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1 **Experimental investigation of the alternate recurrence of**
2 **quasi-static and dynamic crack propagation in PMMA.**

3 **Raphael Heinzmann · Rian Seghir · Syed Yasir**
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7 **Abstract** The alternate (stick-slip) cracking phenomenon in Poly(methyl methacry-
8 late) (PMMA) was investigated using high-speed imaging and digital image correla-
9 tion (DIC). PMMA is known to show a great variety of fracture behaviors by even small
10 changes in loading conditions. With TDCB-shaped samples and under a range of con-
11 stant extension rates, the phenomenon of alternate cracking is observed. Here, loops of
12 successive quasi-static and dynamic crack propagation are found within a single frac-
13 ture experiment suggesting a ‘forbidden’ velocity regime. For the first time, such ma-
14 terial/structural cyclic fracture behavior is examined through the lens of linear elastic
15 fracture mechanics (LEFM) by using in-situ High-Speed (HS) DIC. Energy release rates
16 and crack velocities during fracture experiments are derived from full-field measure-
17 ments using Williams’ series expansion. Fracture surfaces of post-mortem samples have
18 been systematically analyzed using optical microscopy. The investigation of the actual
19 limits of the ‘forbidden’ velocity regime in terms of critical velocity and energy release
20 rate in relation to post-mortem crack length features is achieved by holistic experimen-
21 tal data on alternate cracking. This work provides key experimental data regarding the
22 improved understanding of a unified theoretical framework of crack instabilities.

23 **Keywords** Polymers · Fracture Mechanics · High Speed DIC · Stick-Slip · ‘Forbidden’
24 Velocity

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25 1 Introduction

26 Even the failure of the smallest structural component can be the source of the global col-
27 lapse of an entire engineering structure with severe consequences. Thus, understand-
28 ing the mechanisms of failure by studying the formation and propagation of cracks is
29 of utter importance in preventing such damaging events. A crucial parameter in frac-
30 ture mechanics for experimental investigations of crack propagation is the crack veloc-
31 ity, as they are strongly related to underlying fracture mechanisms. In brittle materi-
32 als, a wide range of velocities can already be explored by relatively minor variations in
33 loading conditions. Within this span of observable velocities, two independent groups
34 of interest—slow (quasi-static) and fast (dynamic) cracking—are covered in literature.
35 Without considering impact tests, at very large constant extension rates, cracks typically
36 propagate at very high velocities (about 0.5 times the Rayleigh wave speed) (Dally et al.,
37 1985; Miller et al., 1999; Scheibert et al., 2010). In that case, cracks are considered unsta-
38 ble, which means that no additional load is required to significantly propagate the crack,
39 eventually resulting in total failure of the sample. On the contrary, with a sufficiently
40 slow loading rate, the crack propagates continuously stable with velocities at least two
41 order of magnitude slower (Marshall et al., 1974; Beaumont and Young, 1975; Vasude-
42 van et al., 2021). Between these two regimes, the phenomenon of alternate cracking
43 (Ravi-Chandar and Balzano, 1988; Hattali et al., 2012) is observed for some materials.
44 Even though the crack is driven by a constant extension rate, the two unique cracking
45 behaviors can be found within one fracture experiment. Between these two propaga-
46 tion types, a range of velocities is suggested that can not be exploited by the crack. This
47 causes a gap between the two well-defined phases of quasi-static and dynamic propa-
48 gation that is commonly described as ‘forbidden’ velocity regime. Due to experimental
49 (crack velocity mismatch of about two orders of magnitude) and theoretical consider-
50 ations (apparent theoretical mismatch regarding the mechanisms), these two regimes
51 have been investigated individually up to now. As a consequence, the ‘forbidden’ veloc-
52 ity regime has only been approached from both sides without clearly quantifying en-
53 ergies and velocities involved during transitions. The experimental design in this work
54 however allows the investigation of the reoccurring appearance of the two particular
55 crack propagation types during one fracture process in PMMA. Better estimations of the
56 ‘forbidden’ velocity regime, considering in-process viscous effects (history-dependent),
57 can be accomplished. Thus, to achieve improved velocity measurements through high
58 temporal and moderate spatial resolution, up to date high-speed imaging has been
59 employed. Energy release rate estimations benefit from direct measurements of crack
60 tip positions and linear elastic fracture mechanic (LEFM) parameters through full-field
61 measurements using digital image correlation (DIC) and Williams’ series expansion.

62 The presented work is introduced with a brief description of the theoretical back-
63 ground and existing literature mainly regarding the experimental exploration of alter-
64 nate cracking. In the following, relevant information related to the experimental pro-
65 gram including short technical explanations of DIC and Williams’ series expansion is
66 presented. The work is then concluded by discussing the results of multiple TDCB ex-
67 periments and a summary of the presented study.

68 2 Alternate (stick-slip) cracking

69 Beforehand, different terminologies are found in literature discussing this fracture phe-
 70 nomenon. The most commonly used expression is stick-slip, due to the theoretical
 71 proximity with observations during the frictional phenomenon of sliding (Feeny et al.,
 72 1998). In this study, however, alternate cracking is used to additionally specify and sep-
 73 arate between sliding and fracture related observations.

74 2.1 Basic theoretical background

75 The specific behavior for materials showing alternate cracking is classically described by
 76 the kinematic law of brittle fracture in the space of energy release rate (G) as a function
 77 of the crack velocity (v) in the log-scale (Figure 1).

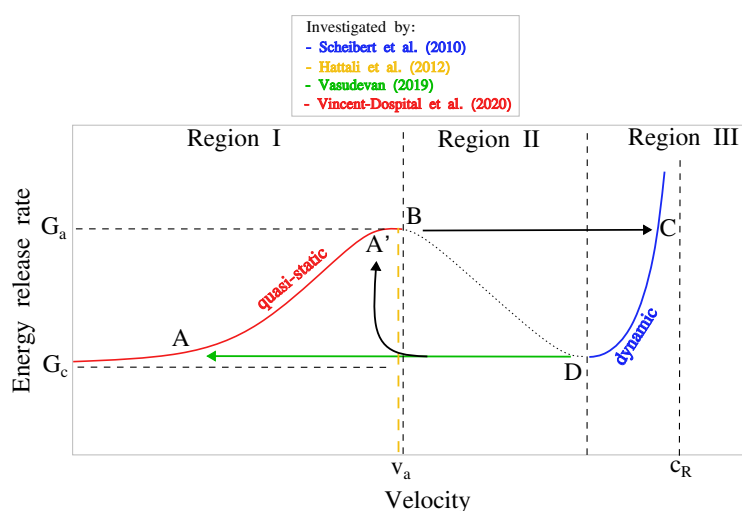


Fig. 1: Kinetic law (G_c - $\log v$ curve) for materials showing alternate cracking (stick-slip) behavior.

78 This kinematic law can be subdivided into three regions. Region I depicts a quasi-
 79 static propagation phase in which the Griffith criterion (Griffith, 1921) should apply.
 80 However, when varying the crack velocity over decades, it has been observed exper-
 81 imentally that the critical energy release rate actually depends on the crack veloc-
 82 ity. Consequently, in this region, the crack speed is driven by its relationship with the
 83 critical energy release rate that varies between G_c for infinitely low crack speeds and
 84 avalanche threshold G_a at the transition with region II (A \rightarrow B). Extreme crack tip accel-
 85 erations take place during the following region II with a supposedly "softening" branch
 86 causing the jump from the region I to the dynamic region III (B \rightarrow C). Hereinafter,
 87 the crack finds itself in the dynamic region where the generic Kanninen's relationship
 88 ($K_{ID} = K_{IC}/(1 - (v/c_R)^m)$) with the dynamic SIF K_{ID} , static SIF K_{IC} , crack tip velocity v ,
 89 Rayleigh wave speed c_R and material constant m) of dynamic fracture (Kanninen and

90 Popelar, 1985) can be applied, and crack tip speeds are restricted by the dynamic limit-
 91 ing speed of Rayleigh wave speed (c_R). However, during the high-velocity branch (region
 92 III), crack tip speeds in alternate cracking slow down after a certain propagation length
 93 to a point of arrest. This arrest marks the beginning of the second jump back to the slow
 94 velocity branch. For this, authors in existing literature generally propose the crack to fol-
 95 low along the reference points D to A. All this, due to the reiterating behavior, explains
 96 the appearance of a hysteresis implying a ‘forbidden’ velocity zone (between v^B and
 97 v^D).

98 A typical example of a fracture surface linked to such cracking is shown in Figure 2.
 99 Here, the two particular surface markings related to quasi-static and dynamic propa-
 100 gating cracks can be observed.

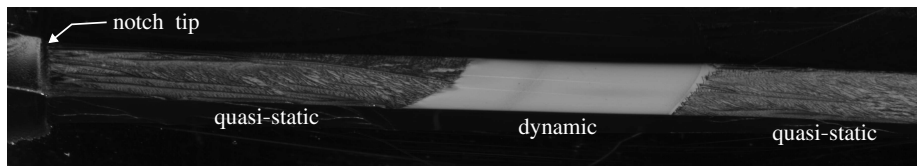


Fig. 2: Crack surface recorded with an inclined high resolution camera through the bulk of a transparent PMMA specimen visualizing the two phases of alternate cracking.

101 2.2 Existing literature

102 While regions I and III are being well defined and investigated independently, insuffi-
 103 cient experimental data and few modeling attempts are provided in the literature to fuse
 104 the two regions with a proper description of the ‘forbidden’ region II. In the following, by
 105 focusing on the most recent modeling attempt and presented experimental methods,
 106 existing literature on alternate cracking will be presented briefly.

107 Most recently, (Vincent-Dospital et al., 2020) provided a predictive model of the frac-
 108 turing behavior in PMMA based on the theoretical concept of thermal softening. In this
 109 model, the failure process is thermally activated and combined with the dissipation and
 110 diffusion of heat in the process zone surrounding the crack tip. Covering the vast range
 111 of different crack tip speeds found in PMMA fracturing, their thermodynamic approach
 112 showed good compliance with the experimental data of quasi-static and dynamic frac-
 113 turing experiments. Furthermore, the two well-defined ‘stable’ regimes were connected
 114 by the model through a continuous description of the ‘softening’ branch. Noteworthy,
 115 no experimental data of alternate cracking was considered. Notice, for terminology pur-
 116 pose, that this unified model introduces the concept of sub-critical failure. Indeed, con-
 117 trary to the discussion made around Figure 1, where velocity domains were classified
 118 through a discontinuous transition from classical Griffith theory to Kanninen’s type rate
 119 dependent relationship, the authors propose that the actual critical energy released rate
 120 is reached at high velocity ($\sim 165 \text{ m}\cdot\text{s}^{-1}$), while initiation and propagation at lower speed
 121 is supposedly sub-critical.

122 To this day, different methods have been used to experimentally investigate the
 123 phenomenon of alternate cracking. In the work of (Ravi-Chandar and Balzano, 1988),
 124 PMMA and Homalite-100 fracturing of single-edge-notched (SEN) and compact ten-
 125 sion (CT) specimens were investigated by focusing on the temporal evolution of the

126 stress intensity factor. Stress intensity factor (SIF) data has been obtained through the
127 method of caustics (Beinert and Kalthoff, 1981) using a camera system at 12 kfps over
128 a duration of one minute. Crack length and K_I measurements were retrieved during
129 quasi-static propagation phases of alternate cracking experiments, while the experi-
130 mental setup did not support dynamic phase measurements.

131 (Hattali et al., 2012) investigated the phenomenon in PMMA, by focusing on crack
132 tip speeds and fracture surfaces, using a high-resolution camera (maximal temporal and
133 spatial resolution of 396 fps and 21 Mpix, respectively) and an in-house image process-
134 ing tool. The system setup only allowed for the investigation of quasi-static crack prop-
135 agation phases. Thus, only the lower limit of the ‘forbidden’ velocity regime, i.e. v_a , was
136 investigated and found to be 3.5 cm.s^{-1} and 4.0 cm.s^{-1} for the two observed dynamic
137 bursts, respectively. However, with an interframe of approx. 2.5 ms, it has to be expected
138 that phase transitional velocities were not captured accurately.

139 (Vasudevan, 2018) contributed an in-depth investigation of cracking in PMMA, ded-
140 icating a chapter of his PhD manuscript to the transition phase from dynamic to quasi-
141 static propagations. During the experiments, load cell and clip gauge have been used
142 to obtain force and displacement data, respectively. Crack tip positions have been ob-
143 tained by using a high-speed camera system (temporal and spatial resolution of 48 kfps
144 and $\sim 175 \text{ kpix}$, respectively) and a home-made algorithm. By combining these re-
145 sults, SIF and thereby fracture energies have been derived through a numerical (FE)
146 based compliance chart relying on load and displacement data. However, difficulties in
147 this setup arise in the high-velocity regime, where the accuracy for load and displace-
148 ment measurements can not be assured. Furthermore, the chosen sample geometry
149 had the unique purpose of studying dynamic to quasi-static crack propagation transi-
150 tions. TDCB specimens with the notch being blunted by a circular hole allow for the
151 initiation of a dynamically propagating crack while using low tensile extension rates
152 ($\sim 2.5 \mu\text{m.s}^{-1}$). Here, with respect to the theoretical concept of fracture in PMMA (Figure
153 1), the crack finds its way back to the master curve far below the value of v_a by following
154 the path D \rightarrow A, which can be explained by the relatively low extension rate. After the
155 arrival of the initiated dynamic crack, by assuming an infinite sample length, the crack
156 would have propagated entirely quasi-static in the wake. Hence, the presented experi-
157 ments are a matter of quasi-static experiments introduced by dynamic crack propaga-
158 tion.

159 Under the scope of providing a model of the fracturing behavior in PMMA, (Vincent-
160 Dospital et al., 2020) performed wedge-splitting fracture experiments on PMMA sam-
161 ples over a wide range of quasi-static driving speeds ($1.6 \mu\text{m.s}^{-1}$ to 1.2 mm.s^{-1}). Force
162 data were monitored in real-time by the load cell, while the propagating crack was fol-
163 lowed by a high-resolution camera. In this case, for quasi-static propagation measure-
164 ments, the temporal resolution of 10 fps provided crack length and subsequently ve-
165 locity measurements by post-treating the images with the in-house crack tip detection
166 algorithm. On the other hand, fast crack propagations were detected, through an os-
167 cilloscope, by the successive rupture of parallel $500 \mu\text{m}$ large chromium or gold lines
168 on the surface. Numerical charts based on FE analysis supported the determination of
169 mode I related stress intensity factor K_I . Energy release rates then were retrieved under
170 plane stress assumption.

171 In summary, literature shows that the investigation of region II has been conducted
172 by experimental procedures only suitable for either of the two other enclosing re-
173 gions. Crack-speeds are usually obtained from camera systems (either high-speed or
174 high-resolution) and tracking algorithms. K_I and subsequently energy release rates are

175 mostly obtained from numerical compliance methods or analytical charts, which are
176 limited due to low accuracies in high loading rate cases. Note that region II is classically
177 referred to as ‘forbidden’ velocity regime, but the model proposed in (Vincent-Dospital
178 et al., 2020) suggest that it could be reached if the simulation or the experiment can
179 be conducted by controlling the crack speeds. However, this is impossible to achieve
180 neither numerically nor experimentally, with the energy release rate being governed by
181 the applied load as the controlled parameter. Beyond the methodological contribution
182 of investigating alternate cracking through high-speed DIC, we address in the follow-
183 ing the experimental investigation of alternate cracking by focusing systematically on
184 crack velocities, SIFs and their relation to the fracture surface. Like this, transitions from
185 dynamic to quasi-static and quasi-static to dynamic can be studied. Hence, we pro-
186 pose a compromised experimental configuration, based on high-speed fullfield mea-
187 surements (DIC), and on asymptotic solutions of kinematic fields to get not only FE
188 independent and chart-free estimations of the SIFs, but also sub-pixel estimations of
189 crack tip positions. Additionally, the contribution of this study can be summed by the
190 attempt of providing experimental data for an improved estimation of the hysteretic
191 fracturing behavior described by the loop following path B, C, D and A’ in Figure 1.

192 3 Experimental program

193 Experiments have been performed during two independent experimental campaigns,
194 called M1 and M2. Setup, material and loading conditions have been kept consistent,
195 while using different sample thicknesses. 5 mm and 8 mm thick plates have been used
196 for the specimen preparation of the M1 and M2 campaign, respectively.

197 3.1 Material and geometry of the specimen

198 In this study, alternate cracking, specific to Poly(methyl methacrylate) (PMMA) (Ravi-
199 Chandar and Yang, 1997), as well as polyesters (Leevers, 1986), epoxy resins (Yamini and
200 Young, 1977), rubber (Isherwood and Williams, 1978) and polymeric adhesives (Maugis
201 and Barquins, 1988), is investigated. Due to its ideal brittle behavior and the presence
202 of different cracking behaviors at varying loading conditions, PMMA is a material of
203 great interest in fracture mechanics research and an ideal model material for this in-
204 vestigation. Here, molded PMMA (Perspex® and Plexiglas® for M1 and M2 campaign,
205 respectively) was the material of choice. For details on material properties and its me-
206 chanical behavior, the authors refer to (Ali et al., 2015). The tapered double-cantilever
207 beam (TDCB) shape was chosen to have a constant specimen compliance w.r.t. the
208 crack length and ensure stable crack propagation. The samples were obtained by using a
209 class 4 laser cutting machine (130 W). Notice that pin holes have been manually drilled
210 to avoid laser beam (conical shape) related clearance angles ($\sim 1.5^\circ$), which would in-
211 duce transverse load, i.e. potential sample bending. The detailed sample geometry can
212 be found in Figure 3.

213 Notice that caution is required regarding sample preparation for achieving good re-
214 peatability. Indeed, PMMA fracture behavior is very sensitive to temperature (Atkins
215 et al., 1975a,b), but also to any significant interaction with organic solvents (Wang et al.,
216 1994). In that context, ethanol (eventually used for surface cleaning) should be avoided,
217 and solvent free paints should be preferred for DIC patterning of the sample surface.

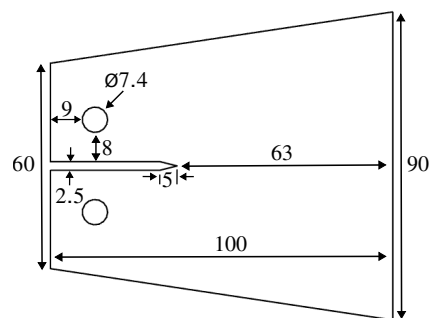


Fig. 3: Sample geometry [mm].

Table 1: Second loading stage extension rates applied during the experiments.

Extension rate [mm.s ⁻¹]	Test
1.0	M1-T2, M2-T15, M2-T19
0.5	M1-T3, M1-T6

218 3.2 Loading and test configuration

219 All experiments were performed using an Instron all electric dynamic and fatigue test-
 220 ing system (ElectroPuls E10000) with a load cell of 1 kN maximum loading capacity in a
 221 temperature controlled (21 °C) room. Beforehand, preliminary experiments were con-
 222 ducted to identify the alternate cracking domain within the experimental environment.
 223 With respect to the used material and geometry, solely quasi-static propagation has
 224 consistently been found below extension rates of 0.1 mm.s⁻¹. First dynamic bursts have
 225 been observed between 0.1 mm.s⁻¹ and 0.5 mm.s⁻¹ with strong variability from one test
 226 to another. Following this domain of inconsistency, alternate cracking was systemati-
 227 cally determined in a repeatable manner (Figure 2). Entirely dynamic cracks have been
 228 found for extension rates above 20 mm.s⁻¹. Extension rates close to the lower end of
 229 the identified alternate cracking domain have been used to provoke higher repetition of
 230 crack arrests while maintaining reasonable quasi-static lengths. Additionally, to achieve
 231 an experimental consistency of the crack line by avoiding influences of the notch tip on
 232 the subsequent cracking and the point of initiation for different cracking experiments,
 233 a two stages loading procedure was established:

- 234 1. *Controlled quasi-static crack propagation.* During this stage a relatively slow exten-
 235 sion rate of 0.01 mm.s⁻¹ was imposed. Stage change has been executed by the help
 236 of the Static Break Detector option within the Instron WaveMatrix software (Figure
 237 4). The tool is designed to trigger a user-selected action when the monitored signal
 238 falls below or rises above a pre-defined value (event value). 25 N was selected for the
 239 event value, which generally led to slow propagating cracks of about 5-10 mm.
- 240 2. *Alternate crack propagation.* A second loading stage was introduced to release the
 241 alternate crack propagation from this fresh pre-crack. During this stage, the exten-
 242 sion rates being used during the different experiments are shown in Table 1.

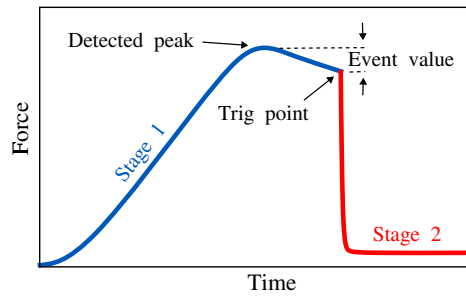


Fig. 4: Example of stage change method (Static Crack Detection).

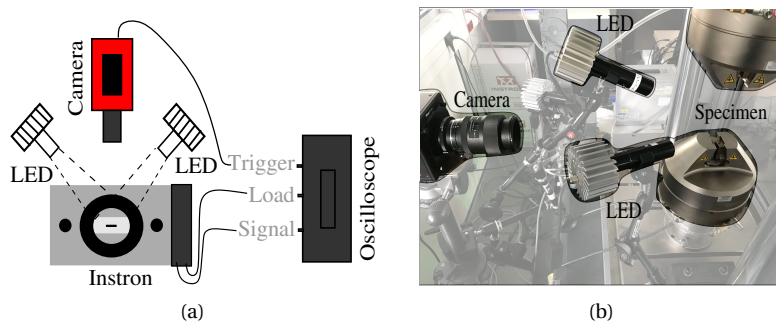


Fig. 5: Experimental setup: (a) sketch, (b) picture.

243 3.3 Experimental setup

244 Alternate cracking is a phenomenon, where two entirely different cracking behaviors
 245 are to be found during solely one fracturing process. Hence, for relevant data acqui-
 246 sition during the dynamic crack propagation phases, the experiments are in need of
 247 high speed imaging. Here, images of the cracking sample were captured using a Photron
 248 FASTCAM Mini AX camera system with an acquisition speed of 40 kfps. Sufficient light-
 249 ing of the sample was ensured by LED projectors (EFFI-Sharp PWR FF by Efilux). Master
 250 device for camera triggering has been the tensile machine, which sent a 5 V TTL trigger
 251 signal to an oscilloscope once the second loading stage has been activated. A thresh-
 252 old in the oscilloscope was then used to trigger the camera system. Hence, only second
 253 loading stage results have been recorded. Picture and sketch of the experimental setup
 254 can be seen in Figure 5. Camera specifications related to the experiments are presented
 255 in Table 2.

256 3.4 Fracture parameters by DIC

257 The foundation for retrieving the sought-after fracture mechanics variables is found in
 258 DIC by providing full-field displacement measurements from sequential images. By fol-
 259 lowing the principle of conservation of brightness between a reference (f) and a de-
 260 formed image (g), i.e. the equation of optical flow, DIC presents an ill-posed non-linear

Table 2: DIC hardware parameters

Campaign	M1	M2
Camera	Photron FASTCAM Mini AX	
Image resolution	896×160 pix	
Acquisition rate	40,000 fps	
Exposure time	25 μ s	
Lens	TOKINA 100 mm	
Aperture	f/2.8	
Field of view	74×13 mm	64.5×11.5 mm
Image scale	1 pix = 83 μ m	1 pix = 72 μ m
Patterning technique	Spray paint	

261 inverse problem:

$$f(\underline{X}) = g(\underline{X} + \underline{u}(\underline{X})) \quad (1)$$

262 with $\underline{u}(\underline{X})$ being the sought displacement field. To bypass this ill-posed problem, the
 263 pixel displacement has to be parameterized using shape functions. Here, FE discretiza-
 264 tion has been used (Besnard et al., 2006). Eventually, the problem can be linearized and
 265 solved iteratively in a least-squares sense. A median regularization is used to mitigate
 266 detrimental impact of noise and smaller elements. By post-treating the displacement
 267 field, variables relevant for fracture mechanics analysis (e.g. SIFs and crack tip posi-
 268 tions) were derived through Williams' series expansion. Traditionally, in the case of a
 269 semi infinite linear elastic isotropic media, stress and displacement fields around the
 270 crack tip depend on SIF, the distance to the crack tip (r) and the angle (θ) in a polar
 271 reference system attached to the crack tip (Williams, 1957). Thus, by knowing the dis-
 272 placement field through DIC and projecting it onto the analytical solution, different
 273 fracture mechanics parameters can be derived through a non-linear inverse problem
 274 (Roux and Hild, 2006; Réthoré, 2015; Roux-Langlois et al., 2015). The displacement field
 275 around the crack tip is written as following:

$$u(r, \theta) = \sum_{i=I,II} \sum_{n=-\infty}^{\infty} A_i^n r^{n/2} g_i^n(\theta) \quad (2)$$

276 with the distance to the crack tip r , the angle θ in a polar reference system attached to
 277 the crack tip, Williams' coefficients A and base function g . The base function $g_i^n(\theta)$ has
 278 the following form:

$$g_I^n(\theta) = \frac{1}{2\mu} \left[\begin{array}{l} (\kappa + n/2 + (-1)^n) \cos[(n/2)\theta] - (n/2) \cos[(n/2 - 2)\theta] \\ (\kappa - n/2 - (-1)^n) \sin[(n/2)\theta] + (n/2) \sin[(n/2 - 2)\theta] \end{array} \right]_{(e_r, e_n)} \quad (3)$$

$$g_{II}^n(\theta) = \frac{1}{2\mu} \left[\begin{array}{l} -(\kappa + n/2 - (-1)^n) \sin[(n/2)\theta] + (n/2) \sin[(n/2 - 2)\theta] \\ (\kappa - n/2 - (-1)^n) \cos[(n/2)\theta] + (n/2) \cos[(n/2 - 2)\theta] \end{array} \right]_{(e_r, e_n)} \quad (4)$$

279 with μ and κ being the shear modulus and the Kolossov's constant, respectively.
 280 Kolossov's constant under plane stress is $\kappa = (3 - \nu)/(1 + \nu)$ with ν being the Poisson's
 281 ratio.

282 Equation 2 states, that displacements can be computed for an infinite sum of
 283 modes. However, limiting the solution to $n_{min} = -3$ and $n_{max} = 7$ is sufficient to re-
 284 tain the relevant crack features. For quasi-brittle medium, good agreement of the me-
 285 chanical fields can be established outside the process zone of the crack for the following
 286 Williams' series coefficient (A_i^n) solutions:

Table 3: DIC & Williams' series parameters

Campaign	M1	M2
Test	2, 3, 6	15, 19
DIC		
Software	UFreckles (Réthoré, 2018)	
Discretization	FE-based global DIC	
Shape function	Bi-linear quadrilateral Lagrange element (Q4P1)	
Element (subset) size	0.581 mm (7 pix)	0.576 mm (8 pix)
Post-filtering	Median filter using 1 st neighbours applied to U	
Williams' series projection		
E	3.5 GPa	
ν	0.32	
R_{min}	1.16 mm	1.15 mm
R_{max}	6.39 mm	6.34 mm

- 287 $n = 0$, in-plane rigid body translations
288 $n = 1$, asymptotic terms K_I and K_{II}
289 $n = 2$, T-stress and in-plane body rotations

290 Notice that the projection zone is defined by R_{min} and R_{max} . On the one hand, the
291 asymptotic behaviour near the crack tip of the fields computed by the super-singular
292 terms ($n < 0$) do not provide any physical meaning and are therefore classically ne-
293 glected. However, when the crack tip position is sought, these super-singular functions
294 appear in the required basis ($n = -1$ being, up to a scaling factor, the derivative of $n = 1$
295 with respect to the assumed crack tip position). To not induce biases due to truncation,
296 terms for n down to -3 are considered. Hence, data at a distance to the crack tip smaller
297 than R_{min} are discarded to maintain reasonable conditioning of the least-squares prob-
298 lem. On the other hand, R_{max} – defining the projection zone size externally – has to be
299 small enough to avoid influences of the free boundary, but large enough to still include
300 a sufficient amount of mesh points in the domain. The right size of projection zone, i.e.
301 defining R_{min} and R_{max} (see Table 3), has been identified through a parametric study.
302 Furthermore, by using a pre-defined crack path, the super-singular term $n = -1$ is used
303 to estimate the position to the equivalent elastic crack tip along this path (Réthoré et al.,
304 2011). In the context of the study, relevant coefficients of the solution imply $n = -1$ and
305 $n = 1$, i.e. crack tip detection and SIFs. It is worthwhile mentioning, that the quasi-static
306 form (not in function of the velocity) of the displacement field around the crack tip is
307 used in line with previous works showing that marginal errors can be assumed for ve-
308 locities below $0.5c_R$ (Lee et al., 2010). Eventually, the outcome is used to derive crack
309 velocities and energy release rates. The second, under the assumption of plane stress,
310 has been computed through Equation 5.

$$G = K_1^2 / E \quad (5)$$

311 with K_1 and E being the SIF related to mode I and Young's Modulus, respectively.
312 DIC and Williams' series projection results have been obtained entirely by accessing the
313 library of the open-source software UFreckles (Réthoré, 2018). All relevant parameters
314 for the two procedures are listed in Table 3.

315 3.5 Post-processing and method synchronization

316 To get derivatives such as the crack velocity, raw crack tip measurements had to be
 317 filtered accordingly. The need for high speed imaging to obtain sufficient amount of
 318 data points during dynamic crack propagation led to comparable huge sets of data for
 319 the quasi-static propagation phases. With a difference of tens (dynamic) to thousands
 320 (quasi-static) of data points between the two regimes, the data set had to be separated
 321 and post-filtered by different kernel sizes. The post-filtering has been performed by us-
 322 ing a Savitzky Golay filter with a second order polynomial. Depending on the regime af-
 323 filiation, a filtering window size of 7.275 ms (291 frames) or 75 μ s (3 frames) has been ap-
 324 plied for quasi-static and dynamic regimes, respectively. The relatively strong and rather
 325 insignificant kernel size for, respectively, the quasi-static and dynamic phases deemed
 326 to be fitting to not drastically influence crack speed evolution in the quasi-static parts
 327 or decrease the accuracy of the dynamic phase measurements.

328 To investigate crack kinetics, not only high-speed recordings but also post-mortem
 329 high resolution microscopy was performed. Therefore, the comparison of data obtained
 330 by techniques with different spatial resolutions are in need of harmonization. Here, as
 331 the notch tip can easily be identified in DIC and microscopic images, it served as com-
 332 mon point of reference. By doing so, DIC obtained crack tip positions were tied to post-
 333 mortem microscopic images. The most dominant error sources related to the process of
 334 spatial matching are the manual selection of the notch position (σ_{notch}) and the pixel to
 335 meter conversion ('pix2m') uncertainty (σ_{pix2m}). Due to camera resolution, used lens
 336 and clearance angle of the sample cut, the uncertainty on the notch tip localization has
 337 been evaluated to be about $\sigma_{notch} = 1.5$ pix. Pixel to meter conversion values have been
 338 identified by analysing ruler images taken before each test, with the help of the Fast
 339 Fourier transform (FFT) approach. The uncertainty of the 'pix2m' value has been eval-
 340 uated to be about $\sigma_{pix2m} = 1.12 \times 10^{-3}$ pix as the FFT sampling was chosen to be the
 341 horizontal pixel size of the image (896 pix). First approximation of the crack tip local-
 342 ization uncertainty ($\sigma_T(a)$) can be evaluated through the following equation:

$$\sigma_T(a) = (a \times \sigma_{pix2m} + \sigma_{notch}) \times pix2m \quad (6)$$

343 with a being the crack length at any position of interest. A maximal error of approx.
 344 160 μ m and 140 μ m have to be expected for the considered field of view (FOV) for ex-
 345 periments related to campaign M1 and M2, respectively.

346 4 Results

347 PMMA samples were subjected to the above described loading condition (see Chapter
 348 3.2) and followed by a high-speed camera system. About 40,000 successive images over
 349 a duration of one second were obtained for each experiment. The images were used
 350 for DIC analysis and subsequently for the retrieving of fracture mechanic parameters.
 351 Five tests have been taken into consideration coming from two different experimen-
 352 tal campaigns (M1 and M2). Representative for all experiments, full-field displacement
 353 measurements (in y-direction) for the first and last recorded frame superimposed on
 354 the appropriate image of experiment M2-T15 are presented in Figure 6(a) and Figure
 355 6(b), respectively. In the same configuration, corresponding Williams' series projection
 356 areas are presented in Figure 6(c) and Figure 6(d). Crack length data, retrieved through
 357 the process of Williams' series expansion, are then presented in Figure 7 in the form

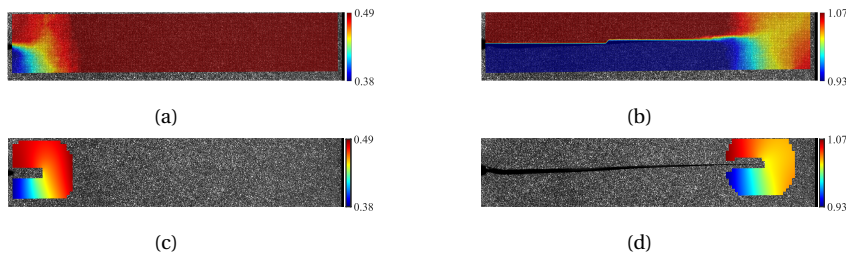


Fig. 6: Entire displacement field in y-direction [mm] and its corresponding Williams' series projection [mm] on the process zone surrounding the crack-tip (defined by R_{min} and R_{max}) of the first ((a) and (c)) and last recorded image ((b) and (d)) for experiment M2-T15. The used Williams' series parameters are listed in Table 3.

358 of the temporal evolution of the normalized crack length. This is defined by the crack
 359 length relative to the length of the samples (63mm from notch tip to sample boundary).
 360 Furthermore, stable first stage crack propagations have been taken into consideration
 361 for the presentation of the normalized crack length. This causes each experiment to
 362 start from its own individual pre-crack length.

363 Each crack length evolution is showing a similar global behavior. After the pre-crack,
 364 almost horizontal lines describe the first quasi-static propagation of the crack. Follow-
 365 ing vertical jumps indicate the dynamic cracking. The combination of the two regimes
 366 is what is referred to as alternate cracking of the sample. This is depicted by the repeat-
 367 ing occurrence of vertical jumps surrounded by quasi-static phases. 4, 3, 2, 2 and 2 ver-
 368 tical lines, i.e. dynamically propagating cracks, are observed for tests M1-T2, M1-T3,
 369 M1-T6, M2-T15 and M2-T19, respectively. The frame rate chosen for this investigation
 370 leads to dynamic crack propagation phases being sampled by 4 to 7 images depending
 371 on their propagation length. It will allow to have good estimations of the mean veloc-
 372 ity, while not being sufficient to accurately probe potential speed variations. Oscilla-
 373 tions in the crack length of the quasi-static rest phases appear always closely following
 374 the dynamic bursts. These fluctuations can be explained by system vibrations due to
 375 history-dependent effects induced by brutal, instantaneous dynamic bursts. It leads to
 376 oscillations of the crack lips (opening and closing) causing the crack tip detection to
 377 record backwards moving cracks. Furthermore, as they become more prominent when
 378 the crack approaches the free end of the sample, the vanishing resistivity of the entire
 379 sample seems to have an enhancing effect on the phenomenon.

380 For sake of clarity, two experiments will be prioritized for the continuation of the
 381 discussion. For this, M1-T2 and M2-T15 have been selected as representatives of the two
 382 experimental campaigns as the tests with the most quasi-static rest repetitions within
 383 their experimental campaign.

384 4.1 Velocities

385 To further differentiate among the crack propagation types observed during alternate
 386 cracking, crack speeds have been derived from the crack length. Figure 8 is showing
 387 the spatial evolution (as a function of crack length) of the crack speed for experiments
 388 M1-T2 and M2-T15. Only effective crack tip positions, i.e. always keeping the most ad-

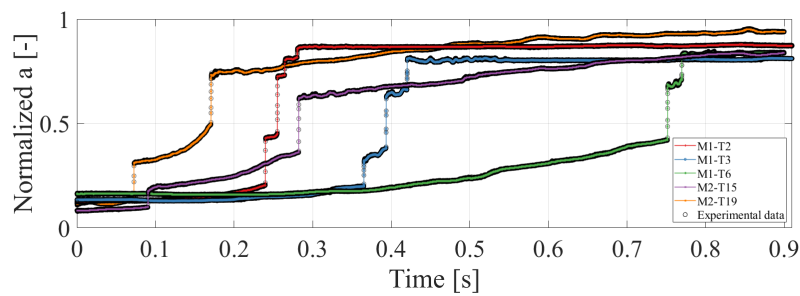


Fig. 7: Normalized crack length (1.0 = 63 mm) as a function of recording time [s] for all five experiments.

389 vanced position as real position of the crack tip, have been taken into consideration to
 390 avoid the influences of crack length oscillations discussed before. Two strongly different
 391 crack speeds are observed in the experiments. Variations in dynamic crack speeds are
 392 rather small, while crack speeds associated to quasi-static propagation show significant
 393 oscillatory behavior. By taking both experiment M1-T2 and M2-T15 into consideration
 394 (Figure 8), average crack tip speeds up to 10.5 cm.s^{-1} and between 14 m.s^{-1} to 107 m.s^{-1}
 395 are observed during quasi-static and dynamic crack propagations, respectively. Average
 396 dynamic crack speeds around $\sim 14 \text{ m.s}^{-1}$ would portray a divergent observation from
 397 the literature (Hattali et al., 2012; Vasudevan, 2018) documented dynamic crack speed
 398 measurements during similar experiments. However, by taking the fracture surface into
 399 consideration, one can see tiny rest bands (see Figure 9) appearing within the flat parts of
 400 some dynamic bursts. The specific surface pattern is regularly found during transition
 401 from dynamic to quasi-static phases (see Detail 2 and Detail 3 in Figure 9). Thus, the
 402 rest bands suggest small pauses of the crack, too short to be visible in the recordings (in-
 403 terframe of $25 \mu\text{s}$) but long enough to significantly influence the crack speed measure-
 404 ments during these dynamic phases considering the frame rate of 40 kfps. The appear-
 405 ance of rest bands can be interpreted as intermittent behavior of dynamic cracks. The
 406 amount of rest band appearances during the different dynamic bursts for all considered
 407 experiments are listed in Table 4. Associated to this Table, Figure 10 provides a rela-
 408 tion between crack speeds and their corresponding propagating length. Here, cracking
 409 phases with biased velocity measurements are marked in red. Globally, two groups of
 410 propagation phases can be observed in the figure. Cracks with rest bands fall all within
 411 the lower velocity range, while the others are found above. It appears that the limit aver-
 412 age velocity separating these two can be allocated at approx. 60 m.s^{-1} . Indeed, with
 413 sufficient temporal resolution to capture velocity variations during dynamic propaga-
 414 tion, rest band deceleration's could have been singled out. This would most certainly
 415 cause the upwards shift in average velocity during these phases with the option of ex-
 416 cluding rest bands. However, by taking all bursts into consideration, a relation of in-
 417 creasing average velocity with increasing burst length can be assumed. A specific case
 418 is found during the first dynamic burst of M2-T15. The particular surface roughening is
 419 observed, even though no rest band is entirely formed. Nevertheless, crack deceleration
 420 influencing the velocity measurement have to be expected. Thus, this burst will be con-
 421 sidered as biased and included to the rest band effected bursts. With an average velocity
 422 of $\sim 60 \text{ m.s}^{-1}$, the dynamic burst seems to be at the velocity threshold for the formation

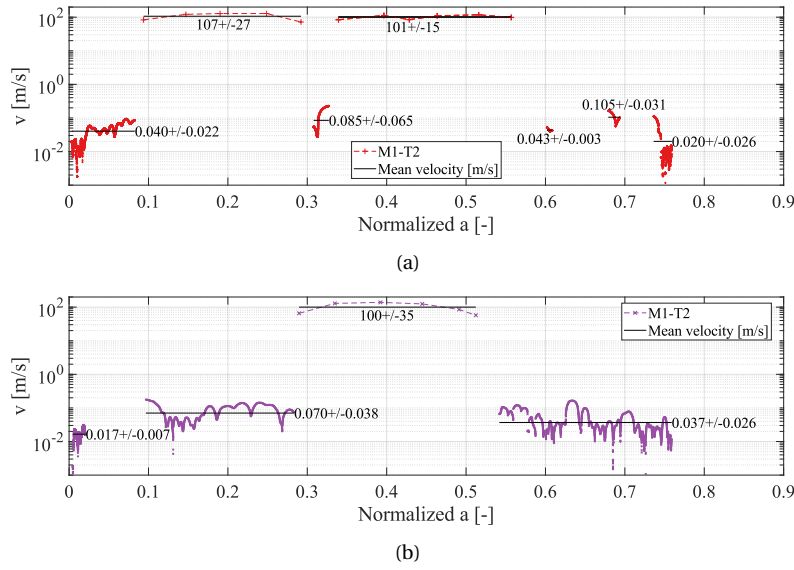


Fig. 8: Crack tip speed (v) [$\text{m}\cdot\text{s}^{-1}$] as a function of the normalized crack length (a) [mm]. Black bars present the phase-dependent mean velocity [$\text{m}\cdot\text{s}^{-1}$], while their value is presented as numbers next to the bar with its standard deviation (std).

423 of rest bands. Hence, if dynamic bursts affected by rest bands (biased velocity measure-
 424 ments) are neglected, a ‘forbidden’ velocity regime of $\sim 36 \text{ cm}\cdot\text{s}^{-1}$ to $\sim 36 \text{ m}\cdot\text{s}^{-1}$ can be
 425 implied taking all conducted experiments into consideration. This is expressed in Figure
 426 11, showing velocities with respect to the normalized crack length for all conducted
 427 experiments excluding dynamic phases that are considered to be biased. Notice, with
 428 average velocities above the identified threshold value ($\sim 60 \text{ m}\cdot\text{s}^{-1}$) for the formation of
 429 rest bands, instantaneous velocity measurements at the beginning and the end of rest
 430 band free dynamic burst can fall below the threshold and thereby present the upper
 431 bound of the ‘forbidden’ velocity regime.

432 4.2 Connections to the fracture surface

433 In the following, more attention is given to the relation of experimental data and frac-
 434 ture surfaces. Microscopic images have been obtained by an optical microscope (Nikon)
 435 with a resolution of $0.67 \mu\text{m}\cdot\text{pix}^{-1}$. K_I and velocity data as a function of the crack length
 436 are presented in face with their corresponding fracture surface in Figure 12. Here, only
 437 the propagation parts between notch tip and the beginning of the second dynamic
 438 phase are presented. Like this, two quasi-static to dynamic and one dynamic to quasi-
 439 static transition can be seen. Crack propagation is from left to right and marked in
 440 the left bottom corner of the figures. DIC (K_I , v) and microscopic observations (frac-
 441 ture surface) have been spatially matched with respect to the notch tip. Fracture sur-
 442 faces were recreated by manually stitching microscopic images with a field of view of
 443 $1.715 \times 1.286 \text{ mm}$. Hence, minor light differences can be seen between the sub-frames.

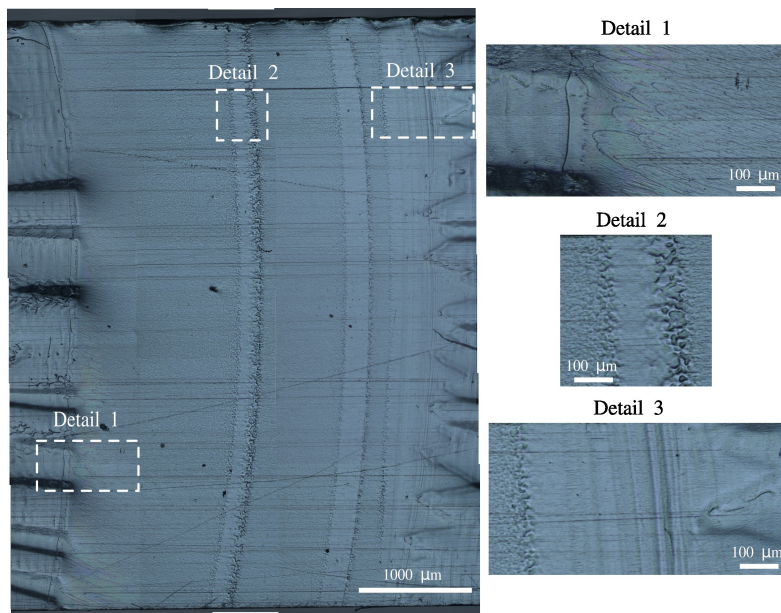
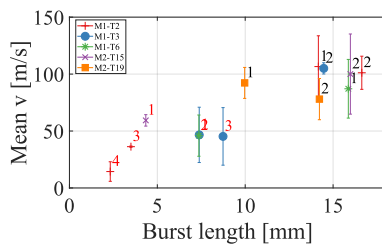


Fig. 9: Example fracture surface showing different particular surface observations. Detail 1, 2 and 3 are showing nucleation phase, rest band and the dynamic to quasi-static deceleration phase with the appearance of Wallner lines, respectively.



Test	DYN-burst			
	1	2	3	4
M1-T2	0	0	2	2
M1-T3	1	0	3	-
M1-T6	0	2	-	-
M2-T15	1*	0	-	-
M2-T19	0	0	-	-

Fig. 10: Mean velocity [$\text{m}\cdot\text{s}^{-1}$] as a function of the burst length [mm] for the dynamic phases of all conducted experiments. The chronological order of the different dynamic burst for each individual test is given by the numbers next to the data points. Red colored numbers indicate the appearance of at least one rest-band during this particular dynamic burst, while the error bars give the variation (std) of the velocities for each dynamic phase.

Table 4: Amount of rest band appearances during the dynamic bursts. (*) means specific roughening of the surface observed, which implies the early formation of a rest band that is yet not fully evolved. However, biased velocity measurements still have to be expected.

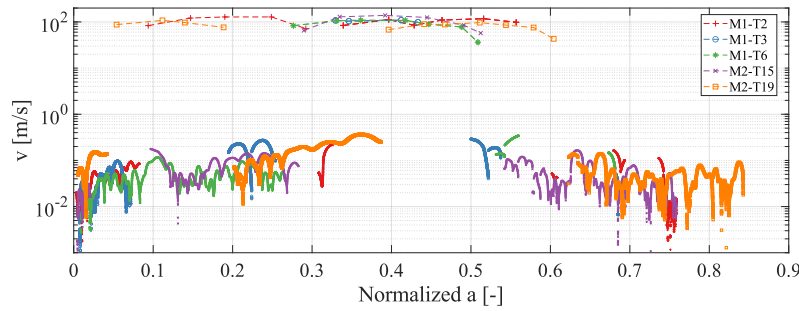


Fig. 11: Velocity [$\text{m}\cdot\text{s}^{-1}$] as a function of the normalized crack length [mm] for all conducted experiments.

444 The most striking observation is the regime and phenomenon-dependent fracture
 445 surface pattern: quasi-static and dynamic fracture processes produce wave-like and flat
 446 surface patterns, respectively. Macroscopically, the two entirely different mechanisms
 447 therefore produce different fracture surfaces. Differences in fracture surface with re-
 448 spect to the experiments coming from the two campaigns are found in height of the
 449 wave-like ligaments and the curvature of the crack fronts. Crack front curvatures arise
 450 by focusing on the transitions between the two phases. Furthermore, the particular
 451 roughening of the fracture surface between 8 to 10 mm in M2-T15 (12(b)) is indicating
 452 the partial formation of a rest band, which has been discussed before. Other particular
 453 observations are the specific rough pattern appearing during transition from dynamic
 454 to quasi-static with Wallner lines (Wallner, H., 1939; Bonamy and Ravi-Chandar, 2003)
 455 (Detail 3 in Figure 9) and the occurrence of conical marks related to micro-crack forma-
 456 tions ahead of the main crack during the quasi-static to dynamic transition phases
 457 (Detail 1 in Figure 9) (Ravi-Chandar and Balzano, 1988). These conical surface marks
 458 are only found in the really beginning of the dynamic bursts suggesting, from (Guerra
 459 et al., 2012), crack speeds above $\sim 165 \text{ m}\cdot\text{s}^{-1}$. Highest velocities can thereby be expected
 460 in the beginning of dynamic bursts. K_I and velocity measurements underline this depen-
 461 dency. With severe differences (around two orders of magnitude) in velocities sep-
 462 arating the two types of crack propagation, K_I is globally showing a decrease as the
 463 crack propagates dynamically and a recovering increase as it rests during the quasi-
 464 static phases. As longer lasting quasi-static propagations are present (see Figure 12(b)),
 465 K_I and v find itself oscillating around a constant value.

466 Special interest is given in the following to the transition phase dynamic to quasi-
 467 static propagation related to the first quasi-static rest of the two experiments. During
 468 the quasi-static rest of the M1-T2 experiment (Figure 12 (a)), K_I consists out of two
 469 phases: Increasing from $\sim 0.9 \text{ MPa}\sqrt{\text{m}}$ to $\sim 1.2 \text{ MPa}\sqrt{\text{m}}$ ($\sim 27 \text{ mm}$ to $\sim 27.6 \text{ mm}$), while
 470 stabilizing around $\sim 1.2 \text{ MPa}\sqrt{\text{m}}$ ($\sim 27.6 \text{ mm}$ to $\sim 28.4 \text{ mm}$) before the occurrence of
 471 the following dynamic burst. By relating these observations to the measured velocities,
 472 one can notice a drop in crack tip speed when K_I has a sudden change and an over-
 473 all decrease of crack speed during this first phase. During the second stable K_I phase,
 474 the velocity increasingly converges towards a stable crack tip speed before the sudden
 475 jump into the dynamic region. Linking the observations to the fracture surface, the for-
 476 mation of triangular surface patterns (see Detail 3 in Figure 9 for zoomed view on an

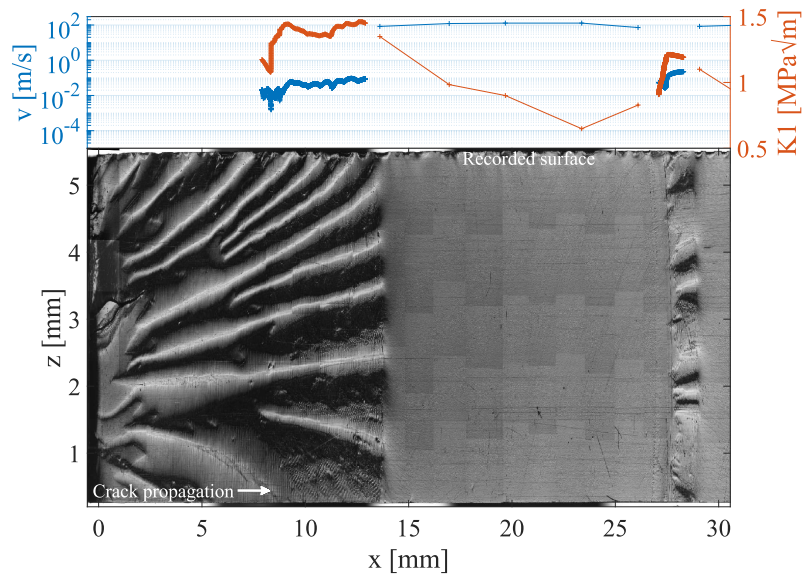
example surface) seems to be related to the first phase, while the stable second phase can be linked to the continuation of the newly formed wavy fracture pattern. Overall, similar trends are found in the quasi-static rest obtained from the M2-T15 experiments (Figure 12 (b)). K_I increases with decreasing crack speeds during the formation of the triangular crack pattern, while both stabilize in the continuation of the wavy pattern. In detail, K_I values increase during the first phase from $\sim 1.1 \text{ MPa}\sqrt{\text{m}}$ to $\sim 1.4 \text{ MPa}\sqrt{\text{m}}$ ($\sim 16 \text{ mm}$ to $\sim 15 \text{ mm}$), while decreasing again until 22 mm to around $\sim 1.3 \text{ MPa}\sqrt{\text{m}}$ with stronger oscillations. However, until $\sim 22.5 \text{ mm}$, a sudden increase back to $1.4 \text{ MPa}\sqrt{\text{m}}$ is observed, followed by a phase of continuity before the crack starts to propagate dynamically again at $\sim 23.7 \text{ mm}$.

When focusing on the fracture surface at the beginning of the quasi-static phase, both specimens show the particular roughening and the appearance of Wallner lines. According to velocity measurements, this area seems to be the result of an intense deceleration zone. Here, a variable e is used to depict the distance of the end of this particular rough pattern and the first appearance of a Wallner line. A relationship of this distance e (Figure 13(b)) with the mean crack tip speed of the preceding dynamic phase is established in Figure 13(a) for the quasi-static rests of all conducted experiments. Red color numbers again indicate the presence of rest bands falsifying the velocity measurement of this particular dynamic burst. Hence, if dynamic bursts with rest band occurrences are neglected, a trend with a growing relationship of dynamic velocity and distance e can be suggested. While first Wallner lines seem to appear after $193 \pm 78 \mu\text{m}$ for mean dynamic crack speeds of $85 \pm 5 \text{ m}\cdot\text{s}^{-1}$, they first appear after $490 \pm 125 \mu\text{m}$ for mean crack tip speeds above $100 \text{ m}\cdot\text{s}^{-1}$. It seems to show that depending on the velocity, a critical distance is required (with almost linear trend) to reach quasi-static propagations.

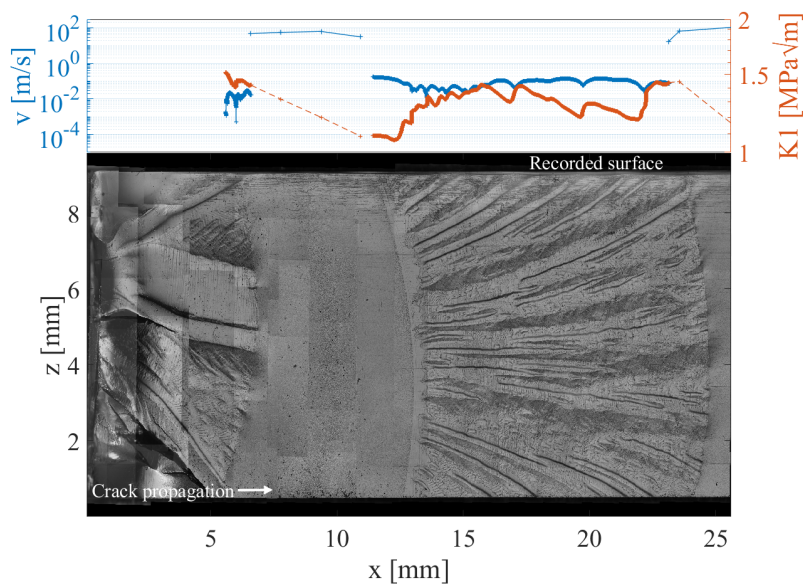
4.3 Energy release rate

Energy release rates have been computed from K_I to reproduce the kinetic law of materials showing alternate cracking (Section 2.1). Again, a detailed presentation of the hysteresis will only be presented for two tests (here, M1-T2 and M2-T19). For clearer visualization, only the crack propagation phases directly associated with the alternate cracking are presented. This means, only the phases surrounded by dynamic bursts are taken into consideration with the hysteresis always starting with the first dynamic burst. In each of the two sub figures, presented in Figure 14, two related graphs are found: the temporal evolution of the crack length ($a(t)$) and the energy release rate as a function of the crack tip speed ($G(v)$). Each individual phase of cracking is colored concordant over the two plots. Also, triangular and square markers indicate the beginning or the end of a phase, respectively.

Let us first look at test M1-T2 (Figure 14(a)). The hysteretic behavior of the fracturing is here introduced by a first dynamic burst (light blue) following the theoretical corner points C to D. During this propagation phase, the energy release rate drops from $\sim 520 \text{ J}\cdot\text{m}^{-2}$ to $\sim 195 \text{ J}\cdot\text{m}^{-2}$, while the crack speed undergoes an increasing, stabilizing and decreasing phase with velocities around $85 \text{ m}\cdot\text{s}^{-1}$, $120 \text{ m}\cdot\text{s}^{-1}$ and $70 \text{ m}\cdot\text{s}^{-1}$, respectively. Subsequently, no data points are acquired until the first points in the quasi-static region, indicating the first jump between the two accessible regions. By arriving at the first quasi-static propagation phase (dark blue), G rises from $\sim 240 \text{ J}\cdot\text{m}^{-2}$ to $\sim 375 \text{ J}\cdot\text{m}^{-2}$, while the crack speed continues to decrease from $5 \text{ cm}\cdot\text{s}^{-1}$ to $3 \text{ cm}\cdot\text{s}^{-1}$. After this, while G continues to grow ($\sim 375 \text{ J}\cdot\text{m}^{-2}$ to $\sim 420 \text{ J}\cdot\text{m}^{-2}$), v finds itself increasing from the low



(a)



(b)

Fig. 12: Velocity [$\text{m}\cdot\text{s}^{-1}$] and K_I [$\text{MPa}\sqrt{\text{m}}$] as a function of crack length [mm] in face with the microscopic image of the fracture surface for the experiments: (a) M1-T2 and (b) M2-T15. A spatial matching error between DIC and microscope measurements of $\sim 160\ \mu\text{m}$ and $\sim 140\ \mu\text{m}$ has to be considered for M1-T2 and M2-T15, respectively.

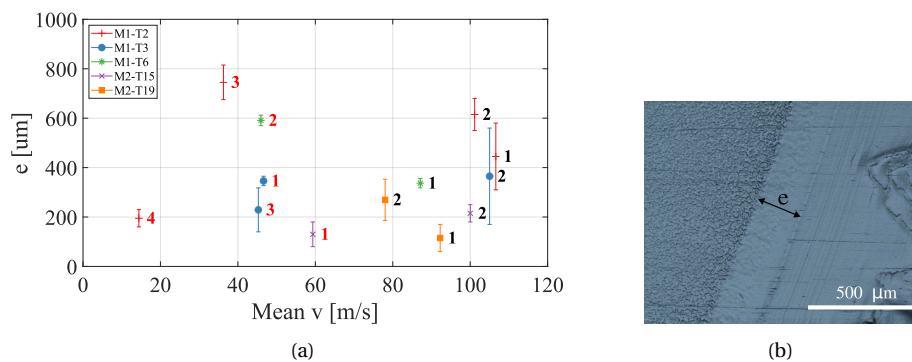


Fig. 13: (a) Mean velocity during dynamic crack propagation [m.s⁻¹] as a function of e -length (i.e. distance from dynamic micro roughening to Wallner lines) [μm] of the subsequent DYN to QS transition phase for all experiments. Error bars present the measurements on the two free edges of the fracture surface over the sample thickness. The chronological order of the different dynamic burst for each individual test is given by the numbers next to the data. Red colored numbers indicate the appearance of at least one rest-band during this particular dynamic burst. (b) Example of the measured distance e on the fracture surface.

523 point to around 15 cm.s⁻¹. With G decreasing after reaching its peak value (~ 420 J.m⁻²),
 524 the crack continues to accelerate to 23 cm.s⁻¹ before jumping back to the dynamic re-
 525 gion. Globally, this repeats itself until full crack arrest. However, the last two dynamic
 526 branches (orange and light red) have to be taken with caution since the measurements
 527 are influenced by the presence of rest bands. Furthermore, the seemingly separation
 528 of the last quasi-static phase (dark red) into two arms can be explained by the post-
 529 treatment done on the crack length to remove post-dynamic oscillation leading to zero
 530 velocities for a short duration. Additionally, it is worthwhile mentioning that the value of
 531 critical fracture energy (G), introducing the ‘forbidden’ region II, is 430 ± 30 J.m⁻² for all
 532 quasi-static phases. Test M2-T19 (Figure 14(b)) generally follows this description with
 533 the difference of a long quasi-static rest resulting in more data points and stronger ve-
 534 locity oscillations during this phase.

535 To conclude the results of this study, all experimental data (excluding phases with
 536 biased velocity measurements) is presented in the same (v, G) space (Figure 15). Ad-
 537 ditionally, an entirely quasi-static experiment, that has been subjected to an extension
 538 rate (0.07 mm.s⁻¹) close to the threshold of the observed appearance of alternate crack-
 539 ing, was added. To place the experimental results into the context of the literature, ex-
 540 perimental data points from (Vincent-Dospital et al., 2020) and (Scheibert et al., 2010)
 541 (quantitatively extracted using online software (Rohatgi, 2022)) were furthermore in-
 542 cluded. While our entirely quasi-static experiment seems to align well with the quasi-
 543 static experiments conducted in (Vincent-Dospital et al., 2020), alternate cracking re-
 544 sults do seem to add the missing experimental data to adequately cover the transition
 545 between quasi-static and dynamic crack propagation in the full spectrum of PMMA
 546 fracturing. Starting with the dynamic branch, experimental data points found in this
 547 study fit well the trend of existing literature. However, the last value of G (before jump-
 548 ing to the quasi-static phase) is systematically found below any G value of the quasi-

static regime, and therefore below the expected value of G_c . In detail, velocity measurements and energy release rates introducing the jump from dynamic to quasi-static propagation are found to be $236 \pm 40 \text{ J.m}^{-2}$ and $67 \pm 31 \text{ m.s}^{-1}$, respectively. During quasi-static phases, with velocities reaching a function of the macroscopic loading rate after the decelerating phase, the crack grows at almost constant speed. Observed variations during quasi-static propagation can be explained partly by physical and partly by post-dynamic effects. Avalanche values for energy release rate and velocity marking the beginning of the ‘forbidden’ region II are found to be $497 \pm 92 \text{ J.m}^{-2}$ and $0.19 \pm 0.15 \text{ m.s}^{-1}$, respectively. Taking both sides into consideration, region II, i.e. ‘forbidden’ velocity regime, is thereby found to be more narrow than presented in literature. Nevertheless, a brutal switch is seen from the rate dependent dynamic branch to the quasi-static regime with only, as transition, the deceleration distance e .

General variations in the energy release rate are found to be in line with the literature. In the quasi-static regime, the material strength increases with increasing strain rates due to the viscosity of PMMA. The resulting increase of G in the quasi-static regime is supported by the experimental data of Vincent-Dospital et al. (Vincent-Dospital et al., 2020). The increase in the energy release rate in the dynamic regime can be explained by the increase of the process zone around the crack tip, which is in line with Kanninen’s theory (Kanninen and Popelar, 1985). However, differing energy release rates for dynamic crack initiation and arrest – where the SIF at dynamic crack initiation $K1D_{ini}$ has been found to be higher than at dynamic crack arrest $K1D_a$ – have also been discussed in Grégoire et al. (Grégoire et al., 2009). Furthermore, influences on the fracture surface of the variations of the dynamic crack propagation and the increase of surface roughness due to the appearance of micro-cracking have also been discussed in the literature (Ravi-Chandar and Balzano, 1988; Grégoire, 2008; Dalmas et al., 2013). This is in line with the presented observations, where conical shapes on the fracture surface (micro-cracking) are found at the beginning of the dynamic propagation (Figure 9, detail 1) with high levels of G . At the same time, they vanish as G decreases to the point of dynamic crack propagation arrest.

5 Conclusions and perspectives

For this study, alternate cracking in PMMA has been observed by high-speed imaging and investigated with the help of DIC and post-mortem analysis. Quasi-static and dynamic crack propagations linked to this specific fracture phenomenon have been investigated by a compromised experimental setup. The setup however allowed for the investigation of the two cracking types within one fracture process. Velocity and SIF measurements obtained through DIC and Williams’ series expansion have been tied to microscopic images to highlight the regime-dependent behavior of alternate cracking. By doing so, it was possible to find relationships through the synchronization of experimental devices. Main observations with potential perspectives can be summed as following:

- ‘Forbidden’ velocity regime. The in literature described ‘forbidden’ velocity regime was not approached from two sides, but studied through holistic alternate cracking experiments with high-speed imaging. Here, the ‘forbidden’ regime has been identified by quasi-static and dynamic velocity threshold measurements of $\sim 36 \text{ cm.s}^{-1}$ and $\sim 36 \text{ m.s}^{-1}$, respectively. Notice that the limit values stem from the lowest/highest measured velocity of all observed phases.

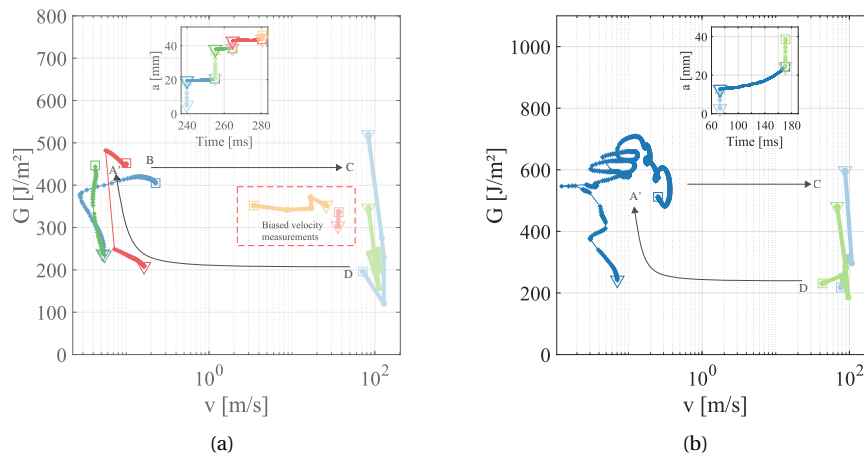


Fig. 14: Energy release rate (G) [$\text{J}\cdot\text{m}^{-2}$] as a function of velocity (v) [$\text{m}\cdot\text{s}^{-1}$] for the alternate cracking phases with its corresponding temporal evolution of the crack for: (a) M1-T2 and (b) M2-T19. Reference to the phases of the theoretical kinematic law (see Figure 1) are added for better comprehension.

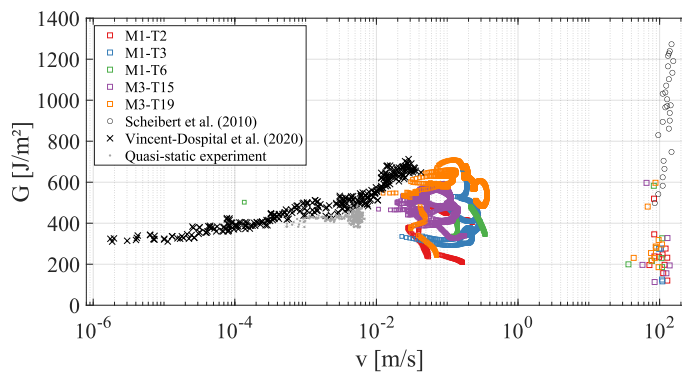


Fig. 15: Energy release rate (G) [$\text{J}\cdot\text{m}^{-2}$] as a function of velocity (v) [$\text{m}\cdot\text{s}^{-1}$] for the alternate cracking and one entirely quasi-static experiment.

- 595 – *Relationship of mean dynamic velocity with burst length.* Dynamic crack speeds and
 596 their burst lengths have been registered and used to establish a relationship. By do-
 597 ing so, it was shown that higher velocities result in longer dynamic bursts.
 598 – *Observation of rest bands.* In line with the previous point, probably for dynamic
 599 propagation phases containing lower levels of energy, dynamic bursts show an in-
 600 termittent behavior with extremely small rest bands. An average velocity threshold
 601 for the potential occurrence of rest bands has been identified at $\sim 60 \text{ m}\cdot\text{s}^{-1}$.
 602 *Perspective:* Higher temporal sampling would allow for the identification of rest
 603 band decelerations influencing the average dynamic velocity measurements and
 604 thereby the exploration of the intermittent behavior of dynamic bursts itself.

- 605 – *Deceleration phase occurring during transition of dynamic to quasi-static propaga-*
606 *tion.* The influence of dynamically propagating cracks arriving at quasi-static phase
607 on the fracture surface led to the definition of a critical deceleration distance be-
608 tween the two phases. Here, an almost linear trend is found between mean crack
609 tip speeds of the dynamic burst and the specific deceleration distance. It is found
610 between 115 and 615 μm for mean velocities ranging from 78 to 106 $\text{m}\cdot\text{s}^{-1}$. It points
611 to the fact that the crack tip undergoes a deceleration of about $10^7 \text{ m}\cdot\text{s}^{-2}$.
612 *Perspective:* The observed extreme accelerations/decelerations potentially induce
613 elastic waves that could be measured with Acoustic Emission (AE) devices. In that
614 context, recent developments in Ultra High Speed (UHS) imaging techniques (tem-
615 poral resolutions in the order of Mfps (Vinel et al., 2021)) would allow for a detailed
616 investigation of the acceleration/deceleration phases and help visualizing crack in-
617 duced waves propagation.
- 618 – *Particular surface markings.* Two particular PMMA-related fracture surface pat-
619 terns, Wallner lines and nucleation phases, have been observed. The second one,
620 only appearing at the beginning of the dynamic bursts, suggests crack speed phases
621 beyond the velocities measured during the experiments. This can be explained by
622 the limits of the chosen interframe.
623 *Perspective:* Again, by stretching the limits of the interframe and reaching higher
624 samplings during dynamic propagation, velocity variations could be explored. Sur-
625 face markings of the nucleation phase, suggesting highest crack tip speeds in the
626 beginning of dynamic bursts, could thereby be confirmed.
- 627 – *Improved sampling of the kinematic law of PMMA fracturing in vicinity of instabil-*
628 *ity.* Kinematic law for fracturing in PMMA was established by supplementing the
629 results of this work with experimental data from existing literature. Here, experi-
630 mental data of the interaction of reoccurring dynamic and quasi-static phases dur-
631 ing alternate cracking experiments was provided. For the first time, direct velocity
632 and SIF measurements have been experimentally identified for both phases dur-
633 ing the same alternate cracking experiments. Results led to a proposed adjustment
634 of the classical description of alternate cracking regarding the transition from dy-
635 namic to quasi-static propagation (path along reference points B-C-D-A' in Figure
636 2.1). On one hand, in view of the chosen extension rates to induce alternate recur-
637 rence of the transitions, different mechanisms could have been activated. On the
638 other hand, the transition of dynamic to quasi-static propagation with extreme de-
639 celerations (potentially being outside of the framework of (Vincent-Dospital et al.,
640 2020)) is not smooth after all. In addition, the path that the crack follows to revert to
641 the master curve in the quasi-static regime differs from the observations in (Vasude-
642 van, 2018). However, in his work, very slow extension rates have been used result-
643 ing in quasi-static experiments initiated by a dynamic crack. No alternate recurrence of
644 the two propagation types was observed. Regarding quasi-static to dynamic transi-
645 tions, even with a temporal resolution of 40 kfps, it was not possible to experimen-
646 tally explore the smooth transition during the 'softening' branch predicted by the
647 thermodynamic model of (Vincent-Dospital et al., 2020). This, however, could po-
648 tentially be explained by a lack of temporal resolution or the feasibility to explore
649 this regime due to the way the experiments were conducted. To do so, an experi-
650 mental configuration enabling decreasing K_I values with increasing crack velocities
651 is needed.
652 *Perspective:* An experimental configuration containing UHS imaging techniques
653 would allow for the investigation of separate dynamic propagation phases during

654 alternate cracking. Consequently, better descriptions of the transition phases could
655 be reached, potentially leading to an exploration of the predicted smooth transition,
656 i.e. the ‘softening’ regime.

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660 Declarations

661 Conflict of interest:

662 The authors declare that they have no conflict of interest.

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