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STRUCTURE OF KÄHLER FOLIATIONS WITH NEGATIVE TRANSVERSE RICCI CURVATURE

BENOÎT CLAUDON AND FRÉDÉRIC TOUZET

ABSTRACT. We investigate the structure of transversely Kähler foliations with quasi-negative transverse Ricci curvature. In particular, we prove a de Rham type theorem decomposition on the leaf space where we characterize each factor.

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1. INTRODUCTION

Preliminary warning. Unless otherwise stated, the objects considered here such as manifolds, foliations, functions, tensors. . . are supposed to be smooth. For the sake of notational simplicity, we will denote by the same symbol (typically \mathcal{F}) a foliation/distribution and its tangent bundle.

1.1. Statement of the main results. Let \mathcal{F} a transversely Kähler foliation of complex codimension n on X compact; it is worth mentioning here that X is not necessarily endowed with a complex structure. We denote by $J_{\mathcal{F}}$ the holomorphic transverse structure (see §3.3) and by \bar{g} the transverse Kähler metric. In some holomorphic coordinates (z_1, \dots, z_n) parameterizing the local space of leaves, it reads

$$\bar{g} = \sum_{i,j} g_{i\bar{j}} dz_i d\bar{z}_j$$

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where $g_{i\bar{j}}$ depends only of the transverse variables (z_1, \dots, z_n) . The foliation is thus equipped with two basic closed $(1, 1)$ forms. Namely the fundamental form ω of \bar{g} and the (transverse) Ricci form $\gamma = \text{Ric}(\bar{g})$ respectively defined in the previous local transverse coordinates as

$$\omega = \sqrt{-1} \sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$$

and

$$\gamma = \text{Ric}(\bar{g}) = -\sqrt{-1} \partial \bar{\partial} \log \left(\frac{\omega^n}{|dz_1 \wedge \dots \wedge dz_n|^2} \right) = -\sqrt{-1} \partial \bar{\partial} \log (\det(g_{i\bar{j}})) .$$

In this paper, we will make use of the following assumptions.

- (A1) The transverse Ricci form $\text{Ric}(\bar{g})$ is *quasi-negative*: $\text{Ric}(\bar{g}) \leq 0$ and $\text{Ric}(\bar{g}) < 0$ in the transverse direction (i.e. has maximal rank n) somewhere.
- (A2) \mathcal{F} is *homologically orientable*, i.e. the top basic cohomology group $H^{2n}(X/\mathcal{F})$ is non-trivial, and then generated over \mathbb{R} by the class of the transverse volume ω^n (see Theorem 3.2).

Then we can prove the following statement.

Theorem A.

- (1) There exist on X two regular foliations \mathcal{G} and $\bar{\mathcal{F}}$ containing \mathcal{F} and such that $\mathcal{F} = \mathcal{G} \cap \bar{\mathcal{F}}$.
- (2) The foliations \mathcal{G} and $\bar{\mathcal{F}}$ are holomorphic with respect to the complex structure $J_{\mathcal{F}}$ on the normal bundle $N\mathcal{F}$. Moreover, \mathcal{G}/\mathcal{F} and $\bar{\mathcal{F}}/\mathcal{F}$ are orthogonal and parallel with respect to \bar{g} (as subbundles of $N\mathcal{F}$).
- (3) The leaves of $\bar{\mathcal{F}}$ are the topological closure of the leaves of \mathcal{F} . In particular, they are closed. The leaf space $X/\bar{\mathcal{F}}$ is a compact Kähler orbifold with quasi-negative Ricci curvature, in particular of general type.

We denote by $\tilde{}$ the lift on the universal cover \tilde{X} of any object previously defined.

Lemma B. *The leaves of $\bar{\mathcal{F}}$ are closed and the space of leaves $\tilde{X}/\bar{\mathcal{F}}$ is a complete Kähler orbifold (with respect to the metric \bar{g}).*

Let \mathcal{G}_1 and \mathcal{G}_2 be the foliations induced on $\tilde{X}/\tilde{\mathcal{F}}$ by $\tilde{\mathcal{G}}$ and $\tilde{\bar{\mathcal{F}}}$. Note that they induce an infinitesimal splitting (in the orbifold category):

$$T(\tilde{X}/\tilde{\mathcal{F}}) = \mathcal{G}_1 \oplus \mathcal{G}_2.$$

We have then the following geometric description of these foliations.

Theorem C.

- (1) The leaves of \mathcal{G}_2 are all isometric to a Hermitian symmetric space \mathcal{H} of the non-compact type.
- (2) The leaves of \mathcal{G}_1 are all isometric to a Kähler complete orbifold \mathcal{K} with quasi-negative Ricci curvature.
- (3) The aforementioned infinitesimal splitting gives rise to a global decomposition (de Rham decomposition)

$$\tilde{X}/\tilde{\mathcal{F}} = \mathcal{K} \times \mathcal{H}.$$

- (4) Let us consider the natural diagonal and isometric action of $\pi_1(X)$ (with respect to the decomposition above). Then this action is minimal (i.e. dense) on the second factor and discrete cocompact on the first factor.

These statements shall be seen as foliated analogues of results obtained by Nadel [Nad90] and Frankel [Fra95]. Both authors have studied the geometry of the universal covering \tilde{X} of X a complex projective manifold with ample canonical bundle.¹ Nadel proved that $\text{Aut}^0(\tilde{X})$ is a semi-simple Lie group having no compact factors and he conjectured the existence of a splitting $\tilde{X} \simeq \mathcal{H} \times \mathcal{H}$ with $\text{Aut}^0(\mathcal{H}) = \{\text{Id}\}$ and \mathcal{H} being a bounded symmetric domain (equivalently a Hermitian symmetric space of the non-compact type). This was first confirmed in the case of surfaces [Nad90, Theorem 0.2] and then in full generality [Fra95, Theorem 0.1]. Theorem C can thus be seen as a foliated version of the above-mentioned splitting.

1.2. Example. Suppose given on a compact manifold N a minimal foliation (*i.e.* with dense leaves) \mathcal{F}_N whose transverse geometry is locally modeled on a Hermitian symmetric space of the non-compact type $\mathcal{H} = G/K$. The foliation carries a natural transverse invariant metric \overline{g}_N induced by the G -invariant Kähler–Einstein metric on \mathcal{H} . Let (M, g_M) be a compact Kähler manifold with quasi-negative Ricci curvature. The product $X = M \times N$ is equipped with an induced foliation \mathcal{F} of the same rank which restricts to each vertical fiber $\{m\} \times N \simeq N$ to \mathcal{F}_N . This foliation is transversely Kähler with respect to the transverse metric $g_M \oplus \overline{g}_N$ and satisfies moreover hypothesis (A1) and (A2). Here, the leaves closure are precisely the vertical fibers. Moreover, if we fixe a Hermitian symmetric space of the non-compact type \mathcal{H} , it is possible to exhibit a compact foliated manifold (N, \mathcal{F}_N) as above. Indeed, let $\mathcal{H} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_p$ be the decomposition of \mathcal{H} into irreducible symmetric factors. It is well known since Borel’s work [Bor63] (see also [Mar91, Section IX.4.7, Theorem C] and [Mor15, Corollary 18.7.4]) that for every i , there exists a discrete torsion free subgroup of holomorphic isometries Γ_i of $\mathcal{H}_i \times \mathcal{H}_i$ acting cocompactly and diagonally and such that Γ_i acts densely on each factor. It is then sufficient to consider the projective manifold $N = (\mathcal{H} \times \mathcal{H}) / \prod_i \Gamma_i$ and to take as \mathcal{F}_N the holomorphic foliation which lift to the horizontal (or vertical) one on the universal cover $\mathcal{H} \times \mathcal{H}$. Apart this use of irreducible uniform lattices, we are not aware of other examples of somewhat different nature. In this setting, it is worth mentioning that, under special circumstances, Zimmer has shown that the “holonomy group” of a minimal Riemannian foliation on a compact manifold with non-compact semi-simple structural Lie algebra is of “arithmetic nature” [Zim88].

1.3. The case where X is complex/Kähler. When X is a compact complex manifold and \mathcal{F} is a holomorphic foliation, the second item of Theorem A implies that \mathcal{F}_1 and \mathcal{F}_2 are also holomorphic (the transverse complex structure being the one induced by the complex structure of the ambient manifold X). Moreover, if X is Kähler, the homological orientability is automatically fulfilled.

1.4. The case where \mathcal{F} has complex codimension one. In this situation the preceding results read as follows²:

- (1) Either the leaves of \mathcal{F} are closed and the leaf space X/\mathcal{F} is a compact Riemann surface hyperbolic in the orbifold sense.
- (2) Either \mathcal{F} is minimal and transversely hyperbolic: the Hermitian symmetric space \mathcal{H} involved in the statement of Theorem C is the upper half-plane \mathbb{H} (\mathcal{H} is reduced to a point).

¹Let us recall that it is equivalent to saying that X admits a Kähler–Einstein metric with negative Ricci curvature.

²This can be proved with simpler arguments than those used in the rest of this paper.

Maybe the simplest instance of such minimal foliation is provided by transversely hyperbolic holomorphic foliations on canonically polarized projective surfaces S which appear in Brunella's classification [Bru97]. In *loc.cit.* Brunella raised the following question: is S necessarily a quotient \mathbb{H}^2/Γ of a bidisk by an irreducible cocompact lattice? Up to our knowledge, this is still an open problem.³

We give now some basic examples of transversely Kähler foliations which have a different behavior when the assumption (A1) or (A2) are dropped.

1.5. Counterexamples.

- (1) The conclusions of Theorem A do not necessarily hold if we only require the Ricci form to be only semi-negative. Actually, it may happen that the topological closure of the leaves of a linear foliation on a complex torus are real hypersurfaces. In this setting, the natural transverse metric \bar{g} is flat and the Ricci form γ vanishes identically.
- (2) For general transversely Kähler holomorphic foliations on compact Kähler manifolds, the dimension of the topological closure of the leaves is likely to vary. A simple instance of this phenomenon is the Riccati foliation constructed on a ruled surface S over a curve C of genus $g \geq 1$ by the datum of a dense representation

$$\pi_1(C) \rightarrow \mathbb{S}^1 \subset \text{Aut}(\mathbb{P}^1).$$

In this situation, there exists exactly two closed leaves and the closure of the other leaves are Levi-Flat hypersurfaces. The natural transverse metric is induced by the Fubini-study metric on \mathbb{P}^1 and thus coincides with its Ricci form. We don't know if this equidimensionality defect can occur when the Ricci form is semi-negative (but not quasi-negative).

- (3) We cannot drop the homological orientability assumption, even if X is complex and \mathcal{F} is holomorphic.

In order to justify this assertion, let us consider the examples of non-Kähler compact complex manifolds associated to number fields as constructed in [OT05] generalizing some examples of Inoue surfaces [Ino74]. We retain the presentation given in [OT05] and we refer to *loc.cit.* for details.

Let K be a number field, let $\sigma_1, \dots, \sigma_s$ be its real embeddings and $\sigma_{s+1}, \dots, \sigma_{s+2t}$ its complex embeddings ($\sigma_{s+t+i} = \overline{\sigma_{s+i}}$). Let us assume that $s, t > 0$. Let \mathbb{H} be the Poincaré upper half-plane. Let $a \in \mathcal{O}_K$ acting on $\mathbb{H}^s \times \mathbb{C}^t$ as a translation by the vector $(\sigma_1(a), \dots, \sigma_{s+t}(a))$. Let $u \in \mathcal{O}_K^{*,+}$ be a totally positive unit (i.e. $\sigma_i(u) > 0$ for all real places). Then u acts on $\mathbb{H}^s \times \mathbb{C}^t$ by $u.(z_1, \dots, z_{s+t}) = (\sigma_1(u)z_1, \dots, \sigma_{s+t}(u)z_{s+t})$. Moreover, as $t > 0$, the set $\{(\sigma_1(a), \dots, \sigma_s(a)) \mid a \in \mathcal{O}_K\}$ is dense in \mathbb{R}^s . For any subgroup U of totally positive units, the semi-direct product $U \ltimes \mathcal{O}_K$ acts freely on $\mathbb{H}^s \times \mathbb{C}^t$. This subgroup U is called admissible if the quotient space $X(K, U)$ is a compact complex manifold. In particular, admissible groups must have rank s . We can always find such admissible subgroups.

The admissible group U being given, the corresponding compact complex manifold support a transversely Hermitian symmetric foliation of complex codimension s transversely locally modeled on \mathbb{H}^s and which lifts to the vertical foliation on the universal cover $\mathbb{H}^s \times \mathbb{C}^t$. The leaves closure are thus codimension s real

³Even if it is known that the monodromy representation of this transverse hyperbolic structure takes values in an arithmetic group (see Theorem 10.1).

submanifolds of X , namely $(2t + s)$ -dimensional real tori fibering over the real s -dimensional torus \mathbb{T}^s . In any cases, these submanifolds fail to be holomorphic.

Here, the representation $\rho_{\mathcal{F}} : \pi_1(X(K, U)) \rightarrow \mathrm{PSL}(2, \mathbb{R})^s$ associated to the transverse hyperbolic structure takes values in the product of affine subgroups $\mathrm{Aff}(2, \mathbb{R})^s$ and its linear part $\rho_{\mathcal{F}}^1 : \pi_1(X(K, U)) \rightarrow (\mathbb{R}_{>0}^s, \times)$ has non-trivial image.

The product of Poincaré metrics (defining the transverse hyperbolic metric) on \mathbb{H}^s is given (up to a multiplicative factor) as $\omega = dd^c (\sum_{i=1}^s \log(\mathrm{Im}(z_i)))$ and thus descends to X as an exact two form, namely the differential of the $\pi_1(X)$ -invariant one form $d^c (\sum_{i=1}^s \log(\mathrm{Im}(z_i)))$. Hence, this foliation is not homologically orientable (albeit satisfying (A1)).

1.6. Transverse action of the group of automorphisms. Consider now a foliated compact complex manifold (X, \mathcal{F}) and let $\mathrm{Aut}(X, \mathcal{F})$ be the group of biholomorphisms of X preserving the foliation \mathcal{F} . This group contains the normal subgroup $\mathrm{Aut}^{\mathrm{Fix}}(X, \mathcal{F})$ of biholomorphisms f preserving the foliation leafwise. That is, $f \in \mathrm{Aut}^{\mathrm{Fix}}(X, \mathcal{F})$ iff for every $x \in X$, $\mathcal{L}_x = \mathcal{L}_{f(x)}$ where \mathcal{L}_x denotes the leaf through x .

Definition 1.1. We will say that the transverse action of $\mathrm{Aut}(X, \mathcal{F})$ is finite whenever the quotient $\mathrm{Aut}(X, \mathcal{F})/\mathrm{Aut}^{\mathrm{Fix}}(X, \mathcal{F})$ is finite.

In other words, the action is transversely finite if the set theoretic action $\mathrm{Aut}(X, \mathcal{F}) \times X/\mathcal{F} \rightarrow X/\mathcal{F}$ on the leaf space X/\mathcal{F} is *finite* (i.e. has finite image in the symmetric group of X/\mathcal{F}).

Assume now that \mathcal{F} is transversely Kähler and satisfies the assumption (A1).

In the classical (unfoliated) situation, the quasi-negativity of the Ricci curvature implies, according to Riemenschneider theorem [Rie73], that X is of general type. As a byproduct, the group $\mathrm{Aut}(X)$ of biholomorphisms of X is necessarily finite. In our foliated setting, it is therefore natural to inquire whether the transverse action is finite. Actually, we cannot expect it to hold in full generality. This is illustrated by the example given in §1.5-(3) above and which was actually given in [LBPRT22, §5.5] to this end. Indeed, retaining the same notations, we know from Dirichlet's units theorem that \mathcal{O}_K^* is a group of rank $s + t - 1$. Elements of $\mathcal{O}_K^{*,+}/U$ induce automorphisms of $X(K, U)$. Therefore as soon as $t > 1$, we obtain automorphisms with infinite transverse order.

We now turn our attention to the particular case where X is itself Kähler. As we can see, the situation is much better.

Theorem D. *Let (X, \mathcal{F}) be a foliated compact Kähler manifold (\mathcal{F} being holomorphic). Assume that \mathcal{F} is transversely Kähler with quasi-negative Ricci curvature. Then the transverse action of $\mathrm{Aut}(X, \mathcal{F})$ is finite.*

Remark 1.2. We are not aware of any example satisfying (A1) and (A2) with X complex non-Kähler such that the transverse action of $\mathrm{Aut}(X, \mathcal{F})$ is infinite.

2. OUTLINE OF THE PROOF

Let us now describe the strategy that we employ to establish the main Theorems A and C.

As already mentioned in the introduction, the techniques are widely inspired by previous results of Nadel [Nad90] and Frankel [Fra95] about the structure of the universal cover of a canonically polarized manifold. In particular, we make use of “foliated twisted harmonic maps”, available in our context.

From a previous result by Touzet [Tou10] (*cf.* Theorem 7.1), we know that the commuting sheaf \mathcal{C} of the foliation (in Molino's theory terminology, see §5.3) is semi-simple without compact factors. Roughly speaking, this commuting sheaf is a locally constant sheaf of Lie algebra (with typical fiber denoted by \mathfrak{g}) of basic Killing vector fields which encodes the dynamic of the foliation and which somehow represent the infinitesimal part of the holonomy pseudo-group (see §6.1, this is a common feature of Riemannian foliations). In our setting, we can derive from semi-simplicity of \mathfrak{g} the existence of a representation (the *monodromy representation* of \mathcal{F} attached to \mathcal{C}):

$$\rho : \pi_1(X) \longrightarrow \text{Aut}^0(\mathfrak{g})$$

with dense image (up to replacing X with a finite étale cover). We can also derive from semi-simplicity that the leaves of $\tilde{\mathcal{F}}$ (the lifted foliation $\tilde{\mathcal{F}}$ on the universal cover \tilde{X}) are closed. The latter being a complete Riemannian foliation, this implies that its space of leaves $\tilde{X}/\tilde{\mathcal{F}}$ is a complete Kähler orbifold as described in Lemma B. The constant sheaf $\tilde{\mathcal{C}}$ defined on \tilde{X} as the lift of \mathcal{C} can be then identified to a Lie subalgebra of the Lie algebra of Killing fields on $\tilde{X}/\tilde{\mathcal{F}}$.

On the other hand, it is well known that $\text{Aut}^0(\mathfrak{g})$ is identified, via the adjoint action, to the unique center-free Lie (and algebraic) group G having \mathfrak{g} as Lie algebra.

Let K be maximal compact subgroup of G . According to a theorem of Corlette [Cor88], there exists a unique ρ -twisted harmonic map $f : \tilde{X} \rightarrow G/K$ (depending of a course on a given Riemannian structure g on X). It is then natural to investigate the existence of such f by requiring f to be constant on the leaves of $\tilde{\mathcal{F}}$. Such a property will be called \mathcal{F} -invariance. It turns out that this can be realized provided the metric g is suitably chosen. Namely, g is bundle-like and \mathcal{F} is *taut*: the leaves of \mathcal{F} are minimal submanifolds with respect to g . Actually, the assumption “homologically orientable” is equivalent to the existence of this kind of metric as stated in Theorem 3.7. Basically, one way to prove the existence of an \mathcal{F} -invariant harmonic map is first to construct a smooth ρ -equivariant and \mathcal{F} -invariant map f_0 and to deform it to a harmonic one via the usual evolution equation. By the result of Section 4 (which we hope has an interest in its own right), the solutions f_t will remain \mathcal{F} -invariant and so will be the sought harmonic map which is obtained by taking the limit when $t \rightarrow \infty$. Actually, we do not proceed exactly in this way because it seems rather delicate to construct directly f_0 from the original manifold X . To circumvent this problem, we work on the transverse orthonormal frame bundle X^\sharp equipped with its natural structure of $U(n)$ -principal bundle over X . According to Molino's theory, the foliation \mathcal{F} lifts to X^\sharp as a transversely parallelizable foliation \mathcal{F}^\sharp of the same rank and the commuting sheaf of \mathcal{F}^\sharp descends on X as the commuting sheaf \mathcal{C} . In particular, the monodromy representation of \mathcal{F}^\sharp is exactly given by ρ . Now, we can exploit the structure of transversely parallelizable foliations to construct on the universal covering \tilde{X}^\sharp (see [Mol88, p. 162]) a ρ -equivariant map F_0 with values in G/K . We can then deform F_0 following the evolution equation

$$\frac{\partial F_t}{\partial t} = -d_{\nabla}^* d(F_t)$$

preserving the ρ -equivariance of the map F_t .

Using that the representation ρ is dense and in particular reductive, we get, taking the limit, a ρ -equivariant and $\tilde{\mathcal{F}}^\sharp$ -invariant harmonic map F_∞ which is by uniqueness aspect of Corlette's Theorem invariant by the isometric action of the structural group $U(n)$ of \tilde{X}^\sharp . This actually implies that F_∞ descends on \tilde{X} as a $\tilde{\mathcal{F}}$ -invariant harmonic map f_∞ (with respect to the original bundle-like metric \tilde{g})

Alternatively, this enables us to consider f_∞ as a ρ -equivariant harmonic map

$$f_\infty : \tilde{X}/\tilde{\mathcal{F}} \longrightarrow G/K$$

which can be easily seen to be a surjective submersion with connected fibers. In addition we can prove that

- (1) $\tilde{\mathcal{C}}$ projects via f_∞ to $\mathfrak{isom}(G/K)$ (the Lie algebra of the isometry group $\mathrm{Isom}(G/K)$ of the symmetric space G/K).
- (2) For every fiber F of f_∞ , for every $x \in F$, there does not exist any $V \in \tilde{\mathcal{C}}$ such that $V(x) \neq 0$ and $V(x) \in T_x F/\tilde{\mathcal{F}}_x$.
- (3) f_∞ is H -equivariant with respect to a subgroup of the isometry group $\mathrm{Isom}(\tilde{X}/\tilde{\mathcal{F}})$ which integrates $\tilde{\mathcal{C}}$.

If we combine this with the rigidity properties of harmonic maps as proved by Carlson–Toledo and Jost–Yau, we obtain that $\mathcal{H} := G/K$ is a Hermitian symmetric space and that f_∞ is indeed holomorphic (up to switching the complex structure of \mathcal{H} to its conjugate). The proof is widely inspired from Frankel’s article [Fra95] and is somehow simpler, at least as far the H -equivariance property is concerned. On the other hand, the item (2) is a byproduct of the existence of solutions of a foliated Monge–Ampère equation in the spirit of what is done in [ElK90] and which requires a fairly more technical analysis.

Again relying on Frankel’s argumentation [Fra89, Fra95], we can deduce the existence of a holomorphic splitting

$$T(\tilde{X}/\tilde{\mathcal{F}}) = \mathcal{G}_1 \oplus \mathcal{G}_2$$

where \mathcal{G}_1 is the foliation induced by the orbits of $\tilde{\mathcal{C}}$ and \mathcal{G}_2 is the fibration defined by f_∞ . Keeping in mind the dynamical meaning of \mathcal{C} , we obtain Theorems A and C.

Theorem D turns to be a consequence of the latter statements together with results concerning linear representations of Kähler groups that we will recall subsequently (see Section 9).

Organization of the article. Let us describe briefly the content of this article. Section 3 gathers standard results on Riemannian foliations and the basic objects attached to them. The (foliated) harmonic flow is studied in Section 4 and more precisely how the tangential energy behaves with respect to the heat operator. This fundamental estimate will be used to produce harmonic maps that are constant along the leaves of the foliation. Using this and Molino’s theory of Riemannian foliations, foliated harmonic maps are constructed in Sections 5 and 6. The Kähler case is studied in the remaining sections. In Section 7, we show the existence of a supplementary foliation and prove the main statements (Theorems A and C). Section 8 is devoted to the proof of a doubly foliated version of Yau’s theorem and Section 9 gives the proof of Theorem D. In Section 10, we formulate final questions and remarks.

3. RECOLLECTIONS/NOTATION

Let (X, \mathcal{F}) be a foliated manifold. One will denote respectively by m and n the *rank* and the *codimension* of \mathcal{F} .⁴ We follow notation from [Mol88, Chapter 2]: $\mathfrak{X}(X)$ (resp.

⁴For the time being, we deal with real foliations. However, from Section 7 for the sake of notational simplicity, we will also denote by n the complex codimension of transversely Kähler foliations (hence of real codimension $2n$). Actually this is the notation used in the introductory part.

$\mathfrak{X}(\mathcal{F})$) stands for the Lie algebra of vector fields on X (resp. tangent to \mathcal{F}). Let us denote by $L(X, \mathcal{F}) \subset \mathfrak{X}(\mathcal{F})$ the Lie algebra of *foliated* vector fields:

$$L(X, \mathcal{F}) = \{v \in \mathfrak{X}(X) \mid [\mathfrak{X}(\mathcal{F}), v] \subset \mathfrak{X}(\mathcal{F})\}.$$

Remark that $L(X, \mathcal{F})$ is a module over the ring $\Omega^0(X/\mathcal{F})$ of basic functions (*i.e.* functions constant on the leaves). Let us finally consider $\ell(X, \mathcal{F})$ the Lie algebra of *basic* vector fields defined by the quotient

$$\ell(X, \mathcal{F}) := L(X, \mathcal{F}) / \mathfrak{X}(\mathcal{F}).$$

It is also a module over $\Omega^0(X/\mathcal{F})$.

Definition 3.1 (*cf.* [Mol88, Chapter 1, §1.1 and 1.4]).

- (1) An open subset $U \subset X$ is said to be *distinguished* (with respect to the foliation \mathcal{F}) if there exists a diffeomorphism $\phi = (x_1, \dots, x_m, y_1, \dots, y_n) : U \rightarrow \Omega$ onto a domain Ω of $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ such that the restriction of \mathcal{F} to U is given by $\{dy_1 = \dots = dy_n = 0\}$.
- (2) The foliation \mathcal{F} is said to be *simple* if its leaves can be defined as the fibers of some surjective submersion $f : X \rightarrow Y$. In particular the space of leaves X/\mathcal{F} (with the quotient topology) is homoeomorphic to Y and thus inherits a natural structure of manifold such that the projection map $X \rightarrow X/\mathcal{F}$ is submersive.

Of course, we can cover X by open subsets U which are both distinguished and simple (with respect to the projection $(x, y) \mapsto y$) for the restricted foliation $\mathcal{F}|_U$. In the sequel, when we will consider *local* foliated/basic objects or *local* space of leaves, it should be understood that this means “in restriction to these peculiar neighborhoods”. The pair (U, ϕ) as above will be referred as a *foliated chart*.

3.1. Basic tensor fields/basic cohomology. Let $N\mathcal{F} = TX/\mathcal{F}$ denote the *normal* bundle to \mathcal{F} . The respective duals \mathcal{F}^* , $N^*\mathcal{F}$ are called the *cotangent* and *conormal* bundle of \mathcal{F} . The (local) sections of the latter are precisely (local) forms of degree one whose restriction to the leaves vanish identically. More generally, a transverse (p, q) tensor field is a section of $\Gamma_{\mathcal{F}}^{p,q} := N\mathcal{F}^{\otimes p} \otimes N^*\mathcal{F}^{\otimes q}$.

We can alternatively define a basic vector field as a section of $N\mathcal{F}$ which is flat with respect to the partial *Bott connection*

$$\nabla^{\mathcal{F}} : N\mathcal{F} \rightarrow N\mathcal{F} \otimes \mathcal{F}^*$$

defined for every local sections X and Y (respectively sections of \mathcal{F} and $N\mathcal{F}$) by $\nabla_X^{\mathcal{F}} Y = [X, Y]$ (well defined as a section of $N\mathcal{F}$ by integrability).

We can more generally define a *basic* (p, q) transverse tensor field as a section s of $\Gamma_{\mathcal{F}}^{p,q}$ such that $\nabla^{\mathcal{F}} s = 0$ (here, we denote by the same symbol the extension of the Bott connection to $\Gamma_{\mathcal{F}}^{p,q}$). In local foliated transverse coordinates $x = (x_1, \dots, x_n)$, this amounts to saying that s can be written as a sum of simple tensors of the form

$$f(x) \frac{\partial}{\partial x_{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x_{i_p}} \otimes dx_{j_1} \otimes \dots \otimes dx_{j_q}.$$

For instance, a (transversely) Riemannian foliation is a foliation equipped with a basic transverse metric \bar{g} , that is a basic section \bar{g} of the symmetric power $\text{Sym}^2 N^*\mathcal{F} \subset N^*\mathcal{F}^{\otimes 2}$ which is positive definite on $N\mathcal{F}$. In the same way, a basic endomorphism of $N\mathcal{F}$ is a basic section of $N\mathcal{F} \otimes N^*\mathcal{F}$, etc. . . We can easily check that a q -form $\theta \in \Gamma(\bigwedge^q T^*X)$ is basic (then in particular is a section of $N^*\mathcal{F}^{\otimes q}$) if and only if we have $i_v \theta = i_v(d\theta) = 0$ for every vector field v tangent to \mathcal{F} .

Note that any automorphism of X preserving the foliation naturally induces an automorphism f_* of the vector space of basic transverse tensor fields.

The algebra $\Omega^\bullet(X/\mathcal{F})$ of basic differential forms is then a subcomplex of $\Omega(X)$ whose cohomology $H^*(X/\mathcal{F})$ is the so-called *basic (de Rham) cohomology*.

In the case where \mathcal{F} is (a codimension n) *Riemannian* foliation on a *compact* manifold X , the basic cohomology turns out to be finite dimensional, according to [ElKH86]. Moreover, the same authors prove in the same setting the following.

Theorem 3.2 (cf. [ElKH86, Théorème de dualité 4.10]). *We have the following alternative:*

- (1) either $H^n(X/\mathcal{F}) = 0$,
- (2) or $H^n(X/\mathcal{F}) \simeq \mathbb{R}$, in which case \mathcal{F} is transversely orientable, $H^n(X/\mathcal{F})$ is generated by the class of the basic volume form and $H^*(X/\mathcal{F})$ satisfies the Poincaré–Hodge duality.

3.2. Bundle-like metrics. Let (X, \mathcal{F}) be a foliated manifold where in addition \mathcal{F} is assumed to be Riemannian. Let us denote by \bar{g} the transverse invariant metric defined on the normal bundle $N\mathcal{F}$ and by $\nu_{\mathcal{F}}$ the associated transverse volume form, assuming that $N\mathcal{F}$ is oriented. A metric g on X is said to be *bundle-like* (with respect to the given Riemannian transverse structure) if the metric induced on $\mathcal{F}^\perp \simeq N\mathcal{F}$ is precisely \bar{g} . This amounts to saying that \mathcal{F} can be locally described by the fibers of a Riemannian submersion where the basis of the fibration is equipped with the metric \bar{g} . It is well known (see [Rei59]) that there always exists a bundle-like metric. The triple (X, \mathcal{F}, g) will be referred as a *Riemannian foliated manifold*. The pair (\mathcal{F}, g) stands for a Riemannian foliation together with a bundle-like metric g on the ambient manifold.

3.3. (Riemannian) extensions/ $J_{\mathcal{F}}$ holomorphic functions.

Definition 3.3 (Extension of a foliation). Let (X, \mathcal{F}) be a foliated manifold. A foliation \mathcal{G} on X is said to be an *extension* of \mathcal{F} if $\mathcal{F} \subset \mathcal{G}$.

Suppose in addition that there exists on X a basic ⁵ symmetric form $\sigma \in \text{Sym}^2 N^*\mathcal{F}$ such that the restriction of σ to \mathcal{G}/\mathcal{F} is positive. This implies that \mathcal{F} is Riemannian in restriction to the leaves of \mathcal{G} with respect to $\sigma' := \sigma|_{\mathcal{G}/\mathcal{F}}$. We will say that σ' is a *\mathcal{F} -basic \mathcal{G} -leafwise metric*. For any (local) basic section v of \mathcal{G}/\mathcal{F} , it is then meaningful to consider the \mathcal{F} -basic \mathcal{G} -leafwise divergence $\text{div}_{\mathcal{G}}(v)$ well defined as a (local) basic function of \mathcal{F} . Similarly, we can consider the \mathcal{F} -basic \mathcal{G} -leafwise gradient $\nabla_{\mathcal{G}}(f)$ (with respect to $\bar{g}|_{\mathcal{G}/\mathcal{F}}$) of a (local) basic function as a (local) basic function of \mathcal{F} . We can thus consider the associated *\mathcal{F} -basic \mathcal{G} -leafwise Laplacian* $\Delta_{\mathcal{G}}$ acting on local basic functions by

$$\Delta_{\mathcal{G}}(f) = -\text{div}_{\mathcal{G}}(\nabla_{\mathcal{G}}(f)).$$

It is well defined on X as a *basic* differential operator of order 2 (see [ElK90]). It will play a prominent role in our work (see §8.1).

Consider now the situation where (X, \mathcal{F}) is a foliated manifold such that \mathcal{F} is transversely holomorphic of complex codimension n . Recall that this means that we can cover X by foliated charts U_i of the form $f_i : U_i \rightarrow V_i \subset \mathbb{R}^m \times \mathbb{C}^n$ such that for every i, j such that $U_i \cap U_j \neq \emptyset$, the local transformation map $f_{ij} = f_j \circ f_i^{-1}$ of $\mathbb{R}^m \times \mathbb{C}^n$ takes the form

$$f_{ij}(x, z) = (g_{ij}(x, z), h_{ij}(z))$$

⁵Here and hereafter, we mean basic with respect to \mathcal{F} .

where h_{ij} is *holomorphic*.

As usual this is equivalent to the datum of a basic endomorphism $J_{\mathcal{F}} : N\mathcal{F} \rightarrow N\mathcal{F}$ satisfying $J_{\mathcal{F}}^2 = -\text{Id}$ (a transverse basic complex structure) fulfilling the Newlander–Nirenberg integrability property:

for all local basic sections v, w of $N\mathcal{F}^{0,1}$, $[v, w]$ is still a section of $N\mathcal{F}^{0,1}$.

Here, $N\mathcal{F}^{0,1}$ is the second summand in the splitting

$$N\mathcal{F} \otimes \mathbb{C} = N\mathcal{F}^{1,0} \oplus N\mathcal{F}^{0,1}$$

determined by the $\pm\sqrt{-1}$ eigenspaces of $J_{\mathcal{F}}$ (more generally, if E is any $J_{\mathcal{F}}$ -stable subbundle of $N\mathcal{F}$, we can analogously consider $E^{1,0}$ and $E^{0,1}$).

Let $f : X \rightarrow Y$ be a smooth *basic* function taking values in a complex manifold Y . Let J be the complex structure on Y . Then f is said to be *transversely-holomorphic* or merely *$J_{\mathcal{F}}$ -holomorphic* if $f_* \circ J_{\mathcal{F}} = J \circ f_*$. This exactly means that the local factorization $\bar{f} : U/\mathcal{F} \rightarrow Y$ of f through the local leaves spaces is *holomorphic*.

Similarly, we will say that an extension \mathcal{G} of \mathcal{F} is *$J_{\mathcal{F}}$ -holomorphic* if the foliation induced by \mathcal{G} on U/\mathcal{F} is *holomorphic*. Alternatively, holomorphicity of \mathcal{G} can be characterized by

- The vector bundle $\mathcal{G}/\mathcal{F} \subset N\mathcal{F}$ is $J_{\mathcal{F}}$ -stable.
- $(\mathcal{G}/\mathcal{F})^{1,0} \subset N\mathcal{F}^{1,0}$ is locally spanned by basic *holomorphic* vector fields (*i.e.* local sections of $N\mathcal{F}^{1,0}$ that project to holomorphic vector fields on U/\mathcal{F}).

3.4. Characteristic and mean curvature forms. Let (X, \mathcal{F}) be a foliated manifold. We also assume that \mathcal{F} is oriented. Let g be a Riemannian metric on X . The characteristic form $\chi_{\mathcal{F}}$ is the m -form (where $m = \text{rk}(\mathcal{F})$) defined by the following properties:

- (1) the restriction of $\chi_{\mathcal{F}}$ to the leaves is the volume form associated to $g_{\mathcal{F}}$ (the leafwise metric induced by g);
- (2) for all $v \in \Gamma(\mathcal{F}^{\perp})$, $i_v(\chi_{\mathcal{F}}) = 0$.

Let $\tau_g \in \Gamma(\mathcal{F}^{\perp})$ be the mean curvature vector of the leaves with respect to g . We can define the *mean curvature form* κ_g by setting

$$\kappa_g(s) = g(\tau_g, s) \quad \text{for } s \in \Gamma(TX).$$

The following fundamental result was proven by Dominguez [Dom98].

Theorem 3.4. *Let (X, \mathcal{F}) be a foliated compact manifold. Assume that \mathcal{F} is Riemannian. Then there exists a bundle-like metric g such that κ_g is basic (or equivalently, τ_g is foliated).*

Remark 3.5. As noticed in [Ton97, Chapter 7], when (X, \mathcal{F}, g) is a compact Riemannian foliated manifold, this basic mean curvature form κ_g is indeed closed and its cohomology class $[\kappa] := [\kappa_g] \in H^1(X, \mathbb{R})$ does not depend on the choice of the bundle-like metric g . Moreover, $[\kappa]$ vanishes if and only if there exists a bundle-like metric g such that κ_g vanishes identically, *i.e.* the leaves are immersed *minimal* submanifolds.

Definition 3.6. We will say that a Riemannian foliation (\mathcal{F}, g) is *tense* if κ_g is basic and (\mathcal{F}, g) is *taut* if κ_g vanishes identically.⁶

⁶This slightly differs from the usual terminology, according to which \mathcal{F} is tense/taut if there exists bundle-like metrics g fulfilling these properties, but which are not necessarily the one we are working with. For this peculiar metrics, tense/taut (in our sense) corresponds to isoparametric/minimal.

Theorem 3.7 (cf. [Mas92]). *Let (X, \mathcal{F}) be a compact foliated manifold such that \mathcal{F} is transversely Riemannian. Suppose that \mathcal{F} is transversely orientable. Then the following two properties are equivalent:*

- *There exists a bundle-like metric g such that (\mathcal{F}, g) is taut.*
- *The foliation \mathcal{F} is homologically orientable.*

3.5. Relationship between characteristic form and mean curvature form. It is given by the so-called Rummmler's formula [Ton97, Corollary 4.26]. We assume here that (X, \mathcal{F}) is a foliated manifold, that \mathcal{F} is oriented and we fix a Riemannian metric g on X . Set as before $m = \text{rk}(\mathcal{F})$, then:

$$(3.1) \quad d\chi_{\mathcal{F}} + \kappa_g \wedge \chi_{\mathcal{F}} \in F^2 A^{m+1}$$

where $F^2 A^{m+1}$ denotes the space of $(m+1)$ -forms φ on X such that $i_V \varphi = 0$ for any m -multivector field tangent to \mathcal{F} (i.e. $V \in \Gamma(\bigwedge^m \mathcal{F})$). In particular, if $\eta \in \Omega^{n-1}(X/\mathcal{F})$ is a basic $(n-1)$ -form where n is the codimension of \mathcal{F} , then $\eta \wedge d\chi_{\mathcal{F}} = -\eta \wedge \kappa_g \wedge \chi_{\mathcal{F}}$.

As a straightforward but fundamental consequence of Stokes' Theorem that we will use repeatedly, we have the following result.

Lemma 3.8. *Let (X, \mathcal{F}) be an oriented compact foliated manifold and g be a Riemannian metric on X . Assume that \mathcal{F} is oriented and that κ_g vanishes identically. If $n = \text{codim}(\mathcal{F})$ and $\eta \in \Omega^{n-1}(X/\mathcal{F})$, we then have:*

$$\int_X d\eta \wedge \chi_{\mathcal{F}} = 0.$$

3.6. Some adapted orthonormal frame bundle. Let (X, \mathcal{F}, g) be a Riemannian foliated manifold (see §3.2). Set $m + n = \dim(X)$ where $m = \text{rk}(\mathcal{F})$. Denote by ∇^g the Levi-Civita connection associated to g . Recall the two basic fundamental properties of ∇_g , namely the torsion freeness and metric compatibility

$$(3.2) \quad \forall u, v \in \Gamma(TX), \quad \nabla_u^g v - \nabla_v^g u = [u, v],$$

$$(3.3) \quad \forall u, v, w \in \Gamma(TX), \quad u \cdot \langle v, w \rangle_g = \langle \nabla_u^g v, w \rangle_g + \langle v, \nabla_u^g w \rangle_g.$$

Definition 3.9. Let $(e_i)_{i=1, \dots, m}$ be a local orthonormal frame of \mathcal{F} near $x \in X$. The family (e_i) is said to be tangentially geodesic (with respect to \mathcal{F}) at the point $x \in X$ if for all $v \in T_x X$, we have:

$$\nabla_v^g e_i(x) \in (\mathcal{F}_x)^\perp.$$

If (e_i) is tangentially geodesic, note that the Lie bracket $[e_i, e_j]$ vanishes at the point x as an immediate consequence of the torsion freeness of ∇^g and the involutivity of \mathcal{F} :

$$\forall u \in \mathcal{F}_x, \quad \langle [e_i, e_j], u \rangle_g = \langle \nabla_{e_i}^g e_j, u \rangle_g - \langle \nabla_{e_j}^g e_i, u \rangle_g = 0.$$

Lemma 3.10. *For every $x \in X$, there exists a tangentially geodesic frame of \mathcal{F} at the point x .*

Proof. Let \mathcal{L} be the leaf of \mathcal{F} through x and $g_{\mathcal{L}}$ be the metric induced by g on \mathcal{L} . Take (f_i) an orthonormal frame with respect to $g_{\mathcal{L}}$ defined in some neighborhood $V \subset \mathcal{L}$ of x and geodesic in the usual sense at x and extend it as a orthonormal frame (ξ_i) of \mathcal{F} in some neighborhood $U \subset X$ of the ambient manifold. For all $v \in \mathcal{F}_x$, we have thus $\nabla_v^g \xi_i(x) \in \mathcal{F}_x^\perp$. We want to extend this property for all $v \in T_x X$ by modifying suitably

the ξ_i 's. To this aim, pick a basis (v_k) , $k = m + 1, \dots, m + n$ of \mathcal{F}_x^\perp . For every $(i, j, k) \in \llbracket 1, m \rrbracket^2 \times \llbracket m + 1, m + n \rrbracket$, there exist $a_{ijk} \in \mathbb{R}$, $v_{ik} \in (T_x \mathcal{F})^\perp$ such that

$$\nabla_{v_k}^g \xi_i(x) = \sum_{i,j} a_{ijk} \xi_j(x) + v_{ik}.$$

From (3.3), we can infer the skew-symmetry property $a_{ijk} = -a_{jik}$ so that there certainly exists a family (A_{ij}) of smooth function $A_{ij} : U \rightarrow \mathbb{R}$ (shrinking U if necessary) fulfilling the followings:

- A_{ij} vanishes along the leaf \mathcal{L} ,
- $v_k(A_{ij})(x) = a_{ijk}$,
- $A_{ij} = -A_{ji}$.

For $i = 1, \dots, m$, set $\epsilon_i = \xi_i - \sum_j A_{ij} \xi_j$. By construction, the family (ϵ_i) forms a local frame of \mathcal{F} around x and satisfies

$$\nabla_v^g \epsilon_i(x) \in \mathcal{F}_x^\perp, \text{ for every } v \in T_x X.$$

Let (e_i) be the local orthonormal frame of \mathcal{F} produced by Gram–Schmidt orthonormalization process applied to the family (ϵ_i) . The properties of the A_{ij} 's listed above guarantee that for $i \neq j$, $\langle \epsilon_i, \epsilon_j \rangle_g$ vanishes at x at order at least 2. This easily implies that the jets of e_i and ϵ_i at the point x coincide up to order 1, thus proving that (e_i) is tangentially geodesic at x . \square

The following lemma will be useful in the proof of the forthcoming Proposition 4.4 to be found in §4.2.

Lemma 3.11. *Let $(e_i)_{i=1, \dots, m+n}$ be a local orthonormal frame of TX near $x \in X$ such that $(e_i)_{i=1, \dots, m}$ is tangentially geodesic at x and e_i is foliated for $i > m$. Then for any indices $i, l \in \llbracket 1, m \rrbracket$ and $k \in \llbracket m + 1, m + n \rrbracket$, we have*

$$\langle [e_i, e_k], e_l \rangle_g(x) = \langle [e_l, e_k], e_i \rangle_g(x)$$

Proof. This is a straightforward consequence of (3.2), (3.3) and the fact that \mathcal{F} is stable under Lie bracket. \square

Concerning the transverse behavior of the Levi-Civita connection, we have the following (see [O'N66, Lemma 1]):

Lemma 3.12. *Let u, v be local foliated vector fields orthogonal to \mathcal{F} . Then, the orthogonal projection of $\nabla_u^g v$ on \mathcal{F}^\perp is still foliated.*

3.7. Tangential energy. Let (X, \mathcal{F}, g) be a foliated Riemannian manifold and $f : X \rightarrow Y$ be a smooth function with values in a Riemannian manifold (Y, h) . Let us denote by f_*^T the restriction of the differential f_* to \mathcal{F} and let us define the *tangential energy density* (with respect to \mathcal{F}) as the function

$$e_T(f) : \begin{cases} X & \longrightarrow \mathbb{R}^+ \\ x & \longmapsto \frac{1}{2} \|f_*^T\|_2^2 = \frac{1}{2} \text{Trace} \left((f_*^T)^* f_*^T \right) = \frac{1}{2} \sum_i |f_*(e_i)|_h^2 \end{cases}$$

where (e_i) is any orthonormal basis of \mathcal{F}_x . Here, the star exponent stands for the adjoint.

3.8. Tension field and basic maps. We maintain notation/assumptions from §3.7. In the sequel, we will denote indifferently the differential of f by df or f_* . Recall that the tension field is the section of $f^*(TY)$ defined by

$$\tau(f) = \text{Trace}(\nabla(df)) = \text{div}(df) = -d_{\nabla}^* df.$$

Here, ∇ is the pull-back by f of the Levi-Civita connection ∇^h on Y and d_{∇}^* is the adjoint operator of the differentiation

$$d_{\nabla} : A^k(X, T^*X \otimes f^*(TY)) \longrightarrow A^{k+1}(X, T^*X \otimes f^*(TY))$$

of forms on X valued in $f^*(TY)$.

In more layman terms, once we have fixed a local orthonormal frame (e_i) of TX ,

$$(3.4) \quad \tau(f) = \sum_i \nabla_{e_i} f_* e_i - f_* \left(\sum_i \nabla_{e_i}^g e_i \right).$$

Let us now focus on the case where f is **basic** (i.e. leafwise constant). Consider the curvature form of the connection ∇ :

$$\mathcal{R}(v, w) = \nabla_v \nabla_w - \nabla_w \nabla_v - \nabla_{[v, w]}$$

where $v, w \in \Gamma(TX)$. If $v \in \Gamma(\mathcal{F})$, $f_* v = 0$ and from the definition of ∇ , we can infer that

$$(3.5) \quad i_v \mathcal{R} = 0$$

for every $v \in \Gamma(\mathcal{F})$. Let us fully justify this vanishing property. Pick $x \in X$ and any local section s of TY defined near $f(x)$. Set $s_f = s \circ f$. This is well defined as a local section of $f^*(TY)$ near x . By multilinearity of the curvature tensor, it is then sufficient to check that $\mathcal{R}(v, w)(s_f) = 0$ for any local foliated vector field w . This last point is just a consequence of the fact that both s_f and $\nabla_w(s_f)$ are constant along the leaves and that $[v, w]$ is tangent to \mathcal{F} , the vector field w being foliated. This yields

$$\nabla_v(s_f) = \nabla_v \nabla_w(s_f) = \nabla_{[v, w]}(s_f) = 0.$$

In the terminology of [KT75, Definition 2.33] (see also [Mol88, §2.6]) $f^*(TY)$ is a *foliated vector bundle* and ∇ is a *basic connection* on it. Equivalently, the pair $f^*(TY)$ belongs to the category of \mathcal{F} -vector bundles as defined in [ElK90, Section 2.2]. In our context, a section s of $f^*(TY)$ is said to be *basic* if $\nabla_v s = 0$ for every $v \in \Gamma(\mathcal{F})$. Local basic sections form a free module over the ring of local basic functions whose rank coincides with that of T_Y .

More generally, denote by $A_b^k \subset A^k := A^k(X, T^*X \otimes f^*(TY))$ the subspace of twisted basic forms of degree k , that is $\alpha \in A_b^k$ iff for every $v \in \Gamma(\mathcal{F})$, we have $i_v(\alpha) = 0$ and $i_v(d_{\nabla} \alpha) = 0$. More explicitly, $\alpha \in A^k$ is basic if and only if it can be written locally as a finite sum of simple tensors of the form $s \otimes \xi$ where s and ξ are respectively (local) basic sections of f^*TY and $\wedge^k N^* \mathcal{F} \subset \wedge^k T^*X$. Thanks to (3.5), the differential d_{∇} induces a differential on the graded algebra A_b^\bullet .

Remark that for every local foliated vector field v , $f_* v$ is constant along the leaves, so that $\nabla_w f_* v = 0$ for every w tangent to \mathcal{F} . In particular the tension field of f takes the simplified expression:

$$(3.6) \quad \tau(f) = \sum_i \nabla_{e_i} f_* e_i - f_* \left(\sum_i \nabla_{e_i}^g e_i \right) - f_*(\tau_g).$$

Here, (e_i) is any local orthonormal frame of \mathcal{F}^\perp . Suppose in addition that \mathcal{F} is *tense* with respect to g (i.e. τ_g is foliated). By choosing the e_i 's to be foliated and thanks to

Lemma 3.12 and (3.5), we immediately check that $\tau(f)$ is *basic* in the previous sense, that is

$$\nabla_w \tau(f) = 0$$

for every $w \in \Gamma(\mathcal{F})$.

As noticed before we can restrict the operator d_∇ to the graded algebra A_b of basic forms. Moreover, when \mathcal{F} is transversely oriented, the basic star operator $\bar{\star}$ defined at the level of local basic forms (cf. [Ton97, Chapter 7]) extends *via* the metric structure to $f^*(TY)$ as an operator that we denote by the same symbol $\bar{\star} : A_b^\bullet \rightarrow A_b^{n-\bullet}$ so that we can consider the adjoint $d_\nabla^\bar{\star} : A_b^\bullet \rightarrow A_b^{\bullet-1}$. As $f \in A_b^0$, we can thus define its *basic tension field*

$$\tau_b(f) = -d_\nabla^\bar{\star} df.$$

More explicitly, let us consider U a sufficiently small distinguished simple open set of X and $\bar{f} : U/\mathcal{F} \rightarrow Y$ the map that factorizes f through the projection $\pi : U \rightarrow U/\mathcal{F}$. Note that π induces a bundle map $\pi_* : f^*(TY) \rightarrow \bar{f}^*(TY)$. The (local) space of leaves U/\mathcal{F} is equipped with the transverse metric \bar{g} and we have

$$\pi_* \tau_b(f) = \tau(\bar{f}) \circ \pi.$$

If (e_i) is a local orthonormal frame of \mathcal{F}^\perp and (\bar{e}_i) the corresponding orthonormal frame with respect to \bar{g} (regarded as vector fields on U/\mathcal{F}), we have

$$\tau(\bar{f}) = \sum_i \bar{\nabla}_{\bar{e}_i} \bar{f}_* \bar{e}_i - \bar{f}_* \left(\sum_i \bar{\nabla}_{\bar{e}_i}^{\bar{g}} \bar{e}_i \right)$$

where $\bar{\nabla} = \bar{f}^*(\nabla^h)$. In view of (3.6) and in the particular case where τ_g vanishes identically, this somehow means that the notion of harmonicity (with respect to g) and transverse harmonicity (with respect to \bar{g}) coincide.

We gather the previous observations in the following.

Proposition 3.13. *Let (X, \mathcal{F}, g) be a Riemannian foliated manifold and $f : X \rightarrow Y$ be a leafwise constant smooth map to a Riemannian manifold (Y, h) . Then the ordinary tension field $\tau(f)$ and the basic tension field $\tau_b(f)$ are related by*

$$\tau(f) = \tau_b(f) - f_*(\tau_g).$$

In particular $\tau(f)$ is basic if (\mathcal{F}, g) is tense and $\tau(f) = \tau_b(f)$ whenever (\mathcal{F}, g) is taut.

Remark 3.14. A special occurrence of Proposition 3.13 is given in [Ton97, Chapter 7], when $Y = \mathbb{R}$ equipped with the standard euclidean metric. In this context, $-\tau$ and $-\tau_b$ are nothing but the usual and basic Laplacian.

4. BEHAVIOR OF THE TANGENTIAL ENERGY UNDER THE EVOLUTION EQUATION

4.1. Setting. Let (X, \mathcal{F}) be a $(m+n)$ -dimensional compact foliated manifold where \mathcal{F} is Riemannian and has rank m . Let us equip \mathcal{F} with a bundle-like metric g such that the mean curvature vector field of the leaves is *foliated*. This can be always achieved thanks to Theorem 3.4.

Let (Y, h) be a Riemannian manifold together with a representation

$$\rho : \pi_1(X) \rightarrow \text{Isom}(Y).$$

of the fundamental group of X , seen as the group of deck transformations of the universal cover \tilde{X} , into the isometry group of Y .

Let $f_0 : \tilde{X} \rightarrow Y$ be a smooth ρ -equivariant mapping and let us also consider a smooth variation $(f_t)_{t \in I}$ of f_0 (with $I = [0, t_0]$), that is a mapping $F \in \mathcal{C}^\infty(I \times \tilde{X})$, $f_t := F(t, \cdot)$ such that

- (1) f_t is ρ -equivariant,
- (2) The family (f_t) satisfies the evolution equation

$$(4.1) \quad \frac{\partial f_t}{\partial t} = \tau(f_t)$$

where $\tau(f_t) \in \Gamma(f^*TY)$ is the *tension field* of f_t (see §3.8).

Finally, let us denote by $e_T : \tilde{X} \times I \rightarrow \mathbb{R}_{\geq 0}$ the function $(x, t) \mapsto e_T(f_t)(x)$ where the tangential energy density (introduced in §3.7) is taken with respect to the foliation and the complete metric on \tilde{X} obtained by respective pull-backs of \mathcal{F} and g . Obviously, e_T descends on $X \times I$ as a function denoted in the same way.

Theorem 4.1. *The tangential energy density e_T is a subsolution of the heat operator, i.e. there exists a positive constant C such that*

$$\left(\frac{\partial}{\partial t} + \Delta^g \right) (e_T) \leq C e_T.$$

Let us introduce the following terminology.

Definition 4.2. A smooth ρ -equivariant mapping $f : \tilde{X} \rightarrow Y$ is said to be \mathcal{F} -invariant whenever f is basic, i.e. leafwise constant (with respect to the lifted foliation $\tilde{\mathcal{F}}$).

Corollary 4.3. *If the initial datum f_0 is \mathcal{F} -invariant, then the maps f_t are \mathcal{F} -invariant as well for all time $t \in I$.*

Here and hereafter Δ^g (resp. ∇^g) denotes indifferently the Laplace–Beltrami operator $\Delta^g(u) = -\operatorname{div}(\operatorname{grad}(u))$ (resp. the Levi-Civita connection) with respect to the metric g or the lifted metric \tilde{g} , depending on whether we work on X or \tilde{X} .

Proof. This is a completely standard application of the maximum principle in the presence of subsolutions of the heat operator $\frac{\partial}{\partial t} + \Delta^g$ (Moser–Harnack’s inequality [Mos64]): we write $a(x, t) = e_T(x, t)e^{-Ct}$ and observe that it satisfies

$$\left(\frac{\partial}{\partial t} + \Delta^g \right) (a) \leq 0$$

so that the maximum principle yields $a(x, t) \leq \sup_{x \in X} a(x, 0)$ or equivalently

$$e_T(x, t) \leq e^{Ct} \sup_{x \in X} e_T(x, 0).$$

Now, it is obvious that the map f_t is \mathcal{F} -invariant if and only if the function $e_T(\cdot, t)$ vanishes identically and the corollary follows. \square

4.2. Proof of Theorem 4.1. Here and henceforth, we will identify \mathcal{F} , \mathcal{F}^\perp and their dual to subbundle of TX and T^*X thanks to the orthogonal decomposition.

Let $f : \tilde{X} \rightarrow Y$ be a smooth function and f_*^T its tangential differential in the direction of the foliation. We can regard f_*^T as a section of $f^*(TY) \otimes \tilde{\mathcal{F}}^* \subset f^*(TY) \otimes T^*\tilde{X}$. Let

$$\tilde{\nabla} : \Gamma(f^*(TY) \otimes T^*\tilde{X}) \longrightarrow \Gamma(f^*(TY) \otimes T^*\tilde{X} \otimes T^*\tilde{X})$$

be the connection induced on the tensor product by ∇^g and $\nabla := f^*(\nabla^h)$. It is defined by the following rule

$$\tilde{\nabla}_w(s \otimes t) = \nabla_w s \otimes t + s \otimes \nabla_w^g t.$$

The connection $\tilde{\nabla}$ splits as $\tilde{\nabla} = \nabla^{\mathcal{F}, \mathcal{F}} + \nabla^{\mathcal{F}^\perp, \mathcal{F}^\perp} + \nabla^{\mathcal{F}, \mathcal{F}^\perp} + \nabla^{\mathcal{F}^\perp, \mathcal{F}}$ according to the canonical decomposition

$$T^* \tilde{X} \otimes T^* \tilde{X} = \left(\tilde{\mathcal{F}}^* \otimes \tilde{\mathcal{F}}^* \right) \oplus \left(\tilde{\mathcal{F}}^{\perp*} \otimes \tilde{\mathcal{F}}^{\perp*} \right) \oplus \left(\tilde{\mathcal{F}}^* \otimes \tilde{\mathcal{F}}^{\perp*} \right) \oplus \left(\tilde{\mathcal{F}}^{\perp*} \otimes \tilde{\mathcal{F}}^* \right)$$

We will also consider $\nabla^{\mathcal{F}, \mathcal{F}^\perp}$ as an operator

$$f^*(TY) \otimes T^* \tilde{X} \longrightarrow f^*(TY) \otimes \tilde{\mathcal{F}}^{\perp*} \otimes \tilde{\mathcal{F}}^*$$

via the canonical isomorphism $\tilde{\mathcal{F}}^{\perp*} \otimes \tilde{\mathcal{F}}^* \simeq \tilde{\mathcal{F}}^* \otimes \tilde{\mathcal{F}}^{\perp*}$. With this at hands, Theorem 4.1 is an easy consequence of the following result.

Proposition 4.4. *When applied to the function e_T the heat operator has the following expression:*

$$(4.2) \quad \begin{aligned} \left(\frac{\partial}{\partial t} + \Delta^g \right) (e_T) = & -\text{Ric}_{\tilde{X}}(f_{t*}^T, f_{t*}^T) + \text{Riem}_Y(f_{t*}^T, f_{t*}^T, f_{t*}^T, f_{t*}^T) \\ & - \left| \nabla^{\mathcal{F}, \mathcal{F}^\perp} f_{t*}^T + \nabla^{\mathcal{F}^\perp, \mathcal{F}} f_{t*}^T \right|^2 - \left| \nabla^{\mathcal{F}, \mathcal{F}} f_{t*}^T \right|^2 \\ & + \left| \nabla^{\mathcal{F}^\perp, \mathcal{F}^\perp} f_{t*}^T \right|^2 + 2 \left| \nabla^{\mathcal{F}^\perp, \mathcal{F}} f_{t*}^T \right|^2. \end{aligned}$$

Before entering into the details of the proof, let us firstly explain the meaning of the two first terms in the right-hand side which involve the Ricci and Riemann curvature tensors and why it implies Theorem 4.1. Both terms $\text{Ric}_{\tilde{X}}$ and Riem_Y in (4.2) are defined respectively from the Ricci tensor of g on X and the full curvature tensor of h on Y and extended in a natural way to tensors in $f^*(TY) \otimes T^* \tilde{X}$. We have chosen to keep notation $\text{Ric}_{\tilde{X}}$ and Riem_Y to avoid cumbersome expressions. In the sequel we use the notation $\langle -, - \rangle$ for the scalar product induced by g (or \tilde{g}) and $\langle -, - \rangle_h$ for the scalar product induced by h . We will denote (e_1, \dots, e_{m+n}) any local orthonormal frame of $T\tilde{X}$ such that (e_1, \dots, e_m) is a local orthonormal frame of $\tilde{\mathcal{F}}$ and $(e_{m+1}^*, \dots, e_{m+n}^*)$ the dual coframe.

If $u_i = s_i \otimes t_i$ (for $i = 1, \dots, 4$) are decomposable tensors in $f^*(TY) \otimes T^* \tilde{X}$, we can set $\text{Ric}_{\tilde{X}}(u_1, u_2) := \langle s_1, s_2 \rangle_h \text{Ric}_{\tilde{X}}(t_1, t_2)$ and extend to general tensors by bilinearity. In particular, in terms of local frame as above, we can easily check that

$$(4.3) \quad \begin{aligned} \text{Ric}_{\tilde{X}}(f_{t*}^T, f_{t*}^T)(x) &= \sum_{i, k \leq m} \langle \text{Ric}_{\tilde{X}}(e_k, e_i) f_{t*} e_k, f_{t*} e_i \rangle_h \\ &= \sum_{\substack{i, k \leq m \\ l \leq m+n}} \left(\langle \nabla_{e_l}^g \nabla_{e_k}^g e_i, e_l \rangle - \langle \nabla_{e_k}^g \nabla_{e_l}^g e_i, e_l \rangle - \langle \nabla_{[e_l, e_k]}^g e_i, e_l \rangle \right) \langle f_{t*} e_k, f_{t*} e_i \rangle_h \\ &= \sum_{\substack{i, k \leq m \\ l \leq m+n}} \left(\langle \nabla_{e_i}^g \nabla_{e_l}^g e_l, e_k \rangle - \langle \nabla_{e_l}^g \nabla_{e_i}^g e_l, e_k \rangle - \langle \nabla_{[e_i, e_l]}^g e_l, e_k \rangle \right) \langle f_{t*} e_k, f_{t*} e_i \rangle_h \end{aligned}$$

where the last equality comes directly from the symmetry property

$$\langle R_{\tilde{X}}(u, v)w, z \rangle = \langle R_{\tilde{X}}(w, z)u, v \rangle$$

of the Riemannian curvature tensor on (\tilde{X}, \tilde{g}) . Let us now focus on the second term in the right hand side of (4.2) and denote by R_Y the Riemannian curvature tensor on the target manifold Y .

For decomposable tensors of $f^*(TY) \otimes T^* \tilde{X}$ as above, we can set

$$R_Y(u_1, u_2)u_3 := \langle t_2, t_3 \rangle_g R_Y(s_1, s_2)s_3 \otimes t_1 \in f^*(TY) \otimes T^* \tilde{X}.$$

This formula allows to extend by multilinearity this map to general tensors $u_i \in f^*(TY) \otimes T^*\tilde{X}$ and also define

$$\text{Riem}_Y(u_1, u_2, u_3, u_4) := \langle R_Y(u_1, u_2)u_3, u_4 \rangle_{h \otimes g}$$

where $h \otimes g$ stands for the metric induced by h and g on $f^*(TY) \otimes T^*\tilde{X}$. In particular, in the local frame (e_1, \dots, e_{m+n}) , we obtain the explicit formula:

(4.4)

$$\begin{aligned} \text{Riem}_Y(f_{t*}^T, f_{t*}, f_{t*}, f_{t*}^T) &= \sum_{\substack{i \leq m \\ k \leq m+n}} \langle R_Y(f_{t*}e_i, f_{t*}e_k)f_{t*}e_k, f_{t*}e_i \rangle_h \\ &= \sum_{\substack{i \leq m \\ k \leq m+n}} \langle \nabla_{e_i} \nabla_{e_k} f_{t*}e_k, f_{t*}e_i \rangle_h - \langle \nabla_{e_k} \nabla_{e_i} f_{t*}e_k, f_{t*}e_i \rangle_h - \langle \nabla_{[e_i, e_k]} f_{t*}e_k, f_{t*}e_i \rangle_h. \end{aligned}$$

Moreover, we have

$$\nabla^{\mathcal{F}^\perp, \mathcal{F}^\perp} f_{t*}^T = \sum_{\substack{i \leq m \\ j, k > m}} \langle \nabla_{e_k} e_i, e_j \rangle_g f_{t*}e_i \otimes e_j^* \otimes e_k^*$$

so that

$$(4.5) \quad \left| \nabla^{\mathcal{F}^\perp, \mathcal{F}^\perp} f_{t*}^T \right|^2 = \sum_{\substack{i, l \leq m \\ k > m}} \langle \nabla_{e_k}^g e_i, \nabla_{e_k}^g e_l \rangle \langle f_{t*}e_l, f_{t*}e_i \rangle_h.$$

In the same vein, we can write

$$(4.6) \quad \nabla^{\mathcal{F}^\perp, \mathcal{F}} f_{t*}^T = \sum_{\substack{i, l \leq m \\ k > m}} \langle \nabla_{e_i}^g e_l, e_k \rangle f_{t*}e_l \otimes e_k^* \otimes e_i^*$$

and infer that

$$(4.7) \quad \left| \nabla^{\mathcal{F}^\perp, \mathcal{F}} f_{t*}^T \right|^2 = \sum_{i, l, k \leq m} \langle \nabla_{e_k}^g e_i, \nabla_{e_k}^g e_l \rangle \langle f_{t*}e_l, f_{t*}e_i \rangle_h.$$

It is now easy (modulo the proof of Proposition 4.4) to get the upper bound given in Theorem 4.1. Indeed, fix a compact fundamental domain $K \subset \tilde{X}$ with respect to the action of $\pi_1(X)$.⁷ It then follows from the compactness of $\bigcup_{t \in [0, t_0]} f_t(K)$ together with the multilinearity of $\text{Ric}_{\tilde{X}}$, Riem_Y and expressions (4.3)–(4.7).

Remark 4.5. When \mathcal{F} has codimension zero, the terms in the right-hand side involving \mathcal{F}^\perp of the equation (4.2) do not appear and we recover the classical Eells–Sampson’s formula [ES64, Chapter II, §8-A] (see also [Don87, Formula 11]).

Proof of Proposition 4.4. Let $(e_i)_{i \leq m+n}$ be a local orthonormal frame centered at $x \in \tilde{X}$ such that e_i is tangent to $\tilde{\mathcal{F}}$ for $i \leq m$ and e_i is foliated otherwise. In particular, $[e_i, e_k]$ remains tangent to \mathcal{F} for every k and every $i \leq m$. We first proceed like the classical calculus of the first variation formula of the energy (see for instance [Ura93, p.130]). To

⁷Recall that the energy density is a well defined function on X . More generally, it is not difficult to see that each of the six terms appearing in the RHS of (4.2) remains unchanged when replacing f_t by $\varphi \circ f_t$, where φ is an isometry of Y . In particular, it descends to X .

this end, let us consider the mapping $F : I \times \tilde{X} \rightarrow Y$ defined by $F(t, x) = f_t(x)$ for $|t| < \varepsilon$. Set $f := f_0$. In *loc. cit.*, it is proven that

$$\frac{\partial}{\partial t} \left(\frac{1}{2} |f_{t*}(e_i)|_h^2 \right) = \langle \nabla_{e_i} \tau(f_t), f_{t*} e_i \rangle_h$$

so that, in view of the local writing of the tension field $\tau(f_t)$ given in (3.4):

$$(4.8) \quad \frac{\partial}{\partial t}(e_T) = \sum_{\substack{i \leq m \\ k \leq m+n}} \langle \nabla_{e_i} \nabla_{e_k} f_{t*} e_k, f_{t*} e_i \rangle_h - \langle \nabla_{e_i} (\langle v, e_k \rangle f_{t*} e_k), f_{t*} e_i \rangle_h$$

where $v = \sum_{l \leq m+n} \nabla_{e_l}^g e_l$ and ∇ still denotes (without specifying the parameter t) the pull-back of the Levi-Civita connection on Y by f_t .

Set $v_1 = \sum_{l \leq m} \nabla_{e_l}^g e_l$ and $v_2 = \sum_{l > m} \nabla_{e_l}^g e_l$. Since the mean curvature vector field τ_g is assumed to be foliated, we derive that $e_i \cdot \langle v_1, e_k \rangle$ **vanishes identically** for $i \leq m$ and $k > m$. This is indeed the **key point** of the calculation. According to Lemma 3.12, the same holds for $e_i \cdot \langle v_2, e_k \rangle$, $i \leq m$, $k > m$.

We also recall that for any vector fields u, w on X , we obtain the identity

$$(4.9) \quad \nabla_u f_*(w) - \nabla_w f_*(u) = f_*([u, w])$$

as a consequence of the torsion-freeness of the Levi-Civita connection on Y (see for instance [Ura93, Lemma 1.16, p.129]). Combining this with the previous vanishing properties, we get

$$(4.10) \quad \begin{aligned} \sum_{\substack{i \leq m \\ k \leq m+n}} \langle \nabla_{e_i} (\langle v, e_k \rangle f_{t*} e_k), f_{t*} e_i \rangle_h &= \sum_{\substack{i, k \leq m \\ l \leq m+n}} \langle \nabla_{e_i}^g \nabla_{e_l}^g e_l, e_k \rangle f_{t*} e_k, f_{t*} e_i \rangle_h \\ &+ \sum_{i, k \leq m} \langle v, \nabla_{e_i}^g e_k \rangle \langle f_{t*} e_k, f_{t*} e_i \rangle_h \\ &+ \sum_{\substack{i \leq m \\ l \leq m+n}} \langle v, e_l \rangle (\langle \nabla_{e_l} f_{t*} e_i, f_{t*} e_i \rangle_h + \langle f_{t*} [e_i, e_l], f_{t*} e_i \rangle_h). \end{aligned}$$

Write $\nabla_{e_i}^g e_k = \sum_{l \leq m+n} \langle \nabla_{e_i}^g e_k, e_l \rangle e_l$. From torsion-freeness (3.2), metric compatibility (3.3) and involutivity of \mathcal{F} , we get

$$\begin{aligned} \sum_{i, k \leq m} \langle v, \nabla_{e_i}^g e_k \rangle \langle f_{t*} e_k, f_{t*} e_i \rangle_h &= \sum_{l \leq m+n} \langle v, e_l \rangle \underbrace{\sum_{i, k \leq m} \langle \nabla_{e_l}^g e_k, e_i \rangle \langle f_{t*} e_k, f_{t*} e_i \rangle_h}_{=0 \text{ thanks to } \langle \nabla_{e_l}^g e_k, e_i \rangle = -\langle \nabla_{e_l}^g e_i, e_k \rangle} \\ &- \sum_{\substack{i \leq m \\ l \leq m+n}} \langle v, e_l \rangle \langle f_{t*} [e_i, e_l], f_{t*} e_i \rangle_h \end{aligned}$$

According to (4.8) and (4.10), this leads to the simplified writing:

$$\begin{aligned} \frac{\partial}{\partial t}(e_T) &= \sum_{\substack{i \leq m \\ k \leq m+n}} \langle \nabla_{e_i} \nabla_{e_k} f_{t*} e_k, f_{t*} e_i \rangle_h - \sum_{\substack{i, k \leq m \\ l \leq m+n}} \langle \nabla_{e_i}^g \nabla_{e_l}^g e_l, e_k \rangle \langle f_{t*} e_k, f_{t*} e_i \rangle_h \\ &- \sum_{\substack{i \leq m \\ k \leq m+n}} \langle v, e_k \rangle \langle \nabla_{e_k} f_{t*} e_i, f_{t*} e_i \rangle_h. \end{aligned}$$

On the other hand, the Laplace operator reads

$$\begin{aligned}\Delta^g(e_T) &= \sum_{\substack{i \leq m \\ k \leq m+n}} \langle v, e_k \rangle \langle \nabla_{e_k} f_{t*} e_i, f_{t*} e_i \rangle_h \\ &\quad - \sum_{\substack{i \leq m \\ k \leq m+n}} \langle \nabla_{e_k}^2 f_{t*} e_i, f_{t*} e_i \rangle_h + |\nabla_{e_k} f_{t*} e_i|_h^2.\end{aligned}$$

Still exploiting repeatedly identity (4.9), we get

$$\nabla_{e_k}^2 f_{t*} e_i = \nabla_{e_k} \nabla_{e_i} f_{t*} e_k - \nabla_{[e_i, e_k]} f_{t*} e_k + f_{t*}([e_k, [e_k, e_i]]).$$

If we combine this with the last lines of (4.3) and (4.4), we can deduce that

$$\begin{aligned}(4.11) \quad & \left(\frac{\partial}{\partial t} + \Delta^g \right) (e_T) = \text{Riem}_Y(f_{t*}^T, f_{t*}, f_{t*}, f_{t*}^T) - \text{Ric}_{\tilde{X}}(f_{t*}^T, f_{t*}^T) \\ & - \sum_{\substack{i \leq m \\ k \leq m+n}} |\nabla_{e_k} f_{t*} e_i|_h^2 - 2 \langle \nabla_{[e_i, e_k]} f_{t*} e_k, f_{t*} e_i \rangle_h + \langle f_{t*}([e_k, [e_k, e_i]]), f_{t*} e_i \rangle_h \\ & - \sum_{\substack{i, l \leq m \\ k \leq m+n}} \langle \nabla_{[e_i, e_k]}^g e_k, e_l \rangle \langle f_{t*} e_i, f_{t*} e_l \rangle_h + \langle \nabla_{e_k}^g \nabla_{e_i}^g e_k, e_l \rangle \langle f_{t*} e_l, f_{t*} e_i \rangle_h.\end{aligned}$$

On the other hand, $[e_k, [e_k, e_i]]$ is tangent to \mathcal{F} for $i \leq m$, so that we can write

$$[e_k, [e_k, e_i]] = \sum_{l \leq m} \langle [e_k, [e_k, e_i]], e_l \rangle e_l.$$

Using (3.2), this yields

$$\begin{aligned}\langle f_{t*}([e_k, [e_k, e_i]]), f_{t*} e_i \rangle_h &= - \sum_{l \leq m} \langle \nabla_{e_k}^g \nabla_{e_i}^g e_k, e_l \rangle \langle f_{t*} e_l, f_{t*} e_i \rangle_h \\ &+ \sum_{l \leq m} \langle \nabla_{[e_i, e_k]}^g e_k, e_l \rangle \langle f_{t*} e_i, f_{t*} e_l \rangle_h + \sum_{l \leq m} \langle \nabla_{e_k}^g \nabla_{e_i}^g e_i, e_l \rangle \langle f_{t*} e_i, f_{t*} e_l \rangle_h\end{aligned}$$

Applying again (3.2) together with (3.3), we can express the last term as

$$\begin{aligned}\sum_{l \leq m} \langle \nabla_{e_k}^g \nabla_{e_i}^g e_i, e_l \rangle \langle f_{t*} e_i, f_{t*} e_l \rangle_h &= - \sum_{l \leq m} \langle \nabla_{e_k}^g e_i, \nabla_{e_k}^g e_l \rangle \langle f_{t*} e_l, f_{t*} e_i \rangle_h \\ &\quad - e_k \cdot \sum_{l \leq m} \langle e_i, \nabla_{e_k}^g e_l \rangle \langle f_{t*} e_i, f_{t*} e_l \rangle_h\end{aligned}$$

whence

$$\begin{aligned}(4.12) \quad & \sum_{\substack{i \leq m \\ k \leq m+n}} \langle f_{t*}([e_k, [e_k, e_i]]), f_{t*} e_i \rangle_h = - \sum_{\substack{i, l \leq m \\ k \leq m+n}} \langle \nabla_{e_k}^g \nabla_{e_i}^g e_k, e_l \rangle \langle f_{t*} e_l, f_{t*} e_i \rangle_h \\ & + \sum_{\substack{i, l \leq m \\ k \leq m+n}} \langle \nabla_{[e_i, e_k]}^g e_k, e_l \rangle \langle f_{t*}(e_i), f_{t*}(e_l) \rangle_h - \langle \nabla_{e_k}^g e_i, \nabla_{e_k}^g e_l \rangle \langle f_{t*} e_l, f_{t*} e_i \rangle_h \\ & - \sum_{k \leq m+n} e_k \cdot \underbrace{\sum_{i, l \leq m} \langle e_i, \nabla_{e_k}^g e_l \rangle \langle f_{t*} e_i, f_{t*} e_l \rangle_h}_{=0 \text{ thanks to } \langle e_i, \nabla_{e_k}^g e_l \rangle = - \langle e_l, \nabla_{e_k}^g e_i \rangle}.\end{aligned}$$

When re-injecting (4.12) into (4.11), we thus find

$$\begin{aligned}
 (4.13) \quad & \left(\frac{\partial}{\partial t} + \Delta^g \right) (e_T) = \text{Riem}_Y(f_{t*}^T, f_{t*}^T, f_{t*}^T f_{t*}^T) - \text{Ric}_{\tilde{X}}(f_{t*}^T, f_{t*}^T) \\
 & + \sum_{\substack{i \leq m \\ k \leq m+n}} 2 \langle \nabla_{[e_i, e_k]} f_{t*} e_k, f_{t*} e_i \rangle_h - |\nabla_{e_k} f_{t*} e_i|_h^2 \\
 & + \sum_{\substack{i, l \leq m \\ k \leq m+n}} \left(\langle \nabla_{e_k}^g e_i, \nabla_{e_k}^g e_l \rangle - 2 \langle \nabla_{[e_i, e_k]}^g e_k, e_l \rangle \right) \langle f_* e_i, f_* e_l \rangle_h.
 \end{aligned}$$

In order to simplify our computations, we will suppose hereafter, without loss of generality, that the orthonormal frame $(e_i)_{i \leq m}$ of \mathcal{F} is *tangentially geodesic* at x (see Lemma 3.10). Because $[e_i, e_l](x) = 0$ for $i, l \leq m$, the formula (4.13) becomes

$$\begin{aligned}
 (4.14) \quad & \left(\frac{\partial}{\partial t} + \Delta^g \right) (e_T) = \text{Riem}_Y(f_{t*}^T, f_{t*}^T, f_{t*}^T f_{t*}^T) - \text{Ric}_X(f_{t*}^T, f_{t*}^T) \\
 & + \sum_{\substack{i \leq m \\ k > m}} 2 \langle \nabla_{[e_i, e_k]} f_{t*} e_k, f_{t*} e_i \rangle_h - \sum_{\substack{i \leq m \\ k \leq m+n}} |\nabla_{e_k} f_{t*} e_i|_h^2 \\
 & + \sum_{\substack{i, l \leq m \\ k \leq m+n}} \langle \nabla_{e_k}^g e_i, \nabla_{e_k}^g e_l \rangle \langle f_{t*} e_i, f_{t*} e_l \rangle_h \\
 & - 2 \sum_{\substack{i, l \leq m \\ k > m}} \langle \nabla_{[e_i, e_k]}^g e_k, e_l \rangle \langle f_* e_i, f_* e_l \rangle_h.
 \end{aligned}$$

Now, in view of proving the equality of Proposition 4.4, let us compute

$$(4.15) \quad \nabla^{\mathcal{F}, \mathcal{F}^\perp} f_{t*}^T = \sum_{\substack{i \leq m \\ k > m}} \nabla_{e_k} f_{t*} e_i \otimes e_k^* \otimes e_i^* + \sum_{l \leq m} \underbrace{\langle \nabla_{e_k}^g e_i, e_l \rangle}_{=0} f_{t*} e_i \otimes e_k^* \otimes e_l^*.$$

On the other hand, by noticing that

$$\begin{aligned}
 \sum_{l \leq m} \langle \nabla_{e_i}^g e_l, e_k \rangle f_{t*} e_l &= - \sum_{l \leq m} \langle e_l, \nabla_{e_i}^g e_k \rangle f_{t*} e_l \\
 &= - \sum_{l \leq m} \langle e_l, [e_i, e_k] \rangle f_{t*} e_l = - f_{t*} [e_i, e_k]
 \end{aligned}$$

and accordingly to (4.6), we get

$$(4.16) \quad \nabla^{\mathcal{F}^\perp, \mathcal{F}} f_{t*}^T = - \sum_{\substack{i \leq m \\ k > m}} f_{t*} [e_i, e_k] \otimes e_k^* \otimes e_i^*.$$

As a consequence of (4.15) and (4.16), we have:

$$\begin{aligned}
 \left| \nabla^{\mathcal{F}, \mathcal{F}^\perp} f_{t*}^T + \nabla^{\mathcal{F}^\perp, \mathcal{F}} f_{t*}^T \right|^2 &= \\
 & \sum_{\substack{i \leq m \\ k > m}} |\nabla_{e_k} f_{t*} e_i|_h^2 + |f_{t*} [e_i, e_k]|_h^2 - 2 \langle \nabla_{e_k} f_{t*} e_i, f_{t*} [e_i, e_k] \rangle_h.
 \end{aligned}$$

By virtue of (4.9), we can rewrite

$$\sum_{\substack{i \leq m \\ k > m}} \langle \nabla_{e_k} f_{t_*} e_i, f_{t_*} [e_i, e_k] \rangle_h = \sum_{\substack{i \leq m \\ k > m}} \langle \nabla_{e_i} f_{t_*} e_k, f_{t_*} [e_i, e_k] \rangle_h - |f_{t_*} [e_i, e_k]|_h^2.$$

Now, thanks to the expansion

$$[e_i, e_k] = \sum_{l \leq m} \langle [e_i, e_k], e_l \rangle e_l$$

and Lemma 3.11, we conclude that

$$\sum_{\substack{i \leq m \\ k > m}} \langle \nabla_{e_i} f_{t_*} e_k, f_{t_*} [e_i, e_k] \rangle_h = \sum_{\substack{i \leq m \\ k > m}} \langle \nabla_{[e_i, e_k]} f_{t_*} e_k, f_{t_*} e_i \rangle_h$$

so that

$$(4.17) \quad \left| \nabla^{\mathcal{F}, \mathcal{F}^\perp} f_{t_*}^T + \nabla^{\mathcal{F}^\perp, \mathcal{F}} f_{t_*}^T \right|^2 = \sum_{\substack{i \leq m \\ k > m}} |\nabla_{e_k} f_{t_*} e_i|_h^2 + 3|f_{t_*} [e_i, e_k]|_h^2 - 2\langle \nabla_{[e_i, e_k]} f_{t_*} e_k, f_{t_*} e_i \rangle_h.$$

On the other hand, comparison of (4.7) and (4.16) yields

$$(4.18) \quad \left| \nabla^{\mathcal{F}^\perp, \mathcal{F}} f_{t_*}^T \right|^2 = \sum_{i, l, k \leq m} \langle \nabla_{e_k}^g e_i, \nabla_{e_k}^g e_l \rangle(x) \langle f_{t_*} e_l, f_{t_*} e_i \rangle_h = \sum_{\substack{i \leq m \\ k > m}} |f_{t_*} [e_i, e_k]|_h^2.$$

Recall also (cf. (4.5)) that

$$(4.19) \quad \left| \nabla^{\mathcal{F}^\perp, \mathcal{F}^\perp} f_{t_*}^T \right|^2 = \sum_{\substack{i, l \leq m \\ k > m}} \langle \nabla_{e_k}^g e_i, \nabla_{e_k}^g e_l \rangle(x) \langle f_{t_*} e_l, f_{t_*} e_i \rangle_h.$$

According to the equations (4.17), (4.18), (4.19) and to the equality

$$\left| \nabla^{\mathcal{F}, \mathcal{F}} f_{t_*}^T \right|^2 = \sum_{i, k \leq m} |\nabla_{e_k} f_{t_*} e_i|_h^2,$$

the equation (4.14) can be rewritten as

$$(4.20) \quad \begin{aligned} & \left(\frac{\partial}{\partial t} + \Delta^g \right) (e_T) = -\text{Ric}_{\tilde{X}}(f_{t_*}^T, f_{t_*}^T) + \text{Riem}_Y(f_{t_*}^T, f_{t_*}^T, f_{t_*}^T, f_{t_*}^T) \\ & - \left| \nabla^{\mathcal{F}, \mathcal{F}^\perp} f_{t_*}^T + \nabla^{\mathcal{F}^\perp, \mathcal{F}} f_{t_*}^T \right|^2 - \left| \nabla^{\mathcal{F}, \mathcal{F}} f_{t_*}^T \right|^2 + \left| \nabla^{\mathcal{F}^\perp, \mathcal{F}^\perp} f_{t_*}^T \right|^2 \\ & - 2 \sum_{\substack{i, l \leq m \\ k > m}} \langle \nabla_{[e_i, e_k]}^g e_k, e_l \rangle \langle f_{t_*} e_i, f_{t_*} e_l \rangle_h + 4 \left| \nabla^{\mathcal{F}^\perp, \mathcal{F}} f_{t_*}^T \right|^2. \end{aligned}$$

It remains to identify the quantity (evaluated at the point x)

$$S := \sum_{\substack{i, l \leq m \\ k > m}} \langle \nabla_{[e_i, e_k]}^g e_k, e_l \rangle \langle f_{t_*} e_i, f_{t_*} e_l \rangle_h.$$

To do so, let us first expand $[e_i, e_k]$ with respect to the basis $(e_j)_{j \leq m}$. This yields:

$$S = \sum_{\substack{i, l, j \leq m \\ k > m}} \langle [e_i, e_k], e_j \rangle \langle \nabla_{e_j}^g e_k, e_l \rangle \langle f_* e_i, f_* e_l \rangle_h.$$

Finally, let us remark that

$$\langle [e_i, e_k], e_j \rangle = -\langle e_k, \nabla_{e_j}^g e_i \rangle(x) \quad \text{and} \quad \langle \nabla_{e_j}^g e_k, e_l \rangle(x) = -\langle e_k, \nabla_{e_j}^g e_l \rangle(x).$$

According to (4.6), we get after summation

$$S = \left| \nabla^{\mathcal{F}^\perp, \mathcal{F}} f_{t*}^T \right|^2.$$

We thus obtain the sought formula of Proposition 4.4 by reporting this identification into the heat operator expression given in the equation (4.20). \square

5. CONSTRUCTION OF EQUIVARIANT FOLIATED HARMONIC MAPS

5.1. Transversely parallelizable and Lie foliations: recollection. Let (X, \mathcal{F}) be a foliated manifold with $\dim(X) = m + n$ and $\text{rk}(\mathcal{F}) = m$. Recall (cf. Section 3) that $\mathfrak{X}(X)$, $\mathfrak{X}(\mathcal{F})$ and $\ell(X, \mathcal{F})$ denote respectively the Lie algebras of vector fields, tangent and basic vector fields.

Definition 5.1 (cf. [Mol88, Chapter 4]). The foliation \mathcal{F} is said to be *transversely parallelizable* or to admit a *transverse parallelism* if there exists $n = \text{codim}(\mathcal{F})$ basic vector fields Y_1, \dots, Y_n that form a global frame of the normal bundle of \mathcal{F} .

Remark that such foliations can be equipped with a holonomy invariant metric \bar{g} . Indeed, it suffices to consider the metric on $N\mathcal{F}$ with respect to which Y_1, \dots, Y_n is an orthonormal frame. Thus, transversely parallelizable foliations turn out to be particular cases of Riemannian foliations.

Definition 5.2. A transversely parallelizable foliation is said to be *complete* (TC for short) if it possesses a transverse frame Y_1, \dots, Y_n such that each Y_i can be represented by a complete vector field $X_i \in \mathfrak{X}(X)$.⁸ Note that this automatically holds whenever X is compact.

Remark 5.3. The lift of a TC foliation to any cover is still TC.

Definition 5.4 (cf. [Mol88, §4.2]). Let \mathcal{F} be a TC foliation and (Y_1, \dots, Y_n) be an associated transverse frame. We will say in addition that \mathcal{F} is a *Lie foliation* (TL for short) if the \mathbb{R} -vector space generated by $\{Y_1, \dots, Y_n\}$ is closed under Lie bracket, hence defines a n -dimensional Lie subalgebra \mathfrak{g} of $\ell(X, \mathcal{F})$. The structure of Lie foliations is described by the following proposition due to Fedida [Fed71].

Proposition 5.5. *Let \mathcal{F} be a TL foliation. Then the lift $\tilde{\mathcal{F}}$ of \mathcal{F} on the universal cover \tilde{X} is defined by the levels of a surjective submersive map with connected fibers (called the developing map):*

$$p : \tilde{X} \longrightarrow G$$

where G is the simply connected Lie group integrating \mathfrak{g} . The map p is in addition equivariant with respect to some representation

$$\rho : \pi_1(X) \longrightarrow G,$$

⁸The definition is a little bit more restrictive than the one given in [Mol88, §4.5] but sufficient for our purposes.

that is

$$\forall x \in \tilde{X}, \forall \gamma \in \pi_1(X), \quad p(\gamma(x)) = \rho(\gamma)p(x).$$

Finally, the left invariant vector fields on G , regarded as $\tilde{\mathcal{F}}$ basic vector fields on \tilde{X} , descend to X as elements of \mathfrak{g} .

Let us briefly explain the proof of the previous statement. The developing map is actually associated to the choice of an abstract isomorphism

$$\varphi : \mathcal{L} := \text{Vect}(Y_1, \dots, Y_n) \longrightarrow \mathfrak{g} := \text{Lie}(G).$$

Let ω be the \mathfrak{g} -valued one form on X such that for $Y \in \mathcal{L}$, $\omega(Y) = \varphi(Y)$ and $\omega(v) = 0$ whenever $v \in \mathfrak{X}(\mathcal{F})$. Let $r : \tilde{X} \rightarrow X$ be the universal covering map. Then p is the map such that $r^*\omega = p^*\omega_{\text{MC}}$ where ω_{MC} is the left invariant Maurer–Cartan form on G . Actually, p is unique up to left composition by left translation L_g . Moreover, the choice of φ is unique modulo left composition by an element of $\text{Aut}(\mathfrak{g}) = \text{Aut}(G)$. Finally, given \mathcal{L} , the choice of the associate developing map is unique modulo left composition by the subgroup $G \rtimes \text{Aut}(G) < \text{Diff}(G)$.

Remark 5.6. A TC minimal foliation is automatically and canonically TL. Actually in that case we have $\mathfrak{g} = \ell(X, \mathcal{F})$ and the representation ρ has dense image.

Remark 5.7. Let \mathcal{F} a TL foliation defined by its developing map. The Lie algebra of right invariant vector fields on G , viewed as basic vector fields on \tilde{X} , descends to X as a locally constant sheaf that we will denote by $\text{Lie}_r(\mathcal{F})$. Note that, by construction, the monodromy of this local system is given by the adjoint action of G on its Lie algebra \mathfrak{g} :

$$v \in \mathfrak{g} \mapsto \text{Ad}(\rho(\gamma))v.$$

5.2. The structure of complete transversely parallelizable foliations. Let (X, \mathcal{F}) be a foliated manifold such that \mathcal{F} is TC. From Molino's theory (see [Mol88, Chapter 4]) the topological closure of the leaves of \mathcal{F} are the fibers of a locally trivial fibration $\pi_{\mathcal{F}} : X \rightarrow W$, the so-called *basic fibration*. In particular, the closedness of a single leaf implies the closedness of the others and in that case the foliation is *simple* (cf. Definition 3.1). The basis W of the fibration π is called the *basic manifold*.

We first have at our disposal the following theorem for transversely parallelizable complete foliations.

Theorem 5.8 (Molino [Mol88, Theorem 4.2]). *The restriction of \mathcal{F} to any fiber of $\pi_{\mathcal{F}}$ is a minimal Lie foliation with typical Lie algebra \mathfrak{g} .*

5.3. The commuting sheaf. We refer to [Mol88, §4.4 and 4.5] for details about this paragraph. Let \mathcal{F} be a TC foliation on X . Let U be an open set of M and denote by $\mathcal{C}(U, \mathcal{F})$ the Lie algebra formed by the basic vector fields of $\mathcal{F}|_U$ that commute with each element of $\ell(X, \mathcal{F})$. The collection of $\mathcal{C}(U, \mathcal{F})$ defines a presheaf. We will denote by $\mathcal{C}_{\mathcal{F}}$ the corresponding sheaf. The main properties of $\mathcal{C}_{\mathcal{F}}$ are summarized in the following proposition.

Proposition 5.9 (cf. [Mol88, Proposition 4.4]). *The sheaf $\mathcal{C}_{\mathcal{F}}$ is a locally constant sheaf of Lie algebra with typical fiber \mathfrak{g} . Moreover, any local section v of $\mathcal{C}_{\mathcal{F}}$ is tangent to the basic fibration and the restriction of $\mathcal{C}_{\mathcal{F}}$ to a fiber F coincides with $(\mathcal{C}_{\mathcal{F}})|_F = (\text{Lie}_r(\mathcal{F}))|_F$.*

The following definition appeared in [Mol88, Theorem 4.2].

Definition 5.10. We will say that \mathfrak{g} is the structural Lie algebra of \mathcal{F} .

Definition 5.11. Let $\rho : \pi_1(X) \rightarrow \text{Aut}(\mathfrak{g})$ the monodromy representation attached to the locally constant sheaf $\mathcal{C}_{\mathcal{F}}$. The covering space X_{ρ} corresponding to $\text{Ker}(\rho)$ is called the *central cover* of X .

5.4. Homogeneous fibration associated to a reduction of the structure group. Let G be a Lie group. Let $\alpha : P \rightarrow W$ be a G -principal bundle over a manifold W . Here, P is supposed to be connected but G is likely to have several connected components.

Let $H < G$ be a closed Lie subgroup. Let us consider $E_H = P \times_G G/H$ the associated homogeneous G/H bundle over W . Assume that there exists a reduction of the structure group G to H , i.e. a section $\sigma : W \rightarrow E_H$. This is equivalent to the datum of a submanifold $H_\sigma \subset P$ such that

$$(5.1) \quad \begin{aligned} & \forall z \in H_\sigma, \forall g \in G, z \cdot g \in H_\sigma \iff g \in H \\ & \alpha|_{H_\sigma} \text{ is a surjective submersion onto } W. \end{aligned}$$

We then build by saturation of H_σ under the G -action a locally trivial fibration

$$\Phi_\sigma : P \longrightarrow G/H$$

with typical fiber H_σ by declaring that Φ_σ maps $H_\sigma \cdot g$ on $g^{-1}H \in G/H$.

Note that Φ_σ satisfies the equivariance property

$$(5.2) \quad \Phi_\sigma(x \cdot g) = g^{-1}\Phi_\sigma(x).$$

We easily check that the converse holds true.

Lemma 5.12. *Let $\Phi \in C^\infty(P, G/H)$ such that Φ satisfies the equivariance property (5.2). Then Φ has the form Φ_σ for some section σ of $E_H \rightarrow W$.*

Proof. Set $o := H \in G/H$ and define $\mathcal{H} = \Phi^*(o)$. The previous equivariance property (5.2) forces Φ to be a surjective submersion in restriction to the fibers of α , so that $\mathcal{H} = H_s$ for some smooth section $\sigma : W \rightarrow E_H$ according to the characterization of H_σ given above. \square

Remark 5.13. By construction, the Lie algebra of fundamental vector fields, i.e. the infinitesimal generators of the right action of G projects via Φ to the Lie algebra of vector fields on G/H generated by the infinitesimal action of G on the homogeneous space G/H .

5.5. The semi-simple case. Note in particular that such a Φ has connected fiber as soon as the homogeneous space G/H is simply connected (write the long exact sequence of homotopy groups for the locally trivial fibration Φ). It happens for instance when $\mathfrak{g} = \text{Lie}(G)$ is semi-simple without compact factors. Indeed, let \mathcal{S} the associate Riemannian symmetric space (cf. for instance [Hel01, Chapter VI] and [Bor98, Chapter IV]). This latter is homeomorphic to an Euclidean space and in particular contractible. Consider the canonical projection $\pi : G \rightarrow G' := G/\text{Ker}(\text{Ad})$. Note that G' is an open subgroup of the real algebraic group $\text{Aut}(\mathfrak{g})$. Let K be a maximal compact subgroup in $G' = G/\text{Ker}(\text{Ad})$ and set $H = \pi^{-1}(K)$. The homogeneous space G/H and G'/K are thus diffeomorphic and according to [Bor98, Chapter IV, Proposition 4.10, Chapter VII, Theorem 3.7], the latter is diffeomorphic to \mathcal{S} (and G' acts faithfully by isometries on \mathcal{S}), whence the existence of a section s of $E_H \rightarrow W$.

5.6. Developability and Almeida-Molino theorem.

Definition 5.14. A foliation \mathcal{F} on X is said to be *developable* if its lift $\tilde{\mathcal{F}}$ to the universal cover \tilde{X} is simple.

We use freely notation from §5.1 and 5.2. Let \mathcal{F} be a TC foliation on X and denote by E the locally free sheaf generated over C_W^∞ (the sheaf of germs of smooth functions on W) by $\ell(X, \mathcal{F})$ that we can alternatively regard as a rank $n = \text{codim}(\mathcal{F})$ vector bundle over W . Note that E can be endowed with the structure of a *transitive Lie algebroid* over

W (see [Mol88, §4.6]). The Lie bracket is the ordinary Lie bracket on $\ell(X, \mathcal{F})$ and the anchor map is given by the projection $\text{an}_E : E \rightarrow TW$.

Theorem 5.15 (cf. [Mol88, Theorem 4.4] and [MM03, Theorem 6.10]). *Let \mathcal{F} a TC foliation on X . Then \mathcal{F} is developable if and only if the Lie algebroid E ⁹ is integrable, i.e. if and only if there exists a principal bundle $P \rightarrow W$ whose structure group G satisfies $\text{Lie}(G) = \mathfrak{g}$ and such that there exists an isomorphism¹⁰ of algebroids $E \simeq TP/G$ where TP/G is the Lie algebroid (over W) defined by the G -invariant vector fields on P .*

Lemma 5.16. *Let \mathcal{F} a TC foliation with centerless structural Lie algebra. Then, \mathcal{F} lifts to a simple foliation on the central cover (as in Definition 5.11). In particular, \mathcal{F} is developable.*

Proof. This is [Mol88, Proposition 4.6]. \square

5.7. Description of P . Let us explain briefly how to construct the desired principal bundle P over W in the case where \mathcal{F} is developable.

Let $\tilde{\mathcal{F}}$ be the pull-back of \mathcal{F} on the universal cover $r : \tilde{X} \rightarrow X$. This is a TC foliation which is by assumption simple and the basic fibration/manifold is thus defined by the projection $\pi_{\tilde{\mathcal{F}}} : \tilde{X} \rightarrow \tilde{X}/\tilde{\mathcal{F}}$. Consider the Lie subalgebra $r^*(\ell(X, \mathcal{F}))$ of $\ell(\tilde{X}, \tilde{\mathcal{F}})$. Denote by P the space of leaves $\tilde{X}/\tilde{\mathcal{F}}$. This is a simply connected manifold. In what follows, we will identify $r^*(\ell(X, \mathcal{F}))$ to a Lie subalgebra of $\mathfrak{X}(P)$.

Let us consider the natural projection $\alpha : P \rightarrow W$. Note that for any $v \in \ell(X, \mathcal{F})$, $\pi_{\mathcal{F}*}v = \alpha_*(r^*v)$, where $\pi_{\mathcal{F}} : X \rightarrow W$ is the original basic fibration. Let G be the group of α -vertical diffeomorphisms of P that fix each element of $r^*\ell(X, \mathcal{F})$. It turns out that G acts simply transitively on each fiber, thus endowing P with a structure of G -principal bundle for which $r^*\ell(X, \mathcal{F})$ is the space of sections of TP/G . The right action of G on P is given by $p \cdot g := g^{-1}(p)$. We refer to [Mol88, MM03, *supra*] for the details.

5.8. Additional observations. Under the same assumptions as above, we list some useful observations.

- (1) The group $\pi_1(X)$ of deck transformations of \tilde{X} acts vertically on P with respect to α and preserves $r^*\ell(X, \mathcal{F})$ pointwise, thus defining a morphism

$$\rho_{\mathcal{F}} : \pi_1(X) \rightarrow G.$$

This morphism maps $\pi_1(X)$ onto a *dense* subgroup of the Lie group G . To see this, pick a point $w \in W$ and let F_w^0 be a connected component of the fiber $\alpha^{-1}(w)$. Let G^0 be the component of the identity in G . As $\pi_1(X)$ acts transitively on the set of connected component of $\alpha^{-1}(w)$, it is sufficient to prove that $\pi_1(X) \cap G^0$ acts densely on F_w^0 . But this last point immediately results from the fact that \mathcal{F} is dense in restriction to the fibers of $\pi_{\mathcal{F}} : X \rightarrow W$.

- (2) The sections of the pull-back $r^*\mathcal{C}_{\mathcal{F}}$ of the commuting sheaf (which becomes a *constant* sheaf of basic vector fields of $\tilde{\mathcal{F}}$) projects isomorphically via the differential of $\pi_{\tilde{\mathcal{F}}} : \tilde{X} \rightarrow P = \tilde{X}/\tilde{\mathcal{F}}$ to the fundamental vector fields associated to the right G -action on P . Indeed, by construction, these sections are vertical vector fields on P commuting with every element of $r^*\ell(X, \mathcal{F}) = \{\text{sections of } TP/G\}$ and they form a \mathbb{R} -vector space of the same dimension as \mathfrak{g} .

⁹See the paragraph after Theorem 5.8.

¹⁰We refer to [Mol88, §4.6] or [MM03, §6.2] for the precise definition of morphisms of Lie algebroids.

- (3) Assume moreover the following: let H be a closed Lie subgroup of G such that the associate bundle $E_H = P \times_G G/H$ (over W) admits a section σ . Let $\Phi_\sigma : P \rightarrow G/H$ be the associated fibration considered in §5.4. Let us consider $\Psi_\sigma = \Phi_\sigma \circ \pi_{\tilde{\mathcal{F}}}$. By construction, $\Psi_\sigma : \tilde{X} \rightarrow G/H$ is an equivariant locally trivial fibration where equivariant means

$$(5.3) \quad \forall \gamma \in \pi_1(X), \forall x \in \tilde{X}, \Psi_\sigma(\gamma(x)) = \rho_{\mathcal{F}}(\gamma) \cdot \Psi_\sigma(x).$$

- (4) Conversely any $\Psi \in \mathcal{C}^\infty(\tilde{X}, G/H)$ constant on the leaves of $\tilde{\mathcal{F}}$ and satisfying the equivariance property (5.3) has the form Ψ_σ . Indeed, let $\Phi \in \mathcal{C}^\infty(P, G/H)$ be the map factorizing Ψ . Then Φ satisfies the equivariance property (5.2) for a dense subgroup of G (namely $\pi_1(X)$), hence for all $g \in G$. We can then conclude using Lemma 5.12.

5.9. The case of a structural Lie algebra without compact factors. We keep notation used in §5.6. Combining the series of results/observations exposed in the preceding paragraphs together with §5.5 we get the following existence theorem.

Theorem 5.17. *Let (X, \mathcal{F}) be a foliated manifold such that \mathcal{F} is TC. Let us denote by $\tilde{\mathcal{F}}$ the lift of \mathcal{F} to the universal cover \tilde{X} . Suppose in addition that the structural Lie algebra \mathfrak{g} is semi-simple without compact factors. Let $\mathcal{S} = G/H$ be the associate Riemannian symmetric space. Denote by $\text{Iso}(\mathcal{S})$ the Lie group of isometries of \mathcal{S} , by $\text{Iso}^0(\mathcal{S})$ its neutral component, by $\mathfrak{iso}(\mathcal{S})$ the corresponding Lie algebra of infinitesimal isometries and by $\pi : G \rightarrow \text{Iso}(\mathcal{S})$ the morphism corresponding to the action of G on \mathcal{S} (and whose image is a union of finitely many connected component of $\text{Iso}(\mathcal{S})$). The following holds true.*

- (1) *The foliation \mathcal{F} is developable.*
- (2) *The closure of the image of $\rho := \pi \circ \rho_{\mathcal{F}}$ contains $\text{Iso}^0(\mathcal{S})$.*
- (3) *There exists on \tilde{X} a basic (with respect to $\tilde{\mathcal{F}}$) smooth map $\Psi : \tilde{X} \rightarrow \mathcal{S}$ such that*

$$(5.4) \quad \forall x \in \tilde{X}, \forall \gamma \in \pi_1(X), \Psi(\gamma(x)) = \rho(\gamma)(\Psi(x)).$$

- (4) *Any smooth basic map $\Psi : \tilde{X} \rightarrow \mathcal{S}$ satisfying the equivariance condition (5.4) is a locally trivial fibration with connected fibers. Moreover we have for every $g \in G$ and for all $x \in \tilde{X}$*

$$\Phi(g \cdot \pi_{\tilde{\mathcal{F}}}(x)) = g \cdot \Psi(x) := \pi(g)(\Psi(x)).$$

- (5) *Let $\Psi : \tilde{X} \rightarrow \mathcal{S}$ as in the previous item. Consider the lift $r^*\mathcal{C}_{\mathcal{F}}$ of the commuting sheaf $\mathcal{C}_{\mathcal{F}}$ on the universal cover \tilde{X} (this is a Lie subalgebra of basic sections of $N\tilde{\mathcal{F}}$). Then, for every $v \in r^*\mathcal{C}_{\mathcal{F}}$, $d\Psi(v)$ is constant along the fibers of Ψ . Moreover, the differential $d\Psi$ induces a Lie algebra isomorphism between $r^*\mathcal{C}_{\mathcal{F}}$ and $\mathfrak{iso}(\mathcal{S})$.*

Remark 5.18. Under the assumptions/notation of the previous theorem, the foliation defined by the submersion Ψ descends to X as a minimal foliation (all leaves are dense) \mathcal{G} whose codimension is the dimension of \mathcal{S} ; this foliation \mathcal{G} is an extension of \mathcal{F} (in the sense of §3.3) and is transversely Riemannian homogeneous.¹¹ Note also that the existence of Ψ , hence \mathcal{G} is directly related to that of a section of $E_H \rightarrow W$ as specified in §5.8. This implies that the morphism $\mathcal{G} \rightarrow TW$ induced by the differential of the basic fibration $\pi_{\mathcal{F}} : X \rightarrow W$ is everywhere surjective (compare with (5.1)).

¹¹We refer to [God91, Chapter III, §3] for the general definition of a transversely homogeneous foliation.

Remark 5.19. From the description of $r^*\mathcal{C}_{\mathcal{F}}$ as the Lie algebra of fundamental vector fields defined by the right G -action and according to Remark 5.13, we can observe that $r^*\mathcal{C}_{\mathcal{F}}$ descends via Ψ to the Lie algebra of infinitesimal symmetries of the symmetric space \mathcal{S} and that the representation ρ is nothing but the monodromy representation of the local system $\mathcal{C}_{\mathcal{F}}$.

When the base manifold X is compact, we can try to construct a harmonic representative of the map Ψ alluded to in Theorem 5.17.

Theorem 5.20. *Let (X, \mathcal{F}) be a foliated manifold with \mathcal{F} being TC and let us assume in addition that X is compact. Let us consider g a metric on X which is bundle-like with respect to \mathcal{F} and such that the mean curvature vector of the leaves is foliated, i.e. (\mathcal{F}, g) is tense (see Theorem 3.4). Then there exists a unique Ψ that is harmonic with respect to the lift \tilde{g} of the metric on \tilde{X} and to the Killing metric on \mathcal{S} .*

Proof. Thanks to the item 2 of Theorem 5.17, there exists a *basic* (in other words \mathcal{F} -invariant) ρ -equivariant map $f_0 : \tilde{X} \rightarrow \mathcal{S}$. We can now apply the existence criterion of twisted harmonic map given in [Lab91] or [Cor88]. As the image of ρ does not fix any point on the boundary of \mathcal{S} , we can apply [Lab91, Théorèmes 0.1 et 0.2] to infer the existence of a ρ -equivariant *harmonic* map $f_\infty : \tilde{X} \rightarrow \mathcal{S}$ which is obtained as the limit of a subsequence of (f_{t_n}) , $t_n \mapsto +\infty$ where f_t is a solution of the evolution equation (4.1) at time t with initial datum $f_{t_0} = f_0$. Moreover, f_∞ remains \mathcal{F} -invariant according to Corollary 4.3. Concerning the uniqueness, we can first assume, up to passing to a finite étale cover of X that the representation takes values in $\text{Iso}^0(\mathcal{S})$, the neutral component of $\text{Iso}(\mathcal{S})$. Let $\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_p$ the decomposition of \mathcal{S} as a Riemannian product of irreducible symmetric spaces. The image of ρ acts diagonally and isometrically with respect to this decomposition. By projection to each factor, we inherit a ρ_i -equivariant harmonic map $\Psi_i : \tilde{X} \rightarrow \mathcal{S}_i$ where $\rho_i : \pi_1(X) \rightarrow \text{Iso}^0(\mathcal{S}_i)$ is the corresponding representation. By density, the image of ρ_i does not preserve any non-trivial subspace of $\mathfrak{g}_i := \text{Lie}(\text{Iso}^0(\mathcal{S}_i))$. By virtue of Corlette's uniqueness Theorem [Cor88, Theorem 3.4], the Ψ_i 's are the only ones ρ_i -equivariant harmonic maps. In particular Ψ is the unique ρ -equivariant harmonic map, as desired. \square

6. APPLICATIONS TO THE EXISTENCE OF FOLIATED HARMONIC MAPS FOR SOME RIEMANNIAN FOLIATIONS

6.1. The associated transverse frame bundle. We first give a brief account on Molino's theory, which allows to study Riemannian foliations from the (simpler) viewpoint of parallelizable ones.

Let (X, \mathcal{F}, g) be a compact Riemannian foliated manifold (see §3.2 for the definition). As usual, we set $m = \text{rk}(\mathcal{F})$ and $m+n = \dim(X)$. Denote by \mathcal{F}^\sharp the foliation constructed as the lift of \mathcal{F} on the direct orthonormal transverse frame bundle X^\sharp (see [Mol88, §2.5]). Both foliations have the same rank, \mathcal{F}^\sharp projects onto \mathcal{F} via the natural projection map

$$p : X^\sharp \longrightarrow X$$

so that the differential p_* induces a surjective morphism \overline{p}_* between the normal bundles $N\mathcal{F}^\sharp$ and $p^*N\mathcal{F}$.

The space X^\sharp is naturally endowed with a structure of $\text{SO}(n)$ -principal bundle. As in the classical setting, the transverse Levi-Civita connection associated to \mathcal{F} is defined on X^\sharp by a horizontal distribution \mathcal{H} on X^\sharp . By construction, the foliation \mathcal{F}^\sharp is tangent to \mathcal{H} , so that $\overline{\mathcal{H}} := \mathcal{H}/\mathcal{F}^\sharp$ is a subbundle of $N\mathcal{F}^\sharp$, and both \mathcal{H} and \mathcal{F}^\sharp are invariant under the right

action of $\mathrm{SO}(n)$. Moreover, the foliated manifold $(X^\sharp, \mathcal{F}^\sharp)$ is equipped with a canonical transverse parallellism. More precisely let us first fix a basis $(\lambda_1, \dots, \lambda_k)$ of $\mathfrak{so}(n)$ (with $k = \frac{n(n-1)}{2}$) identified with the Lie algebra of fundamental vector fields (with respect to the action of $\mathrm{SO}(n)$). On the other hand, pick a basis (e_1, \dots, e_n) of \mathbb{R}^n and denote by $\overline{u_i} \in \Gamma(\overline{\mathcal{H}})$ the horizontal transverse vector field on X^\sharp such that for every $z \in X^\sharp$, the projection $\overline{p}_*(\overline{u_i}(z))$ has coordinate vector e_i in the transverse frame z . Let $\overline{\lambda_i}$ be the projection of λ_i on $N\mathcal{F}^\sharp$. It turns out that the transverse vector fields $\overline{\lambda_1}, \dots, \overline{\lambda_k}, \overline{u_1}, \dots, \overline{u_n}$ are *basic* and thus define a transverse parallelism for the lifted foliation \mathcal{F}^\sharp (cf. [Mol88, §3.3]).

Definition 6.1. The structural Lie algebra of \mathcal{F} is by definition the structural Lie algebra of the transversely parallelizable foliation \mathcal{F}^\sharp .

The transverse invariant metric \overline{g} on $N\mathcal{F}$ induces a canonical transverse invariant metric $\overline{g}^\sharp = p^*\overline{g} \oplus_{\overline{\mathcal{H}}} \vartheta$ on $N\mathcal{F}^\sharp$. The latter is obtained as the orthogonal sum (with respect to the splitting $N\mathcal{F}^\sharp = \overline{\mathcal{H}} \oplus \mathrm{Ker}(dp)$) of the lifting of \overline{g} on $\mathcal{H}/\mathcal{F}^\sharp$ and the metric ϑ on vertical fibers induced by the unique bi-invariant metric of volume 1 on $\mathrm{SO}(n)$. Over the local Riemannian leaf space, $(U/\mathcal{F}, \overline{g})$, $(p^{-1}(U)/\mathcal{F}^\sharp, \overline{g}^\sharp)$ is nothing but the orthonormal frame bundle equipped with its canonical metric. In particular the vertical fibers are totally geodesic (see for instance [O’N66, §5, p. 466]).

The bundle-like metric g induces a canonical $\mathrm{SO}(n)$ -invariant bundle-like metric $g^\sharp = p^*g \oplus_{\mathcal{H}} \vartheta$ for \mathcal{F}^\sharp on X^\sharp . Indeed, g^\sharp is constructed as the orthogonal sum of the lifting of g on the horizontal distribution $T\mathcal{H}$ and ϑ . In particular $p : (X^\sharp, g^\sharp) \rightarrow (X, g)$ is a Riemannian submersion with totally geodesic fibers. According to [Noz10, Proof of Lemma 7], the mean curvature forms (see §3.4) of (X, \mathcal{F}, g) and $(X^\sharp, \mathcal{F}^\sharp, g^\sharp)$ are simply related by

$$(6.1) \quad \kappa_{g^\sharp} = p^*(\kappa_g).$$

In particular, if (\mathcal{F}, g) is *tense* (resp. *taut*), then $(\mathcal{F}^\sharp, g^\sharp)$ is *tense* (resp. *taut*).

Let \mathfrak{g} and $C_{\mathcal{F}^\sharp}$ be respectively the structural Lie algebra and the commuting sheaf attached to \mathcal{F}^\sharp . Note that $C_{\mathcal{F}^\sharp}$ is invariant under the right action of $\mathrm{SO}(n)$, so that we can define $C_{\mathcal{F}} := p_*C_{\mathcal{F}^\sharp}$, the *commuting sheaf* associated to \mathcal{F} . According to [Mol88, §5.3], $C_{\mathcal{F}}$ is a locally constant sheaf of Lie algebras with typical fiber \mathfrak{g} formed by local basic Killing vector fields whose local flows describe the leaves closure. This commuting sheaf can be also alternatively defined as the Lie algebra of the closure of the holonomy pseudo-group which turns out to be a Lie pseudo-group [Mol88, É. Salem’s Appendix D].

The locally constant sheaf $C_{\mathcal{F}}$ gives rise to a representation of the fundamental group

$$\rho : \pi_1(X) \longrightarrow \mathrm{Aut}(\mathfrak{g})$$

such that $\rho^\sharp = \rho \circ p_*$ where $p : X^\sharp \rightarrow X$ is the natural projection and where the representation

$$\rho^\sharp : \pi_1(X^\sharp) \longrightarrow \mathrm{Aut}(\mathfrak{g})$$

is the one associated to $C_{\mathcal{F}^\sharp}$.

Consider the universal cover map $r^\sharp : \widetilde{X}^\sharp \rightarrow X^\sharp$ and the $\mathrm{SO}(n)$ -principal bundle $Y := r^{-1}(X^\sharp)$ over \widetilde{X} . This is actually the Galois cover of X^\sharp associated to the kernel of the natural projection morphism $\pi_1(X^\sharp) \rightarrow \pi_1(X)$. It also coincides with the transverse orthonormal frame bundle associated to $(\widetilde{X}, \widetilde{\mathcal{F}})$ and we have a natural projection map $q : \widetilde{X}^\sharp \rightarrow Y \rightarrow \widetilde{X}$ which factors through $r^{-1}(p) : Y \rightarrow \widetilde{X}$. The space \widetilde{X}^\sharp inherits from

Y a structure of \mathcal{S}_n -principal bundle over \tilde{X} where \mathcal{S}_n is some connected covering group of $\mathrm{SO}(n)$.

In view of the forthcoming Theorem 6.4, we complete the picture with the following result.

Theorem 6.2 (cf. [Noz10, Theorem 2]). *Let (X, \mathcal{F}) a compact Riemannian foliated manifold and let us assume in addition that the structural Lie algebra of \mathcal{F} is semi-simple.*

Then \mathcal{F} is minimizable, i.e. there exists a bundle-like metric g such that (\mathcal{F}, g) is taut.

We will also make use of the following observation, which is a consequence of the definition of $C_{\mathcal{F}}$ in term of $C_{\mathcal{F}}^{\sharp}$, the identification specified in Remark 5.19 between $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Iso}(\mathcal{S})$, and the fact that ρ^{\sharp} factors through ρ .

Lemma 6.3. *Suppose that the structural Lie algebra \mathfrak{g} is semi-simple without compact factors and let \mathcal{S} be the associated symmetric space. Assume that there exists a $\tilde{\mathcal{F}}$ -invariant smooth ρ -equivariant map $\Psi : \tilde{X} \rightarrow \mathcal{S}$. Then the composed map $\Psi^{\sharp} := \Psi \circ q : \tilde{X}^{\sharp} \rightarrow \mathcal{S}$ satisfies the equivariance property (5.4) of Theorem 5.17:*

$$\forall x \in \tilde{X}^{\sharp}, \forall \gamma \in \pi_1(X^{\sharp}), \Psi^{\sharp}(\gamma(x)) = \rho^{\sharp}(\gamma)(\Psi(x)).$$

Using the results from Section 5 in the setting of the transverse frame bundle, we can prove the following result.

Theorem 6.4. *Let \mathcal{F} be a transversely orientable Riemannian foliation on a compact manifold X . Let g be a bundle-like metric with vanishing mean curvature vector field (it exists by virtue of Theorem 6.2). Denote by $\tilde{\mathcal{F}}$ the lifted foliation on the universal cover \tilde{X} . Assume that the structural Lie algebra of \mathcal{F} is semi-simple without compact factors and let \mathcal{S} be the associated symmetric space.*

Let $\rho : \pi_1(X) \rightarrow \mathrm{Aut}(\mathfrak{g}) \simeq \mathrm{Iso}(\mathcal{S})$ be the monodromy representation attached to the commuting sheaf $C_{\mathcal{F}}$. Then there exists a surjective and submersive ρ -equivariant harmonic map $\Psi : \tilde{X} \rightarrow \mathcal{S}$ with connected fibers which is constant on the leaves of $\tilde{\mathcal{F}}$.

Proof. Consider the lift \mathcal{F}^{\sharp} of \mathcal{F} on the orthonormal frame bundle X^{\sharp} together with its bundle like metric g^{\sharp} . Following (6.1), $(\mathcal{F}^{\sharp}, g^{\sharp})$ is also *taut*, so that the conclusion of Theorem 5.20 is valid (taking also into account Remark 5.19): there exists a unique $\tilde{\mathcal{F}}^{\sharp}$ -invariant and ρ^{\sharp} -equivariant harmonic map $\Psi_0^{\sharp} : \tilde{X}^{\sharp} \rightarrow \mathcal{S}$. A priori, it is not completely obvious that Ψ_0^{\sharp} factors through a ρ -equivariant map defined on \tilde{X} . Actually, this holds true according to the following trick which is inspired from [EIKGG96]. The connected structure group \mathcal{S}_n acts by isometries on \tilde{X}^{\sharp} (with respect to the induced bundle-like metric). Consequently, if τ is the isometric transformation associated to an element of \mathcal{S}_n , the mapping $\Psi_0^{\sharp} \circ \tau$ is also ρ^{\sharp} -equivariant and harmonic. By the uniqueness part of Theorem 5.20, Ψ_0^{\sharp} is \mathcal{S}_n -invariant, hence factors through a ρ -equivariant and $\tilde{\mathcal{F}}$ -invariant map $\Psi_0 : \tilde{X} \rightarrow \mathcal{S}$. We can now apply the same deformation process as in the proof of Theorem 5.20 and thus obtain the expected (and necessarily unique) ρ -equivariant and $\tilde{\mathcal{F}}$ -invariant harmonic map Ψ .¹² By Lemma 6.3, $\Psi^{\sharp} = \Psi \circ q$ satisfies property (5.4) of Theorem 5.17, so that Ψ^{\sharp} is a surjective submersion with connected fibers by the item (4) of the same theorem. This immediately implies that Ψ is so. \square

As a direct consequence of the above construction of Ψ and the item (5) of Theorem 4.1, we have established the following statement.

¹²In fact, it is not difficult to observe, using that the fibers of $X^{\sharp} \rightarrow X$ are totally geodesic combined with Proposition 3.13, that we can directly take $\Psi = \Psi_0$.

Corollary 6.5. *Keeping notation as above, let us consider the lift $r^*\mathcal{C}_{\mathcal{F}}$ of the commuting sheaf $\mathcal{C}_{\mathcal{F}}$ on the universal cover \tilde{X} (this is a Lie subalgebra of basic sections of $N\tilde{\mathcal{F}}$). Then, for every $v \in r^*\mathcal{C}_{\mathcal{F}}$, $d\Psi(v)$ is constant along the fibers of Ψ . Moreover, $d\Psi$ induced a Lie algebra isomorphism between $r^*\mathcal{C}_{\mathcal{F}}$ and $\mathfrak{iso}(\mathcal{S})$.*

6.2. The case of a transversely Kähler foliation. In the Kähler setting, rigidity of harmonic maps can be used to strengthen the conclusion of Theorem 6.4. Let us start with a Riemannian compact foliated manifold (X, \mathcal{F}, g) where (\mathcal{F}, g) is transversely Kähler with semi-simple structural Lie algebra without compact factors. As before (with a small shift), m and $2n$ denote respectively the real dimension and codimension of \mathcal{F} .¹³ The following two theorems, as well as their proofs, are strongly related to [Fra95, Proposition 4.3 and Corollary 4.4]

Theorem 6.6. *Let (X, \mathcal{F}, g) be a Riemannian compact foliated manifold where (\mathcal{F}, g) is transversely Kähler, its structural Lie algebra \mathfrak{g} being semi-simple without compact factors. Assume moreover that (\mathcal{F}, g) is taut (as in Theorem 6.2). Let $\Psi : \tilde{X} \rightarrow \mathcal{S}$ be the (unique) ρ -equivariant harmonic map provided by Theorem 6.4. Then \mathcal{S} is Hermitian symmetric (of the non-compact type).*

Proof. We follow closely Toledo's survey [Tol99]. Here, we have to think that the relevant substitute for the complexification of the tangent bundle of the source manifold is the *complexified normal bundle* together with its splitting into $(1, 0)$ and $(0, 1)$ parts provided by the transverse complex structure $J := J_{\mathcal{F}}$:

$$N_{\mathbb{C}}\mathcal{F} = N^{1,0}\mathcal{F} \oplus N^{0,1}\mathcal{F}$$

and similarly for the lifted foliation $\tilde{\mathcal{F}}$.

Let us pick a bundle-like metric g such that (\mathcal{F}, g) is taut. We begin with some useful observations. Recall that the tension field $\tau(\Psi)$ of Ψ coincides with the basic one $\tau_b(\Psi)$. Consequently the harmonic equation $\tau(\Psi) = \tau_b(\Psi) = 0$ reads

$$\tilde{\omega}^{n-1} \wedge d_{\nabla} d^c \Psi = 0$$

where $\tilde{\omega}$ is the lift of the transverse Kähler metric, d_{∇} is the differentiation operator

$$A^k(\tilde{X}, f^*T\mathcal{S}) \longrightarrow A^{k+1}(\tilde{X}, f^*T\mathcal{S})$$

which extends the Levi-Civita connection on $f^*T\mathcal{S}$ and which also acts (by restriction) on the complex of twisted basic forms A_b^{\bullet} . The term $d^c \Psi$ is a basic twisted one form and stands for $Jd\Psi$. Consider the basic scalar-valued 3-form

$$\eta_{\Psi} = \|d^c \Psi \wedge d_{\nabla} d^c \Psi\|^2$$

where the norm is taken with respect to the scalar product $f^*T\mathcal{S} \otimes f^*T\mathcal{S} \rightarrow \mathbb{R}$ induced by the Killing metric on \mathcal{S} . Note also that $\eta_{\Psi} = \eta_{\varphi \circ \Psi}$ for any isometry φ of \mathcal{S} ; in particular, $\eta = \eta_{\Psi}$ is actually well defined on X .

Up to passing to a double cover, we can assume that \mathcal{F} is oriented. The volume form defined by g is thus $\nu = \omega^n \wedge \chi$ where $\chi := \chi_{\mathcal{F}}$ is the characteristic form associated to the bundle-like metric g (cf. §3.4). By Stokes' Theorem, we have

$$\int_X d(\eta \wedge \omega^{n-2} \wedge \chi) = 0.$$

¹³For transversely Kähler foliation, the $\mathrm{SO}(2n)$ -principal bundle $X^{\#}$ admits a reduction to a $U(n)$ -principal bundle, but we will not use this property in the sequel.

On the other hand, we know that $d(\eta \wedge \omega^{n-2})$ is basic and that the mean curvature κ vanishes identically. From Lemma 3.8, we obtain that

$$d(\eta \wedge \omega^{n-2} \wedge \chi) = d(\eta \wedge \omega^{n-2}) \wedge \chi.$$

We can then apply the punctual Hodge index theorem combined with the harmonic equation to deduce that Siu's vanishing theorem is still valid in our setting (see the proof of Theorem 3.1 in [Tol99] and Generalization 1 in *loc. cit.*), namely

$$(6.2) \quad d_{\nabla} d^c \Psi = 0 \quad \text{and} \quad R(d\Psi(V), d\Psi(W), d\Psi(\bar{V}), d\Psi(\bar{W})) = 0$$

for any (local) basic vector field V and W of type $(1, 0)$. Here $R = -d_{\nabla}^2$ denotes the complexification of the curvature tensor on $T\mathcal{S} \otimes \mathbb{C}$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra \mathfrak{g} . The following reasoning is again borrowed from [Tol99, §4]. Pick a point $x \in \tilde{X}$. From the previous vanishing properties and the fact that Ψ is submersive, we deduce that $d\Psi(N_x^{1,0} \tilde{\mathcal{F}})$ is an abelian subalgebra of $T_{\Psi(x)}\mathcal{S} \otimes \mathbb{C} \simeq \mathfrak{p} \otimes \mathbb{C}$. With this at hands and recalling that Ψ is submersive and \mathcal{S} is a symmetric space of the non-compact type, we can conclude as in the discussion at the beginning of §4 in [Tol99] that \mathcal{S} is actually *Hermitian symmetric*. \square

The fact that Ψ has maximum rank also implies the following statement (*cf.* [Tol99, Theorem 4.2]).

Theorem 6.7. *Up to replacing the complex structure by its conjugate on each irreducible factor of \mathcal{S} , the map Ψ is (transversely) holomorphic.*

Proof. Up to replacing X by a finite étale cover, we can suppose that the image of the representation ρ lies in the identity component $\text{Iso}^0(\mathcal{S})$ of the isometry group of \mathcal{S} . We have an isometric splitting $\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_p$ into irreducible Hermitian symmetric spaces ordered in such a way that for some p' , $\mathcal{S}_{p'} \times \cdots \times \mathcal{S}_p$ is the polydisk factor (maybe empty). The fundamental group $\pi_1(X)$ (viewed as the group of deck transformations) acts via the representation ρ diagonally and isometrically on \mathcal{S} . In particular, it inherits for every $i = 1, \dots, p$ a representation $\rho_i : \pi_1(X) \rightarrow \text{Iso}^0(\mathcal{S}_i)$ with dense image together with the foliated ρ_i -equivariant harmonic map $\Psi_i : \tilde{X} \rightarrow \mathcal{S}_i$ induced by Ψ by projection. We need exactly to prove that for every factor \mathcal{S}_i , $\Psi_* \circ J_{\tilde{\mathcal{F}}} = J_i \circ \Psi_*$, where J_i is one of the two $\text{Iso}^0(\mathcal{S}_i)$ -invariant complex structures on \mathcal{S}_i . Because Ψ_i has maximal rank, this automatically holds in the case $i < p'$ according to Siu–Carlson–Toledo's rigidity results [Tol99, Theorem 4.2] which can be derived from (6.2).

In the case where $i \geq p'$ and $\mathcal{S}_i = \mathbb{D}$ is the Poincaré disk, we resort to the analysis developed in [JY83]. Because of the maximal rank condition and the property (6.2) are fulfilled, the local levels of Φ are given by $\{z_1 = \text{const}\}$ where z_1 is a suitable holomorphic transverse coordinate (with respect to $\tilde{\mathcal{F}}$). Then, by connectedness of the fibers of Φ , there exists on \mathcal{S}_i a well defined and necessarily unique complex structure J such that Φ become (transversely) holomorphic with respect to J . Moreover, J is invariant under the action of the image of ρ_i by equivariance, hence by the whole action of $\text{Iso}^0(\mathcal{S}_i)$ by density of the representation. The proof of Theorem 6.7 is thus complete. \square

7. DE RHAM DECOMPOSITION FOR TRANSVERSELY KÄHLER FOLIATION OF QUASI-NEGATIVE TRANSVERSE RICCI CURVATURE

Let (X, \mathcal{F}, g) be a compact Riemannian foliated manifold satisfying the assumptions of Section 1, that is

- \mathcal{F} is transversely Kähler and from now on the integer n will denote its complex codimension
- The transverse Ricci form $\gamma = \text{Ric}(\bar{g})$ is quasi-negative. (7.1)
- \mathcal{F} is homologically orientable: the basic de Rham cohomology class of the basic volume form induced by \bar{g} is non-trivial.

We will denote by ω the fundamental form of the transverse Kähler metric \bar{g} . This is a basic $(1, 1)$ form which is positive in the transverse direction. As in the classical setting of complex manifolds, the restriction of the differential d to complex valued basic forms decomposes as the sum of two operators ∂ and $\bar{\partial}$ of respective bi-degrees $(1, 0)$ and $(0, 1)$.

Also, as orbifolds enter into the picture at the end this section, we refer to [MM03] or [Car19] (and references therein) for the related basic definitions/properties.

7.1. Preparatory material. The following result was proven in [Tou10].

Theorem 7.1. *The structural Lie algebra \mathfrak{g} of \mathcal{F} is semi-simple without compact factors.*

In particular we can (and we will) assume that (\mathcal{F}, g) is *taut* by Nosawa's Theorem 6.2. Actually, tautness can be directly derived from the homological orientability assumption according to Masa's criterion recalled in Theorem 3.7. According to Theorem 6.7, \mathcal{F} admits an extension \mathcal{G} (a priori not Riemannian) which is defined on the universal cover \tilde{X} of X by the levels of a $J_{\mathcal{F}}$ -holomorphic map $\Psi : \tilde{X} \rightarrow \mathcal{S}$ to a bounded symmetric domain and whose additional properties are listed in Theorem 6.4. Let $h_{\mathcal{S}}$ be the Killing metric on \mathcal{S} . This is a Kähler–Einstein metric with negative Ricci form $\text{Ric}(h_{\mathcal{S}})$. After normalization, we can suppose that $\text{Ric}(h_{\mathcal{S}}) = -\omega_{\mathcal{S}}$ where $\omega_{\mathcal{S}}$ is the fundamental form of $h_{\mathcal{S}}$. The pull-back of $\omega_{\mathcal{S}}$ by Ψ is $\pi_1(X)$ -invariant and then descends to X as a basic (with respect to both \mathcal{F} and \mathcal{G}) $(1, 1)$ -forms that we will denote by Ω .¹⁴

Set $p = \text{rk}_{\mathbb{C}} \mathcal{G}/\mathcal{F}$. The short exact sequence of complex vector bundles¹⁵

$$0 \rightarrow \mathcal{G}/\mathcal{F} \rightarrow N\mathcal{F} \rightarrow N\mathcal{G} \rightarrow 0$$

implies that the corresponding Chern classes are related by

$$c_1(N\mathcal{F}) = c_1(\mathcal{G}/\mathcal{F}) + c_1(N\mathcal{G}) \in H^2(X, \mathbb{Z}).$$

Consequently, $c_1(\mathcal{G}/\mathcal{F})$ is represented by the basic closed $(1, 1)$ -form $\alpha = \frac{1}{2\pi}(\gamma + \Omega)$. Note that α coincides with $\frac{1}{2\pi}\gamma$ in restriction to the leaves of \mathcal{G} and in particular is quasi-negative on \mathcal{G}/\mathcal{F} .

On the other hand, $c_1(\mathcal{G}/\mathcal{F})$ is also represented by the Chern curvature of the metric ω^p induced by ω on \mathcal{G}/\mathcal{F} , that is $c_1(\mathcal{G}/\mathcal{F}) = [\frac{1}{2\pi}\alpha']$ with

$$\alpha' = -\sqrt{-1}\partial\bar{\partial}\log\left(\frac{\omega^p \wedge \Omega^{n-p}}{|dz_1 \wedge \cdots \wedge dz_p|^2 \wedge \Omega^{n-p}}\right)$$

and where the local transverse holomorphic coordinates (z_1, \dots, z_n) are chosen in such a way that $|dz_1 \wedge \cdots \wedge dz_p|^2$ does not vanish in restriction to the leaves of \mathcal{G} . Actually, both α and α' are closed basic $(1, 1)$ -forms and they are related by

$$(7.2) \quad \alpha = \alpha' + \sqrt{-1}\partial\bar{\partial}f$$

¹⁴Unless otherwise specified, “basic” means basic with respect to the original foliation \mathcal{F} ; of course, any covariant tensor which is basic with respect to \mathcal{G} is automatically basic with respect to \mathcal{F}

¹⁵After complexification and identification with their $(1, 0)$ parts.

where f_0 is the basic function such that

$$\omega^p \wedge \Omega^{n-p} = e^{f_0} \omega^n.$$

7.2. Existence of a special \mathcal{F} -basic \mathcal{G} -leafwise Kähler metric. The following definition concerns (holomorphic) extensions of foliations (cf. §3.3) and makes precise the notion of “invariant Kähler metric along the leaves of \mathcal{G} ”.

Definition 7.2. A basic (with respect to \mathcal{F}) and closed $(1, 1)$ -form ω' is said to be a \mathcal{G} -leafwise Kähler metric whenever is positive in restriction to \mathcal{G}/\mathcal{F} .

Actually, the restriction $\mathcal{F}|_{\mathcal{L}_{\mathcal{G}}}$ to any leaf $\mathcal{L}_{\mathcal{G}}$ of \mathcal{G} is transversely holomorphic on the (non compact) manifold $\mathcal{L}_{\mathcal{G}}$, so that an ω' satisfying the property above induces an invariant transverse Kähler metric with respect to $\mathcal{F}|_{\mathcal{L}_{\mathcal{G}}}$, whence the terminology used.

The following result guarantees the existence of a special \mathcal{F} -basic \mathcal{G} -leafwise Kähler metric and can be thought as a variant of the famous Yau’s existence Theorem of solutions to the complex Monge–Ampère equation. It is motivated by the relationship between α and α' given in (7.2).

Lemma E. *Let \mathcal{F} be a foliation satisfying the assumptions listed in (7.1). Assume in addition that \mathcal{F} is orientable. Let χ be the characteristic form associated to the bundle-like metric g . Let f be a real basic smooth function such that*

$$\int_X e^f \omega^p \wedge \Omega^{n-p} \wedge \chi = \int_X \omega^p \wedge \Omega^{n-p} \wedge \chi.$$

Then there exists a basic real smooth function φ which solves

$$(7.3) \quad (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^p \wedge \Omega^{n-p} = e^f \omega^p \wedge \Omega^{n-p}$$

and such that $\omega + \sqrt{-1}\partial\bar{\partial}\varphi$ is a \mathcal{F} -basic \mathcal{G} -leafwise Kähler metric. Moreover, such a φ is unique up to an additive constant.

The proof being quite involved (particularly the existence part), we postpone it to Section 8. Here we simply formulate some useful remarks and conclude the proofs of the main results of this article.

Remark 7.3. Observe that when \mathcal{F} is minimal, i.e. has dense leaves, f is automatically constant, hence identically zero by the normalization condition, so that existence part of the lemma is obvious.

Remark 7.4. When \mathcal{F} is still supposed minimal, $\overline{\gamma_h} := -\text{Ric}(\overline{g})$ is the fundamental form of a transverse Kähler metric h . Now, $\text{Ric}(h)$ represents the same cohomology class than $-\gamma_h$ (namely $c_1(N\mathcal{F})$). By El-Kacimi’s basic $\partial\bar{\partial}$ -Lemma [ElK90, Proposition 3.5.1], we can conclude that $\text{Ric}(h) = -\gamma_h$. That is h is (transversely) Kähler–Einstein.

By adding the observations made at the end of §7.1 and more specially the equality (7.2), we can formulate the following corollary.

Corollary 7.5. *Let φ_0 a solution of (7.3) with $f = f_0 + c$ (c a suitable normalizing constant) and set $\omega_{\varphi_0} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_0$. Let $\overline{\mathcal{L}}$ be a leaf of the holomorphic foliation $\overline{\mathcal{G}}$ induced by \mathcal{G} on the local space of leaves U/\mathcal{F} . The restriction of ω_{φ_0} to $\overline{\mathcal{L}}$ defines on $\overline{\mathcal{L}}$ a Kähler metric whose Ricci form coincide with $\alpha|_{\overline{\mathcal{L}}}$.*

7.3. Vanishing loci of isotropy subalgebras of the commuting sheaf. Recall that the differential $d\Psi$ provides a Lie algebra isomorphism between the lift $r^*\mathcal{C}_{\mathcal{F}}$ of the commuting sheaf $r^*\mathcal{C}_{\mathcal{F}}$ to the universal cover $r : \widetilde{X} \rightarrow X$ and the Lie algebra $\mathfrak{iso}(\mathcal{S})$ of infinitesimal isometries of \mathcal{S} . For any $s \in \mathcal{S}$, we will denote by $\mathfrak{iso}_s(\mathcal{S}) \subset \mathfrak{iso}(\mathcal{S})$ the isotropy Lie subalgebra:

$$\mathfrak{iso}_s(\mathcal{S}) := \{w \in \mathfrak{iso}(\mathcal{S}) \mid w(s) = 0\}.$$

Theorem 7.6. *Let $s \in \mathcal{S}$ and $w \in \mathfrak{iso}_s(\mathcal{S})$. Let v be the unique element of $r^*\mathcal{C}_{\mathcal{F}}$ such that $d\Psi(v) = w$. Then v vanishes identically on the fiber $F_s = \Psi^{-1}(s)$.*

Proof. Let $r^\sharp : \widetilde{X}^\sharp \rightarrow X^\sharp$ be the universal cover and $q : \widetilde{X}^\sharp \rightarrow \widetilde{X}$ be the natural projection. Set $\Psi^\sharp = \Psi \circ q$. Let us also consider:

$$r^*\mathcal{C}_{\mathcal{F}} \supset r^*(\mathcal{C}_{\mathcal{F},s}) := d\Psi^{-1}(\mathfrak{iso}_s(\mathcal{S}))$$

and similarly

$$(r^\sharp)^*\mathcal{C}_{\mathcal{F}^\sharp} \supset (r^\sharp)^*(\mathcal{C}_{\mathcal{F}^\sharp,s}) = d\Psi^{\sharp-1}(\mathfrak{iso}_s(\mathcal{S})).$$

Let $\pi_{\mathcal{F}^\sharp} : \widetilde{X}^\sharp \rightarrow P := X^\sharp/\mathcal{F}^\sharp$ be the projection map onto the space of leaves. Recall that P has the structure of a G -principal bundle (where $\mathrm{Lie}(G) = \mathfrak{g}$) over the basic manifold W and such that the Lie algebra of fundamental vector fields coincides with $(r^\sharp)^*\mathcal{C}_{\mathcal{F}^\sharp}$ (identified with its projection via $\pi_{\mathcal{F}^\sharp}$). Recall also that G acts on the Hermitian symmetric space \mathcal{S} via $\pi : G \rightarrow \mathrm{Iso}(\mathcal{S})$. Moreover, Ψ^\sharp and its differential $d\Psi^\sharp$ are “equivariant” with respect to the representation ρ^\sharp and the adjoint action respectively; it means that the following identities hold for every $x \in \widetilde{X}^\sharp$, $\gamma \in \pi_1(X)$, $h \in G$, and $v \in (r^\sharp)^*(\mathcal{C}_{\mathcal{F}^\sharp})$:

$$\begin{aligned} \Psi^\sharp(\gamma(x)) &= \rho^\sharp(\gamma)(\Psi^\sharp(x)), \\ d\Psi^\sharp(\mathrm{Ad}(h)(v)) &= \mathrm{Ad}(\pi(h))(d\Psi^\sharp(v)), \quad \text{and} \\ d\gamma(v) &= \mathrm{Ad}(\rho_{\mathcal{F}^\sharp}(\gamma))(v). \end{aligned}$$

Let ξ be a transverse symmetric $(0,2)$ tensor on X (e.g. $\xi = \overline{g}$, the transverse invariant metric of \mathcal{F}), and $\widetilde{\xi}$, ξ^\sharp , $\widetilde{\xi}^\sharp$ be their respective pull-backs on \widetilde{X} , X^\sharp and \widetilde{X}^\sharp . Those are basic transverse symmetric $(0,2)$ tensors (respectively with respect to $\widetilde{\mathcal{F}}$, \mathcal{F}^\sharp and $\widetilde{\mathcal{F}^\sharp}$), ξ^\sharp and $\widetilde{\xi}^\sharp$ being in addition $\mathrm{SO}(2n)$ and \mathcal{S}_{2n} -invariant respectively. Denote by B the Killing form on $\mathfrak{g} \simeq (r^\sharp)^*\mathcal{C}_{\mathcal{F}^\sharp} \simeq r^*\mathcal{C}_{\mathcal{F}}$ and by B_s its restriction to the maximal compact subalgebra $\mathfrak{g}_s \simeq (r^\sharp)^*(\mathcal{C}_{\mathcal{F}^\sharp,s}) \simeq r^*(\mathcal{C}_{\mathcal{F},s})$. In particular B_s is negative definite on \mathfrak{g}_s and $\mathrm{Ad}(h^{-1})$ provides an isometry between (\mathfrak{g}_s, B_s) and $(\mathfrak{g}_{h(s)}, B_{h(s)})$ (here, we identify \mathfrak{g} with $(r^\sharp)^*\mathcal{C}_{\mathcal{F}^\sharp}$). In particular, if we fix s and an orthonormal basis (v_i) of \mathfrak{g}_s (with respect to the scalar product $-B_s$), the function $\|\mathfrak{g}_s\| : \widetilde{X}^\sharp \rightarrow \mathbb{R}$ defined by $\|\mathfrak{g}_s\|(x) := \sum_i \widetilde{\xi}^\sharp(v_i(x), v_i(x))$ does not depend on the choice of (v_i) and is a basic function which satisfies

$$(7.4) \quad \|\mathfrak{g}_s\| = \|\mathfrak{g}_{\rho^\sharp(\gamma)(s)}\| \circ \gamma$$

for any $\gamma \in \pi_1(X^\sharp)$. In addition $\|\mathfrak{g}_s\|$ is \mathcal{S}_{2n} -invariant (because $(r^\sharp)^*\mathcal{C}_{\mathcal{F}^\sharp}$ is so).

Define $\mathcal{D}^\sharp : \widetilde{X}^\sharp \rightarrow \mathbb{R}$ by $\mathcal{D}^\sharp(x) = \|\mathfrak{g}_{\Psi^\sharp(x)}\|(x)$. From the equation (7.4) we can deduce that \mathcal{D}^\sharp is a basic function, that is moreover invariant under the actions of $\pi_1(X^\sharp)$ and \mathcal{S}_{2n} . Actually, the \mathcal{S}_{2n} -invariance is inherited from that of $\|\mathfrak{g}_s\|$, taking into account that the \mathcal{S}_{2n} -action on \widetilde{X}^\sharp makes the map Ψ^\sharp equivariant. Now, as the projection $q : \widetilde{X}^\sharp \rightarrow \widetilde{X}$ induces a surjective morphism from $\pi_1(\widetilde{X}^\sharp)$ onto $\pi_1(X)$, we can conclude that there exists a basic $\pi_1(X)$ -invariant function $\mathcal{D} : \widetilde{X} \rightarrow \mathbb{R}$ such that $\mathcal{D}^\sharp = \mathcal{D} \circ q$.

Note also that \mathcal{D}^\sharp (and consequently \mathcal{D}) is smooth. Indeed, pick a fiber $F_s = (\Psi^\sharp)^{-1}(s)$ and remark that the restriction of $\|\mathfrak{g}_s\|$ to F_s is obviously smooth. Let $\Upsilon_s \subset G$ be a small transverse to the stabilizer of G at $s \in \mathcal{S}$ passing through e_G . By the implicit function Theorem, we can fill out a small neighborhood U of s by $\{\pi(h)(s) \mid h \in \Upsilon_s\}$ such that the correspondence $h \rightarrow \pi(h)(s)$ induces a diffeomorphism between Υ_s and U . To any element $v \in \mathfrak{g}_s$, we can thus associate on the neighborhood $(\Psi^\sharp)^{-1}(U)$ of F_s a unique smooth basic vector field $\tau(v)$ characterized by

- for every $h \in \Upsilon_s$, $\tau(v)|_{F_{\pi(h)(s)}}$ is tangent to $F_{\pi(h)(s)} = (\Psi^\sharp)^{-1}(\pi(h)(s))$
- $\tau(v)|_{F_{\pi(h)(s)}} = \text{Ad}(h^{-1})(v)|_{F_{\pi(h)(s)}}$.

Roughly speaking, $\tau(v)$ is the “transport” of $v|_{F_s}$ along Υ_s . With this in mind, we thus observe that \mathcal{D}^\sharp coincides on $(\Psi^\sharp)^{-1}(U)$ with $\sum_i \tilde{\xi}^\sharp(\tau(v_i), \tau(v_i))$ where v_i is any orthonormal basis of $(\mathfrak{g}_s, -B_s)$, thus proving the smoothness of \mathcal{D}^\sharp and \mathcal{D} .

By construction, \mathcal{D} only depends on the choice of the original tensor ξ . In the sequel, we will focus on basic symmetric tensors of the forms $\xi = \bar{g}_\varphi = \omega_\varphi(\cdot, J_{\mathcal{F}}(\cdot))$ where $\varphi : X \rightarrow \mathbb{R}$ is basic and $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$. We will restrict our attention for those φ for which the restriction of ξ to \mathcal{G}/\mathcal{F} is positive definite. Let $s \in \mathcal{S}$ and denote by \mathcal{F}_s the restriction of $\tilde{\mathcal{F}}$ to F_s . We obtain in this way a foliated Riemannian manifold (F_s, \mathcal{F}_s) where the transverse Riemannian structure is defined by the transverse Kähler metric ξ_s , setting $\xi_s = \tilde{\xi}|_{F_s}$. Let w a basic vector field on F_s , such that v is the real part of a *holomorphic* basic vector field (equivalently the Lie derivative $\mathcal{L}_v J_{\mathcal{F}}$ vanishes). The transverse formulation of the classical Weitzenböck’s formula (see for instance [Kob87, Proposition 3.1.8]) reads as

$$(7.5) \quad -\frac{1}{2}\Delta_s \left(\|w\|_s^2 \right) = \|\nabla_s w\|_s^2 - \text{Ric}_s(w, w)$$

where $\Delta_s, \nabla_s, \text{Ric}_s, \|\cdot\|_s$ are respectively the (transverse) Laplacian, the $(1, 0)$ part of the Levi-Civita connection, the Ricci form and the norm with respect to ξ_s . In particular, this formula holds whenever w is the restriction v_s of an element of $r^*(\mathcal{C}_{\mathcal{F}, s})$ to F_s .

Now, Lemma E and Corollary 7.5 provide the existence of φ_0 such that for every s and every $x \in F_s$, Ric_s is negative semi-definite and is moreover negative somewhere. Fix φ_0 from now on. By semi-negativity of the Ricci form, *i.e.* the semi-negativity of α on \mathcal{G}/\mathcal{F} , the right-hand side of (7.5) is non-negative. Moreover, as \mathcal{D} descends to X , it reaches its maximum at $x_0 \in \tilde{X}$. Let $s_0 = \Psi(x_0)$. As the restriction of \mathcal{D} to F_{s_0} is given by $\mathcal{D}_{s_0} = \sum_i \|v_{i s_0}\|_{s_0}^2$, (v_i) an orthonormal basis of $r^*(\mathcal{C}_{\mathcal{F}, s_0})$, we can derive from (7.5) and the Hopf’s maximum principle that \mathcal{D}_{s_0} is constant. As \mathcal{D} descends to X and is continuous, this implies that \mathcal{D} is indeed constant on the whole of \tilde{X} by minimality of the foliation \mathcal{G} and is finally identically zero by quasi-negativity of α on \mathcal{G}/\mathcal{F} . This concludes the proof of Theorem 7.6. \square

Remark 7.7. The use of the Weitzenböck formula above and the resulting vanishing property given above is the substitute in our setting to [Fra95, Corollary 4.5(3)].

7.4. Two complementary parallel holomorphic foliations and proof of Theorem A. By Theorem 7.6, X is now endowed with a foliation $\overline{\mathcal{F}}$ satisfying the following properties:

- $\overline{\mathcal{F}}$ is an extension of \mathcal{F} .
- $\overline{\mathcal{F}}$ is locally generated by \mathcal{F} and $\mathcal{C}_{\mathcal{F}}$. In particular, the leaves of $\overline{\mathcal{F}}$ are the topological closure of the leaves of \mathcal{F} .

- Let $x \in X$. Locally, the vanishing locus of the set $\mathcal{C}_{\mathcal{F},x}$ of local sections of $\mathcal{C}_{\mathcal{F}}$ vanishing at x coincides with the leaf of \mathcal{G} through x . In particular, the rank of $\overline{\mathcal{F}}$ is equal to $\text{rk}(\mathcal{F}) + \dim(\mathcal{S})$ and $\overline{\mathcal{F}}$ intersects \mathcal{G} transversely along \mathcal{F} :

$$\overline{\mathcal{F}} \cap \mathcal{G} = \mathcal{F}.$$

Using in addition that the local sections v of $\mathcal{C}_{\mathcal{F}}$ are real part of *holomorphic* basic vector fields $v^{1,0}$, we can apply the statement of [Fra89, Lemma 12.1] to conclude that we have a \overline{g} -orthogonal decomposition

$$(7.6) \quad N\mathcal{F} = \mathcal{G}/\mathcal{F} \oplus \overline{\mathcal{F}}/\mathcal{F}$$

whence the following result.

Lemma 7.8. *The foliation $\overline{\mathcal{F}}$ is $J_{\mathcal{F}}$ -holomorphic and the aforementioned orthogonal splitting is parallel with respect to the transverse Levi-Civita connection of the transverse Kähler metric \overline{g} .*

Proof. Observe first that $\overline{\mathcal{F}}/\mathcal{F}$ is $J_{\mathcal{F}}$ -invariant as \mathcal{G}/\mathcal{F} is so. Now, $\overline{\mathcal{F}}/\mathcal{F}$ is locally spanned by sections v of $\mathcal{C}_{\mathcal{F}}$. Hence $(\overline{\mathcal{F}}/\mathcal{F})^{1,0}$ is locally spanned by the *holomorphic* vector fields $v^{1,0} = v - \sqrt{-1}J_{\mathcal{F}}(v)$. This proves the first claim. Concerning the second point, this amounts to showing that on the local space of leaves U/\mathcal{F} , the orthogonal and holomorphic distributions induced by $\overline{\mathcal{F}}$ and \mathcal{G} are parallel. This last property is a general fact in Kähler geometry (see for instance [Joh80, Theorem 2.1]). \square

As $\overline{\mathcal{F}}$ is a Riemannian foliation with closed leaves and the transverse metric is Kähler with quasi-negative Ricci curvature, we conclude that the leaf space $X/\overline{\mathcal{F}}$ is a Kähler orbifold with quasi-negative canonical bundle, hence of the general type according to [Puc18]. The statement of Theorem A follows directly. \square

The decomposition into parallel subbundles given in (7.6) provides the existence of two transverse invariant Kähler metrics with quasi-negative Ricci curvature $\overline{g}_1, \overline{g}_2$ for the foliations \mathcal{G} and $\overline{\mathcal{F}}$ and such that $\overline{g} = \overline{g}_1 \oplus \overline{g}_2$.

7.5. Proof of Lemma B. Consider the universal cover map $r^\sharp : \widetilde{X}^\sharp \rightarrow X^\sharp$ and the $\text{SO}(n)$ -principal bundle $Y := (r^\sharp)^{-1}(X^\sharp)$ over \widetilde{X} . The latter coincides with the transverse orthonormal frame bundle associated to the complete Riemannian foliated manifold $(\widetilde{X}, \widetilde{\mathcal{F}}, \widetilde{g})$. By virtue of Lemma 5.16, the lift $\widetilde{\mathcal{F}}^\sharp$ of $\widetilde{\mathcal{F}}$ to Y is *simple*. According to Molino's theory, this implies that the space of leaves $(\widetilde{X}/\widetilde{\mathcal{F}}, \widetilde{g})$ has a (unique) orbifold structure such that the projection map $\widetilde{X} \rightarrow \widetilde{X}/\widetilde{\mathcal{F}}$ is a smooth orbifold map. Indeed, $\widetilde{X}/\widetilde{\mathcal{F}}$ is canonically identified with the space of orbits of the locally free action of the compact Lie group $\text{SO}(n)$ on the manifold $Y/\widetilde{\mathcal{F}}^\sharp$ and the points of $\widetilde{X}/\widetilde{\mathcal{F}}$ with non-trivial isotropy correspond to leaves with non-trivial holonomy. In our setting, the leaf space $\widetilde{X}/\widetilde{\mathcal{F}}$ inherits a Kähler metric from the transverse structure of $\widetilde{\mathcal{F}}$. \square

7.6. Proof of Theorem C. Let $\widetilde{\mathcal{G}}$ and $\widetilde{\overline{\mathcal{F}}}$ be the respective pull-backs of \mathcal{G} and $\overline{\mathcal{F}}$ on \widetilde{X} . Denote by $(\widetilde{\mathcal{G}}_1, \widetilde{g}_1), (\widetilde{\mathcal{G}}_2, \widetilde{g}_2)$ the respective foliations induced on $\widetilde{X}/\widetilde{\mathcal{F}}$ together with their transverse Kähler metric \widetilde{g}_i coming from the orthogonal and parallel splitting

$$(N\widetilde{\mathcal{F}}, \widetilde{g}) = (\widetilde{\mathcal{G}}/\widetilde{\mathcal{F}}, \widetilde{g}_2) \oplus (\widetilde{\overline{\mathcal{F}}}/\widetilde{\mathcal{F}}, \widetilde{g}_1)$$

where $\widetilde{-}$ stands for the pull-back of the transverse metrics under consideration on the universal cover. Equivalently

$$(T(\widetilde{X}/\widetilde{\mathcal{F}}), \widetilde{g}) = (\mathcal{G}_1, \overline{g}_2) \oplus (\mathcal{G}_2, \overline{g}_1)$$

in the orbifold setting.

Note that the orbifold fundamental group $\pi_1^{\text{orb}}(\widetilde{X}/\widetilde{\mathcal{F}})$ is nothing but the fundamental group of the holonomy pseudogroup of $\widetilde{\mathcal{F}}$ (cf. [Mol88, Appendix D]). Because we have a natural *surjective* morphism $\pi_1(\widetilde{X}) = \{1\} \rightarrow \pi_1^{\text{orb}}(\widetilde{X}/\widetilde{\mathcal{F}})$ (*loc.cit*), we conclude that $\widetilde{X}/\widetilde{\mathcal{F}}$ is a *simply connected* orbifold. The orbifold version of de Rham decomposition Theorem [KL14, Lemma 2.19] then yields:

$$(7.7) \quad (\widetilde{X}/\widetilde{\mathcal{F}}, \widetilde{g}) \simeq (\mathcal{L}_1/\widetilde{\mathcal{F}}, \overline{g}_2) \times (\mathcal{L}_2/\widetilde{\mathcal{F}}, \overline{g}_1)$$

where $\mathcal{L}_1, \mathcal{L}_2$ are (arbitrary) leaves of $\widetilde{\mathcal{G}}$ and $\widetilde{\mathcal{F}}$. Of course, the (local) isotropy groups acts diagonally with respect to this splitting. Note also that the map Ψ descends on $\widetilde{X}/\widetilde{\mathcal{F}}$ as an orbifold holomorphic map $\overline{\Psi} : \widetilde{X}/\widetilde{\mathcal{F}} \rightarrow \mathcal{S}$. Set $\mathcal{H} = \mathcal{L}_1/\widetilde{\mathcal{F}}$ and $\mathcal{H}' = \mathcal{L}_2/\widetilde{\mathcal{F}}$. The levels of $\overline{\Psi}$ are precisely the leaves of the horizontal foliation in the Kählerian product (7.7). As Ψ is submersive, the restriction $\overline{\Psi}|_{\mathcal{H}} : \mathcal{H} \rightarrow \mathcal{S}$ is a local diffeomorphism. In particular \mathcal{H} is smooth (the isotropy groups acts trivially on the second factor). Moreover $\overline{\Psi}|_{\mathcal{H}}$ is a local isometry between complete Riemannian manifolds (for a suitable invariant Kähler metric on \mathcal{S}). The target \mathcal{S} being simply connected, we can conclude that $\overline{\Psi}|_{\mathcal{H}}$ induces an isometric biholomorphism between \mathcal{H} and \mathcal{S} . This proves the three first items of Theorem C. The item (4) follows from the fact that $\widetilde{\mathcal{F}}$ has compact leaves and that the action of $\pi_1(X)$ on \mathcal{S} is dense. \square

7.7. Remark on the case where \mathcal{F} is minimal. We can give a simple proof (without resorting to the technical material developed in this article) of Theorem C in the situation where \mathcal{F} is minimal, *i.e.* in the situation where every leaf is dense.¹⁶ In this case $\widetilde{X}/\widetilde{\mathcal{F}}$ should be reduced to \mathcal{H} . Actually, according to [Tou10], the structural Lie algebra \mathfrak{g} (cf. §5.3) of \mathcal{F} is semi-simple without compact factors. We can then derive from the works of Haefliger [Hae88, Theorem 6.4.1] that \mathcal{F} is transversely homogeneous. It precisely means that $\widetilde{\mathcal{F}}$ is given by the fibers of a submersion $\Phi : \widetilde{X} \rightarrow H$ onto a homogeneous Kähler manifold (H, h) such that the Ricci curvature of h is negative and such that Φ is ρ -equivariant with respect to a representation

$$\rho : \pi_1(X) \rightarrow \text{Iso}(H)$$

whose image consists in a subgroup of holomorphic isometries acting densely on H . By a result of Borel [Bor54, Theorem 4], H admits a structure of a homogeneous holomorphic fibre bundle whose base is a homogeneous bounded domain (*i.e.* a Hermitian symmetric space) \mathcal{H} and whose fiber is a flag manifold F . As F is rational algebraic, it is reduced to a point, due to the negativity of the Ricci curvature. Finally we get that $H = \mathcal{H}$ as desired.

8. PROOF OF LEMMA E

Let us recall that we aim at proving that the equation (7.3) has a unique solution φ (up to adding a constant). The proof of existence is a non-trivial adaptation of the continuity

¹⁶In the case of Riemannian foliation, this is equivalent to saying that at least one leaf is dense.

method in the foliated setting and relies on subtle estimates. The uniqueness of the solution is easier to establish and we first explain this part of the proof.

Proof of the uniqueness in Lemma E. Let φ_1, φ_2 two solutions of (7.3) such that $\omega + i\partial\bar{\partial}\varphi_i$ is positive in restriction to \mathcal{G}/\mathcal{F} (for $i = 1, 2$). Set $\omega_i = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_i$. Let $\varphi = \varphi_1 - \varphi_2$. Because forms of even degree commute, we obtain

$$(8.1) \quad \partial\bar{\partial}\varphi \wedge \sum_{k=0}^{p-1} \omega_1^k \wedge \omega_2^{p-k-1} \wedge \Omega^{n-p} = 0.$$

On the other hand, and thanks to Rummier's formula (3.1), we have $\beta \wedge d\chi = 0$ for every basic $(n-1)$ -form β . Combined with the closedness of ω_i and Ω , Stokes' Theorem yields

$$\int_X d(\varphi d^c \varphi) \wedge \left(\sum_{k=0}^{p-1} \omega_1^k \wedge \omega_2^{p-k-1} \right) \wedge \Omega^{n-p} \wedge \chi = 0$$

When expanding the integrand and according to (8.1), we get

$$(8.2) \quad \int_X \sum_{k=0}^{p-1} d\varphi \wedge d^c \varphi \wedge \omega_1^k \wedge \omega_2^{p-k-1} \wedge \Omega^{n-p} \wedge \chi = 0$$

Let $x \in X$. On some distinguished neighborhood of x , we can find holomorphic transverse coordinates (z_1, \dots, z_n) such the foliation \mathcal{G} is defined by $\{dz_\alpha = 0, \alpha > p\}$. Because ω_i ($i = 1, 2$) are positive in restriction to \mathcal{G}/\mathcal{F} , we can moreover choose the p first coordinates z_1, \dots, z_p such that

$$\begin{aligned} \omega_1(x) &= \sqrt{-1} \left(\sum_{\alpha=1}^p dz_\alpha \wedge d\bar{z}_\alpha + \xi_1 \right) (x) \quad \text{and} \\ \omega_2(x) &= \sqrt{-1} \left(\sum_{\alpha=1}^p \mu_\alpha dz_\alpha \wedge d\bar{z}_\alpha + \xi_2 \right) (x) \end{aligned}$$

where the $\xi_i(x)$'s vanish identically in restriction to $\mathcal{G}/\mathcal{F}_x$ and the μ_α 's are positive real numbers. Thanks to this writing, we infer that there exist p positive numbers ν_1, \dots, ν_p such that

$$\left(\sum_{k=0}^{p-1} d\varphi \wedge d^c \varphi \wedge \omega_1^k \wedge \omega_2^{p-k-1} \wedge \Omega^{n-p} \right) (x) = \sum_{\alpha=1}^p \nu_\alpha \left| \frac{\partial \varphi}{\partial z_\alpha} \right|^2 |dz_1 \wedge \dots \wedge dz_n|^2 (x).$$

Together with (8.2), this implies that the real function $\varphi = \varphi_1 - \varphi_2$ is constant on the leaves of \mathcal{G} , hence constant on the whole of X as \mathcal{G} is minimal. \square

Remark 8.1. Actually, by the same proof, the uniqueness holds as soon as $\varphi \in \mathcal{C}^2(X)$.

8.1. Preliminaries. We now turn to the proof of the existence part in Lemma E. It strongly relies on El-Kacimi's work [EIK90], even if the context and presentation differ at multiple places.

The original foliation is equipped with the transverse volume form $\xi = \omega^p \wedge \Omega^{n-p}$ and X is itself equipped with the volume form $dV = \xi \wedge \chi$ where χ is the characteristic form of \mathcal{F} . We can use them to endow the space $\mathcal{C}^0(X/\mathcal{F})$ of *continuous* basic functions with a scalar product $\langle \cdot, \cdot \rangle_X$, namely

$$\langle f_1, f_2 \rangle_X = \int_X f_1 f_2 dV$$

We can of course assume that $\text{Vol}(X) := \int_X dV = 1$ (up to multiplying ω by a suitable constant).

The transverse invariant metric \bar{g} induces by restriction a metric on $\mathcal{G}/\mathcal{F} \subset N\mathcal{F}$ that we denote by the same symbol. One can then consider the corresponding \mathcal{F} -basic \mathcal{G} -leafwise Laplacian $\Delta_{\mathcal{G}}$ as defined in §3.3. Thanks to the choice of dV , we can derive, as in the classical setting, the following statement.

Lemma 8.2. *For $f, f_1, f_2 \in \mathcal{C}^0(X/\mathcal{F}) \cap \mathcal{C}^2(X)$, we have:*

- (1) $\int_X \Delta_{\mathcal{G}}(f) dV = 0$. In particular, f is constant if and only if $\Delta_{\mathcal{G}}(f) \geq 0$.
- (2) $\langle f_1, \Delta_{\mathcal{G}}(f_2) \rangle = \langle \Delta_{\mathcal{G}}(f_1), f_2 \rangle$

Proof. Let v be a basic section of class \mathcal{C}^1 of \mathcal{G}/\mathcal{F} . The usual calculation yields $\text{div}_{\mathcal{G}}(v)\xi = d(i_v\omega^p) \wedge \Omega^{n-p} = d(i_v\xi)$. According to Lemma 3.8, this implies that

$$(8.3) \quad \int_X \text{div}_{\mathcal{G}}(v) dV = 0.$$

The item (1) is then obtained by choosing $v = \Delta_{\mathcal{G}}(f)$. The item (2) is a consequence of the standard identities

$$\text{div}_{\mathcal{G}}(f_i v) = \langle \nabla_{\mathcal{G}}(f_i), v \rangle_{\bar{g}} + f_i \text{div}_{\mathcal{G}}(v), \quad i = 1, 2$$

and the vanishing property (8.3) where we take alternatively $v = \nabla_{\mathcal{G}}(f_2)$ and $v = \nabla_{\mathcal{G}}(f_1)$. \square

We now perform an analogous construction on the foliated transverse frame bundle $p : (X^{\sharp}, \mathcal{F}^{\sharp}) \rightarrow (X, \mathcal{F})$, $\pi_{\mathcal{F}^{\sharp}} : X^{\sharp} \rightarrow W$ together with its projection map p , its basic fibration $\pi_{\mathcal{F}^{\sharp}}$ and its bundle-like metric $g^{\sharp} = p^*g \oplus_{\mathcal{H}} \vartheta$ as defined in §6.1 from which we maintain notation. Let $\chi^{\sharp} := p^*\chi$ be the induced characteristic form attached to \mathcal{F}^{\sharp} . Let ν be the characteristic form of the vertical fibration ϑ . Note that, thanks to the transverse \mathcal{F}^{\sharp} -parallelism (see §6.1) $i_v d\nu = 0$, ν is *basic* with respect to \mathcal{F}^{\sharp} . Let us consider $\omega^{\sharp} = p^*\omega$ and $\Omega^{\sharp} = p^*\Omega$. They can be used to endow \mathcal{F}^{\sharp} with a transverse volume form $\xi^{\sharp} = \omega^{\sharp p} \wedge \nu \wedge \Omega^{\sharp n-p}$ that can be completed to a volume form dV^{\sharp} on X^{\sharp} by setting $dV^{\sharp} = \xi^{\sharp} \wedge \chi^{\sharp}$. As before we can consider the \mathcal{F}^{\sharp} -basic \mathcal{G}^{\sharp} -leafwise metric induced by the restriction to $\mathcal{G}^{\sharp}/\mathcal{F}^{\sharp}$ of the orthogonal sum $\bar{g}^{\sharp} := p^*\bar{g} \oplus_{\bar{\mathcal{H}}} \vartheta$ (see §6.1). Let $\Delta_{\mathcal{G}^{\sharp}}$ be the corresponding \mathcal{F}^{\sharp} -basic \mathcal{G}^{\sharp} -leafwise Laplacian. Every basic function of class \mathcal{C}^r ($r \in \llbracket 0, \infty \rrbracket$) defined on X^{\sharp} is of the form $f \circ \pi_{\mathcal{F}^{\sharp}}$, $f \in \mathcal{C}^r(W)$ and the $\text{SO}(2n)$ -action on the principal bundle X^{\sharp} projects to an $\text{SO}(2n)$ action on W . We can identify the space of orbits $W/\text{SO}(2n)$ with the space of leaves closure of \mathcal{F} and the subspace $\mathcal{C}_{\text{SO}(2n)}^r(W) \subset \mathcal{C}^r(W)$ of $\text{SO}(2n)$ -invariant functions to the space $\mathcal{C}^0(X/\mathcal{F}) \cap \mathcal{C}^r(X)$. We will denote by D_W^2 the linear differential operator induced on W by $\Delta_{\mathcal{G}^{\sharp}}$ (as in [ElK90, Proposition 2.7.7]). It is given, for every $f \in \mathcal{C}^2(W)$, by the formula:

$$D_W^2(f) = \Delta_{\mathcal{G}^{\sharp}}(f \circ \pi).$$

Arguing as above, we can prove that the equality

$$(8.4) \quad \int_{X^{\sharp}} \Delta_{\mathcal{G}^{\sharp}}(f_1 \circ \pi_{\mathcal{F}^{\sharp}})(f_2 \circ \pi_{\mathcal{F}^{\sharp}}) dV^{\sharp} = \int_{X^{\sharp}} \Delta_{\mathcal{G}^{\sharp}}(f_2 \circ \pi_{\mathcal{F}^{\sharp}})(f_1 \circ \pi_{\mathcal{F}^{\sharp}}) dV^{\sharp}$$

holds true for any $f_1, f_2 \in \mathcal{C}^2(W)$. Let dw be the volume form¹⁷ on W such that we have

$$(8.5) \quad \int_{X^\sharp} (f_1 \circ \pi_{\mathcal{F}^\sharp})(f_2 \circ \pi_{\mathcal{F}^\sharp}) dV^\sharp = \int_W f_1 f_2 dw =: \langle f_1, f_2 \rangle_W.$$

for every $f_1, f_2 \in \mathcal{C}^0(W)$. In particular, since ν assigns volume 1 to the fibers of $X^\sharp \rightarrow X$, we have

$$\langle f_1, f_2 \rangle_X = \langle f_1, f_2 \rangle_W$$

whenever f_1 and f_2 are $\mathrm{SO}(2n)$ -invariant.

The following lemma establishes some “basic” properties of D_W^2 .

Lemma 8.3.

- (1) $D_W^2(1) = 0$.
- (2) D_W^2 is a strongly elliptic operator of order 2, in particular D_W^2 satisfies the Hopf’s maximum principle and then $\mathrm{Ker}(D_W^2) = \mathbb{R}$.
- (3) D_W^2 is $\mathrm{SO}(2n)$ -invariant and for every $f \in \mathcal{C}_{\mathrm{SO}(2n)}^2(W) (= \mathcal{C}^0(X/\mathcal{F}) \cap \mathcal{C}^2(X))$, $D_W^2(f) = \Delta_{\mathcal{G}}(f)$.
- (4) D_W^2 is self-adjoint with respect to the scalar product $\langle \cdot, \cdot \rangle_W$, i.e, for every $f_1, f_2 \in \mathcal{C}^2(W)$, $\langle f_1, D_W^2(f_2) \rangle_W = \langle D_W^2(f_1), f_2 \rangle_W$.

Proof.

(1): Obvious.

(2): The differential operator D_W^2 has obviously order ≤ 2 (= order of the basic operator $\Delta_{\mathcal{G}}$).

Let $w \in W$ and $u \in \pi_{\mathcal{F}^\sharp}^{-1}(w)$. Let $\xi \in T_x^*W - \{0\}$ and $f \in \mathcal{C}^2(W)$ such that $f(w) = 0$ and $df_w = \xi$. Set $f^\sharp = f \circ \pi_{\mathcal{F}^\sharp}$. The differential $d\pi_{\mathcal{F}^\sharp}$ vanishes on \mathcal{F}^\sharp and induces a surjective morphism of vector bundles: $\mathcal{G}^\sharp/\mathcal{F}^\sharp \rightarrow TW$ (see Remark 5.18); then the differential df_u^\sharp induces a non-trivial linear form on $(\mathcal{G}^\sharp/\mathcal{F}^\sharp)_u$. Consequently $D_W^2(f^2)(w) = \Delta_{\mathcal{G}^\sharp}((f^\sharp)^2)(u) < 0$ by strong ellipticity of the Laplacian. The terms involved in this equality are thus nothing but the principal symbol $\sigma(D_W^2)(w, \xi)$ (up to a factor $\frac{1}{2}$), showing that D_W^2 is strongly elliptic of order 2.

(3): Recall firstly that the structural group $\mathrm{SO}(2n)$ acts on X^\sharp by basic transformation with respect to both \mathcal{F}^\sharp and \mathcal{G}^\sharp (more precisely the action is tangent to the latter). Moreover this action preserves the transverse metric \bar{g}^\sharp . This implies the $\mathrm{SO}(2n)$ -invariance of $\Delta_{\mathcal{G}^\sharp}$ and consequently that of D_W^2 .

Concerning the last point, we maintain notation/observations from §6.1. Let us denote by $p_{U/\mathcal{F}} : (p^{-1}(U)/\mathcal{F}^\sharp, \bar{g}^\sharp) \rightarrow (U/\mathcal{F}, \bar{g})$ the Riemannian submersion induced by p and which allows us to identify the source as the orthonormal frame bundle of the target. Denote by $\bar{\mathcal{G}}$ (resp. by $\bar{\mathcal{G}}^\sharp$) the foliation induced by \mathcal{G} (resp. by \mathcal{G}^\sharp) on U/\mathcal{F} (resp. on $p^{-1}(U)/\mathcal{F}^\sharp$). Both are related by $\bar{\mathcal{G}}^\sharp = p_{U/\mathcal{F}}^* \bar{\mathcal{G}}$. Now the $\mathrm{SO}(2n)$ -action preserves the horizontal distribution $\bar{\mathcal{H}}$, so that the vertical fibers of $p_{U/\mathcal{F}}$ are totally geodesic (with respect to \bar{g}^\sharp) and in particular minimal. From Remark 3.14 applied to the vertical foliation, we can deduce that $D_W^2(f) = \Delta_{\mathcal{G}^\sharp}(f \circ \pi_{\mathcal{F}^\sharp}) = \Delta_{\mathcal{G}}(f)$, as desired.

(4): Straightforward consequence of the self-adjointness of (8.4) and (8.5). \square

In the sequel, we will also have to replace the transverse reference metric \bar{g} by the “metric” \bar{g}_φ with fundamental form $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ where φ is a basic function of

¹⁷We assume here that W is oriented; this can be achieved by replacing if necessary X by a double cover. Note also that for any finite étale cover of $r : Y \rightarrow X$, the commuting sheaves are simply related by $\mathcal{C}_{r^*\mathcal{F}} = r^*\mathcal{C}_{\mathcal{F}}$.

class \mathcal{C}^k , $k \gg 0$ such that ω_φ is positive in restriction to \mathcal{G}/\mathcal{F} , so that \overline{g}_φ is *a priori* only a genuine metric of class \mathcal{C}^{k-2} in restriction to \mathcal{G}/\mathcal{F} .¹⁸ This is however enough to construct a $\mathrm{SO}(2n)$ -invariant \mathcal{F}^\sharp -basic \mathcal{G}^\sharp -leafwise metric on $\mathcal{G}^\sharp/\mathcal{F}^\sharp$ by taking the restriction of the orthogonal sum $p^*\overline{g}_\varphi \oplus_{\overline{\mathcal{H}}} \vartheta$.¹⁹ As before, we can attach to it the \mathcal{F}^\sharp -basic \mathcal{G}^\sharp -leafwise (resp. \mathcal{F} -basic \mathcal{G} -leafwise) Laplacian $\Delta_{\mathcal{G}^\sharp}^\varphi$ (resp. $\Delta_{\mathcal{G}}^\varphi$). In this setting, the relevant volume form on X and X^\sharp are respectively $dV_\varphi := \omega_\varphi^p \wedge \Omega^{n-p} \wedge \chi$ and $dV_\varphi^\sharp := \xi_\varphi^\sharp \wedge \chi^\sharp$ where $\xi_\varphi^\sharp = \omega_\varphi^\sharp{}^p \wedge \nu \wedge \Omega^{n-p}$ and $\omega_\varphi^\sharp = p^*\omega_\varphi$. Let us denote by $\langle \cdot, \cdot \rangle_{\varphi, W}$ the corresponding scalar product on $\mathcal{C}^0(W)$. As before, $\Delta_{\mathcal{G}^\sharp}^\varphi$ descends to W as a second order differential $D_W^{2, \varphi}$ fulfilling all the items of Lemma 8.3, replacing $\Delta_{\mathcal{G}}$ by $\Delta_{\mathcal{G}^\sharp}^\varphi$ and $\langle \cdot, \cdot \rangle_W$ by $\langle \cdot, \cdot \rangle_{\varphi, W}$.²⁰

As in the classical context of complex Monge–Ampère equations, we want to apply the continuity method in order to prove Lemma E, that is to solve for every $t \in [0, 1]$ the family of equations

$$\begin{aligned} (\mathrm{MA})_t : \quad & (\omega + \sqrt{-1}\partial\bar{\partial}\varphi_t)^p \wedge \Omega^{n-p} = e^{f_t} \omega^p \wedge \Omega^{n-p} \\ & = e^{tf} \frac{\int_X \omega^p \wedge \Omega^{n-p} \wedge \chi}{\int_X e^{tf} \omega^p \wedge \Omega^{n-p} \wedge \chi} \omega^p \wedge \Omega^{n-p} \end{aligned}$$

which consists in replacing f by

$$f_t := tf + \log \left(\int_X \omega^p \wedge \Omega^{n-p} \wedge \chi \right) - \log \left(\int_X e^{tf} \omega^p \wedge \Omega^{n-p} \wedge \chi \right)$$

in the right-hand side of (7.3).

Let k be a non-negative integer and $\alpha \in (0, 1)$. Consider the Banach space $\mathcal{C}^{k, \alpha}(W)$ of functions of class \mathcal{C}^k and Hölder exponent α and the closed subspace $\mathcal{C}_{\mathrm{SO}(2n)}^{k, \alpha}(W)$ of $\mathrm{SO}(2n)$ -invariant functions. The latter can be identified with the space of basic functions on X of class $\mathcal{C}^{k, \alpha}$. On the local leaf space U/\mathcal{F} , observe that φ_t is a solution to $(\mathrm{MA})_t$ iff for every leaf $\overline{\mathcal{L}}$ of the foliation $\overline{\mathcal{G}}$ induced by \mathcal{G} on U/\mathcal{F} , the following equation holds:

$$(8.6) \quad (\omega + \sqrt{-1}\partial\bar{\partial}\varphi_t)^p \Big|_{\overline{\mathcal{L}}} = e^{f_t} \omega^p \Big|_{\overline{\mathcal{L}}}.$$

Let \mathcal{W} be a connected open subset of $\overline{\mathcal{L}}$ relatively compact in U/\mathcal{F} . The following observation will turn out to be useful.

Lemma 8.4. *Let φ_t be a solution of $(\mathrm{MA})_t$. Assume that φ_t belongs to the class $\mathcal{C}^{k, \alpha}$ in restriction to \mathcal{W} . Then $\varphi_t \in \mathcal{C}^{k, \alpha}(\mathcal{V})$ where \mathcal{V} is an open neighborhood of \mathcal{W} in U/\mathcal{F} .*

Proof. Consider the constant sheaf $\mathcal{C}_{\mathcal{F}|U}$, viewed as a Lie algebra \mathfrak{g} of Killing vector fields on U/\mathcal{F} . These vector fields are also foliated with respect to $\overline{\mathcal{G}}$. Among those vector fields, those that vanish along $\overline{\mathcal{L}}$ form a maximal compact subalgebra \mathfrak{k} of \mathfrak{g} . Take a decomposition of \mathfrak{g} as a direct sum of linear subspaces $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (for instance a Cartan decomposition). Note that the dimension of \mathfrak{p} is the real codimension $d = 2(n - p)$ of $\overline{\mathcal{G}}$. Let $\mathcal{B} =$

¹⁸In fact, φ is necessarily constant along the leaves of \mathcal{G} (see the proof of Lemma 8.4), so that \overline{g}_φ really is a transverse invariant metric on $N\mathcal{F}$.

¹⁹Recall that the $\mathrm{SO}(2n)$ -action on X^\sharp preserve both \mathcal{F}^\sharp and \mathcal{G}^\sharp (and is actually tangent to the latter) and consequently induces an action on $\mathcal{G}^\sharp/\mathcal{F}^\sharp$.

²⁰In view of §8.2, one could also argue by considering the space of transverse orthonormal frames attached to the new metric \overline{g}_φ .

$\{v_1, \dots, v_d\}$ be a basis of \mathfrak{p} . In particular, \mathcal{B} provides along $\overline{\mathcal{L}}$ a trivialization of the normal bundle $N\mathcal{L}$. For $(t_1, \dots, t_d) \in]-\varepsilon, \varepsilon[^d$, the map

$$\Psi : \begin{cases} \mathcal{W} \times]-\varepsilon, \varepsilon[^d & \longrightarrow U/\mathcal{F} \\ (x, (t_1, \dots, t_d)) & \longmapsto e^{t_1 v_1} \circ e^{t_2 v_2} \circ \dots \circ e^{t_d v_d}(x) \end{cases}$$

is a smooth diffeomorphism onto an open subset \mathcal{V} of U/\mathcal{F} and $\mathcal{W} = \Psi(\mathcal{W} \times \{0\})$. The lemma is then a straightforward consequence of this trivialization and the fact φ_t is constant along the orbit of any $v \in \mathfrak{g}$. \square

We will also make use of the spaces:

$$\begin{aligned} E_{k,\alpha} &= \left\{ \varphi \in \mathcal{C}_{\mathrm{SO}(2n)}^{k,\alpha}(W) \mid \langle \varphi, 1 \rangle_W = 0 \right\} \quad \text{and} \\ H_{k,\alpha} &= \left\{ h \in \mathcal{C}_{\mathrm{SO}(2n)}^{k-2,\alpha}(W) \mid \langle h, 1 \rangle_W = \langle 1, 1 \rangle_W \right\} \end{aligned}$$

with $k \gg 0$. They are respectively a closed linear subspace of $\mathcal{C}_{\mathrm{SO}(2n)}^{k,\alpha}(W)$ and a closed affine subspace of $\mathcal{C}_{\mathrm{SO}(2n)}^{k-2,\alpha}(W)$. Let us denote by $A_{k,\alpha}$ the subset of $t \in [0, 1]$ such that $(\mathrm{MA})_t$ admits a solution $\varphi_t \in \mathcal{C}_{\mathrm{SO}(2n)}^{k,\alpha}(W)$ with the property that $\omega_{\varphi_t}|_{\mathcal{G}/\mathcal{F}} > 0$. We must prove that $A_{k,\alpha}$ contains 1 for every (k, α) . To this end, let us consider the open set subset of $E_{k,\alpha}$,

$$\mathcal{U}_{k,\alpha} := \left\{ \varphi \in E_{k,\alpha} \mid \omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \text{ in restriction to } \mathcal{G}/\mathcal{F} \right\}$$

and the map $\mathcal{C} : \mathcal{U}_{k,\alpha} \rightarrow H_{k,\alpha}$ defined by

$$\mathcal{C}(\varphi) = \frac{(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^p \wedge \Omega^{n-p}}{\omega^p \wedge \Omega^{n-p}}.$$

8.2. Openness of $A_{k,\alpha}$. A straightforward computation shows that \mathcal{C} is differentiable and that the differential at φ is given by:

$$d\mathcal{C}_\varphi = -\mathcal{C}(\varphi) \Delta_\mathcal{G}^\varphi.$$

Now, recall that

$$\Delta_\mathcal{G}^\varphi : \mathcal{C}_{\mathrm{SO}(2n)}^{k,\alpha}(W) \longrightarrow \mathcal{C}_{\mathrm{SO}(2n)}^{k-2,\alpha}(W)$$

is the restriction of

$$D_W^{2,\varphi} : \mathcal{C}^{k,\alpha}(W) \longrightarrow \mathcal{C}^{k-2,\alpha}(W).$$

By standard elliptic theory and Schauder estimates, the image of the latter coincides with

$$\mathcal{E}_0 := \left\{ h \in \mathcal{C}^{k-2,\alpha}(W) \mid \langle h, 1 \rangle_{\varphi, W} = 0 \right\}$$

(see [Con78, Exposé VI]). We can infer that the image of the former is exactly $\mathcal{E}_0 \cap \mathcal{C}_{\mathrm{SO}(2n)}^{k-2,\alpha}(W)$. Indeed, the image of $\mathcal{C}_{\mathrm{SO}(2n)}^{k,\alpha}(W)$ is contained in $\mathcal{E}_0 \cap \mathcal{C}_{\mathrm{SO}(2n)}^{k-2,\alpha}(W)$ thanks to the $\mathrm{SO}(2n)$ -invariance of $D_W^{2,\varphi}$. Conversely and still by $\mathrm{SO}(2n)$ -invariance, we can express every element of $\mathcal{E}_0 \cap \mathcal{C}_{\mathrm{SO}(2n)}^{k-2,\alpha}(W)$ as $D_W^{2,\varphi}(f)$ where $f \in \mathcal{C}^{k,\alpha}(W)$ verifies

$$D_W^{2,\varphi}(v(f)) = 0$$

for every fundamental vector field v of the $\mathrm{SO}(2n)$ -action. This shows that $v(f)$ is constant, hence identically zero by compactness of W , proving our assertion. We have thus established that $d\mathcal{C}_\varphi$ is an isomorphism between the tangent spaces $T_\varphi \mathcal{U}_{k,\alpha}$ and $T_{\mathcal{C}(\varphi)} H_{k,\alpha}$. By the inverse function Theorem, we can conclude that $A_{k,\alpha}$ is an *open* (non-empty) subset of $[0, 1]$.

It remains to show that $A_{k,\alpha}$ is *closed*. We can mimick without fundamental changes the classical proof given by Yau as explained below.

8.3. Closedness of $A_{k,\alpha}$. Let $t \in A_{k,\alpha}$ and $\varphi_t \in \mathcal{C}_{\text{SO}(2n)}^{k,\alpha}(W)$ such that $\omega_{\varphi_t}|_{\mathcal{G}/\mathcal{F}} > 0$ the solution of $(\text{MA})_t$ together with the normalization $\langle \varphi_t, 1 \rangle_W = 0 = \langle \varphi_t, 1 \rangle_X$, so that φ_t is necessarily *unique*. It is worth noticing that the properties of the operator D_W^2 are mainly used in the item (2) of Step 1 below and that the remaining part only needs to work directly on the original manifold X without any reference to the normal frame bundle and the basic fibration on it.

8.3.1. Step 1: L^1 estimate. The solution φ_t satisfies the following estimates.

- (1) $\Delta_{\mathcal{G}}(\varphi_t) < p$ ($= \dim_{\mathbb{C}}(\mathcal{G}/\mathcal{F})$): this is a straightforward adaptation of the proof of [Con78, Exposé VII, Lemme 2.1(i)].
- (2) the upper-bound $\sup_{X \times A_{k,\alpha}} \varphi_t \leq C_1$ follows from the estimates for the Green function of D_W^2 (see [Con78, Exposé VII] and for instance [AS13, Appendix A]).
- (3) We get the L^1 uniform bound $\int_X |\varphi_t| dV \leq 2C_1$ as a trivial consequence.

8.3.2. Step 2: L^q estimate. This is [Con78, Exposé VII, Lemme 3.3] where the L^q norm under consideration is computed with respect to the volume form $dV = \omega^p \wedge \Omega^{n-p} \wedge \chi$. The proof is a direct adaptation of the strategy depicted in *loc. cit.* (repeated use of L^1 estimates and Sobolev embeddings).

8.3.3. Step 3: uniform \mathcal{C}^0 estimate. This is again obtained as a direct adaptation of the classical situation [Con78, Exposé VII]. Indeed it follows from L^q estimates that we have:

$$\text{Sup}_{A_{k,\alpha}} \|\varphi_t\|_{\infty} \leq C_2.$$

8.3.4. Step 4: Laplacian \mathcal{C}^0 estimate. We follow *verbatim* the strategy and computation made in [Con78, Exposé VIII] and obtain

$$\text{Sup}_{A_{k,\alpha}} \|\Delta_{\mathcal{G}}(\varphi_t)\|_{\infty} \leq C_3.$$

8.3.5. Last step: higher order estimates. This part follows closely the strategy²¹ developed in [GZ17, §14.3]. Unlike the preceding steps, this is a purely local result to which we can reduce by considering the equation (8.6) and Lemma 8.4 (see the comments at the end of [GZ17, §14.1.2]).

In the first place, we can derive from Evans–Krilov theory the upper bound:

$$\exists 0 < \beta < \alpha, \text{Sup}_{A_{k,\alpha}} \|\varphi_t\|_{\mathcal{C}^{2,\beta}} \leq C_4.$$

As the injection $\mathcal{C}^{2,\alpha}(X) \hookrightarrow \mathcal{C}^{2,\beta}(X)$ is compact, we can find for any cluster point $t_0 \in [0, 1]$ a sequence $(t_n) \in A_{k,\alpha}$, $t_n \rightarrow t_0$ and $\varphi \in \mathcal{C}^{2,\beta}(X)$ such that φ_{t_n} tends to φ with respect to the $\mathcal{C}^{2,\beta}$ norm. In particular we get that $t_0 \in A_{2,\beta}$. We can then deduce by applying inductively Schauder's estimates that $\varphi \in \mathcal{C}^{k',\beta}(X)$ for every positive integer k' . In particular t_0 belongs to $A_{k,\alpha}$. Eventually $A_{k,\alpha} = [0, 1]$ and, as k is arbitrarily large, this provides the sough solution. This concludes the proof of Lemma E. \square

²¹We could alternatively use [Con78, Exposé XI].

9. AUTOMORPHISM GROUP OF THE FOLIATION IN THE KÄHLER CASE: PROOF OF THEOREM D

In this section, (X, \mathcal{F}) is a foliated compact Kähler manifold (\mathcal{F} holomorphic) which satisfies the hypothesis of Theorem D stated in the Introduction. Recall that $\text{Aut}(X, \mathcal{F})$ denote the group of analytic diffeomorphisms of X preserving the foliation \mathcal{F} . Let \bar{g} be the invariant transverse Kähler metric.

The following preliminary and simple observations will be proved to be useful.

Lemma 9.1. *Let V be an irreducible analytic complex space. Let G a countable group of biholomorphisms of V . Let f be a biholomorphism of V such that for every $x \in V$, $f(x) \in G \cdot x$, the G -orbit of x . Then f is an element of G .*

Proof. For every $g \in G$, set $Z_g := \{x \in V \mid g(x) = f(x)\}$. We have $\bigcup_{g \in G} Z_g = V$ by assumptions. According, to Baire's lemma, there exists $h \in G$ such that Z_h has non-empty interior. We then conclude by analytic continuation that $f = h$, as wanted. \square

Lemma 9.2. *If $f \in \text{Aut}(X, \mathcal{F})$, then $f_*(\mathcal{C}_{\mathcal{F}}) = \mathcal{C}_{\mathcal{F}}$.*

Proof. Firstly, note that $\bar{h} = f_*\bar{g}$ is also a transverse Kähler metric. On the other hand, $\mathcal{C}_{\mathcal{F}}$ is independant of the Riemannian transverse structure, as noticed in [Mol88, Proposition 5.1], whence the result. \square

Corollary 9.3. *The following inclusion holds true:*

$$\text{Aut}(X, \mathcal{F}) \subset \text{Aut}(X, \bar{\mathcal{F}}) \cap \text{Aut}(X, \mathcal{G}).$$

Proof. The first inclusion $\text{Aut}(X, \mathcal{F}) \subset \text{Aut}(X, \bar{\mathcal{F}})$ is a consequence of the topological characterization of $\bar{\mathcal{F}}$. The second one $\text{Aut}(X, \mathcal{F}) \subset \text{Aut}(X, \mathcal{G})$ is due to the fact that \mathcal{G} is defined by the vanishing locus of the isotropy Lie subalgebra of $\mathcal{C}_{\mathcal{F}}$ (cf. §7.3) combined with Lemma 9.2. \square

During the proof of Theorem D, we will be led to replace the initial manifold X with finite étale covers. They can be chosen in such a way that the action of the group G lifts, as shown in the next result.

Lemma 9.4. *Let X_1 be a compact complex manifold and X_2 be a finite étale cover of X_1 . Let $G = \text{Aut}(X_1)$ be the group of biholomorphisms of X_1 . Then, there exists a finite étale cover X_3 of X_2 such that every $g \in G$ lifts to a biholomorphism of X_3 . In particular, Theorem D holds on X iff it holds on a finite étale cover.*

Proof. It is clearly sufficient to find a finite index characteristic subgroup H of $\pi_1(X_1)$ contained in $\pi_1(X_2)$. Because $\pi_1(X_1)$ is finitely generated, we can take H to be the (finite) intersection of all subgroups of $\pi_1(X_1)$ of the given index $[\pi_1(X_1) : \pi_1(X_2)]$. \square

Proof of Theorem D. Let $G = \text{Aut}(X, \mathcal{F})$. Denote by \tilde{G} the group of biholomorphisms of \tilde{X} which descend to X as an element of $\text{Aut}(X, \mathcal{F})$. Remark that \tilde{g} acts by biholomorphism on the complex orbifold $\tilde{X}/\tilde{\mathcal{F}} = \mathcal{H} \times \mathcal{H}$. According to Corollary 9.3, G both preserves $\bar{\mathcal{F}}$ and \mathcal{G} , so that \tilde{G} acts diagonally on $\tilde{X}/\tilde{\mathcal{F}}$ with respect to the above decomposition. We can also suppose, up to taking a finite index subgroup that G acts trivially on the general type orbifold $X/\bar{\mathcal{F}}$. Let us first show that the transverse action of G on X/\mathcal{G} is finite. For this, recall that up to replacing X by a finite étale cover and according to Lemma 9.4, we can assume that the projection of the diagonal action of $\pi_1(X)$ on $\tilde{X}/\tilde{\mathcal{F}} = \mathcal{H} \times \mathcal{H}$ given by the morphism $(\rho_{\mathcal{H}}, \rho_{\mathcal{H}}) : \pi_1(X) \rightarrow \text{Aut}(\mathcal{H}) \times \text{Aut}(\mathcal{H})$

is a dense subgroup H (in the Euclidean topology) of the connected semi-simple real algebraic group $\text{Aut}^0(\mathfrak{g}) = \text{Aut}^0(\mathcal{H})$. We can even assume (Selberg's Lemma) that H does not contain any torsion elements. We can now exploit the results of [Zuo96] and [CCE15, Theorem 1] according to which there exists a dominant quasi-holomorphic map with connected fibers $\text{Sh}_{\rho_{\mathcal{H}}} : X \rightarrow V$ factorizing the representation $\rho_{\mathcal{H}}$ and such that V is a manifold of general type. Here, $\text{Sh}_{\rho_{\mathcal{H}}}$ is nothing but the Shafarevich map associated to the representation $\rho_{\mathcal{H}}$. Now from Lemma 9.2 and uniqueness of the fibration defined by $\text{Sh}_{\rho_{\mathcal{H}}}$, we can infer that every $f \in G$ descend to V as a birational transformation of V . This immediately implies that G acts trivially on V (again up to extracting a finite index subgroup). It remains to observe that in a suitable neighborhood U of a general fiber F of $\text{Sh}_{\rho_{\mathcal{H}}}$ (where the restriction of $\rho_{\mathcal{H}}$ is thus trivial), \mathcal{G} is defined by the levels of a holomorphic submersion $\varphi_U : U \rightarrow \mathcal{H}$ which takes its values in the bounded domain \mathcal{H} . This map is thus constant on F and we can then conclude that F is contained in a leaf of \mathcal{G} and eventually that G acts trivially on X/\mathcal{G} .

At this stage, we have thus showed that up to finite index, G acts both trivially on the space of leaves $X/\overline{\mathcal{F}}$ and X/\mathcal{G} . However, as the leaves intersection of these two foliations may consist of infinitely many leaves of \mathcal{F} , one cannot conclude directly.

Let us consider the lift of this action $\lambda_G : \tilde{G} \rightarrow \text{Aut}(\mathcal{H}) \times \text{Aut}(\mathcal{H})$. According to Lemma 9.1, the image of λ_G lies in $\rho_{\mathcal{H}}(\pi_1(X)) \times \rho_{\mathcal{H}}(\pi_1(X))$. In particular, we can restrict λ_G to \tilde{G}_0 where

$$\tilde{G}_0 = \lambda_G^{-1}(\{\text{Id}_{\mathcal{H}}\} \times \rho_{\mathcal{H}}(\pi_1(X))).$$

Indeed, \tilde{G}_0 still projects to G . Let o in \mathcal{H} and let \mathfrak{L}_o be the leaf of $\tilde{\mathcal{F}}$ through o . Let q be the image of o by the covering map $\mathcal{H} \rightarrow X/\overline{\mathcal{F}}$ and let \mathcal{L}_q be the leaf of $\overline{\mathcal{F}}$ through q . The covering map $\mathfrak{L}_o \rightarrow \mathcal{L}_q$ is Galois with deck transformations group the stabilizer $\pi_1(X)_o$ of \mathfrak{L}_o with respect to the $\pi_1(X)$ -action on the leaves of $\tilde{\mathcal{F}}$. The pull-back on \mathfrak{L}_o of the foliation $\mathcal{F}|_{\mathcal{L}_q}$ is thus given by the fibers of the restriction of the surjective submersion $\Phi : \tilde{X} \rightarrow \mathcal{H}$ defining $\tilde{\mathcal{G}}$. Let us denote it by $\Phi_o : \mathfrak{L}_o \rightarrow \mathcal{H}$. Indeed, Φ_o is a topologically trivial fibration over the 1-connected space \mathcal{H} , so that the fibers are connected. By construction Φ_o is $\pi_1(X)_o$ -equivariant, that is

$$\forall \gamma \in \pi_1(X)_o, \forall x \in \mathfrak{L}_o, \Phi_o(\gamma(x)) = \rho_{\mathcal{H}}(\gamma)(\Phi_o(x)).$$

Denote by $\rho_{\mathcal{H}}^q$ the restriction of $\rho_{\mathcal{H}}$ to $\pi_1(\mathcal{L}_q)$. By definition of \mathcal{L}_q , this implies that the image of $\rho_{\mathcal{H}}^q (= \rho_{\mathcal{H}}(\pi_1(X)_o))$ intersects $\text{Aut}^0(\mathcal{H})$ as a dense subgroup. On the other hand, the space of leaves $X/\overline{\mathcal{F}}$ is an orbifold, so that we can choose o such the corresponding point $q \in X/\overline{\mathcal{F}}$ has trivial isotropy or, in other words, such that the leaf \mathcal{L}_q has trivial holonomy. This also amounts to saying that the $\pi_1(X)_o$ -action on $\tilde{X}/\tilde{\mathcal{F}} = \mathcal{H} \times \mathcal{H}$ is trivial on the first factor.

It is thus sufficient to show that the action of G on $\mathcal{L}_q/\mathcal{F}$ is finite. Actually, thanks to the expression of \tilde{G}_0 and Lemma 9.1, any subgroup of G acting trivially on $\mathfrak{L}_q/\mathcal{F}$ necessarily acts trivially on X/\mathcal{F} .

As before the latter fact can be established by considering the Shafarevich morphism $\text{Sh}_{\rho_{\mathcal{H}}^q} : \mathcal{L}_q \rightarrow V_q$, following exactly the same line of ideas. \square

10. SOME FINAL REMARKS/QUESTIONS

10.1. If the ambient manifold X is Kähler, the foliations under consideration are “smooth occurrences” of numerical Bogomolov sheaves $L \subset \Omega_X^p$. These are saturated rank one

subsheaves of $L \subset \Omega_X^p$ with numerical dimension $\text{nd}(L) = p$ (the maximum possible value according to [Bou02, Théorème 3.2.12]). A result of Demailly [Dem02] asserts that $\text{Ker}(L)$ is an integrable distribution and that moreover $\Theta \wedge \omega = 0$ where ω is a local generator of L and Θ is any closed positive current representing $c_1(L)$. The foliations we deal with include those for which we can choose Θ to be a closed positive $(1, 1)$ form of rank p everywhere, *i.e.* those with negative transverse Ricci curvature.

Indeed the vanishing of $L \wedge \Theta = 0$ implies that the kernel of Θ is exactly the tangent bundle to \mathcal{F} and in particular that Θ is (the fundamental form of) a holonomy transverse invariant Kähler metric for \mathcal{F} . We get thus another real basic $(1, 1)$ -form, namely the transverse Ricci form $r = -\text{Ricci}(\Theta)$. Note that $-r$ also represents $c_1(N_{\mathcal{F}}^*)$, so that there exists, by the dd^c -lemma, a smooth function $f : X \rightarrow \mathbb{R}$ such that $-r = \eta + dd^c f$. Moreover, $dd^c f$ is basic as it is a sum of two basic forms. This implies that f is pluriharmonic along the leaves of \mathcal{F} . It turns out that f is basic: indeed, let \mathcal{L} be a leaf and $\overline{\mathcal{L}}$ its topological closure. Let $x \in \overline{\mathcal{L}}$ such that $f|_{\overline{\mathcal{L}}}$ reaches its maximum at x and let \mathcal{L}_x be the leaf passing through x . By the maximum principle for pluriharmonic functions, f is constant on \mathcal{L}_x hence on $\overline{\mathcal{L}_x}$. On the other hand, the leaves closure form a partition of X , a common feature for Riemannian foliations [Mol88, Theorem 5.1]. In particular, $\overline{\mathcal{L}} = \overline{\mathcal{L}_x}$. As the original leaf \mathcal{L} has been chosen arbitrarily, we can conclude that f is leafwise constant, as wanted. Then, r and η are not only cohomologous in the ordinary $\partial\bar{\partial}$ cohomology, but also in the *basic* $\partial\bar{\partial}$ cohomology. By the foliated version of Yau's solution to Calabi conjecture [ElK90, Section 3.5], there exists for \mathcal{F} an invariant transverse Kähler metric whose Ricci form is equal to $-\Theta$. In fact, these simple minded models could provide an insight of what is likely to happen for general numerical Bogomolov sheaves for which we expect a singular version of the decomposition theorem to hold. For codimension one foliations, this vague expectation can be turned into reality, and the picture is even more precise according to the

Theorem 10.1 (*cf.* [Tou16] and [PRT22, Theorem D]). *Let (X, \mathcal{F}) be a foliated Kähler manifold such that \mathcal{F} is a holomorphic codimension one foliation given as the annihilator of a numerical Bogomolov subsheaf $L \subset \Omega_X^1$. Assume that \mathcal{F} is not algebraically integrable. Then, up to replacing X by a non-singular Kähler modification, there exists a morphism $\Psi : X \rightarrow \mathbb{D}^N/\Gamma$ whose image has dimension $p \geq 2$ such that $\mathcal{F} = \Psi^*\mathcal{G}$ where \mathcal{G} is one of the tautological foliation on \mathbb{D}^N/Γ .*

It is worth mentioning that in this situation, the analogue of the representation provided by the item 4 of Theorem C is given by a morphism

$$\rho : \pi_1(X - H) \longrightarrow \text{pr}(\Gamma) \subset \text{Aut}(\mathbb{D})$$

with dense image and where H is an \mathcal{F} -invariant hypersurface. It is in particular of arithmetic nature (see the comments in §1.2).

10.2. It is very likely that we can enlarge the setting of Theorem D by considering the group $\text{Bim}(X, \mathcal{F})$ of bimeromorphic transformations preserving the foliation \mathcal{F} with essentially the same proof. Here, $f \in \text{Bim}(X, \mathcal{F})$ is said to preserve the foliation leafwise if there exists a non-empty Zariski open subset of X such that the restriction of f to U is a biholomorphism onto its image and such that for every $x \in U$, the leaf \mathcal{L}_x through x coincides with $\mathcal{L}_{f(x)}$.

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UNIV RENNES, CNRS, IRMAR - UMR 6625, F-35000 RENNES, FRANCE ET INSTITUT UNIVERSITAIRE DE FRANCE

Email address: benoit.claudon@univ-rennes1.fr

UNIV RENNES, CNRS, IRMAR - UMR 6625, F-35000 RENNES, FRANCE

Email address: frederic.touzet@univ-rennes1.fr