On Gaussian interpolation inequalities

Giovanni Brigati, Jean Dolbeault, Nikita Simonov

On Gaussian interpolation inequalities

Sur des inégalités d'interpolation Gaussiennes

Giovanni Brigati [a], **Jean Dolbeault** [b] **and Nikita Simonov** [c]

[a] CEREMADE (CNRS UMR n° 7534), PSL University, Université Paris-Dauphine, Place de Lattre de Tassigny, 75775 Paris 16, France

[b] CEREMADE (CNRS UMR n° 7534), PSL University, Université Paris-Dauphine, Place de Lattre de Tassigny, 75775 Paris 16, France

[c] LJLL (CNRS UMR n° 7598) Sorbonne Université, 4 place Jussieu, 75005 Paris, France

E-mails: brigati@ceremade.dauphine.fr (G. Brigati), dolbeaul@ceremade.dauphine.fr (J. Dolbeault), nikita.simonov@sorbonne-universite.fr (N. Simonov)

Abstract. This paper is devoted to Gaussian interpolation inequalities with endpoint cases corresponding to the Gaussian Poincaré and the logarithmic Sobolev inequalities, seen as limits in large dimensions of Gagliardo-Nirenberg-Sobolev inequalities on spheres. Entropy methods are investigated using not only heat flow techniques but also nonlinear diffusion equations as on spheres. A new stability result is established for the Gaussian measure, which is directly inspired by recent results for spheres.

Résumé. Cet article est consacré à des inégalités d'interpolation Gaussiennes, avec comme cas extrêmes l'inégalité de Poincaré Gaussienne et l'inégalité de Sobolev logarithmique, vues comme limites en grandes dimensions des inégalités de Gagliardo-Nirenberg-Sobolev sur les sphères. Les méthodes d'entropie sont abordées en utilisant non seulement des techniques basée sur l'équation de la chaleur mais aussi sur des équations de diffusion non-linéaires, comme pour les sphères. Un nouveau résultat de stabilité est établi pour les mesures Gaussiennes, qui s'inspire directement de résultats récents sur les sphères.

Keywords. logarithmic Sobolev inequality, Gagliardo-Nirenberg-Sobolev inequalities, Gaussian Poincaré inequality, sphere, spectral decomposition, entropy methods, Ornstein-Uhlenbeck operator, nonlinear diffusions, improved inequalities, stability.

2020 Mathematics Subject Classification. 26D10, 46T12, 46E35, 39B62, 43A90, 49J40, 58E35.

Date: 8 February 2023. *File:* BDS2023-PLS.tex

1. Introduction and main results

Let us consider the *Gagliardo-Nirenberg-Sobolev inequalities* on the unit d-dimensional sphere

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} \ge \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)}\right) \quad \forall\, u\in \mathrm{H}^1(\mathbb{S}^d,d\mu_d) \tag{1}$$

for any $p\in[1,2)\cup(2,+\infty)$ if $d=1$, 2, and for any $p\in[1,2)\cup(2,2^*]$ if $d\ge 3$. Here $d\mu_d$ denotes the uniform probability measure on $\mathbb{S}^d\subset\mathbb{R}^{d+1}$ and, if $d\ge 3$, $2^*=2\,d/(d-2)$ is the critical Sobolev

exponent. By convention, we take $2^*=+\infty$ if $d=1$ or 2. The purpose of this paper is to clarify the links of these interpolation inequalities with the family of *Gaussian interpolation inequalities*

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\ge\frac{1}{2-p}\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}-\|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right)\quad\forall\,v\in\mathrm{H}^1(\mathbb{R}^n,d\gamma)\tag{2}$$

where the exponent p is taken in the range $1\le p<2$. Inequality (2) is intermediate between the Poincaré inequality corresponding to $p=1$ and the *Gaussian logarithmic Sobolev inequality*

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\ge\frac12\int_{\mathbb{R}^n}|v|^2\log\left(\frac{|v|^2}{\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right)d\gamma\quad\forall\,v\in\mathrm{H}^1(\mathbb{R}^n,d\gamma)$$

obtained as a limit case of (2) as $p\to2_-$. Here $d\gamma(y):=(2\pi)^{-n/2}\,e^{-\frac12|y|^2}\,dy$ denotes the centred normalized Gaussian probability measure and the dimension n is any positive integer.

It is somewhat classical that, if we consider the Sobolev inequality on the sphere, *i.e.*, Inequality (1) with $p=2^*$ and $d\ge3$, rescale it to the sphere of radius $\sqrt{d}$ and fix a function depending on n variables only on this sphere, since the curvature tends to 0 at the right order as one takes the limit as $d\to+\infty$, then the sphere tends to the flat space with Gaussian measure. For instance, we read (with adapted notations) from [9, p. 4818] that *if we rescale this inequality so as to be on a sphere of radius $\sqrt{d}$ and take the limit $d\to\infty$ or $p\to2$ we obtain in the Poincaré limit the Gross logarithmic inequality for the Gaussian measure since $-(1/d)\,\Delta$ on $\mathbb{S}^d$ goes to $-\Delta+x\cdot\nabla$ on the infinite-dimensional limit.* The last part of the sentence refers to a result known as the Maxwell–Poincaré lemma : see [37] and [39, Remark 4, p. 254] for some historical comments. The statement of [9] has been made more precise later in [11, 12] using a slightly different limit. However, to our knowledge, the infinite-dimensional limit has not been considered in the subcritical range $p<2^*$.

On the sphere, Inequality (1) follows from [4, 6] for any $p\ge1$ if $d=1$ and any $p\in[1,2^\#]$ with $2^\#:=(2\,d^2+1)/(d-1)^2$ if $d\ge2$. The proof in the range $p\in(2^\#,+\infty)$ if $d=2$ and $p\in(2^\#,2^*]$ if $d\ge3$ can be found in [15, Corollary 6.1], [16] and [10]. Also see [35] in the case $p=2^*$. In the case of the Gaussian measure, we refer to [8] (also see [32]) for a first proof of Inequality (2). The formal analogy of (1) and (2) is striking. Although computations are somewhat standard, our first purpose is to make this point rigorous and recover (2) as a special limit of (1) as $d\to+\infty$.

Theorem 1. *Let n be a positive integer, $p\in[1,2)$ and consider a function $v\in\mathrm{H}^1(\mathbb{R}^n,d\gamma)$ with compact support. For any $d\ge n$, large enough, if $u_d(\omega)=v\big(\omega_1/\sqrt{d},\omega_2/\sqrt{d},\dots,\omega_n/\sqrt{d}\big)$ where $\omega=(\omega_1,\omega_2,\dots,\omega_{d+1})\in\mathbb{R}^{d+1}\supset\mathbb{S}^d$ is such that $|\omega|=1$, then*

$$\lim_{d\to+\infty}d\left(\|\nabla u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)}-\frac{d}{2-p}\left(\|u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)}-\|u_d\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)}\right)\right)$$
$$=\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}-\frac{1}{2-p}\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}-\|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right).$$

The *carré du champ* method has frequently been applied to prove Gaussian interpolation inequalities ranging between the logarithmic Sobolev inequality and the Poincaré inequality like (2) using the linear flow associated with the Ornstein-Uhlenbeck operator; see [4], and [7] for an overview. Still in the case of the Gaussian measure, we adopt here a new point of view by using *nonlinear diffusion equations* in order to prove the same inequalities, but with different remainder terms. This is a very natural point of view when dealing with inequalities like (1) on the sphere, as shown for instance in [24] (see earlier references therein). In that case, linear flows are indeed limited to exponents $p\le2^\#$ if $d\ge2$. To overcome this difficulty if either $d=2$ and $p>2^\#$ or $d\ge3$ and $p\in(2^\#,2^*]$, one has to consider fast diffusion flows. Before explaining the results for the Gaussian measure, let us summarize the main known results on the sphere.

On $\mathbb{S}^d$, let us consider a positive solution u of

$$\frac{\partial u}{\partial t} = u^{-p(1-m)}\left(\Delta u + (m\,p-1)\,\frac{|\nabla u|^2}{u}\right) \tag{3}$$

where Δ denotes the Laplace-Beltrami operator on $\mathbb{S}^d$. In the case $m = 1$, u^p solves the heat equation and for this reason, we shall call it the *linear case*. Otherwise u^p solves a nonlinear diffusion equation corresponding either to a fast diffusion flow with $m < 1$ or to a porous media equation with $m > 1$. In any case, we claim that

$$\frac{d}{dt}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} = 0 \quad \text{and} \quad \frac{d}{dt}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = 2\,(p-2)\int_{\mathbb{S}^d} u^{-p(1-m)}\,|\nabla u|^2\,d\mu_d\,.$$

Let us define

$$m_\pm(d,p) := \frac{1}{(d+2)\,p}\left(d\,p+2 \pm \sqrt{d\,(p-1)\,\big(2\,d-(d-2)\,p\big)}\right). \tag{4}$$

The following result can be found in [20, 24] with additional details in [17, 22, 25].

Proposition 2 ([20, 24]). *Assume that $d \ge 1$, with either $p \in [1,2)\cup(2,+\infty)$ if $d = 2$ or $p \in [1,2)\cup(2,2^*]$ if $d \ge 3$, and let $m \in [m_-(d,p), m_+(d,p)]$. If $u > 0$ solves* (3) *with an initial datum in $\mathrm{L}^2\cap\mathrm{L}^p(\mathbb{S}^d, d\mu_d)$, then*

$$\frac{d}{dt}\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)}\right)\right) \le 0 \quad \forall\, t > 0\,.$$

The limit, as $t \to +\infty$, of any solution of (3) is a constant. This means that the *deficit*, that is, the difference of the two sides in Inequality (1), converges to 0. Then, by Proposition 2, it follows that the deficit is non-negative, which directly proves (1). The same monotonicity property applies to the deficit of the *logarithmic Sobolev inequality on the sphere*

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} \ge \frac{d}{2}\int_{\mathbb{S}^d} |u|^2\,\log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)d\mu_d \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu_d)$$

in the limit case corresponding to $p = 2$. The admissible values of the parameters are limited to $m_-(d,p) \le m \le m_+(d,p)$ and $1 \le p \le 2^*$ if $d \ge 3$. Moreover, at the endpoints, we have $m_\pm(d,1) = 1$ and $m_\pm(d,2^*) = (d-1)/d$ if $d \ge 3$, while $m_+(d,2^\#) = 1$ so that $m = 1$ is admissible if and only if $1 \le p \le 2^\#$ when $d \ge 2$. See Fig. 1. For appropriate initial data, it is shown in [25] that the monotonicity property of the deficit along the flow of (3) is violated for any $p \in [2,2^*)$ or $p = 2^*$ if $d \ge 3$ as soon as either $m < m_-(d,p)$ or $m > m_+(d,p)$.

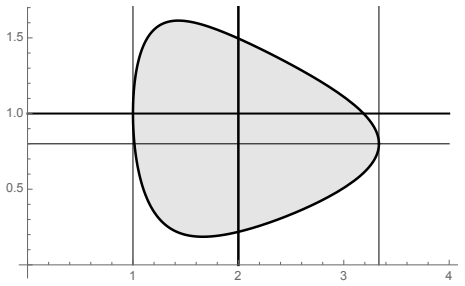

Figure 1. Case $d = 5$. The admissible parameters p and m correspond to the grey area. The boundary of the admissible set is tangent to the vertical lines $p = 1$ at $(m,p) = (1,1)$ and $p = 2^* = 10/3$ at $(m,p) = (4/5, 10/3)$. Qualitatively, this figure does not change as d increases but gets squeezed in the interval $1 \le p \le 2$ as $d \to +\infty$.

In view of the results of Theorem 1, it is natural to ask whether there is also a monotonicity property of the deficit associated to the Gaussian interpolation inequalities (2) when we rely on a nonlinear diffusion flow on $\mathbb{R}^n$. Let $m_\pm(p) := \lim_{d\to+\infty} m_\pm(d,p)$ and notice that

$$m_\pm(p) = 1 \pm \frac{1}{p}\sqrt{(p-1)\,(2-p)}\,. \tag{5}$$

The diffusion operator associated to the Gaussian measure is the *Ornstein-Uhlenbeck operator* $\mathscr{L} = \Delta - x\cdot\nabla$ and we consider now the nonlinear parabolic equation

$$\frac{\partial v}{\partial t} = v^{-p\,(1-m)}\left(\mathscr{L}\,v + (m\,p-1)\,\frac{|\nabla v|^2}{v}\right). \tag{6}$$

In the definition of $\mathscr{L}$, Δ denotes the standard Laplacian on $\mathbb{R}^n$. The following result is new for $m \neq 1$ while the case $m = 1$ follows from the method of the *carré du champ* developed by Bakry and Emery in [4].

Theorem 3. *Assume that $n \geq 1$, $p \in [1,2)$. If $v > 0$ solves (6) with $m \in [m_-(p), m_+(p)]$ for an initial datum in $\mathrm{L}^2 \cap \mathrm{L}^p(\mathbb{R}^n, d\gamma)$, then*

$$\frac{d}{dt}\left(\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \frac{1}{p-2}\left(\|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)} - \|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\right)\right) \leq 0 \quad \forall\, t > 0\,.$$

The limiting case $p = 2$ corresponding to the Gaussian logarithmic Sobolev inequality is also covered but it is obtained as a standard application of the linear *carré du champ* method known from [4] because $m_\pm(2) = 1$. See Fig. 2.

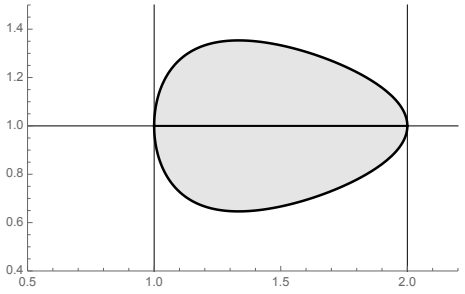

Figure 2. The admissible parameters p and m correspond to the grey area and are independent of the dimension n. The boundary of the admissible set is tangent to the vertical lines $p = 1$ at $(m,p) = (1,1)$ and $p = 2$ at $(m,p) = (1,2)$. It is the limit set of the admissible parameters for Proposition 2 as $d \to +\infty$.

Said in simple words, the result of Theorem 3 is that the admissible range of exponents of the nonlinear flow, for which the deficit associated to (2) is monotone non-increasing, is obtained as the limit of the range of the corresponding exponents on the sphere, in the large dimensions limit. Moreover, $p = 2$ appears as a *critical exponent* for the Gaussian measure.

Let us now focus on stability results. The main result of [17] for the sphere is a constructive stability estimate for Inequality (1), limited to the subcritical range $p \in (1, 2^*)$, which measures the distance to optimal functions, and distinguishes the subspaces generated by constant functions, spherical harmonic functions associated to the first positive eigenvalue of the Laplace-Beltrami operator, and the orthogonal directions. Optimal exponents in the stability estimate measuring the distance to the set of optimal functions differ, depending on the directions. Here we have the exact counterpart in the Gaussian case. Let Π_1 denote the orthogonal projection of $\mathrm{L}^2(\mathbb{R}^n, d\gamma)$ onto the $(n+1)$-dimensional function space generated by 1 and x_i with $i = 1, 2, \ldots, n$.

Theorem 4. *For all $n \ge 1$, and all $p \in (1,2)$, there is an explicit constant $c_{n,p} > 0$ such that, for all $v \in \mathrm{H}^1(d\gamma)$, it holds*

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \frac{1}{p-2}\left(\|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)} - \|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\right) \ge c_{n,p}\left(\|\nabla(\mathrm{Id}-\Pi_1)\,v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} + \frac{\|\nabla\Pi_1 v\|^4_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}{\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} + \|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right).$$

Exponents 2 and 4 which appear in the right-hand side of the inequality are sharp and the constant $c_{n,p}$ has an explicit although complicated expression given in the proof. If $p = 1$, the last term drops and the distance to optimal functions is measured only by

$$\|\nabla(\mathrm{Id}-\Pi_1)\,v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)},$$

and a decomposition on Hermite polynomials shows that the estimate cannot be improved. If $p = 2^*$ and $n \ge 3$, the recent stability result on Sobolev's inequality on $\mathbb{R}^n$ of [23], which quantifies the estimate of Bianchi and Egnell in [14] can be translated into a stability result for (1) on $\mathbb{S}^d$, that can be recast in the form of [17, Theorem 6]. By a large dimension argument, a stability on the Gaussian logarithmic Sobolev inequality is also shown in [23], although the distance is measured only by an $\mathrm{L}^2(\mathbb{R}^n,d\gamma)$ norm. Whether a stronger estimate can be obtained in the limiting case $p = 2$, eventually under some restriction, is therefore so far an open question.

This paper is organized as follows. In Section 2, we give a new proof of Inequality (2) as a consequence of Inequality (1) by taking a large dimensions limit, applied to the inequality written for a function depending only on a fixed number n of real variables. To our knowledge, this is new except for the limit case $p = 2$ of the logarithmic Sobolev inequality. Section 3 is devoted to the proof of Theorem 3: we characterize the nonlinear diffusion flows of porous medium or fast diffusion type such that the deficit is monotone non-increasing and recover the picture known on the sphere in the large dimensions limit. Moreover, by the *carré du champ* method, we establish improved inequalities that provide us with first stability results. The stability result of Theorem 4 for the Gaussian measure is proved in Section 4 using a detailed Taylor expansion and the improved inequalities of Section 3.

2. From subcritical interpolation inequalities on the sphere to Gaussian interpolation

In this section we explain how Inequality (2) can be seen as the limit of Inequality (1) in the large dimensions limit, that is, as $d \to +\infty$, and prove Theorem 1. Comments on the limit case $p = 2$ can be found at the end of this section.

The unit sphere $\mathbb{S}^d$ is parametrized in terms of the *stereographic coordinates* by

$$\omega_j = \frac{2\,x_j}{1+|x|^2} \quad \text{if} \quad 1 \le j \le d \quad \text{and} \quad \omega_{d+1} = \frac{1-|x|^2}{1+|x|^2}$$

where $\omega = (\omega_1,\omega_2,\dots,\omega_{d+1})$ denote the coordinates in $\mathbb{R}^{d+1} \supset \mathbb{S}^d$ and $x = (x_1,x_2,\dots,x_d)$ are Cartesian coordinates in $\mathbb{R}^d$. To a function u on $\mathbb{S}^d$, we associate a function w on $\mathbb{R}^d$ using the stereographic projection such that

$$\left(\frac{2}{1+|x|^2}\right)^{\frac{d-2}{2}} u(\omega) = w(x) \quad \forall\, x \in \mathbb{R}^d.$$

It is a standard result that

$$\int_{\mathbb{S}^d} |u|^p \, d\mu_d = 2^{\frac{\delta(p)}{2}} \left|\mathbb{S}^d\right|^{-1} \int_{\mathbb{R}^n} \langle x\rangle^{-\delta(p)} \, |w|^p \, dx$$

and that

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\mu_d + \tfrac{1}{4}\, d\,(d-2) \int_{\mathbb{S}^d} |u|^2 \, d\mu_d = \left|\mathbb{S}^d\right|^{-1} \int_{\mathbb{R}^n} |\nabla w|^2 \, dx .$$

where $\langle x \rangle := \sqrt{1+|x|^2}$ and $\delta(p) := 2\,d - p\,(d-2)$. Using the stereographic projection, Inequality (1) can be written on the Euclidean space $\mathbb{R}^d$ as the weighted interpolation inequality

$$\int_{\mathbb{R}^n} |\nabla w|^2 \, dx + \frac{d\,\delta(p)}{p-2} \int_{\mathbb{R}^n} \frac{|w|^2}{\langle x \rangle^4} \, dx \ge \frac{\mathscr{C}_{d,p}}{p-2} \left(\int_{\mathbb{R}^n} \frac{|w|^p}{\langle x \rangle^{\delta(p)}} \, dx \right)^{\frac{2}{p}} \quad \text{with} \quad \mathscr{C}_{d,p} = 2^{\frac{\delta(p)}{p}}\, d \left|\mathbb{S}^d\right|^{1-\frac{2}{p}}. \tag{7}$$

See [22] for details. Equality is achieved by the Aubin-Talenti function $w_\star(x) := \langle x \rangle^{2-d}$. Assume that $d \ge 4$. Let us consider $f = w / w_\star$ and notice that the inequality rewritten in terms of f is

$$\int_{\mathbb{R}^n} |\nabla f|^2 \, w_\star^2 \, dx + \frac{4\,d}{p-2} \int_{\mathbb{R}^n} |f|^2 \, w_\star^{2^*} \, dx \ge \frac{\mathscr{C}_{d,p}}{p-2} \left(\int_{\mathbb{R}^n} |f|^p \, w_\star^{2^*} \, dx \right)^{\frac{2}{p}} .$$

At this point, we may notice that for any $d \ge 3$, we can choose freely any $p \in [1,2) \cup (2,2^*]$ if $d \ge 3$. Since we are interested in the limit as $d \to +\infty$, in order to consider a fixed, given value of p, we have no choice but to restrict p to the case $1 \le p < 2$.

With $g(x) = f\big(x/\sqrt{d}\big)$, we obtain after changing variables that

$$\frac{1}{4} \int_{\mathbb{R}^d} |\nabla g|^2 \, \frac{dx}{\left(1+\frac{1}{d}\,|x|^2\right)^{d-2}} + \frac{1}{p-2} \int_{\mathbb{R}^d} |g|^2 \, \frac{dx}{\left(1+\frac{1}{d}\,|x|^2\right)^{d}} \ge \frac{\mathscr{C}_{d,p}\, d^{d\,\frac{p-2}{2p}}}{4\,d\,(p-2)} \left(\int_{\mathbb{R}^d} |g|^p \, \frac{dx}{\left(1+\frac{1}{d}\,|x|^2\right)^{d}} \right)^{\frac{2}{p}} .$$

Let us assume that $n \ge 1$ is a given integer and take $d > \max\{n,3\}$. With $x = (y,z) \in \mathbb{R}^n \times \mathbb{R}^{d-n} \approx \mathbb{R}^d$, we also assume that the function g depends only on y. In other words, we write $g = g_d$ where $g_d(y,z) = v(y)$ for some function v defined on $\mathbb{R}^n$, that is,

$$g_d(y,z) = v(y) \quad \forall\, (y,z) \in \mathbb{R}^n \times \mathbb{R}^{d-n} \tag{8}$$

Here we use an index d in order to emphasize that g_d has to be considered as a function on $\mathbb{R}^d$. Let us define

$$c_d := (d\,\pi)^{\frac{d}{2}}\, \frac{\Gamma(d/2)}{\Gamma(d)} .$$

Lemma 5. *Let n be a positive integer, $p \in [1,2)$, consider a function $v \in \mathrm{H}^1(\mathbb{R}^n, d\gamma)$ with compact support and define g_d according to* (8)*. Then we have*

$$\begin{aligned} &\lim_{d\to+\infty} \frac{1}{4\,c_d} \int_{\mathbb{R}^d} |\nabla g_d|^2 \, \frac{dx}{\left(1+\frac{1}{d}\,|x|^2\right)^{d-2}} \\ &\quad + \frac{1}{2-p} \lim_{d\to+\infty} \frac{1}{c_d} \left(\frac{\mathscr{C}_{d,p}\, d^{d\,\frac{p-2}{2p}}}{4\,d} \left(\int_{\mathbb{R}^d} |g_d|^p \, \frac{dx}{\left(1+\frac{1}{d}\,|x|^2\right)^{d}} \right)^{\frac{2}{p}} - \int_{\mathbb{R}^d} |g_d|^2 \, \frac{dx}{\left(1+\frac{1}{d}\,|x|^2\right)^{d}} \right) \\ &\qquad = \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n, d\gamma)} - \frac{1}{2-p} \left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n, d\gamma)} - \|v\|^2_{\mathrm{L}^p(\mathbb{R}^n, d\gamma)} \right) . \end{aligned}$$

In other words, we prove that the infinite dimensional limit of (7), for functions depending only on a finite number n of real variables, is (2). The assumption of compact support can be removed if g_d is square integrable with respect to the measure $\left(1+\frac{1}{d}\,|x|^2\right)^{-d} dx$ and ∇g_d is square integrable with respect to the measure $\left(1+\frac{1}{d}\,|x|^2\right)^{2-d} dx$, at least for some $d \in \mathbb{N}$ large enough.

Proof. Using

$$\left(1+\tfrac{1}{d}\,|x|^2\right)^{2-d} = \left(1+\tfrac{1}{d}\left(|y|^2+|z|^2\right)\right)^{2-d} = \left(1+\tfrac{1}{d}\,|y|^2\right)^{2-d} \left(1+\tfrac{1}{d}\,|\zeta|^2\right)^{2-d} \quad \text{with} \quad \zeta = \frac{z}{\sqrt{1+\frac{1}{d}\,|y|^2}} ,$$

we can integrate with respect to z and obtain

$$\int_{\mathbb{R}^{d-n}} \left(1+\tfrac{1}{d}\,|x|^2\right)^{2-d}\,dz = \left(1+\tfrac{1}{d}\,|y|^2\right)^{2-\frac{d+n}{2}} \int_{\mathbb{R}^{d-n}} \left(1+\tfrac{1}{d}\,|\zeta|^2\right)^{2-d}\,d\zeta\,.$$

Let $d\gamma(y) := (2\pi)^{-n/2}\,e^{-\frac{1}{2}\,|y|^2}\,dy$ be the centred normalized Gaussian probability measure. We recall that $|\mathbb{S}^{k-1}| = 2\,\pi^{k/2}/\Gamma(k/2)$ for any $k \in \mathbb{N}\setminus\{0\}$ and

$$\int_0^{+\infty} r^{a-1}\left(1+\tfrac{1}{d}\,r^2\right)^{-b}\,dr = d^{\frac{a}{2}}\,\frac{\Gamma\left(\frac{a}{2}\right)\Gamma\left(b-\frac{a}{2}\right)}{2\,\Gamma(b)}$$

if $0 < a < 2\,b$. Applying these formulas with $a = k = d-n$ and $b = d-2 > 2-n$, we find that

$$\int_{\mathbb{R}^{d-n}} \left(1+\tfrac{1}{d}\,|\zeta|^2\right)^{2-d}\,d\zeta = (d\,\pi)^{\frac{d-n}{2}}\,\frac{\Gamma\left(\frac{d+n}{2}-2\right)}{\Gamma(d-2)}\,.$$

Applying these formulas with $a = b = k = d \geq 2$, we find that

$$\int_{\mathbb{R}^n} \left(1+\tfrac{1}{d}\,|x|^2\right)^{-d}\,dx = c_d$$

and, as a consequence

$$\lim_{d\to+\infty} \frac{1}{c_d} \int_{\mathbb{R}^{d-n}} \left(1+\tfrac{1}{d}\,|\zeta|^2\right)^{2-d}\,d\zeta = \frac{4}{(2\,\pi)^{n/2}}$$

using Stirling's formula. Since

$$\lim_{d\to+\infty} \left(1+\tfrac{1}{d}\,|y|^2\right)^{2-\frac{d+n}{2}} = e^{-\frac{1}{2}\,|y|^2}\,,$$

we obtain

$$\lim_{d\to+\infty} \frac{1}{c_d} \int_{\mathbb{R}^n} |\nabla g_d(y)|^2 \left(1+\tfrac{1}{d}\,|x|^2\right)^{2-d}\,dx = 4\int_{\mathbb{R}^n} |\nabla v|^2\,d\gamma\,.$$

Similar computations show that

$$\int_{\mathbb{R}^n} |g_d(y)|^2 \left(1+\tfrac{1}{d}\,|x|^2\right)^{-d}\,dx = \int_{\mathbb{R}^n} |g_d(y)|^2 \left(1+\tfrac{1}{d}\,|y|^2\right)^{-\frac{d+n}{2}}\,dy \int_{\mathbb{R}^{d-n}} \left(1+\tfrac{1}{d}\,|\zeta|^2\right)^{-d}\,d\zeta\,,$$

$$\lim_{d\to+\infty} \frac{1}{c_d} \int_{\mathbb{R}^n} |g_d(y)|^2 \left(1+\tfrac{1}{d}\,|x|^2\right)^{-d}\,dx = \int_{\mathbb{R}^n} |v|^2\,d\gamma\,,$$

and

$$\int_{\mathbb{R}^n} |g_d(y)|^p \left(1+\tfrac{1}{d}\,|x|^2\right)^{-d}\,dx = \int_{\mathbb{R}^n} |g_d(y)|^p \left(1+\tfrac{1}{d}\,|y|^2\right)^{-\frac{d+n}{2}}\,dy \int_{\mathbb{R}^{d-n}} \left(1+\tfrac{1}{d}\,|\zeta|^2\right)^{-d}\,d\zeta\,,$$

$$\lim_{d\to+\infty} \frac{\mathscr{C}_{d,p}\,d^{d\,\frac{p-2}{2p}}}{4\,d\,c_d} \left(\int_{\mathbb{R}^n} |g_d(y)|^p \left(1+\tfrac{1}{d}\,|x|^2\right)^{-d}\,dx\right)^{\frac{2}{p}} = \left(\int_{\mathbb{R}^n} |v|^p\,d\gamma\right)^{\frac{2}{p}}\,.$$

This completes the proof of Lemma 5. □

Proof of Theorem 1. Applied to the function u_d, Inequality (1) is transformed into Inequality (7) applied to

$$g_d(x) = u_d\left(\frac{2\,y}{1+\frac{1}{d}\,|x|^2}\right) \quad \forall\, x = (y,z) \in \mathbb{R}^n \times \mathbb{R}^{d-n}\,.$$

Since the right-hand side uniformly converges to $v(y)$ for any smooth and compactly supported function v, the same conclusion holds for Theorem 1 as for Lemma 5. □

It is a natural question to ask what happens in (1) to the marginals depending only on a finite number n of variables if $p = 2$ or in the case $2 < p \leq 2^* = 2\,d/(d-2)$. We may notice that $\lim_{d\to+\infty} 2\,d/(d-2) = 2$ and it is known, for instance from [9], that one recovers the *Gaussian logarithmic Sobolev inequality* as a limit case of Sobolev's inequality on $\mathbb{S}^d$ corresponding to $p = 2^*$ when $d \to +\infty$. This is also true if we consider a sequence $(p_d)_{d\in\mathbb{N}}$ with $1 < p_d < 2^*$,

depending on d, if its limit is also 2, as shown next. By convention, when $p_d = 2$, we consider the Gaussian logarithmic Sobolev inequality instead of (2).

Proposition 6. *Let n be a positive integer and consider a function $v \in \mathrm{H}^1(\mathbb{R}^n, d\gamma)$ with compact support. For any $d \ge n$, large enough, let $u_d(\omega) = v\big(\omega_1/\sqrt{d}, \omega_2/\sqrt{d}, \ldots, \omega_n/\sqrt{d}\big)$ where $\omega = (\omega_1, \omega_2, \ldots, \omega_d) \in \mathbb{R}^{d+1} \supset \mathbb{S}^d$ is such that $|\omega| = 1$, as in Theorem 1. Then we have*

$$\lim_{d\to+\infty} d\left(\|\nabla u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \frac{d}{2}\int_{\mathbb{S}^d} |u_d|^2 \log\left(\frac{|u_d|^2}{\|u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu_d\right)$$
$$= \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \frac{1}{2}\int_{\mathbb{R}^n} |v|^2 \log\left(\frac{|v|^2}{\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right) d\gamma\,.$$

If $(p_d)_{d\ge3}$ is a sequence of real numbers such that $p_d \in (1,2)\cup(2,2^*)$ and $\lim_{d\to+\infty} p_d = 2$, then

$$\lim_{d\to+\infty} d\left(\|\nabla u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \frac{d}{2-p_d}\left(\|u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \|u_d\|^2_{\mathrm{L}^{p_d}(\mathbb{S}^d,d\mu_d)}\right)\right)$$
$$= \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \frac{1}{2}\int_{\mathbb{R}^n} |v|^2 \log\left(\frac{|v|^2}{\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right) d\gamma\,.$$

Proof. The proof is an adaptation of the proof of Theorem 1 and, in the case $p_d \neq 2$, relies on the standard observation that

$$\lim_{p\to2} \frac{\|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)} - \|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}{p-2} = \frac{1}{2}\int_{\mathbb{R}^n} |v|^2 \log\left(\frac{|v|^2}{\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right) d\gamma\,.$$

As this computation raises no special difficulty, details are omitted. □

3. Entropy methods and nonlinear flows for Gaussian measures

In this section, we prove the result of Theorem 3 for the Gaussian measure $d\gamma$ and extend it to the slightly more general framework of a uniformly strictly log-concave measure $d\mu$, before drawing some consequences. Most of the results are similar to computations usually done on the sphere, but we are not aware of the use of nonlinear flows ($m \neq 1$) in the context of Gaussian measures. This approach is very natural in the perspective of spheres in the large-dimensional limit.

3.1. *Gaussian interpolation inequalities: a proof by the* carré du champ *method*

On $\mathbb{R}^n$, let us consider the probability measure

$$d\mu = Z^{-1}\, e^{-\phi}\, dy \quad \text{with} \quad Z = \int_{\mathbb{R}^n} e^{-\phi}\, dy \tag{9}$$

and redefine the *Ornstein-Uhlenbeck operator* by

$$\mathscr{L} := \Delta - \nabla\phi\cdot\nabla \tag{10}$$

on $\mathrm{L}^2(\mathbb{R}^n, d\mu)$. This generalizes the case of the harmonic potential $\phi(y) = \frac{1}{2}|y|^2$ considered in the introduction. We assume that ϕ satisfies the *Bakry-Emery condition*

$$\mathrm{Hess}\,\phi \ge \lambda_\star\,\mathrm{Id} \quad \text{a.e.} \tag{11}$$

for some $\lambda_\star > 0$. The harmonic potential corresponds to the equality case with $\lambda_\star = 1$. Under Assumption (11), it is well known (see for instance [7, Section 7.6.2]) that, with $\lambda = \lambda_\star$ and for any $p \in [1,2)$,

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)} \ge \frac{\lambda}{2-p}\left(\|f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)} - \|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\mu)}\right) \quad \forall\, f \in \mathrm{H}^1(\mathbb{R}^n, d\mu) \tag{12}$$

and also, by taking the limit as $p \to 2_-$, that

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)} \ge \frac{\lambda}{2}\int_{\mathbb{R}^n} |f|^2 \log\left(\frac{|f|^2}{\|f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)}}\right) d\mu \quad \forall\, f \in \mathrm{H}^1(\mathbb{R}^n,d\mu)\,.$$

The classical proof by the carré du champ, as in [3–5], relies on the *Ornstein-Uhlenbeck flow* $\partial_t \rho = \mathscr{L}\rho$ applied to the solution with initial datum $\rho(t=0,\cdot) = |f|^p$. Here we consider a more general strategy and compute as in the carré du champ method using the *nonlinear diffusion flow*

$$\frac{\partial \rho}{\partial t} = \frac{1}{m}\,\mathscr{L}\rho^m \quad \forall\, t \ge 0\,, \quad \rho(t=0,\cdot) = |f|^p\,. \tag{13}$$

Our goal is to understand the range of m for which we have a monotonicity property of the deficit as in the case $m = 1$. Let $m_\pm(p)$ be defined as in (5).

Theorem 7. *Assume that $n \ge 1$, $p \in [1,2)$ and $m \in [m_-(p), m_+(p)]$. We consider the measure $d\mu$ as in* (9) *such that* (11) *holds for some $\lambda_\star > 0$. If $\rho > 0$ solves* (13) *with $\mathscr{L}$ defined by* (10) *for an initial datum $\rho(t=0,\cdot) = |f|^p$ in $\mathrm{L}^{2/p} \cap \mathrm{L}^1(\mathbb{R}^n,d\mu)$, then*

$$\frac{d}{dt}\left(\|\nabla \rho^{1/p}\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)} - \frac{\lambda_\star}{2-p}\left(\|\rho^{1/p}\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)} - \|\rho\|^{2/p}_{\mathrm{L}^1(\mathbb{R}^n,d\mu)}\right)\right) \le 0 \quad \forall\, t > 0\,.$$

The result of Theorem 7 is new for $m \neq 1$. Theorem 3 corresponds to the special case of the harmonic potential $\phi(y) = \frac{1}{2}|y|^2$ with $v = \rho^{1/p}$ in Theorem 7. The range of the admissible parameters (m,p) is the same in Theorem 7 and shown in Fig. 2. With the additional observation that $\rho(t,\cdot)$ converges to a constant as $t \to +\infty$ so that the limit of the deficit is 0, the monotonicity of the deficit of Theorem 7 provides us with a proof of (12).

Proof of Theorem 7. In order to do computations, a very convenient reformulation is obtained with the flow

$$\frac{\partial w}{\partial t} = w^{2-2\beta}\left(\mathscr{L} w + \kappa\,\frac{|\nabla w|^2}{w}\right) \tag{14}$$

for any $t \ge 0$, with initial datum $w(t=0,\cdot) = |f|^{1/\beta}$, where

$$\kappa := \beta\,(p-2) + 1 \quad \text{and} \quad \beta = \frac{2}{2 - p\,(1-m)}\,.$$

A first computation shows that $\int_{\mathbb{R}^n} w^{\beta p}\,d\mu = \int_{\mathbb{R}^n} |f|^p\,d\mu$ is independent of t because

$$\frac{d}{dt}\int_{\mathbb{R}^n} w^{\beta p}\,d\mu = \beta\,p\int_{\mathbb{R}^n} w^\kappa\left(\mathscr{L} w + \kappa\,\frac{|\nabla w|^2}{w}\right) d\mu = 0\,.$$

A second useful computation goes as follows:

$$\begin{aligned}
&-\frac{1}{2\,\beta^2}\,\frac{d}{dt}\int_{\mathbb{R}^n}\left(\left|\nabla w^\beta\right|^2 + \frac{\lambda_\star}{p-2}\,w^{2\beta}\right) d\mu \\
&\quad= \int_{\mathbb{R}^n}\left(\mathscr{L} w + (\beta-1)\,\frac{|\nabla w|^2}{w} - \frac{\lambda_\star\, w}{\beta\,(p-2)}\right)\left(\mathscr{L} w + \kappa\,\frac{|\nabla w|^2}{w}\right) d\mu \\
&\quad= \int_{\mathbb{R}^n} (\mathscr{L} w)^2\,d\mu + (\kappa+\beta-1)\int_{\mathbb{R}^n} (\mathscr{L} w)\,\frac{|\nabla w|^2}{w}\,d\mu + \kappa\,(\beta-1)\int_{\mathbb{R}^n}\frac{|\nabla w|^4}{w^2}\,d\mu - \lambda_\star\int_{\mathbb{R}^n}|\nabla w|^2\,d\mu\,.
\end{aligned}$$

The purely technical purpose of introducing the exponent β is to make the last line of the above computation 2-homogeneous in w, which makes the discussion easier to read. Inserting the two following estimates,

$$\begin{aligned}\int_{\mathbb{R}^n} (\mathscr{L} w)^2 \, d\mu &= -\int_{\mathbb{R}^n} \nabla w \cdot \nabla(\mathscr{L} w) \, d\mu = -\int_{\mathbb{R}^n} \nabla w \cdot (\mathscr{L} \nabla w) \, d\mu + \int_{\mathbb{R}^n} \nabla w \cdot [\mathscr{L}, \nabla] \, w \, d\mu \\ &= \int_{\mathbb{R}^n} \|\mathrm{Hess}\, w\|^2 \, d\mu + \int_{\mathbb{R}^n} \nabla w \cdot [\mathscr{L}, \nabla] \, w \, d\mu \\ &= \int_{\mathbb{R}^n} \|\mathrm{Hess}\, w\|^2 \, d\mu + \int_{\mathbb{R}^n} \mathrm{Hess}\, \phi : \nabla w \otimes \nabla w \, d\mu \\ &\geq \int_{\mathbb{R}^n} \|\mathrm{Hess}\, w\|^2 \, d\mu + \lambda_\star \int_{\mathbb{R}^n} |\nabla w|^2 \, d\mu\end{aligned}$$

and

$$\int_{\mathbb{R}^n} (\mathscr{L} w) \, \frac{|\nabla w|^2}{w} \, d\mu = -2 \int_{\mathbb{R}^n} \mathrm{Hess}\, w : \frac{\nabla w \otimes \nabla w}{w} \, d\mu + \int_{\mathbb{R}^n} \frac{|\nabla w|^4}{w^2} \, d\mu ,$$

we obtain that

$$\frac{d}{dt} \int_{\mathbb{R}^n} \left(\left| \nabla w^\beta \right|^2 + \frac{\lambda_\star}{p-2} \, w^{2\beta} \right) d\mu \leq 0$$

if, for any function w, we have

$$\begin{aligned}\mathscr{Q}_\beta[w] := \int_{\mathbb{R}^n} \|\mathrm{Hess}\, w\|^2 \, d\mu - 2\,(\kappa + \beta - 1) \int_{\mathbb{R}^n} \mathrm{Hess}\, w : \frac{\nabla w \otimes \nabla w}{w} \, d\mu \\ + \big(\kappa\,(\beta-1) + \kappa + \beta - 1\big) \int_{\mathbb{R}^n} \frac{|\nabla w|^4}{w^2} \, d\mu \geq 0 .\end{aligned}$$

A sufficient condition is obtained if the reduced discriminant is negative, that is, if

$$(\kappa + \beta - 1)^2 - \big(\kappa\,(\beta-1) + \kappa + \beta - 1\big) \leq 0 .$$

Altogether, this gives the condition

$$\beta_-(p) \leq \beta \leq \beta_+(p) \quad \text{with} \quad \beta_\pm(p) := \frac{1 \pm \sqrt{(p-1)\,(2-p)}}{1 - (p-1)\,(2-p)} . \tag{15}$$

See Fig. 3.

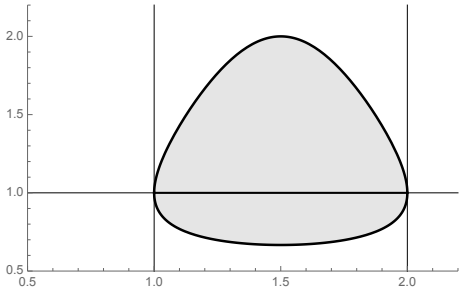

Figure 3. The admissible parameters p and β correspond to the grey area and are independent of the dimension n. The boundary of the admissible set is tangent to the vertical lines $p = 1$ at $(\beta, p) = (1, 1)$ and $p = 2$ at $(\beta, p) = (1, 2)$. This figure corresponds to Fig. 2 up to the transformation of $m \to \beta = 2/\big(2 - p\,(1-m)\big)$.

Equivalently, written in terms of m, the condition is $m_-(p) \leq m \leq m_+(p)$ with $m_\pm(p)$ defined by (5). Summarizing our computations, we learn that

$$\frac{d}{dt} \left(\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^n, d\mu)} - \frac{\lambda_\star}{p-2} \left(\|w\|^2_{\mathrm{L}^p(\mathbb{R}^n, d\mu)} - \|w\|^2_{\mathrm{L}^2(\mathbb{R}^n, d\mu)} \right) \right) = -2\,\beta^2\, \mathscr{Q}_\beta[w] \leq 0$$

because $\mathscr{Q}_\beta[w]$ is non-negative under Condition (15).

To make these computations rigorous, one has to justify all integrations by parts. This is by now rather standard and can be done using the following scheme.

(1) Let $h>0$ be large enough and consider $\Omega_h:=\{x\in\mathbb{R}^n:\phi(x)<h\}$. We can consider the evolution equation restricted to Ω_h, with no flux boundary conditions. Then apply the *carré du champ* method and keep track of the boundary terms. Since Ω_h is a bounded convex domain, these terms have a sign due to Grisvard's lemma: see for instance [27, Lemma 5.2], [36, Proposition 4.2], [21, 29] and [31, Lemma A.3].

(2) Extend Inequality (12) written on Ω_h to $\mathbb{R}^d$ by taking the limit as $h\to+\infty$ and then argue by density.

This completes the proof of Theorem 7. □

Remark 8. For any $p\in[1,2)$, if the condition $m_-(p)\le m\le m_+(p)$ is not satisfied, then one can find a positive initial datum such that the solution v of (6) is such that

$$\frac{d}{dt}\left(\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)}-\frac{d}{p-2}\left(\|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\mu)}-\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)}\right)\right)_{|t=0}>0\,.$$

See [25] for a similar statement on the sphere and its proof.

3.2. *Improved inequalities based on the* carré du champ *method*

In the proof of Theorem 7, using only $\mathscr{Q}_\beta[w]\ge0$ is a crude estimate. Let us explain how one can obtain improved estimates by making a better use of $\mathscr{Q}_\beta[w]\ge0$. Under Condition (15), we can indeed rewrite $\mathscr{Q}_\beta[w]$ as an integral of a sum of squares,

$$\mathscr{Q}_\beta[w]=\int_{\mathbb{R}^n}\left\|\mathrm{Hess}\,w-(\kappa+\beta-1)\,\frac{\nabla w\otimes\nabla w}{w}\right\|^2d\mu+\delta\int_{\mathbb{R}^n}\frac{|\nabla w|^4}{w^2}\,d\mu \tag{16}$$

with

$$\delta:=\kappa\,(\beta-1)+\kappa+\beta-1-(\kappa+\beta-1)^2=\big(\beta-\beta_-(p)\big)\big(\beta_+(p)-\beta\big)>0\,. \tag{17}$$

As in [2, Theorem 2], let us consider the special case $m=\beta=1$ of the linear flow. Let us define the *entropy* and the *Fisher information* by

$$\mathscr{E}[w]:=\frac{1}{2-p}\left(\|w\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)}-\|w\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\mu)}\right)\quad\text{and}\quad\mathscr{I}[w]:=\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)}\,.$$

Inequality (12) amounts simply to

$$\mathscr{I}[w]-\lambda_\star\,\mathscr{E}[w]\ge0\,.$$

We can now state a first *improved entropy–entropy production inequality*.

Proposition 9. *Let n be any positive integer. We consider the measure $d\mu$ as in* (9) *such that* (11) *holds for some $\lambda_\star>0$. For any $p\in(1,2)$, let $\varphi(s):=1+s-(1+s)^{p-1}$. With the above notations, we have*

$$\mathscr{I}[f]\ge\frac{\lambda_\star}{(2-p)^2}\,\|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\mu)}\,\varphi\left(\frac{(2-p)\,\mathscr{E}[f]}{\|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\mu)}}\right)\quad\forall\,f\in\mathrm{H}^1(\mathbb{R}^n,d\mu)\,. \tag{18}$$

Since $p\in(1,2)$, the function φ is convex with $\varphi(0)=0$ and $\varphi'(0)=2-p$, which implies in particular that $\varphi(s)\ge(2-p)\,s$ for any $s\ge0$, so that Inequality (18) is stronger than (12). Inequality (18) amounts to

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)}\ge\frac{\lambda_\star}{(2-p)^2}\left(\|f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)}-\|f\|^{2\,(p-1)}_{\mathrm{L}^2(\mathbb{R}^n,d\mu)}\,\|f\|^{2\,(2-p)}_{\mathrm{L}^p(\mathbb{R}^n,d\mu)}\right)$$

and can also be rewritten as

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)} - \frac{\lambda_\star}{2-p}\left(\|f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)} - \|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\mu)}\right)$$
$$\geq \frac{\lambda_\star}{(2-p)^2}\left((p-1)\,\|f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)} + (2-p)\,\|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\mu)} - \|f\|^{2\,(p-1)}_{\mathrm{L}^2(\mathbb{R}^n,d\mu)}\,\|f\|^{2\,(2-p)}_{\mathrm{L}^p(\mathbb{R}^n,d\mu)}\right). \quad (19)$$

We claim no originality in either Proposition 9 or Inequality (19) and refer to [2, Theorem 2] and [1, Ineq. (3.3)] for earlier results. Also see [22, Theorem 2.1] for an improved inequality like (19) in the case of the sphere. However, let us give a proof for completeness.

Proof. With $\beta = 1$, notice that (16) holds with $\delta = (2-p)\,(p-1)$. Using the Cauchy-Schwarz inequality, we obtain

$$\big(\mathscr{I}[w]\big)^2 = \left(\int_{\mathbb{R}^n} |\nabla w|^2\,d\mu\right)^2 \leq \int_{\mathbb{R}^n} |w|^2\,d\mu \int_{\mathbb{R}^n} \frac{|\nabla w|^4}{w^2}\,d\mu$$

and, with $M := \big(\int_{\mathbb{R}^n} |w|^p\,d\mu\big)^{2/p}$, we can also write that

$$\int_{\mathbb{R}^n} |w|^2\,d\mu = (2-p)\,\mathscr{E}[w] + M\,.$$

Altogether, with $\mathsf{e}(t) := (2-p)\,M^{-1}\,\mathscr{E}[w(t,\cdot)]$, if w solves (14) with $\beta = 1$, then we have the differential inequality

$$\mathsf{e}'' + 2\,\lambda_\star\,\mathsf{e}' - (p-1)\,\frac{(\mathsf{e}')^2}{1+\mathsf{e}} \geq 0\,.$$

We claim that $(2-p)\,\mathsf{e}' + 2\,\lambda_\star\left(1+\mathsf{e}-(1+\mathsf{e})^{p-1}\right) \leq 0$, which follows from the observation that the equation

$$y'' + \mathsf{a}\,y' - \mathsf{b}\,\frac{(y')^2}{1+y} = 0$$

can be solved using the ansatz $y' = \frac{\mathsf{a}}{\mathsf{b}-1}\,\varphi(y)$ with $\varphi(0) = 0$ if

$$\varphi' - \mathsf{b}\,\frac{\varphi}{1+s} = 1-\mathsf{b}\,. \quad (20)$$

It is straightforward to check that the unique solution is $\varphi(s) = 1+s-(1+s)^{\mathsf{b}}$. With $\mathsf{a} = 2\,\lambda_\star$ and $\mathsf{b} = p-1$, we obtain

$$\left(\mathsf{e}' + \frac{2\,\lambda_\star}{2-p}\,\varphi(\mathsf{e})\right)' \geq (p-1)\,\frac{\mathsf{e}'}{1+\mathsf{e}}\left(\mathsf{e}' + \frac{2\,\lambda_\star}{2-p}\,\varphi(\mathsf{e})\right).$$

By integrating from $t=0$ to $+\infty$ using the facts that $\mathsf{e}' \leq 0$, $\mathsf{e}' + \frac{2\,\lambda_\star}{2-p}\,\mathsf{e} \leq 0$ and $\lim_{t\to+\infty} \mathsf{e}'(t) = \lim_{t\to+\infty} \mathsf{e}(t) = 0$, we conclude with $M = \|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\mu)}$ that

$$\frac{2-p}{M}\left(\mathscr{I}[w] - \frac{\lambda_\star\,M}{(2-p)^2}\,\varphi\left(\frac{(2-p)\,\mathscr{E}[w]}{M}\right)\right) = -\,\frac{1}{2}\left(\mathsf{e}' + \frac{2\,\lambda_\star}{2-p}\,\varphi(\mathsf{e})\right) \geq 0 \quad \forall\, t \geq 0$$

See [17, Appendix B.4] for a more detailed proof in the similar case of the sphere. □

Inspired once more by the results on the sphere (see [22] and earlier references therein), it is a very natural question to wonder if *improved entropy–entropy production inequalities* can be achieved with $\beta \neq 1$. The answer goes as follows. Let us consider the function φ_β given by

$$\chi_\beta(s) = \big(1-\mathsf{b}(\beta)\big)\left(1+s-(1+s)^{\mathsf{b}(\beta)}\right) \quad \text{where} \quad \mathsf{b}(\beta) = \frac{\delta(\beta)}{\beta^2}\,\frac{2-p}{\lambda_\star}$$

and $\delta(\beta)$ is defined by (17).

Proposition 10. *Let n be any positive integer. We consider the measure $d\mu$ as in* (9) *such that* (11) *holds for some $\lambda_\star > 0$. For any $p \in (1,2)$, take $\beta \in \big[1, \beta_+(p)\big]$. With the same notations as in Proposition 9 and χ_β as above, we have*

$$\|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\mu)}\,\chi_\beta\left(\frac{(2-p)\,\mathscr{E}[f]}{\|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\mu)}}\right)\le \mathscr{I}[f] \quad \forall\, f\in \mathrm{H}^1(\mathbb{R}^n,d\mu)\,. \tag{21}$$

Proof. With the notations

$$\mathsf{e}:=\frac{\big\|w^\beta\big\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)}}{\big\|w^\beta\big\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\mu)}} \quad\text{and}\quad \mathsf{i}:=\frac{\big\|\nabla\big(w^\beta\big)\big\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)}}{\big\|w^\beta\big\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\mu)}}\,,$$

we have now to consider the differential inequality

$$\left(\mathsf{i}'-\frac{\lambda_\star}{2-p}\,\mathsf{e}\right)'\le-\delta\,\beta^2\int_{\mathbb{R}^n}\frac{|\nabla w|^4}{w^2}\,d\mu\,.$$

The key ingredient is to replace an estimate due to Demange in the case of the sphere for $p>2$ (see [19,20] and also [Lemma 15, (iii)]) by its counterpart for log-concave measures and $p\in(1,2)$. Compared to the linear case, the Cauchy-Schwarz inequality in the proof of Proposition 9 has to be replaced by the Hölder inequalities

$$\int_{\mathbb{R}^n}|\nabla w|^2\,d\mu=\int_{\mathbb{R}^n}\left(\frac{|\nabla w|^2}{w}\,w\,1\right)d\mu\le\left(\int_{\mathbb{R}^n}\frac{|\nabla w|^4}{w^2}\,d\mu\right)^{\frac12}\left(\int_{\mathbb{R}^n}|w|^{2\,\beta}\,d\mu\right)^{\frac{1}{2\beta}}1^{\frac{\beta-1}{2\beta}}\,,$$

$$\frac{1}{\beta^2}\int_{\mathbb{R}^n}\left|\nabla w^\beta\right|^2d\mu=\int_{\mathbb{R}^n}\left(\frac{|\nabla w|^2}{w}\,w^{2\,\beta-1}\,1\right)d\mu\le\left(\int_{\mathbb{R}^n}\frac{|\nabla w|^4}{w^2}\,d\mu\right)^{\frac12}\left(\int_{\mathbb{R}^n}|w|^{2\,\beta}\,d\mu\right)^{\frac{2\beta-1}{2\beta}}1^{\frac{1}{2\beta}}\,,$$

after observing that $\beta\ge\beta_-(p)\ge\beta_-(3/2)=2/3>1/2$, so that

$$\int_{\mathbb{R}^n}\frac{|\nabla w|^4}{w^2}\,d\mu\ge\frac{\int_{\mathbb{R}^n}|\nabla w|^2\,d\mu\,\int_{\mathbb{R}^n}\left|\nabla w^\beta\right|^2d\mu}{\beta^2\int_{\mathbb{R}^n}w^{2\beta}\,d\mu}=-\,\frac{\mathsf{i}\,\mathsf{e}'}{\beta^4\,(1+\mathsf{e})}\,.$$

Hence

$$\left(\mathsf{i}-\frac{\lambda_\star}{2-p}\,\mathsf{e}\right)'\le\frac{\delta}{\beta^2}\,\frac{\mathsf{i}\,\mathsf{e}'}{1+\mathsf{e}}\,.$$

Let us compute

$$\left(\mathsf{i}-\frac{\lambda_\star}{2-p}\,\chi(\mathsf{e})\right)'=\left(\mathsf{i}-\frac{\lambda_\star}{2-p}\,\mathsf{e}\right)'+\frac{\lambda_\star}{2-p}\,\big(\mathsf{e}-\chi(\mathsf{e})\big)'\le\frac{\delta}{\beta^2}\,\frac{\mathsf{e}'}{1+\mathsf{e}}\left(\mathsf{i}-\frac{\lambda_\star}{2-p}\,\chi(\mathsf{e})\right)$$

on the condition that χ solves

$$\chi'(s)=1+\frac{\delta}{\beta^2}\,\frac{2-p}{\lambda_\star}\,\frac{\chi(s)}{1+s}\,,\quad\chi(0)=0\,.$$

The solution $\chi(s)=(1-\mathsf{b})\,\varphi(s)$ is such that φ solves (20) with $\mathsf{b}=\delta\,\beta^{-2}\,(2-p)/\lambda_\star$, which shows that $\chi=\chi_\beta$. The proof then follows from the same considerations as for Proposition 9 (also see [17, Appendix B.4] for details). □

If $\phi(y)=|y|^2/2$ is the harmonic potential so that $d\mu=d\gamma$, by testing (2) with $f_\varepsilon(y)=1+\varepsilon\,y_1$ where y_1 denotes the first coordinate of $y=(y_1,y_2,\dots,y_n)\in\mathbb{R}^n$, we find that

$$\|\nabla f_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}-\frac{1}{2-p}\left(\|f_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}-\|f_\varepsilon\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right)=\frac12\,(p-1)\,\varepsilon^4+O(\varepsilon^5)\quad\text{as}\quad\varepsilon\to0\,. \tag{22}$$

This is the standard computation for checking that $\lambda=\lambda_\star=1$ is the optimal constant in (2). Since

$$\mathscr{I}[f_\varepsilon]-\frac{1}{(2-p)^2}\,\|f_\varepsilon\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\,\varphi\left(\frac{(2-p)\,\mathscr{E}[f_\varepsilon]}{\|f_\varepsilon\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}}\right)=\frac12\,(p-1)^2\,\varepsilon^4+O(\varepsilon^5)\quad\text{as}\quad\varepsilon\to0\,,$$

we also learn that (18) involves the optimal exponent at least in the limit as $\varepsilon \to 0$. After observing that $\|\nabla f_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} = \varepsilon^2$, we may wonder whether the *deficit*

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \frac{1}{2-p}\left(\|f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right)$$

measures the distance in terms of $\|\nabla f\|^4_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}$. The detailed answer is not limited to the case $\mu = \gamma$ and goes as follows. For simplicity, we take $\beta = 1$ and consider $\varphi(s) := 1 + s - (1+s)^{p-1}$ as in Proposition 9. We recall that φ is monotone increasing and convex on $\mathbb{R}^+$, such that $\varphi'(0) = 2-p$, hence invertible of inverse φ^{-1} such that $\psi(t) := t - (2-p)\,\varphi^{-1}(t)$ is also a convex, non-negative, monotone increasing function.

Corollary 11. *Let $p \in (1,2)$ and n be a positive integer. We consider the measure $d\mu$ as in* (9) *such that* (11) *holds for some $\lambda_\star > 0$. With ψ as above, for any $f \in \mathrm{H}^1(d\gamma)$ we have*

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)} - \frac{\lambda_\star}{2-p}\left(\|f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)} - \|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\mu)}\right) \ge \frac{\lambda_\star}{2-p}\,\|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\mu)}\,\psi\left(\frac{2-p}{\lambda_\star}\,\frac{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)}}{\|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\mu)}}\right).$$

Moreover, there is some $\kappa > 0$ such that

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)} - \frac{\lambda_\star}{2-p}\left(\|f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)} - \|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\mu)}\right) \ge \frac{\kappa\,\|\nabla f\|^4_{\mathrm{L}^2(\mathbb{R}^n,d\mu)}}{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)} + \frac{\lambda_\star}{2-p}\,\|f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)}}.$$

The constant κ depends only on p and its value is estimated in the proof below.

Proof. Let $M = \|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\mu)}$. We deduce from Proposition 9 that

$$\mathsf{i} := \frac{(2-p)^2}{\lambda_\star\,M}\,\mathscr{I}[f] \ge \varphi(\mathsf{e}) \quad \text{where} \quad \mathsf{e} := \frac{(2-p)\,\mathscr{E}[f]}{\|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\mu)}},$$

which is equivalent to $-\mathsf{e} \ge -\varphi^{-1}(\mathsf{i})$, so that

$$\mathsf{i} - (2-p)\,\mathsf{e} \ge \mathsf{i} - (2-p)\,\varphi^{-1}(\mathsf{i}) = \psi(\mathsf{i})$$

and, as a consequence,

$$\mathscr{I}[f] - \lambda_\star\,\mathscr{E}[f] = \frac{\lambda_\star\,M}{(2-p)^2}\left(\mathsf{i} - (2-p)\,\mathsf{e}\right) \ge \frac{\lambda_\star\,M}{(2-p)^2}\,\psi(\mathsf{i}) = \frac{\lambda_\star\,M}{(2-p)^2}\,\psi\left(\frac{2-p}{\lambda_\star\,M}\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\mu)}\right).$$

Since $\varphi(s) \sim s$ as $s \to +\infty$, we deduce that $\psi(t) \sim (p-1)\,t$ as $t \to +\infty$. On the other hand, since

$$\psi''(t) = (2-p)\,\frac{\varphi''(s)}{\big(\varphi'(s)\big)^3} \quad \text{with} \quad s = \varphi^{-1}(t),$$

we learn that $\psi''(0) = (p-1)/(2-p) > 0$ and $t \mapsto \psi''(t)$ is non-increasing because $\varphi'(s)^5\,\psi'''(t) = \varphi'(s)\,\varphi'''(s) - \varphi''(s)^2 < 0$. This allows us to define

$$\kappa := \inf_{t>0} t^{-2}\,(1+t)\,\psi(t).$$

Using $\psi(0) = \psi'(0) = 0$, we know that $\kappa > 0$ and $\psi(t) \ge \kappa\,t^2/(1+t)$ concludes the proof using $\|f\|_{\mathrm{L}^p(\mathbb{R}^n,d\mu)} \le \|f\|_{\mathrm{L}^2(\mathbb{R}^n,d\mu)}$. □

Remark 12. If $\phi(y) = |y|^2/2$ is the harmonic potential so that $d\mu = d\gamma$, then $\lambda = \lambda_\star = 1$ is the optimal constant in (2) and the results of Proposition 9 and Corollary 11 are both stability estimates. Even in the general case of a measure $d\mu$ as in (9) such that (11) holds for some $\lambda_\star > 0$, Proposition 9 and Corollary 11 provide improvements to the basic inequality with a (generically non-optimal) constant $\lambda_\star$.

3.3. *From the* carré du champ *method to Obata's theorem*

As a side result, we consider an improvement of the *carré du champ* method as in [34, Theorem 2.1] or [24], which goes as follows. Let us consider the optimal constant $\lambda_1 > 0$ in the Poincaré inequality

$$\int_{\mathbb{R}^n} |\nabla w|^2 \, d\mu \ge \lambda_1 \int_{\mathbb{R}^n} |w - \bar{w}|^2 \, d\mu \quad \forall\, w \in \mathrm{H}^1(\mathbb{R}^n, d\mu)\,, \tag{23}$$

where $\bar{w} := \int_{\mathbb{R}^n} w \, d\mu$. By expanding $\int_{\mathbb{R}^n} \left(\mathscr{L} w + \lambda_1 (w - \bar{w})\right)^2 d\mu \ge 0$, we obtain

$$\int_{\mathbb{R}^n} (\mathscr{L} w)^2 \, d\mu \ge \lambda_1 \int_{\mathbb{R}^n} |\nabla w|^2 \, d\mu\,.$$

On the other hand, by the computation of Section 3.1 with $\beta = p = 1$, we know that

$$-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\left(|\nabla w|^2 + \frac{\lambda_\star}{p-2}\, w^2\right) d\mu = \int_{\mathbb{R}^n} (\mathscr{L} w)^2 \, d\mu - \lambda_\star \int_{\mathbb{R}^n} |\nabla w|^2 \, d\mu = \mathscr{Q}_1[w] \ge 0\,,$$

which proves that

$$\lambda_1 \ge \lambda_\star\,.$$

Lemma 13. *Assume that $n \ge 1$, $p \in [1,2)$ and consider the measure $d\mu$ as in* (9) *such that* (11) *holds for some $\lambda_\star > 0$. Then* (12) *holds with*

$$\lambda = (2-p)\,\lambda_1 + (p-1)\,\lambda_\star\,. \tag{24}$$

As a consequence, we have $\lambda \ge \lambda_\star$ with equality for the optimal value of λ in (12) *if and only if $\lambda_\star = \lambda_1$ and $\phi(y) = \lambda_\star\,|y - y_0|^2/2$ for some $y_0 \in \mathbb{R}^n$.*

Proof. The *carré du champ* method applied as in Section 3.1 with $p = 1$ shows that $\lambda_1 \ge \lambda_\star$. Coming back to the computations of Section 3.1, we can rearrange the integrals in the expression $\mathscr{Q}_\beta[w]$ differently and get

$$\begin{aligned}
&-\frac{1}{2\beta^2}\frac{d}{dt}\int_{\mathbb{R}^n}\left(\left|\nabla w^\beta\right|^2 + \frac{\lambda}{p-2}\, w^{2\beta}\right) d\mu \\
&\quad = (1-\theta)\int_{\mathbb{R}^n} (\mathscr{L} w)^2 \, d\mu - \lambda \int_{\mathbb{R}^n} |\nabla w|^2 \, d\mu \\
&\qquad\quad + \theta \int_{\mathbb{R}^n} (\mathscr{L} w)^2 \, d\mu + (\kappa+\beta-1)\int_{\mathbb{R}^n} (\mathscr{L} w)\,\frac{|\nabla w|^2}{w}\, d\mu + \kappa\,(\beta-1)\int_{\mathbb{R}^n} \frac{|\nabla w|^4}{w^2}\, d\mu \\
&\quad = \big((1-\theta)\,\lambda_1 + \theta\,\lambda_\star - \lambda\big)\int_{\mathbb{R}^n} |\nabla w|^2 \, d\mu \\
&\qquad\quad + \theta\left(\int_{\mathbb{R}^n} \|\mathrm{Hess}\, w\|^2 \, d\mu + \lambda_\star \int_{\mathbb{R}^n} |\nabla w|^2 \, d\mu\right) \\
&\qquad\quad - 2\,(\kappa+\beta-1)\int_{\mathbb{R}^n} \mathrm{Hess}\, w : \frac{\nabla w \otimes \nabla w}{w}\, d\mu + \big(\kappa\,(\beta-1) + \kappa + \beta - 1\big)\int_{\mathbb{R}^n} \frac{|\nabla w|^4}{w^2}\, d\mu\,.
\end{aligned}$$

With the choice of θ such that

$$(\kappa+\beta-1)^2 - \theta\,\big(\kappa\,(\beta-1) + \kappa + \beta - 1\big) = 0\,,$$

which means $\theta = \theta(\beta)$ with

$$\theta(\beta) := \frac{(p-1)^2\,\beta^2}{(p-2)\,\beta^2 + 2\,\beta - 1}\,,$$

we can write that

$$\begin{aligned}
\theta \int_{\mathbb{R}^n} \|\mathrm{Hess}\, w\|^2 \, d\mu - 2\,(\kappa+\beta-1)\int_{\mathbb{R}^n} \mathrm{Hess}\, w : \frac{\nabla w \otimes \nabla w}{w}\, d\mu + \big(\kappa\,(\beta-1) + \kappa + \beta - 1\big)\int_{\mathbb{R}^n} \frac{|\nabla w|^4}{w^2}\, d\mu \\
= \theta \int_{\mathbb{R}^n} \left\| \mathrm{Hess}\, w - \frac{\beta\,(p-1)}{\theta}\, \frac{\nabla w \otimes \nabla w}{w} \right\|^2 d\mu \ge 0\,.
\end{aligned}$$

Altogether, we have shown that

$$\frac{d}{dt}\int_{\mathbb{R}^n}\left(\left|\nabla w^\beta\right|^2+\frac{\lambda}{p-2}\,w^{2\beta}\right)d\mu\le 0$$

if $\lambda=(1-\theta)\,\lambda_1+\theta\,\lambda_\star$. Notice that $\theta\big(\beta_\pm(p)\big)=1$ for any $p\in[1,2)$. The observation that $\min_{\beta\in[\beta_-(p),\beta_+(p)]}\theta(\beta)=\theta(1)=p-1$ completes the proof of (24).

With $\beta=1$ and $p=1$, the computation in the proof of Theorem 7 shows that, if the initial datum $w(t=0,\cdot)$ is an optimal function for the Poincaré inequality (23), then

$$0=\frac{d}{dt}\int_{\mathbb{R}^n}\left(|\nabla w|^2-\lambda_\star\,w^2\right)d\mu=-2\left(\mathscr{Q}_1[w]+\int_{\mathbb{R}^n}\big(\mathrm{Hess}\,\phi-\lambda_\star\mathrm{Id}\big):\nabla w\otimes\nabla w\,d\mu\right)$$

at $t=0$. Here we keep all terms and in particular do not use the fact that $\mathrm{Hess}\,\phi-\lambda_\star\mathrm{Id}\ge 0$ a.e. in the sense of positive matrices. Since $\nabla w\neq 0$ a.e. and $\mathscr{Q}_1[w]\ge 0$, we find that $\mathrm{Hess}\,\phi-\lambda_\star\mathrm{Id}=0$ a.e. This completes the proof in the equality case $\lambda_\star=\lambda_1$. □

Remark 14. The proof of Lemma 13 is reminiscent of [24, 34]. The result when $\lambda_\star=\lambda_1$ points in the direction of *Obata's theorem* (also known as the *Obata–Lichnerowicz theorem*) and in some sense, it is the analogue for Gaussian measures of the result of [33, p. 135] (also see for instance [13, p. 179]) on the sphere. The case $\lambda_\star=\lambda_1$ is easy to understand in dimension $d=1$: with $\beta=1$ and $p=1$, we apply the computation of the proof of Lemma 13 to a function u in the eigenspace associated with λ_1 and obtain that $u''=0$ almost everywhere. This means that $u(y)=a\,y+b$ for some real constants $a\neq 0$ and b, and there is no loss of generality if we take $a=1$. Using now the eigenvalue equation $\mathscr{L}u+\lambda_1\,u=0$, we read that $\phi'(y)=\lambda_1\,(y-b)$, which means that ϕ is an harmonic potential. In higher dimensions, one has to remember that Inequality (12) can be tensorized on product spaces: see for instance [18, 26, 32]. This is however responsible for some technicalities, which are dealt with in greatest generality, *e.g.*, in [28].

3.4. *Improved inequalities under orthogonality conditions*

Let Π_1 be the $\mathrm{L}^2(\mathbb{R}^n,d\gamma)$orthogonal projection onto the space generated by the constants and the coordinate functions, corresponding to the Hermite polynomials of order less or equal than 1. The following result was recently proved in [17, Appendix A] on the basis of Nelson's hypercontractivity estimate in [38, Theorem 3] and its relation with Gross' logarithmic Sobolev inequality in [30] (also see [1, 32] for earlier results).

Proposition 15. *Let $n\ge 1$ and $p\in[1,2)$. For any $f\in\mathrm{H}^1(\mathbb{R}^n,d\gamma)$, we have*

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}-\frac{1}{p-2}\left(\|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}-\|f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\right)\ge\frac{1}{2}\,(2-p)\,\|\nabla(\mathrm{Id}-\Pi_1)f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\,.$$

Compared to (2), this result provides us with an *improved entropy–entropy production inequalities* under orthogonality conditions. As noted in [17], such an improvement is not optimal. There are other possible approaches. For instance, a finer analysis of entropy methods has been used in [25] on the sphere, that could probably be adapted to the case of the Gaussian measure. Alternatively, the convex interpolation of [32], with the possible advantage that the result would not degenerate in the limit as $p\to 2$ using the recent stability result of [23, Theorem 2].

4. Stability results for the Gaussian measure in the subcritical range

The whole Section is devoted to the proof of Theorem 4. We split it in four lemmas. The key estimate is obtained in Lemma 19.

① Let us start with the easy case, far away from the optimizers of (2) in the sense that for some $\theta>0$, we assume

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\ge\theta\,\|f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\,.\tag{25}$$

By homogenity of the inequalities, we can fix $\|f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}=1$ without loss of generality.

Lemma 16. *Let $n\ge1$ and $\theta\in(0,1)$. For any function $f\in\mathrm{H}^1(\mathbb{R}^n,d\gamma)$ such that $\|f\|_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}=1$ and $\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\ge\theta$, we have the estimate*

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}-\frac{1}{2-p}\left(\|f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}-\|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right)\ge\kappa_\star(\theta)\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}$$

In case (25), this already proves the result of Theorem 4 with $c_{n,p}\le c^{(1)}_{n,p}:=\kappa_\star(\theta)$.

Proof. From Corollary 11 and its proof applied with $\lambda_\star=1$, we obtain

$$\begin{aligned}\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}-\frac{1}{2-p}&\left(\|f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}-\|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right)\\&\ge\frac{\|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}}{2-p}\,\frac12\,\psi\big((2-p)\,\theta\big)\left((2-p)\,\frac{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}{\|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}}\right)^2\\&\ge\frac12\,(2-p)\,\psi\big((2-p)\,\theta\big)\,\frac{\|\nabla f\|^4_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}{\|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}}\ge\kappa_\star(\theta)\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\end{aligned}$$

with $\kappa_\star(\theta)=\frac12\,(2-p)\,\psi\big((2-p)\,\theta\big)\,\theta$ because $t\mapsto\psi''(t)$ is non-increasing, $\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\ge\theta$ and $\|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\le\|f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}$. □

② From now on we work in a neighbourhood of the constants which, by homogeneity of the inequalities, is defined as

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\le\theta\,\|f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\,. \tag{26}$$

With $\theta>0$ small, we claim that $\int_{\mathbb{R}^n}f\,d\gamma$ is close to 1 if $\|f\|_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}=1$.

Lemma 17. *Let $n\ge1$ and $\theta\in(0,1)$. For any function*

$$f\in\mathrm{H}^1(\mathbb{R}^n,d\gamma)\quad\textit{such that}\quad\|f\|_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}=1\quad\textit{and}\quad\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\le\theta\,, \tag{27}$$

we have the estimate

$$\sqrt{1-\theta}\le\int_{\mathbb{R}^n}f\,d\gamma\le1\,.$$

Proof. With $\bar f:=\int_{\mathbb{R}^n}f\,d\gamma$, the result follows from the Gaussian Poincaré inequality according to

$$1=\|f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}=\int_{\mathbb{R}^n}|f-\bar f|^2\,d\gamma+\bar f^2\le\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}+\bar f^2\le\theta+\bar f^2\,.$$

□

③ Assume that f is as in (27) and let us decompose $u_f(x):=f(x)/\int_{\mathbb{R}^n}f\,d\gamma$ as

$$u_f(x)=1+\varepsilon\,x\cdot v+\eta\,r(x)$$

where $v\in\mathbb{S}^{n-1}$ is such that $\varepsilon\,v=\int_{\mathbb{R}^n}x\,u_f(x)\,d\gamma$ with $\varepsilon>0$, η is a positive number and r is a function in $\mathrm{H}^1(\mathbb{R}^n,d\gamma)\cap(\mathrm{Id}-\Pi_1)\mathrm{L}^2(\mathbb{R}^n,d\gamma)$ such that $\|\nabla r\|_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}=1$ and $\|r\|_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\le1/2$ by the Gaussian Poincaré inequality after taking into account the additional orthogonality condition $\int_{\mathbb{R}^n}r\,x_i\,d\gamma=0$ for any $i=1,2,\dots,n$.

Lemma 18. *Let $n\ge1$ and $\theta\in(0,1)$. Let $f\in\mathrm{H}^1(\mathbb{R}^n,d\gamma)$ be such that (27) holds. With the above notations, we have*

$$\|u_f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}=1+\varepsilon^2+\eta^2\,\|r\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\le1+\theta\quad\textit{and}\quad\|\nabla u_f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}=\varepsilon^2+\eta^2\le\frac{\theta}{1-\theta}$$

and, if $\eta>t\,\varepsilon^2$ for some $t>0$, then

$$\|\nabla u_f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}-\frac{1}{2-p}\left(\|u_f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}-\|u_f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right)\ge\frac14\,(2-p)\left(\eta^2+\frac{t^2\,\varepsilon^4}{1+\varepsilon^2+\eta^2}\right).$$

By homogeneity, if (27) holds and $\eta > t\,\varepsilon^2$ for some given $t > 0$, using Lemma 17, we obtain

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \frac{1}{2-p}\left(\|f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \|f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right)$$
$$\geq \frac{1}{4}\,(2-p)\,(1-\theta)\left(\|\nabla(\mathrm{Id}-\Pi_1)\,f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} + \frac{t^2\,\|\nabla(\Pi_1 f)\|^4_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} + \|f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right).$$

This case already covers the result of Theorem 4 with $c_{n,p} \leq c^{(2)}_{n,p} := \frac{1}{4}\,(2-p)\,(1-\theta)\min\{t,1\}$.

Proof. The result follows from Proposition 15 and from the chain of elementary inequalities

$$\eta^2 \geq \frac{1}{2}\left(\eta^2 + t^2\,\varepsilon^4\right) \geq \frac{1}{2}\left(\eta^2 + \frac{t^2\,\varepsilon^4}{1+\varepsilon^2+\eta^2}\right).$$

□

④ The next part of the proof relies on a Taylor expansion of $\|u_f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}$. With no loss of generality, by rotational invariance, we can assume that $\nu = (1,0,\ldots,0)$ so that with Cartesian coordinates $x = (x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$, we write $u_f(x) = 1 + \varepsilon\,x_1 + \eta\,r(x)$. The following result is at the core of our strategy. It heavily relies on the Gaussian logarithmic Sobolev inequality and new estimates for the remainder terms based on the boundedness of $\int_{\mathbb{R}^n} r^2 \log r^2\,d\gamma$.

Lemma 19. *Let $n \geq 1$ and $f \in \mathrm{H}^1(\mathbb{R}^n,d\gamma)$ be a non-negative function such that* (27) *holds. We keep the same notations as in Lemma 18 and further assume that $\eta \leq t\,\varepsilon^2$ for some $t > 0$. Then there is a constant $\mathscr{C} > 0$, depending only on n, p and t, such that*

$$\|u_f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)} \geq 1 + (p-1)\left(\varepsilon^2 + \eta^2\|r\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\right)$$
$$+ (p-1)\,(2-p)\left(\frac{1}{2}\,\varepsilon^4 + \varepsilon^2\,\eta\int_{\mathbb{R}^n} x_1^2\,r(x)\,d\gamma\right) - \frac{\mathscr{C}\,\varepsilon^4}{\log\left(\frac{1}{\varepsilon}\right)} \quad \text{as} \quad \varepsilon \to 0_+.$$

Proof. This proof is elementary although a little bit lengthy. Let us split it into three steps.

Step 1. Let us start with a list of preliminary remarks.

• Let B_ε be the centred ball of radius $1/(2\,\varepsilon)$, that is,

$$B_\varepsilon := \left\{x \in \mathbb{R}^n : 2\,\varepsilon\,|x| < 1\right\}$$

and let $B^c_\varepsilon = \mathbb{R}^n \setminus B_\varepsilon$. We observe that

$$\gamma(B^c_\varepsilon) = \left|\mathbb{S}^{n-1}\right| \int_{1/(2\,\varepsilon)}^{+\infty} r^{n-1}\,e^{-\frac{r^2}{2}}\,dr = c_n\,\varepsilon^{2-n}\,e^{-\frac{1}{8\varepsilon^2}}\left(1 + O\left(\varepsilon^2\right)\right) \quad \text{as} \quad \varepsilon \to 0_+$$

with $c_n = 2^{3\,(2-n)/2}/\Gamma(n/2)$. Let $\xi_p := \sup_{\varepsilon\in(0,1/2)} \varepsilon^{-5}\left(\gamma(B^c_\varepsilon)\right)^{(2-p)/2}$. Hence we have

$$\int_{B^c_\varepsilon} |g|^p\,d\gamma \leq \|g\|^p_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\left(\gamma(B^c_\varepsilon)\right)^{(2-p)/2} \leq \xi_p\,\|g\|^p_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\,\varepsilon^5$$

for any $g \in \mathrm{L}^2(\mathbb{R}^n,d\gamma)$, by Hölder's inequality and, as a consequence,

$$\int_{B_\varepsilon} |g|^p\,d\gamma \leq \int_{\mathbb{R}^n} |g|^p\,d\gamma \leq \int_{B_\varepsilon} |g|^p\,d\gamma + \xi_p\,\|g\|^p_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\,\varepsilon^5 \quad \forall\,\varepsilon \in (0,1/2)\,. \tag{28}$$

From now on, we assume without further notice that $x \in B_\varepsilon$ unless it is specified.

• An expansion in Taylor series of $(1+s)^p$ for $s \leq 0$ shows that all terms are non-negative:

$$(1+s)^p \geq 1 + p\,s + \frac{1}{2}\,p\,(p-1)\,s^2 \quad \forall\,s \in (-1,0]\,.$$

Applied to $u_f = 1+\varepsilon\,x_1+\eta\,r$ whenever $1+\varepsilon\,x_1>0$ and $r\le 0$, we obtain

$$\begin{aligned}|u_f|^p = \big(1+\varepsilon\,x_1+\eta\,r\big)^p &= (1+\varepsilon\,x_1)^p\left(1+\frac{\eta\,r}{1+\varepsilon\,x_1}\right)^p\\ &\ge (1+\varepsilon\,x_1)^p\left(1+p\,\frac{\eta\,r}{1+\varepsilon\,x_1}+\frac12\,p\,(p-1)\left(\frac{\eta\,r}{1+\varepsilon\,x_1}\right)^2\right)\\ &= (1+\varepsilon\,x_1)^p+p\,(1+\varepsilon\,x_1)^{p-1}\,\eta\,r+\frac12\,p\,(p-1)\,(1+\varepsilon\,x_1)^{p-2}\,\eta^2\,r^2\,.\end{aligned}$$

• Let us consider the case $1+\varepsilon\,x_1>0$ and $r>0$. The function

$$\rho(s):=\frac{1}{s^2}\left((1+s)^p-1-p\,s-\frac12\,p\,(p-1)\,s^2\right)\quad\forall\,s\ge0$$

is bounded. Let us extend ρ by 0 on $(-1,0)$ and define

$$\psi_{\varepsilon,\eta,r}(x):=(1+\varepsilon\,x_1)^{p-2}\,\rho\left(\frac{\eta\,r(x)}{1+\varepsilon\,x_1}\right).$$

With this definition, using $u_f\ge0$ by hypothesis, we obtain

$$\begin{aligned}|u_f|^p &\ge \big(1+\varepsilon\,x_1+\eta\,r\big)^p\\ &= (1+\varepsilon\,x_1)^p+p\,(1+\varepsilon\,x_1)^{p-1}\,\eta\,r+\frac12\,p\,(p-1)\,(1+\varepsilon\,x_1)^{p-2}\,\eta^2\,r^2+\frac12\,p\,(p-1)\,\eta^2\,r(x)^2\,\psi_{\varepsilon,\eta,r}(x)\end{aligned}$$

with equality whenever $r\ge0$.

As $\eta\to0_+$, $\psi_{\varepsilon,\eta,r}$ converges a.e. to 0 on B_ε uniformly with respect to $\varepsilon\in(0,1/2)$. The dominated convergence theorem is enough to conclude that

$$\lim_{\eta\to0_+}\int_{B_\varepsilon} r(x)^2\,\psi_{\varepsilon,\eta,r}(x)\,d\gamma=0$$

for a given function r, but this is not enough to conclude uniformly with respect to r. To do this, we need more detailed estimates. Notice however that

$$\|\psi_{\varepsilon,\eta,r}\|_{\mathrm{L}^\infty(B_\varepsilon)}\le M:=2^{2-p}\,\|\rho\|_{\mathrm{L}^\infty(\mathbb{R}^+)}$$

where M is independent of ε, η and r.

• Since $\int_{B_\varepsilon}x_1^{2\,k}\,d\gamma\ge\int_{B_\varepsilon}x_1^{2\,k+1}\,d\gamma=0$ and $\int_{\mathbb{R}^n}x_1^{2\,k}\,d\gamma\ge\int_{\mathbb{R}^n}x_1^{2\,k+1}\,d\gamma=0$ for any $k\in\mathbb{N}$, an expansion in Taylor series of $(1+\varepsilon\,x_1)^p$ gives

$$\begin{aligned}\int_{\mathbb{R}^n}(1+\varepsilon\,x_1)^p\,d\gamma &\ge \int_{\mathbb{R}^n}\left(1+\frac12\,p\,(p-1)\,\varepsilon^2\,x_1^2+\frac{1}{24}\,p\,(p-1)\,(p-2)\,(p-3)\,x_1^4\,\varepsilon^4\right)d\gamma\\ &= 1+\frac12\,p\,(p-1)\,\varepsilon^2+\frac18\,p\,(p-1)\,(p-2)\,(p-3)\,\varepsilon^4\,.\end{aligned}$$

By applying (28) with $g=1+\varepsilon\,x_1$, we obtain

$$\int_{\mathbb{R}^n}|1+\varepsilon\,x_1|^p\,d\gamma\le\int_{B_\varepsilon}|1+\varepsilon\,x_1|^p\,d\gamma+\xi_p\left(1+\varepsilon^2\right)^{p/2}\varepsilon^5\,.$$

Summing up with $\varepsilon^2\le\theta$, we have

$$\int_{B_\varepsilon}|1+\varepsilon\,x_1|^p\,d\gamma\ge1+\frac12\,p\,(p-1)\,\varepsilon^2+\frac18\,p\,(p-1)\,(p-2)\,(p-3)\,\varepsilon^4-\xi_p\,(1+\theta)^{p/2}\,\varepsilon^5\,.$$

• Let us estimate $\|u_f\|^p_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}$ using

$$\int_{\mathbb{R}^n}|u_f|^p\,d\gamma\ge\int_{B_\varepsilon}|u_f|^p\,d\gamma$$

and

$$\int_{B_\varepsilon} |u_f|^p \, d\gamma \ge \int_{B_\varepsilon} |1+\varepsilon\, x_1|^p \, d\gamma + p\,\eta \int_{B_\varepsilon} (1+\varepsilon\, x_1)^{p-1}\, r\, d\gamma + \frac{1}{2}\, p\,(p-1)\,\eta^2 \int_{B_\varepsilon} (1+\varepsilon\, x_1)^{p-2}\, r^2\, d\gamma + \frac{1}{2}\, p\,(p-1)\,\eta^2 \int_{B_\varepsilon} r(x)^2\, \psi_{\varepsilon,\eta,r}(x)\, d\gamma .$$

We obtain

$$\int_{\mathbb{R}^n} |u_f|^p \, d\gamma \ge 1 + \frac{1}{2}\, p\,(p-1)\,\varepsilon^2 + \frac{1}{8}\, p\,(p-1)\,(p-2)\,(p-3)\,\varepsilon^4 - \xi_p\,(1+\theta)^{p/2}\,\varepsilon^5 + p\,\eta \int_{B_\varepsilon} (1+\varepsilon\, x_1)^{p-1}\, r\, d\gamma + \frac{1}{2}\, p\,(p-1)\,\eta^2 \int_{B_\varepsilon} (1+\varepsilon\, x_1)^{p-2}\, r^2\, d\gamma + \frac{1}{2}\, p\,(p-1)\,\eta^2 \int_{B_\varepsilon} r(x)^2\, \psi_{\varepsilon,\eta,r}(x)\, d\gamma .$$

Step 2. We prove that $\eta^2 \int_{B_\varepsilon} r(x)^2\, \psi_{\varepsilon,\eta,r}(x)\, d\gamma$ is of order $o\big(\varepsilon^4\big)$ as $\eta \le t\,\varepsilon^2 \to 0$ for a given $t > 0$.

• By the logarithmic Sobolev inequality,

$$\int_{\mathbb{R}^n} h^2 \log h^2 \, d\gamma \le 2 \int_{\mathbb{R}^n} |\nabla h|^2 \, d\gamma + \int_{\mathbb{R}^n} h^2 \, d\gamma \log\left(\int_{\mathbb{R}^n} h^2 \, d\gamma \right),$$

applied to $h = 1 + (r-1)_+$, we learn that

$$\int_{\mathbb{R}^n} h^2 \log h^2 \, d\gamma \le 2 \int_{\mathbb{R}^n} |\nabla r|^2 \, d\gamma + \left(1 + \int_{\mathbb{R}^n} r^2 \, d\gamma \right) \log\left(1 + \int_{\mathbb{R}^n} r^2 \, d\gamma \right) \le 2 \log(2\,e) .$$

Let $\chi := \mathbb{1}_{\{h > s_0\}}$ for any $s_0 > 1$ and consider $A_{\varepsilon,\eta,r,s_0} := \big\{ x \in B_\varepsilon : \eta\, r(x) \le s_0 \big\}$. Then we have

$$\int_{\mathbb{R}^n} h^2 \log h^2 \, d\gamma \ge \int_{B_\varepsilon \setminus A_{\varepsilon,\eta,r,s_0}} h^2 \log h^2 \, d\gamma = \int_{B_\varepsilon} \chi\, h^2 \log h^2 \, d\gamma \ge \log\left(\frac{s_0}{\eta} \right)^2 \int_{B_\varepsilon} \chi\, r^2 \, d\gamma$$

because $h \ge 1$ a.e. and $h > s_0/\eta$ on $B_\varepsilon \setminus A_{\varepsilon,\eta,r,s_0}$ and, as a consequence,

$$\eta^2 \int_{B_\varepsilon \setminus A_{\varepsilon,\eta,r,s_0}} r(x)^2\, \psi_{\varepsilon,\eta,r}(x)\, d\gamma \ge - \frac{\log(2\,e)\, M}{\log s_0 + \log\left(\frac{1}{\eta} \right)}\, \eta^2 .$$

• Let us notice that

$$\mathscr{M} = \sup_{s>0} \frac{|\rho(s)|}{\log(1+s)}$$

is finite. This allows us to write that

$$\eta^2 \int_{A_{\varepsilon,\eta,r,s_0}} r(x)^2\, \psi_{\varepsilon,\eta,r}(x)\, d\gamma \ge - 2^{2-p}\, \mathscr{M} \int_{A_{\varepsilon,\eta,r,s_0}} \eta^2\, r^2 \log(1 + 2\,\eta\, r)\, \mathbb{1}_{\{r>0\}}\, d\gamma ,$$

where the restriction to the set $\{r > 0\}$ comes from the fact that $\psi_{\varepsilon,\eta,r}(x) = 0$ whenever $r(x) \le 0$. Now we estimate $\log(1 + 2\,\eta\, r)$ by

$$\log(1 + 2\,\eta\, r) \le \log\big(1 + 2\,\sqrt{\eta}\big) \qquad \text{if} \quad 0 \le r \le \frac{1}{\sqrt{\eta}} ,$$

$$\log(1 + 2\,\eta\, r) \le \frac{\log(1 + 2\, s_0)}{\log\left(\frac{1}{\eta} \right)} \log r^2 \qquad \text{if} \quad \frac{1}{\sqrt{\eta}} \le r \le \frac{s_0}{\eta} ,$$

and conclude using $\int_{\mathbb{R}^n} r^2\, d\gamma \le 1$ and $\int_{\mathbb{R}^n} r^2 \log r^2\, d\gamma \le 2 \int_{\mathbb{R}^n} |\nabla r|^2\, d\gamma = 2$ by the logarithmic Sobolev inequality that

$$\eta^2 \int_{A_{\varepsilon,\eta,r,s_0}} r(x)^2\, \psi_{\varepsilon,\eta,r}(x)\, d\gamma \ge - 2^{2-p}\, \mathscr{M}\, \eta^2 \left(\theta \log\big(1 + 2\,\sqrt{\eta}\big) + \frac{2 \log(1 + 2\, s_0)}{\log\left(\frac{1}{\eta} \right)} \right),$$

Step 3. We compute the contribution of

$$p\,\eta\int_{B_\varepsilon}(1+\varepsilon\,x_1)^{p-1}\,r\,d\gamma\quad\text{and}\quad\frac12\,p\,(p-1)\,\eta^2\int_{B_\varepsilon}(1+\varepsilon\,x_1)^{p-2}\,r^2\,d\gamma$$

to the expansion of $\int_{\mathbb{R}^n}|u_f|^p\,d\gamma$.

• Using (28) applied to $g=|r|^{1/p}$ and the orthogonality constraints on r, we obtain

$$\begin{aligned}p\,\eta\int_{B_\varepsilon}(1+\varepsilon\,x_1)^{p-1}\,r\,d\gamma&\ge p\,\eta\left(\int_{\mathbb{R}^n}(1+\varepsilon\,x_1)^{p-1}\,r\,d\gamma-2^{1-p}\,\xi_p\,\varepsilon^5\right)\\&\ge p\,\eta\,\varepsilon^2\left(\frac12\,(p-1)\,(p-2)\int_{\mathbb{R}^n}x_1^2\,r\,d\gamma-\mathsf{c}_1\,\varepsilon^3-2^{1-p}\,\xi_p\,\varepsilon^5\right)\end{aligned}$$

where $\mathsf{c}_1:=\sqrt{15}\,\sup_{s\in(-1,0)\cup(0,+\infty)}\left|(1+s)^{p-1}-1-(p-1)\,s-\frac12\,(p-1)\,(p-2)\,s^2\right|/s^3$

• A similar computation shows that

$$\frac12\,p\,(p-1)\,\eta^2\int_{B_\varepsilon}(1+\varepsilon\,x_1)^{p-2}\,r^2\,d\gamma=\frac12\,p\,(p-1)\,\eta^2\left(\|r\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}-\mathsf{c}_2\,\varepsilon\right)$$

where $\mathsf{c}_2:=\sup_{s\in(-1,0)\cup(0,+\infty)}\left|(1+s)^{p-2}-1\right|/s$

Step 4. Collecting all terms, we have

$$\begin{aligned}\int_{\mathbb{R}^n}|u_f|^p\,d\gamma\ge1+\frac12\,p\,(p-1)\,\varepsilon^2+\frac18\,p\,(p-1)\,(p-2)\,(p-3)\,\varepsilon^4+\frac12\,p\,(p-1)\,\eta^2\,\|r\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\\+\frac12\,p\,(p-1)\,(p-2)\,\eta\,\varepsilon^2\int_{\mathbb{R}^n}x_1^2\,r\,d\gamma-C\,\frac{\varepsilon^4}{\log\left(\frac1\varepsilon\right)}\end{aligned}$$

for some constant $C>0$ that is explicitly given in terms of t, ξ_p, M, $\mathscr{M}$, c_1 and c_2. In order to conclude, we notice that for any $s>-1$, $\frac{d^4}{ds^4}(1+s)^{\frac2p}>0$ implies that

$$(1+s)^{\frac2p}\ge1+\frac2p\,s+\frac{1}{p^2}\,(2-p)\,s^2-\frac{1}{3\,p^3}\,(2-p)\,(4-p)\,s^3\,.$$

Applied to

$$\|u_f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}=\left(\int_{\mathbb{R}^n}\left(1+\varepsilon\,x_1+\eta\,r\right)^p\,d\gamma\right)^{\frac2p},$$

this completes the proof of Lemma 19. □

Proof of Theorem 4. Up to the replacement of f by $|f|$, there is no restriction in assuming that f is non-negative: we can rely on Lemma 19, which is the main ingredient of the proof. The strategy is now very similar to the proof of [17, Theorem 7]. Let is consider the Hermite polynomial $h_1(x):=x_1^2-1$ and decompose r according to

$$r(x):=\alpha\,h_1(x)+\beta\,\tilde r(x)$$

with $\|\tilde r\|_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}=1$ so that $\|r\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}=\alpha^2+\beta^2\le1/2$ and $\int_{\mathbb{R}^n}x_1^2\,r(x)\,d\gamma=\alpha$. With these notations we have

$$\begin{aligned}&\|\nabla u_f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}=\varepsilon^2+\eta^2\,,\\&\|u_f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}=1+\varepsilon^2+\eta^2\left(\alpha^2+\beta^2\right),\\&\|u_f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\ge1+(p-1)\left(\varepsilon^2+\eta^2\left(\alpha^2+\beta^2\right)\right)+(p-1)\,(2-p)\left(\frac12\,\varepsilon^4+\alpha\,\varepsilon^2\,\eta\right)-\frac{\mathscr{C}\,\varepsilon^4}{\log\left(\frac1\varepsilon\right)},\end{aligned}$$

where the estimate of $\|u_f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}$ comes from Lemma 19. Hence, for some $\lambda>0$ to be fixed,

$$\begin{aligned}
&\|\nabla u_f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}-\frac{1}{2-p}\left(\|u_f\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}-\|u_f\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right)-\lambda\left(\eta^2+\frac{t^2\,\varepsilon^4}{1+\varepsilon^2+\eta^2}\right)\\
&\ge\eta^2\left(1-\alpha^2-\beta^2-\lambda\right)+\left(\frac{p-1}{2}-\frac{\lambda\,t^2}{1+\varepsilon^2+\eta^2}\right)\varepsilon^4+(p-1)\,\alpha\,\varepsilon^2\,\eta-\frac{1}{2-p}\,\frac{\mathscr{C}\,\varepsilon^4}{\log\left(\frac{1}{\varepsilon}\right)}\\
&\ge\left(t^2\left(1-\alpha^2-\beta^2-2\,\lambda\right)+(p-1)\,\alpha\,t+\frac{1}{2}\,(p-1)-\frac{1}{2-p}\,\frac{\mathscr{C}}{\log\left(\frac{1}{\varepsilon}\right)}\right)\varepsilon^4\,.
\end{aligned}$$

By writing

$$\begin{aligned}
t^2\left(1-\alpha^2-\beta^2-2\,\lambda\right)+(p-1)\,\alpha\,t+\frac{1}{2}\,(p-1)&\\
=t^2\left(1-\frac{p+1}{2}\,\alpha^2-\beta^2-2\,\lambda\right)+\frac{1}{2}\,(p-1)(1+\alpha\,t)^2&\\
\ge\frac{1}{4}\,t^2\left(3-p-8\,\lambda\right)+\frac{1}{2}\,(p-1)(1+\alpha\,t)^2\,,&
\end{aligned}$$

for any given $t>0$ and $\lambda<(3-p)/8$, if $\varepsilon>0$ is small enough, we obtain the result with $c_{n,p}=(1-\theta)\,\lambda$. □

Acknowledgements

G.B. has been funded by the European Union's Horizon 2020 research and innovation program under the Marie Sklodowska-Curie grant agreement No. 754362. The authors thank M.J. Esteban for useful comments.

References

[1] A. Arnold, J.-P. Bartier, J. Dolbeault, "Interpolation between logarithmic Sobolev and Poincaré inequalities", *Commun. Math. Sci.* **5** (2007), no. 4, p. 971-979, http://projecteuclid.org/euclid.cms/1199377560.

[2] A. Arnold, J. Dolbeault, "Refined convex Sobolev inequalities", *J. Funct. Anal.* **225** (2005), no. 2, p. 337-351, https://doi.org/10.1016/j.jfa.2005.05.003.

[3] A. Arnold, P. Markowich, G. Toscani, A. Unterreiter, "On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations", *Comm. Partial Differential Equations* **26** (2001), no. 1-2, p. 43-100, https://doi.org/10.1081/PDE-100002246.

[4] D. Bakry, M. Émery, "Diffusions hypercontractives", in *Séminaire de probabilités, XIX, 1983/84*, Lecture Notes in Math., vol. 1123, Springer, Berlin, 1985, p. 177-206, https://doi.org/10.1007/BFb0075847.

[5] D. Bakry, M. Émery, "Hypercontractivité de semi-groupes de diffusion", *C. R. Acad. Sci. Paris Sér. I Math.* **299** (1984), no. 15, p. 775-778, https://gallica.bnf.fr/ark:/12148/bpt6k6288170c/f789.image.r=Bakry.

[6] D. Bakry, M. Émery, "Inégalités de Sobolev pour un semi-groupe symétrique", *C. R. Acad. Sci. Paris Sér. I Math.* **301** (1985), no. 8, p. 411-413, https://gallica.bnf.fr/ark:/12148/bpt6k54942041/f25.item.r=BakryEmery.

[7] D. Bakry, I. Gentil, M. Ledoux, *Analysis and geometry of Markov diffusion operators*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 348, Springer, Cham, 2014, https://doi.org/10.1007/978-3-319-00227-9, xx+552 pages.

[8] W. Beckner, "A generalized Poincaré inequality for Gaussian measures", *Proc. Amer. Math. Soc.* **105** (1989), no. 2, p. 397-400, https://doi.org/10.2307/2046956.

[9] ———, "Sobolev inequalities, the Poisson semigroup, and analysis on the sphere $\mathbb{S}^n$", *Proc. Nat. Acad. Sci. U.S.A.* **89** (1992), no. 11, p. 4816-4819, https://doi.org/10.1073/pnas.89.11.4816.

[10] ———, "Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality", *Ann. of Math. (2)* **138** (1993), no. 1, p. 213-242, https://doi.org/10.2307/2946638.

[11] ———, "Geometric asymptotics and the logarithmic Sobolev inequality", *Forum Math.* **11** (1999), no. 1, p. 105-137, https://doi.org/10.1515/form.11.1.105.

[12] W. Beckner, M. Pearson, "On sharp Sobolev embedding and the logarithmic Sobolev inequality", *Bull. London Math. Soc.* **30** (1998), no. 1, p. 80-84, https://doi.org/10.1112/S0024609397003901.

[13] M. Berger, P. Gauduchon, E. Mazet, *Le spectre d'une variété riemannienne*, Lecture Notes in Mathematics, Vol. 194, Springer-Verlag, Berlin, 1971, vii+251 pages.

[14] G. Bianchi, H. Egnell, "A note on the Sobolev inequality", *J. Funct. Anal.* **100** (1991), no. 1, p. 18-24, https://doi.org/10.1016/0022-1236(91)90099-Q.

[15] M.-F. Bidaut-Véron, L. Véron, "Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations", *Invent. Math.* **106** (1991), no. 3, p. 489-539, https://doi.org/10.1007/BF01243922.

[16] ———, "Erratum: "Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations" [Invent. Math. **106** (1991), no. 3, 489–539; MR1134481 (93a:35045)]", *Invent. Math.* **112** (1993), no. 2, p. 445, https://doi.org/10.1007/BF01232442.

[17] G. Brigati, J. Dolbeault, N. Simonov, "Logarithmic Sobolev and interpolation inequalities on the sphere: constructive stability results", 2022, https://arxiv.org/abs/2211.13180.

[18] D. Chafaï, "Entropies, convexity, and functional inequalities: on Φ-entropies and Φ-Sobolev inequalities", *J. Math. Kyoto Univ.* **44** (2004), no. 2, p. 325-363.

[19] J. Demange, "Des équations à diffusion rapide aux inégalités de Sobolev sur les modèles de la géométrie", PhD Thesis, Université Paul Sabatier Toulouse 3, 2005.

[20] J. Demange, "Improved Gagliardo-Nirenberg-Sobolev inequalities on manifolds with positive curvature", *J. Funct. Anal.* **254** (2008), no. 3, p. 593-611, https://doi.org/10.1016/j.jfa.2007.01.017.

[21] J. Dolbeault, B. Nazaret, G. Savaré, "On the Bakry-Emery criterion for linear diffusions and weighted porous media equations", *Commun. Math. Sci.* **6** (2008), no. 2, p. 477-494, http://projecteuclid.org/euclid.cms/1214949932.

[22] J. Dolbeault, M. J. Esteban, "Improved interpolation inequalities and stability", *Adv. Nonlinear Stud.* **20** (2020), no. 2, p. 277-291, https://doi.org/10.1515/ans-2020-2080.

[23] J. Dolbeault, M. J. Esteban, A. Figalli, R. L. Frank, M. Loss, "Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence", 2022, https://arxiv.org/abs/2209.08651.

[24] J. Dolbeault, M. J. Esteban, M. Loss, "Nonlinear flows and rigidity results on compact manifolds", *J. Funct. Anal.* **267** (2014), no. 5, p. 1338-1363, https://doi.org/10.1016/j.jfa.2014.05.021.

[25] ———, "Interpolation inequalities on the sphere: linear vs. nonlinear flows", *Ann. Fac. Sci. Toulouse Math. (6)* **26** (2017), no. 2, p. 351-379, https://doi.org/10.5802/afst.1536.

[26] J. Dolbeault, X. Li, "φ-entropies: convexity, coercivity and hypocoercivity for Fokker-Planck and kinetic Fokker-Planck equations", *Math. Models Methods Appl. Sci.* **28** (2018), no. 13, p. 2637-2666, https://doi.org/10.1142/S0218202518500574.

[27] U. Gianazza, G. Savaré, G. Toscani, "The Wasserstein gradient flow of the Fisher information and the quantum drift-diffusion equation", *Arch. Ration. Mech. Anal.* **194** (2009), no. 1, p. 133-220, https://doi.org/10.1007/s00205-008-0186-5.

[28] N. Gigli, C. Ketterer, K. Kuwada, S.-i. Ohta, "Rigidity for the spectral gap on RCD(K,∞)-spaces", *Amer. J. Math.* **142** (2020), no. 5, p. 1559-1594, https://doi.org/10.1353/ajm.2020.0039.

[29] P. Grisvard, *Elliptic problems in nonsmooth domains*, Classics in Applied Mathematics, vol. 69, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011, https://doi.org/10.1137/1.9781611972030.ch1, Reprint of the 1985 original edition, With a foreword by Susanne C. Brenner, xx+410 pages.

[30] L. Gross, "Logarithmic Sobolev inequalities", *Amer. J. Math.* **97** (1975), no. 4, p. 1061-1083, https://doi.org/10.2307/2373688.

[31] A. Jüngel, *Entropy methods for diffusive partial differential equations*, SpringerBriefs in Mathematics, Springer, [Cham], 2016, https://doi.org/10.1007/978-3-319-34219-1, viii+139 pages.

[32] R. Latała, K. Oleszkiewicz, "Between Sobolev and Poincaré", in *Geometric aspects of functional analysis*, Lecture Notes in Math., vol. 1745, Springer, Berlin, 2000, p. 147-168, https://doi.org/10.1007/BFb0107213.

[33] A. Lichnerowicz, *Géométrie des groupes de transformations*, Travaux et Recherches Mathématiques, III. Dunod, Paris, 1958, ix+193 pages.

[34] J. R. Licois, L. Véron, "A class of nonlinear conservative elliptic equations in cylinders", *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **26** (1998), no. 2, p. 249-283, http://www.numdam.org/item?id=ASNSP_1998_4_26_2_249_0.

[35] E. H. Lieb, "Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities", *Ann. of Math. (2)* **118** (1983), no. 2, p. 349-374, https://doi.org/10.2307/2007032.

[36] M. Liero, A. Mielke, "Gradient structures and geodesic convexity for reaction-diffusion systems", *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **371** (2013), no. 2005, p. 20120346, 1-28, https://doi.org/10.1098/rsta.2012.0346.

[37] H. P. McKean, "Geometry of differential space", *Ann. Probability* **1** (1973), p. 197-206, https://doi.org/10.1214/aop/1176996973.

[38] E. Nelson, "The free Markoff field", *J. Funct. Anal.* **12** (1973), p. 211-227, https://doi.org/10.1016/0022-1236(73)90025-6.

[39] A. M. Vershik, "Does there exist a Lebesgue measure in the infinite-dimensional space ?", *Proceedings of the Steklov Institute of Mathematics* **259** (2007), no. 1, p. 248-272, https://doi.org/10.1134/S0081543807040153.