_2 \leq^g \dot{\mathfrak{g}}_2$ and $\dot{\mathfrak{g}}_2\{\dot{\mathfrak{g}}'_2 \leftarrow \mathfrak{g}'_3\} = \mathfrak{g}_3$, thus $\dot{\mathfrak{g}}_1 \approx \mathfrak{g}_2 \models \dot{\mathfrak{g}}_2 \approx \mathfrak{g}_3$. By a similar reasoning, we may show that $\dot{\mathfrak{g}}_2 \approx \mathfrak{g}_3 \models \dot{\mathfrak{g}}_3 \approx \mathfrak{g}_1$ and $\dot{\mathfrak{g}}_3 \approx \mathfrak{g}_1 \models \dot{\mathfrak{g}}_1 \approx \mathfrak{g}_2$, so that the equations $\dot{\mathfrak{g}}_1 \approx \mathfrak{g}_2$, $\dot{\mathfrak{g}}_2 \approx \mathfrak{g}_3$, and $\dot{\mathfrak{g}}_3 \approx \mathfrak{g}_1$, are actually pairwise equivalent, which entails that Γ is unsatisfiable. However, $\square$ cannot be derived from Γ if the clauses containing complementary literals are discarded.

Thus, the conditions that ensure that a clause is redundant must be stronger than those of the usual superposition calculus. The definition proposed below covers usual deletion rules such as subsumption. Actually, two different criteria will be used, namely non-strict and strict redundancy, depending on whether the considered clauses are Horn or not. Indeed, in the former case a slightly less restrictive definition can be used, which permits the deletion of (some) tautological clauses.

Definition 24. *Let C, D be two clauses and let Γ be a set of clauses. We say that C is* subsumed *by D and write $C \geq^{sub} D$ if $C = D \vee C'$, up to associativity and commutativity of $\vee$ and isomorphism. We write $C \rightarrow_\Gamma D$ (C* demodulates *to D w.r.t. Γ) if C is of the form $\mathfrak{g} \bowtie \mathfrak{h} \vee E$ (with $\bowtie \in \{\approx, \not\approx\}$), $D = \mathfrak{g}\{\mathfrak{i} \leftarrow \mathfrak{j}\} \bowtie \mathfrak{h} \vee E$, and there exists a clause $F \in \Gamma$ such that $F = (\mathfrak{i} \approx \mathfrak{j}) \vee F'$, with $F' \leq^{sub} E$, $\mathfrak{i} > \mathfrak{j}$, $F' < (\mathfrak{i} \approx \mathfrak{j})$ and $(\mathfrak{i} \approx \mathfrak{j}) < (\mathfrak{g} \bowtie \mathfrak{h})$.*

The set of clauses that are redundant w.r.t. a set of clauses Γ is defined inductively as follows. A clause C is redundant *w.r.t. Γ iff one of the following conditions holds: (1) C contains two literals $\mathfrak{g}_1 \approx \mathfrak{g}_2$ and $\mathfrak{g}'_1 \not\approx \mathfrak{g}'_2$, with $\mathfrak{g}_i \equiv \mathfrak{g}'_i$ for $i = 1, 2$; (2) C contains a literal of the form $\mathfrak{g} \approx \mathfrak{h}$ with $\mathfrak{g} \equiv \mathfrak{h}$; (3) $C \geq^{sub} D$, for some $D \in \Gamma$; (4) $C \rightarrow_\Gamma D$ and D is redundant. The set of* strictly redundant *ground clauses is defined in a similar way, except that Item 1 is removed.*

Intuitively, the conditions ensuring that C demodulates to D in Definition 24 are meant to ensure that D may be deduced from C by applying the rule $\mathtt{Sp}^+$ or $\mathtt{Sp}^-$ using the clause F (with $D < C$ and $F < C$) and that $\{D\} \cup \Gamma$ is equivalent to $\{C\} \cup \Gamma$. In particular, the condition $F' \leq^{sub} E$ ensures that all the literals added by the inference already occur in C.

Definition 25. *A set of clauses Γ is* saturated *(resp.* strictly saturated*) if every clause that can be deduced from premises in Γ using one of the rules of* SC *(in one step) is redundant (resp. strictly redundant) w.r.t. Γ.*

We prove that SC is refutationally complete. We actually establish two completeness results, the first one for general clauses and the second one for Horn clauses. The latter is stronger since it uses the weaker non-strict saturatedness criterion instead of strict saturatedness.

Theorem 26. *Let Γ be a set of clauses. If $\Box \notin \Gamma$ and Γ is strictly saturated or both Horn and saturated then Γ is satisfiable.*

5 A Constrained Graph Superposition Calculus

We now lift the calculus SC defined in Section 4 into a constrained calculus. The goal is to handle graphs labeled by terms interpreted in some arbitrary theory, and possibly containing variables. To this aim, we attach constraints to the clauses, which are formulas interpreted in the considered theory, asserting conditions on the labels. Such constraints will be updated when inference rules will be applied, by asserting the conditions that are required by the rule applications.

5.1 Constrained Clauses

Let $\mathcal{V}$ be a countably infinite set of *variables* and let Σ be a set of *function symbols*[1]. Each symbol f in Σ is associated with a unique *arity* $\#(f)$. We denote by $\mathcal{T}$ the set of *terms* built inductively as usual on $\mathcal{V}$ and Σ, and by $\mathcal{C}$ the set of first-order formulas, called *constraints*, built inductively as usual on atoms of the form $t \doteq s$, where $t, s \in \mathcal{T}$ using the logical connectives $\vee, \wedge, \neg, \Rightarrow, \Leftrightarrow$, the quantifiers $\exists, \forall$ and two logical constants $\bot$ and $\top$.

A *substitution* σ is a function mapping all variables x to a term $x\sigma$. The *domain* $dom(\sigma)$ of σ is the set of variables x such that $x\sigma \neq x$. For every term or formula e, we denote by $e\sigma$ the term or formula obtained from e by replacing every (free) variable x by $x\sigma$. A term is *ground* if it contains no variables, and a substitution σ is *ground* if $x\sigma$ is ground for all $x \in dom(\sigma)$.

*$\mathcal{T}$-graph*s are $\mathcal{L}$-graphs with labels in $\mathcal{T}$. A *$\mathcal{T}$-clause* is a clause defined on $\mathcal{T}$-graphs. Substitutions are extended to $\mathcal{T}$-graphs and $\mathcal{T}$-clauses as follows. For every $\mathcal{T}$-graph $\mathfrak{g}$, we denote by $\mathfrak{g}\sigma$ the $\mathcal{T}$-graph such that: $F_{\mathfrak{g}\sigma} = F_{\mathfrak{g}}$ for all $F \in \{N, E, R\}$ and $L_{\mathfrak{g}\sigma}(\alpha) = L_{\mathfrak{g}}(\alpha)\sigma$, for all $\alpha \in \widehat{N}_{\mathfrak{g}}$. Then: $(\mathfrak{g} \approx \mathfrak{h})\sigma \stackrel{def}{=} \mathfrak{g}\sigma \approx \mathfrak{h}\sigma$, $(\mathfrak{g} \not\approx \mathfrak{h})\sigma \stackrel{def}{=} \mathfrak{g}\sigma \not\approx \mathfrak{h}\sigma$ and $(C \vee D)\sigma \stackrel{def}{=} C\sigma \vee D\sigma$. A $\mathcal{T}$-graph $\mathfrak{g}$ is *ground* if for all $\alpha \in \widehat{N}_{\mathfrak{g}}$, $L_{\mathfrak{g}}(\alpha)$ is ground. A $\mathcal{T}$-clause is *ground* if all the $\mathcal{T}$-graphs occurring in it are ground. For every expression (term, $\mathcal{T}$-graph, constraint or $\mathcal{T}$-clause) E, we denote by $\mathcal{V}(E)$ the set of variables (freely) occurring in E.

Definition 27. *A* constrained clause *(or c-clause) is a pair $[C \mid \phi]$, where C is a $\mathcal{T}$-clause and $\phi \in \mathcal{C}$.*

[1] As usual, predicates may be encoded as functions.

Let $\mathcal{I}$ be some fixed set of first-order interpretations on the signature Σ. For all $I \in \mathcal{I}$, we denote by $dom(I)$ the domain of I and by f^I the interpretation of the function f (with $f \in \Sigma$). For every ground term t and for all $I \in \mathcal{I}$, we denote by $[t]^I$ the value of t in I, inductively defined as usual. To simplify notations, we assume that for every $I \in \mathcal{I}$ and for every $e \in dom(I)$, there exists a ground term t such that $[t]^I = e$.

The satisfiability relation $\models$ relating interpretations in $\mathcal{I}$ and constraints in $\mathcal{C}$ is defined as usual, where $\doteq$ is interpreted as the identity, and $\bot$ and $\top$ are interpreted as false and true, respectively. We write $\phi \models_{\mathcal{I}} \psi$ if the implication $I \models \phi\sigma \implies I \models \psi\sigma$ holds for all $I \in \mathcal{I}$ and for all ground substitutions of domain $\mathcal{V}(\phi) \cup \mathcal{V}(\psi)$; and $\phi \equiv_{\mathcal{I}} \psi$ iff $\phi \models_{\mathcal{I}} \psi$ and $\psi \models_{\mathcal{I}} \phi$. For any set of constraints, we write $I \models S$ iff $I \models \phi$ for all $\phi \in S$. For any constraint (or set of constraints) ϕ, if there exists a ground substitution σ with domain $\mathcal{V}(\phi)$ and an interpretation $I \in \mathcal{I}$ such that $I \models \phi\sigma$, then ϕ is *$\mathcal{I}$-satisfiable* (and *$\mathcal{I}$-unsatisfiable* otherwise). For instance, the fixed set of first-order interpretations may be the set $\mathcal{I}_1$ of first-order interpretations on Σ that satisfy the above condition on the domain (this is not restrictive provided there are infinitely many ground terms), in which case $\mathcal{I}$-satisfiability is simply the standard satisfiability in first-order clausal logic, or the set $\mathcal{I}_{\mathbb{N}}$ of interpretations of domain $\mathbb{N}$ interpreting the functions $0, 1, +$ as usual. We say that $\mathcal{I}$ is *compact* if for every $\mathcal{I}$-unsatisfiable set of constraints S there exists a finite set $S' \subseteq S$ such that S' is $\mathcal{I}$-unsatisfiable. It is well-known that $\mathcal{I}_1$ is compact [22] and that $\mathcal{I}_{\mathbb{N}}$ is not compact[2].

Any ground $\mathcal{T}$-graph may be transformed into a $dom(I)$-graph by replacing the labels by their interpretations in I. More formally:

Definition 28. *For all $I \in \mathcal{I}$ and for all ground $\mathcal{T}$-graphs $\mathfrak{g}$ we denote by $[\mathfrak{g}]^I$ the graph such that $F_{[\mathfrak{g}]^I} = F_{\mathfrak{g}}$ for all $F \in \{N, E, R\}$ and $L_{[\mathfrak{g}]^I}(\alpha) = [L_{\mathfrak{g}}(\alpha)]^I$, for all $\alpha \in \widehat{N}_{\mathfrak{g}}$. For every ground $\mathcal{T}$-clause C, we denote by $[C]^I$ the clause obtained from C by replacing every $\mathcal{T}$-graph $\mathfrak{g}$ by $[\mathfrak{g}]^I$. For all sets of c-clauses Γ, we denote by $[\Gamma]^I$ the set of clauses of the form $[C\sigma]^I$, where $C \in \Gamma$ and σ is a substitution mapping every variable in C to a ground term.*

Note that by definition, all the labels of $[\mathfrak{g}]^I$ are elements of the domain of I. Proposition 29 follows immediately from Definition 28.

Proposition 29. *Let $\mathfrak{g}, \mathfrak{h}$ be $\mathcal{T}$-graphs, let $I \in \mathcal{I}$ and let σ be a ground substitution with domain $\mathcal{V}(\mathfrak{g}) \cup \mathcal{V}(\mathfrak{h})$. If $\mathfrak{g} \equiv \mathfrak{h}$ then $[\mathfrak{g}\sigma]^I \equiv [\mathfrak{h}\sigma]^I$.*

Definition 30. *An* $\mathcal{I}$-interpretation *is a pair $(I, \sim)$, where $I \in \mathcal{I}$ and $\sim$ is a congruence on $dom(I)$-graphs. An $\mathcal{I}$-interpretation* $(I, \sim)$ validates *a set of c-clauses Γ (written $(I, \sim) \models \Gamma$) if $\sim \models [\Gamma]^I$.*

5.2 Lifting the Calculus

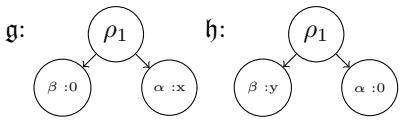
In the constrained calculus, the equality of labels will not be checked when an inference rule is applied. Instead, the corresponding conditions will be extracted

[2] For instance, the set $\{n \doteq i \mid i \in \mathbb{N}\}$ is unsatisfiable if n is interpreted as a natural number, but admits no finite unsatisfiable subset.

from the considered graphs and added to the constraints of the conclusion. We first introduce a relation stating that two $\mathcal{T}$-graphs are identical up to their labels. This relation is parameterized by a constraint that asserts conditions on the labels ensuring that the graphs are identical (modulo $\mathcal{I}$).

Definition 31. *Let $\mathfrak{g}, \mathfrak{h}$ be two $\mathcal{T}$-graphs and let $\phi \in \mathcal{C}$. We write $\mathfrak{g} =_\phi \mathfrak{h}$ if $N_\mathfrak{g} = N_\mathfrak{h}$, $E_\mathfrak{g} = E_\mathfrak{h}$, $R_\mathfrak{g} = R_\mathfrak{h}$, and $\phi = \bigwedge_{\alpha \in \widehat{N}_\mathfrak{g}} (L_\mathfrak{g}(\alpha) \doteq L_\mathfrak{h}(\alpha))$ (up to associativity and commutativity of $\wedge$).*

Example 32. Consider the $\mathcal{T}$-graphs $\mathfrak{g}$ and $\mathfrak{h}$ below, of root (ρ_1). We have $\mathfrak{g} =_\phi \mathfrak{h}$, with $\phi = (x \doteq 0 \wedge 0 \doteq y)$.



Every relation between $\mathcal{T}$-graphs or $\mathcal{T}$-clauses may be adapted in a similar way, keeping the conditions on the nodes, edges and roots, and asserting conditions ensuring that the label of every given node is unique (up to equality modulo $\mathcal{I}$). Definitions 33 and 34 lift the subgraph and subsumption relations, respectively:

Definition 33. *We write $\mathfrak{h} \leq^g_\phi \mathfrak{g}$ if $N_\mathfrak{h} \subseteq N_\mathfrak{g}$; $E_\mathfrak{h} \sqsubseteq E_\mathfrak{g}$; every node $\alpha \in N_\mathfrak{h}$ occurring in $R_\mathfrak{g}$ also occurs in $R_\mathfrak{h}$; if $\alpha \in N_\mathfrak{h}$ occurs in an edge in $E_\mathfrak{g} \setminus E_\mathfrak{h}$ then $\alpha \in R_\mathfrak{h}$, and $\phi = \bigwedge_{\alpha \in \widehat{N}_\mathfrak{h}} L_\mathfrak{h}(\alpha) \doteq L_\mathfrak{g}(\alpha)$. The notation $\mathfrak{g}\{\mathfrak{h} \leftarrow \mathfrak{i}\}$ may be extended to the case where $\mathfrak{h} \leq^g_\phi \mathfrak{g}$ (following Definition 7). Orthogonality is extended accordingly (as it does not depend on labels).*

Definition 34. *We write $C \leq^{sub}_\phi D$ if C and D are respectively of the form (up to associativity and commutativity of $\vee$ and isomorphism): $\bigvee_{i=1}^n \mathfrak{g}_i \bowtie_i \mathfrak{h}_i$, and $\bigvee_{i=1}^n \mathfrak{g}'_i \bowtie_i \mathfrak{h}'_i \vee D'$, with $\mathfrak{g}_i =_{\phi_i} \mathfrak{g}'_i$, $\mathfrak{h}_i =_{\psi_i} \mathfrak{h}'_i$ (for all $i = 1, \ldots, n$) and $\phi = \bigwedge_{i=1}^n (\phi_i \wedge \psi_i)$.*

The notion of a merge is extended analogously:

Definition 35. *A* ϕ-merge *of two $\mathcal{T}$-graphs $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is a $\mathcal{T}$-graph $\mathfrak{h}$ such that:*
- *$N_\mathfrak{h} = N_{\mathfrak{g}_1} \cup N_{\mathfrak{g}_2}$, $E_\mathfrak{h} = E_{\mathfrak{g}_1} \sqcup E_{\mathfrak{g}_2}$, and $\widehat{N}_\mathfrak{h} = \widehat{N}_{\mathfrak{g}_1} \cup \widehat{N}_{\mathfrak{g}_2}$.*
- *For every node $\alpha \in \widehat{N}_\mathfrak{h}$, we have $L_\mathfrak{h}(\alpha) = L_{\mathfrak{g}_i}(\alpha)$, for some (arbitrarily chosen) $i = 1, 2$ such that $L_{\mathfrak{g}_i}(\alpha)$ is defined.*
- *$\phi = \bigwedge_{\alpha \in \widehat{N}_{\mathfrak{g}_1} \cap \widehat{N}_{\mathfrak{g}_2}} L_{\mathfrak{g}_1}(\alpha) \doteq L_{\mathfrak{g}_2}(\alpha)$.*

We now lift the order relation. Let $\leq_I$ (for $I \in \mathcal{I}$) be a family of well-founded preorders on $dom(I)$-$\mathcal{T}$-graphs that are closed under isomorphisms and embeddings and contain $\leq^g$. Let $\leq_\phi$ (for $\phi \in \mathcal{C}$) be a family of pre-orders on $\mathcal{T}$-graphs satisfying the following conditions: $\mathfrak{g} >_\phi \mathfrak{h} \implies \mathfrak{g} >_\psi \mathfrak{h}$, for all constraints ϕ, ψ such that $\psi \models_\mathcal{I} \phi$, and $(I \models \phi \wedge \mathfrak{g} >_\phi \mathfrak{h}) \implies [\mathfrak{g}]^I >_I [\mathfrak{h}]^I$. The simplest solution in practice is to order $\mathcal{T}$-graphs according to their number of

nodes, in which case the order does not depend on I or ϕ: $\mathfrak{g} \leq_I \mathfrak{h} \iff \mathfrak{g} \leq_\phi \mathfrak{h} \iff \mathit{card}(N_\mathfrak{g}) \leq \mathit{card}(N_\mathfrak{h})$. However, our framework is meant to be general enough to cope with orders that take labels into account.

A literal L is *maximal* in a c-clause $[C \mid \phi]$ if there is no literal $L' \in C$ such that $L' >_\phi L$. It is *eligible* in a c-clause $[C \mid \phi]$ if L is a $>_\phi$-maximal literal in C.

We are now in the position to define the constrained inference rules. As for the rules in Section 4.1, they apply modulo isomorphism. We assume as for the standard resolution or superposition calculus that the premises share no variables. In every rule, the conclusion inherits the constraints of the premises together with additional conditions on the labels which makes the inference valid. In all rules, the eligibility condition is tested after adding all the constraints enabling the inference, as this yields the most restrictive condition, thus reducing the branching factor.

$$\mathtt{Sp}^+ : \frac{[\mathfrak{g}_1 \approx \mathfrak{h}_1 \vee C_1 \mid \phi_1] \quad [\mathfrak{g}_2 \approx \mathfrak{h}_2 \vee C_2 \mid \phi_2]}{[\mathfrak{i}\{\mathfrak{g}_1 \leftarrow \mathfrak{h}_1\} \approx \mathfrak{i}\{\mathfrak{g}_2 \leftarrow \mathfrak{h}_2\} \vee C_1 \vee C_2 \mid \phi_1 \wedge \phi_2 \wedge \psi]}$$

where:

1. $\mathfrak{i}$ is a ψ-merge of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ and $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are not orthogonal;
2. $\mathfrak{g}_i \approx \mathfrak{h}_i$ is eligible in $[\mathfrak{g}_i \approx \mathfrak{h}_i \vee C_i \mid \phi_1 \wedge \phi_2 \wedge \psi]$ (for all $i = 1, 2$);
3. $\mathfrak{g}_i \not<_{\phi_1 \wedge \phi_2 \wedge \psi} \mathfrak{h}_i$ (for all $i = 1, 2$).

$$\mathtt{Sp}^- : \frac{[\mathfrak{g} \approx \mathfrak{h} \vee C \mid \phi] \quad [\mathfrak{i} \not\approx \mathfrak{j} \vee D \mid \psi]}{[\mathfrak{i}\{\mathfrak{g} \leftarrow \mathfrak{h}\} \not\approx \mathfrak{j} \vee C \vee D \mid \phi \wedge \psi \wedge \xi]}$$

where:

1. $\mathfrak{g} \leq^g_\xi \mathfrak{i}$ (note that ξ is uniquely defined by Definition 33);
2. $\mathfrak{g} \approx \mathfrak{h}$ and $\mathfrak{i} \not\approx \mathfrak{j}$ are eligible in $[\mathfrak{g} \approx \mathfrak{h} \vee C \mid \phi \wedge \psi \wedge \xi]$ and $[\mathfrak{i} \not\approx \mathfrak{j} \vee D \mid \phi \wedge \psi \wedge \xi]$, respectively;
3. $\mathfrak{g} \not<_{\phi \wedge \psi \wedge \xi} \mathfrak{h}$ and $\mathfrak{i} \not<_{\phi \wedge \psi \wedge \xi} \mathfrak{j}$.

$$\mathtt{F} : \frac{[\mathfrak{g} \approx \mathfrak{h} \vee \mathfrak{g}' \approx \mathfrak{h}' \vee C \mid \phi]}{[\mathfrak{g} \approx \mathfrak{h} \vee C \mid \phi \wedge \psi \wedge \psi']}$$

where $\mathfrak{g} \approx \mathfrak{h}$ is eligible in $[\mathfrak{g} \approx \mathfrak{h} \vee \mathfrak{g}' \approx \mathfrak{h}' \vee C \mid \phi \wedge \psi \wedge \psi']$, $\mathfrak{g} =_\psi \mathfrak{g}'$, and $\mathfrak{h} =_{\psi'} \mathfrak{h}'$.

$$\mathtt{R} : \frac{[\mathfrak{g} \not\approx \mathfrak{h} \vee C \mid \phi]}{[C \mid \phi \wedge \psi]}$$

where $\mathfrak{g} \not\approx \mathfrak{h}$ is eligible in $[\mathfrak{g} \not\approx \mathfrak{h} \vee C \mid \phi \wedge \psi]$ and $\mathfrak{g} =_\psi \mathfrak{h}$.

5.3 Soundness and Refutational Completeness

We establish the soundness and completeness of the constrained calculus, by lifting the corresponding properties for the base calculus. Note that semi decidability holds only if the base theory is semi-decidable[3] and compact (otherwise it is easy to see that unsatisfiability is not semi-decidable in general).

Lemma 36. *The rules* $\mathtt{Sp}^+$, $\mathtt{Sp}^-$, $\mathtt{F}$ *and* $\mathtt{R}$ *(applied on c-clauses) are sound, i.e., for all* $\mathcal{I}$*-interpretations* $(I, \sim)$ *and for all c-clauses* $[C \mid \phi]$ *deducible for a set of premises* Γ*, we have* $(I, \sim) \models \Gamma \implies (I, \sim) \models [C \mid \phi]$.

The redundancy criterion may be lifted as follows:

Definition 37. *A c-clause* $[C \mid \phi]$ *is (strictly)* $\mathcal{I}$-redundant *in a set of c-clauses* Γ *if for all ground substitutions* σ *of domain* $\mathcal{V}(C) \cup \mathcal{V}(\phi)$ *and for all* $I \in \mathcal{I}$ *such that* $I \models \phi\sigma$*, the clause* $[C\sigma]^I$ *is (strictly) redundant in* $[\Gamma]^I$.

A set of c-clauses Γ *is (strictly)* saturated *if every c-clause that is deducible from* Γ *by the rules above is (strictly)* $\mathcal{I}$*-redundant in* Γ.

Theorem 38. *Let* Γ *be a set of c-clauses. If* Γ *is unsatisfiable and strictly saturated or Horn and saturated, then* Γ *contains a set of c-clauses* $\{[\Box \mid \phi_I] \mid I \in \mathcal{I}\}$ *such that for every* $I \in \mathcal{I}$, $I \models \exists \boldsymbol{x}_I.\phi_i$*, with* $\boldsymbol{x}_I = \mathcal{V}(\phi_I)$*. If, moreover,* $\mathcal{I}$ *is compact, then* Γ *contains a finite set of c-clauses* $\{[\Box \mid \phi_i] \mid i = 1, \ldots, n\}$ *such that* $\bigwedge_{i=1}^{n} \neg(\exists \boldsymbol{x}.\phi_i)$ *is* $\mathcal{I}$*-unsatisfiable, with* $\boldsymbol{x}_i = \mathcal{V}(\phi_i)$.

5.4 Redundancy Testing

The redundancy criterion in Definition 37 is very general, but it may be difficult to test in practice. We thus introduce a second notion of redundancy, defined directly on constrained clauses, that is stronger and easier to decide.

Definition 39. *Let* $[C \mid \phi], [D \mid \psi]$ *be two clauses and let* Γ *be a set of clauses. Let* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *be the vectors of variables occurring in* $[C \mid \phi]$ *and* $[D \mid \psi]$*, respectively (we assume by renaming that* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *share no variable).*

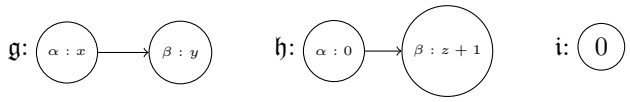
We say that $[C \mid \phi]$ *is* subsumed *by* $[D \mid \psi]$ *and we write* $[C \mid \phi] \geq^{sub} [D \mid \psi]$ *if there exists* $\xi \in \mathcal{C}$ *such that* $D \leq_\xi^{sub} C$ *and* $\phi \models_{\mathcal{I}} \exists \boldsymbol{y}.(\psi \wedge \xi)$.

We write $[C \mid \phi] \to_\Gamma [D \mid \psi]$ *(*$[C \mid \phi]$ demodulates *to* $[D \mid \psi]$ *w.r.t.* Γ*) if* C *is of the form* $\mathfrak{g} \bowtie \mathfrak{h} \vee E$, $D = \mathfrak{g}\{\mathfrak{i} \leftarrow \mathfrak{j}\} \bowtie \mathfrak{h} \vee E$*, and there exists a c-clause* $[F \mid \xi] \in \Gamma$ *(with free variables* $\boldsymbol{z}$*) such that* $F = (\mathfrak{i} \approx \mathfrak{j}) \vee F'$, $\mathfrak{i} \leq_{\xi'}^{g} \mathfrak{g}$, $F' \leq_{\xi''}^{sub} E$, $\phi \models_{\mathcal{I}} \exists \boldsymbol{y}.\exists \boldsymbol{z}.(\psi \wedge \xi \wedge \xi' \wedge \xi'')$, $\mathfrak{i} >_\xi \mathfrak{j}$, $F' <_\xi (\mathfrak{i} \approx \mathfrak{j})$ *and* $(\mathfrak{i} \approx \mathfrak{j}) <_\xi (\mathfrak{g} \bowtie \mathfrak{h})$.

A c-clause $[C \mid \phi]$ *is* redundant *w.r.t.* Γ *iff one of the following conditions holds: (1)* $\exists \boldsymbol{x}.\phi$ *is* $\mathcal{I}$*-unsatisfiable, with* $\boldsymbol{x} = \mathcal{V}(\phi)$*. (2)* C *contains two literals* $\mathfrak{g}_1 \approx \mathfrak{g}_2$ *and* $\mathfrak{g}_1' \not\approx \mathfrak{g}_2'$*, with* $\mathfrak{g}_i =_{\phi_i} \mathfrak{g}_i'$*, and* $\phi \models_{\mathcal{I}} \phi_i$ *(for all* $i = 1, 2$*); (3)* C *contains a literal of the form* $\mathfrak{g} \approx \mathfrak{h}$ *with* $\mathfrak{g} =_\psi \mathfrak{h}$ *and* $\phi \models_{\mathcal{I}} \psi$*; (4)* $[C \mid \phi] \geq^{sub} [D \mid \psi]$*, for some* $[D \mid \psi] \in \Gamma$*; (5)* $[C \mid \phi] \to_\Gamma [D \mid \psi]$ *and* $[D \mid \psi]$ *is redundant.*
The notion of strictly redundant c-clause is defined in a similar way, removing Item 2.

[3] in the sense that there exists a semi-decision procedure to check whether a formula in $\mathcal{C}$ is unsatisfiable.

Example 40. Consider the following $\mathcal{T}$-graphs, of root ():



We have $\mathfrak{g} \approx \mathfrak{i} \leq_{\phi}^{sub} \mathfrak{h} \approx \mathfrak{i}$, with $\phi = (x \doteq 0 \wedge y \doteq z + 1 \wedge 0 \doteq 0)$. Thus, if $\mathcal{I}$ only contains the standard model of Presburger arithmetic, then $[\mathfrak{g} \approx \mathfrak{i} \mid y \not\approx 0]$ subsumes $[\mathfrak{h} \approx \mathfrak{i} \mid \top]$.

The following lemma states the relation between the new notion of redundancy and $\mathcal{I}$-redundancy (as defined in Definition 37).

Lemma 41. *Let Γ be a set of c-clauses. If $[C \mid \phi]$ is (strictly) redundant w.r.t. Γ then it is (strictly) $\mathcal{I}$-redundant w.r.t. Γ.*

Remark 42. By the previous definitions, checking whether a given c-clause is (strictly) redundant involves testing the validity of entailments of the form $\phi \models_{\mathcal{I}} \exists \boldsymbol{y}.\psi$, which may be infeasible in practice (for instance the problem is undecidable if $\mathcal{I}$ contains all interpretations). Stronger conditions may be used instead, e.g., one may check whether there exists a substitution σ such that ϕ is of the form $\psi\sigma \wedge \psi$, which is decidable.

6 Conclusion

We devised a constrained superposition calculus to test the satisfiability of sets of clauses defined over graphs. Its soundness and refutational completeness was established, modulo a redundancy criterion that captures the usual deletion and simplification rules: subsumption, demodulation, deletion of clauses with trivial equations and – in the case of Horn clauses only – deletion of clauses containing complementary literals. The considered structures are rooted directed labeled graphs, which are general enough to capture most existing equational graph theories, such as those developed for quantum circuits. In contrast to [15], the calculus is able to handle disjunctions as well as interpreted labels, and in contrast to [23], our solution avoids any encoding of graphs into terms, by defining inference rules operating directly on graphs.

From a practical point of view, it would be interesting to get more general redundancy criteria, to reduce the branching factor and improve the efficiency of the procedure. In particular, is it possible to define a version of the calculus in which tautology deletion is allowed, even for non Horn clauses? As evidenced by Example 23, this would require to define a new equational factorization rule, allowing for non trivial superposition inferences within a single clause.

Another interesting issue is to add variables denoting not only labels, but also graphs. This would allow for instance to synthesize graphs satisfying some properties. As graphs can be viewed as functions with multiple inputs and outputs (denoted by the roots) such an addition would yield a second order logic.

Finally, it would be interesting to identify fragments for which the calculus terminates, ensuring decidability of the satisfiability problem. In contrast to terms, the calculus does not terminate (and the satisfiability problem is undecidable) for ground unit clauses [15], hence strong restrictions on the shape of the graphs are required to ensure termination.

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A Some Results On Sets and Multisets

We state two useful propositions about sets and multisets.

Proposition 43. *Let S, S_1, S_2, T_1, T_2 be sets such that $S_1 \cup S_2 \subseteq S$, $T_1 \cap S_2 \subseteq T_2$ and $T_2 \cap S_1 \subseteq T_1$. Then $((S \setminus S_1) \cup T_1) \setminus S_2) \cup T_2 = ((S \setminus S_2) \cup T_2) \setminus S_1) \cup T_1$.*

Proof. By symmetry, it is sufficient to prove that $((S \setminus S_1) \cup T_1) \setminus S_2) \cup T_2 \subseteq ((S \setminus S_2) \cup T_2) \setminus S_1) \cup T_1$. Let $e \in ((S \setminus S_1) \cup T_1) \setminus S_2) \cup T_2$. We distinguish several cases.

- If $e \in S$ and $e \notin S_1 \cup S_2$, then by definition $e \in (S \setminus S_2)$, thus $e \in ((S \setminus S_2) \cup T_2)$ and $e \in ((S \setminus S_2) \cup T_2) \setminus S_1) \cup T_1$.
- If $e \in T_1$ and $e \notin S_2$ then it is clear that $e \in ((S \setminus S_2) \cup T_2) \setminus S_1) \cup T_1$.
- Now assume that $e \in T_2$. If $e \notin S_1$ then $e \in ((S \setminus S_2) \cup T_2) \setminus S_1) \cup T_1$. Otherwise, we have $e \in T_2 \cap S_1$, thus by hypothesis $e \in T_1$ and we deduce that $e \in ((S \setminus S_2) \cup T_2) \setminus S_1) \cup T_1$.

Proposition 44. *Let M, M_1, N_1, M_2, N_2 be multisets, with $M_1 + M_2 \sqsubseteq M$. Then:*

$$((M - M_1) + N_1) - M_2) + N_2 = ((M - M_2) + N_2) - M_1) + N_1$$

Proof. Let e be an element, and let $i \in \{1, 2\}$ and $j \stackrel{def}{=} 3 - i$. First assume that $e \notin M$. Then $(M - M_i)(e) = M_j(e) = 0$, thus $((M - M_i) + N_i) - M_j)(e) = N_i(e)$ and $((M - M_i) + N_i) - M_j) + N_j)(e) = N_i(e) + N_j(e)$.

Now assume that $e \in M$. Then $(M - M_i)(e) = M(e) - M_i(e)$ because $M_i \sqsubseteq M$, and $((M - M_i) + N_i) = M(e) + N_i(e) - M_i(e)$. Since $M_1 + M_2 \sqsubseteq M$, we have $M(e) - M_i(e) \geq M_j(e)$, and therefore $((M - M_i) + N_i) - M_j)(e) = M(e) + N_i(e) - M_i(e) - M_j(e)$. Therefore $((M - M_i) + N_i) - M_j) + N_j)(e) = M(e) + N_i(e) + N_j(e) - M_i(e) - M_j(e)$.

In both cases, we have $(((M - M_1) + N_1) - M_2) + N_2)(e) = (((M - M_2) + N_2) - M_1) + N_1)(e)$.

B Some Results on Graph Transformations

Proposition 45. *Let $\mathfrak{g}$ be an $\mathcal{L}$-graph and let $\mathfrak{h}$ be a subgraph of $\mathfrak{g}$. If $\mathfrak{i}$ is substitutable for $\mathfrak{h}$ in $\mathfrak{g}$ then $\mathfrak{g}\{\mathfrak{h} \leftarrow \mathfrak{i}\}$ is an $\mathcal{L}$-graph.*

Proof. Let $\mathfrak{g}' = \mathfrak{g}\{\mathfrak{h} \leftarrow \mathfrak{i}\}$. We have to check that the nodes occurring in edges and as roots of $\mathfrak{g}'$ are all in $N_{\mathfrak{g}'}$, that $dom(L_{\mathfrak{g}'}) = \widehat{N}_{\mathfrak{g}'}$ and that $L_{\mathfrak{g}'}(\widehat{N}_{\mathfrak{g}'}) \subseteq \mathcal{L}$. Consider a node α occurring in an edge in $E_{\mathfrak{g}'}$. Then either α occurs in an edge from $E_{\mathfrak{i}}$, in which case $\alpha \in N_{\mathfrak{i}} \subseteq N_{\mathfrak{g}'}$, or α occurs in an edge from $E_{\mathfrak{g}} - E_{\mathfrak{h}}$. In the latter case, if $\alpha \notin N_{\mathfrak{h}}$ then $\alpha \in N_{\mathfrak{g}} \setminus N_{\mathfrak{h}} \subseteq N_{\mathfrak{g}'}$. Otherwise, since $\mathfrak{h} \leq^g \mathfrak{g}$, by Definition 5 we must have $\alpha \in R_{\mathfrak{h}} = R_{\mathfrak{i}}$, so that $\alpha \in N_{\mathfrak{i}} \subseteq N_{\mathfrak{g}'}$. Now consider a node $\alpha \in R_{\mathfrak{g}'} = R_{\mathfrak{g}}$. Then either $\alpha \notin N_{\mathfrak{h}}$, in which case $\alpha \in N_{\mathfrak{g}'}$, or (because $\widehat{N}_{\mathfrak{h}} \subseteq \widehat{N}_{\mathfrak{g}}$) $\alpha \in R_{\mathfrak{h}} = R_{\mathfrak{i}} \subseteq N_{\mathfrak{g}'}$.

Let $\alpha \in dom(L_{\mathfrak{g}'})$, we distinguish two cases. If $\alpha \in \widehat{N}_{\mathfrak{i}}$, then $L_{\mathfrak{g}'}(\alpha) \in \mathcal{L}$ and $\alpha \notin R_{\mathfrak{i}} = R_{\mathfrak{h}}$, which entails by Definition 5 that $\alpha \notin R_{\mathfrak{g}} = R_{\mathfrak{g}'}$, thus $\alpha \in \widehat{N}_{\mathfrak{g}'}$. Otherwise, we must have $\alpha \in \widehat{N}_{\mathfrak{g}}$, so that $L_{\mathfrak{g}'}(\alpha) \in \mathcal{L}$ and $\alpha \notin R_{\mathfrak{g}} = R_{\mathfrak{g}'}$.

The following proposition states that subgraph replacement is compatible with the isomorphism relation.

Proposition 46. *Let $\mathfrak{g}$ be an $\mathcal{L}$-graph and let $\mathfrak{h}$ be a subgraph of $\mathfrak{g}$. If $\mathfrak{i}$ and $\mathfrak{i}'$ are both substitutable for $\mathfrak{h}$ in $\mathfrak{g}$ and $\mathfrak{i} \equiv \mathfrak{i}'$ then $\mathfrak{g}\{\mathfrak{h} \leftarrow \mathfrak{i}\} \equiv \mathfrak{g}\{\mathfrak{h} \leftarrow \mathfrak{i}'\}$.*

Proof. Let μ' be an $\mathcal{N}$-renaming such that $\mu'(\mathfrak{i}) = \mathfrak{i}'$. Then we have $\mu'(R_{\mathfrak{i}}) = R_{\mathfrak{i}'} = R_{\mathfrak{h}} = R_{\mathfrak{i}}$. Since $N_{\mathfrak{g}} \cap (N_{\mathfrak{i}} \cup N_{\mathfrak{i}'}) = \emptyset$, there exists an $\mathcal{N}$-renaming μ such that $\mu(\alpha) = \mu'(\alpha)$ for all $\alpha \in \widehat{N}_{\mathfrak{i}}$ and $\mu(\alpha) = \alpha$ for all $\alpha \in N_{\mathfrak{g}}$. Then $\mu(\mathfrak{g}) = \mathfrak{g}$ and $\mu(\mathfrak{i}) = \mathfrak{i}'$, and it is straightforward to verify that $\mu(\mathfrak{g}\{\mathfrak{h} \leftarrow \mathfrak{i}\}) = \mu(\mathfrak{g}\{\mathfrak{h} \leftarrow \mathfrak{i}'\})$.

C On the Completeness of the Ground Calculus

C.1 A Confluence Criterion for Graph Rewriting

In this section, we introduce some notions that will be used to represent models of satisfiable sets. If $\rightarrow$ is a binary relation, then we denote by $\rightarrow^*$ its reflexive and transitive closure, and we write $x \rightarrow^{0|1} y$ if either $x = y$ or $x \rightarrow y$. A relation is *confluent* (resp. *locally confluent*) if for all triples (x, y, z) such that $x \rightarrow^* y$ and $x \rightarrow^* z$, (resp. $x \rightarrow y$ and $x \rightarrow z$) there exists u such that $y \rightarrow^* u$ and $z \rightarrow^* u$. It is *subcommutative* if for all triples (x, y, z) such that $x \rightarrow y$ and $x \rightarrow z$, there exists u such that $y \rightarrow^{0|1} u$ and $z \rightarrow^{0|1} u$. It is well-known that every well-founded binary relation that is locally confluent is confluent and that every (possibly non well-founded) binary relation that is subcommutative is confluent (see [1, Lemma 2.7.4]). Building on these results, we now devise a new confluence criterion.

Definition 47. *A relation $\rightarrow$ is* $\leq$-subcommutative *if $\rightarrow \subseteq \geq$ and for all triples (x, y_1, y_2) such that $x \rightarrow y_1$ and $x \rightarrow y_2$, there exists u such that for every $i = 1, 2$, either $y_i \rightarrow^{0|1} u$ or there exists $v < x$ such that $y_i \rightarrow v$ and $v \rightarrow^* u$.*

Lemma 48. *Let $\leq$ be a total preorder and let $<$ be the associated strict order. If $\rightarrow$ is $\leq$-subcommutative and $<$ is well-founded then $\rightarrow$ is confluent.*

Proof. Assume that $\rightarrow$ is not confluent, and let x be a minimal (w.r.t. $\leq$) element such that $x \rightarrow^* y_i$ holds for $i = 1, 2$, and for every u, at least one $y_i \rightarrow^* u$ does not hold. Let $\hookrightarrow$ be the relation defined as follows: $x' \hookrightarrow y \iff x' \leq x \wedge (x' \rightarrow y \vee \exists z.(z < x \wedge x' \rightarrow z \wedge z \rightarrow^* y))$. We have $x \hookrightarrow^* y_i$ for $i = 1, 2$, since $x \rightarrow^* y_i$, and $\rightarrow \subseteq \geq$. By construction $\hookrightarrow \subseteq \rightarrow^*$, hence $\hookrightarrow^* \subseteq \rightarrow^*$ and there is no u such that $y_i \hookrightarrow^* u$ holds for $i = 1, 2$. Consequently, $\hookrightarrow$ is not confluent. We prove that $\hookrightarrow$ is subcommutative, which yields a contradiction. Consider x', y_i' such that $x' \hookrightarrow y_i'$, for all $i = 1, 2$, we prove that there exists u such that $y_i' \hookrightarrow^{0|1} u$. By definition of $\hookrightarrow$, we have $x' \leq x$ and for $i = 1, 2$ either $x' \rightarrow y_i'$ or there exists

$x_i < x$ such that $x' \to x_i \to^* y'_i$. If $x' < x$, then by minimality of x, there exists u such that $y'_i \to^* u$, for all $i = 1, 2$. Since $\to \subseteq \geq$ and $x > x'$, we deduce that $y'_i \leq x' < x$, so that all the elements in the derivation from y'_i to u are strictly smaller than x, and therefore $y'_i \hookrightarrow^{0|1} u$.

We now assume that $x' \not< x$, i.e., that $x' \geq x$, which entails that x and x' are in the same equivalence class of the relation induced by the preorder $\leq$. We distinguish several cases.

- If $x' \to y_i$ holds for all $i = 1, 2$, then, since $\to$ is $\leq$-subcommutative by hypothesis, there exists u such that for every $i = 1, 2$, either $y'_i \to^{0|1} u$, or there exists $v_i < x'$ such that $y'_i \to v_i$ and $v_i \to^* u$. Since $y'_i \leq x' \leq x$, this entails that for every $i = 1, 2$ $y'_i \hookrightarrow^{0|1} u$.
- Assume that $x' \to y'_1$ and $x' \to x_2 \to^* y'_2$, where $x_2 < x$. Since $\to$ is $\leq$-subcommutative, we deduce as in the previous case that there exists u' such that $y'_1 \hookrightarrow^{0|1} u'$ and $x_2 \hookrightarrow^{0|1} u'$. This entails that $x_2 \to^* u'$ and $x_2 \to^* y'_2$, and by minimality of x, since $x_2 < x$, we deduce that there exists u'' such that $u' \to^* u''$ and $y'_2 \to^* u''$. We have $y'_1 \hookrightarrow u' \to^* u''$ and $y'_2 \to^* u''$. Since $\to \subseteq \geq$, all the elements in the derivation from y'_2 to u'' are smaller than y'_2, hence strictly smaller than $x_2 < x$, thus we get $y'_2 \hookrightarrow^{0|1} u''$. Now consider the derivation from y'_1 to u''. If $y'_1 = u''$ then $y'_1 \hookrightarrow^{0|1} u''$. If $y'_1 \to z_1 \to^* u'$, for some $z_1 < x$, then we have $y'_1 \hookrightarrow u''$ and the proof is completed. Otherwise, by definition of $\hookrightarrow$, we must have $y'_1 \to u' \to^* u''$, and since $\to \subseteq \geq$ we have $u' \leq x_2 < x$. Thus $y'_1 \hookrightarrow u''$ also holds in this case.
- The case where $x \to y'_2$ and $x \to x_1 \to^* y'_1$ is symmetric.
- Assume that $x' \to x_i \to^* y'_i$ holds for all $i = 1, 2$, with $x_i < x$. Since $\to$ is $\leq$-subcommutative, we deduce as in the previous cases that there exists u' such that $x_i \hookrightarrow^{0|1} u'$ for all $i = 1, 2$, which entails that $x_i \to^* u'$. By minimality of x, we deduce that there exist u_i (for all $i = 1, 2$) such that $u' \to^* u_i$ and $y'_i \to^* u_i$. Since $\to \subseteq \geq$ we have $u' \leq x_i < x$, and by using again the minimality of x, we deduce that there exists u such that $u_i \to^* u$ (for $i = 1, 2$), so that $y'_i \to^* u$. Since $y'_i \leq x_i < x$, this entails that $y'_i \hookrightarrow^{0|1} u$.

Definition 49. *For every set of literals S, we denote by $\to_S$ the relation defined as follows: $\mathfrak{g} \to_S \mathfrak{h}$ iff S contains an equation $\mathfrak{i} \approx \mathfrak{j}$ (modulo isomorphism) such that $\mathfrak{i} \geq \mathfrak{j}$ and $\mathfrak{h} = \mathfrak{g}\{\mathfrak{i} \leftarrow \mathfrak{j}\}$. We write $\mathfrak{g} \downarrow_S \mathfrak{h}$ if there exists an $\mathcal{L}$-graph $\mathfrak{i}$ such that $\mathfrak{g} \to_S^* \mathfrak{i}$ and $\mathfrak{h} \to_S^* \mathfrak{i}$. If C and D are clauses, we write $C \to_S D$ if $C = (\mathfrak{g} \bowtie \mathfrak{h}) \vee E$ (with $\bowtie \in \{\approx, \not\approx\}$), $D = (\mathfrak{g}' \bowtie \mathfrak{h}) \vee E$, and $\mathfrak{g} \to_S \mathfrak{g}'$.*

Note that $\to_S$ is not well-founded.

Proposition 50. *For every set of literals S, $\to_S$ is closed under isomorphisms and embeddings.*

Proof. Immediate.

With a slight abuse of notations, we shall consider $\to_S$ as a relation between $\equiv$-equivalence classes of $\mathcal{L}$-graphs (e.g., we assume that $\mathfrak{g} \to_S^* \mathfrak{h}$ if $\mathfrak{g} \equiv \mathfrak{h}$). We now introduce two useful restrictions of the relation $\to_S$.

Definition 51. *We write* $\mathfrak{h} \to_{S|\mathfrak{g}} \mathfrak{i}$ *if* $\mathfrak{h} \to_S \mathfrak{i}$, $\mathfrak{h} \leq \mathfrak{g}$ *and* $\mathfrak{i} \leq \mathfrak{g}$*; and* $\mathfrak{h} \to_{S|<\mathfrak{g}} \mathfrak{i}$ *if* $\mathfrak{h} \to_S \mathfrak{i}$ *and* $\mathfrak{g} > \mathfrak{i}$.

Intuitively, $\to_{S|\mathfrak{g}}$ is the restriction of $\to_S$ to graphs that are smaller or equal to $\mathfrak{g}$ (w.r.t. $\leq$), while $\to_{S|<\mathfrak{g}}$ considers only reductions yielding a graph that is strictly smaller than $\mathfrak{g}$.

C.2 Model Construction

To establish refutational completeness, we show that a model can be constructed for every (strictly) saturated set of clauses. We shall consider either strictly saturated sets of arbitrary clauses, or saturated sets of Horn clauses, the difference being that in the latter case, clauses containing two complementary literals can be dismissed as redundant. For the sake of conciseness, the two cases will be handled simultaneously, as both constructions follow the same pattern and only differ at some key points. We consider a pre-order $\preceq$ on $\mathcal{L}$-graphs satisfying the following conditions:

1. $\preceq$ is total;
2. the associated strict order $\prec$ is well-founded;
3. $\leq \subseteq \preceq$;
4. $\mathfrak{g} \preceq \mathfrak{h} \wedge \mathfrak{h} \preceq \mathfrak{g} \implies \mathfrak{g} \equiv \mathfrak{h}$.

Note that $\prec$ is not required to be closed under embeddings, since by Example 20 no such order would possibly exist. It is easy to check that $\prec$ exists: it suffices for example to extend the relation $<$ by ordering the $\mathcal{L}$-graphs occurring in the same equivalence classes of $\simeq$ arbitrarily. Since these equivalence classes contain finitely many elements and $<$ is well-founded by hypothesis, $\prec$ is well-founded.

We use sets of equations to represent interpretations. The satisfiability relation is defined in the following way (note it does not in general satisfy the law of excluded middle, i.e., we may have $S \not\models^\Gamma \mathfrak{g} \approx \mathfrak{h}$ and $S \not\models^\Gamma \mathfrak{g} \not\approx \mathfrak{h}$, if the considered set of clauses is not Horn):

Definition 52. *Let S be a set of equations and let Γ be a set of clauses. We write $S \models^\Gamma E$ if and only if one of the following conditions holds:*

1. *E is a negative literal $\mathfrak{g} \not\approx \mathfrak{h}$ and $\mathfrak{g} \not\downarrow_S \mathfrak{h}$.*
2. *E is a positive literal $\mathfrak{g} \approx \mathfrak{h}$, Γ is Horn and $\mathfrak{g} \downarrow_S \mathfrak{h}$.*
3. *E is a positive literal $\mathfrak{g} \approx \mathfrak{h}$, and one of the following holds: (i) $\mathfrak{g} \equiv \mathfrak{h}$; (ii) $\mathfrak{g} \approx \mathfrak{h} \in S$; (iii) S contains an equation $\mathfrak{i} \approx \mathfrak{j}$ such that $(\mathfrak{i} \approx \mathfrak{j}) < (\mathfrak{g} \approx \mathfrak{h})$, $\mathfrak{j} < \mathfrak{i}$, $\mathfrak{i} \leq^g \mathfrak{g}$ and $S \models^\Gamma \mathfrak{g}\{\mathfrak{i} \leftarrow \mathfrak{j}\} \approx \mathfrak{h}$.*
4. *E is a clause and contains a literal L such that $S \models^\Gamma L$.*
5. *E is set of clauses and for all $C \in E$, $S \models^\Gamma C$.*

We associate every set of clauses Γ with a set of equations $\mathcal{E}_\Gamma$ as follows:

Definition 53. *Let Γ be a set of clauses. For every equation $\mathfrak{g} \approx \mathfrak{h}$, we have $\mathfrak{g} \approx \mathfrak{h} \in \mathcal{E}_\Gamma$ iff Γ contains a clause $C = \mathfrak{g} \approx \mathfrak{h} \vee D$ with:*

– $\mathfrak{g} \approx \mathfrak{h} \succ D$;
– $\mathcal{E}_\Gamma \not\models^\Gamma D$.

Observe that $\mathcal{E}_\Gamma$ is well-defined, since the condition $\mathcal{E}_\Gamma \not\models^\Gamma D$ only depends on the equations in $\mathcal{E}_\Gamma$ that are strictly $\prec$-smaller than $\mathfrak{g} \approx \mathfrak{h}$. Indeed, for all literals $\mathfrak{i} \bowtie \mathfrak{j}$ occurring in D, two cases may occur:

– $\bowtie$ is $\approx$, and in this case Γ cannot be Horn, hence by definition of $\models^\Gamma$ the condition $\mathcal{E}_\Gamma \not\models^\Gamma \mathfrak{i} \approx \mathfrak{j}$ depends only on the equations in $\mathcal{E}_\Gamma$ that are $\prec$-smaller than $\mathfrak{i} \approx \mathfrak{j} \prec \mathfrak{g} \approx \mathfrak{h}$.
– $\bowtie$ is $\not\approx$ and $(\mathfrak{i} \not\approx \mathfrak{j}) \not\succ (\mathfrak{g} \approx \mathfrak{h})$ (otherwise $\mathfrak{g} \approx \mathfrak{h} \not\succ D$). By definition of the function $mset()$ and of the order $<$ on literals, this entails that the $\mathcal{L}$-graphs $\mathfrak{i}, \mathfrak{j}$ are both $<$-smaller than $\mathfrak{g}$ and $\mathfrak{h}$, so that all the $\mathcal{L}$-graphs occurring in any derivation $\mathfrak{i} \to^*_{\mathcal{E}_\Gamma} \mathfrak{k}$ or $\mathfrak{j} \to^*_{\mathcal{E}_\Gamma} \mathfrak{k}$ are also $<$-smaller than $\mathfrak{g}$ and $\mathfrak{h}$. Consequently, the condition $\mathcal{E}_\Gamma \not\models^\Gamma \mathfrak{i} \approx \mathfrak{j}$, that is equivalent to $\mathfrak{i} \not\downarrow_{\mathcal{E}_\Gamma} \mathfrak{j}$, depends only on equations in $\mathcal{E}_\Gamma$ that are $\prec$-smaller than $\mathfrak{g} \approx \mathfrak{h}$.

Definition 54. *For every set of clauses Γ, we denote by $\Omega(\Gamma)$ the set of $\mathcal{L}$-graphs $\mathfrak{g}$ such that for all $\mathcal{L}$-graphs $\mathfrak{h} < \mathfrak{g}$, $\to_{\mathcal{E}_\Gamma|\mathfrak{h}}$ is confluent.*

Definition 55. *For every $\mathcal{L}$-graph $\mathfrak{g}$ and for every set of clauses C, we write $\mathfrak{g} \gg C$ iff for every L in C, $L \not\succ (\mathfrak{g} \approx \mathfrak{g})$. For every set of clauses Γ, we denote by $\Gamma_\mathfrak{g}$ the set of clauses $C \in \Gamma$ such that $\mathfrak{g} \gg C$.*

Note that by definition of the order on literals, if $\mathfrak{g} \gg C$ then the negative (resp. positive) literals in C contain no $\mathcal{L}$-graph $\mathfrak{h}$ such that $\mathfrak{g} \leq \mathfrak{h}$ (resp. $\mathfrak{g} < \mathfrak{h}$).

Proposition 56. *If $S \models^\Gamma \mathfrak{h}_1 \approx \mathfrak{h}_2$, $\mathfrak{g} \gg (\mathfrak{h}_1 \approx \mathfrak{h}_2)$ and S is not Horn, then there exist $\mathcal{L}$-graphs $\mathfrak{i}_1, \mathfrak{i}_2$ such that $\mathfrak{h}_i \to^*_{S|<\mathfrak{g}} \mathfrak{i}_i$ for $i = 1, 2$ and either $\mathfrak{i}_1 \equiv \mathfrak{i}_2$ or $\mathfrak{i}_1 \approx \mathfrak{i}_2 \in S$.*

Proof. By a straightforward induction on the relation $\models^\Gamma$, using Condition 3 in Definition 52.

Lemma 57. *Let $\trianglelefteq$ be any ordering on literals such that $< \subseteq \trianglelefteq$, extended to clauses using the multiset extension. Let Γ be a set of clauses, let $\mathfrak{g} \in \Omega(\Gamma)$ and let C be a clause such that $\mathfrak{g} \gg C$ and $\mathcal{E}_\Gamma \not\models^\Gamma C$. If C is strictly redundant w.r.t. Γ then Γ contains a clause $E \trianglelefteq C$ such that $\mathcal{E}_\Gamma \not\models^\Gamma E$. Moreover, if Γ is Horn, the same property holds if C is redundant.*

Proof. We establish the two results simultaneously, by induction on the set of strictly redundant (resp. redundant) clauses. We distinguish several cases, following Definition 24. The first item is specific to the case of redundant and Horn clauses, the other items are shared.

1. Assume that C contains two literals $\mathfrak{g}_1 \approx \mathfrak{g}_2$ and $\mathfrak{g}'_1 \not\approx \mathfrak{g}'_2$, with $\mathfrak{g}_i \equiv \mathfrak{g}'_i$ for $i = 1, 2$. Since $\mathcal{E}_\Gamma \not\models^\Gamma C$, we must have $\mathcal{E}_\Gamma \not\models^\Gamma \mathfrak{g}_1 \approx \mathfrak{g}_2$ and $\mathcal{E}_\Gamma \not\models^\Gamma \mathfrak{g}'_1 \not\approx \mathfrak{g}'_2$. The latter statement is equivalent to $\mathfrak{g}'_1 \downarrow_{\mathcal{E}_\Gamma} \mathfrak{g}'_2$. This case is specific to the case where C is redundant, hence by the hypothesis of the lemma, S must be Horn, thus by definition of $\models^\Gamma$, the former statement entails that $\mathfrak{g}_1 \not\downarrow_{\mathcal{E}_\Gamma} \mathfrak{g}_2$. Since $\to_{\mathcal{E}_\Gamma}$ is closed under isomorphisms (Proposition 50), we deduce that $\mathfrak{g}'_1 \not\downarrow_{\mathcal{E}_\Gamma} \mathfrak{g}'_2$ a contradiction.

2. Assume C contains a literal of the form $\mathfrak{g}' \approx \mathfrak{h}$ with $\mathfrak{g}' \equiv \mathfrak{h}$. Then $\mathcal{E}_\Gamma \models^\Gamma \mathfrak{g}' \approx \mathfrak{h}$ by Definition 52, thus $\mathcal{E}_\Gamma \models^\Gamma C$, which contradicts the hypothesis of the lemma.
3. Assume $C \geq^{sub} D$, for some $D \in \Gamma$. We have $\mathcal{E}_\Gamma \not\models^\Gamma C$, hence, by definition of $\models^\Gamma$, necessarily $\mathcal{E}_\Gamma \not\models^\Gamma D$. By the induction hypothesis, Γ contains a clause E such that $E \trianglelefteq D$ and $\mathcal{E}_\Gamma \not\models^\Gamma E$. Since $C \geq^{sub} D$ necessarily $C \trianglerighteq D$, hence $E \trianglelefteq C$ and the proof is completed.
4. Assume $C \rightarrow_\Gamma D$ and D is strictly redundant (resp. redundant). Since $C \rightarrow_\Gamma D$, C and D are respectively of the form $(\mathfrak{g}' \bowtie \mathfrak{h}) \vee C'$ and $(\mathfrak{g}'\{\mathfrak{i} \leftarrow \mathfrak{j}\} \bowtie \mathfrak{h}) \vee C'$, where $\bowtie \in \{\approx, \not\approx\}$, $\mathfrak{i} > \mathfrak{j}$ and $(\mathfrak{g}' \bowtie \mathfrak{h}) > (\mathfrak{i} \approx \mathfrak{j})$. Furthermore, Γ contains a clause $F = (\mathfrak{i} \approx \mathfrak{j}) \vee F'$, with $F' < (\mathfrak{i} \approx \mathfrak{j})$ and $F' \leq^{sub} C'$. Since $\mathcal{E}_\Gamma \not\models^\Gamma C$ we deduce that $\mathcal{E}_\Gamma \not\models^\Gamma C'$, thus $\mathcal{E}_\Gamma \not\models^\Gamma F'$, so that $(\mathfrak{i} \approx \mathfrak{j}) \in \mathcal{E}_\Gamma$ (because $F' < \mathfrak{i} \approx \mathfrak{j}$). Since $\geq$ is closed under embeddings and $\mathfrak{i} > \mathfrak{j}$, we have $\mathfrak{g}' > \mathfrak{g}'\{\mathfrak{i} \leftarrow \mathfrak{j}\}$, so that $\mathfrak{g}' \trianglerighteq \mathfrak{g}'\{\mathfrak{i} \leftarrow \mathfrak{j}\}$ and $C \trianglerighteq D$. Thus $\mathfrak{g} \gg D$. If $\mathcal{E}_\Gamma \not\models^\Gamma D$, then, by the induction hypothesis, Γ contains a clause E such that $E \trianglelefteq D \trianglelefteq C$ and $\mathcal{E}_\Gamma \not\models^\Gamma E$, thus the proof is completed. Now, we assume that $\mathcal{E}_\Gamma \models^\Gamma D$ and we derive a contradiction. Since $\mathcal{E}_\Gamma \not\models^\Gamma C'$, necessarily $\mathcal{E}_\Gamma \models^\Gamma (\mathfrak{g}'\{\mathfrak{i} \leftarrow \mathfrak{j}\} \bowtie \mathfrak{h})$. We distinguish several cases.
   - If $\bowtie = \not\approx$ then since $\mathfrak{g} \gg C$, we must have $\mathfrak{g} > \mathfrak{g}'$ and $\mathfrak{g} > \mathfrak{h}$. By definition of $\Omega(\mathfrak{g})$, this entails that $\rightarrow_{\mathcal{E}_\Gamma|\mathfrak{g}'}$ is confluent. Also, since $\mathcal{E}_\Gamma \not\models^\Gamma C$, by definition of $\models^\Gamma$ we have $\mathfrak{g}' \downarrow_{\mathcal{E}_\Gamma} \mathfrak{h}$, so that $\mathfrak{g}' \leftrightarrow^*_{\mathcal{E}_\Gamma|\mathfrak{g}'} \mathfrak{h}$ (because $\mathfrak{h} \leq \mathfrak{g}'$ and $\rightarrow_{\mathcal{E}_\Gamma} \subseteq \geq$). Since $(\mathfrak{i} \approx \mathfrak{j}) \in \mathcal{E}_\Gamma$ and $\mathfrak{i} > \mathfrak{j}$ we have $\mathfrak{g}' \rightarrow_{\mathcal{E}_\Gamma} \mathfrak{g}'\{\mathfrak{i} \leftarrow \mathfrak{j}\}$. We deduce that $\mathfrak{g}'\{\mathfrak{i} \leftarrow \mathfrak{j}\} \leftrightarrow^*_{\mathcal{E}_\Gamma|\mathfrak{g}'} \mathfrak{h}$, and since $\rightarrow_{\mathcal{E}_\Gamma|\mathfrak{g}'}$ is confluent, this entails that $\mathfrak{g}'\{\mathfrak{i} \leftarrow \mathfrak{j}\} \downarrow_{\mathcal{E}_\Gamma} \mathfrak{h}$, which contradicts the fact that $\mathcal{E}_\Gamma \models^\Gamma \mathfrak{g}'\{\mathfrak{i} \leftarrow \mathfrak{j}\} \not\approx \mathfrak{h}$.
   - If $\bowtie = \approx$ and Γ is Horn, then by Definition 52, $\mathfrak{g}'\{\mathfrak{i} \leftarrow \mathfrak{j}\} \downarrow_{\mathcal{E}_\Gamma} \mathfrak{h}$. Since $(\mathfrak{i} \approx \mathfrak{j}) \in \mathcal{E}_\Gamma$ and $\mathfrak{i} > \mathfrak{j}$ we have $\mathfrak{g}' \rightarrow_{\mathcal{E}_\Gamma} \mathfrak{g}'\{\mathfrak{i} \leftarrow \mathfrak{j}\}$, thus $\mathfrak{g}' \downarrow_{\mathcal{E}_\Gamma} \mathfrak{h}$, which contradicts the fact that $\mathcal{E}_\Gamma \not\models^\Gamma C$.
   - If $\bowtie = \approx$ and Γ is not Horn, then since $\mathcal{E}_\Gamma \not\models^\Gamma C$, we have $\mathcal{E}_\Gamma \not\models^\Gamma \mathfrak{g}' \approx \mathfrak{h}$. Since $\mathfrak{i} \approx \mathfrak{j} \in \mathcal{E}_\Gamma$, $(\mathfrak{i} \approx \mathfrak{j}) < (\mathfrak{g}' \approx \mathfrak{h})$ and $\mathfrak{j} < \mathfrak{i}$, this entails in particular that $\mathcal{E}_\Gamma \not\models^\Gamma \mathfrak{g}'\{\mathfrak{i} \leftarrow \mathfrak{j}\} \approx \mathfrak{h}$.

Lemma 58. *Let Γ be a set of clauses such that Γ is strictly saturated or both Horn and saturated, and $\Box \notin \Gamma$. If $\mathfrak{g} \in \Omega(\Gamma)$ then $\mathcal{E}_\Gamma \models^\Gamma \Gamma_\mathfrak{g}$.*

Proof. For every negative literal $L = (\mathfrak{h} \not\approx \mathfrak{i})$ such that $\mathfrak{h} \downarrow_{\mathcal{E}_\Gamma} \mathfrak{i}$ we denote by $\pi(L)$ the minimal (w.r.t. the multiset extension of the usual order on natural numbers) unordered pair of natural numbers $\{n, m\}$ such that there exists an $\mathcal{L}$-graph $\mathfrak{j}$ with $\mathfrak{h} \rightarrow^n_{\mathcal{E}_\Gamma} \mathfrak{j}$ and $\mathfrak{i} \rightarrow^m_{\mathcal{E}_\Gamma} \mathfrak{j}$. If $\mathfrak{h} \downarrow_{\mathcal{E}_\Gamma} \mathfrak{i}$ does not hold or if L is positive then $\pi(L)$ is defined as $\{0, 0\}$. We define a strict order $\triangleright$ on literals as follows: $L \triangleright M$ iff one of the following conditions holds: (i) $L > M$; (ii) $L \simeq M$ and $\pi(L) > \pi(M)$; (iii) $L \simeq M$, $\pi(L) = \pi(M)$ and $L \succ M$. The order $\triangleright$ is extended to clauses using the multiset extension. Since $<$ and $\prec$ are well-founded, it is easy to check that $\triangleright$ is a well-founded strict order that is total on clauses. Assume that $\mathcal{E}_\Gamma \not\models^\Gamma \Gamma_\mathfrak{g}$ and let C be the $\triangleright$-minimal clause in $\Gamma_\mathfrak{g}$ such that $\mathcal{E}_\Gamma \not\models^\Gamma C$. Since by hypothesis $\Box \notin \Gamma$, C cannot be empty, hence must be of the form

$L \vee D$, where L is $\prec$-maximal in C. Note that this entails that L is eligible in C because $< \subseteq \prec$. We distinguish several cases.

- Assume that L is positive. If $L \succ D$, then since $\mathcal{E}_\Gamma \not\models^\Gamma C$ we must have $\mathcal{E}_\Gamma \not\models^\Gamma D$ hence $L \in \mathcal{E}_\Gamma$, by definition of $\mathcal{E}_\Gamma$. This entails that $\mathcal{E}_\Gamma \models L$, hence $\mathcal{E}_\Gamma \models C$, which contradicts our assumption. Consequently, $L \not\succ D$, and since L is $\prec$-maximal in C and $\preceq$ is total, this implies that D contains a literal L' such that $L' \equiv L$. By renaming (since the rules are defined modulo isomorphism), we may assume that $D = L \vee E$. Then the rule $\mathtt{F}$ applies and generates the clause $L \vee E$. We have $L \vee E \lhd L \vee L \vee E = C$, and since $\mathcal{E}_\Gamma \not\models^\Gamma C$, necessarily $\mathcal{E}_\Gamma \not\models^\Gamma L \vee E$. As Γ is strictly saturated (resp. Horn and saturated), $L \vee E$ must be strictly redundant (resp. redundant) in Γ. Moreover, $\mathfrak{g} \gg L \vee E$, since $\mathfrak{g} \gg C$. By Lemma 57, we deduce that Γ contains a clause C' such that $C' \trianglelefteq L \vee E$ and $\mathcal{E}_\Gamma \not\models^\Gamma C'$. As $L \vee E \lhd C$, we have $C' \lhd C$, which contradicts the minimality of C.
- Assume that L is a negative literal of the form $\mathfrak{h} \not\approx \mathfrak{h}'$, with $\mathfrak{h} \equiv \mathfrak{h}'$. By renaming, we assume that $\mathfrak{h}' = \mathfrak{h}$. Then the rule $\mathtt{R}$ applies, yielding the conclusion D. Since $\mathcal{E}_\Gamma \not\models^\Gamma C = L \vee D$, it is clear that $\mathcal{E}_\Gamma \not\models^\Gamma D$. Furthermore, $\mathfrak{g} \gg D$ because $\mathfrak{g} \gg C$. By Lemma 57, $\mathcal{E}_\Gamma$ contains a clause E such that $\mathcal{E}_\Gamma \not\models^\Gamma E$ and $E \trianglelefteq D \lhd C$, which contradicts the minimality of C.
- Assume that L is a negative literal $\mathfrak{h} \not\approx \mathfrak{i}$, with $\mathfrak{h} \not\equiv \mathfrak{i}$. Since $\prec$ is total on $\mathcal{L}$-graphs , we may assume by symmetry that $\mathfrak{h} \succ \mathfrak{i}$, which entails that $\mathfrak{h} \not< \mathfrak{i}$. Since $\mathcal{E}_\Gamma \not\models^\Gamma L$, necessarily $\mathfrak{h} \downarrow_{\mathcal{E}_\Gamma} \mathfrak{i}$ by Definition 52, so that $\pi(L) = \{\!\{n, m\}\!\}$ with either $n > 0$ or $m > 0$ (because $\mathfrak{h} \not\equiv \mathfrak{i}$). By definition, we have $\mathfrak{h} \to^n_{\mathcal{E}_\Gamma} \mathfrak{j}$ and $\mathfrak{i} \to^m_{\mathcal{E}_\Gamma} \mathfrak{j}$, for some $\mathcal{L}$-graph $\mathfrak{j}$. We distinguish two cases.
  - Assume that $n > 0$. Then we have $\mathfrak{h} \to_{\mathcal{E}_\Gamma} \mathfrak{h}' \to^{n-1}_{\mathcal{E}_\Gamma} \mathfrak{j}$, and by definition $\mathfrak{h}'$ is of the form $\mathfrak{h}\{\mathfrak{k} \leftarrow \mathfrak{k}'\}$, where $\mathfrak{k}' \leq \mathfrak{k}$ and $\mathfrak{k} \approx \mathfrak{k}' \in \mathcal{E}_\Gamma$. This entails that $\mathcal{E}_\Gamma$ contains a clause of the form $(\mathfrak{k} \approx \mathfrak{k}') \vee E$, where $\mathcal{E}_\Gamma \not\models^\Gamma E$, and $\mathfrak{k} \approx \mathfrak{k}' \succ E$. Hence, in particular, $\mathfrak{k} \approx \mathfrak{k}' \not< E$. Then the rule $\mathtt{Sp}^-$ applies from $(\mathfrak{k} \approx \mathfrak{k}') \vee E$ into C, yielding the conclusion $(\mathfrak{h}\{\mathfrak{k} \leftarrow \mathfrak{k}'\} \not\approx \mathfrak{i}) \vee D$. Since $\mathfrak{k} \geq \mathfrak{k}'$ and $\geq$ is closed under embeddings, we deduce that $L \geq (\mathfrak{h}\{\mathfrak{k} \leftarrow \mathfrak{k}'\} \approx \mathfrak{i})$, Now $\mathcal{E}_\Gamma \not\models^\Gamma (\mathfrak{h}\{\mathfrak{k} \leftarrow \mathfrak{k}'\} \not\approx \mathfrak{i}) \vee D$ because $\mathfrak{h}' \downarrow_{\mathcal{E}_\Gamma} \mathfrak{i}$, and by definition of π, we have $\pi(\mathfrak{h}\{\mathfrak{k} \leftarrow \mathfrak{k}'\} \approx \mathfrak{i}) = \{\!\{n-1, m\}\!\} < \{\!\{n, m\}\!\}$. Thus $L \rhd (\mathfrak{h}\{\mathfrak{k} \leftarrow \mathfrak{k}'\} \approx \mathfrak{i})$ and therefore $C \rhd (\mathfrak{h}\{\mathfrak{k} \leftarrow \mathfrak{k}'\} \not\approx \mathfrak{i}) \vee D)$. The latter property entails that $\mathfrak{g} \gg (\mathfrak{h}\{\mathfrak{k} \leftarrow \mathfrak{k}'\} \not\approx \mathfrak{i}) \vee D)$, and by Lemma 57, Γ contains a clause $E \trianglelefteq (\mathfrak{h}\{\mathfrak{k} \leftarrow \mathfrak{k}'\} \not\approx \mathfrak{i}) \vee D) \lhd C$ such that $\mathcal{E}_\Gamma \not\models^\Gamma E$. This contradicts the minimality of C.
  - If $n = 0$ then $\mathfrak{i} \to_{\mathcal{E}_\Gamma} \mathfrak{h}$, thus $\mathfrak{i} \geq \mathfrak{h}$ and $\mathfrak{i} \not< \mathfrak{g}$. Then $\mathfrak{i}$ is $<$-maximal in L and we may apply the same reasoning as in the previous case.

Corollary 59. *Let Γ be a set of clauses such that Γ is either strictly saturated or both Horn and saturated, and $\square \notin \Gamma$. If $\mathfrak{g} \in \Omega(\Gamma)$, $\mathfrak{g} \gg C$ and C is strictly redundant w.r.t. Γ then $\mathcal{E}_\Gamma \models^\Gamma C$. Furthermore, if Γ is Horn, the same property holds when C is redundant.*

Proof. The result is an immediate consequence of Lemmata 57 and 58.

Lemma 60. *Let Γ be a set of clauses and let $\mathfrak{g} \in \Omega(\Gamma)$. Let $(\mathfrak{h} \approx \mathfrak{i}) \vee C$ be a clause such that $\mathfrak{g} \gg (\mathfrak{h} \approx \mathfrak{i}) \vee C$ and $\mathcal{E}_\Gamma \not\models^\Gamma C$. If Γ is not Horn and $(\mathfrak{h} \approx \mathfrak{i}) \vee C$ is strictly redundant w.r.t. Γ then there exist $\mathcal{L}$-graphs $\mathfrak{h}'$ and $\mathfrak{i}'$ such that $\mathfrak{h} \to^*_{\mathcal{E}_\Gamma|<\mathfrak{g}} \mathfrak{h}'$, $\mathfrak{i} \to^*_{\mathcal{E}_\Gamma|<\mathfrak{g}} \mathfrak{i}'$ and either $\mathfrak{h}' \equiv \mathfrak{i}'$ or $\mathfrak{h}' \approx \mathfrak{i}' \in \mathcal{E}_\Gamma$. The same property holds if Γ is Horn, C is negative and $(\mathfrak{h} \approx \mathfrak{i}) \vee C$ is redundant w.r.t. Γ.*

Proof. By Corollary 59 we have $\mathcal{E}_\Gamma \models^\Gamma (\mathfrak{h} \approx \mathfrak{i}) \vee C$. Since by hypothesis $\mathcal{E}_\Gamma \not\models^\Gamma C$, necessarily $\mathcal{E}_\Gamma \models^\Gamma (\mathfrak{h} \approx \mathfrak{i})$. If Γ is not Horn, then this entails by Proposition 56 that the $\mathcal{L}$-graphs $\mathfrak{h}', \mathfrak{i}'$ satisfying the property of the lemma exist. Now, assume that Γ is Horn, which entails by hypothesis that C is negative. The proof is by induction on the set of redundant clauses. By definition of the relation $\models^\Gamma$ (in the Horn case), we have $\mathfrak{h} \downarrow_{\mathcal{E}_\Gamma} \mathfrak{i}$, hence there exists an $\mathcal{L}$-graph $\mathfrak{g}'$ such that $\mathfrak{h} \to^*_{\mathcal{E}_\Gamma} \mathfrak{g}'$ and $\mathfrak{i} \to^*_{\mathcal{E}_\Gamma} \mathfrak{g}'$. If $\mathfrak{g} > \mathfrak{h}$ and $\mathfrak{g} > \mathfrak{i}$, this entails (as $\to_{\mathcal{E}_\Gamma} \subseteq \geq$) that $\mathfrak{h} \to^*_{\mathcal{E}_\Gamma|<\mathfrak{g}} \mathfrak{g}'$ and $\mathfrak{i} \to^*_{\mathcal{E}_\Gamma|<\mathfrak{g}} \mathfrak{g}'$, hence the proof is completed. We now assume that either $\mathfrak{g} \leq \mathfrak{h}$ or $\mathfrak{g} \leq \mathfrak{i}$. Since $\mathfrak{g} \gg (\mathfrak{h} \approx \mathfrak{i}) \vee C$, and C is negative, all the $\mathcal{L}$-graphs occurring in C are $>$-smaller than $\mathfrak{g}$. By definition of the order on literals, this entails that $(\mathfrak{h} \approx \mathfrak{i}) > C$. We distinguish several cases, following Definition 24.

- Assume that $(\mathfrak{h} \approx \mathfrak{i}) \vee C$ contains two literals $\mathfrak{g}_1 \approx \mathfrak{g}_2$ and $\mathfrak{g}'_1 \not\approx \mathfrak{g}'_2$ with $\mathfrak{g}_i \equiv \mathfrak{g}'_i$ (for all $i = 1, 2$). If $(\mathfrak{g}_1 \approx \mathfrak{g}_2)$ and $(\mathfrak{g}'_1 \not\approx \mathfrak{g}'_2)$ both occur in C, then we get a contradiction with the hypothesis $\mathcal{E}_\Gamma \not\models^\Gamma C$, as it is done in the proof of Lemma 57 (first item). Otherwise, we must have $(\mathfrak{g}'_1 \not\approx \mathfrak{g}'_2) > (\mathfrak{g}_1 \approx \mathfrak{g}_2)$, by definition of the order on literals, which contradicts the fact that $\mathfrak{h} \approx \mathfrak{i} > C$.
- Assume that $(\mathfrak{h} \approx \mathfrak{i}) \vee C$ contains a literal $\mathfrak{g}' \approx \mathfrak{g}''$, with $\mathfrak{g}' \equiv \mathfrak{g}''$. If $(\mathfrak{h} \approx \mathfrak{i}) = (\mathfrak{g}' \approx \mathfrak{g}'')$ then $\mathfrak{h} \equiv \mathfrak{i}$, hence the proof is completed (with $\mathfrak{h}' = \mathfrak{h}$ and $\mathfrak{i}' = \mathfrak{i}$). Otherwise $\mathfrak{g}' \approx \mathfrak{g}''$ occurs in C, and $\mathcal{E}_\Gamma \models^\Gamma \mathfrak{g}' \approx \mathfrak{g}''$ by definition of $\models^\Gamma$, thus $\mathcal{E}_\Gamma \models^\Gamma C$, which contradicts the hypothesis of the lemma.
- Assume that $(\mathfrak{h} \approx \mathfrak{i}) \vee C \geq^{sub} D$, with $D \in \mathcal{E}_\Gamma$. If $D \leq^{sub} C$, then we get $\mathcal{E}_\Gamma \not\models^\Gamma D$ (since $\mathcal{E}_\Gamma \not\models^\Gamma C$), which contradicts Lemma 58. Otherwise, D is of the form $\mathfrak{h} \approx \mathfrak{i} \vee C'$ with $C' \leq^{sub} C$. We deduce that $\mathcal{E}_\Gamma \not\models^\Gamma C'$ which, as $(\mathfrak{h} \approx \mathfrak{i}) > C$, entails, by definition of $\mathcal{E}_\Gamma$, that $\mathfrak{h} \approx \mathfrak{i} \in \mathcal{E}_\Gamma$. This completes the proof, with $\mathfrak{h}' = \mathfrak{h}$ and $\mathfrak{i}' = \mathfrak{i}$.
- Assume that $(\mathfrak{h} \approx \mathfrak{i}) \vee C \to_\Gamma D$, and that D is redundant w.r.t. Γ. By definition $(\mathfrak{h} \approx \mathfrak{i}) \vee C$ and D are respectively of the forms $\mathfrak{j}_1 \approx \mathfrak{j}_2 \vee C'$ and $\mathfrak{j}_1\{\mathfrak{k}_1 \leftarrow \mathfrak{k}_2\} \approx \mathfrak{j}_2 \vee C'$, and Γ contains a clause $\mathfrak{k} \approx \mathfrak{k}' \vee E$ with $\mathfrak{k} > \mathfrak{k}'$, $(\mathfrak{k} \approx \mathfrak{k}') > E'$ and $E \leq^{sub} C'$.
  If $(\mathfrak{h} \approx \mathfrak{i}) \neq (\mathfrak{j}_1 \approx \mathfrak{j}_2)$, then we have $D = (\mathfrak{h} \approx \mathfrak{i}) \vee D'$, with $D' < D < \mathfrak{h} \approx \mathfrak{i}$, so that $\mathfrak{g} \gg D$, and we can prove, as it is done in the proof of Lemma 58, that $\mathcal{E}_\Gamma \not\models^\Gamma D'$. Then the proof follows by the induction hypothesis. Now assume that $(\mathfrak{h} \approx \mathfrak{i}) = (\mathfrak{j}_1 \approx \mathfrak{j}_2)$. By symmetry, we only consider the case where $\mathfrak{h} = \mathfrak{j}_1$ and $\mathfrak{i} = \mathfrak{j}_2$. Note that we have $C' = C$ in this case, thus $E \leq^{sub} C$, and therefore $\mathcal{E}_\Gamma \not\models^\Gamma E$. This entails, by definition of $\mathcal{E}_\Gamma$, that $\mathfrak{k} \approx \mathfrak{k}' \in \mathcal{E}_\Gamma$. Since $\mathfrak{j}_1\{\mathfrak{k}_1 \leftarrow \mathfrak{k}_2\} \approx \mathfrak{j}_2 < \mathfrak{h} \approx \mathfrak{i}$, we have $\mathfrak{g} \gg D$. By the induction hypothesis, there exist $\mathfrak{h}', \mathfrak{i}'$ such that $\mathfrak{j}_1\{\mathfrak{k}_1 \leftarrow \mathfrak{k}_2\} \to^*_{\Gamma|<\mathfrak{g}} \mathfrak{h}'$ and $\mathfrak{i} \to^*_{\Gamma|<\mathfrak{g}} \mathfrak{i}'$ and either $\mathfrak{h}' \equiv \mathfrak{i}'$ or $\mathfrak{h}' \approx \mathfrak{i}' \in \mathcal{E}_\Gamma$. This entails that $\mathfrak{h} \to^*_{\Gamma|<\mathfrak{g}} \mathfrak{h}'$ (since $\mathfrak{g} \geq \mathfrak{h}$, $\mathfrak{k} \approx \mathfrak{k}' \in \mathcal{E}_\Gamma$ and $\mathfrak{k} > \mathfrak{k}'$) hence the proof is completed.

Lemma 61. *Let Γ be a set of clauses. Assume that Γ is strictly saturated or both Horn and saturated, and that $\Box \notin \Gamma$. If $\mathfrak{g} \in \Omega(\Gamma)$ then $\rightarrow_{\mathcal{E}_\Gamma|\mathfrak{g}}$ is confluent.*

Proof. We prove that $\rightarrow_{\mathcal{E}_\Gamma|\mathfrak{g}}$ is $\geq$-subcommutative, which entails the result by Lemma 48. Let $\mathfrak{h}, \mathfrak{h}_1, \mathfrak{h}_2$ be $\mathcal{L}$-graphs such that $\mathfrak{h} \rightarrow_{\mathcal{E}_\Gamma|\mathfrak{g}} \mathfrak{h}_i$ for $i = 1, 2$. Note that this entails that $\mathfrak{g} \geq \mathfrak{h}$ and $\mathfrak{g} \geq \mathfrak{h}_i$. We have to prove that there exists a $\mathcal{L}$-graph $\mathfrak{h}'$ such that for all $i = 1, 2$, either $\mathfrak{h}_i \rightarrow^{0|1}_{\mathcal{E}_\Gamma|\mathfrak{g}} \mathfrak{h}'$ or $\mathfrak{h}_i \rightarrow_{\mathcal{E}_\Gamma|\mathfrak{g}} \mathfrak{h}'_i \rightarrow^*_{\mathcal{E}_\Gamma|\mathfrak{g}} \mathfrak{h}'$ for some $\mathcal{L}$-graph $\mathfrak{h}'_i$ with $\mathfrak{h}'_i < \mathfrak{h}$. By definition, there exist equations $\mathfrak{i}_i \approx \mathfrak{j}_i$ in $\mathcal{E}_\Gamma$ such that $\mathfrak{h}_i = \mathfrak{h}\{\mathfrak{i}_i \leftarrow \mathfrak{j}_i\}$ and $\mathfrak{i}_i \geq \mathfrak{j}_i$.

- Assume that $\mathfrak{i}_1$ and $\mathfrak{i}_2$ are orthogonal in $\mathfrak{h}$. By Proposition 12, we deduce that $\mathfrak{h}_i\{\mathfrak{i}_j \leftarrow \mathfrak{j}_j\} = \mathfrak{h}_j\{\mathfrak{i}_i \leftarrow \mathfrak{j}_i\}$, for all $(i, j) \in \{(1, 2), (2, 1)\}$. Let $\mathfrak{h}' = \mathfrak{h}_1\{\mathfrak{i}_2 \leftarrow \mathfrak{j}_2\}$. By definition, we have $\mathfrak{h}_i \rightarrow_{\mathcal{E}_\Gamma} \mathfrak{h}'$ for $i = 1, 2$, and $\mathfrak{h}_i \rightarrow_{\mathcal{E}_\Gamma|\mathfrak{g}} \mathfrak{h}'$ since $\rightarrow_{\mathcal{E}_\Gamma} \subseteq \geq$. Hence the proof is completed.
- Now, assume that $\mathfrak{i}_1$ and $\mathfrak{i}_2$ are not orthogonal. By Lemma 16, $\mathfrak{i}_1$ and $\mathfrak{i}_2$ admit a merge $\mathfrak{i}$, and $\mathfrak{i} \leq^g \mathfrak{g}$. By definition of $\mathcal{E}_\Gamma$, Γ contains clauses of the form $\mathfrak{i}_i \approx \mathfrak{j}_i \vee C_i$, with $\mathcal{E}_\Gamma \not\models^\Gamma C_i$ for $i = 1, 2$ and $(\mathfrak{i}_i \approx \mathfrak{j}_i) \succ C_i$, so that $(\mathfrak{i}_i \approx \mathfrak{j}_i) \not\prec C_i$ (since $< \subseteq \prec$). Therefore $\mathfrak{i}_i \approx \mathfrak{j}_i$ is eligible in $\mathfrak{i}_i \approx \mathfrak{j}_i \vee C_i$ and (as $\mathfrak{i}_1$ and $\mathfrak{i}_2$ are not orthogonal) the rule $\mathtt{Sp}^+$ applies, yielding $D = \mathfrak{i}\{\mathfrak{i}_1 \leftarrow \mathfrak{j}_1\} \approx \mathfrak{i}\{\mathfrak{i}_2 \leftarrow \mathfrak{j}_2\} \vee C_1 \vee C_2$. As Γ is strictly saturated (resp. Horn and saturated), D must be strictly redundant (resp. redundant). Note that $\mathfrak{i} \gg D$, since $\mathfrak{i} \geq \mathfrak{i}_i$ for $i = 1, 2$. Since $\mathcal{E}_\Gamma \not\models^\Gamma C_i$, this entails by Lemma 60 that there exist $\mathcal{L}$-graphs $\mathfrak{k}_i$ (for $i = 1, 2$) such that $\mathfrak{i}\{\mathfrak{i}_i \leftarrow \mathfrak{j}_i\} \rightarrow^*_{\mathcal{E}_\Gamma|<\mathfrak{i}} \mathfrak{k}_i$ and either $\mathfrak{k}_1 \equiv \mathfrak{k}_2$ or $\mathfrak{k}_1 \approx \mathfrak{k}_2 \in \mathcal{E}_\Gamma$. Assume by symmetry that $\mathfrak{k}_1 \geq \mathfrak{k}_2$. We get $\mathfrak{i}\{\mathfrak{i}_1 \leftarrow \mathfrak{j}_1\} \rightarrow^*_{\mathcal{E}_\Gamma|<\mathfrak{i}} \mathfrak{k}_1 \rightarrow^{0|1}_{\mathcal{E}_\Gamma} \mathfrak{k}_2$, furthermore, if the length of the derivation from $\mathfrak{i}\{\mathfrak{i}_1 \leftarrow \mathfrak{j}_1\}$ to $\mathfrak{k}_2$ is strictly greater than 1, then the second $\mathcal{L}$-graph in the derivation is necessarily strictly $>$-smaller than $\mathfrak{i}$, by definition of $\rightarrow_{\mathcal{E}_\Gamma|<\mathfrak{i}}$. Similarly, if the length of the derivation from $\mathfrak{i}\{\mathfrak{i}_2 \leftarrow \mathfrak{j}_2\}$ to $\mathfrak{k}_2$ is strictly greater than 1, then the second $\mathcal{L}$-graph in the derivation is strictly $>$-smaller than $\mathfrak{i}$. Since $\rightarrow_{\mathcal{E}_\Gamma}$ is closed under embeddings, this entails that for every $i = 1, 2$: $\mathfrak{h}\{\mathfrak{i} \leftarrow \mathfrak{i}\{\mathfrak{i}_i \leftarrow \mathfrak{j}_i\}\} \rightarrow^*_{\mathcal{E}_\Gamma} \mathfrak{h}\{\mathfrak{i} \leftarrow \mathfrak{k}_2\}$. By Proposition 9, $\mathfrak{h}\{\mathfrak{i} \leftarrow \mathfrak{i}\{\mathfrak{i}_i \leftarrow \mathfrak{j}_i\}\} = \mathfrak{h}\{\mathfrak{i}_i \leftarrow \mathfrak{j}_i\} = \mathfrak{h}_i$. Moreover, if the derivation from $\mathfrak{h}_i$ to $\mathfrak{g}\{\mathfrak{i} \leftarrow \mathfrak{k}_2\}$ is of length strictly greater than 1, then the second $\mathcal{L}$-graph occurring in it must be strictly $>$-lower than $\mathfrak{h}$ (since the subgraph $\mathfrak{i}$ is replaced by a strictly lower $\mathcal{L}$-graph and $>$ is closed under embeddings). Thus the proof is completed (with $\mathfrak{h}' = \mathfrak{g}\{\mathfrak{i} \leftarrow \mathfrak{k}_2\}$).

D Lifting

The following propositions state that the lifted relations satisfy the expected properties:

Proposition 62. *For all $\mathcal{T}$-graphs $\mathfrak{g}, \mathfrak{h}$, for all $\mathcal{I}$-interpretations I and for all ground substitutions of domain $\mathcal{V}(\mathfrak{g}) \cup \mathcal{V}(\mathfrak{h})$: $[\mathfrak{g}\sigma]^I = [\mathfrak{h}\sigma]^I \iff \exists \phi. (\mathfrak{g} =_\phi \mathfrak{h} \wedge I \models \phi\sigma)$.*

Proposition 63. *For all $\mathcal{T}$-graphs $\mathfrak{g}, \mathfrak{h}$, for all $\mathcal{I}$-interpretations I and for all ground substitutions of domain $\mathcal{V}(\mathfrak{g}) \cup \mathcal{V}(\mathfrak{h})$: $[\mathfrak{g}\sigma]^I \leq^g [\mathfrak{h}\sigma]^I \iff \exists \phi.(\mathfrak{g} \leq^g_\phi \mathfrak{h} \wedge I \models \phi\sigma)$.*

Proposition 64. *Let $\mathfrak{g}, \mathfrak{h}, \mathfrak{i}$ be $\mathcal{T}$-graphs, let $\phi \in \mathcal{C}$ and let $I \in \mathcal{I}$. Let σ be a ground substitution of domain $\mathcal{V}(\mathfrak{g}) \cup \mathcal{V}(\mathfrak{h}) \cup \mathcal{V}(\mathfrak{i})$. If $\mathfrak{h} \leq^g_\phi \mathfrak{g}$ and $pr(\mathfrak{h}) = pr(\mathfrak{i})$ then: $[\mathfrak{g}\{\mathfrak{h} \leftarrow \mathfrak{i}\}\sigma]^I = [\mathfrak{g}\sigma]^I\{[\mathfrak{h}\sigma]^I \leftarrow [\mathfrak{i}\sigma]^I\}$.*

Proof. Let $\mathfrak{g}_1 = \mathfrak{g}\{\mathfrak{h} \leftarrow \mathfrak{i}\}$ and $\mathfrak{g}_2 = [\mathfrak{g}\sigma]^I\{[\mathfrak{h}\sigma]^I \leftarrow [\mathfrak{i}\sigma]^I\}$. We assume by renaming that the sets $N_\mathfrak{g}$, $\widehat{N}_\mathfrak{h}$ and $\widehat{N}_\mathfrak{i}$ are disjoint, and that $R_\mathfrak{h} = R_\mathfrak{i}$. As substitutions and interpretations affect only labels, we have $F_{[\mathfrak{j}\sigma]^I} = F_\mathfrak{j}$, for all $F \in \{N, E, R, \widehat{N}\}$ and for all $\mathcal{T}$-graphs $\mathfrak{j}$. By Definition 7, we get $N_{\mathfrak{g}_2} = (N_{[\mathfrak{g}\sigma]^I} \setminus N_{[\mathfrak{h}\sigma]^I}) \cup N_{[\mathfrak{i}\sigma]^I} = (N_\mathfrak{g} \setminus N_\mathfrak{h}) \cup N_\mathfrak{i} = N_{\mathfrak{g}_1} = N_{[\mathfrak{g}_1\sigma]^I}$. Similarly, $E_{\mathfrak{g}_2} = E_{[\mathfrak{g}_1\sigma]^I}$. Moreover, still by Definition 7: $R_{\mathfrak{g}_2} = R_{[\mathfrak{g}\sigma]^I} = R_\mathfrak{g} = R_{\mathfrak{g}_1} = R_{[\mathfrak{g}_1\sigma]^I}$. Finally, consider a node $\alpha \in N_{\mathfrak{g}_2}$. If $\alpha \in \widehat{N}_{[\mathfrak{i}\sigma]^I} = \widehat{N}_\mathfrak{i}$, we have: $L_{\mathfrak{g}_2}(\alpha) = L_{[\mathfrak{i}\sigma]^I}(\alpha) = [L_\mathfrak{i}(\alpha)\sigma]^I = [L_{\mathfrak{g}_1}(\alpha)\sigma]^I = L_{[\mathfrak{g}_1\sigma]^I}(\alpha)$. Otherwise, we get $L_{\mathfrak{g}_2}(\alpha) = L_{[\mathfrak{g}\sigma]^I}(\alpha) = [L_\mathfrak{g}(\alpha)\sigma]^I = [L_{\mathfrak{g}_1}(\alpha)\sigma]^I = L_{[\mathfrak{g}_1\sigma]^I}(\alpha)$.

Proposition 65. *For all $\mathcal{T}$-graphs $\mathfrak{g}, \mathfrak{h}$, for all $\mathcal{I}$-interpretations I and for all ground substitutions of domain $\mathcal{V}(\mathfrak{g}) \cup \mathcal{V}(\mathfrak{h})$, $[\mathfrak{g}\sigma]^I$ and $[\mathfrak{h}\sigma]^I$ admit a merge $\mathfrak{i}$ iff $\mathfrak{g}$ and $\mathfrak{h}$ admit a ϕ-merge $\mathfrak{j}$ such that $I \models \phi\sigma$.*

Proof. Assume that $\mathfrak{i}$ is a merge of $[\mathfrak{g}\sigma]^I$ and $[\mathfrak{h}\sigma]^I$. Then: $N_\mathfrak{i} = N_{[\mathfrak{g}\sigma]^I} \cup N_{[\mathfrak{h}\sigma]^I} = N_\mathfrak{g} \cup N_\mathfrak{h}$, $E_\mathfrak{i} = E_{[\mathfrak{g}\sigma]^I} \sqcup E_{[\mathfrak{h}\sigma]^I} = E_\mathfrak{g} \sqcup E_\mathfrak{h}$, $\widehat{N}_\mathfrak{i} = \widehat{N}_{[\mathfrak{g}\sigma]^I} \cup \widehat{N}_{[\mathfrak{h}\sigma]^I} = \widehat{N}_\mathfrak{g} \cup \widehat{N}_\mathfrak{h}$, and $L_{[\mathfrak{g}\sigma]^I}(\alpha) = L_{[\mathfrak{h}\sigma]^I}(\alpha)$, for all $\alpha \in \widehat{N}_\mathfrak{g} \cap \widehat{N}_\mathfrak{h}$. Consequently, $[L_{\mathfrak{g}\sigma}(\alpha)]^I = [L_{\mathfrak{h}\sigma}(\alpha)]^I$, for all $\alpha \in \widehat{N}_\mathfrak{g} \cap \widehat{N}_\mathfrak{h}$. Let $\mathfrak{j}$ the $\mathcal{T}$-graph defined as follows: $F_\mathfrak{j} = F_\mathfrak{i}$ for all $F \in \{N, E, R, \widehat{N}\}$, and for all $\alpha \in \widehat{N}_\mathfrak{i}$, $L_\mathfrak{j}(\alpha) = L_\mathfrak{g}(\alpha)$ if $\alpha \in \widehat{N}_\mathfrak{g}$, otherwise $L_{\mathfrak{g}'}(\alpha) = L_\mathfrak{h}(\alpha)$. Let $\phi = \bigwedge_{\alpha \in \widehat{N}_\mathfrak{g} \cap \widehat{N}_\mathfrak{h}} (L_\mathfrak{g}(\alpha) \doteq L_\mathfrak{h}(\alpha))$. By construction, $\mathfrak{j}$ is a ϕ-merge of $\mathfrak{g}$ and $\mathfrak{h}$, $I \models \phi\sigma$ and $[\mathfrak{j}\sigma]^I = \mathfrak{i}$. The converse is straightforward.

E On the Completeness of the Constrained Calculus

Proposition 66. *Let $L = \mathfrak{g} \bowtie \mathfrak{h}$ be a literal, and let $[C \mid \phi]$ be a c-clause. Let $I \in \mathcal{I}$ and let σ be a ground substitution of domain $\mathcal{V}(C) \cup \mathcal{V}(\phi)$. If $[L\sigma]^I$ is eligible in $[(L \vee C)\sigma]^I$ and $I \models \phi\sigma$, then L is eligible in $[L \vee C \mid \phi]$.*

Proof. By definition $[L\sigma]^I$ is $<_I$-maximal in $[(L \vee C)\sigma]^I$, and by definition of the order $<_\phi$, since $I \models \phi\sigma$, L must be $<_\phi$-maximal in $L \vee C$, so that L is eligible in $[L \vee C \mid \phi]$.

Lemma 67. *Let Γ be a set of c-clauses. If Γ is (strictly) saturated then for every $I \in \mathcal{I}$, $[\Gamma]^I$ is (strictly) saturated (w.r.t. the order $<_I$).*

Proof. Let C be a clause deducible from $[\Gamma]^I$ by a single application of one of the rules $\mathtt{Sp}^+$, $\mathtt{Sp}^-$, $\mathtt{F}$ or $\mathtt{R}$. We show that C is (strictly) redundant w.r.t. $[\Gamma]^I$. We

provide the proof only for the rule $\mathtt{Sp}^+$, the other cases are handled in a similar way. By definition, $[\Gamma]^I$ contains premises $\mathfrak{g}_i \approx \mathfrak{h}_i \vee C_i$ (for $i = 1, 2$), where every literal $\mathfrak{g}_i \approx \mathfrak{h}_i$ is eligible in its clause, $\mathfrak{g}_i \not<_I \mathfrak{h}_i$, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are not orthogonal and admit a merge $\mathfrak{g}$, and $C = (\mathfrak{g}\{\mathfrak{g}_1 \leftarrow \mathfrak{h}_1\} \approx \mathfrak{g}\{\mathfrak{g}_1 \leftarrow \mathfrak{h}_1\}) \vee C_1 \vee C_2$. By definition of $[\Gamma]^I$, this entails that Γ contains two c-clauses $[\mathfrak{g}'_i \approx \mathfrak{h}'_i \vee C'_i \mid \phi_i]$ (for $i = 1, 2$), with $\mathfrak{g}_i = [\mathfrak{g}'_i\sigma_i]^I$, $\mathfrak{h}_i = [\mathfrak{h}'_i\sigma_i]^I$, $C_i = [C'_i\sigma_i]^I$ and $I \models \phi_i\sigma_i$ (where σ_i is a ground substitution of domain $\mathcal{V}(\mathfrak{g}'_i) \cup \mathcal{V}(\mathfrak{h}'_i) \cup \mathcal{V}(C'_i) \cup \mathcal{V}(\phi_i)$). We may assume by α-renaming that the c-clauses $[\mathfrak{g}'_i \approx \mathfrak{h}'_i \vee C'_i \mid \phi_i]$ share no variable, and we denote by σ the composition of σ_1 and σ_2. Since $\mathfrak{g}$ is a merge of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, by Proposition 65, there exists a ϕ-merge $\mathfrak{g}'$ of $\mathfrak{g}'_1$ and $\mathfrak{g}'_2$ such that $[\mathfrak{g}'\sigma]^I = \mathfrak{g}$ and $I \models \phi\sigma$. By Proposition 66, $\mathfrak{g}'_i \approx \mathfrak{h}'_i$ is eligible in $[\mathfrak{g}'_i \approx \mathfrak{h}'_i \vee C'_i \mid \phi_1 \wedge \phi_2 \wedge \psi]$ (for all $i = 1, 2$). If $\mathfrak{g}'_i <_{\phi\wedge\phi_1\wedge\phi_2} \mathfrak{h}'_i$ then we get (by definition of the order $<_{\phi\wedge\phi_1\wedge\phi_2}$) $[\mathfrak{g}'_i]^I < [\mathfrak{h}'_i\sigma]^I$, i.e., $\mathfrak{g}_i < \mathfrak{h}_i$, which contradicts our assumption. Thus $\mathfrak{g}'_i \not<_{\phi\wedge\phi_1\wedge\phi_2} \mathfrak{h}'_i$. Since $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are not orthogonal, $\mathfrak{g}'_1$ and $\mathfrak{g}'_2$ cannot be orthogonal (as the notion of orthogonality does not depend on labels). Consequently, $\mathtt{Sp}^+$ applies, yielding: $C' = [\mathfrak{g}'\{\mathfrak{g}'_1 \leftarrow \mathfrak{h}'_1\} \approx \mathfrak{g}'\{\mathfrak{g}'_2 \leftarrow \mathfrak{h}'_2\} \vee C'_1 \vee C'_2 \mid \phi \wedge \phi_1 \wedge \phi_2]$. Using Proposition 64, we get $[\mathfrak{g}'\{\mathfrak{g}'_i \leftarrow \mathfrak{h}'_i\}\sigma]^I = \mathfrak{g}\{\mathfrak{g}_i \leftarrow \mathfrak{h}_i\}$, for all $i = 1, 2$, so that $[C'\sigma]^I = C$. By Definition 37, this entails that C is (strictly) redundant w.r.t. $[\Gamma]^I$.