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# Mean field kinetic theory

X. Garbet

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## Abstract

This note presents the theoretical framework to predict the evolution of a mean distribution function in an Hamiltonian dynamical system. Assumptions are an integrable unperturbed Hamiltonian and a scale separation between mean and perturbed distribution functions. The first result shows that the evolution of the mean distribution function, which depends on action only, can be written in a conservative form. In other words the advection contribution to the kinetic equation can be written as the divergence of a flux in the action space. Moreover fluxes can be computed via a principle of minimum entropy production rate. The special case of a single perturbation is treated first. It appears that the set of constant energy surfaces bear the shape of an island near the resonant surface in the phase space. The distribution function gets flat within the island, while strong gradients develop near its separatrix. Not surprisingly the entropy production rate is maximum in the separatrix region. The case of multiple perturbations is then addressed. A Chirikov overlap parameter is defined that measures the degree of island overlap, i.e. compares the island width with the distance between adjacent resonant surfaces. Whenever the Chirikov overlap parameter exceeds 1, trajectories become chaotic. Under some conditions a quasilinear transport theory can then be used. Finally the case where two perturbations are involved, but do not produce chaos is also investigated.

Sections labelled with a star “\*” can be skipped in a first reading.

## 1 Introduction

The aim of this note is to compute the mean distribution function of a population of charged particles in a magnetised plasma. The word “mean” is somewhat vague at this point. It refers to mean field theories, which assume that a field, here the distribution function, can be split into mean and fluctuating parts. It relies on scale separation in time, i.e. assumes that the mean part of the distribution function evolves on time scales much longer than fluctuations - also the latter are supposed to be much smaller in amplitude than the mean component. The problem at hand consists in calculating a kinetic equation that governs the distribution function for a given electromagnetic field, which can itself be split into a mean field and perturbations. In other words, the objective for now is to compute a statistical distribution of charged particles submitted to a known electromagnetic field.

## 2 Statistical equilibrium in a collisionless plasma

### 2.1 Unperturbed distribution function

Let us remind that the distribution function  $F(\mathbf{x}, \mathbf{p}, t)$  of a given species is solution of the Fokker-Planck equation

$$\frac{\partial F}{\partial t} - \{H, F\} = C[F] \quad (1)$$

where  $(\mathbf{x}, \mathbf{p})$  are the particle position-momentum coordinates,  $H$  the particle Hamiltonian,  $C[F]$  a collision operator, and  $\{ , \}$  the Poisson bracket written in  $(\mathbf{x}, \mathbf{p})$  variables

$$\{H, F\} = \frac{\partial H}{\partial \mathbf{x}} \cdot \frac{\partial F}{\partial \mathbf{p}} - \frac{\partial H}{\partial \mathbf{p}} \cdot \frac{\partial F}{\partial \mathbf{x}}$$

The collision operator  $C[F]$  is a complex object: it is quadratic in  $F$  and involves multi-species collisions. Its general form is given in Appendix A.1. However, it will be seen that a simplified form is sufficient for our purpose in the following. We look for time independent solutions of the Fokker-Planck, i.e., loosely speaking an “equilibrium”. The analysis is restricted to the case where the unperturbed Hamiltonian is integrable. It was seen in the lecture on particle motion that this situation occurs whenever 3 invariants of motion can be found [1]. Obviously, the particle energy  $H_{eq}$  is an invariant of motion if the electromagnetic field is static. If the magnetic field is strong enough, a second invariant is the particle magnetic moment  $\mu$ . Finally, if the system is left invariant by a rotational symmetry, a third invariant can be built according to the Noether theorem, for instance the canonical toroidal momentum in an axisymmetric configuration like a tokamak. If trajectories are bounded in the phase space, a set of action/angle variables  $(\boldsymbol{\alpha}, \mathbf{J})$  can be built such that the equations of motion bear a symplectic form<sup>1</sup>

$$\begin{aligned} \frac{d\alpha_i}{dt} &= \frac{\partial H_{eq}}{\partial J_i} = \Omega_i \\ \frac{dJ_i}{dt} &= -\frac{\partial H_{eq}}{\partial \alpha_i} = 0 \end{aligned}$$

where  $H_{eq}(\mathbf{J})$  is the unperturbed Hamiltonian and  $\Omega_i = \partial_{J_i} H_{eq}$  are the resonant angular frequencies. The unperturbed distribution function  $F_{eq}$  in a collisionless plasma is solution of a Vlasov equation for each species

$$\{H_{eq}, F_{eq}\} = 0$$

where the Poisson bracket is now expressed in the new conjugate variables  $(\boldsymbol{\alpha}, \mathbf{J})$

$$\{H_{eq}, F_{eq}\} = \frac{\partial H_{eq}}{\partial \boldsymbol{\alpha}} \cdot \frac{\partial F_{eq}}{\partial \mathbf{J}} - \frac{\partial H_{eq}}{\partial \mathbf{J}} \cdot \frac{\partial F_{eq}}{\partial \boldsymbol{\alpha}}$$

The variables  $\boldsymbol{\alpha}$  are angles so that the distribution function  $F_{eq}(\boldsymbol{\alpha}, \mathbf{J})$  can be expressed as a Fourier series

$$F_{eq}(\boldsymbol{\alpha}, \mathbf{J}) = \sum_{\mathbf{n}} F_{\mathbf{n}}(\mathbf{J}) \exp[i(\mathbf{n} \cdot \boldsymbol{\alpha})]$$

where  $\mathbf{n}$  designates a triplet of integers  $(n_1, n_2, n_3)$ . It then appears that the unperturbed distribution function  $F_{eq}$ , the “equilibrium” distribution function, is in fact an average of the distribution function over the angles  $\boldsymbol{\alpha}$ . Hence the word “mean” takes now a precise mathematical definition. Plugging this Fourier series in the Vlasov equation implies

$$[\mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{J})] F_{\mathbf{n}}(\mathbf{J}) = 0$$

If  $\mathbf{n} \neq \mathbf{0}$ , the solution is  $F_{\mathbf{n}}(\mathbf{J}) \sim \delta[\mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{J})]$ , where  $\delta$  is the Kronecker function. The surface in the action space such that  $\mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{J}) = 0$  is called a resonant surface. It depends uniquely on the choice of the triplet  $\mathbf{n}$ . This result implies that  $F_{\mathbf{n}}(\mathbf{J})$  vanishes everywhere, except on the resonant surface associated with the triplet  $\mathbf{n}$ . If the distribution function is smooth in the phase space, then all  $F_{\mathbf{n}}(\mathbf{J})$  are null, except  $F_{\mathbf{0}}(\mathbf{J})$ . Hence the unperturbed, or “equilibrium”, distribution function must be a function of the invariants of motion  $F_{eq}(\mathbf{J})$ . The shape of  $F_{eq}(\mathbf{J})$  is however arbitrary. This means that an infinity of

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<sup>1</sup>Strictly speaking, angles should be written in a contravariant form, and action in covariant form to ensure some coherence. We will avoid this complication since no formula of differential geometry will be needed.

equilibrium solutions exists. This finding is surprising since a single well defined statistical equilibrium is usually observed in a physical system<sup>2</sup>. The answer to this conundrum is that collisions should be added to reduce the number of possibilities down to one. This conclusion is reinforced by the assumption of smooth distribution function in the phase space to eliminate singular solutions on resonant surfaces. Adding collisions regularises the solution and wipes out singular solutions. The need for dissipation processes can be better clarified by computing the response to small perturbations.

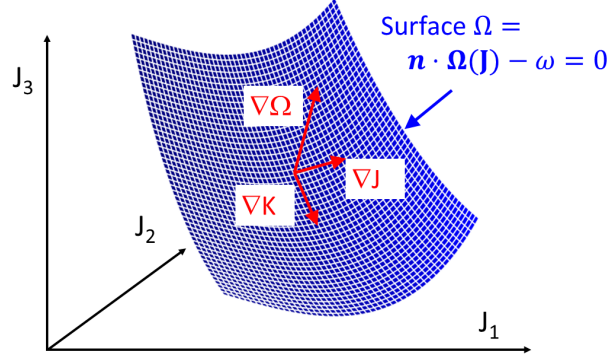


Figure 1: Resonant surface in the action space

## 2.2 Response to a small perturbation - resonances

Let us assume that the Hamiltonian is the sum of its unperturbed value  $H_{eq}(\mathbf{J})$ , plus a set of time dependent perturbations

$$\tilde{H}(\boldsymbol{\alpha}, \mathbf{J}, t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \sum_{\mathbf{n}} H_{\mathbf{n}\omega}(\mathbf{J}) \exp[i(\mathbf{n} \cdot \boldsymbol{\alpha} - \omega t)] \quad (2)$$

If the perturbed distribution function is small, the Vlasov equation can be linearised<sup>3</sup> to give an expression for each Fourier harmonic  $F_{\mathbf{n}\omega}(\mathbf{J})$

$$F_{\mathbf{n}\omega}(\mathbf{J}) = \frac{\mathbf{n} \cdot \frac{\partial F_{eq}}{\partial \mathbf{J}}}{\mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{J}) - \omega} H_{\mathbf{n}\omega}(\mathbf{J}) \quad (3)$$

This result should be marked. A kinetic response to an arbitrary perturbation that is analytic is indeed a quite remarkable property, even if linearised. The structure of this linear response is worth being commented. It contains three ingredients: a perturbed Hamiltonian, a numerator proportional to the gradient of the distribution function in the action space, and a resonant denominator. We will see that the gradient  $\partial_{\mathbf{J}} F_{eq}$  of the distribution function measures the departure from thermodynamic equilibrium as long as the distribution function does not depend uniquely on the unperturbed Hamiltonian  $H_{eq}$ . The resonant denominator deserves some attention. The linear response is infinite whenever  $\omega - \mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{J}) = 0$ . This condition defines a “resonant surface” characterised by the angular frequency  $\omega$  and the triplet  $\mathbf{n}$  - resonant surfaces associated with a static perturbation form a special case  $\mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{J}) = 0$  (see Fig.1). A resonance is met when particles stay in phase with the perturbation along their motion and thus respond in an optimum way. Indeed, the derivative in time of the phase  $\mathbf{n} \cdot \boldsymbol{\alpha} - \omega t$  along the motion

<sup>2</sup>There are exceptions of course, like bistable systems.

<sup>3</sup>Here “linearise” means keeping order 1 contribution in a small parameter that measures the relative perturbation.

of a particle is nothing else than  $\mathbf{n} \cdot \boldsymbol{\Omega} - \omega$ . Hence the resonant condition can be seen as a condition of constant phase. If a particle stays in phase with a force, the exchange of energy is optimal. This is the basic process that underlies a Landau resonance. Describing a Landau resonance is not in the scope of the present note. The reader is sent to the abundant literature on this subject [2, 3, 4, 5]. Nevertheless, an urgent matter for our

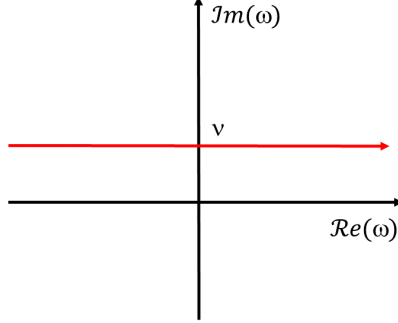


Figure 2: Integration path for an inverse Fourier-Laplace transform

purpose is to elucidate how a singular response can be handled. There exists at least 2 ways to solve the singularity across a resonance  $\omega - \mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{J}) = 0$ , with real  $\omega$ . One radical solution is to add collisions - for instance, if one adds a Krook operator<sup>4</sup>  $-\nu(F - F_{eq})$ , where  $\nu$  is a collision frequency, to the right hand side of the Vlasov equation, then the resonant denominator becomes  $\omega - \mathbf{n} \cdot \boldsymbol{\Omega} + i\nu$ . The singularity is thus resolved since the plasma response is now well defined all over the phase space. This recipe is however questionable as treating collisionless problems is desirable. In this case, the original path followed by Landau [2] is more appropriate. It can be summarised as follows. Let us continue to expand the perturbed distribution function and Hamiltonian in angles, but not in time. The Vlasov equation reads

$$\frac{\partial F_{\mathbf{n}}(\mathbf{J}, t)}{\partial t} + i\mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{J})F_{\mathbf{n}}(\mathbf{J}, t) = -i \left( \mathbf{n} \cdot \frac{\partial F_{eq}}{\partial \mathbf{J}} \right) H_{\mathbf{n}}(\mathbf{J}, t)$$

A Fourier-Laplace transform in time is then introduced instead of a conventional Fourier transform<sup>5</sup>

$$F_{\mathbf{n}\omega}(\mathbf{J}) = \int_0^{+\infty} dt F_{\mathbf{n}}(\mathbf{J}, t) e^{i\omega t}$$

This definition holds in the frequency upper half-plane  $\Im(\omega) > 0$ . The Fourier-Laplace transform of the Vlasov equation is performed using the property

$$\int_0^{+\infty} dt \frac{\partial F_{\mathbf{n}}(\mathbf{J}, t)}{\partial t} e^{i\omega t} = i\omega F_{\mathbf{n}\omega}(\mathbf{J}) - F_{\mathbf{n}}(\mathbf{J}, t=0)$$

This result is obtained via an integration by parts, and using  $F_{\mathbf{n}}(\mathbf{J}, t = +\infty) = 0$ , which is granted in the frequency upper half-plane  $\Im(\omega) > 0$ . The actual domain of integration actually depends on the asymptotic behaviour of  $F_{\mathbf{n}}(\mathbf{J}, t \rightarrow +\infty)$ . The solution of the linearised Vlasov equation now reads

$$F_{\mathbf{n}\omega}(\mathbf{J}) = \frac{1}{\mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{J}) - \omega} \left[ \mathbf{n} \cdot \frac{\partial F_{eq}}{\partial \mathbf{J}} - iF_{\mathbf{n}}(\mathbf{J}, t=0) \right] H_{\mathbf{n}\omega}(\mathbf{J}) \quad (4)$$

<sup>4</sup>Strictly speaking one should also remove the “adiabatic” response, i.e. use  $-\nu(F - F_{eq} - H_{\mathbf{n}\omega} \frac{\partial F_{eq}}{\partial E})$ . We will come back to this point

<sup>5</sup>Let us stress that a conventional Laplace transform would involve an integral over  $e^{-st}$  instead of  $e^{i\omega t}$  here, hence the name “Fourier-Laplace”. The essential difference with a conventional Fourier transform is of course the lower bound in time, 0 in Laplace in place of  $-\infty$  for Fourier.

A key step is to reconstruct the time dependent functions  $F_{\mathbf{n}}(\mathbf{J}, t)$  and  $H_{\mathbf{n}}(\mathbf{J}, t)$ , knowing their Fourier-Laplace transforms. This is done thanks to the inverse Fourier-Laplace transform

$$F_{\mathbf{n}}(\mathbf{J}, t) = \int_{-\infty+i\nu}^{+\infty+i\nu} \frac{d\omega}{2\pi} F_{\mathbf{n}\omega}(\mathbf{J}) e^{-i\omega t}$$

where  $\nu$  is a small positive number. The latter grants that the integral is computed in the definition domain of  $F_{\mathbf{n}\omega}$ , i.e. the upper half-plane  $\Im(\omega) > 0$ , see Fig.2. A similar expression rules  $H_{\mathbf{n}}(\mathbf{J}, t)$  vs  $H_{\mathbf{n}\omega}(\mathbf{J})$ . The asymptotic value of  $H_{\mathbf{n}}(\mathbf{J}, t)$  when  $t \rightarrow +\infty$  is often of interest, to get information on stability for instance. Given the definition of the inverse Fourier-Laplace transform, asymptotic values  $t \rightarrow +\infty$  are expediently calculated by deforming the contour of integration to the lower half  $\Im(\omega) < 0$  of the space of complex values of  $\omega$ . This requires an analytic continuation of  $H_{\mathbf{n}\omega}$  in the lower half-plane. In this lecture, quantities of interest involve integrals over the phase space and time, e.g.

$$\mathcal{A} = \int d^3\mathbf{J} d^3\boldsymbol{\alpha} dt F_{\mathbf{n}}(\boldsymbol{\alpha}, \mathbf{J}, t) H_{\mathbf{n}}(\boldsymbol{\alpha}, \mathbf{J}, t) = \sum_{\mathbf{n}} \int_{\mathcal{C}} \frac{d\omega}{2\pi} \int d^3\mathbf{J} H_{\mathbf{n}\omega}(\mathbf{J}) F_{-\mathbf{n}-\omega}(\mathbf{J}) \quad (5)$$

where the Parseval's theorem has been used. Examples of this type of integrals are the action functional to compute the electromagnetic field, or entropy production rates. The contour  $\mathcal{C}$  has to be chosen in the domain of convergence of  $H_{\mathbf{n}\omega}(\mathbf{J})$  and  $F_{-\mathbf{n}-\omega}(\mathbf{J})$ . Since  $(\omega, -\omega)$  with  $\Im(\omega) > 0$  covers the whole complex plane, the simplest procedure is to manipulate exclusively analytic functions in the frequency space. At first sight, this would require an analytic continuation of  $F_{\mathbf{n}\omega}$  and  $H_{\mathbf{n}\omega}$  in the frequency lower half-plane. However, an analysis of Eq.(5) combined with Eq.(4) points out the need for *integrals of  $F_{\mathbf{n}\omega}(\mathbf{J})$  over the action space*, to be continued analytically in the frequency lower half-plane. This property greatly simplifies the process of continuation since an integral over the action space smooths the singularity. The integrals of interest are of the form

$$I(\omega) = \int d^3\mathbf{J} \frac{\Lambda(\mathbf{J})}{\mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{J}) - \omega}$$

where  $\Lambda(\mathbf{J})$  is a smooth function. It is useful to replace one of the actions by a new variable  $\Omega = \mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{J})$ , better suited to resolve the singularity. The variable  $\Omega$  is complemented by two variables  $(J, K)$  that spans the resonant surface (see Fig.1). The integral  $I(\omega)$  then becomes

$$I(\omega) = \int d^2S \int_{-\infty}^{+\infty} d\Omega \frac{\Lambda(\Omega, J, K)}{\Omega - \omega}$$

where  $d^2S = dJdK/\mathcal{J}(\Omega, J, K)$ , and  $\mathcal{J}(\Omega, J, K)$  is the Jacobian of the new set of variables. The integral over the variables  $(J, K)$  does not raise any difficulty. So we focus on the difficult point, i.e. the treatment of the singularity, by analysing functions of the form

$$Z(\omega) = \int_{-\infty}^{+\infty} d\Omega \frac{\Lambda(\Omega)}{\Omega - \omega}$$

Function  $Z$  can be seen as a prototype of the integrals that are looked for. It is requested to be analytic in  $\omega$ . The regularisation is done in three steps: first choose a half-plane where the function  $Z$  behaves well without further modification, second resolve the singularity on the real axis in  $\omega$ , third perform an analytic continuation towards the other half-plane. The process can be understood from Fig.3 and is detailed as follows. Let us take the simple example of a function  $\Lambda(\Omega)$  that converges at infinity in the upper-half plane. In this case, a method of residues can be used to compute  $Z(\omega)$ <sup>6</sup>. The contour is an

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<sup>6</sup>Note that the function  $\Lambda(\Omega)$  may not satisfy the conditions for ignoring the integral on a outer circle at infinity. A classical counter-example is  $\Lambda(\Omega) = \frac{1}{\sqrt{\pi}} e^{-\Omega^2}$  which generates a  $Z$  function called the plasma dispersion function, or Fried and Conte function.  $\Lambda(\Omega)$  does not converge to 0 in the left-upper quadrant. In

integration line along the real axis in  $\Omega$ , from  $-\infty$  to  $+\infty$ , complemented by a half-circle at infinity. If the pole  $\Omega = \omega$  is in the upper half-plane,  $\Im(\omega) > 0$  (panel a), the integral is well resolved - no special care is required provided that  $\Lambda(\Omega)$  decreases at least as  $1/\Omega$  at infinity. This is the situation when dealing with the Fourier-Laplace transform. If the pole is in the lower half-plane  $\Im(\omega) < 0$ , and the integration contour still the same, a difficulty occurs since the pole has now crossed the real line so that the value of  $Z(\omega)$  is shifted by  $-2i\pi\Lambda(\omega)$  - this means that  $Z(\omega)$  is discontinuous. This unwelcome difficulty is overcome by deforming the integration contour as shown in Fig.3.c, which consists in adding  $2i\pi\Lambda(\omega)$  times the residue to the principal part<sup>7</sup>. The case where the pole is just below the real axis is special, since the jump would be  $-i\pi\Lambda(\omega)$  when the pole barely crosses the real axis. The contour is deformed accordingly, as shown in Fig.3.b. Finally the analytic continuation of  $Z(\omega)$  reads

$$Z(\omega) = \begin{cases} \int_{-\infty}^{+\infty} d\Omega \frac{\Lambda(\Omega)}{\Omega - \omega} & \text{if } \Im(\omega) > 0 \\ P.P. \int_{-\infty}^{+\infty} d\Omega \frac{\Lambda(\Omega)}{\Omega - \omega} + i\pi\Lambda(\omega) & \text{if } \Im(\omega) = 0 \\ \int_{-\infty}^{+\infty} d\Omega \frac{\Lambda(\Omega)}{\Omega - \omega} + 2i\pi\Lambda(\omega) & \text{if } \Im(\omega) < 0 \end{cases}$$

where it is reminded that a principal part, noted P.P., is calculated as follows

$$P.P. \int_{-\infty}^{+\infty} d\Omega \frac{\Lambda(\Omega)}{\Omega - \omega} = \lim_{\epsilon \rightarrow 0^+} \left[ \int_{-\infty}^{\omega - \epsilon} d\Omega \frac{\Lambda(\Omega)}{\Omega - \omega} + \int_{\omega + \epsilon}^{+\infty} d\Omega \frac{\Lambda(\Omega)}{\Omega - \omega} \right]$$

This choice is consistent with the Plemelj formula

$$\lim_{\nu \rightarrow 0^+} \frac{\Lambda(\Omega)}{\Omega - \omega - i\nu} = P.P. \frac{\Lambda(\Omega)}{\Omega - \omega} + i\pi\delta(\Omega - \omega)$$

In the following we will use massively the Fourier-Laplace transform. Whenever expressions of the form Eq.(3) are used, the frequency  $\omega$  will be noted  $\omega + i\nu$  to recall the above treatment of integrals.

## 3 Transport equations

### 3.1 Thermodynamic potential

The next step is to write an evolution equation over the mean distribution function  $F_{eq}$ . The analysis is greatly simplified by using an extended phase space. Let us first introduce an angular variable  $\alpha_0 = \Omega_0 t + \alpha_{00}$ , where  $\alpha_{00}$  is an initial condition,  $\Omega_0$  the lowest frequency of fluctuations, and  $J_0$  an action conjugate to  $\alpha_0$ . The Hamiltonian character of the dynamics is preserved by using a unperturbed Hamiltonian  $H_{eq} + J_0\Omega_0$ <sup>8</sup>. In practice  $J_0$

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this case the contour must be deformed accordingly, or another technique be used. A powerful method consists in using the relationship  $\frac{1}{\Omega - \omega} = i \int_0^{+\infty} d\sigma e^{-i\sigma(\Omega - \omega)}$ , valid for  $\Im(\omega) > 0$ . In the special case of the plasma dispersion function, a few manipulations lead to the relationship  $Z(\omega) = i\sqrt{\pi}e^{-\omega^2}\text{erfc}(-i\omega)$ . The function  $\text{erfc}(-i\omega)$  is analytical over the whole complex plane in  $\omega$ , so that there is no further need to worry about the singularity!

<sup>7</sup>Perhaps this point can be clarified in the following way. When the pole  $\omega$  gets near the real axis, it reads  $\omega = \omega_r + i\gamma$ , where  $\omega_r$  is the real part of  $\omega$ , and  $\gamma$  its positive imaginary part. The singularity can be isolated by using the relation  $\frac{1}{\Omega - \omega} = \frac{\Omega - \omega_r + i\gamma}{(\Omega - \omega_r)^2 + \gamma^2}$ . When  $\gamma \ll \omega_r$ , the imaginary part is close a Kronecker  $\delta$  function  $+i\pi\delta(\Omega - \omega_r)$ . When  $\omega$  crosses the real axis, than  $\omega = \omega_r - i\gamma$ , with  $\gamma > 0$ . All the formulas are the same, with  $\gamma$  changed in  $-\gamma$ . Hence the imaginary part of  $\frac{1}{\Omega - \omega}$  gets close to  $-i\pi\delta(\Omega - \omega_r)$ . Hence the real part of  $Z$  remains unchanged, while the imaginary part jumps by  $-2i\pi\delta(\Omega - \omega_r)\Lambda(\omega_r)$ . Therefore  $Z(\omega)$  is discontinuous unless a corrective factor  $+2i\pi\delta(\Omega - \omega_r)\Lambda(\omega_r)$  is added to compensate this jump. If one stops the process right at the real axis, the jump is just  $-i\pi\delta(\Omega - \omega_r)\Lambda(\omega_r)$ .

<sup>8</sup>Strictly speaking, other notations should be used in the spirit of what was done in the note on gyrokinetics. For the sake of simplicity, we will keep the same notation in the 8D extended phase space.

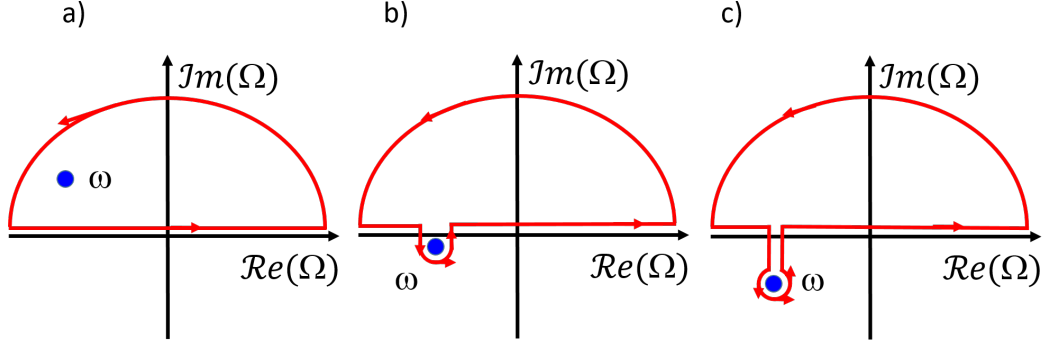


Figure 3: Analytic continuation of a resonant function  $\int_{-\infty}^{+\infty} d\Omega \frac{\Lambda(\Omega)}{\Omega - \omega}$ . The integration contour is shown in red. Panel a):  $\omega$  is in the upper half-plane - the resonant function is well defined. Panel b)  $\omega$  is just below the real axis: the integration contour must be deformed to incorporate the pole  $\Omega = \omega$  - a phase shift  $-i\pi$  appears that must be compensated to ensure analyticity. b)  $\omega$  is well below the real axis: the phase shift is  $-2i\pi$  and must be compensated accordingly (inspired from references [3, 5]).

appears as a dummy variable that disappears at the end of the calculation. This trick allows treating time on the same footing as the other angles. The Hamiltonian equations read

$$\begin{aligned} \frac{d\alpha_i}{dt} &= \frac{\partial H_{eq}}{\partial J_i} = \Omega_i \\ \frac{dJ_i}{dt} &= -\frac{\partial H_{eq}}{\partial \alpha_i} = 0 \end{aligned}$$

where now the index  $i$  runs from 0 to 3. Of course time does not play a symmetrical role as the other variables, because of the resonant character of the perturbed distribution function. Nevertheless we know how to solve this difficulty thanks to the Landau methodology.

We would like to allow collisions in the analysis. The full collision operator in a plasma is intricate, in particular in the multi-species case. For now we simplify the problem by considering one species only, and replace the full collision operator, which is quadratic in  $F$ , by a linearised version noted  $C[F]$ . Thermodynamic equilibrium of an isolated plasma in contact with a thermostat at temperature  $T_0$  is associated with a mean distribution function of the form  $\exp(-H_{eq}/T_0)$ , where  $H_{eq}$  is the mean Hamiltonian in the extended phase space. This reference distribution must be in the kernel of  $C[F]$ , i.e.  $C[\exp(-H_{eq}/T_0)] = 0$ , since collisions enforce a relaxation of the plasma towards a thermodynamic equilibrium. Of course plasmas of interest are not in full thermodynamic equilibrium - in particular they are inhomogeneous, and moving. Solutions of  $C[F_M] = 0$  are local Maxwellians [6]

$$F_M(\mathbf{x}, \mathbf{v}) = N_{eq}(\mathbf{x}) \left( \frac{m}{2\pi T_{eq}(\mathbf{x})} \right)^{3/2} \exp \left\{ -\frac{m(\mathbf{v} - \mathbf{V}_{eq}(\mathbf{x}))^2}{2T_{eq}(\mathbf{x})} \right\}$$

where  $N_{eq}$ ,  $T_{eq}$ , and  $\mathbf{V}_{eq}$  are respectively the density, temperature and velocity. However a local Maxwellian is usually not a solution of the Fokker-Planck equation that rules the distribution function  $F^9$

$$-\{H_{eq}, F\} = C[F] \quad (6)$$

Indeed the left-hand side of Eq.(6) does not generally satisfy  $\{H_{eq}, F_M\} = 0$ , while the r.h.s. vanishes,  $C[F_M] = 0$ . As seen before, in absence of fluctuations, the solution  $F_{eq}$  of  $\{H_{eq}, F_{eq}\} = 0$  is a function of the invariants of motion. It can thus be anticipated

<sup>9</sup>Note that the distribution function  $F$  does not depend explicitly on “time”  $t$  since time dependencies are already contained in the angle variable  $\alpha_0 = \Omega_0 t + \alpha_{00}$ .



that the mean distribution function is some compromise between a local Maxwellian and a function of the actions.

Studying the processes that lead to a relaxation of the mean distribution function

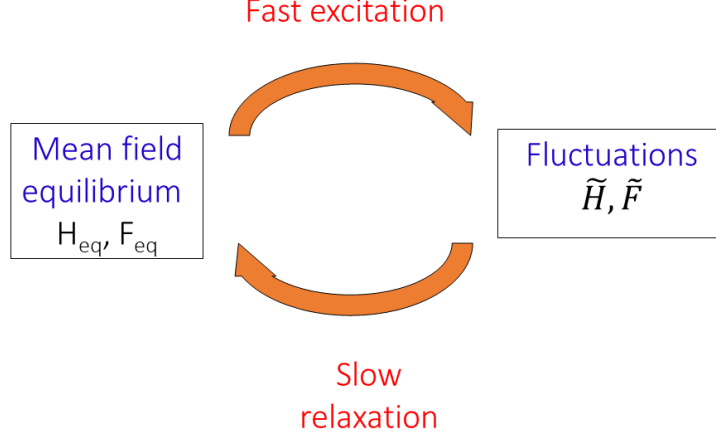


Figure 4: Schematic rationale of a kinetic mean field theory.

towards its equilibrium value is the main objective of this lecture note. Hence a slow time variation of the mean fields must be allowed. This is done by writing the mean distribution function as  $F_{eq}(\mathbf{J}, \epsilon_D t)$ , where  $\epsilon_D \ll 1$ , in spirit of a multi-scale approach. In the following, the parameter  $\epsilon_D$  will not appear explicitly, but one should keep in mind that explicit derivatives of  $F_{eq}$  with time are small compared with the characteristic time scale of fluctuations. If  $\tau_c$  denotes the fluctuation typical time scale, and  $\tau_D$  the slow relaxation time evolution of the mean distribution function, a mean field approach is appropriate whenever  $\tau_c \ll \tau_D$ . In view of the discussion above, it makes sense to look for a distribution function of the form

$$F_{eq} = \exp \left\{ -\frac{H_{eq} - U_{eq}}{T_0} \right\}$$

where  $U_{eq}$  is a thermodynamic potential. The ratio  $U_{eq}/T_0$  measures the distance to a local thermodynamic equilibrium. We assume now that the system is close to thermodynamic equilibrium, i.e.  $U_{eq}/T_0 \ll 1$ .

### 3.2 Mean field theory

Both thermodynamic potential and Hamiltonian are split into mean and perturbed parts

$$\begin{aligned} F(\boldsymbol{\alpha}, \mathbf{J}, t) &= F_{eq}(\mathbf{J}, t) + \tilde{F}(\boldsymbol{\alpha}, \mathbf{J}) \\ H(\boldsymbol{\alpha}, \mathbf{J}) &= H_{eq}(\mathbf{J}) + \tilde{H}(\boldsymbol{\alpha}, \mathbf{J}) \end{aligned}$$

The collision operator  $\mathcal{C}[F]$  is linearised in  $F$  near a Maxwellian. A rough justification is that collisions regularise fast variations in velocity space over a collision time, and thus maintain the distribution function close to a Maxwellian. This approximation has to be verified a posteriori depending on the problem at hand and the definition of the model of collision operator that is chosen. The equations over  $F_{eq}$  and  $\tilde{F}$  are then readily obtained

$$\frac{\partial F_{eq}}{\partial t} - \left\langle \left\{ \tilde{H}, \tilde{F} \right\} \right\rangle = \mathcal{C}[F_{eq}] \quad (7)$$

$$-\left\{ H, \tilde{F} \right\} - \mathcal{C}[\tilde{F}] = \left\{ \tilde{H}, F_{eq} \right\} \quad (8)$$

where the brackets denote an average over the angles  $\alpha$ . Let us note that a term  $-\langle \{\tilde{H}, \tilde{F}\} \rangle$  was neglected in Eq.(8). It also appears in the evolution equation over  $F_{eq}$  Eq.(7). The reason why it is kept in  $F_{eq}$  Eq.(7), and not in Eq.(8), is that it is responsible for the relaxation of  $F_{eq}$ , i.e. of the order of  $F_{eq}/\tau_D$ , hence smaller than  $\tilde{F}/\tau_c$ , provided that

$$\frac{\tau_c}{\tau_D} \ll \frac{\tilde{F}}{F_{eq}}$$

This condition has to be verified a posteriori. In fact, the term  $-\langle \{\tilde{H}, \tilde{F}\} \rangle$  is the divergence of a flux  $\mathbf{\Gamma}$ , i.e.

$$\nabla_{\mathbf{J}} \cdot \mathbf{\Gamma} = \frac{\partial \Gamma_i}{\partial J_i} = -\langle \{\tilde{H}, \tilde{F}\} \rangle$$

When the direct effect of collisions on  $F_{eq}$  is small, the equation eq.(7) then bears a conservative form

$$\frac{\partial F_{eq}}{\partial t} + \nabla_{\mathbf{J}} \cdot \mathbf{\Gamma} = 0 \quad (9)$$

where

$$\Gamma_i = \int \frac{d^4 \alpha}{(2\pi)^4} \left( -\frac{\partial \tilde{H}}{\partial \alpha_i} \right) \tilde{F} \quad (10)$$

The set of equations Eqs.(7, 8) is in line with the rationale of a mean field theory: the unperturbed system drives fluctuations because the plasma is not at full thermodynamic equilibrium - this excitation occurs on fast time scales. Fluctuations produce in turn a slow relaxation of the distribution function towards its equilibrium value, see Fig.4. A key question is then to relate the flux  $\mathbf{\Gamma}$  to gradients of the mean distribution  $F_{eq}$ , which measure the distance from equilibrium. This is called a constitutive relation. If a constitutive relation can be built, the equation that rules  $F_{eq}$  is autonomous. It is called a transport equation. A simple constitutive equation is of diffusive type, viz.

$$\Gamma_i = -D_{ik} \frac{\partial F_{eq}}{\partial J_k} \quad (11)$$

This relation is in fact a generalisation of Fick's and Fourier's laws<sup>10</sup>. The procedure that consists in splitting thermodynamic fields into a mean and fluctuation parts is classic and abundantly commented in textbooks [7, 8, 9]. The construction of a mean field via projectors is called sometimes Zwanzig-Mori approach, after their seminal papers [10, 11] (see also [12] for the relationship with plasma physics).

### 3.3 Entropy production and variational principle

#### 3.3.1 Definition of entropy and entropy production rate

This methodology allows computing an entropy production rate that is minimum at equilibrium. This is the basis for a more general variational principle, associated with transport equations as described in the previous section. The advantages of a variational

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<sup>10</sup>Fick's law relates a particle flux to a density gradient. It is a special case of Eq.(11). Indeed let us consider an isothermal plasma with no flow, so that  $F_{eq} = N_{eq} \exp(-H/T_0)$ . Let us also consider the case of a plasma submitted to a uniform magnetic field  $B_0$  directed along  $z$  - we use a set of Cartesian coordinates  $(x, y, z)$ . The density  $N_{eq}$  is supposed to depend on  $x$  only. The vector potential can be chosen as  $\mathbf{A}_{eq} = B_0 x \mathbf{e}_y$ . The canonical momentum  $p_y = eB_0 x + mv_y$  is conserved since there is no explicit dependence of the particle Lagrangian on the  $y$  coordinate. The action  $J$  is equal to the momentum  $p_y$  up to a spatial period  $L_y$  in the  $y$  direction supposed periodic - more precisely  $2\pi J = \oint p_y dy = p_y L_y$ . If the magnetic field  $B_0$  is strong enough so that  $eB_0 x \gg mv_y$ , one has  $J \simeq eB_0 x L_y / 2\pi$ . The Eq.(7) combined with Eq.(11) restricted to the sole variation along the action  $J$  can then be seen as a diffusion equation in the  $x$  direction on the density  $N_{eq}$ , similar to the Fick's equation. A similar exercise can be done for the temperature by relaxing the isothermal assumption.

principle are numerous. It is usually more compact than the whole set of transport equations - quite a number in a multi-species plasma. Its variational character offers a recipe to compute a reduced set of transport equations. In the case of transport equations, it appears that it can be used to demonstrate Onsager symmetry relations. In the simplest case of diffusive constitutive relations Eq.(11), this implies that the matrix  $D_{ik}$  is symmetric. The entropy is defined here in its conventional form, i.e.

$$S = - \int d\gamma F \ln F$$

where  $d\gamma = d^4\alpha d^4\mathbf{J}$  is the element of volume in the phase space, and  $F$  the distribution function of the considered species, which depends on  $(\alpha, \mathbf{J}, t)$ . For a multi-species plasma, the total entropy is the sum of entropies over all species. For a fixed total number of particles and energy in the plasma, the entropy production rate reads

$$\dot{S} = - \int d\gamma \frac{\partial F}{\partial t} \ln F$$

The entropy production rate is exactly zero in absence of collisions. This is an important property, which results from the cancellation of the average of a Poisson bracket over the phase space, here  $\{H, s\}$ , with  $s = F \ln F - F$ , provided that  $\nabla \Sigma \cdot \frac{\partial s}{\partial \mathbf{J}} = 0$ , where  $\Sigma(\mathbf{J}) = 0$  defines the plasma boundary in the action space. This is equivalent to assuming that the system is isolated (no flux of entropy through the plasma boundary). Hence the important conclusion that the total entropy increases because of collisions only. Let us write the full distribution in the form

$$F = \exp \left( - \frac{H - U}{T_0} \right) \quad (12)$$

where  $U$  is a generalisation of the thermodynamic potential that was introduced earlier. The entropy production rate can be recast as

$$\dot{S} = - \frac{1}{T_0} \int d\gamma \frac{\partial F}{\partial t} U \quad (13)$$

The contribution  $\partial_t FH$  has been ignored. This is a consequence of the identity

$$H \frac{\partial F}{\partial t} = \frac{1}{2} \{H^2, F\} = HC[F]$$

that comes from the Fokker-Planck equation Eq.(1). As mentioned before, the average of a Poisson bracket over the whole phase space vanishes with appropriate boundary conditions. Besides the collision operator conserves the Hamiltonian<sup>11</sup>.

### 3.3.2 A mean field entropy production rate

Let us consider the implications for a mean field theory. The distribution function  $F$  and the potential  $U$  are expanded as sums of unperturbed and perturbed contributions. Hence the entropy production rate Eq.(13) can be split into contributions associated with mean and perturbed parts of the thermodynamic potential, keeping in mind that all perturbed quantities average out to zero because of the integration over the angle variables. Let us focus on the entropy associated with the unperturbed distribution function

$$\dot{S}_{eq} = - \frac{1}{T_0} \int d\gamma \frac{\partial F_{eq}}{\partial t} U_{eq}$$

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<sup>11</sup>Strictly speaking it conserves the total energy, i.e. once summed over species. We will assume for now equal temperatures for all species. If it is not the case, it suffices to add an equipartition term in the transport equations.

where

$$F_{eq} = \exp \left[ -\frac{1}{T_0} (H_{eq} - U_{eq}) \right]$$

It reads as well

$$\dot{S}_{eq} = -\frac{1}{T_0^2} \int d\gamma F_{eq} \frac{\partial U_{eq}}{\partial t} U_{eq}$$

since the unperturbed Hamiltonian does not depend on the slow time scale  $t$ . Using Eq.(7), it appears that  $\dot{S}_{eq}$  contains two contributions that come respectively from transport and collisions

$$\dot{S}_{eq} = \dot{S}_{res} + \dot{S}_{coll}$$

where

$$\dot{S}_{res} = -\frac{1}{T_0^2} \int d\gamma \mathbf{\Gamma}_U \cdot \frac{\partial U_{eq}}{\partial \mathbf{J}} \quad (14)$$

$$\dot{S}_{coll} = -\frac{1}{T_0^2} \int d\gamma F_{eq} U_{eq} \mathcal{C}[U_{eq}] \quad (15)$$

where an integration by parts has been performed, assuming a thermodynamically insulated plasma<sup>12</sup>. The flux  $\mathbf{\Gamma}_U$  is defined as

$$\mathbf{\Gamma}_U = T_0 \mathbf{\Gamma}$$

and the collision operator is written formally

$$\mathcal{C}[U_{eq}] = \frac{T_0}{F_{eq}} C[F_{eq}]$$

A justification of this form comes from the general expression of the linearised Boltzmann collision operator, which is detailed in Appendix A.1. The label “res” refers to the resonant character of the perturbed distribution function in the limit of weak collisions, while  $\dot{S}_{coll}$  is the entropy production rate directly related to collisions. This terminology is somewhat stretched since collisions also participate in the resonant contribution via resonance broadening. In some cases  $\dot{S}_{res}$  can also be seen as collisional transport enhanced by resonances. This is the essence of “neoclassical” transport in fusion devices. Let us consider the generic case where the flux is linearly related to the gradient of the thermodynamic potential via the diffusive relation Eq.(11). The resonant entropy production rate then becomes quadratic in  $\partial_{\mathbf{J}} U_{eq}$

$$\dot{S}_{res}[U_{eq}, \partial_{\mathbf{J}} U_{eq}] = \frac{1}{T_0^2} \int d\gamma F_{eq} \frac{\partial U_{eq}}{\partial \mathbf{J}} \cdot \mathbf{D} \cdot \frac{\partial U_{eq}}{\partial \mathbf{J}} \quad (16)$$

where  $\mathbf{D}$  is the diffusion matrix. In the simple case of a diagonal matrix,  $\dot{S}_{res}$  is always positive since a diagonal diffusion coefficient is positive, except in some special situations. One may then argue that the entropy production rate associated with the mean thermodynamic potential should be positive. There is no general rigorous proof of this property, which should be demonstrated for each case of interest<sup>13</sup>. If the mean thermodynamic potential is eligible to the second principle, the diffusion matrix should grant a positive  $\dot{S}_{res}$ . In other words  $\mathbf{D}$  should be definite-positive<sup>14</sup>. This raises an apparent paradox. It was seen that entropy is conserved in absence of collisions. How could it be that the resonant entropy production is positive? The response is that the entropy production associated

<sup>12</sup> $\mathbf{\Gamma} \cdot \nabla \Sigma = 0$  on the plasma boundary  $\Sigma(\mathbf{J}) = 0$ , since  $\nabla \Sigma$  is a vector normal to the enclosing surface.

<sup>13</sup>In fact the anti-hermitian character of the Poisson bracket in the Vlasov equation prevents deriving a general result. Mean field theory is a notable exception, under the restrictions that are described in this note.

<sup>14</sup>Since the operator  $\Lambda$  is requested to be self-adjoint, the matrix  $\mathbf{D}$  is symmetric, a property sometimes satisfied, related to Onsager symmetries. A sufficient condition for definite-positiveness is  $D_{ii} > \sum_{j \neq i} |D_{ij}|$ .

with  $\tilde{U}$  decreases in time. Landau resonances indeed produces smaller and smaller scales in angle variables which are wiped out by the averaging procedure that defines  $U_{eq}$ . In contrast, the increasing complexity in  $\tilde{F}$  leads to a decrease of the corresponding entropy. This is equivalent to a coarse-graining procedure. This subtle point is discussed in [13]. The general properties of a collision operator grant a positive entropy production as long as the operator is of the Boltzmann-type. Also this operator is self-adjoint, a property that will often be used. If the properties of the operator that relates the flux  $\mathbf{\Gamma}$  to the gradient of the thermodynamical potential  $\partial_{\mathbf{J}}U_{eq}$  are the same as those of the collision operator  $\mathcal{C}$ , then properties of the entropy production functional will be the same as the case with collisions only. In particular, the mean field entropy production is positive, viz.  $\dot{S}_{eq} > 0$ . Equilibrium is reached whenever  $\dot{S}_{eq} = 0$ , i.e. when the entropy production reaches a minimum. This is called a principle of minimum of entropy production. In practice it can be used to compute  $U_{eq}$  by requesting that the functional  $\dot{S}_{eq}[U_{eq}, \partial_{\mathbf{J}}U_{eq}]$  exhibits a minimum with respect to  $\partial_{\mathbf{J}}U_{eq}$  i.e.<sup>15</sup>

$$\frac{\delta \dot{S}_{eq}}{\delta(\partial_{\mathbf{J}}U_{eq})} = 0$$

This result relies unfortunately on severe assumptions: proximity to thermodynamic equilibrium, self-adjointness of the constitutive relations, positiveness of the resonant entropy production rate. Nevertheless it turns out that it applies to many practical situations, and is thus useful. We now consider a more general variational principle, that contains a principle of minimum entropy production.

### 3.3.3 Entropy variational principle

The minimum of entropy production provides the steady-state value of the mean field potential  $U_{eq}$ . However it does not provide information on its (slow) evolution. To this aim a functional  $\mathcal{S}(U_{eq}, U_{eq}^\dagger)$  is built and reads [14, 15, 16]

$$\mathcal{S} = \mathcal{S}_t + \mathcal{S}_{res} + \mathcal{S}_{coll}$$

where

$$\mathcal{S}_t(U_{eq}, U_{eq}^\dagger) = \frac{2}{T_0^2} \int d\gamma F_{eq} U_{eq}^\dagger \frac{\partial U_{eq}}{\partial t} \quad (17)$$

$$\mathcal{S}_{res}(U_{eq}, U_{eq}^\dagger) = -\frac{2}{T_0^2} \int d\gamma \frac{\partial U_{eq}^\dagger}{\partial \mathbf{J}} \cdot \mathbf{\Gamma}_U \quad (18)$$

$$\mathcal{S}_{coll}(U_{eq}, U_{eq}^\dagger) = -\frac{2}{T_0^2} \int d\gamma F_{eq} U_{eq}^\dagger \mathcal{C}[U_{eq}] \quad (19)$$

The extremum of the functional  $\mathcal{S}$  for all variations of  $U_{eq}^\dagger$  near  $U_{eq}$  yields the transport equation Eq.(7). This is readily demonstrated by writing  $U_{eq}^\dagger = U_{eq} + \delta U_{eq}^\dagger$  and recasting the functional variation  $\delta \mathcal{S}$  in the form

$$\delta \mathcal{S} = \frac{2}{T_0^2} \int d\gamma \delta U_{eq}^\dagger \left( \frac{\partial F_{eq}}{\partial t} + \nabla_{\mathbf{J}} \cdot \mathbf{\Gamma}_U - \mathcal{C}[U_{eq}] \right)$$

where an integration by parts has been performed using the boundary conditions aforementioned. Stating  $\delta \mathcal{S} = 0$  for all  $\delta U_{eq}^\dagger$  is equivalent to Eq.(7). This procedure is sometimes called “weak formulation” of Eq.(7), keeping in mind Eqs.(9,11). In the frequent case where the flux  $\mathbf{\Gamma}$  is diffusive Eq.(11), and the collision operator  $\mathcal{C}$  is self-adjoint in

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<sup>15</sup>  $\frac{\delta \dot{S}_{eq}}{\delta \partial_{\mathbf{J}}U_{eq}}$  is a functional derivative, already met in gyrokinetics. It is performed like a conventional derivative, i.e. by adding an increment  $\delta \partial_{\mathbf{J}}U_{eq}$  to  $\partial_{\mathbf{J}}U_{eq}$ .

the velocity space, a convenient formulation of the functionals  $\mathcal{S}_{res}$  and  $\mathcal{S}_{coll}$  can be found, namely

$$\mathcal{S}_{res}(U_{eq}, \partial_{\mathbf{J}} U_{eq}^\dagger) = -\frac{1}{T_0^2} \int d\gamma F_{eq} \frac{\partial U_{eq}^\dagger}{\partial \mathbf{J}} \cdot \mathbf{D} \cdot \frac{\partial U_{eq}^\dagger}{\partial \mathbf{J}} \quad (20)$$

$$\mathcal{S}_{coll}(U_{eq}, \partial_{\mathbf{J}} U_{eq}^\dagger) = -\frac{1}{T_0^2} \int d\gamma F_{eq} U_{eq}^\dagger \mathcal{C}[U_{eq}^\dagger] \quad (21)$$

The notation  $\partial_{\mathbf{J}} U_{eq}^\dagger$  in the functional indicates that  $U_{eq}^\dagger$  appears only via its gradient in the action space - the latter measures the distance to thermodynamic equilibrium. Note the pre-factor 2 has become unity in both  $\mathcal{S}_{res}$ ,  $\mathcal{S}_{coll}$  by virtue of the self-adjointness of the corresponding operators. The functional  $\mathcal{S}[U_{eq}, \partial_{\mathbf{J}} U_{eq}^\dagger]$  is equal to zero when  $U_{eq}^\dagger = U_{eq}$ , i.e. is solution of the transport equation Eq.(7). It then appears that  $\mathcal{S}_{res} + \mathcal{S}_{coll} = \dot{S}_{eq}$ , i.e. the entropy function is equal to the entropy production rate, a convenient property. Hence for time-independent potentials  $U_{eq}$ , finding an extremum of  $\mathcal{S}$  is equivalent to finding a minimum of entropy production.

## 4 Non linear equilibrium for a single perturbation

The case of a single perturbation is worth being analysed in detail, to see how the resonant singularity is non linearly resolved.

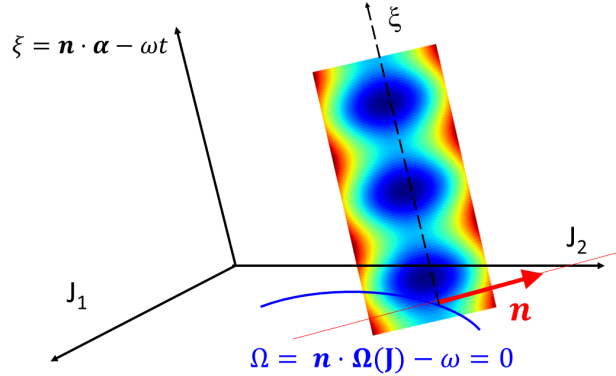


Figure 5: Schematic view of an island in the phase space. The action space is reduced to a 2D space  $(J_1, J_2)$  and the angle space to the island phase  $\xi = \mathbf{n} \cdot \boldsymbol{\alpha} - \omega t$ . In that case, the resonant surface  $\Omega = \mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{J}) - \omega = 0$  reduces to a curve. The island displacement  $I$  is directed along the vector  $\mathbf{n}$ .

### 4.1 Island in the phase space

#### 4.1.1 Reduced dynamics

Let us consider a single perturbation

$$\tilde{H}(\boldsymbol{\alpha}, \mathbf{J}, t) = -h(\mathbf{J}) \cos[(\mathbf{n} \cdot \boldsymbol{\alpha} - \omega t)]$$

To simplify  $h(\mathbf{J})$  is supposed constant. Actions evolve as follows

$$\frac{dJ_i}{dt} = -\frac{\partial \tilde{H}}{\partial \alpha_i} = -h n_i \sin \xi \quad (22)$$

where  $\xi = \mathbf{n} \cdot \boldsymbol{\alpha} - \omega t$ . The phase  $\xi$  is ruled by the equation

$$\frac{d\xi}{dt} = n_i \frac{\partial \tilde{H}}{\partial J_i} = \Omega(\mathbf{J})$$

where  $\Omega = \mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{J}) - \omega$ . The dynamics is scrutinised near the resonant surface  $\Omega = 0$ . The resonant surface is labelled by two variables  $(J, K)$ . An approximate solution of Eq.(22) is

$$\mathbf{J} = \mathbf{J}_R(J, K) + I\mathbf{n} \quad (23)$$

with

$$\frac{dI}{dt} = -h \sin \xi$$

and  $\mathbf{J}_R$  an action vector that lies on the resonant surface, hence  $\Omega(\mathbf{J}_R) = 0$ . The frequency  $\Omega$  can be Taylor expanded near  $\mathbf{J}_R$  as  $\Omega(\mathbf{J}) = C(\mathbf{J}_R)I$ , where

$$C = n_i n_j \left. \frac{\partial^2 H_{eq}}{\partial J_i \partial J_j} \right|_{\mathbf{J}=\mathbf{J}_R} \quad (24)$$

The scalar  $C$  is called here, somewhat abusively, Hamiltonian “curvature”. The evolution equation of the phase is then simply

$$\frac{d\xi}{dt} = CI$$

A new invariant of motion  $H_I$ , an Hamiltonian in fact<sup>16</sup>, is built, namely

$$H_I = \frac{1}{2}CI^2 - h \cos \xi$$

A slightly different reformulation is obtained by introducing a frequency  $\omega_b = (Ch)^{1/2}$ , and a normalised Hamiltonian  $H_\Omega = CH_I = \frac{1}{2}\Omega^2 - \omega_b^2 \cos(\xi)$  that is homogeneous to the square of a frequency. The total Hamiltonian is therefore  $H = H(\mathbf{J}_R) + \frac{H_\Omega}{C}$ . The equations of motion for the coordinates  $(\xi, \Omega)$  read

$$\begin{aligned} \frac{d\xi}{dt} &= \frac{\partial H_\Omega}{\partial I} = \Omega \\ \frac{d\Omega}{dt} &= -\frac{\partial H_\Omega}{\partial \xi} = -\omega_b^2 \sin \xi \end{aligned}$$

This formulation bears this advantage that quantities of interest  $(\Omega, \omega_b)$  are frequencies. The frequency  $\omega_b$  provides a natural frequency unit. Its meaning is elucidated in the following.

#### 4.1.2 Island characteristics

The Hamiltonian  $H_\Omega$  is similar to the one found for a pendulum. The lines of constant energy  $H_\Omega$  draw the usual shape of an island (or cat-eye) in the phase space (see Fig.5 and Fig.6). This kind of Hamiltonian was met when analysing the trajectories of charged particles in a tokamak. Hence the analysis is the same, and the main results are summarised here (more details can be found in the Appendix B). Particles are “trapped” whenever  $-\omega_b^2 \leq H_\Omega \leq \omega_b^2$ , and are passing when  $\omega_b^2 \leq H_\Omega \leq +\infty$ . Trapped particles move back and forth between two angle positions  $-\xi_b$  and  $\xi_b$  (modulo  $2\pi$ ) such that  $H_\Omega = -\omega_b^2 \cos \xi_b$ . In contrast, passing particles span the whole  $\xi$  domain. The two domains are isolated by a separatrix defined in the space  $(\xi, \Omega)$  by the condition  $H_\Omega = \omega_b^2$ . The separatrix equation is thus  $\Omega = \pm 2\omega_b \cos(\xi/2)$ . The island half-width is therefore  $\delta_\Omega = 2\omega_b$  in frequency unit, and  $\delta_I = 2\sqrt{h/C}$  in action units.

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<sup>16</sup>It is an Hamiltonian because the equations of motion in  $(\xi, I)$  bear a symplectic form.

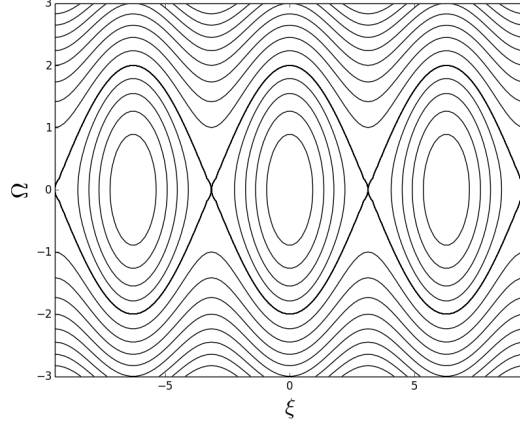


Figure 6: Contour lines of the Hamiltonian  $H_\Omega = \frac{1}{2}\Omega^2 - \omega_b^2 \cos(\xi)$ .

## 4.2 Kinetic equilibrium near an island

### 4.2.1 Reduction of the Fokker-Planck equation

The first step to characterise the thermodynamic equilibrium near an island in the phase space is to solve Eq.(8)

$$-\left\{H_\Omega, \tilde{F}\right\} - \mathcal{C}[\tilde{F}] = \omega_b^2 F'_{eq} \sin \xi$$

where the Poisson bracket is calculated in the variables  $(\xi, \Omega)$  and

$$F'_{eq} = \frac{1}{C} \left( \mathbf{n} \cdot \frac{\partial F_{eq}}{\partial \mathbf{J}} \right) \Big|_{\mathbf{J}=\mathbf{J}_R}$$

is the gradient of the unperturbed distribution function on the resonant surface along the vector  $\mathbf{n}$ . It is convenient to further reduce this equation by introducing a new set of conjugate variables  $(\xi, p = \frac{\Omega}{\omega_b})$  associated with the Hamiltonian for a pendulum submitted to gravitation

$$k = \frac{p^2}{2} - \cos \xi$$

The variable  $p$  measures the distance from the resonance  $\Omega = 0$  in unit of island width  $\omega_b$  ( $\omega_b$  is in fact 1/4 of the island width). A normalised distribution function  $f = \tilde{f}(\xi, p) + p$  is defined as

$$\tilde{f} = \frac{\tilde{F}}{\omega_b F'_{eq}}$$

The physical meaning of  $\tilde{f}$  gets clearer when remembering that the island half-width is  $\delta_\Omega = 2\omega_b$  in frequency units. The function  $f$  then appears related to the distribution  $F$  via the relation

$$f = \frac{1}{\omega_b F'_{eq}} [F(\boldsymbol{\alpha}, \mathbf{J}) - F_{eq}(\mathbf{J}_R)]$$

where

$$F(\boldsymbol{\alpha}, \mathbf{J}) = F_{eq}(\mathbf{J}_R) + F'_{eq}(\mathbf{J}_R)\Omega + \tilde{F}(\boldsymbol{\alpha}, \mathbf{J})$$

is the Taylor development of the distribution function near  $\Omega = 0$ , see Fig.7. Collisions are expected to regularise the resonant singularity of the distribution function near  $\Omega = 0$ . Hence it makes sense to use a simple diffusion expression

$$\mathcal{C}[\tilde{F}] = \frac{\partial}{\partial \Omega} \left( D_\Omega \frac{\partial \tilde{F}}{\partial \Omega} \right)$$



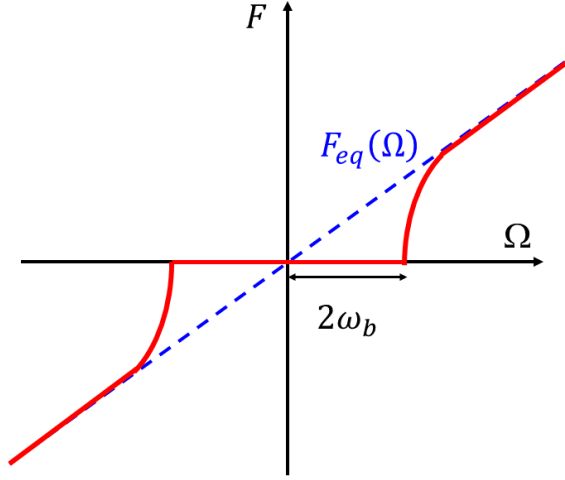


Figure 7: Typical shape of a distribution function near an island in the action space.

where  $D_\Omega = \left\langle \frac{\Delta\Omega^2}{2\Delta t} \right\rangle$  is a diffusion coefficient and  $\left\langle \frac{\Delta\Omega^2}{2\Delta t} \right\rangle$  a Fokker-Planck scattering term in  $\Omega$ , homogeneous to the cubic power of a frequency. The latter is related to collision frequencies via the relationship between  $\Omega$  and the particle velocity. Let us note that  $D_\Omega$  scales as the cubic power of a frequency. This coefficient is calculated at the resonant surface  $\Omega = 0$  since its variations play usually no role. This allows dependences of the  $\eta$  parameter on the actions  $\mathbf{J}_R$  on the resonant surface. The equation over the normalised potential  $f$  then bears a simple form, i.e.

$$\{k, \tilde{f}\} + \eta \frac{\partial^2 \tilde{f}}{\partial p^2} = -\sin \xi \quad (25)$$

where the Poisson bracket is calculated in variables  $(\xi, p)$ . The dimensionless parameter  $\eta$  is defined as

$$\eta = \frac{D_\Omega}{\omega_b^3}$$

Using the relationship  $f = \tilde{f}(\xi, p) + p$ , this equation can be written as well

$$\{k, f\} + \eta \frac{\partial^2 f}{\partial p^2} = 0 \quad (26)$$

The physical meaning of the parameter  $\eta$  is quite clear. Since collisions are responsible for a random walk in the phase space, the displacement  $\delta\Omega = \Omega(t) - \Omega(0)$  of a particle that leaves the resonant surface  $\Omega = 0$  at the initial time  $t = 0$  evolves as  $\delta\Omega^2(t) \sim D_\Omega t$ . Since the island half-width is  $2\omega_b$ , it gets out of the island after a typical detrapping time  $\tau_d \sim \omega_b^2/D_\Omega$ . Hence  $\eta = \tau_b/\tau_d$  is the ratio of a bounce time  $\tau_b = 1/\omega_b$  to the detrapping time  $\tau_d$ . If the particle is detrapped by collisions before exploring the island, then  $\eta \gg 1$ . In the opposite case  $\eta \ll 1$ , the particle experiences many bounce times before it gets out of the island. In this case, no gradient can be sustained within the island because of a mixing due to the motion of trapped particles that explore the whole energy surface in the island before enduring a collision. The result is a flattening of the distribution function in the island, as shown in Fig.7.

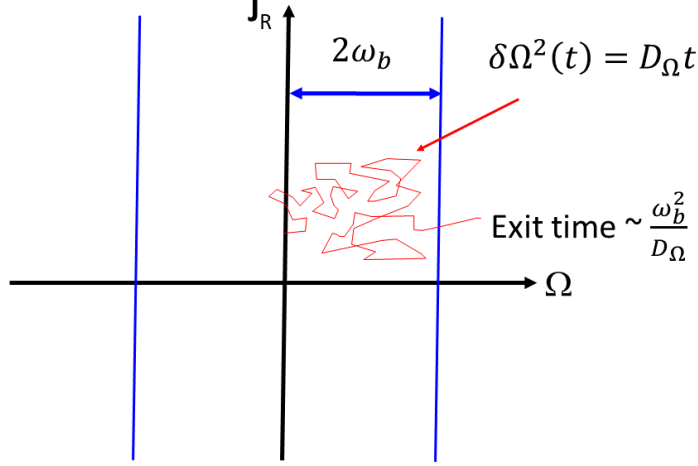


Figure 8: Schematic random motion of a particle due to collisions. A particle that starts from the resonant surface  $\Omega = 0$  diffuses as  $\delta\Omega^2(t) \sim D_\Omega t$ . The island half-width is  $2\omega_b$ , so that it gets out of the island after a typical detrapping time  $\tau_d \sim \omega_b^2/D_\Omega$ .

#### 4.2.2 Compact formulation of the entropy production rate

Hence the resonant entropy functional as defined in Eq.(20) becomes

$$\dot{S}_{res} [U_{eq}, \partial \mathbf{J} U_{eq}^\dagger] = \frac{1}{T_0^2} \int d\gamma F_{eq}(\mathbf{J}) \delta(\mathbf{n} \cdot \boldsymbol{\Omega}) h^2 \Lambda(\mathbf{J}) \left( \mathbf{n} \cdot \frac{\partial U_{eq}^\dagger}{\partial \mathbf{J}} \right)^2 \quad (27)$$

where

$$\Lambda(\mathbf{J}) = \eta \int_{-\infty}^{+\infty} dp \int_{-\pi}^{\pi} \frac{d\xi}{2\pi} \left( \frac{\partial \tilde{f}(\xi, p)}{\partial p} \right)^2$$

is always positive. The value of  $\Lambda(\mathbf{J}_R)$  can be found by solving the differential equation that rules  $f$  (see Appendix D). The final result is quite simple, though the derivation is not, and reads

$$\Lambda(\mathbf{J}) = \begin{cases} \frac{\pi}{2} & \eta \gg 1 \\ 2\mathcal{I}\eta & \eta \ll 1 \end{cases}$$

where  $\mathcal{I} \simeq 1.38$ . The expression for  $\eta \gg 1$  breaks down for large values of  $\eta$ . It ultimately decreases as  $1/\eta$ .

## 5 Quasi-linear theory

### 5.1 Transition to chaos

#### 5.1.1 Island overlap criterion

A different situation is now investigated, where several perturbations are present in the plasma. This case is tractable under some conditions. Since time plays a specific role in quasi-linear theory, we go back to the original description of the phase space with a set of 3 angle/action variables  $(\boldsymbol{\alpha}, \mathbf{J})$ . The perturbed Hamiltonian is now a sum over a large number of components in the Fourier series Eq.(2). Intuition suggests that chaos sets on whenever several perturbations are present. At this point a definition of chaos is needed. One possible, and practical, approach is to analyse particle trajectories. Chaotic

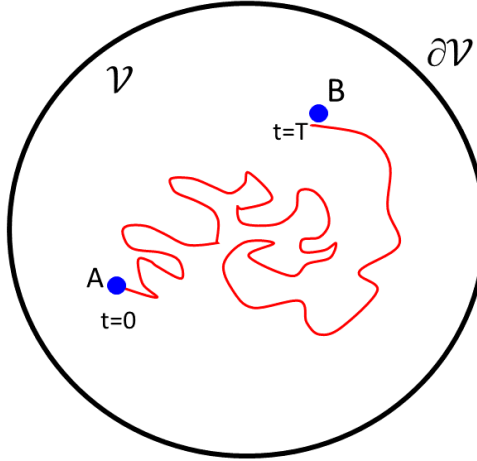


Figure 9: Schematics of the ergodic property: a particle that starts at position A at  $t = 0$  in a bounded phase space gets as close as desired to another point B in the phase space.

trajectories are characterised by a great sensitivity to initial conditions, which make them hardly predictable since the slightest change of initial conditions leads to different paths. In this section, we call stochastic a subset of chaotic systems that are requested to bear two properties:

- ergodicity: a trajectory gets as near as wanted to any point of the phase space when time goes on. In this sense a trajectory covers the whole phase space, see Fig.9.
- exponentiation: two trajectories initially nearby depart exponentially from each other with time, see Fig.10. This process is responsible for sensitivity on initial conditions.

These two criteria are in fact quite difficult to fulfil. Many systems are ergodic only locally. A powerful way to assess ergodicity is to draw a Poincaré map, i.e. a 2D plane in the action space where the position of a particle is marked every time it goes through the selected plane. Integrability usually materialises via regular lines in the plane (island is an example), whereas chaos corresponds to a random walk of this point (see [17] for an overview). Ergodicity is related to a property shared by many stochastic systems, i.e. mixing. Mixing is the fact that two regions of the phase space characterised by a different macroscopic quantity, e.g. different temperatures, get mixed after some time. Note that mixing may occur in non chaotic systems (an island for instance). Also the concept of exponentiation has to be understood asymptotically - usually the distance between two trajectories that start from two points located nearby increase only algebraically for small times. The exponential increase occurs after a long enough period. The exponential coefficient is called Lyapunov exponent. Obviously exponentiation is related to sensitivity to initial conditions since it implies that two initially nearby trajectories become uncorrelated after a time that is measured by the inverse of the greatest Lyapunov exponent. A dynamical system where several perturbations coexist can be formally represented by a perturbed Hamiltonian that is expanded in a Fourier-Laplace series, as in Eq.(2). An island can be formally built near each resonant surface  $\omega - \mathbf{n} \cdot \mathbf{\Omega}(\mathbf{J}) = 0$ . However, while a situation where islands are far from each other and thus well separated makes sense, one may expect that a situation where islands are densely packed leads to chaos, since a particle located in between two resonant surfaces feels equally the action of forces with different helicities. This behaviour is formalised by the Kolmogorov-Arnold-Moser

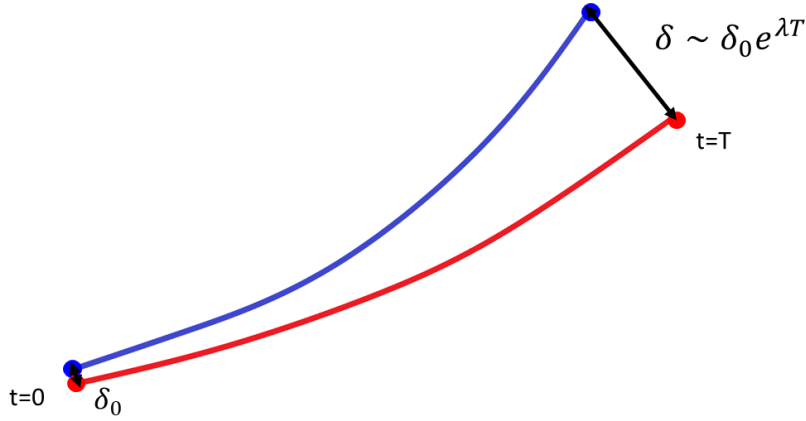


Figure 10: Schematics of the exponentiation property: the distance  $\delta_0$  between two nearby particles at  $t = 0$  increases exponentially in time as  $\delta_0 e^{\lambda t}$ . The coefficient  $\lambda$  is a Lyapunov exponent.

(KAM) theorem [18, 19, 20], which roughly states that an integrable system stays integrable when perturbations of the kind Eq.(2) are added, provided their amplitude remains small. The theorem actually says a bit more since it appears that the onset of chaos takes place in the form of trajectories that span larger and larger volume of the phase space. KAM surfaces are surfaces which cannot be crossed by trajectories and can thus be considered as barriers. While increasing the amplitude of perturbations, KAM surfaces are “broken”, i.e. some trajectories get through these surfaces, starting with low order rational surfaces, i.e. resonant surfaces associated with low order wave numbers  $(n_1, n_2, n_3)$ . When the last KAM surface falls down, global chaos takes place, i.e. trajectories fill in the phase space. Let us stress that chaos is generic, whereas integrability is not. For instance chaos appears generically for two perturbations only whenever their wave numbers and frequencies are incommensurate. Nevertheless, the lesson of the KAM theorem is that confinement has a better chance to be granted when starting from an integrable system, provided perturbations are small. This is of course of extreme importance for magnetic confinement fusion. The question that remains to clarify is to determine a criterion for the onset of chaos. The interested reader is sent to monographs that address this question in details [21, 4]. We give here a schematic picture. A working rule of thumb is provided by the Chirikov overlap criterion [22]. For a couple of perturbations  $(\mathbf{n}, \omega)$  and  $(\mathbf{n}', \omega')$ , the Chirikov parameter,  $S_{ch}$ , is defined as the sum of the half-widths of the corresponding islands divided by the distance between resonant surfaces. The overlap criterion states that chaos sets on whenever  $S_{ch} \geq 1$ . A slightly more accurate threshold, based on simulations of various systems, is  $S_{ch} \geq 2/3$ . Of course this criterion is difficult to state for a broad spectrum of wave numbers and frequencies, since there then exists a multitude of (virtual) islands of various size and associated distances between resonant surfaces. To grasp some feeling of it, let us consider a highly simplified situation, where the perturbation  $\tilde{H}$  does not depend on time nor  $\alpha_1$ . We consider two adjacent perturbations with wave numbers  $\mathbf{n} = (n_2, n_3)$  and  $\mathbf{n}' = (n_2 + 1, n_3)$ , see Fig.11. The corresponding resonant frequencies are  $\Omega = n_2 \Omega_2 + n_3 \Omega_3$ , and  $\Omega' = (n_2 + 1) \Omega_2 + n_3 \Omega_3 = \Omega + \Omega_2$ . Hence the distance between the 2 resonant surfaces  $\Omega = 0$  and  $\Omega' = 0$  in unit of  $\Omega$  is just  $d_\Omega = \Omega_2$ . Moreover the island half-width for the first perturbation is

$$\delta_{\Omega \mathbf{n}} = 2\omega_b = 2(|C_{\mathbf{n}} H_{\mathbf{n}\omega}|)^{1/2}$$

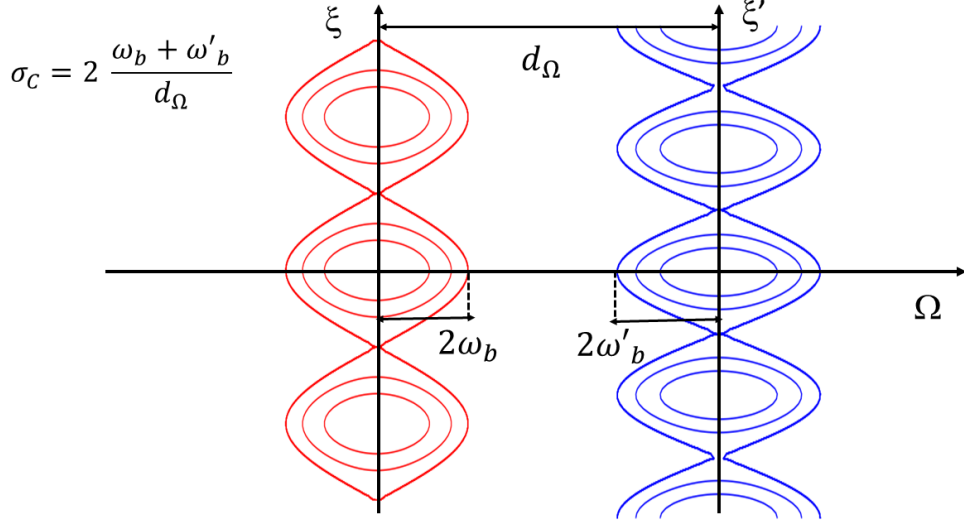


Figure 11: Schematics of the Chirikov overlap parameter for a simple example of two islands with nearby wave numbers.

where  $C_{\mathbf{n}}$  is given by Eq.(24). A similar expression yields the second island half-width  $\delta_{\Omega\mathbf{n}'} = 2\omega'_b$ . The Chirikov overlap parameter then reads

$$S_{ch} = \frac{2}{d_\Omega} (\omega_b + \omega'_b)$$

This estimate provides a fast way to assess the degree of stochasticity. In other words it is related to the greatest Lyapunov exponent.

An example of a transition to chaos is shown for a standard map on Fig.(12). The standard map is the discrete version of the kicked rotator, whose Hamiltonian is

$$H(\theta, p, t) = \frac{1}{2}p^2 + K \cos(\theta) \sum_{n=-\infty}^{+\infty} \delta(t - n)$$

where  $\theta$  is an angle, and  $p$  a momentum conjugate to  $\theta$ . The discrete version is obtained after an integration over time, and yields the recurrence

$$\begin{cases} p_{n+1} = p_n + K \sin(\theta_n) \\ \theta_{n+1} = \theta_n + p_{n+1} \end{cases}$$

Chaos appears when increasing the  $K$  parameter - the critical value is  $K_c = 0.971635...$  [23, 24].

The demonstration of the KAM theorem is quite technical and will not be reproduced here. The reader is sent to the original article and to textbooks on the subject and related matter [1, 21]. A pedagogical and pleasant lecture on the transition to chaos can be found in [25]. The relationship of Hamiltonian chaos with statistical mechanics, notably phase transitions, is commented in [26]. Finally the interested reader will find an account on quantum chaos based on Hamiltonian approach in [27].

### 5.1.2 A pedestrian approach

Let us remind that the mean distribution function is ruled by Eq.(7), where the flux is

$$\Gamma = \int_0^T \frac{dt}{T} \int \frac{d^3\alpha}{(2\pi)^3} \left( -\frac{\partial \tilde{H}}{\partial \alpha} \right) \tilde{F} = \sum_{\mathbf{n}\omega} i \mathbf{n} H_{\mathbf{n}\omega}^*(\mathbf{J}) F_{\mathbf{n}\omega}(\mathbf{J}) \quad (28)$$

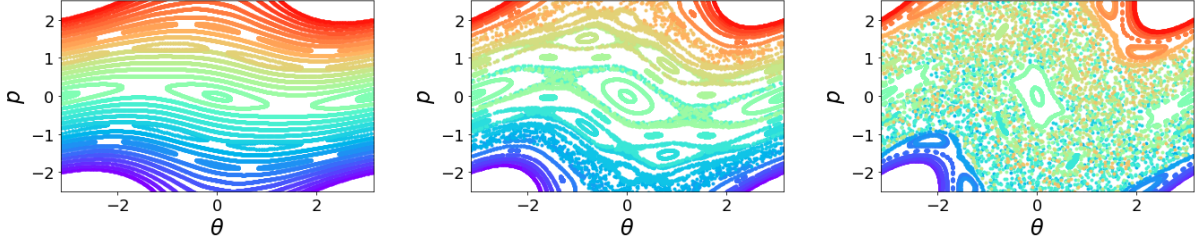


Figure 12: Poincaré map of trajectories for the standard map. The index  $n$  runs from 0 to  $N_t = 1000$ . Trajectories are initiated with 36 initial conditions that are equally spaced over the quadrant  $-2 \leq p \leq 2$  and  $-\pi \leq \theta \leq \pi$ . Left panel:  $K = 0.5$ , trajectories lie on lines (“KAM surfaces”), some island are visible - the system remains integrable. Centre panel:  $K = 1.0$ , some trajectories start being chaotic - however some KAM surfaces resist, preventing particles to cross the domain in the  $p$  direction. Right panel  $K = 1.5$ , global chaos is now established over almost the whole interval in  $p$ .

where  $T$  is a time larger than a turbulence correlation time, and smaller than a time that characterises the evolution of the mean distribution function. At this stage,  $F_{\mathbf{n}\omega}(\mathbf{J})$  is unknown and should be computed. Let us recall the Vlasov equation over the perturbed distribution function

$$\frac{\partial \tilde{F}}{\partial t} - \{H_{eq} + \tilde{H}, \tilde{F}\} = \frac{\partial F_{eq}}{\partial \mathbf{J}} \cdot \frac{\partial \tilde{H}}{\partial \boldsymbol{\alpha}}$$

In an ideal world, the response  $\tilde{F}$  is expected to be proportional to the gradient of the equilibrium distribution function  $\partial_{\mathbf{J}} F_{eq}$  (departure from thermodynamic equilibrium), and the perturbed field  $\partial_{\boldsymbol{\alpha}} \tilde{H}$  (drive). This is obviously not so, because of the non quadratic term  $\{\tilde{H}, \tilde{F}\}$  that appears in the l.h.s. of the Vlasov equation. Let us nevertheless postulate a response in the Fourier space of the form

$$F_{\mathbf{n}\omega}(\mathbf{J}) = -i R_{\mathbf{n}\omega}(\mathbf{J}) \left( \mathbf{n} \cdot \frac{\partial F_{eq}}{\partial \mathbf{J}} \right) H_{\mathbf{n}\omega}(\mathbf{J})$$

where  $R_{\mathbf{n}\omega}$  is dubbed “resonant response function”. The flux is then

$$\Gamma = \sum_{\mathbf{n}} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \Im [R_{\mathbf{n}\omega}(\mathbf{J})] |H_{\mathbf{n}\omega}(\mathbf{J})|^2 \left( \mathbf{n} \cdot \frac{\partial F_{eq}}{\partial \mathbf{J}} \right) \mathbf{n}$$

The resonant response is a priori some complicated function of the perturbed Hamiltonian  $\tilde{H}$ . Let us suppose for now that it is given by the linear response Eq.(3) computed in the upper half of the complex plane

$$R_{\mathbf{n}\omega}^{(0)} = \frac{1}{-i(\omega - \mathbf{n} \cdot \boldsymbol{\Omega} + i\nu)} \quad (29)$$

where  $\nu > 0$ . Then

$$\Gamma = \sum_{\mathbf{n}\omega} |H_{\mathbf{n}\omega}(\mathbf{J})|^2 \frac{\nu}{(\omega - \mathbf{n} \cdot \boldsymbol{\Omega})^2 + \nu^2} \left( \mathbf{n} \cdot \frac{\partial F_{eq}}{\partial \mathbf{J}} \right) \mathbf{n}$$

The Lorentzian function can be replaced by a Kronecker delta function  $\pi \delta(\omega - \mathbf{n} \cdot \boldsymbol{\Omega})$  whenever  $\nu \ll \omega$ . Hence the flux is of the diffusive form Eq.(11), with

$$D_{ik} = \pi \sum_{\mathbf{n}\omega} |H_{\mathbf{n}\omega}(\mathbf{J})|^2 \delta(\omega - \mathbf{n} \cdot \boldsymbol{\Omega}) n_i n_k \quad (30)$$

The entropy production rate is given by Eq.(20)

$$\dot{S}_{res}(U_{eq}, \partial_{\mathbf{J}} U_{eq}) = \frac{\pi}{T_0^2} \sum_{\mathbf{n}\omega} \int d\gamma F_{eq} |H_{\mathbf{n}\omega}(\mathbf{J})|^2 \delta(\omega - \mathbf{n} \cdot \boldsymbol{\Omega}) \left( \mathbf{n} \cdot \frac{\partial U_{eq}}{\partial \mathbf{J}} \right)^2$$

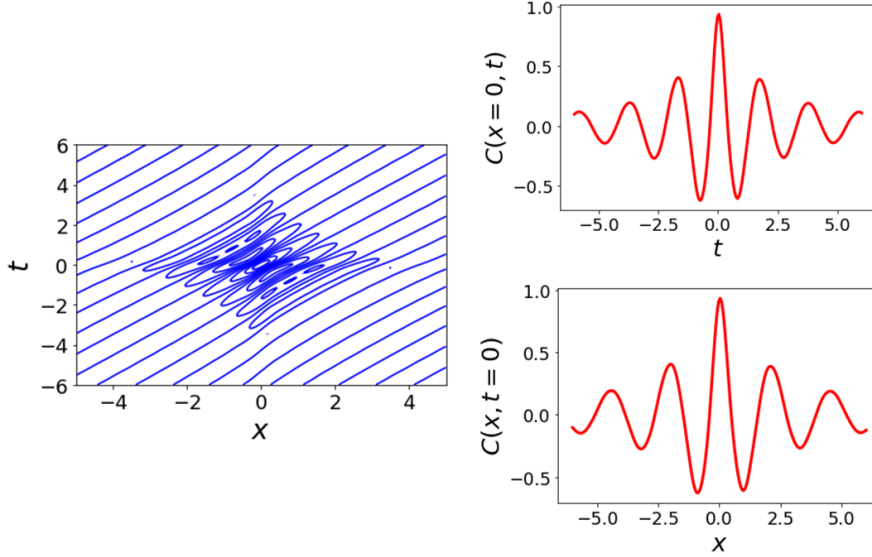


Figure 13: Eulerian correlation function  $\mathcal{C}_E(x, t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \mathcal{I}_{k\omega} \exp(ikx - i\omega(k)t)$ , where  $\omega(k) = \omega_0 + v_g(k - k_0)$  and the spectrum power is  $\mathcal{I}_{k\omega} = \frac{\Delta\omega}{(\omega - \omega_0)^2 + \Delta\omega^2} \frac{\Delta k}{(k - k_0)^2 + \Delta k^2}$ . Parameters are  $k_0 = 3$ ,  $\omega_0 = 3$ ,  $\Delta k = 1$ ,  $\Delta\omega = 1$ . The group velocity  $v_g = \left. \frac{\partial\omega}{\partial k} \right|_{k=k_0}$  is set to 0.

## 5.2 Derivation and conditions of validity

### 5.2.1 Correlation functions

The derivation of the quasilinear theory was originally done in [28, 29]. A subsequent copious literature is available on this question, including experimental proof. An overview can be found in [4]. The version that follows is inspired by another overview on quasilinear theory [30] (in French), that was developed for the special case of beam-plasma instabilities. The generalisation to Hamiltonian systems does not raise difficulties. This methodology relies on a perturbative method that enables a precise identification of the validity criteria. We consider here an idealised situation where the perturbed Hamiltonian  $\tilde{H}(\alpha, \mathbf{J}, t)$  is a turbulent random field, thus allowing a statistical approach. In a first attempt, we ignore the dependence of the perturbed Hamiltonian on the action  $\mathbf{J}$ . Also collisions are neglected. We will come back to these approximations. Turbulence is assumed homogeneous, i.e. the correlation functions are independent of the choice of initial position and time<sup>17</sup>. This means that the Eulerian correlation function of the perturbed Hamiltonian is of the form

$$\langle \tilde{H}(\alpha, t) \tilde{H}(\alpha', t') \rangle = \mathcal{C}_E(\alpha - \alpha', t - t') \quad (31)$$

where the bracket  $\langle \dots \rangle$  denotes a statistical average. The correlation function  $\mathcal{C}_E$  is typically a decaying function of its arguments, thus defining Eulerian correlation lengths and times. The decay of its absolute value is often exponential, so that the correlation time (length) is defined as an e-folding time (length). The Eulerian correlation time is noted  $\tau_{cE}$ , and the 3 correlation lengths (one for each angle direction) define a vector  $\mathbf{L}_c$ . Turbulence homogeneity implies that correlation times and lengths are the same whatever the time and location where they are computed. On the other hand, turbulence is not supposed isotropic since correlation lengths can differ depending on the considered direction due to the presence of a magnetic field. Let us stress immediately that a condition of time decay of the Eulerian correlation function Eq.(31) is not indispensable to use a

<sup>17</sup>This approximation is not valid for the original problem of the beam-plasma instability, which by definition is unsteady since the electric field grows exponentially in time

quasilinear theory. In fact, there exists many instances where quasilinear theory applies to a static Hamiltonian. It will be seen indeed that another efficient decorrelation process comes from stochasticity. In terms of the spectral components of the turbulent fields, the above condition reads

$$\left\langle \tilde{H}_{\mathbf{n}\omega} \tilde{H}_{\mathbf{n}'\omega'}^* \right\rangle = \mathcal{I}_{\mathbf{n}\omega} \delta_{\mathbf{n}\mathbf{n}'} \delta(\omega - \omega')$$

where  $\delta$  is a Kronecker delta function and the “turbulent” intensity is defined as

$$\mathcal{I}_{\mathbf{n}\omega} = \left\langle \left| \tilde{H}_{\mathbf{n}\omega} \right|^2 \right\rangle$$

The intensity  $\mathcal{I}_{\mathbf{n}\omega}$  is the Fourier transform of the correlation function  $\mathcal{C}_E(\boldsymbol{\alpha} - \boldsymbol{\alpha}', t - t')$ , i.e.

$$\mathcal{C}_E(\boldsymbol{\alpha} - \boldsymbol{\alpha}', t - t') = \sum_{\mathbf{n}} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \mathcal{I}_{\mathbf{n}\omega} \exp[i\mathbf{n} \cdot (\boldsymbol{\alpha} - \boldsymbol{\alpha}') - i\omega(t - t')] \quad (32)$$

Examples are shown on Fig.13 and Fig.14. The notion of statistical average may sound a bit vague. Ideally a statistical average involves multiple realisations of the dynamics by changing initial conditions. In practice this is rarely possible since experiments or simulations cannot be reproduced a large number of times. One solution for a stationary turbulent state is to cut the times series of one experiment/simulation in several subsets, each one one much longer than a correlation time. In this case, a statistical average is equivalent to a time average. Moreover, in view of the heuristic approach described above, it makes sense to extend the statistical average to an average over the angle variables. Hence a statistical average is in fact a mean over time and angles. In that respect, the flux Eq.(28) can be understood as  $\mathbf{\Gamma} = \left\langle (-\partial_{\boldsymbol{\alpha}} \tilde{H}) \tilde{F} \right\rangle$ , while  $F_{eq} = \langle F \rangle$ . It may occur that time an angle averaging is not sufficient to smooth out the distribution functions. This difficulty is cured by an average in the action space over a few fluctuation correlation lengths. Hence a mean field theory often requires a heavy coarse-graining procedure in all direction to be relevant.

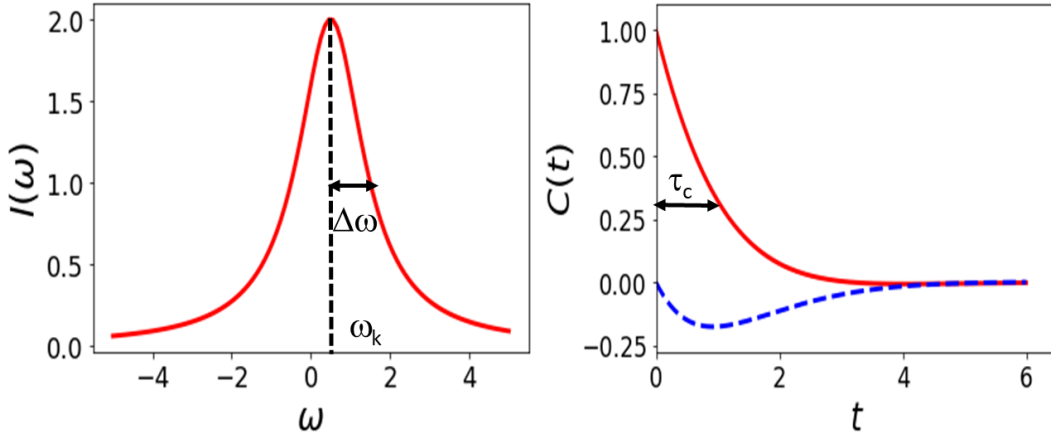


Figure 14: Example of a Lorentzian spectral power  $I(\omega) = \frac{2\Delta\omega}{(\omega - \omega_k)^2 + \Delta\omega^2}$  and corresponding correlation function  $C(t) = \exp(-t/\tau_c) \exp(-i\omega_k t)$  (real part in solid red line, imaginary part in dashed blue line). The frequency broadening  $\delta\omega$  and correlation time  $\tau_c$  are related through the relationship  $\tau_c = \frac{1}{\delta\omega}$ .

The Hamiltonian correlation function can be further detailed by assuming a Lorentzian frequency spectrum

$$\mathcal{I}_{\mathbf{n}\omega} = \mathcal{I}_{\mathbf{n}} \frac{\Delta\omega_{\mathbf{n}}}{(\omega - \omega_{\mathbf{n}})^2 + \Delta\omega_{\mathbf{n}}^2}$$



where  $\Delta\omega_{\mathbf{n}}$  is a spectral broadening frequency, and  $\omega_{\mathbf{n}}$  an angular frequency that is often close to the frequency of the most unstable linear mode at wave number  $\mathbf{n}$ . The frequency integral in Eq.(32) is readily done by using the residue theorem

$$\mathcal{C}_E(\boldsymbol{\alpha} - \boldsymbol{\alpha}', t - t') = \sum_{\mathbf{n}} \mathcal{I}_{\mathbf{n}} \exp [i\mathbf{n} \cdot (\boldsymbol{\alpha} - \boldsymbol{\alpha}') - i\omega_{\mathbf{n}}(t - t')] \exp [-\Delta\omega_{\mathbf{n}} |t - t'|]$$

The Eulerian correlation time  $\tau_{cE,\mathbf{n}}$  at wave number  $\mathbf{n}$  can thus be identified to the inverse of the spectral width  $1/\Delta\omega_{\mathbf{n}}$ . This property is widely used when analysing experimental data. A similar calculation can be done by assuming a wave number spectra of the form

$$\mathcal{I}_{\mathbf{n}} = I_0 \prod_{i=1,2,3} \frac{\Delta n_i}{(n_i - n_{0i})^2 + \Delta n_i^2}$$

Under some approximations, and after some calculations described in Appendix E, the Eulerian correlation time is

$$\tau_{cE,\mathbf{n}} = \frac{1}{\Delta\omega_{\mathbf{n}} + |\Delta\mathbf{n} \cdot \mathbf{v}_{g\mathbf{n}_0}|}$$

where  $\mathbf{v}_{g\mathbf{n}_0} = \frac{\partial\omega_{\mathbf{n}}}{\partial\mathbf{n}}|_{\mathbf{n}=\mathbf{n}_0}$  is a group velocity calculated at  $\mathbf{n} = \mathbf{n}_0$ . Therefore, even in absence of a frequency broadening width  $\Delta\omega_{\mathbf{n}}$ , decorrelation occurs due to the group velocity combined with wave number spectral broadening. This point is illustrated in Fig.13 and Fig.15, which show an example of Eulerian correlation functions without and with a finite group velocity. This rule provides a useful estimate of the correlation time, and is often met in the literature. The reader should be warned however that the wave number spectrum for a turbulence rarely looks like a Lorentzian. It is rather a power Law, e.g. the celebrated Kolmogorov spectrum  $k^{-5/3}$  in 3D hydrodynamic turbulence, or exponential.

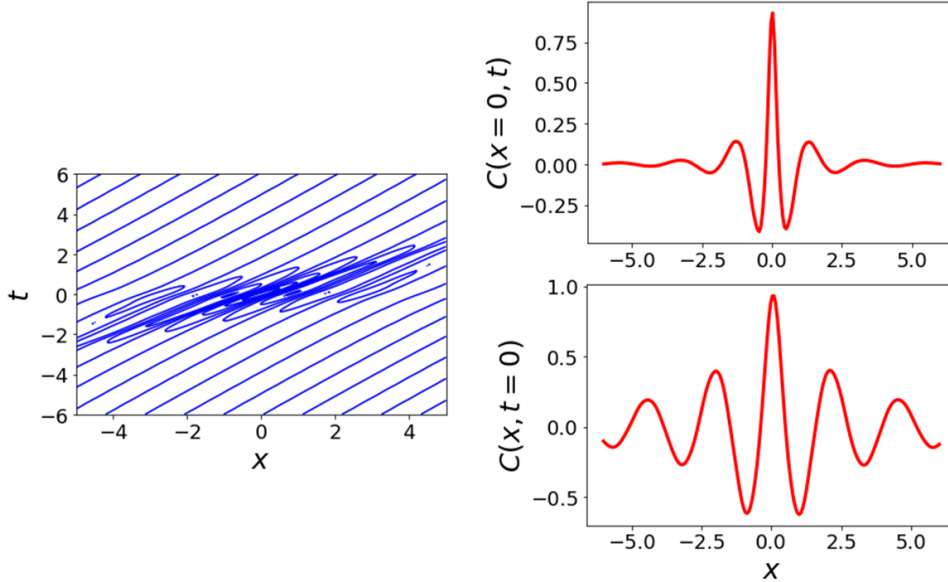


Figure 15: Eulerian correlation function as defined in Fig.13 The group velocity  $v_g = \frac{\partial\omega}{\partial k}|_{k=k_0}$  is set to 2. Note that the correlation function in time has shrunk.

### 5.2.2 Perturbative treatment of the Vlasov equation

The Vlasov equation over the total distribution function Eq.(1) reads explicitly

$$\frac{\partial F}{\partial t} + \boldsymbol{\Omega} \cdot \frac{\partial F}{\partial \boldsymbol{\alpha}} = \mathbf{E} \cdot \frac{\partial F}{\partial \mathbf{J}} \quad (33)$$

where

$$\mathbf{E} = \frac{\partial \tilde{H}}{\partial \boldsymbol{\alpha}}$$

The l.h.s. of Eq.(33) corresponds to the unperturbed ballistic motion of a particle, such that  $d_t \boldsymbol{\alpha} = \boldsymbol{\Omega}$  and  $d_t \mathbf{J} = 0$ . The r.h.s. of Eq.(33) is proportional to the random field  $\mathbf{E}$ . The operator  $\mathcal{E} = \mathbf{E} \cdot \partial_{\mathbf{J}}$  is sometimes called a stochastic operator, and Eq.(33) a stochastic equation. Let us introduce the propagator (or Green's function)  $G_0(\boldsymbol{\alpha}, \mathbf{J}, t / \boldsymbol{\alpha}_0, \mathbf{J}_0, t_0)$  associated with the l.h.s. of Eq.(33), and defined as

$$\begin{aligned} \frac{\partial G_0}{\partial t} + \boldsymbol{\Omega} \cdot \frac{\partial G_0}{\partial \boldsymbol{\alpha}} &= 0 \\ G_0(\boldsymbol{\alpha}, \mathbf{J}, t_0 / \boldsymbol{\alpha}_0, \mathbf{J}_0, t_0) &= \delta(\boldsymbol{\alpha} - \boldsymbol{\alpha}_0) \delta(\mathbf{J} - \mathbf{J}_0) \end{aligned}$$

The propagator  $G_0$  can be computed explicitly as

$$G_0(\boldsymbol{\alpha}, \mathbf{J}, t_0 / \boldsymbol{\alpha}_0, \mathbf{J}_0, t_0) = \delta(\mathbf{J} - \mathbf{J}_0) \delta(\boldsymbol{\alpha} - \boldsymbol{\alpha}_0 - \boldsymbol{\Omega}(t - t_0)) \quad (34)$$

The formal solution of Eq.(33) is

$$F = G_0 * F_0 + G_0 * \mathcal{E}[F] \quad (35)$$

where

$$F_0(\boldsymbol{\alpha}_0, \mathbf{J}_0, t) = \delta(t_0) F(\boldsymbol{\alpha}_0, \mathbf{J}_0, t = 0)$$

is related to the initial distribution function. The symbol  $*$  designates a convolution

$$(G_0 * S)(\boldsymbol{\alpha}, \mathbf{J}, t) = \int_0^t dt_0 \int d\boldsymbol{\alpha}_0 \int d\mathbf{J}_0 G_0(\boldsymbol{\alpha}, \mathbf{J}, t_0 / \boldsymbol{\alpha}_0, \mathbf{J}_0, t_0) S(\boldsymbol{\alpha}_0, \mathbf{J}_0, t_0)$$

for any function  $S(\boldsymbol{\alpha}, \mathbf{J}, t)$ . Using the expression Eq.(34), this convolution can be made more explicit

$$(G_0 * S)(\boldsymbol{\alpha}, \mathbf{J}, t) = \int_0^t dt_0 S(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_0), \mathbf{J}, t_0)$$

The formal solution of Eq.(35) can be expanded as

$$F = G_0 * F_0 + G_0 * \mathcal{E}[G_0 * F_0] + \quad (36)$$

$$\begin{aligned} &G_0 * \mathcal{E}[G_0 * \mathcal{E}[G_0 * F_0]] + \dots + \\ &G_0 * \mathcal{E}[G_0 * \mathcal{E}[\dots * \mathcal{E}[G_0 * F_0]\dots]] + \dots \end{aligned} \quad (37)$$

To illustrate this rather abstract expression, the  $n$ th term in this series is given explicitly as

$$\begin{aligned} G_0 * \mathcal{E}[G_0 * \mathcal{E}[\dots * \mathcal{E}[G_0 * F_0]\dots]] &= \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \dots \int_0^{t_{n-1}} dt_n \\ &\mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_1), t_1) \cdot \frac{\partial}{\partial \mathbf{J}} \mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_2), t_2) \cdot \frac{\partial}{\partial \mathbf{J}} \dots \\ &\mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_{n-1}), t_{n-1}) \cdot \frac{\partial}{\partial \mathbf{J}} \mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_n), t_n) \cdot \frac{\partial}{\partial \mathbf{J}} F_0(\boldsymbol{\alpha} - \boldsymbol{\Omega}t, \mathbf{J}, 0) \end{aligned}$$

Eq.(37) is a rather cumbersome expression, and apparently not very useful. However it appears that after a statistical average,  $F_0$  can be replaced by  $\langle F \rangle$ , with some modification of the various terms in the series. This result is not obvious and requires a bit of care. It is demonstrated in Appendix F, using a diagrammatic representation. It appears that after a statistical average, Eq.(37) can be reformulated as

$$\langle F \rangle = - F_0 + - M - \langle F \rangle \quad (38)$$

where  $M$  is a sum of products of two-field correlated operators, which is sometimes called a “mass” operator. In this representation, each horizontal line is a convolution  $G_0*$ , while each knot is an operator, here  $M$ . This identity is called a Dyson equation, who was the first to prove it in the context of quantum field theory. The first non zero contribution to the mass operator is

$$\langle G_0 * \mathcal{E}[G_0 * \mathcal{E}[* \langle F \rangle]] \rangle = \int_0^t dt_1 \int_0^{t_1} dt_2 \left\langle \mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_1), t_1) \cdot \frac{\partial}{\partial \mathbf{J}} \mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_2), t_2) \cdot \frac{\partial}{\partial \mathbf{J}} \langle F \rangle(\boldsymbol{\alpha} - \boldsymbol{\Omega}t, \mathbf{J}, t_2) \right\rangle$$

Let us now apply the operator  $\partial_t + \boldsymbol{\Omega} \cdot \partial_{\boldsymbol{\alpha}}$  to the expansion Eq.(38). The first term  $(\partial_t \boldsymbol{\Omega} \cdot \partial_{\boldsymbol{\alpha}})(G_0 * F_0)$  is equal to  $F_0$  which is zero everywhere except on the initial time  $t = 0$ . Using that  $\langle F \rangle$  does not depend on the angle  $\boldsymbol{\alpha}$ , and employing a statistical average that encompasses an average over the angles, one gets

$$\frac{\partial \langle F \rangle}{\partial t} = \frac{\partial}{\partial J_i} \int_0^t dt' \mathcal{C}_{L,ik}(\mathbf{J}, t - t') \frac{\partial}{\partial J_k} \langle F \rangle(\mathbf{J}, t')$$

where

$$\mathcal{C}_{L,ik}(\mathbf{J}, t - t') = \langle \mathbf{E}_i(\boldsymbol{\alpha}, t) \mathbf{E}_k(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t'), t') \rangle$$

is a Lagrangian correlation function<sup>18</sup>. Its Fourier transform is readily shown to be

$$\mathcal{C}_{L,ik}(\mathbf{J}, t - t') = \sum_{\mathbf{n}} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} n_i n_k \mathcal{I}_{\mathbf{n}\omega} \exp(-i(\omega - \mathbf{n} \cdot \boldsymbol{\Omega})(t - t')) \quad (39)$$

It is reminded that the correlation time, whether Lagrangian or Eulerian, is supposed much smaller than the time scale that characterises the evolution of the distribution function  $\langle F \rangle$ , noted  $\tau_D$ <sup>19</sup>, i.e. we consider the situation  $\tau_c \ll \tau_D$  and  $\tau_c \ll t$ . In this case,  $\langle F \rangle(\mathbf{J}, t')$  can safely be replaced by  $\langle F \rangle(\mathbf{J}, t)$ <sup>20</sup>, and the integral  $\int_0^t dt'$  can be replaced by  $\int_0^{+\infty} dt'$ . This crucial step is called “markovianisation”, that can be translated as “loss of memory”. The quasi-linear equation is then of the form Eq.(9) with a flux of diffusive type Eq.(11) and

$$D_{ik} = \int_0^{+\infty} d\tau \mathcal{C}_{L,ik}(\mathbf{J}, \tau) = \pi \sum_{\mathbf{n}\omega} n_i n_k \mathcal{I}_{\mathbf{n}\omega} \delta(\omega - \mathbf{n} \cdot \boldsymbol{\Omega})$$

This diffusion matrix is the same as the one found in a pedestrian approach Eq.(30).

### 5.2.3 Validity condition of the quasi-linear equation

The reader will probably find this derivation quite lengthy and cumbersome. After all the pedestrian approach was more rewarding ... and faster. The crucial point though is that an estimate of the error that is made when applying the quasi-linear theory is now within reach. The quasi-linear equation is obtained by keeping the first diagram in the Dyson expansion. Since in the quasi-Gaussian framework all moments can be expressed as products of two-point correlation functions, the error, noted  $K$ , is the ratio of the 4th diagram to the second one, so typically

$$K = \int_0^t dt_3 \int_0^{t_3} dt_4 \left\langle \frac{\partial}{\partial \mathbf{J}} \cdot \mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_3), t_3) \cdot \frac{\partial}{\partial \mathbf{J}} \mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_4), t_4) \right\rangle$$

<sup>18</sup>In fact a mix of Eulerian and Lagrangian correlation functions

<sup>19</sup> $\tau_D$  is essentially a diffusion time, typically  $a^2/D$ , where  $a$  is the system size, and  $D$  the largest of the quasi-linear diffusion coefficients  $D_{ik}$ .

<sup>20</sup>This is verified via a Taylor expansion of  $\langle F \rangle(\mathbf{J}, t')$  in  $t' - t$ , and using for instance a correlation function that behaves as  $\exp(-(t - t')/\tau_c)$

An estimate is done as follows. Contributions from derivatives with respect to the actions come essentially from derivatives of the Hamiltonian angular frequencies  $\mathbf{\Omega}$ . The error is easier to compute in the Fourier space

$$K = \sum_{\mathbf{n}\omega} \mathcal{I}_{\mathbf{n}\omega} \left( \mathbf{n} \cdot \frac{\partial}{\partial \mathbf{J}} (\mathbf{n} \cdot \mathbf{\Omega}) \right) \int_0^t d\tau \tau \int_0^\tau d\tau' (\tau + \tau') \exp(-i(\omega - \mathbf{n} \cdot \mathbf{\Omega}) \tau')$$

The integrals in time are regularised by the summations in  $\mathbf{n}$  and  $\omega$ . This actually yields a somewhat more accurate definition of the Eulerian and Lagrangian correlation times. The Eulerian correlation time at wave number  $\mathbf{n}$  is the inverse of the frequency broadening  $\Delta\omega_{\mathbf{n}}$ , i.e.  $\tau_{cL} \sim 1/\Delta\omega_{\mathbf{n}}$ . The Lagrangian decorrelation time is rather the product of the wave number spectrum width  $\Delta\mathbf{n}$  times the Hamiltonian frequency vector  $\mathbf{\Omega}$  minus the group velocity

$$\tau_{cL} = \frac{1}{|\Delta\mathbf{n} \cdot (\frac{\partial\omega_{\mathbf{n}}}{\partial\mathbf{n}} - \mathbf{\Omega})|}$$

The resonant frequency derivative  $\partial_{\mathbf{J}}(\mathbf{n} \cdot \mathbf{\Omega})$  was already met in the case of a single perturbation - it is the Hamiltonian curvature  $C_{\mathbf{n}\omega}$ . The product  $C_{\mathbf{n}\omega}H_{\mathbf{n}\omega}$  is the bounce frequency  $\omega_b$  of the  $\mathbf{n}\omega$  perturbation. The double integral in time is of order  $\tau_c^4$ . Hence the error is of the order of  $K \sim (\text{Ku})^2$ , where

$$\text{Ku} = \omega_b^2 \tau_c^2$$

is the Kubo number. The alert reader may complain that there was in fact another assumption: the perturbed Hamiltonian  $\tilde{H}$  was supposed not to depend on the action variables. This can be formally resolved by replacing the stochastic operator  $E\partial_{\mathbf{J}}$  by a Lagrange bracket  $\{\tilde{H}, \cdot\}$ . The demonstration is the same. However the derivative  $\partial_{\mathbf{J}}$  applied on the Hamiltonian  $\tilde{H}(\boldsymbol{\alpha} - \mathbf{\Omega}(t - t_j), \mathbf{J}, t_j)$  in each diagram not only acts on the Hamiltonian frequencies  $\mathbf{\Omega}$ , but also on the explicit dependence on  $\mathbf{J}$ . Fourier modes  $H_{\mathbf{n}\omega}(\mathbf{J})$  exhibit usually a final extent  $L_I$  in the action space near a resonant surface in the direction  $\mathbf{n}$ . This introduces a new decorrelation process that matters if the particle gets out of a  $\mathbf{n}\omega$  Hamiltonian mode well before a bounce time. The corresponding decorrelation time is  $\tau_w = L_I/h$ . The error made by neglecting the 4th order diagram contribution is  $K \sim (Ku)^2$ , where the Kubo number is  $\text{Ku} = \tau_c/\tau_w$ , i.e. the ratio of the decorrelation time to the time needed by a particle to move over a distance  $L_I$  under the effect of the perturbed Hamiltonian. In summary, one should take for  $\tau_c$  the smallest of all times

$$\tau_c = \min(\tau_{cE}, \tau_{cL})$$

and for the Kubo number the smallest of both definitions

$$\text{Ku} = \min\left(\omega_b^2 \tau_c^2, \frac{\tau_c}{\tau_w}\right)$$

The Kubo number represents physically the ratio of a turbulence correlation time to the time needed to cross a turbulent structure. If the Kubo number is smaller than 1, a particle leaves a turbulent structure (it can be an island, a vortex, etc..) before fully exploring it. If the Kubo number is larger than 1, then a particle has enough time to explore an island/vortex before being subject to a decorrelation process. The condition  $\text{Ku} \ll 1$  is needed to apply the quasilinear theory. The conditions that we used for deriving the quasilinear theory can be summarised as follows, see summary in Table 1,

- the dynamics must be chaotic. The condition is that the Chirikov overlap parameter should be larger than 1,  $S_{ch} \gg 1$ .
- the field should be a quasi-gaussian random field. This condition is not mandatory.

Condition	Criterion
Hamiltonian chaos	$S_{ch} \gg 1$
Correlation time vs diffusion time	$\tau_c \ll \tau_D$
Weak perturbation	$Ku = \min(\omega_b^2 \tau_c^2, \tau_c / \tau_w) \ll 1$

Table 1: Conditions of validity for the quasi-linear theory. The correlation time  $\tau_c$  should be chosen as the minimum of the Eulerian and Lagrangian times.

- the correlation time  $\tau_c = \min(\tau_{cE}, \tau_{cL})$  should be smaller than a quasi-linear diffusion time  $\tau_c \ll \tau_D$ . This condition is also at the heart of a mean field theory - so in some sense a prerequisite.
- the Kubo number should be smaller than 1,  $Ku = \min(\omega_b^2 \tau_c^2, \frac{\tau_c}{\tau_w}) \ll 1$ . This condition can be seen as a weak perturbation requirement.

The correlation time  $\tau_c$  is defined as  $\tau_c = \min(\tau_{cE}, \tau_{cL})$  with  $\tau_{cE, \mathbf{n}} = [\Delta\omega_{\mathbf{n}} + |\Delta\mathbf{n} \cdot \mathbf{v}_{g\mathbf{n}0}|]^{-1}$  and  $\tau_{cL} = [|\Delta\mathbf{n} \cdot (\frac{\partial\omega_{\mathbf{n}}}{\partial\mathbf{n}} - \boldsymbol{\Omega})|]^{-1}$ , where  $\Delta\omega$  and  $\Delta\mathbf{n}$  are the frequency and wave number spectral widths.

One may expect that the condition of random field is automatically fulfilled if trajectories are stochastic, i.e. if the previous condition  $S_{ch} \gg 1$  is fulfilled. However, the demonstration of this point requires to solve the Maxwell equations, which relate the perturbed Hamiltonian to the perturbed distribution function. This delicate question is beyond the present overview.

### 5.3 Renormalised quasi-linear theory

The average distribution function  $\langle F \rangle$  can be derived from the initial distribution function via a Green function noted  $\langle G \rangle$ , i.e.  $\langle F \rangle = \langle G \rangle F_0$ <sup>21</sup>. The Dyson equation Eq.(38) can then be written as

$$\langle G \rangle = G_0 + G_0 M \langle G \rangle$$

One would be tempted to write formally the solution as  $\langle G \rangle = (1 - M)^{-1} G_0$ , problem being of course to invert the mass operator  $M$  - a formidable task. However a proxy is built when using a normal distribution of random fields  $\mathbf{E}$ . In this case, the mass operator  $M$  is a sum of powers of two-field correlation functions, i.e. of diffusion operators in the framework of the quasi-linear theory. However diffusion was obtained by using a markovianisation procedure that is only valid for the average distribution function, and a priori not for fast evolving functions that appear in course of the Dyson expansion. Let us reformulate the objective in a another way. In the pedestrian approach, a key step was the resonant response function Eq.(29). This function is the Fourier-Laplace transform of the propagator  $G_0(\boldsymbol{\alpha}, \mathbf{J}, t/0, 0, 0) = \delta(\mathbf{J})\delta(\boldsymbol{\alpha} - \boldsymbol{\Omega}t)$ <sup>22</sup>, i.e.

$$G_{0\mathbf{n}\omega} = \int_0^{+\infty} dt \int \frac{d\boldsymbol{\alpha}}{(2\pi)^3} \exp(-i\mathbf{n} \cdot \boldsymbol{\alpha} + i(\omega + i\nu)t) \delta(\mathbf{J})\delta(\boldsymbol{\alpha} - \boldsymbol{\Omega}t) = R_0\delta(\mathbf{J})$$

The resonant function  $R_{\mathbf{n}\omega}^{(0)}$  gave rise to the Kronecker delta function  $\pi\delta(\omega - \mathbf{n} \cdot \boldsymbol{\Omega})$  in the limit  $\nu \ll \omega$ . This singular function is regularised after a summation over the fluctuation spectra. One may speculate that higher order terms in the Dyson equation may allow replacing the resonant function  $R_{\mathbf{n}\omega}^{(0)}$  by a regularised function

$$R_{\mathbf{n}\omega} = \frac{1}{-i(\omega - \mathbf{n} \cdot \boldsymbol{\Omega} + i\nu_d)}$$

<sup>21</sup>To lighten notations, the convolution symbol  $*$  is omitted.

<sup>22</sup>The Fourier-Laplace transform is performed by integrating over positive times  $t \geq 0$  instead of  $-\infty < t < +\infty$  in the conventional Fourier transform, and shifting  $\omega$  by a  $i\nu$  with  $\nu > 0$ .

where  $\nu_d$  comes from diffusive process of the quasilinear type. This is in practice a non linear resonance broadening process. It appears that this broadening resonant term can be calculated explicitly, under some reasonable assumptions. The procedure is described in Appendix G. The main result is the resonant function

$$R_{\mathbf{n}\omega} = \int_0^{+\infty} d\tau \exp \left[ i(\omega + i\nu - \mathbf{n} \cdot \boldsymbol{\Omega}) \tau - \frac{\tau^3}{3\tau_d^3} \right]$$

where

$$\frac{1}{\tau_d^3} = \sum_{\mathbf{n}'} \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \left( \mathbf{n}' \cdot \frac{\partial \boldsymbol{\Omega}}{\partial \mathbf{J}} \right) \frac{\mathcal{I}_{\mathbf{n}'\omega'}}{-i[\omega - \omega' - (\mathbf{n} - \mathbf{n}') \cdot \boldsymbol{\Omega} + i\nu]}$$

depends weakly on  $\boldsymbol{\Omega}$  in the vicinity of  $\boldsymbol{\Omega} = 0$ . The Dupree time  $\tau_d$  depends on  $(\omega, \mathbf{n})$ . Nevertheless, for low frequency  $\omega$  and wave numbers  $\mathbf{n}$  compared with typical fluctuation parameters, it gets close to  $\tau_d \simeq [D_{ik} \partial_{J_i} \Omega \partial_{J_k} \Omega]^{-1/3}$ , where  $D_{ik}$  is the quasilinear diffusion matrix. This simple result can be understood as follows. The phase of a wave is  $\phi = \mathbf{n} \cdot \boldsymbol{\alpha} - \omega t$ . For a particle that moves ballistically in the wave  $\boldsymbol{\alpha} = \boldsymbol{\Omega}(\mathbf{J})t + \boldsymbol{\alpha}_0$ , this phase is just  $\phi = \Omega t$ , where  $\Omega = \mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{J}) - \omega$  up to a constant. Resonance occurs when the phase remains constant, i.e. when  $\Omega = 0$ . If the particle diffuses under the effect of fluctuations other than the considered wave (typically at smaller scales), then  $\langle \delta J_i \delta J_k \rangle = D_{ik} t$ , so that the phase variation evolves in time as  $\langle \delta \phi^2 \rangle = D_{ik} \partial_{J_i} \Omega \partial_{J_k} \Omega t^3$ . Decorrelation occurs when  $\delta \phi \simeq \pi$ , hence after a time of the order of the Dupree time - see Fig.16 for an illustration.

This important result calls for several remarks:

- a new time  $\tau_d$  appears, called Dupree time after the physicist who introduced it first. Resonant particles can now be defined properly as particles such that  $\Omega \tau_d \ll 1$ .
- the regularising parameter  $\nu$  can be neglected as long as  $\nu \tau_d \ll 1$ .
- the singular resonant function has now be regularised by a turbulent diffusion matrix via the Dupree time.
- the diffusion coefficient now depends on a resonant function, which depends itself on a quasi-linear-like diffusion coefficient. Hence an iterative procedure must be used, which is called “renormalisation”. It is described in Appendix G. As long as the Kubo number stays smaller than 1, the quasi-linear diffusion coefficient is not much affected by this effect.

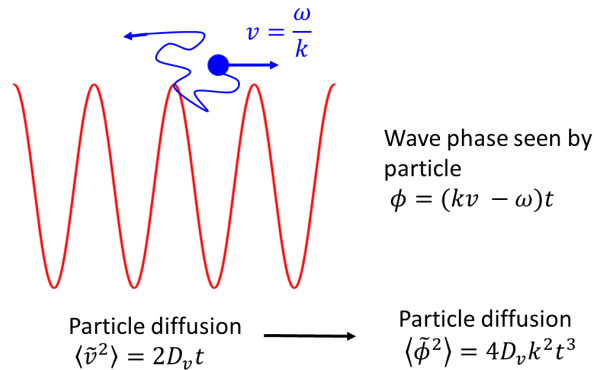


Figure 16: Physical meaning of the Dupree time for the case of a 1D wave  $\exp[i(kx - \omega t)]$ . The phase for a particle that moves ballistically in the wave is  $\phi = kv - \omega t$ . Resonance is met when  $v = \omega/k$ . However a diffusion in velocity implies that  $\langle \delta v^2 \rangle = D_v t$ , so that the phase variation evolves as  $\langle \delta \phi^2 \rangle = D_v t^3$ . Decorrelation occurs when  $\delta \phi \simeq \pi$  after a time  $t \simeq \tau_d = (Dk^2)^{-1/3}$ .

## 6 Nearly degenerate Hamiltonian perturbations\*

### 6.1 General considerations

So far, two extreme cases have been addressed:

- a single Hamiltonian perturbation. This situation can be extended to the case where several perturbations are present, but the corresponding resonant surfaces are far away from each other. Collisions play an essential role in the production of entropy. An island forms near each resonant surface, which strongly reduces the production of entropy in the weakly collisional regime. This is a consequence of particle trapping within the island, and subsequent flattening of the distribution function.
- several Hamiltonian perturbations whose resonant surfaces are close to each other. A Chirikov overlap parameter is defined that measures the degree of island overlap, i.e. compares the island width with the distance between resonant surfaces. Whenever the Chirikov overlap parameter exceeds 1, trajectories become chaotic. Under some conditions quasilinear theory applies, and the motion is diffusive. This “stochastic diffusion” is responsible for entropy production.

A third situation is frequently met, where several perturbations resonate on nearby resonant surfaces, but the respective islands are oriented differently. We call this case “degenerate” since the resonant surfaces are almost the same. The simplest prototype for this situation is the case of two perturbations that resonate at the same place (or nearby) in the phase space - see Fig.(17). In this case, the entropy production can be due to collisions or stochastic diffusion, depending on the parameters. This is a complex question, that we treat first in a simplified manner. To identify the difficulty, let us consider two perturbations with frequencies and wave numbers  $(\omega, \mathbf{n})$  and  $(\omega', \mathbf{n}')$ . Each single perturbation is associated with an island near its own resonant perturbation. Intuition suggests that strong non linear interactions are expected if the islands “overlap”: in this case a particle trajectory cannot be dictated by a single perturbation. The Chirikov parameter is defined as the sum of the island half-widths divided by the distance between resonant surfaces. A chaotic motion is expected when this parameter is greater than one, as seen in Section 5. However peculiar situations must be considered. Let us remember that near resonant surfaces, the displacements in the action space are respectively oriented along the triplets  $\mathbf{n}$  and  $\mathbf{n}'$ . Also the phases are  $\xi = \omega t - \mathbf{n} \cdot \boldsymbol{\alpha}$  and  $\xi' = \omega' t - \mathbf{n}' \cdot \boldsymbol{\alpha}$ . If the islands are oriented very differently, a chaotic motion may never set on, even if the perturbation amplitudes are large. Let us illustrate this point with the case where  $\omega = \omega' = 0$  and  $n_1 = n'_1 = 0$ , a situation of relevance in fusion devices. Indeed this situation is met in tokamaks when the magnetic field corrugations due to the finite number of coils (“magnetic ripple”) are accounted for, or in stellarators where a helical magnetic field coexists with a toroidal field. In these two examples, perturbations do not depend on time (angle  $\alpha_0$ , nor on the cyclotron angle  $\alpha_1$ ). This limit case is modelled by considering an Hamiltonian of the form

$$H(\boldsymbol{\alpha}, \mathbf{J}, t) = H_{eq}(\mathbf{J}) + \tilde{H}(\boldsymbol{\alpha}, \mathbf{J}, t) + \tilde{H}'(\boldsymbol{\alpha}, \mathbf{J}, t)$$

where

$$\tilde{H}(\boldsymbol{\alpha}, \mathbf{J}) = -h \cos(\mathbf{n} \cdot \boldsymbol{\alpha})$$

and

$$\tilde{H}'(\boldsymbol{\alpha}, \mathbf{J}) = -h' \cos(\mathbf{n}' \cdot \boldsymbol{\alpha})$$

where  $\mathbf{n} = (0, 0, n_2, n_3)$  and  $\mathbf{n}' = (0, 0, n'_2, n'_3)$ . The amplitude perturbations  $h$  and  $h'$  are assumed to depend slowly on the action variables. The phases reduce to  $\xi = n_2 \alpha_2 + n_3 \alpha_3$ , and  $\xi' = n'_2 \alpha_2 + n'_3 \alpha_3$ . The resonant surface associated with the first perturbation is determined by the equation  $\Omega = n_2 \Omega_2 + n_3 \Omega_3 = 0$ . This condition defines a surface in

the action space, such that<sup>23</sup>  $J_3 = G(J_2)$ . A iota function is also defined as minus the gradient  $dJ_3/dJ_2$  along the resonant surface

$$\iota(J_2) = -\frac{dG}{dJ_2}$$

The second resonant surface is defined in the same way, with  $\mathbf{n}$  replaced by  $\mathbf{n}'$ , hence  $\Omega' = n'_2\Omega_2 + n'_3\Omega_3 = 0$ . This second resonant surface is close to the first one, and may actually intercept or be identical to the first one.

Let us consider also the extreme case where the two resonant surfaces are the same. The considerations above suggest that chaos should be maximum. This will be the case for instance if  $n'_2 = n_2$ , and  $n'_3 = n_3 + 1$ . However, this is not obviously true if  $n_3 = 0$  and  $n'_2 = 0$  since in this case the islands are oriented very differently, and overlap only in some limited area of the phase space. In the latter case, the dominant transport mechanism is related to particle trapping in a primary perturbation, say  $(n_2, n_3)$ , that is affected by a secondary perturbation, here  $(n'_2, n'_3)$ . Hence the particle of motion remains regular in this case, and a calculation of the Zakharov-Karpman type should be conducted to calculate the entropy production rate.

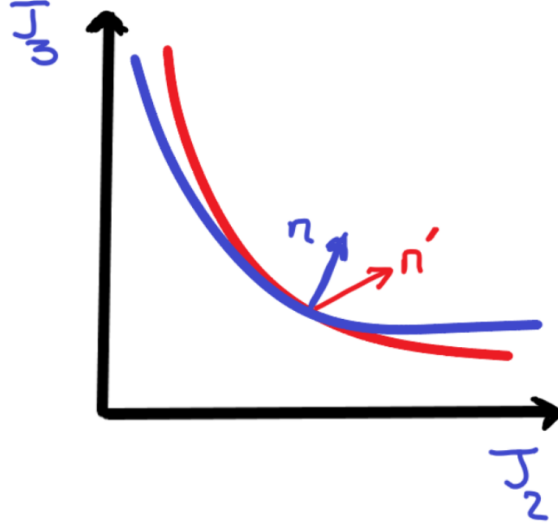


Figure 17: Example of 2 neighbouring resonant surfaces in the action space.

## 6.2 Condition for particle trapping

A first difficulty is the modification of the condition for trapping in a secondary perturbation due to the primary one. This can be understood as follows. The phase evolves in the primary perturbation as  $\frac{d\xi}{dt} = \Omega$ , while the secondary phase obeys to the equation  $\frac{d\xi'}{dt} = \Omega'$ . Near the first resonant surface,  $\mathbf{J} = \mathbf{J}_R + \mathbf{n}I$ , and  $\Omega = CI$ ,  $\Omega' = C^*I$ , where  $C$  is given by Eq.(24), and  $C^*$  is an “hybrid” curvature

$$C^* = n_i n'_j \left. \frac{\partial^2 H_{eq}}{\partial J_i \partial J_j} \right|_{\mathbf{J}=\mathbf{J}_R} \quad (40)$$

Let us now consider the adiabatic limit, where the dynamics along the resonant surface is slow compared with the one across the resonant surfaces. The two phases are then related

<sup>23</sup>This explicit relation excludes de facto multivalued function. Though such a situation may occur - in fusion devices for instance - it is not addressed here.



via the relationship  $\xi' - \xi'_0 = \sigma(\xi - \xi_0)$ , where  $\sigma = C^*/C = \Omega'/\Omega$ , and  $(\xi_0, \xi'_0)$  evolve slowly in time. Let us suppose that  $\sigma \gg 1$ . The full Hamiltonian perturbation can be expressed versus the phase  $\xi'$  as

$$\tilde{H}(\mathbf{J}, \xi') = -h \cos \left( \xi_0 + \frac{1}{\sigma}(\xi' - \xi'_0) \right) - h' \cos(\xi')$$

Since  $\sigma \gg 1$ , the condition for trapping in the primary perturbation is not much changed since  $\xi = \xi_0 + \frac{1}{\sigma}(\xi' - \xi'_0) \simeq \xi_0$ . However the condition for trapping in the secondary perturbation is modified by the first one. Indeed trapping is subject to the existence of local extrema of  $\tilde{H}$  when expressed as a function of  $\xi'$  (see Fig. 18). The existence of a vanishing derivative of  $\tilde{H}$  in  $\xi'$  imposes that

$$\sin(\xi') = -\alpha^* \sin(\xi_0) \quad (41)$$

up to corrections of order  $1/\sigma$ . The parameter  $\alpha^*$  measures the relative amplitude of perturbations and is defined as

$$\alpha^* = \frac{h}{\sigma h'}$$

It appears immediately that trapping in the secondary perturbation requires  $Y \leq 1$ , where  $Y$  is defined as

$$Y = |\alpha^* \sin(\xi_0)|$$

This condition defines a domain  $\mathcal{D}$  in the phase space. If  $\sigma \ll 1$ , the role of primary and secondary perturbations is exchanged, and the condition becomes  $Y = |\frac{1}{\alpha^*} \sin(\xi'_0)| \leq 1$ .

The depth of the Hamiltonian perturbation can be calculated as follows. If  $Y = 0$ , the minima and maximum correspond respectively to  $\xi'_1 = 0$  and  $\xi'_2 = \pi$ , and the depths is just  $\Delta H = 2h'$ . Let us now assume that  $Y$  is small but finite. Eq.(41) implies that minima and maxima of  $\tilde{H}(\mathbf{J}, \xi')$  are reached for respectively  $\xi' = \xi'_1 = \arcsin Y$  and  $\xi' = \xi'_2 = \pi - \arcsin Y$ . The depth of the Hamiltonian well  $\Delta H$  is computed by making the difference between  $\tilde{H}$  computed at  $\xi' = \xi'_2$  and  $\xi' = \xi'_1$ . A straightforward calculation provides

$$\Delta H = \tilde{H}(\mathbf{J}, \xi'_2) - \tilde{H}(\mathbf{J}, \xi'_1) \simeq 2h' \left[ \sqrt{1 - Y^2} - Y \arccos Y \right]$$

It may be more accurate in some cases to replace  $h'$  with  $\Delta H/2$ . This complication will be avoided, but it will be kept in mind that some of the expressions derived below are valid only in the domain  $\mathcal{D}$  or its complementary.

## 6.3 Entropy production rate for a double Hamiltonian perturbation

The objective of this section is the computation of the entropy production rate due to collisions. Schematically, one can define a primary perturbation, say  $\tilde{H}$ , and a secondary perturbation,  $\tilde{H}'$ . The entropy production rate associated with the primary perturbation has already been calculated, and is given by Eq.(27). Hence the entropy production rate due to the secondary perturbation remains to be computed. Obviously the role of the primary and secondary perturbations can be exchanged, so that the total entropy production rate contains 4 contributions. The calculation is somewhat cumbersome and is detailed in Appendices H and I. A short summary is produced in the following sections.

### 6.3.1 Key parameters

Let us define first the main parameters that determine the validity domain for each regime. It is important to realise that the main contribution to transport comes from particles trapped in the primary perturbation, the motion of which is affected by the secondary

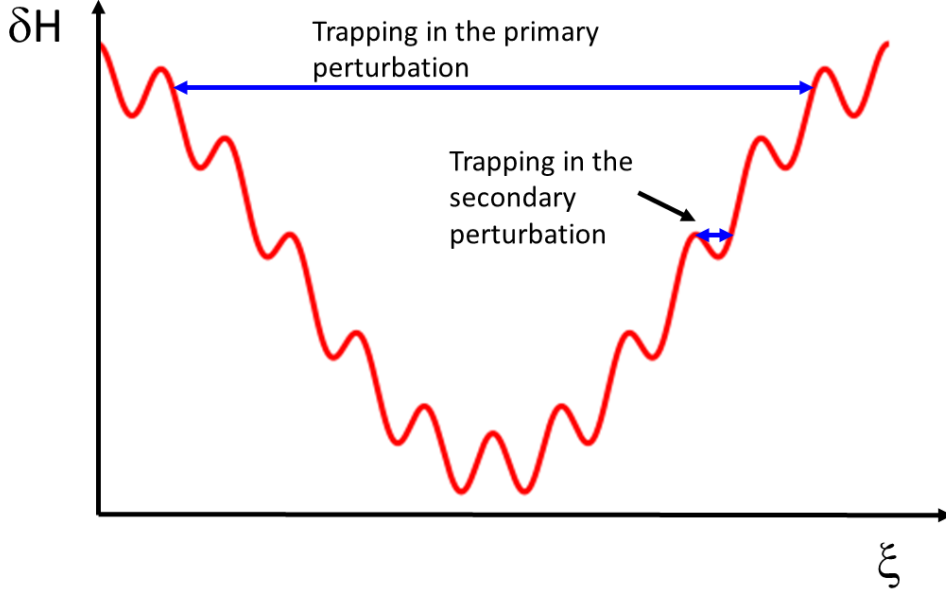


Figure 18: Particle trapping in primary and secondary perturbations.

perturbation in presence of collisions. An important parameter is the effective detrapping collision frequency in the primary perturbation, which is potentially modified by the secondary perturbation. A change of sign of the displacement in the action space due to  $\tilde{H}'$  occurs when the phase  $\xi'$  changes by an amount of  $\pi$ . It is this change of sign that matters for the random walk responsible for diffusion.

Let us recall that the two phases are related via the relationship  $\xi' - \xi'_0 = \sigma(\xi - \xi_0)$ . The phases  $(\xi_0, \xi'_0)$  evolve slowly in time, so that their exact values does not matter much in this discussion. We set  $\xi_0 = \xi'_0 = 0$  to simplify the discussion. When  $\sigma \ll 1$ , the effect of the secondary perturbation on the force felt by the particle in the primary perturbation is weak since decorrelation occurs when  $\xi' \simeq \pi$ , i.e. when  $\xi \simeq \pi/\sigma \gg \pi$ , i.e. well after the change of sign in the first perturbation  $\xi \simeq \pi$ . The effective detrapping collision time in the primary perturbation is therefore unchanged: decorrelation in the secondary perturbation occurs when the primary phase is shifted by  $\xi \simeq \pi$ . The relation  $H_\Omega = \frac{1}{2}\Omega^2 - \omega_b^2 \cos \xi$  implies that this corresponds to a variation  $\delta\Omega \simeq \omega_b$ . The effective collision frequency, defined as the inverse of the time needed by a particle to cross the trapped domain associated with the primary perturbation, is therefore  $\nu_{eff} \simeq D_\Omega/\omega_b^2$ , where  $D_\Omega = \left\langle \frac{\Delta\Omega^2}{2\Delta t} \right\rangle$  is a diffusion coefficient in the variable  $\Omega$  that scales as a cubic frequency. In the opposite case,  $\sigma \gg 1$ , the secondary perturbation plays an important role since decorrelation occurs when  $\xi' \simeq \pi$ , and thus  $\xi \simeq \pi/\sigma \ll \pi$ . A phase shift  $\xi \simeq \pi/\sigma$  implies a shift  $\delta\Omega \simeq \omega_b/\sigma$  for deeply trapped particles. The effective collision frequency is thus enhanced by a factor  $\sigma^2$ , i.e.  $\nu_{eff} \simeq \sigma^2 D_\Omega/\omega_b^2$ . So the general expression of the effective collision frequency is

$$\nu_{eff} \simeq \frac{D_\Omega}{\omega_b^2} \max(1, \sigma^2) \quad (42)$$

Another key quantity is the drift frequency of the secondary phase on the primary resonant surface. This comes from the fact that in the relation  $\xi' - \xi'_0 = \sigma(\xi - \xi_0)$ ,  $\xi_0$  and  $\xi'_0$  can move slowly in time, related to the dynamics along the resonant surface, while the part  $\sigma\xi$  is rather related to the motion across the resonant surface. It is shown in Appendix H that this drift frequency, noted  $\omega_d$ , is

$$\omega_d = (\mathbf{n}' - \sigma \mathbf{n}) \cdot \boldsymbol{\Omega}$$

The third characteristic frequency is the bounce frequency of particles trapped in the primary perturbation  $\omega_b = \sqrt{Ch}$ . So in terms of typical frequency, one can define two frequency ratios, e.g.  $\omega_d/\omega_b$ , and  $\nu_{eff}/\omega_b$ . Other dimensionless parameters are the curvature ratio  $\sigma = C^*/C$  and the ratio of Hamiltonian perturbations  $h'/h$ . This set fully characterises the collisional entropy production (see Fig. 19). The stochastic case involves a few additional parameters, which will be discussed later on.

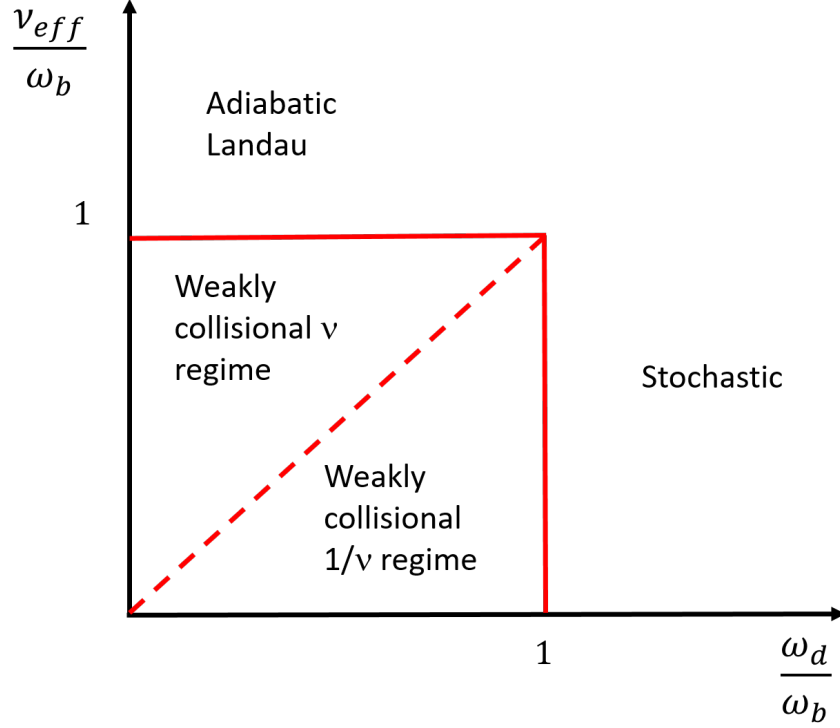


Figure 19: Various transport regimes for a double resonant perturbation in the space  $\omega_d/\omega_b$ , and  $\nu_{eff}/\omega_b$ .

### 6.3.2 Entropy production rate for a double perturbation

All calculations done, the final expression of the entropy production rate due to the secondary perturbation reads as follows. In the collisional case  $\nu_{eff} \gg \omega_b$ , the production rate is of the Landau type, similar to the single perturbation case in the same collisional regime

$$\dot{S}_{res} [U_{eq}, \partial_{\mathbf{J}} U_{eq}^{\dagger}] = \frac{1}{T_0^2} \frac{\pi}{2} \int d\gamma \delta(\mathbf{n}' \cdot \boldsymbol{\Omega}) F_{eq}(\mathbf{J}) h'^2 \left( (\mathbf{n}' - \sigma \mathbf{n}) \cdot \frac{\partial U_{eq}^{\dagger}}{\partial \mathbf{J}} \right)^2$$

In the weakly collisional regime  $\nu_{eff} \ll \omega_b$ , two situations may be met :

i) an adiabatic regime where  $n_d \Omega_d \ll \Omega_b$

$$\begin{aligned} \dot{S}_{res} (U_{eq}, \partial_{\mathbf{J}} U_{eq}^{\dagger}) &= \frac{1}{T_0^2} \int d\gamma \delta(\mathbf{n} \cdot \boldsymbol{\Omega}) F_{eq}(\mathbf{J}) \Lambda(\mathbf{J}) h'^2 \\ &\quad \frac{\nu_{eff} \omega_b}{[(\mathbf{n}' - \mathbf{n} \sigma) \cdot \boldsymbol{\Omega}]^2 + \nu_{eff}^2} \left( (\mathbf{n}' - \mathbf{n} \sigma) \cdot \frac{\partial U_{eq}^{\dagger}}{\partial \mathbf{J}} \right)^2 \end{aligned}$$

where

$$\Lambda(\mathbf{J}) = \begin{cases} \frac{16}{9\pi} & \text{if } \sigma \ll 1 \\ \frac{1}{\pi\sigma} & \text{if } \sigma \gg 1 \end{cases}$$

and

$$\sigma = \frac{n'_2\iota + n'_3}{n_2\iota + n_3}$$

$$\nu_{eff} = \text{Max}(1, \sigma^2) \frac{D_\Omega}{4\omega_b^2}$$

ii) a stochastic regime where  $n_d\Omega_d \gg \Omega_b$ . It is useful to define the following frequency and time

$$\omega_I = \left[ C_\Lambda \sqrt{\frac{2}{\pi\sigma\theta_b}} h' \right]^{1/2} \quad ; \quad \tau = \frac{K_\Lambda}{C_\Lambda}$$

where

$$K_\Lambda = n_b \left. \frac{\partial(\sigma\theta_b)}{\partial J_b} \right|_{\mathbf{J}=\mathbf{J}_R} + n_d \left. \frac{\partial(\sigma\theta_b)}{\partial J_d} \right|_{\mathbf{J}=\mathbf{J}_R}$$

$$C_\Lambda = n_b \left. \frac{\partial\omega_{n_b}}{\partial J_b} \right|_{\mathbf{J}=\mathbf{J}_R} + n_d \left. \frac{\partial\omega_{n_b}}{\partial J_d} \right|_{\mathbf{J}=\mathbf{J}_R}$$

and  $\omega_{n_b} = n_b\omega_b + n_d\omega_d$ .

Depending on the value of  $\tau\omega_I$ , two sub-regimes can be identified [31]

- the weak perturbation regime defined by the condition  $\tau\omega_I \leq 2$ . The Chirikov overlap parameter is  $S_I = \frac{4\omega_I}{\Omega_b}$  and trajectories are stochastic whenever  $S_I \geq S_c$ , where  $S_c$  is a number of order 1.
- the strong perturbation regime  $\tau\omega_I \geq 2$ . The Chirikov parameter becomes  $S_{II} = 2\frac{\tau\omega_I^2}{\Omega_b}$ . Stochasticity occurs whenever  $S_{II} \geq S_c$  [32].

Let us note the relationship

$$S_{II} = \frac{\tau\omega_I}{2} S_I$$

This relationship shows that the stochasticity condition  $S_I \geq S_c$  *implies* that  $S_{II} \geq S_c$  in the strong perturbation regime since the latter is conditioned to  $\tau\omega_I \geq 2$ . The condition  $\tau\omega_I = 2$  defines a critical value of  $h'$  that depends on  $h$  and other geometrical and plasma parameters.

For values of  $S_I$  much larger than the stochasticity threshold  $S_c$ , and therefore  $S_{II} \geq S_c$  in the strong perturbation regime, a diffusion coefficient can be computed under some hypothesis detailed in Appendix J. The corresponding entropy production rate is

$$\dot{S}_{res} \left( U_{eq}, \partial_{\mathbf{J}} U_{eq}^\dagger \right) = \frac{1}{T_0^2} \frac{\pi}{8} \int d\gamma F_{eq} \frac{h'^2}{\Omega_b} \Lambda(\mathbf{J}) \left( (\mathbf{n}' - \sigma\mathbf{n}) \cdot \frac{\partial U_{eq}^\dagger}{\partial \mathbf{J}} \right)^2$$

where the form factor  $\Lambda(\mathbf{J})$  is such that

- if  $\sigma \ll 1$ , then  $\Lambda(\mathbf{J}) = 1$
- if  $\sigma \gg 1$ , then

$$\Lambda(\mathbf{J}) = \frac{2}{\pi\sigma\theta_b} \min \left( 1, \frac{S_{st}}{S_{II}} \right) \simeq \frac{2}{\pi\sigma\theta_b} \frac{1}{1 + \frac{S_{II}}{S_{st}}}$$

where

$$S_{II} = \sqrt{\frac{2}{\pi\sigma\theta_b}} \frac{K_\Lambda h'}{\Omega_b}$$

This expression bears the advantage of being continuous across the transition from weak to strong perturbation regime. However it predicts a transition when

$$S_{II} = \frac{\tau\omega_I}{2} S_I \simeq S_{st}$$

Consistency imposes that  $S_{st} > S_c$  - Grua et al. [31] recommends  $S_{st} = 2$ . Given the expression of the form factor  $\Lambda(\mathbf{J})$ , the quasilinear diffusion  $D_{QL}$  that applies in the weak perturbation regime reads

$$D_I = D_{QL} = \frac{1}{2\sigma\theta_b} \frac{h'^2}{\Omega_b}$$

The diffusion coefficient in the strong regime is just  $D_I = D_{QL}S_{st}/S_{II}$ . As expected the quasilinear diffusion coefficient behaves as  $h'^2$  in the weak perturbation regime, and  $h'$  in the strong perturbation regime. It is sketched on Fig.20.

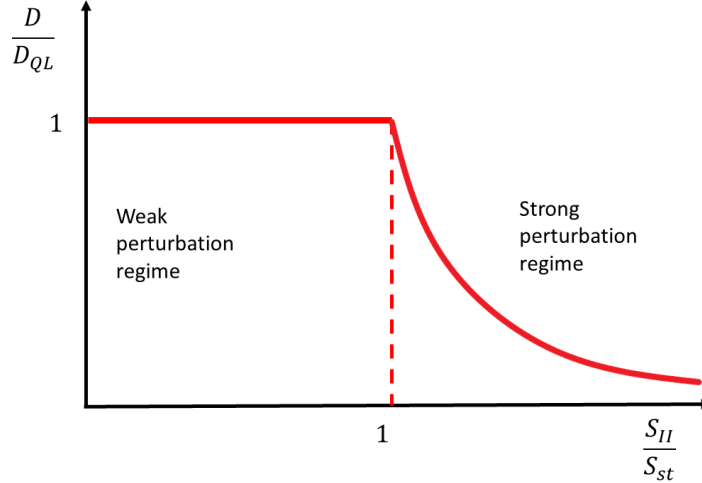


Figure 20: Diffusion coefficient normalised to the quasilinear value versus the Chirikov overlap parameter.

## 7 Conclusion

It is time to wrap up the main results of this lecture note. What is left of the initial objective, which was to predict the time evolution of the mean distribution function of an Hamiltonian dynamical system? Let us remind first the assumptions :

- the unperturbed Hamiltonian is integrable and the phase space is bounded. This allows defining a set of angle and action variables. The actions are invariant of motions, and the time derivatives of the angle variables are resonant angular frequencies, which depend on the actions only.
- a scale separation allows a clear separation between the mean and perturbed distribution function. More precisely the Hamiltonian can be separated in a mean integrable part plus fluctuations. The mean distribution function, defined as an average over the angles and time, evolves on time scales that are much larger than those of fluctuations.

The first result deals with the mean distribution function. In absence of collisions, it has to be a function of the invariants of motions, thus justifying its definition as an average of the full distribution function over the angle variables. With collisions only, the distribution function is known to be a local Maxwellian, usually not a function of the invariants of motion. Hence the steady-state “equilibrium” distribution function has to be a compromise between a local Maxwellian and a function of the actions. The problem of finding the mean distribution is expressed in a variational form, which physically expresses that the mean distribution function minimises the entropy production rate. This formulation can be extended to compute the slow time evolution of the mean distribution function. In the latter case, it appears that the evolution of the mean distribution function can be written

in a conservative form, the contribution from the motion of particles in the Fokker-Planck equation can be written as the divergence of a flux in the action space. This is an essential result: this methodology gives access to particle transport in the phase space. More precisely, each moment of the distribution function is associated with a flux. Fluxes are related to thermodynamic forces via a transport matrix. Under some conditions, this matrix is symmetric, consistently with Onsager symmetry prescription. It remains to compute these fluxes explicitly, or equivalently to compute the entropy production rate.

The special case of a single perturbation is treated first. The Vlasov equation can be solved case when it is linearised. It appears that the solution is singular near a resonant surface in the action space, where the phase of the Hamiltonian perturbation stays constant in time, hence optimising the exchange of energy between the particles and the perturbation. A closer look at the trajectories shows that they lie on surfaces of constant energy. The set of constant energy surfaces bear the shape of an island near the resonant surface in the phase space. Within the island, the perturbation phase felt by particles bounces back and forth in time - these are called trapped particles. Outside the island, the phase felt by particles move forward or backward without bouncing - these are called passing particles. These two domains in the action space are separated by a separatrix, that defines the island boundary. An exact solution of the Fokker-Planck equation, i.e. with collisions, can be computed. The mean distribution is found to flatten within the island, a consequence of mixing due to particles that spans surfaces of constant energy on a short time scale, combined with a slow collisional diffusion across these iso-energy surfaces. On the contrary, the mean distribution function develops strong gradients at the separatrix between trapped and passing particle domains in the phase space, i.e. near the island separatrix. Not surprisingly the entropy production rate is maximum in the vicinity of the island separatrix, where strong gradients develop.

The case of multiple perturbations is then addressed. A Chirikov overlap parameter is defined that measures the degree of island overlap, i.e. compares the island width with the distance between adjacent resonant surfaces. Whenever the Chirikov overlap parameter exceeds 1, trajectories become chaotic. 3 limits are then identified:

- the resonant surfaces associated with perturbations are far away from each other in the phase space, i.e. Chirikov overlap parameter is smaller than 1. This is equivalent to treat several single perturbations independently. Collisions play an essential role in the production of entropy, which is just the sum of individual entropy production rates near each island.
- resonant surfaces are close to each other, i.e. the Chirikov overlap parameter is larger than 1. Under some condition quasilinear theory applies, and the motion is diffusive. This “stochastic diffusion” is responsible for entropy production. It does satisfy the Onsager symmetry constraint.
- resonant surfaces are close, but the direction of Hamiltonian perturbations across resonant surfaces are not aligned. This is the most intricate situation since collisions or chaos may prevail depending on the parameters. The case of two Hamiltonian perturbations whose resonant surfaces are close in the action space is addressed as a prototype of this peculiar configuration. After some effort, the entropy production rate is calculated. Collisions or chaos can indeed contribute most to transport across the resonant surfaces, depending on collisionality and the hierarchy of resonant frequencies. The parameter domain that defines the various regimes is made available.

At this stage, a sceptical reader may question the relevance of all these efforts. The reason is that these various cases cover most of the situations encountered in plasma physics. Here a major issue is to compute the transport due to collisions and/or turbulence. It appears that in many cases, transport coefficients can actually be computed from the expressions given in this lecture, *provided the spectrum of fluctuations is given*.

A second criticism lies with the almost infinite range of possibilities offered by numerical simulations. So why bothering with analytical formula? One reason is that codes, even the most powerful ones, need be verified. Analytical formula allows a thorough verification, i.e. check they provide the right answer whenever an exact solution is known. Besides, it appears that the computation time may be unreasonably large. This is quite frequently the case for turbulence simulations. It is then useful to develop reduced models, based on known analytical formula. Reason why quasi-linear models underlie most successful turbulent transport models. Finally, deep understanding of transport processes provides us with an intuition which is certainly useful when running experiments.

# APPENDICES

## A Collision operator

### A.1 Fokker-Planck operator

We are interested here in the evolution of the distribution function  $F_a(\mathbf{z})$  of a species  $a$ , where  $\mathbf{z} = (\mathbf{x}, \mathbf{p}, t)$  ( $\mathbf{x}$  and  $\mathbf{p}$  are the particle position and momentum). This evolution is dictated by the motion of particles  $a$ , and collisions with other species  $b$  (including other particles of species  $a$ ). This evolution is ruled by the Fokker-Planck equation

$$\frac{\partial F_a}{\partial t} - \{H_a, F_a\} = \sum_b C_{ab}(F_a, F_b)$$

where

$$\{H_a, F_a\} = \frac{\partial H_a}{\partial \mathbf{x}} \cdot \frac{\partial F_a}{\partial \mathbf{p}} - \frac{\partial H_a}{\partial \mathbf{p}} \cdot \frac{\partial F_a}{\partial \mathbf{x}}$$

is a Poisson bracket,  $H_a(\mathbf{z})$  the Hamiltonian of species  $a$ ,

$$C_a(F_a) = \sum_b C_{ab}(F_a, F_b)$$

is the collision operator of species  $a$ , and  $C_{ab}(F_a, F_b)$  is the collisional operator between particles of species  $a$  and  $b$

$$C_{ab}(F_a, F_b) = \frac{1}{2} \gamma_{ab} \frac{\partial}{\partial \mathbf{p}} \cdot \left\{ \int d^3 \mathbf{p}' \mathfrak{L} \cdot \left[ F_b(\mathbf{z}') \frac{\partial F_a(\mathbf{z})}{\partial \mathbf{p}} - F_a(\mathbf{z}) \frac{\partial F_b(\mathbf{z}')}{\partial \mathbf{p}'} \right] \right\}$$

with<sup>24</sup>  $\mathbf{z}' = (\mathbf{x}, \mathbf{p}', t)$

$$\gamma_{ab} = 4\pi \frac{e_a^2 e_b^2}{(4\pi\epsilon_0)^2} \ln \Lambda$$

and

$$\mathfrak{L} = \frac{u^2 \mathbf{I} - \mathbf{u} \mathbf{u}}{u^3}$$

is a tensor. Here  $e_a$  is the charge,  $\ln \Lambda$  the Coulombian logarithm and

$$\mathbf{u}(\mathbf{x}, \mathbf{p}, \mathbf{p}', t) = \frac{\partial H_a(\mathbf{z})}{\partial \mathbf{p}} - \frac{\partial H_b(\mathbf{z}')}{\partial \mathbf{p}'}$$

is the relative velocity. The structure of the collision operator  $C_{ab}$  is of the Fokker-Planck type

$$C_{ab}(F_a, F_b) = -\frac{\partial}{\partial p_i} (\mathfrak{A}_{ab,i} F_a) + \frac{1}{2} \frac{\partial^2}{\partial p_i \partial p_j} (\mathfrak{D}_{ab,ij} F_a)$$

where

$$\begin{aligned} \mathfrak{A}_{ab,i} &= \left( \frac{1}{m_a} + \frac{1}{m_b} \right) \gamma_{ab} m_b \frac{\partial K_b}{\partial p_i} \\ \mathfrak{D}_{ab,ij} &= \gamma_{ab} m_a^2 \frac{\partial^2 G_b}{\partial p_i \partial p_j} \end{aligned}$$

and  $\mathcal{H}_b$  and  $\mathcal{G}_b$  are the Rosenbluth potentials defined in Ref.[33]

$$\begin{aligned} K_b(\mathbf{z}) &= \int d^3 \mathbf{p}' F_b(\mathbf{z}') \frac{1}{u}(\mathbf{z}, \mathbf{p}') \\ G_b(\mathbf{z}) &= \int d^3 \mathbf{p}' F_b(\mathbf{z}') u(\mathbf{z}, \mathbf{p}') \end{aligned}$$

The collisional operator in Eq.(43) is the starting point for expressing the Fokker-Planck equation in a variational form.

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<sup>24</sup>Note that the collision operator is local in space and time, but not in momentum



## A.2 Thermodynamical potentials

Let us now write each distribution function in the form Eq.(12), i.e.

$$F_a = \exp\left(-\frac{H_a - U_a}{T_0}\right)$$

where  $T_0$  is a constant<sup>25</sup>. Here the function  $U_a(\mathbf{z})$  is a thermodynamical potential that measures the departure from thermodynamical equilibrium. Using the property  $\mathfrak{L} \cdot \mathbf{u} = 0$ , it appears readily that  $C_{ab}(F_{Ha}, F_{Hb}) = 0$ , where

$$F_{H_a} = \exp\left(-\frac{H_a}{T_0}\right)$$

It is then convenient to define a collision operator on the potentials  $(U_a, U_b)$  as follows

$$C_{ab}(F_a, F_b) = F_a \frac{1}{T_0} C_{ab}(U_a, U_b)$$

or equivalently

$$C_{ab}(U_a, U_b) = \frac{1}{F_a} \frac{1}{2} \gamma_{ab} \frac{\partial}{\partial \mathbf{p}} \cdot \int d^3 \mathbf{p}' F_a F_b \mathfrak{L} \cdot \left[ \frac{\partial U_a}{\partial \mathbf{p}} - \frac{\partial U_b}{\partial \mathbf{p}'} \right]$$

where  $U_a$  is a function of  $\mathbf{z} = (\mathbf{x}, \mathbf{p}, t)$ , while  $U_b$  is a function of  $\mathbf{z}' = (\mathbf{x}, \mathbf{p}', t)$ . Note that at this stage, the operators  $C_{ab}$  are not linearised with respect to the functions  $U_a$ . With these notations, and assuming that the system is static, the Fokker-Planck equation can be recast as

$$\frac{\partial U_a}{\partial t} - \{H_a, U_a\} = \frac{\partial H_a}{\partial t} + \sum_b C_{ab}(U_a, U_b)$$

## A.3 Properties of the collision operator

Collision operators verify some important properties which are summarised here - demonstrations can be found in [6]. First a collision operator grants the positivity of distribution functions provided that the initial distribution functions are positive (and of course ignoring the Poisson bracket in the evolution equation). Second, collision operators conserve the density for each species, and also total momentum and energy (total means here after summation over all species). Also functions in the kernel of collision operators, i.e. such that  $C_{ab}(U_a, U_b) = 0$ , are local Maxwellian, i.e. their thermodynamic potential read

$$U_a(\mathbf{x}, \mathbf{p}) = A_a(\mathbf{x}) + \mathbf{V}_a(\mathbf{x}) \cdot \mathbf{p} + \beta_a(\mathbf{x}) H_a(\mathbf{x}, \mathbf{p}) \quad (43)$$

As a consequence, a collision operator induces a relaxation of the distribution towards a local Maxwellian. Finally let us define an entropy as

$$S_a = - \int d\gamma F_a \ln F_a \quad (44)$$

where  $d\gamma = d^3 \mathbf{x} d^3 \mathbf{p}$  is the volume element in the phase space. It appears that collision operators ensure that entropy increases in time, in accordance with the Boltzmann H-theorem.

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<sup>25</sup>Note that the same reference temperature  $T_0$  is chosen for all species

## A.4 Collisional entropy variational principle

The set of Fokker-Planck equations for all species is equivalent to find an extremum of the following entropy functional for all variations of  $U_a^\dagger$

$$\delta\dot{S} = \frac{2}{T_0^2} \sum_a \int d\gamma F_a U_a^\dagger \left( \frac{\partial U_a}{\partial t} - \{H_a, U_a\} \right) + \sum_{ab} \dot{S}_{ab}^{coll}(U_a, U_b^\dagger) \quad (45)$$

where the functional  $\dot{S}_{ab}^{coll}$  is defined as

$$\begin{aligned} \dot{S}_{ab}^{coll}(U_a, U_b^\dagger) &= \frac{1}{T_0^2} \frac{\gamma_{ab}}{2} \int d^3\mathbf{x} d^3\mathbf{p} d^3\mathbf{p}' F_a F_b \\ &\quad \left[ \frac{\partial U_a}{\partial \mathbf{p}} - \frac{\partial U_b}{\partial \mathbf{p}'} \right] \cdot \mathfrak{L} \cdot \left[ \frac{\partial U_a^\dagger}{\partial \mathbf{p}} - \frac{\partial U_b^\dagger}{\partial \mathbf{p}'} \right] \end{aligned} \quad (46)$$

which is close to the expression given in Ref.[34]. This definition is consistent with the standard entropy for a species  $a$ , as defined in Eq.(44). It yields the production rate

$$\dot{S}_a = -\frac{1}{T_0^2} \int d\gamma F_a U_a \frac{\partial U_a}{\partial t}$$

at constant number of particles and energy. The collisional contribution  $\dot{S}_{ab}^{coll}(U, U)$  to the entropy production rate Eq.(46) is always positive (consistently with the Boltzmann H-theorem). Moreover it is equal to zero if and only if  $U_a(\mathbf{x}, \mathbf{p})$  is of the form Eq.(43). When  $U_a \ll H_a$ , the functional Eq.(46) can be replaced by the symmetrical form

$$\begin{aligned} \dot{S}_{ab}^{coll}(U_a^\dagger, U_b^\dagger) &= \frac{1}{T_0^2} \frac{\gamma_{ab}}{4} \int d^3\mathbf{x} d^3\mathbf{p} d^3\mathbf{p}' F_{H_a,eq} F_{H_b,eq} \\ &\quad \left[ \frac{\partial U_a^\dagger}{\partial \mathbf{p}} - \frac{\partial U_b^\dagger}{\partial \mathbf{p}'} \right] \cdot \mathfrak{L} \cdot \left[ \frac{\partial U_a^\dagger}{\partial \mathbf{p}} - \frac{\partial U_b^\dagger}{\partial \mathbf{p}'} \right] \end{aligned} \quad (47)$$

Many interesting problems are time-dependent so that the Hamiltonian is not an invariant of motion. In that case, the variational principle Eq.(45) does not hold.

## B Angle and action of an island in the phase space

It is useful to introduce the trapping parameter

$$\kappa^2 = \frac{2\omega_b^2}{H_\Omega + \omega_b^2}$$

Particles are passing if  $0 \leq \kappa \leq 1$  and trapped if  $1 \leq \kappa \leq +\infty$ . The motion is periodic, which allows defining a “bounce/transit” frequency  $\Omega_b$ , equal to  $2\pi/T_b$ , where  $T_b$  is the period

$$T_b = \frac{2\pi}{\Omega_b} = \oint \frac{d\xi}{\Omega} = \begin{cases} \int_{-\pi}^{\pi} \frac{\xi}{\Omega} & 0 \leq \kappa \leq 1 \\ 2 \int_{-\xi_b}^{\xi_b} \frac{\xi}{\Omega} & 1 \leq \kappa \leq +\infty \end{cases} \quad (48)$$

For trapped particles, a period covers a forward motion from  $-\xi_b$  to  $\xi_b$ , and then back from  $\xi_b$  to  $-\xi_b$ . After some algebra, the angular frequency  $\Omega_b$  reads

$$\Omega_b = \frac{2\omega_b}{\kappa\tau(\kappa)}$$

with

$$\tau(\kappa) = \frac{2}{\pi} \begin{cases} \mathbb{K}(\kappa^2) & 0 \leq \kappa \leq 1 \\ \frac{2}{\kappa} \mathbb{K}\left(\frac{1}{\kappa^2}\right) & 1 \leq \kappa \leq +\infty \end{cases}$$

and  $\mathbb{K}$  is the complete elliptic function of the first kind

$$\mathbb{K}(m) = \int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{1 - m \sin^2 u}}$$

Since  $\mathbb{K}(0) = \pi/2$ , it appears that  $\Omega_b$  coincides with the reference value  $\omega_b$  for deeply trapped particles  $\kappa \rightarrow \infty$ , and  $\sqrt{2H_\Omega}$  for freely passing particles  $\kappa \rightarrow 0$ . The bounce/transit period becomes large near the passing/trapped boundary  $\kappa = 1$  since  $\mathbb{K}(\kappa) \simeq -\frac{1}{2} \ln |1 - \kappa|$  for  $|1 - \kappa| \ll 1$ . The angle  $\xi$  can be expressed as a function of the angular variable  $\alpha_b = \Omega_b t$

$$\sin\left(\frac{\xi}{2}\right) = \begin{cases} \text{sn}\left(\frac{1}{2}\tau(\kappa)\alpha_b, \kappa^2\right) & 0 \leq \kappa < 1 \\ \frac{1}{\kappa} \text{sn}\left(\frac{1}{2}\kappa\tau(\kappa)\alpha_b, \frac{1}{\kappa^2}\right) & 1 < \kappa < +\infty \end{cases}$$

valid for all  $\alpha_b, \xi$ . The function  $\text{sn}(\delta, m)$  is the Jacobian elliptic function that coincides with the trigonometric  $\sin \delta$  function for  $m \ll 1$ , more precisely

$$\text{sn}(\delta, m) = \sin\left[\left(1 - \frac{m}{4}\right)\delta + \frac{m}{8} \sin 2\delta\right] + o(m^2)$$

One recovers that  $\xi = \alpha_b + \frac{1}{4}\kappa^2 \sin(\alpha_b)$  for quasi-freely passing particles  $\kappa \rightarrow 0$ . For deeply trapped particles  $\kappa \rightarrow \infty$ ,  $\xi = \xi_b \sin \alpha_b$  where  $\xi_b$  is the bounce angle,  $\sin(\xi_b/2) = 1/\kappa$ .

An action  $J_b$  can be built by using the invariance of the Poincaré-Cartan integrals under a canonical change of variables (see lecture on particle trajectories)

$$2\pi J_b = \oint \Omega d\xi \quad (49)$$

Practical expressions are found after a bit of algebra

$$J_b = 4\omega_b \frac{2}{\pi} \begin{cases} \frac{1}{2\kappa} \mathbb{E}(\kappa^2) & 0 \leq \kappa \leq 1 \\ \mathbb{E}\left(\frac{1}{\kappa^2}\right) - \mathbb{K}\left(\frac{1}{\kappa^2}\right) + \frac{1}{\kappa^2} \mathbb{K}\left(\frac{1}{\kappa^2}\right) & 1 \leq \kappa \leq +\infty \end{cases} \quad (50)$$

where  $\mathbb{E}$  is the complete elliptic integral of the second kind

$$\mathbb{E}(m) = \int_0^{\frac{\pi}{2}} du \sqrt{1 - m \sin^2 u}$$

For deeply trapped particles  $\kappa \rightarrow \infty$ , a Taylor development of the elliptic functions near  $m = 0$  gives the expression  $J_b = 2\omega_b/\kappa^2 = (H_\Omega + \omega_b^2)/\omega_b$ . A combination with  $\Omega_b = \omega_b$  provides the relationship  $H_\Omega = -\omega_b^2 + J_b\Omega_b$ . For freely passing particles  $\kappa \rightarrow 0$ , one finds  $J_b = \omega_b/\kappa = \sqrt{H_\Omega/2}$ , which combined with  $\Omega_b = \sqrt{2H_\Omega}$  yields  $H_\Omega = J_b\Omega_b$ . In both case,  $\Omega_b = dH_\Omega/dJ_b$ , as expected. This property can be verified for arbitrary values of  $\kappa$  by using the definitions of  $\Omega_b$  and  $J_b$ , Eqs.(48,49).

## C More on islands

### C.1 Island and conjugate variables

The description of an island in the phase space can be made more accurate by building a new set of conjugate variables. The perturbed Hamiltonian in the extended phase space reads

$$\mathcal{H}(\boldsymbol{\alpha}, \mathbf{J}) = \mathcal{H}_{eq}(\mathbf{J}) - h(\mathbf{J}) \cos[\mathbf{n} \cdot \boldsymbol{\alpha}]$$

To keep the analysis as simple as possible, we assume that the resonant condition  $\Omega = \mathbf{n} \cdot \boldsymbol{\Omega} = 0$ , which defines a surface in the phase space, can be written as  $J_3 = R(\mathbf{J}^*)$ , where

$\mathbf{J}^* = (J_0, J_1, J_2)$ , and  $R$  is a single-valued and smooth function (see Fig.21). We wish to construct a new set of conjugate variables  $(\boldsymbol{\theta}, \mathbf{I})$  that decouple the island dynamics from the dynamics over the resonant surface. Let us remember that one way to construct such a set of variables is to use the invariance of the action

$$\mathcal{A} = \int \mathbf{J} \cdot d\boldsymbol{\alpha} - \mathcal{H}d\tau = \int \mathbf{I} \cdot d\boldsymbol{\theta} - \mathcal{H}'d\tau + dS$$

under a canonical change of variables. Here  $\tau$  is the time in the extended phase space, and  $S(\boldsymbol{\theta}, \mathbf{I}, \boldsymbol{\alpha}, \mathbf{J})$  is a generating function chosen here as

$$S(\boldsymbol{\theta}, \mathbf{I}, \boldsymbol{\alpha}, \mathbf{J}) = -\mathbf{I} \cdot \boldsymbol{\theta} + I_3 \mathbf{n} \cdot \boldsymbol{\alpha} + \mathbf{J}_R(\mathbf{I}^*)$$

where  $\mathbf{J}_R = (\mathbf{I}^*, R(\mathbf{I}^*))$ , and  $\mathbf{I}^* = (I_0, I_1, I_2)$ . Calculating the partial derivatives of  $S$  provides the link between the old and new set of variables, i.e.

$$\begin{aligned} J_k &= I_k + n_k I_3 & ; & \quad \theta_k = \alpha_k + \frac{\partial R}{\partial I_k}(I_0, I_1, I_2) \\ J_3 &= R(I_0, I_1, I_2) + n_3 I_3 & ; & \quad \theta_3 = \mathbf{n} \cdot \boldsymbol{\alpha} \end{aligned} \quad (51)$$

where the index  $k$  runs from 0 to 2. This is consistent with the relation  $\mathbf{J} = \mathbf{J}_R + I_3 \mathbf{n}$  found above, where  $I_3$  coincides with  $I$ , and  $\alpha_3$  with  $\xi$ , with a clean definition of  $\mathbf{J}_R$ . The last constraint that comes from the action invariance relates the old and new Hamiltonians

$$\mathcal{H}'(\mathbf{I}, \theta_3) = \mathcal{H}(\mathbf{J}_R + I_3 \mathbf{n}) - h(\mathbf{J}_R + I_3 \mathbf{n}) \cos(\theta_3)$$

A Taylor development for small values of  $J_3$  provides  $\mathcal{H}' = \mathcal{H}(\mathbf{J}_R) + \frac{1}{2} C I_3^2 - h(\mathbf{J}_R) \cos(\theta_3)$ , where  $C$  is the Hamiltonian curvature. Hence our previous result is recovered. As mentioned above the couple of variables  $(I_3, \theta_3)$  can itself be transformed into a new set of action/angle variables such that the Hamiltonian  $\mathcal{H}'$  depends on the actions only. In other words the system stays integrable. Note that the variables  $\theta_k$  are not strictly angles since when the angle  $\alpha_k$  spans  $2\pi$ , the angle  $\theta^k$  is equal to  $2\pi + \partial_{I_k} R$  which differs from  $2\pi$  unless the partial derivative  $\partial_{I_k} R$  vanishes.

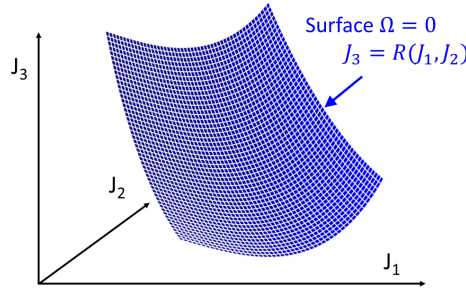


Figure 21: Example of a resonant surface where the third action is a single-valued function of the two others  $J_3 = R(J_1, J_2)$

## C.2 Angle/action variables

### C.2.1 Another recipe

The change of variables Eq.(51) is asymmetric, i.e. the new variables  $\boldsymbol{\theta}$  are related to the old ones  $\boldsymbol{\alpha}$ , but the old variables  $\mathbf{J}$  are functions of the new ones  $\mathbf{I}$ . Obviously a relationship that relates new variables to old variables is preferable. Also manipulating a set of angle/variables offers some comfort. To this aim, another recipe is used to build

up a new set of conjugate variables  $(\boldsymbol{\theta}, \mathbf{I})$  from old ones  $(\boldsymbol{\alpha}, \mathbf{J})$ . If the following properties can be proved

$$\{\alpha^i, \alpha^j\} = 0 \quad \{\alpha^i, J_j\} = \delta_j^i \quad \{J^i, J_j\} = 0 \quad (52)$$

then  $(\boldsymbol{\theta}, \mathbf{I})$  are conjugate. All Poisson brackets in Eqs.(52) are calculated in  $(\boldsymbol{\alpha}, \mathbf{J})$  variables. The latter also satisfy the same constraints. One can then look for new variables linearly related to the old ones, and use the constraints above to find the coefficients

### C.2.2 Angle/action variables in a 2D subset

This programme can be executed in the case of 2 degrees of freedom, i.e. when two pairs of conjugate variables can be separated from the others, and can be extended to higher dimensions. This is for instance the case whenever the considered perturbation does not depend on time  $\alpha_0$  nor on the gyroangle  $\alpha^1$ . One example is the calculation of collisional (“neoclassical”) transport in fusion devices, or more generally the effect of a departure of the magnetic field from its required value (“error field”). The objective is then to find two angles  $(\theta_2, \theta_3)$  and conjugate actions  $(I_2, I_3)$  versus  $(\alpha_2, \alpha_3, J_2, J_3)$ . We assume that the integers  $(n_2, n_3)$  are coprime numbers<sup>26</sup>. In this case, there exists a pair of two other prime numbers  $(p_2, p_3)$  (Bézout’s identity) such that

$$p_2 n_2 + p_3 n_3 = 1$$

The conditions Eq.(52) produce in practice 4 independent relations. An additional constraint is provided if one chooses  $\theta_3 = n_2 \alpha_2 + n_3 \alpha_3$ . All the other variables are then fully determined. The result is

$$\begin{pmatrix} \theta_2 \\ \theta_3 \end{pmatrix} \begin{pmatrix} p_3 & -p_2 \\ n_2 & n_3 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_3 \end{pmatrix}$$

for angles, and

$$\begin{pmatrix} I_2 \\ I_3 \end{pmatrix} \begin{pmatrix} n_3 & -n_2 \\ p_2 & p_3 \end{pmatrix} \begin{pmatrix} J_2 \\ J_3 \end{pmatrix}$$

for actions. The reader can verify directly that the constraints on Poisson brackets Eqs.(52) are verified. These relations can be easily inverted as

$$\begin{pmatrix} \alpha_2 \\ \alpha_3 \end{pmatrix} \begin{pmatrix} n_3 & p_2 \\ -n_2 & p_3 \end{pmatrix} \begin{pmatrix} \theta_2 \\ \theta_3 \end{pmatrix}$$

and

$$\begin{pmatrix} J_2 \\ J_3 \end{pmatrix} \begin{pmatrix} p_3 & n_2 \\ -p_2 & n_3 \end{pmatrix} \begin{pmatrix} I_2 \\ I_3 \end{pmatrix}$$

Let us stress that the variables  $(\theta_2, \theta_3)$  are angles since when the angles  $(\alpha_2, \alpha_3)$  are equal to a multiple of  $2\pi$ ,  $(\theta_2, \theta_3)$  are also equal to multiples of  $2\pi$ . This change of variables turns out to be very useful for practical calculations.

### C.2.3 Extension to 3D

The 3D case is handled as follows. Let us define the change of variables

$$\boldsymbol{\theta} = \begin{pmatrix} \mathbf{b} \times \mathbf{p} \\ \mathbf{p} \times \boldsymbol{\tau} \\ \mathbf{n} \end{pmatrix} \cdot \boldsymbol{\alpha}$$

---

<sup>26</sup>If not,  $(n_2, n_3)$  can be written as  $k(n'_2, n'_3)$ , where  $k$  is an integer, and  $(n'_2, n'_3)$  are coprime numbers. The analysis that follows can then be generalised without any difficulty.

for angles, and

$$\mathbf{I} = \begin{pmatrix} \frac{\boldsymbol{\tau}}{\mathbf{b} \cdot (\mathbf{p} \times \boldsymbol{\tau})} \\ \frac{\mathbf{b}}{\mathbf{b} \cdot (\mathbf{p} \times \boldsymbol{\tau})} \\ \mathbf{p} \end{pmatrix} \cdot \mathbf{J}$$

The parenthesis are to be read as matrices whose components on each row is given by a vector. The basis  $(\boldsymbol{\tau}, \mathbf{b}, \mathbf{n})$  is a set of 3 orthogonal and unitary vectors whose components are integers<sup>27</sup>. The vector  $\mathbf{p}$  satisfies the Bezout's identity  $\mathbf{p} \cdot \mathbf{n} = 1$ . It then appears that Eqs.(52) are verified since Poisson brackets are just scalar products between the lines of the two matrices that relate the new to the old variables. Let us stress that the components that relate  $\boldsymbol{\theta}$  to  $\boldsymbol{\alpha}$  are all integers, which guarantees that the variables  $\theta_i$  are angles (but with a periodicity that is usually not  $2\pi$ ). On the other hand, the coefficients that relates the actions are not integers. This methodology can be extended to higher dimensions.

## D Solution of the Fokker-Planck equation

### D.1 Solution in the collisional regime $\eta \gg 1$

The collisional regime is treated by solving Eq.(25) that reads

$$-p \frac{\partial \tilde{f}}{\partial \xi} + \sin \xi \frac{\partial \tilde{f}}{\partial p} + \eta \frac{\partial^2 \tilde{f}}{\partial p^2} = -\sin \xi \quad (53)$$

The function  $\tilde{f}(\xi, p)$  is then developed as a Fourier series

$$\tilde{f}(\xi, p) = \sum_{\ell=-\infty}^{+\infty} \tilde{f}_{\ell}(p) e^{i\ell\xi}$$

The term  $-\sin \xi$  in the r.h.s. of Eq.(53) acts as a source and involves only the  $\ell = \pm 1$  harmonics. The first and third terms in the l.h.s. are “linear” in the sense that they respond to the drive with the same harmonics  $\ell = \pm 1$ . In contrast the second term in the l.h.s. is “non linear” as it involves beating terms with harmonics other than  $\ell = \pm 1$ . The key point is that the linear response involves a typical value of  $p$  of the order  $\eta^{1/3}$ . This property is obtained by balancing the linear terms in the l.h.s. of Eq.(53). Balancing the first and third term in the l.h.s. yields  $p \sim 1/\eta^{1/3}$ , while balancing one of those with the drive (r.h.s of Eq.(53)) implies  $\tilde{f}_{\pm 1} \sim 1/\eta^{1/3}$ . As a consequence, the non linear term scales as  $1/\eta^{4/3}$ . It is thus subdominant when  $\eta \gg 1$  and can be neglected. The resulting linear equation is readily solved in  $\tilde{f}_{\pm 1}(p)$  via a Fourier transform in the variable  $p$ . The final result is

$$\tilde{f}(\xi, p) = \int_0^{+\infty} d\rho \exp\left(-\frac{\eta\rho^3}{3}\right) \sin(\xi - p\rho) \quad (54)$$

The resonant entropy production rate can now be calculated. Using Eq.(10), it appears that

$$\boldsymbol{\Gamma} = \Gamma h F'_{eq} \omega_b \mathbf{n}$$

where

$$\Gamma = \int_{-\pi}^{\pi} \frac{d\xi}{2\pi} \sin \xi \tilde{f}(\xi, p)$$

---

<sup>27</sup>For instance, given a Cartesian basis  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$  and a vector  $\mathbf{n}$  not aligned with  $\mathbf{e}_3$ , i.e.  $\mathbf{n} = n_1 \mathbf{e}_1 + n_2 \mathbf{e}_2 + n_3 \mathbf{e}_3$ , an example of such a basis is  $\boldsymbol{\tau} = \mathbf{e}_3 \times \mathbf{n}$  and  $\mathbf{b} = \mathbf{n} \times \boldsymbol{\tau}$ .

Hence the resonant entropy functional as defined in Eq.(20) becomes<sup>28</sup>

$$\dot{S}_{res} [U_{eq}, \partial_{\mathbf{J}} U_{eq}^\dagger] = \frac{1}{T_0^2} \int d\gamma F_{eq}(\mathbf{J}) \delta(\mathbf{n} \cdot \boldsymbol{\Omega}) \Lambda(\mathbf{J}_R) h^2 \left( \mathbf{n} \cdot \frac{\partial U_{eq}^\dagger}{\partial \mathbf{J}} \right)^2 \quad (55)$$

where

$$\Lambda(\mathbf{J}_R) = - \int_{-\infty}^{+\infty} dp \int_{-\pi}^{\pi} \frac{d\xi}{2\pi} \sin \xi \tilde{f}(\xi, p) \quad (56)$$

A useful equivalent form is found by multiplying Eq.(25) by  $\tilde{f}$  and integrating over the phase space<sup>29</sup>

$$\Lambda(\mathbf{J}_R) = \eta \int_{-\infty}^{+\infty} dp \int_{-\pi}^{\pi} \frac{d\xi}{2\pi} \left( \frac{\partial \tilde{f}(\xi, p)}{\partial p} \right)^2 \quad (57)$$

This formulation grants positivity of the entropy production rate. Plugging the collisional solution Eq.(54) in Eq.(56), or equivalently Eq.(57), one finds  $\Lambda(\mathbf{J}_R) = \frac{\pi}{2}$ . The Kronecker  $\delta(\Omega)$  function is just a trick to deal with a surface integral, i.e.

$$\int d\gamma G(\mathbf{J}) \delta(\mathbf{n} \cdot \boldsymbol{\Omega}) = \int d\mathcal{S}^* dIG(\mathbf{J}) \delta(CI) = \frac{1}{C} \int d\mathcal{S}^* G(\mathbf{J}_R)$$

where  $d\mathcal{S}^*$  is a surface element on the resonant surface  $\Omega = \mathbf{n} \cdot \boldsymbol{\Omega} = 0$ . The structure of the resonant production rate is quite interesting:

- It appears that it is a quadratic function of the gradient of the thermodynamic potential along the vector  $\mathbf{n}$ , namely  $\mathbf{n} \cdot \frac{\partial U_{eq}}{\partial \mathbf{J}}$ . Therefore gradients of the thermodynamic potential along action variables measure departures from thermodynamic equilibrium.
- The production of entropy is localised near the resonant surface  $\Omega = 0$ , thus justifying the name “resonant”. the localising Kronecker  $\delta(\Omega)$  function is a bit misleading in that regard, since the entropy is produced within a layer, not just at  $\Omega = 0$ . The width of this boundary layer is  $\eta^{1/3} \omega_b$ . So it increases slowly with collisionality.
- Quite remarkably the form factor  $\Lambda(\mathbf{J}_R)$  is a constant, and hence the resonant production rate does not depend on the collision diffusion  $D_\Omega$ . This result is quite general and depends weakly on the details of the collision operator. A simple Krook operator produces the same result. However this is no longer true in the weakly collisional regime  $\eta \ll 1$ .

## D.2 Solution in the weakly collisional regime $\eta \ll 1$

Access to the weakly collisional regime turns out to be easier by solving Eq.(26) in the limit  $\eta \ll 1$ . In absence of dissipation  $\eta = 0$ , the solution  $f_0(k, \sigma)$  is a function of the invariant of motion  $k$  and the sign  $\sigma$  of  $p$ . However the exact shape of  $f_0(k, \sigma)$  is unknown. Thus dissipation is needed to compute the entropy production rate, as expected. This is readily done by looking for a solution  $f$  that is a function of  $(k, \xi, \sigma)$ . The distribution function  $f$  is then expanded in powers of  $\eta$ , namely

$$f(k, \xi, \sigma) = f_0(k, \sigma) + \eta f_1(k, \xi, \sigma) + o(\eta^2)$$

<sup>28</sup>One must be cautious that the flux given above is in  $\Omega$  coordinate. The flux in action space is obtained after a multiplication by  $\omega_b$ . This is the one that must be used to compute the entropy production rate. The factor  $\omega_b^2 = Ch$  provides an extra perturbed Hamiltonian  $h$ , while the curvature  $C$  recombines with  $F'_{eq}$ .

<sup>29</sup>Note that the integral over the phase space of the Poisson bracket  $\{k, \tilde{f}^2\}$  vanishes. This is a general property of Poisson bracket with appropriate boundary conditions.

Eq.(26) becomes at first order in  $\eta$

$$\frac{\partial f_1}{\partial \xi} = \frac{\partial}{\partial k} \left( p \frac{\partial u_0}{\partial k} \right) \quad (58)$$

where  $p$  is here a function of  $(k, \xi, \sigma)$ , namely  $p = \sigma \sqrt{2}(k + \cos \xi)^{1/2}$ . All functions are  $2\pi$  periodic in  $\xi$ . Integrating Eq.(58) yields a solvability equation that rules  $f_0(k, \sigma)$

$$\frac{\partial}{\partial k} \left[ \left( \oint \frac{d\xi}{2\pi} p(k, \xi) \right) \frac{\partial f_0}{\partial k} \right] = 0$$

This equation is in fact an equation of conservation of the flux across the island. It must be solved separately inside and outside the island, because of the change of topology at the island separatrix. The meaning of the contour integral is given in Appendix B (see Eq.(48)). The solution of this equation is

$$\frac{\partial f_0}{\partial k}(k, \sigma) = \sigma \frac{A}{Q(k)}$$

where

$$Q(k) = \sqrt{2} \oint \frac{d\xi}{2\pi} (k + \cos \xi)^{1/2}$$

The function  $Q$  is close to the bounce action  $J_b$  defined in Appendix B (see Eq.50). The integration constant  $A$  has to be determined inside the island  $-1 \leq k < 1$ , and outside the island  $k > 1$  for both  $\sigma = +1$  and  $\sigma = -1$ . Since  $Q(k)$  vanishes at the island O point  $k = -1$ , the constant  $A$  must be zero within the island, otherwise  $f_0$  would be singular at the O point. Outside the island, and far away from the island  $k \rightarrow \infty$ ,  $f_0$  must match the unperturbed solution  $f_0 \rightarrow p$ . This imposes that  $A = 1$  whatever the sign of  $p$ . Hence the thermodynamic potential is at lowest order in  $\eta$

$$\tilde{f}_0(\xi, p) = \sigma \Theta(k - 1) \int_1^k \frac{dk'}{Q(k')} - p \quad (59)$$

where  $\Theta$  is a Heaviside function and  $k = \frac{p^2}{2} - \cos \xi$ . The resonant entropy production is still given by Eq.(55) with  $\Lambda(\mathbf{J}_R)$  given by Eq.(56). The calculation is not difficult but requires some care. The equivalent expression of  $\Lambda(\mathbf{J}_R)$  Eq.57 is easier to manipulate in this case. At lowest order in  $\eta$ ,  $\tilde{f}$  can be replaced by  $\tilde{u}_0$ . The integrand is an even function of  $p$ , so that  $\Lambda$  is twice the integral over positive values of  $p$ . Eq.(59) yields

$$\frac{\partial \tilde{f}(\xi, p)}{\partial p} = \Theta(k - 1) \frac{|p|}{Q(k)} - 1 \quad (60)$$

Note that the derivative of the Heaviside function with  $k$  yields a  $\eta(k - 1)$  contribution that vanishes since multiplied by a function that vanishes at  $k = 1$ . Moving to the set of coordinates  $(k, \xi)$  is then timely. It is found that

$$\Lambda(\mathbf{J}_R) = 2\mathcal{I}\eta$$

where  $\mathcal{I}$  is a number

$$\mathcal{I} = \lim_{\ell \rightarrow +\infty} \int_{-1}^{\ell} dk \oint \frac{d\xi}{2\pi} \frac{1}{|p|} \left( \Theta(k - 1) \frac{|p|}{Q(k)} - 1 \right)^2$$

The limit has been introduced to anticipate cancellations between large contributions. Noting that

$$\frac{dQ}{dk} = \oint \frac{d\xi}{2\pi} \frac{1}{|p|}$$

and  $Q(-1) = 0$ , one gets the final expression

$$\mathcal{I} = \lim_{\ell \rightarrow +\infty} \left( Q(\ell) - \int_1^{\ell} \frac{dk}{Q(k)} \right) \simeq 1.38$$



## E Eulerian correlation time for a broad spectrum

The objective is to compute the correlation function  $\mathcal{C}_E(\boldsymbol{\alpha} - \boldsymbol{\alpha}', t - t')$ , i.e.

$$\mathcal{C}_E(\boldsymbol{\alpha} - \boldsymbol{\alpha}', t - t') = \sum_{\mathbf{n}} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \mathcal{I}_{\mathbf{n}\omega} \exp [i\mathbf{n} \cdot (\boldsymbol{\alpha} - \boldsymbol{\alpha}') - i\omega(t - t')] \quad (61)$$

when the fluctuation spectrum is Lorentzian

$$\mathcal{I}_{\mathbf{n}\omega} = I_0 \prod_{i=1,2,3} \frac{\Delta n_i}{(n_i - n_{0i})^2 + \Delta n_i^2} \frac{\Delta \omega_{\mathbf{n}}}{(\omega - \omega_{\mathbf{n}})^2 + \Delta \omega_{\mathbf{n}}^2}$$

where  $\Delta \omega_{\mathbf{n}}$  is a spectral broadening frequency, and  $\omega_{\mathbf{n}}$  an angular frequency that is often close to the frequency of the most unstable linear mode at wave number  $\mathbf{n}$ . The summation over  $\mathbf{n}$  is performed by using several approximations:

- the summation over  $\mathbf{n}$  is replaced by an integral  $\int dn_1 dn_2 dn_3$ . The procedure is valid as long as  $\Delta n_i \gg 1$ . It is equivalent to using a trapezoidal rule (assuming  $\mathcal{I}_{\mathbf{n}} \sim 0$  for  $|\mathbf{n}| \rightarrow \infty$ ).
- the frequency  $\omega_{\mathbf{n}}$  is expanded linearly near  $\mathbf{n} = \mathbf{n}_0$ , i.e.  $\omega_{\mathbf{n}} = \omega_{\mathbf{n}_0} + \mathbf{v}_{g\mathbf{n}_0} \cdot (\mathbf{n} - \mathbf{n}_0)$ , where  $\mathbf{v}_{g\mathbf{n}_0} = \frac{\partial \omega_{\mathbf{n}}}{\partial \mathbf{n}} \big|_{\mathbf{n}=\mathbf{n}_0}$  is a group velocity calculated at  $\mathbf{n} = \mathbf{n}_0$ .
- $\Delta \omega_{\mathbf{n}} \simeq \Delta \omega_{\mathbf{n}_0}$ , i.e. the frequency spectrum width is assumed constant near the peak wave number  $\mathbf{n} = \mathbf{n}_0$ .

The result is

$$\begin{aligned} \mathcal{C}_E(\boldsymbol{\alpha} - \boldsymbol{\alpha}', t - t') &= \mathcal{I}_0 \exp [i\mathbf{n}_0 \cdot (\boldsymbol{\alpha} - \boldsymbol{\alpha}') - i\omega_{\mathbf{n}_0}(t - t')] \\ &\quad \exp [-\Delta \omega_{\mathbf{n}} |t - t'|] \\ &\quad \exp [-|\Delta \mathbf{n} \cdot (\boldsymbol{\alpha} - \boldsymbol{\alpha}' - \mathbf{v}_{g\mathbf{n}_0}(t - t'))|] \end{aligned}$$

Let us now call  $\mathcal{C}_E(t - t')$  the correlation function  $\mathcal{C}_E(\boldsymbol{\alpha} - \boldsymbol{\alpha}', t - t')$  calculated at the position  $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$ . It appears readily that

$$\begin{aligned} \mathcal{C}_E(t - t') &= \mathcal{I}_0 \exp [-i\omega_{\mathbf{n}_0}(t - t')] \\ &\quad \exp [-(\Delta \omega_{\mathbf{n}} + |\Delta \mathbf{n} \cdot \mathbf{v}_{g\mathbf{n}_0}|) |t - t'|] \end{aligned}$$

Hence the Eulerian correlation time is

$$\tau_{cE,\mathbf{n}} = \frac{1}{\Delta \omega_{\mathbf{n}} + |\Delta \mathbf{n} \cdot \mathbf{v}_{g\mathbf{n}_0}|}$$

## F Diagrammatic representation and Dyson equation

Let us remind the structure of the formal solution of Eq.(35), which can be expanded as

$$\begin{aligned} F &= G_0 * F_0 + G_0 * \mathcal{E}[G_0 * F_0] + \\ &\quad G_0 * \mathcal{E}[G_0 * \mathcal{E}[G_0 * F_0]] + \dots + \\ &\quad G_0 * \mathcal{E}[G_0 * \mathcal{E}[\dots * \mathcal{E}[G_0 * F_0]\dots]] + \dots \end{aligned} \quad (62)$$

It is quite useful to make use of a diagrammatic representation to better see the fate of the various terms in Eq.(37). Each convolution  $G_0*$  is represented by an horizontal bar, and the stochastic operator  $\mathcal{E}[]$  is represented by  $\mathcal{E}$ . The formal solution Eq.(37) in Fig.22. The  $n$ th diagram in this series is

$$\begin{aligned}
F = & \text{--- } \mathcal{E} \text{ --- } F_0 + \text{--- } \mathcal{E} \text{ --- } \mathcal{E} \text{ --- } F_0 \\
& + \text{--- } \mathcal{E} \text{ --- } \mathcal{E} \text{ --- } \mathcal{E} \text{ --- } F_0 + \\
& + \text{--- } \mathcal{E} \text{ --- } \mathcal{E} \text{ --- } \mathcal{E} \text{ --- } \mathcal{E} \text{ --- } F_0 + \dots \\
& + \text{--- } \mathcal{E} \text{ --- } \mathcal{E} \dots \text{--- } \mathcal{E} \text{ --- } F_0 + \dots
\end{aligned}$$

Figure 22: Diagrammatic representation of the formal solution Eq.(37) of the Vlasov equation. Each horizontal bar is a convolution  $G_0*$ . The stochastic operator  $\mathcal{E}[\ ]$  is represented by the symbol  $\mathcal{E}$ .  $F_0$  is the initial distribution function.

$$\begin{aligned}
G_0 * \mathcal{E}[G_0 * \mathcal{E}[\dots * \mathcal{E}[G_0 * F_0]\dots]] &= \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \dots \int_0^{t_{n-1}} dt_n \\
&\mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_1), t_1) \cdot \frac{\partial}{\partial \mathbf{J}} \mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_2), t_2) \cdot \frac{\partial}{\partial \mathbf{J}} \dots \\
&\mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_{n-1}), t_{n-1}) \cdot \frac{\partial}{\partial \mathbf{J}} \mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_n), t_n) \cdot \frac{\partial}{\partial \mathbf{J}} F_0(\boldsymbol{\alpha} - \boldsymbol{\Omega}t, \mathbf{J}, 0)
\end{aligned}$$

Note that the derivative  $\partial_{\mathbf{J}}$  with respect to the action variables do not commute with the random fields  $\mathbf{E}$  even if they do not depend on  $\mathbf{J}$ . The reason is that the Hamiltonian resonant frequencies  $\boldsymbol{\Omega}$  are action dependent.

## F.1 Statistical average

We now make use of the random character of the turbulent field  $\mathbf{E}$  by computing the statistical average of Eq.(37). The structure of the correlation function Eq.(31) implies that fields are correlated if their argument are within a correlation time. Two types of correlation times appear, the Eulerian one  $\tau_{cL}$ , and Lagrangian correlation times that will be defined as  $L_{ci}/\Omega_i$  in each 3 angle direction  $i = 1, 2, 3$ . We note  $\tau_c$  the longest time of all. For a given diagram of length  $n$ , two adjacent fields computed at time  $t_i$  and  $t_{i-1}$  may be within a correlation time  $|t_i - t_{i-1}| < \tau_c$  or not. If the times  $t_i$  and  $t_{i-1}$  are within a correlation time, the corresponding diagram is called “connected”, and represented with a hat that connects the two fields. If not, the diagram is “disconnected”. An essential property of a Markov process is that disconnected diagram can be decomposed

$$\begin{aligned}
\left\langle \text{--- } \mathcal{E} \text{ --- } \mathcal{E} \text{ --- } \mathcal{E} \text{ --- } \mathcal{E} \right\rangle = & \\
& \text{--- } \mathcal{E} \text{ --- } \mathcal{E} \text{ --- } \mathcal{E} \text{ --- } \mathcal{E} + \\
& \text{--- } \mathcal{E} \text{ --- } \mathcal{E} \text{ --- } \mathcal{E} \text{ --- } \mathcal{E} + \\
& \text{--- } \mathcal{E} \text{ --- } \mathcal{E} \text{ --- } \mathcal{E} \text{ --- } \mathcal{E}
\end{aligned}$$

Figure 23: Decomposition in connected diagrams of a 4th order contribution to the diagrammatic expansion of Fig.22.

as a product of connected diagrams. Moreover, the analysis is restricted to a Gaussian distribution of random fields. A field Gaussian (or normal) is defined as follows. To ease the understanding, we  $\mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_i), t_i)$  as  $\mathbf{E}(i)$ , and derivatives with respect to the action

are ignored. A Gaussian random field distribution is such that the statistical average of an odd number of fields vanish, e.g.  $\langle \mathbf{E}(1)\mathbf{E}(2)\mathbf{E}(3) \rangle = 0$ . Conversely, the statistical average of an even number can be expressed as a sum of product of two point correlation functions, e.g.

$$\begin{aligned} \langle \mathbf{E}(1)\mathbf{E}(2)\mathbf{E}(3)\mathbf{E}(4) \rangle &= \langle \mathbf{E}(1)\mathbf{E}(2) \rangle \langle \mathbf{E}(3)\mathbf{E}(4) \rangle + \\ &\quad \langle \mathbf{E}(1)\mathbf{E}(3) \rangle \langle \mathbf{E}(2)\mathbf{E}(4) \rangle + \langle \mathbf{E}(1)\mathbf{E}(4) \rangle \langle \mathbf{E}(2)\mathbf{E}(3) \rangle \end{aligned}$$

A two-point correlation function does not vanish as long as the corresponding times are within a correlation time  $\tau_c$ . The advantage of a Gaussian random field is that all correlation functions can be reduced down to products of two-point correlation functions. However a realistic turbulent field is usually not Gaussian. But it is often “quasi-Gaussian”, in the sense that odd moments can be expressed as function of even moments via a closure. The even moments can still be reduced to two-point correlation function. For this reason we will keep odd moments in the development, provided of course that the corresponding times cluster within a correlation time. Note however that the 1st moment  $\langle \mathbf{E}(i) \rangle = 0$  is always zero. For instance, let us consider the full 4th order diagram

$$\begin{aligned} \langle G_0 * \mathcal{E}[G_0 * \mathcal{E}[G_0 * \mathcal{E}[*\mathcal{E}[G_0 * F_0]]]] \rangle &= \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \\ &\left\langle \mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_1), t_1) \cdot \frac{\partial}{\partial \mathbf{J}} \mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_2), t_2) \cdot \frac{\partial}{\partial \mathbf{J}} \right. \\ &\quad \left. \mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_3), t_3) \cdot \frac{\partial}{\partial \mathbf{J}} \mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_4), t_4) \right\rangle \cdot \frac{\partial}{\partial \mathbf{J}} F(\boldsymbol{\alpha} - \boldsymbol{\Omega}t, \mathbf{J}, 0) \end{aligned}$$

where the initial distribution function has been put outside the bracket since it is not a random field. Let us now assume that  $t_2 - t_3 \gg \tau_c$ , then one has

$$\begin{aligned} \langle G_0 * \mathcal{E}[G_0 * \mathcal{E}[G_0 * \mathcal{E}[*\mathcal{E}[G_0 * F_0]]]] \rangle &= \int_0^t dt_1 \int_0^{t_1} dt_2 \\ &\left\langle \mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_1), t_1) \cdot \frac{\partial}{\partial \mathbf{J}} \mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_2), t_2) \right\rangle \cdot \frac{\partial}{\partial \mathbf{J}} \\ &\int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \left\langle \mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_3), t_3) \cdot \frac{\partial}{\partial \mathbf{J}} \mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_4), t_4) \right\rangle \cdot \frac{\partial}{\partial \mathbf{J}} F(\boldsymbol{\alpha} - \boldsymbol{\Omega}t, \mathbf{J}, 0) \end{aligned}$$

Of course one has to look in all position combinations of the various times. The final result is represented graphically in Fig.23. We now call a  $n$ th order diagram “irreducible” if no line can be drawn that would cut this diagram in correlated subparts. With these conventions, the distribution function can be expanded as a function of irreducible diagrams as shown on Fig.24.

$$\langle F \rangle = -F_0 + -\mathcal{D} - F_0 \quad (63)$$

where  $\mathcal{D}$  is the sum of all possible products of irreducible diagrams (products meaning convolutions).

## F.2 Dyson equation

We arrive now to an essential result. The Eq.(63) can be reformulated as

$$\langle F \rangle = -F_0 + -M - \langle F \rangle \quad (64)$$

where  $M$  is the sum of all irreducible diagrams, sometimes called a “mass” operator. This identity is called Dyson equation, who was the first to prove it in the context of quantum

field theory. This demonstrated by iterating Eq.(64) and iterating with respect to  $F_0$ . It can then be proved that the two expansions are identical. The first non zero diagram is

$$\langle G_0 * \mathcal{E}[G_0 * \mathcal{E}[* \langle F \rangle]] \rangle = \int_0^t dt_1 \int_0^{t_1} dt_2 \left\langle \mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_1), t_1) \cdot \frac{\partial}{\partial \mathbf{J}} \mathbf{E}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t_2), t_2) \cdot \right\rangle \frac{\partial}{\partial \mathbf{J}} \langle F \rangle (\boldsymbol{\alpha} - \boldsymbol{\Omega}t, \mathbf{J}, t_2)$$

Figure 24: Expansion of the formal solution Eq.(37) in connected diagrams.

## G Renormalisation procedure

### G.1 Equation over the resonant function

A proxy for the propagator  $\langle G \rangle$ , that is called quasi-linear propagator, is obtained by keeping only the first term in the Dyson expansion

$$\mathcal{G} = G_0 + G_0 \langle \mathbf{E} G_0 \mathbf{E} \rangle \mathcal{G}$$

Applying the operator  $\partial_t + \boldsymbol{\Omega} \cdot \partial_{\boldsymbol{\alpha}}$  on this relation, and using  $(\partial_t + \boldsymbol{\Omega} \cdot \partial_{\boldsymbol{\alpha}})G_0 = 0$ , one gets the equation

$$\left( \frac{\partial}{\partial t} + \boldsymbol{\Omega} \cdot \frac{\partial}{\partial \boldsymbol{\alpha}} \right) \mathcal{G} = \frac{\partial}{\partial J_i} \int_0^t dt' \mathcal{C}_{L,ik}(\mathbf{J}, t - t') \frac{\partial}{\partial J_k} \mathcal{G}(\boldsymbol{\alpha} - \boldsymbol{\Omega}(t - t'), \mathbf{J}, t') \quad (65)$$

where the partial derivative  $\partial_{J_k}$  applies only on the explicit dependence of  $\mathcal{G}$  on  $\mathbf{J}$ , and not on its indirect dependence via  $\boldsymbol{\Omega}(\mathbf{J})$ . Let us now compute the Fourier-Laplace of this equation. First an integration by part shows that

$$\int_0^{+\infty} dt \exp(i(\omega + i\nu)t) \int \frac{d\boldsymbol{\alpha}}{(2\pi)^3} \exp(-i\mathbf{n} \cdot \boldsymbol{\alpha}) \left( \frac{\partial}{\partial t} + \boldsymbol{\Omega} \cdot \frac{\partial}{\partial \boldsymbol{\alpha}} \right) \mathcal{G} = -i(\omega + i\nu - \mathbf{n} \cdot \boldsymbol{\Omega}) \mathcal{G}_{\mathbf{n}\omega} - \delta(\mathbf{J})$$

Since the function  $\mathcal{C}_{L,ik}(\mathbf{J}, t - t')$  does not depend on  $\boldsymbol{\alpha}$ , the Fourier-Laplace transform (noted FLT) of the r.h.s. of Eq.(65) reads

$$\begin{aligned} FLT(r.h.s.) &= \int_{-\infty}^{+\infty} \frac{d\omega_1}{2\pi} \int_0^{+\infty} dt \exp[i(\omega + i\nu - \omega_1)t] \\ &\quad \frac{\partial}{\partial J_i} \int_0^t d\tau \mathcal{C}_{L,ik}(\mathbf{J}, \tau) \exp[i(\omega_1 - \mathbf{n} \cdot \boldsymbol{\Omega})\tau] \\ &\quad \frac{\partial}{\partial J_k} \mathcal{G}_{\mathbf{n}\omega_1}(\mathbf{J}) \end{aligned}$$

Since correlation functions decrease fast in time, the upper bound in the integral on  $\tau$  can be expanded to the interval  $+\infty$ . The two integrals in  $t$  and  $\tau$  can then be decoupled. The expression Eq.(39) of  $\mathcal{C}_{L,ik}(\mathbf{J}, \tau)$  is then used to find

$$\begin{aligned} FLT(r.h.s.) &= \int_{-\infty}^{+\infty} \frac{d\omega_1}{2\pi} \frac{1}{i(\omega_1 - \omega - i\nu)} \\ &\quad \frac{\partial}{\partial J_i} \sum_{\mathbf{n}'} \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \frac{n'_i n'_k \mathcal{I}_{\mathbf{n}'\omega'}}{-i[\omega_1 - \omega' - (\mathbf{n} - \mathbf{n}') \cdot \boldsymbol{\Omega}]} \\ &\quad \frac{\partial}{\partial J_k} \mathcal{G}_{\mathbf{n}\omega_1}(\mathbf{J}) \end{aligned}$$

The integration over  $\omega_1$  by the residue theorem yields the final result

$$-i(\omega - \mathbf{n} \cdot \boldsymbol{\Omega} + i\nu) \mathcal{G}_{\mathbf{n}\omega} - \frac{\partial}{\partial J_i} \left[ \tilde{D}_{\mathbf{n}\omega, ik} \frac{\partial \mathcal{G}_{\mathbf{n}\omega}}{\partial J_k} \right] = \delta(\mathbf{J}) \quad (66)$$

where

$$\tilde{D}_{\mathbf{n}\omega, ik} = \sum_{\mathbf{n}'} \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \frac{n'_i n'_k \mathcal{I}_{\mathbf{n}'\omega'}}{-i[\omega - \omega' - (\mathbf{n} - \mathbf{n}') \cdot \boldsymbol{\Omega} + i\nu]}$$

Let us stress that the diffusion matrix  $\tilde{D}_{\mathbf{n}\omega, ik}$  depends on  $\mathbf{n}\omega$ . Since the propagator  $\mathcal{G}_{\mathbf{n}\omega}$  yields the resonant response function  $R_{\mathbf{n}\omega}$ , it is worth being compared with the linear resonant function  $R_{\mathbf{n}\omega}^{(0)}$  Eq.(29). Dividing the equation Eq.(66) by  $\omega - \mathbf{n} \cdot \boldsymbol{\Omega} + i\nu$  to find  $\mathcal{G}_{\mathbf{n}\omega}$  gives rise to a double pole (not located at the same position though). This is the simplest example of a non linear Landau resonance. We can nevertheless simplify this equation further, by noting that quasilinear transport is essentially a transport of resonant particles such that  $\omega = \mathbf{n} \cdot \boldsymbol{\Omega}$ . For these particles the resonance in the r.h.s. of Eq.(66) is in fact  $\omega_1 = \mathbf{n}_1 \cdot \boldsymbol{\Omega}$ . It then appears that the real part of the diffusion matrix  $\tilde{D}_{\mathbf{n}\omega, ik}$  coincides with the quasilinear diffusion matrix  $D_{ik}$  given by Eq.(30). The other limit where the  $\tilde{D}_{\mathbf{n}\omega, ik} \sim D_{ik}$  is the low frequency limit  $\omega \rightarrow 0$ ,  $\mathbf{n}_1 = \mathbf{0}$  that corresponds to the evolution of the average distribution function  $\langle F \rangle$ . The resonant function is solution of the equation

$$-i(\omega - \mathbf{n} \cdot \boldsymbol{\Omega} + i\nu) R_{\mathbf{n}\omega} - \frac{\partial}{\partial J_i} \left[ \tilde{D}_{\mathbf{n}\omega, ik} \frac{\partial R_{\mathbf{n}\omega}}{\partial J_k} \right] = 1 \quad (67)$$

## G.2 Calculation of the resonant function

Solving Eq.(66) is feasible but somewhat cumbersome - it requires diagonalising the matrix  $\tilde{D}_{\mathbf{n}\omega, ik}$ . Nevertheless a reasonable proxy can be found via an educated guess the resonant function  $R_{\mathbf{n}\omega}$  should evolve rapidly across the resonant surface  $\Omega = \mathbf{n} \cdot \boldsymbol{\Omega} - \omega = 0$ . In addition we assume that the resonances that matter in the diffusion matrix  $\tilde{D}_{\mathbf{n}\omega, ik}$  are close to  $\Omega = 0$ . This is a consequence of the Chirikov overlap parameter. As for the case of a single perturbation, we use a change variables in the action space  $\mathbf{J} = \mathbf{J}_R(J, K) + I\mathbf{n}$  (see Eq.(23) for definitions). If the variations are fast in the  $\Omega = CI$  direction, and slow along the resonant surface  $\mathbf{J} = \mathbf{J}_R$ , the diffusion regularisation builds down to a second derivative with respect to  $I$  (or equivalently  $\Omega = CI$ ). With this approximations, the derivatives  $\mathbf{n} \cdot \partial_{\mathbf{J}}$  can be replaced by  $C\partial_{\Omega}$ , and Eq.(67) becomes

$$i(\Omega - i\nu) R_{\mathbf{n}\omega} - \frac{\partial}{\partial \Omega} \left[ \frac{1}{\tau_d^3} \frac{\partial R_{\mathbf{n}\omega}}{\partial \Omega} \right] = 1 \quad (68)$$

where

$$\frac{1}{\tau_d^3} = \sum_{\mathbf{n}'} \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \left( \mathbf{n}' \cdot \frac{\partial \Omega}{\partial \mathbf{J}} \right) \frac{\mathcal{I}_{\mathbf{n}'\omega'}}{-i[\omega - \omega' - (\mathbf{n} - \mathbf{n}') \cdot \boldsymbol{\Omega} + i\nu]}$$

depends weakly on  $\Omega$  in the vicinity of  $\Omega = 0$ . The solution is readily found to be

$$R_{\mathbf{n}\omega} = \int_0^{+\infty} d\tau \exp \left[ i(\omega + i\nu - \mathbf{n} \cdot \boldsymbol{\Omega}) \tau - \frac{\tau^3}{3\tau_d^3} \right]$$

This important result calls for several remarks. First a new time  $\tau_d$  appears, called Dupree time after the physicist who introduced it first. Resonant particles can now be defined properly as particles such that  $\Omega\tau_d \ll 1$ . Second the regularizing parameter  $\nu$  can be neglected as long as  $\nu\tau_d \ll 1$ . Third, the singular resonant function has now be regularised by a turbulent diffusion matrix  $\tilde{D}_{\mathbf{n}\omega,ik}$  via the Dupree time. This regularisation is therefore a non linear process.

### G.3 Iterative procedure

Let us wrap up this lengthy procedure. The quasi-linear flux is related to the resonant function and sources and drive via Eq.(29). The resonant function is solution of the diffusion equation Eq.(67, where the diffusion matrix  $\tilde{D}_{\mathbf{n}\omega,ik}$  is given by Eq.(67). One can then establish the following iterative procedure, labelled with the integer index  $m$ ,

$$\begin{aligned} \tilde{D}_{\mathbf{n}\omega,ik}^{(m)} &= \sum_{\mathbf{n}'} \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \Im \left[ R_{\mathbf{n}-\mathbf{n}',\omega-\omega'}^{(m)} \right] n'_i n'_k \mathcal{I}_{\mathbf{n}'\omega'} \\ -i(\omega - \mathbf{n} \cdot \boldsymbol{\Omega} + i\nu) R_{\mathbf{n}\omega}^{(m+1)} - \frac{\partial}{\partial J_i} \left[ \tilde{D}_{\mathbf{n}\omega,ik}^{(m)} \frac{\partial R_{\mathbf{n}\omega}^{(m+1)}}{\partial J_k} \right] &= 1 \\ \tilde{D}_{\mathbf{n}\omega,ik}^{(m+1)} &= \sum_{\mathbf{n}} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \Im \left[ R_{\mathbf{n}\omega}^{(m+1)} \right] n_i n_k \mathcal{I}_{\mathbf{n}\omega} \end{aligned}$$

The procedure is started with  $R_{\mathbf{n}\omega}^{(m+1)} = R_{\mathbf{n}\omega}^{(0)}$ , which is the linear resonant response. This recurrence can be seen as a renormalisation procedure. In the case where  $\tilde{D}_{\mathbf{n}\omega,ik}$  is close to  $D_{ik}$ , reason being that transport is dominated by resonant particles, this recurrence is a series over  $D_{ik}$ . The quasilinear diffusion coefficient is the fixed point of this iteration.

Let us note that unless the regime is strongly non linear, the final result is in fact close to the pedestrian one.

## H Hamiltonian perturbations on nearby resonant surfaces

### H.1 Primary and secondary perturbations

A special case is considered where two Hamiltonian perturbations resonate on nearby surfaces in the action space. This situation is met in tokamaks when the magnetic field corrugations due to the finite number of coils (“magnetic ripple”) are accounted for, or in stellarators where a helical magnetic field coexist with a toroidal field. In these two examples, perturbations do not depend on time (angle  $\alpha_0$ , nor on the cyclotron angle  $\alpha_1$ ). This limit case is modelled by considering an Hamiltonian of the form

$$H(\boldsymbol{\alpha}, \mathbf{J}, t) = H_{eq}(\mathbf{J}) + \tilde{H}(\boldsymbol{\alpha}, \mathbf{J}, t) + \tilde{H}'(\boldsymbol{\alpha}, \mathbf{J}, t)$$

where

$$\tilde{H}(\boldsymbol{\alpha}, \mathbf{J}) = -h \cos(\mathbf{n} \cdot \boldsymbol{\alpha})$$

and

$$\tilde{H}'(\boldsymbol{\alpha}, \mathbf{J}) = -h' \cos(\mathbf{n}' \cdot \boldsymbol{\alpha})$$

where  $\mathbf{n} = (0, 0, n_2, n_3)$  and  $\mathbf{n}' = (0, 0, n'_2, n'_3)$ . The amplitude perturbations  $h$  and  $h'$  are assumed independent of the action variables. The resonant surface associated with the first perturbation is determined by the equation  $\Omega = n_2\Omega_2 + n_3\Omega_3 = 0$ . The second resonant surface is defined in the same way, with  $\mathbf{n}$  replaced by  $\mathbf{n}'$ , hence  $\Omega' = n'_2\Omega_2 + n'_3\Omega_3 = 0$ . In the following, the variables  $(\alpha_0, \alpha_1)$  and the corresponding actions  $(J_0, J_1)$  are not mentioned explicitly, except in calculations that involve an integration over the phase space volume.

The objective of this section is to calculate the entropy production rate due to collisions or chaotic motion. Schematically, one can define a primary perturbation, say  $\tilde{H}$ , and a secondary perturbation,  $\tilde{H}'$ . The entropy production rate associated with the primary perturbation has already been calculated, and is given by Eq.(27). Hence the entropy production rate due to the secondary perturbation remains to be computed. Obviously the role of the primary and secondary perturbations can be exchanged, so that the total entropy production rate contains 4 contributions.

## H.2 Primary perturbation

We assume that the equation that rules the primary surface  $\Omega(J_2, J_3) = 0$  can be written in an explicit form  $J_3 = G(J_2)$ , where  $G$  is a function that such  $\Omega(J_2, G(J_2)) = 0$ . To avoid unnecessary complications,  $G$  is assumed single-valued. The derivative of the  $G$  plays an important role. It is useful to define a iota function as

$$\iota(J_2) = -\frac{dG}{dJ_2}$$

Any point on the resonant surface can then be identified by its coordinates  $(J_2, J_3)$ , noted  $J_{R2}$ , and  $J_{R3} = G(J_{R2})$ . A function  $F(J_2, J_3)$  is calculated on the resonant surface via the expression  $F_R(J_{R2}) = F(J_{R2}, G(J_{R2}))$ . Let us now introduce a new set of variables  $\boldsymbol{\theta}$  and  $\mathbf{I}$ , defined in the vicinity of a specific point  $\mathbf{J}^*$  that lies on the resonant surface. This new set of variables is defined as

$$\begin{aligned}\theta_2 &= n_2\alpha_2 + n_3\alpha_3 \\ \theta_3 &= \frac{\alpha_2 - \iota\alpha_3}{\iota n_2 + n_3}\end{aligned}$$

$$I_2 = \frac{\iota\tilde{J}_2 + \tilde{J}_3}{\iota n_2 + n_3} \quad (69)$$

$$I_3 = n_3\tilde{J}_2 - n_2\tilde{J}_3 \quad (70)$$

where  $\tilde{\mathbf{J}} = \mathbf{J} - \mathbf{J}^*$ , and  $\iota$  is calculated at  $\mathbf{J}^*$ , i.e.  $\iota = \iota(J_2^*)$  - hence it is now a parameter. It can be verified that  $(\boldsymbol{\theta}, \mathbf{I})$  is a set of conjugate variables since  $\{I_j, I_k\} = 0$ ,  $\{\theta_j, \theta_k\} = 0$  and  $\{\theta_j, I_k\} = \delta_{jk}$  whatever the indices  $(j, k) \in [2, 3]$ . This property can also be demonstrated by constructing the generating function that relates  $(\boldsymbol{\theta}, \mathbf{I})$  to  $(\boldsymbol{\alpha}, \mathbf{J})$ . Note however that  $\theta_2$  is not an angle since  $\iota$  is not necessarily a rational number. However it can be made an angle via a small adjustment in the choice of  $\mathbf{J}^*$  since the space of rational numbers is dense in the space of real numbers. In the following Fourier series in  $\theta_2$  instead of integrals, to simplify the notations. The relationship between angles can be inverted easily

$$\begin{aligned}\alpha_2 &= n_3\theta_3 + \frac{\iota\theta_2}{\iota n_2 + n_3} \\ \alpha_3 &= -n_2\theta_3 + \frac{\theta_2}{\iota n_2 + n_3}\end{aligned}$$

In a similar way, the relationship on action variables is inverted in a convenient form

$$\mathbf{J} = \mathbf{J}^* + I_2\mathbf{n} + I_3\mathbf{t} \quad (71)$$

where the vector

$$\mathbf{t} = \frac{1}{n_2\iota + n_3} \begin{pmatrix} 1 \\ -\iota \end{pmatrix}$$

is tangent to the resonant surface at  $\mathbf{J} = \mathbf{J}^*$  in the  $(J_2, J_3)$  space<sup>30</sup>. Defining  $\mathbf{J}_R$  as  $\mathbf{J}_R = \mathbf{J}^* + I_3\mathbf{t}$ , it appears that Eq.(71) can be recast as  $\mathbf{J} = \mathbf{J}_R + I_2\mathbf{n}$ , where  $\mathbf{J}_R$  lies on the resonant surface. This relation is identical to Eq.(23). The nature of the actions  $(I_2, I_3)$  becomes clear:  $I_3$  characterises the dynamics along the resonant surface in the vicinity of  $\mathbf{J}^*$ , whereas  $I_2$  measures the dynamics across the resonant surface. Turning our gaze to the angles,  $\theta_2$  appears to be the argument of the primary perturbation, while the variable  $\theta_3$  characterises the missing dimension in a convenient way such that the new system of coordinates is conjugate. The set of variables  $(\boldsymbol{\theta}, \mathbf{I})$  is invertible, thus allowing moving back and forth from old to new set of conjugate coordinates. Still in the vicinity of the resonant surface, the Hamiltonian reads

$$H(\boldsymbol{\alpha}, \mathbf{J}, t) = H_{eq}(\mathbf{J}_R) + \frac{1}{2}CI_2^2 - h \cos \theta_2 \quad (72)$$

where the Hamiltonian curvature was already met in the section 4.1.1 and is given by Eq.(24). Its expression is reproduced here

$$C = n_i n_j \left. \frac{\partial^2 H_{eq}}{\partial J_i \partial J_j} \right|_{\mathbf{J}=\mathbf{J}_R} \quad (73)$$

It was seen that a new set of action/angle variables  $(\alpha_b, J_b)$  can be constructed such that  $J_b$  is an invariant of motion, and  $d_t \alpha_b = \Omega_b(J_b, I_3)$ . The label “b” stands for “bounce”. As done before, the action  $I_2$  can be replaced by the pulsation  $\Omega = CI_2$ , so that  $d_t \theta_2 = \Omega$ . The Hamiltonian can be recast as  $H = H_{eqR} + \frac{1}{C}H_\Omega$ , where

$$H_\Omega = \frac{1}{2}\Omega^2 - \omega_b^2 \cos \theta_2$$

It is convenient to introduce the trapping parameter  $0 \leq K \leq 1$  defined as

$$K = \frac{1}{2\omega_b^2} (H_\Omega + \omega_b^2)$$

The limit  $K \rightarrow 0$  corresponds to deeply trapped particles, while  $K \rightarrow 1$  for barely trapped particles. Let us note that

$$K = \frac{\Omega^2}{4\omega_b^2} + \sin^2 \left( \frac{\theta_2}{2} \right) \quad (74)$$

The invariance of Poincaré integrals under change of conjugate variables implies that

$$2\pi J_b = \int_0^{2\pi} d\alpha_2 I_2$$

An expression of the action  $J_b$ , valid for all values of  $K$ , reads

$$J_b = \lambda_m \oint \frac{d\theta_2}{2\pi} \sqrt{K - \sin^2 \left( \frac{\theta_2}{2} \right)} \quad (75)$$

where the integral covers the interval  $-\theta_b \leq \theta_2 \leq +\theta_b$  back and forth, i.e. twice, and  $\theta_b$  is the bounce angle defined by  $K = \sin^2 \left( \frac{\theta_b}{2} \right)$ . The pulsation reads

$$\Omega_b = \frac{2\omega_b}{\oint \frac{d\theta_2}{2\pi} \frac{1}{\sqrt{K - \sin^2 \left( \frac{\theta_2}{2} \right)}}} \quad (76)$$

---

<sup>30</sup>This is really verified by noting that  $\partial_{\mathbf{J}}\Omega$  is a vector normal to the resonant surface. Since  $\Omega_R(J_2) = 0$  all along the resonant surface, it readily appears that  $(\partial_{J_2}\Omega - \iota\partial_{J_3}\Omega)|_{\mathbf{J}=\mathbf{J}^*} = 0$ , hence  $\mathbf{t} \cdot \partial_{\mathbf{J}}\Omega|_{\mathbf{J}=\mathbf{J}^*} = 0$ , which demonstrates that  $\mathbf{t}$  is tangent to the resonant surface at  $\mathbf{J} = \mathbf{J}^*$ .



For deeply trapped particles (small bounce angle  $\theta_b \ll 1$ ), one has  $K \simeq \frac{\theta_b^2}{4}$  and

$$\begin{aligned}\theta_2 &= \theta_b \sin \alpha_b \\ \Omega &= \omega_b \theta_b \cos \alpha_b\end{aligned}$$

where  $\Omega_b = \omega_b = \sqrt{Ch}$ . The Hamiltonian  $H = H_\Omega$  reads

$$H_\Omega = \frac{1}{2} \omega_b^2 \theta_b^2 = C J_b \Omega_b$$

Hence the trapped/passing limit  $K = 1$  corresponds to  $\theta_b = \theta_m = 2$  if one stays in the deeply trapped limit, whereas obviously  $\theta_m = \pi$  in the general case, consistently with Eq.(74). Note also that in the deeply trapped limit

$$J_b \simeq \lambda_m \frac{\theta_b^2}{4} \quad (77)$$

with  $\lambda_m = 2\sqrt{\frac{h}{C}}$ . The corresponding resonant frequency is the bounce frequency  $\Omega_b$ . The angles  $(\theta_3, I_3)$  are re-labeled  $(\alpha_d, J_d)$ , where the subscript “d” stands for “drift”. The associated resonant angular frequency is noted  $\Omega_d$ . Then the new set of variables  $(\alpha_b, \alpha_d, J_b, J_d)$  is conjugate to  $(\alpha_2, \alpha_3, J_2, J_3)$  and provides a convenient set of coordinates to treat the secondary perturbation. The system is integrable, and the new “unperturbed” Hamiltonian is noted  $\mathfrak{H}_{eq}(J_b, J_d)$ .

### H.3 Secondary perturbation

The phase  $\xi' = \mathbf{n}' \cdot \boldsymbol{\alpha}$  of the secondary perturbation can be expressed as function of the variables  $\boldsymbol{\theta}$  using the relationship between  $\boldsymbol{\alpha}$  and  $\boldsymbol{\theta}$ , i.e.

$$\xi' = \sigma \theta_2 + n_d \theta_3$$

where

$$n_d = n'_2 n_3 - n_2 n'_3$$

and

$$\sigma = \frac{n'_2 \iota + n'_3}{n_2 \iota + n_3}$$

where  $\iota$  is calculated on the resonant surface (strictly speaking at  $\mathbf{J}^*$ ). The parameter  $\sigma$  can be given a physical interpretation as follows. Let us use the expansion Eq.(72) of  $\mathbf{J}$  near the specific point  $\mathbf{J}^*$  on the primary resonant surface, hence  $\Omega(\mathbf{J}^*)$ . The resonant frequency  $\Omega$  reads

$$\Omega = \Omega(\mathbf{J}^*) + I_2 \left( \mathbf{n} \cdot \frac{\partial \Omega}{\partial \mathbf{J}} \Big|_{\mathbf{J}=\mathbf{J}^*} \right) + I_3 \left( \mathbf{t} \cdot \frac{\partial \Omega}{\partial \mathbf{J}} \Big|_{\mathbf{J}=\mathbf{J}^*} \right)$$

where  $\mathbf{t}$  is the tangent vector given by Eq.(72). Since  $\Omega(\mathbf{J}_R) = 0$  on the primary resonant surface for any  $\mathbf{J}_R$ , and therefore any  $I_3$ , the tangent derivative  $\mathbf{t} \cdot \partial_{\mathbf{J}} \Omega|_{\mathbf{J}=\mathbf{J}^*}$  must vanish<sup>31</sup>. This imposes the following constraint

$$\frac{\partial \Omega}{\partial J_2} \Big|_{\mathbf{J}=\mathbf{J}^*} = \iota(\mathbf{J}^*) \frac{\partial \Omega}{\partial J_3} \Big|_{\mathbf{J}=\mathbf{J}^*}$$

Let us now expand the resonant frequency  $\Omega'$  in a similar way, than

$$\Omega = \Omega'(\mathbf{J}^*) + I_2 \left( \mathbf{n} \cdot \frac{\partial \Omega'}{\partial \mathbf{J}} \Big|_{\mathbf{J}=\mathbf{J}^*} \right) + I_3 \left( \mathbf{t} \cdot \frac{\partial \Omega'}{\partial \mathbf{J}} \Big|_{\mathbf{J}=\mathbf{J}^*} \right)$$

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<sup>31</sup>Another way to derive this relation is note that  $\mathbf{t}$  is tangent to the resonant surface at  $\mathbf{J} = \mathbf{J}^*$ , while  $\partial_{\mathbf{J}} \Omega|_{\mathbf{J}=\mathbf{J}^*}$  is normal to the resonant surface. Hence their scalar product must be zero.

Combining these relations, one finds the following constraint

$$\Omega'(\mathbf{J}) = \Omega'(\mathbf{J}_R) + \sigma\Omega(\mathbf{J}) \quad (78)$$

where  $\mathbf{J}_R = \mathbf{J}^* + I_3\mathbf{t}$ . This property holds nearby  $\mathbf{J} = \mathbf{J}^*$ . It was also seen that

$$\begin{aligned} \Omega &= CI_2 \\ \Omega' &= \Omega'(\mathbf{J}_R) + C^*I_2 \end{aligned}$$

where

$$C = n_j \frac{\partial \Omega}{\partial J_j} \Big|_{\mathbf{J}=\mathbf{J}^*} = (n_2 \iota(J_2^*) + n_3) \frac{\partial \Omega}{\partial J_3} \Big|_{\mathbf{J}=\mathbf{J}^*}$$

and

$$C^* = n'_j \frac{\partial \Omega}{\partial J_j} \Big|_{\mathbf{J}=\mathbf{J}^*} = (n'_2 \iota(J_2^*) + n'_3) \frac{\partial \Omega}{\partial J_3} \Big|_{\mathbf{J}=\mathbf{J}^*}$$

Hence the important property

$$\sigma = \frac{C^*}{C}$$

These relationships turn out to be quite useful when interpreting the results. In particular the relation  $\xi' = n_d\theta_3 + \sigma\theta_2$  can be re-examined as follows. Since the phase of the primary perturbation is  $\xi = \mathbf{n} \cdot \boldsymbol{\alpha}$ , the time derivative of this relation reads  $\Omega' = n_d\Omega_d + \sigma\Omega$ , consistently with Eq.(78). Moreover it allows the identification of  $n_d\Omega_d$  with  $\Omega'(\mathbf{J}_R)$ , and  $n_d\theta_3$  with a reference phase  $\xi'_0$  such that  $d_t\xi'_0 = \Omega'(\mathbf{J}_R) = n_d\Omega_d$ . Since the number  $n_d$  is unique and fully defined by  $(\mathbf{n}, \mathbf{n}')$ , we introduce a drift frequency

$$\omega_d = n_d\Omega_d$$

Also the bounce frequency of deeply trapped particles  $\omega_b$  will often be taken as a proxy for the bounce frequency  $\Omega_b$ . This may actually be quite wrong, in particular for barely trapped particles for which  $\Omega_b \simeq 0$ . Nevertheless several calculations will be done in the limit of deeply trapped particles  $\Omega_b \simeq \omega_b$ , thus justifying this estimate.

The perturbation  $\tilde{H}'(\boldsymbol{\alpha}, \mathbf{J})$  can be re-expressed as a function of the new angles and actions, periodic in  $(\alpha_b, \alpha_d)$ . Its expansion in Fourier series reads

$$\tilde{H}'(\alpha_b, \alpha_d, J_b, J_d) = \sum_{n_b=-\infty}^{+\infty} H'_{n_b}(J_b, J_d) e^{i(n_b\alpha_b + n_d\alpha_d)}$$

where  $n_d = n'_2n_3 - n_2n'_3$ . In the limit where only deeply trapped particles within the primary perturbation are considered, one gets

$$H'_{n_b}(J_b, J_d) = -\frac{h'}{2} J_{n_b}(\sigma\theta_b)$$

where  $\theta_b$  must be understood as a function of  $(J_b, J_d)$ , and  $\sigma$  is a function of  $J_d$ . Large values of  $\sigma$ , i.e.  $\sigma \gg 1$ , allow using the following asymptotic form of the Bessel functions

$$J_{n_b}(\sigma\theta_b) \simeq \sqrt{\frac{2}{\pi\sigma\theta_b}} \cos\left(\sigma\theta_b - n_b\frac{\pi}{2} - \frac{\pi}{4}\right) \quad (79)$$

## H.4 Solving the Fokker-Planck equation

### H.4.1 Formulation of the Fokker-Planck equation

The effect of the primary perturbation is to produce an new equilibrium such that the distribution function, noted  $\mathfrak{F}_{eq}$ , is a function of the invariants of motion on the resonant

surface, i.e.  $\mathfrak{F}_{eq}$ , and therefore the corresponding thermodynamic potential  $\mathfrak{U}_{eq}$ , is a function of  $\mathbf{J}_R = (J_0, J_1, J_d = I_3)$ . At low collisionality, the effect of the secondary perturbation is essentially to perturb the motion of trapped particles in the primary one. This motion can be stochastic, and thus produce some transport. When the collision frequency gets higher, the effect of trapping vanishes, and the secondary perturbation mainly affects passing particles - the response is essentially of the Landau type. The equation to be solved is the Fokker-Planck equation

$$-\left\{\tilde{H}', \mathfrak{F}_{eq}\right\} - \left\{H_{eq}, \tilde{F}'\right\} = \mathcal{C}[\tilde{F}'] \quad (80)$$

written in the initial system of variables  $(\alpha, \mathbf{J})$ . The Fokker-Planck equation can be expressed as well in the new set of variables  $(\alpha_b, \alpha_d, J_b, J_d)$ . If the amplitude of the secondary perturbation is small enough, the Fokker-Planck equation becomes

$$-\left\{\tilde{H}', \mathfrak{F}_{eq}\right\} - \left\{\mathfrak{H}_{eq}, \tilde{F}'\right\} = \mathcal{C}[\tilde{F}'] \quad (81)$$

where the collision operator is linearised in  $\tilde{F}'$ <sup>32</sup>.

#### H.4.2 Collisional regime

Let us remind that the effective detrapping collision frequency in the primary perturbation modified by the effect of the secondary perturbation is given by Eq.(42), i.e.  $\nu_{eff} \simeq \langle \Delta \Omega^2 \rangle / \omega_b^2 \max(1, \sigma^2)$ . If the regime is collisional  $\nu_{eff} \gg \omega_b$ , the Fokker-Planck equation Eq.(80) is readily solved via a method already used for a single perturbation. Indeed it was seen in Appendix D.1 that in this regime the full collision operator can be replaced by a simple Krook operator, to yield the Fourier components of the distribution function

$$F'_{\mathbf{n}'} = -H'_{\mathbf{n}'} \frac{\mathbf{n}' \cdot \frac{\partial \mathfrak{F}_{eq}}{\partial \mathbf{J}}}{\mathbf{n}' \cdot \boldsymbol{\Omega} - i0^+} \quad (82)$$

It is the same as the linear response Eq.(3) translated to the secondary perturbation. However, one important difference is that the distribution function  $\mathfrak{F}_{eq}$  is the one for a modified equilibrium in presence of the primary perturbation.

#### H.4.3 Weakly collisional regime and adiabatic limit $\omega_b \gg \omega_d$

In the weakly collisional regime  $\nu_{eff} \ll \omega_b$ , it is better to solve the Fokker-Planck equation in the new set of conjugate variables  $(\alpha_b, \alpha_d, J_b, J_d)$  Eq.(81). The perturbed distribution function is expanded in Fourier series, as was done for the Hamiltonian Eq.(79)

$$\tilde{F}' = \sum_{n_b=-\infty}^{+\infty} F'_{n_b} e^{i(n_b \alpha_b + n_d \alpha_d)}$$

If the frequency  $\Omega_b \sim \omega_b$  is much larger than  $\omega_d = n_d \Omega_d$ , the response of the harmonics  $n_b \neq 0$  are smaller than those with  $n_b = 0$  in the ratio  $\omega_d / \omega_b$ . In this limit, only the single Fourier component ( $n_b = 0, n_d$ ) of the distribution function, i.e.  $F'_0$ , needs be calculated, i.e.

$$-in_d \frac{\partial \mathfrak{F}_{eq}}{\partial J_d} H'_0 + i\omega_d F'_0 = \mathcal{C}[\tilde{F}']_0$$

As done previously, it is anticipated that most of the entropy production comes from the fast variation of the distribution near the primary resonant surface. The collision operator is essentially a diffusion operator

$$\mathcal{C}[F] = \frac{1}{2} \frac{\partial}{\partial I_2} \left( \frac{\langle \Delta I_2^2 \rangle}{\Delta t} \frac{\partial F}{\partial I_2} \right) = \frac{1}{2} \frac{\partial}{\partial \Omega} \left( \frac{\langle \Delta \Omega^2 \rangle}{\Delta t} \frac{\partial F}{\partial \Omega} \right)$$

---

<sup>32</sup>Strictly speaking, a notation other than  $\tilde{F}'$  should be used since the variables have been changed - we will keep up with only one label to avoid excessive technicalities.

It is recalled that calculating an harmonic  $n_b = 0$  is equivalent to compute an average over the angle  $\alpha_b$ . Hence

$$\begin{aligned}
\mathcal{C}[\tilde{F}']_0 &= \int_0^{2\pi} \frac{d\alpha_b}{2\pi} \frac{\partial}{\partial \Omega} \left( D_\Omega \frac{\partial F'_0}{\partial \Omega} \right) \\
&= D_\Omega \Omega_b \frac{\partial}{\partial H_\Omega} \left[ \left( \int_0^{2\pi} \frac{d\alpha_b}{2\pi} \Omega \right) \frac{\partial F'_0}{\partial H_\Omega} \right] \\
&\simeq D_\Omega \frac{\partial}{\partial H_\Omega} \left( H_\Omega \frac{\partial F'_0}{\partial H_\Omega} \right)
\end{aligned} \tag{83}$$

where  $D_\Omega = \lim_{\Delta t \rightarrow 0} \frac{\langle \Delta \Omega^2 \rangle}{2\Delta t}$ . Several operations and approximations have been done to get there. First it is recalled that the Fourier component  $F'_0$  is much larger than the other components. This allows ignoring terms in the collision operator that involves time average over these components. Second the diffusion coefficient  $\langle \Delta \Omega^2 \rangle$  is supposed not to depend on the angle  $\alpha_b$ , a reasonable approximation in general. The average over  $\alpha_b$  is restated as  $d\alpha_b = \Omega_b d\alpha_2 / \Omega$ . When moving to the variable  $H_\Omega$  instead of  $\Omega$ , one has  $\partial_\Omega = \Omega \partial_{H_\Omega}$ . Besides it has been used that  $F'_0$  does not depend on  $\alpha_b$  and that  $2\pi C J_b = \int_0^{2\pi} d\alpha_2 \Omega$  - both are exact results. The third approximation consists in assuming that  $\Omega_b \simeq \omega_b$  is a constant, and hence can be moved inside the derivative with respect to  $H_\Omega$ . The identity  $H_\Omega = C J_b \Omega_b$  was then used. These two statements are correct only for particles that are deeply trapped in the primary perturbation.

The Hamiltonian  $H_\Omega$  is not convenient for practical calculations. It is replaced by the bounce angle  $\theta_b$ . Using  $H_\Omega = \frac{1}{2} \omega_b^2 \theta_b^2$ , the following expression is obtained

$$\mathcal{C}[\tilde{F}']_0 = \frac{D_\Omega}{2\omega_b^2} \frac{1}{\theta_b} \frac{\partial}{\partial \theta_b} \left( \theta_b \frac{\partial F'_0}{\partial \theta_b} \right)$$

The Fokker-Planck equation of interest finally reads

$$\omega_d F'_0 + i \frac{D_\Omega}{\omega_b^2} \frac{1}{\theta_b} \frac{\partial}{\partial \theta_b} \left( \theta_b \frac{\partial F'_0}{\partial \theta_b} \right) = -n_d \frac{\partial \mathfrak{F}_{eq}}{\partial J_d} \frac{h'}{2} J_0(\sigma \theta_b)$$

The solution  $F'_0$  must vanish at the trapped/passing boundary  $\theta_b = \theta_m$ . In principle this limit sits at  $\theta_m = \pi$ . However, in the approximation of deeply trapped particles  $\theta_b \ll 1$ , this upper bound must be modified. Two limits can be solved - approximate solutions are

- $\sigma \ll 1$

$$F'_0 = -\frac{h'}{2} \frac{n_d \frac{\partial \mathfrak{F}_{eq}}{\partial J_d}}{\omega_d - i\nu_{eff}} \left( 1 - \frac{\theta_b^2}{4} \right) \tag{84}$$

- $\sigma \gg 1$

$$F'_0 = -\frac{h'}{2} \frac{n_d \frac{\partial \mathfrak{F}_{eq}}{\partial J_d}}{\omega_d - i\nu_{eff}} J_0(\sigma \theta_b) \tag{85}$$

where

$$\nu_{eff} = \text{Max}(1, \sigma^2) \frac{D_\Omega}{2\omega_b^2}$$

The solution  $F'_0$  found in the limit  $\sigma \rightarrow 0$  satisfies the requested boundary condition  $F'_0 = 0$  at  $\theta_b = \theta_m$ , provided that  $\theta_m = 2$  is chosen (see also discussion in Appendix I),  $\omega_d \rightarrow 0$ , and also by using  $J_0(\sigma \theta_b) \rightarrow 1$ . On the contrary, in the case  $\sigma \gg 1$ ,  $J_0(\sigma \theta_b)$  is an exact solution of the differential equation. However it only satisfies approximately the boundary condition  $J_0(\sigma \theta_m) \simeq 0$ . Indeed an approximate expression of the Bessel function  $J_0$  for large arguments is obtained from Eq.(79)

$$J_0(\sigma \theta_b) \simeq \sqrt{\frac{2}{\pi \sigma \theta_b}} \cos \left( \sigma \theta_b - \frac{\pi}{4} \right) \tag{86}$$

So it oscillates fast and is equal to zero via a change of the upper bound  $\theta_m$  by a small shift of order<sup>33</sup>  $1/\sigma$ .

#### H.4.4 Weakly collisional regime and stochastic limit $\omega_b \simeq \omega_d$

If the frequency  $\Omega_b \sim \omega_b$  scales as  $\omega_d = n_d \Omega_d$ , the response of all harmonics  $n_b \neq 0$  must be accounted for. The detailed calculation is quite complex and is detailed in Appendix J.

### H.5 Entropy production rate due to the secondary perturbation

The resonant entropy production rate is given by Eqs.(18,9)

$$\dot{S}_{res}(\mathfrak{U}_{eq}, \partial_{\mathbf{J}} \mathfrak{U}_{eq}^\dagger) = -\frac{2}{T_0} \int d\gamma U_{eq}^\dagger \{ \tilde{H}', \tilde{F}' \} = \frac{2}{T_0^2} \int d\gamma \mathfrak{F}_{eq} \{ \tilde{H}', \mathfrak{U}_{eq}^\dagger \} \tilde{U}'$$

The functional can be as well expressed as a bi-linear function of  $\mathfrak{U}_{eq}^\dagger$  just by removing the pre-factor 2. The volume element in the integral is just

$$d\gamma = d^4 \mathbf{x} d^4 \mathbf{p} = d^4 \boldsymbol{\alpha} d^4 \mathbf{J} \delta(J_0 - H) = (2\pi)^2 d\alpha_b d\alpha_d dJ_0 dJ_1 dJ_b dJ_d \delta(J_0 - H)$$

where the dummy variables  $J_0$  and  $J_1$  have been re-established to compute correctly the integral over the whole phase space.

In the collisional case  $\nu_{eff} \gg \omega_b$ , one uses the set of variables  $(\boldsymbol{\alpha}, \mathbf{J})$  to find

$$\dot{S}_{res}(\mathfrak{U}_{eq}, \partial_{\mathbf{J}} \mathfrak{U}_{eq}^\dagger) = \frac{\pi}{2} \frac{1}{T_0^2} \int d\gamma F_{eq}(\mathbf{J}) \delta(\mathbf{n}' \cdot \boldsymbol{\Omega}) h'^2 \left( \mathbf{n}' \cdot \frac{\partial \mathfrak{U}_{eq}^\dagger}{\partial \mathbf{J}} \right)^2 \quad (87)$$

In the weakly collisional case  $\nu_{eff} \ll \omega_b$ , it is better to work with the new set of variables  $(\alpha_b, \alpha_d, J_b, J_d)$ . Using

$$\{ \tilde{H}', \mathfrak{U}_{eq}^\dagger \} = \frac{\partial \tilde{H}'_0}{\partial \alpha_d} \frac{\partial \mathfrak{U}_{eq}^\dagger}{\partial J_d}$$

and expanding all perturbed fields in Fourier series while keeping only the  $\pm n_d$  and  $n_b = 0$  components, one finds

$$\dot{S}_{res}(\mathfrak{U}_{eq}, \partial_{\mathbf{J}} \mathfrak{U}_{eq}^\dagger) = \frac{1}{T_0^2} \int d\gamma \mathfrak{F}_{eq} \left[ -in_d \frac{\partial \mathfrak{U}_{eq}^\dagger}{\partial J_d} \frac{h'}{2} J_0(\sigma \theta_b) \right]^* U'_0 + c.c.$$

The limit solutions of the Fokker-Planck equation Eqs(84,85) can now be used, to yield

$$\dot{S}_{res}(\mathfrak{U}_{eq}, \partial_{\mathbf{J}} \mathfrak{U}_{eq}^\dagger) = \frac{1}{2T_0^2} \Re \int d\gamma \mathfrak{F}_{eq} h'^2 \frac{-i \left( n_d \frac{\partial \mathfrak{U}_{eq}^\dagger}{\partial J_3} \right)^2}{\omega_d - i\nu_{eff}} J_0^2(\sigma \theta_b)$$

where  $J_0^2(\sigma \theta_b)$  is replaced by  $1 - \frac{\theta_b^2}{4}$  when  $\sigma \ll 1$ . Let us spend some time on the integration volume. It reads

$$d\gamma = dS dJ_b = dS \frac{\theta_b d\theta_b}{2} \lambda_m \quad (88)$$

---

<sup>33</sup>More precisely, if  $k$  is the closest integer to  $\frac{2}{\pi} \sigma \theta_m - \frac{1}{2}$ , then  $\theta_m$  should be chosen equal to  $\frac{\pi}{2\sigma} (k + \frac{1}{2})$ . Hence  $\theta_m$  is shifted by  $\nu \frac{\pi}{2\sigma}$ , where  $\nu$  is a number between 0 and 1.

where the integration element along the resonant surface is  $dS = (2\pi)^4 dJ_0 dJ_1 dI_3 \delta(J_0 - H)$ . The second equality is obtained by using  $J_b = \lambda_m \theta_b^2/4$ , where  $\lambda_m = 2\sqrt{h/C}$ . A tractable expression of the entropy production rate is therefore

$$\dot{S}_{res}(\mathfrak{U}_{eq}, \partial_{\mathbf{J}} \mathfrak{U}_{eq}^\dagger) = \frac{1}{2T_0^2} \Re \int dS \lambda_m \mathfrak{F}_{eq} h'^2 \frac{\left( n_d \frac{\partial \mathfrak{U}_{eq}^\dagger}{\partial J_d} \right)^2}{i\omega_d + \nu_{eff}} \Lambda(\sigma) \quad (89)$$

where

$$\Lambda(\sigma) = \frac{1}{2} \int_0^{\theta_m} d\theta_b \theta_b \begin{cases} \left(1 - \frac{\theta_b^2}{4}\right) & \text{if } \sigma \ll 1 \\ J_0^2(\sigma \theta_b) & \text{if } \sigma \gg 1 \end{cases} \quad (90)$$

Using the asymptotic expression Eq.(86) of the Bessel function, one gets  $\Lambda(\sigma) = \frac{1}{\pi\sigma}$  when  $\sigma \gg 1$ . In the limit  $\sigma \ll 1$ , one finds  $\Lambda(\sigma) = \frac{1}{2}$ . A more accurate calculation yields  $\Lambda(\sigma) = \frac{16}{9\pi}$  (see Appendix I).

## H.6 Covariant expression of the entropy production rate

The expression Eq.(89) is an attractive result, but still depends explicitly on the new set of variables that was constructed to derive it. It is of interest to derive an expression that is independent of the choice of variables, that is call here “covariant”. Let us note first that the element of surface along a resonant surface can be written<sup>34</sup>

$$\lambda_m dS = 2d\gamma \delta(\Omega) \omega_b$$

Of course the  $\delta$  function enforces all functions to be calculated at  $I_2 = 0$ , which leaves the action  $I_3$  as an integration variable. It is convenient to replace the variable  $I_3$  by a variable closer to the action  $J_{R2}$ , or equivalently  $\tilde{J}_2$ , on the resonant surface. The relation  $\mathbf{J}_R = \mathbf{J}^* + I_3 \mathbf{t}$  imposes the relation

$$\tilde{J}_{R2} = \frac{I_3}{\iota n_2 + n_3} \quad (91)$$

The change of variables Eq.(70), and its reverse Eq.(71) becomes

$$I_2 = \frac{\iota \tilde{J}_2 + \tilde{J}_3}{\iota n_2 + n_3} \quad (92)$$

$$\tilde{J}_{R2} = \frac{n_3 \tilde{J}_2 - n_2 \tilde{J}_3}{\iota n_2 + n_3} \quad (93)$$

and

$$\mathbf{J} = \mathbf{J}^* + I_2 \mathbf{n} + \tilde{J}_{R2} \mathbf{t}' \quad (94)$$

where  $\mathbf{t}'$  is a new tangent vector defined as

$$\mathbf{t}' = \begin{pmatrix} 1 \\ -\iota \end{pmatrix} = (\iota n_2 + n_3) \mathbf{t}$$

---

<sup>34</sup>The sequence of steps is

$$\begin{aligned} \lambda_m dS &= (2\pi)^4 dJ_0 dJ_1 dI_2 \delta(J_0 - H) 2\sqrt{\frac{h}{C}} \\ &= (2\pi)^4 dJ_0 dJ_1 \delta(J_0 - H) dI_2 dI_3 \delta(\Omega) C 2\sqrt{\frac{h}{C}} \\ &= (2\pi)^4 dJ_0 dJ_1 \delta(J_0 - H) dI_2 dI_3 \delta(\Omega) 2\omega_b \\ &= 2d\gamma \delta(\Omega) \omega_b \end{aligned}$$

Let us note also that  $\mathbf{n}' - \mathbf{n}\sigma = n_d \mathbf{t}^{35}$ . It then appears that for any function  $\mathfrak{U}_{eq}$ , its derivative on a resonant surface reads

$$n_d \left. \frac{\partial \mathfrak{U}_{eq}}{\partial J_d} \right|_{I_2=0} = n_d \left. \frac{\partial \mathfrak{U}_{eq}}{\partial I_3} \right|_{I_2=0} = \frac{n_d}{n_2\iota + n_3} \left. \frac{\partial U_{eq}}{\partial J_{R2}} \right|_{\mathbf{J}=\mathbf{J}_R} = (\mathbf{n}' - \mathbf{n}\sigma) \cdot \left. \frac{\partial U_{eq}}{\partial \mathbf{J}} \right|_{\mathbf{J}=\mathbf{J}_R} \quad (95)$$

where the relation Eq.(91) has been used, and also the expression of the derivative of a function  $U(J_2, J_3)$  along the resonant surface

$$\frac{dU_R}{dJ_{R2}} = \left( \frac{\partial U}{\partial J_2} - \iota(J_2) \frac{\partial U}{\partial J_3} \right) \Big|_{\mathbf{J}=\mathbf{J}_R} = \mathbf{t}' \cdot \left. \frac{\partial U}{\partial \mathbf{J}} \right|_{\mathbf{J}=\mathbf{J}_R}$$

where  $\mathbf{J}_R = (J_{R2}, G(J_{R2}))$  and  $U_R = U(\mathbf{J}_R)$ . Applying Eq.(95) to the special case  $\mathfrak{U}_{eq} = \mathfrak{H}_{eq}$ , one finds a practical expression of the drift frequency  $\omega_d$

$$\omega_d = n_d \left. \frac{\partial \mathfrak{H}_{eq}}{\partial J_d} \right|_{I_2=0} = (\mathbf{n}' - \mathbf{n}\sigma) \cdot \left. \frac{\partial H_{eq}}{\partial \mathbf{J}} \right|_{\mathbf{J}=\mathbf{J}_R} \quad (96)$$

Combining all these expressions, a covariant expression of the entropy production rate reads

$$\dot{S}_{res} \left( U_{eq}, \partial_{\mathbf{J}} U_{eq}^\dagger \right) = \frac{1}{T_0^2} \int d\gamma \delta(\Omega) F_{eq} h'^2 \Lambda(\sigma) \frac{\nu_{eff} \omega_b}{[(\mathbf{n}' - \mathbf{n}\sigma) \cdot \boldsymbol{\Omega}]^2 + \nu_{eff}^2} \left( (\mathbf{n}' - \mathbf{n}\sigma) \cdot \left. \frac{\partial U_{eq}^\dagger}{\partial \mathbf{J}} \right| \right)^2 \quad (97)$$

where  $\omega_b = \sqrt{Ch}$ ,

$$\Lambda(\sigma) = \begin{cases} \frac{16}{9\pi} & \text{if } \sigma \ll 1 \\ \frac{1}{\pi\sigma} & \text{if } \sigma \gg 1 \end{cases}$$

and

$$\sigma = \frac{n'_2 \iota + n'_3}{n_2 \iota + n_3}$$

$$\nu_{eff} = \text{Max}(1, \sigma^2) \frac{D_\Omega}{2\omega_b^2}$$

Here  $\mathfrak{F}_{eq}$  has been replaced by  $F_{eq}$  since the integration over the resonant response function  $\delta(\mathbf{n} \cdot \boldsymbol{\Omega})$  enforces  $F_{eq}$  to be function of  $\mathbf{J}_R$  only, i.e. a function of the actions that lay on the resonant surface. Note also that the entropy production rate goes like the square of the gradient of the thermodynamic potential, as required.

The entropy production rate Eq.(97) allows identifying 2 subregimes: when  $\nu_{eff} \leq \omega_d$ , it is proportional to the collision frequency  $\nu_{eff}$  - this is called the “ $\nu$  regime”. When  $\nu_{eff} \geq \omega_d$ , the entropy production rate is rather proportional to  $1/\nu$  - not surprisingly this defines the “ $1/\nu$  regime”.

A similar procedure can be followed for the entropy production rate in the collisional case Eq.(87). Here the objective is to compute a covariant form of  $\mathbf{n}' \cdot \partial_{\mathbf{J}} U_{eq}^\dagger$ . The function  $\mathfrak{U}_{eq}$  must be a function  $U_{eq}$  of the actions on the primary resonant surface only  $\mathbf{J} = \mathbf{J}_R$ . Using the relation  $\mathbf{J} = \mathbf{J}_R + \mathbf{n}I_2$ , it appears that

$$\mathfrak{U}_{eq}(\mathbf{J}) = U_{eq}(\mathbf{J} - \mathbf{n}I_2(\mathbf{J})) \simeq U_{eq}(\mathbf{J}) - I_2(\mathbf{J}) (\mathbf{n} \cdot \nabla_{\mathbf{J}} U_{eq}(\mathbf{J})) \quad (98)$$

One then has

$$\mathbf{n}' \cdot \frac{\partial \mathfrak{U}_{eq}}{\partial \mathbf{J}} = \mathbf{n}' \cdot \frac{\partial U_{eq}}{\partial \mathbf{J}} - \left( \mathbf{n} \cdot \frac{\partial U_{eq}}{\partial \mathbf{J}} \right) \left( \mathbf{n}' \cdot \frac{\partial I_2}{\partial \mathbf{J}} \right) = \mathbf{n}' \cdot \frac{\partial U_{eq}}{\partial \mathbf{J}} - \sigma \mathbf{n} \cdot \frac{\partial U_{eq}}{\partial \mathbf{J}}$$

where the expression Eq.(93) of  $I_2(\mathbf{J})$  has been used. This leads to the following expression of the entropy production rate

$$\dot{S}_{res} \left[ U_{eq}, \partial_{\mathbf{J}} U_{eq}^\dagger \right] = \frac{\pi}{2} \frac{1}{T_0^2} \int d\gamma F_{eq}(\mathbf{J}) \delta(\mathbf{n}' \cdot \boldsymbol{\Omega}) h'^2 \left( (\mathbf{n}' - \sigma \mathbf{n}) \cdot \left. \frac{\partial U_{eq}^\dagger}{\partial \mathbf{J}} \right| \right)^2 \quad (99)$$

---

<sup>35</sup>This relation is expediently demonstrated by computing  $n'_2 - \sigma n_2$  and  $n'_3 - \sigma n_3$  and using  $n_d = n'_2 n_3 - n_2 n'_3$  and  $\sigma = \frac{n'_2 \iota + n'_3}{n_2 \iota + n_3}$ .

# I Entropy production of a secondary perturbation: alternative calculation

This appendix shows that a more precise calculation can be done for the entropy production due to a secondary perturbation in the case  $\sigma \ll 1$ , i.e. relaxing the assumption of deeply trapped particles. Let us now reconsider the derivation of the collision operator Eq.(83), and stops at the second line

$$\mathcal{C}[F']_0 = CD_\Omega \Omega_b \frac{\partial}{\partial H_\Omega} \left[ J_b \frac{\partial F'_0}{\partial H_\Omega} \right]$$

In principle, it is not possible to use the same recipe as for deeply trapped particles, i.e. moving  $\Omega_b$  inside the partial derivative in  $H_\Omega$ , since  $\Omega_b$  is a function of  $H_\Omega$  in the general case. However, for the special case of a function  $F'_0$  proportional  $1 - K$ , it appears that

$$\mathcal{C}[\tilde{F}']_0 = -\nu_{eff} F'_0$$

where it is reminded that

$$\nu_{eff} = \frac{D_\Omega}{2\omega_b^2}$$

This result holds because

$$\Omega_b = \frac{1}{C} \frac{\partial H_\Omega}{\partial J_b}$$

This is a general property of Hamiltonian systems (recall that the Hamiltonian is  $H = H_{eqR} + \frac{1}{C}H_\Omega$ ), which can also be verified directly from Eq.(75,76). It then appears that the solution Eq.(84) still holds, in the form

$$F'_0 = -\frac{h'}{2} \frac{n_d \frac{\partial \mathfrak{F}_{eq}}{\partial J_d}}{\omega_d - i\nu_{eff}} (1 - K) \quad (100)$$

It is in fact an exact solution of the Fokker-Planck equation in the limit  $\omega_d \rightarrow 0$ , and an acceptable one when  $K \leq 1$  if  $\omega_d \neq 0$ .

The expression of the entropy production rate Eq.(89) is the same. However the integration over the phase space must be changed. Indeed the expression Eq.(88)  $d\gamma = dS \frac{\theta_b d\theta_b}{2} \lambda_m$  must be changed in

$$d\gamma = dS dJ_b = dS dK \int_{-\theta_b}^{+\theta_b} \frac{d\theta_2}{2\pi} \frac{1}{\sqrt{K - \sin^2 \left( \frac{\theta_2}{2} \right)}}$$

where the expression of  $J_b$  Eq.(75) has been used<sup>36</sup>. Hence the expression of  $\Lambda(\sigma)$  Eq.(90) becomes

$$\Lambda(\sigma) = \int_0^1 dK (1 - K) \int_{-\theta_b}^{+\theta_b} \frac{d\theta_2}{2\pi} \frac{1}{\sqrt{K - \sin^2 \left( \frac{\theta_2}{2} \right)}}$$

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<sup>36</sup>One must be careful that both signs of  $J_b$  must be included, which is contained in both branches in the loop integral in  $\theta_2$ . This introduces a factor 2 that is balanced by factor 1/2 that comes out of the square root derivative.



The integrals can be permuted to yield the following sequence of steps

$$\begin{aligned}
\Lambda(\sigma) &= \int_{-\pi}^{\pi} \frac{d\xi}{2\pi} \int_{\sin^2(\frac{\xi}{2})}^1 dK \frac{1-K}{\sqrt{K - \sin^2(\frac{\xi}{2})}} \\
&= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} du \int_{\sin^2 u}^1 dK \left( \frac{1 - \sin^2 u}{\sqrt{K - \sin^2 u}} - \sqrt{K - \sin^2 u} \right) \\
&= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} du \left[ 2(1 - \sin^2 u) \sqrt{K - \sin^2 u} - \frac{2}{3} (K - \sin^2 u)^{3/2} \right]_{\sin^2 u}^1 \\
&= \frac{8}{3\pi} \int_0^{\frac{\pi}{2}} du \cos^3 u \\
&= \frac{16}{9\pi}
\end{aligned}$$

## J Stochastic regime

### J.1 Onset of Hamiltonian chaos

We consider the case where  $\Omega_b$  is of the same order of magnitude as  $n_d \Omega_d$ . Then all Hamiltonian harmonics  $n_b$  must be accounted for in the Hamiltonian expansion Eq.(79), that is repeated here for completion

$$\tilde{H}' = \sum_{n_b=-\infty}^{+\infty} H'_{n_b} e^{i(n_b \alpha_b + n_d \alpha_d)}$$

where  $n_d = n'_2 n_3 - n_2 n'_3$ . The analysis is restricted to deeply trapped particles within the primary perturbation, so that

$$H'_{n_b} = -\frac{h'}{2} J_{n_b}(\sigma \theta_b) \quad (101)$$

Whenever the parameter  $\sigma$  is large against 1, i.e.  $\sigma \gg 1$ , the asymptotic form of the Bessel functions is used

$$J_{n_b}(\sigma \theta_b) \simeq \sqrt{\frac{2}{\pi \sigma \theta_b}} \cos \left( \sigma \theta_b - n_b \frac{\pi}{2} - \frac{\pi}{4} \right) \quad (102)$$

Each perturbation  $H'_{n_b}$  is associated with a resonant surface

$$\omega_{n_b}(J_b, J_d) = n_b \Omega_b + n_d \Omega_d = 0$$

As seen many times now, trajectories for a single harmonic  $(n_b, n_d)$  are bound to lie on an island. More precisely, the displacement  $\Lambda$  near the resonant surface in the action coordinates  $(J_b, J_d)$  is such that  $\mathbf{J} = \mathbf{J}_R + \Lambda(n_b \hat{\mathbf{e}}_b + n_d \hat{\mathbf{e}}_d)$  where  $\omega_{n_b}(\mathbf{J}_R) = 0$ . Trajectories must stay on the island prescribed by the invariant of motion  $H_\Lambda$  defined as

$$H_\Lambda = \frac{1}{2} C_\Lambda \Lambda^2 + H'_{n_b} \cos(\zeta)$$

where  $\zeta = n_b \alpha_b + n_d \alpha_d$ ,  $H'_{n_b}$  is given by Eq.(101) and the Hamiltonian curvature  $C_\Lambda$  is

$$C_\Lambda = n_b \left. \frac{\partial \omega}{\partial J_b} \right|_{\mathbf{J}=\mathbf{J}_R} + n_d \left. \frac{\partial \omega}{\partial J_d} \right|_{\mathbf{J}=\mathbf{J}_R}$$

An extra complication comes from the dependence on the perturbed Hamiltonian component  $H'_{n_b}$  on the action  $\Lambda$ . As above, we consider two extreme cases:  $\sigma \ll 1$ , where the Bessel function is approximated by 1 and thus  $H'_{n_b} \simeq -h'/2$ . In the opposite case where

$\sigma \gg 1$ , the asymptotic form of the Bessel function Eq.(102) is used. A Taylor expansion yields the relation

$$\sigma\theta_b = \sigma\theta_b(\mathbf{J}_R) + K_\Lambda\Lambda$$

where

$$K_\Lambda = n_b \left. \frac{\partial(\sigma\theta_b)}{\partial J_b} \right|_{\mathbf{J}=\mathbf{J}_R} + n_d \left. \frac{\partial(\sigma\theta_b)}{\partial J_d} \right|_{\mathbf{J}=\mathbf{J}_R}$$

is homogeneous to the inverse of an action, and characterises the degree of variation of the perturbed Hamiltonian across the resonant surface. Hence the Hamiltonian Fourier harmonic reads

$$H'_{n_b} = -\frac{h'}{2} \sqrt{\frac{2}{\pi\sigma\theta_b}} \cos(K_\Lambda(\Lambda - \Lambda_0))$$

where

$$\Lambda_0 = -\frac{1}{K_\Lambda} \left[ (\sigma\theta_b(\mathbf{J}_R)) - n_b \frac{\pi}{2} - \frac{\pi}{4} \right] \simeq -\frac{1}{K_\Lambda} (\sigma\theta_b)(\mathbf{J}_R)$$

The quantity  $\Lambda_0$  has no reason to vanish and can be seen as a shift of the resonant surface in the  $\Lambda$  direction. This shift is quite the same for all resonant surfaces because of the weak dependence of  $\Lambda_0$  on the resonant index  $n_b$ . The constant shift  $\Lambda_0$  will thus be ignored in the following since it does not affect the threshold for stochasticity. We now focus our intention on the dynamics for a given value of  $n_b$ , say  $n_b = 0$  - thus the label  $n_b$  is dropped to lighten notations. As done previously, a reformulation of the island invariant of motion in term of the frequency  $\omega = C_\Lambda\Lambda$  makes life easier

$$H_\omega = \frac{1}{2}\omega^2 - \omega_I^2 \cos(\tau\omega) \cos(\zeta)$$

where  $H_\omega = C_\Lambda H_\Lambda$  is a normalised Hamiltonian,  $\tau = \frac{K_\Lambda}{C_\Lambda}$  is a time that characterises the dependence of the perturbed Hamiltonian on  $\Lambda$ , and

$$\omega_I = \left[ C_\Lambda \sqrt{\frac{2}{\pi\sigma\theta_b}} h' \right]^{1/2} \quad (103)$$

The equations of motion with this set of conventions read

$$\frac{d\zeta}{dt} = \frac{\partial H_\omega}{\partial \omega} = \omega + \omega_I^2 \tau \sin(\tau\omega) \cos(\zeta) \quad (104)$$

$$\frac{d\omega}{dt} = -\frac{\partial H_\omega}{\partial \zeta} = -\omega_I^2 \cos(\tau\omega) \sin(\zeta) \quad (105)$$

A key parameter is the product  $\omega_I\tau = \frac{\omega_{II}}{\omega_I}$ , where  $\omega_{II}$  is a frequency that characterises the motion on the structures that emerge in the phase space due to the dependence of the perturbed Hamiltonian on the action (the  $\cos(\tau\omega)$  prefactor)

$$\omega_{II} = K_\Lambda h' = \tau\omega_I^2$$

Depending on the value of  $\omega_I\tau$ , two regimes can be identified [31]

- When  $\tau\omega_I = \frac{\omega_{II}}{\omega_I} \leq S_0$ , where  $S_0$  is a number to be determined, the surfaces  $H_\omega = cte$  reproduce the classical shape of a cat's eye, Fig.26. The integer  $n_b$  varies while  $n_d$  is fixed, so that the distance between resonant surfaces is  $\Omega_b$ . The island size (in frequency) is  $4\omega_I$ . The Chirikov parameter is  $S_I = \frac{4\omega_I}{\Omega_b}$  and trajectories are stochastic whenever  $S_I \geq S_c$ , where  $S_c$  is a number of order 1. This limit is called "weak perturbation regime".
- In the opposite case  $\tau\omega_I = \frac{\omega_{II}}{\omega_I} \geq S_0$ , the surfaces  $H_\omega = cte$  exhibit a more complex shape, with multiple lobes. The broadening and increasing complexity of the set of iso- $H_\omega$  lines is illustrated on Fig.26. This figure allows a physical interpretation of the frequency  $\omega_{II}$ , which in fact is the inverse of the time needed by a particle

to explore a secondary island (or lobe). The whole size of this set of lobes is no longer the island width  $4\omega_I$ , but rather  $2\tau\omega_I^2 = 2\omega_{II}$  so that the Chirikov parameter becomes  $S_{II} = 2\frac{\tau\omega_I^2}{\Omega_b} = 2\frac{\omega_{II}}{\Omega_b}$ . Stochasticity occurs whenever  $S_{II} \geq S_c$ , admitting that the critical number for transition to stochasticity is the same in the weak and strong perturbation regime. This limit is called “strong perturbation regime”.

The critical value  $S_0$  of  $\tau\omega_I$  can now be computed. It determines the transition from weak to strong perturbation regimes, and is defined by the condition  $S_{II} = S_I$ , hence  $S_0 = 2$ . Let us note the relationship

$$S_{II} = \frac{\tau\omega_I}{S_0} S_I = 2\frac{\tau\omega_I^2}{\Omega_b}$$

This relationship shows that the stochasticity condition  $S_I \geq 1$  *implies* that  $S_{II} \geq 1$  in the strong perturbation regime since the latter is conditioned to  $\tau\omega_I \geq 2$ . The time  $\tau$  depends on the amplitude  $h$  of the primary perturbation via the frequencies  $(\Omega_b, \Omega_d)$ , while the frequency  $\omega_I$  depends on the amplitude  $h'$  of the secondary perturbation. Hence the condition  $\tau\omega_I = 2$  defines a critical value of  $h'$  that depends on  $h$  and other geometrical and plasma parameters.

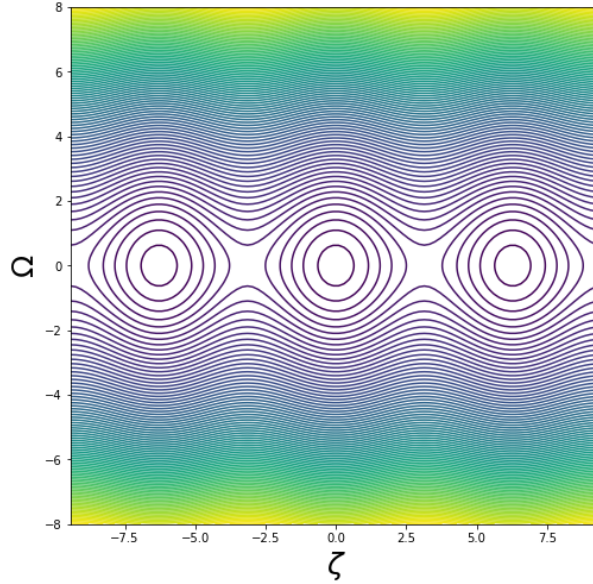


Figure 25: Contour lines of the Hamiltonian  $H_\omega(\omega, \zeta) = \frac{1}{2}\omega^2 - \omega_I^2 \cos(\tau\omega) \cos(\zeta)$ . Parameters are  $\omega_I = 1$ , and  $\omega_I\tau = 0$ .

## J.2 Entropy production rate in the the stochastic regime

Computing the transport in the phase space in the stochastic regime is a difficult task. Let us start with the simplest approach, i.e. the case where the Chirikov overlap parameter is high enough for the quasi-linear theory to applies. Based on the discussion above, the diffusion coefficient reads

$$D_{ij} = \pi \sum_{n_b=-\infty}^{+\infty} |H'_{n_b}|^2 n_i n_j \delta(n_b \Omega_b + n_d \Omega_d)$$

where the index  $i$  can label the direction  $b$  or  $d$ , and

$$H'_{n_b} = -\frac{h'}{2} J_{n_b}(\sigma\theta_b)$$

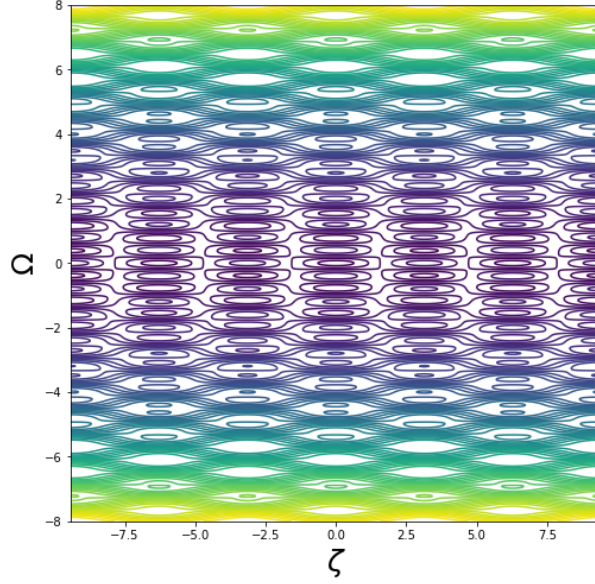


Figure 26: Contour lines of the Hamiltonian  $H_\omega(\omega, \zeta) = \frac{1}{2}\omega^2 - \omega_I^2 \cos(\tau\omega) \cos(\zeta)$ . Parameters are  $\omega_I = 1$ , and  $\omega_I\tau = 8$ .

The entropy production rate as given by Eq.(20) reads

$$\begin{aligned} \dot{S}_{res} \left( U_{eq}, \partial_{\mathbf{J}} U_{eq}^\dagger \right) &= \frac{\pi}{2T_0^2} \sum_{n_b=-\infty}^{+\infty} \int d\gamma \mathfrak{F}_{eq} |H'_{n_b}|^2 \\ &\quad \delta(n_b\Omega_b + n_d\Omega_d) \left( n_b \frac{\partial \mathfrak{U}_{eq}^\dagger}{\partial J_b} + n_d \frac{\partial \mathfrak{U}_{eq}^\dagger}{\partial J_d} \right) \bigg|_{\mathbf{J}=\mathbf{J}_R}^2 \end{aligned}$$

The pre-factor 1/2 was introduced to deal with the function  $\mathfrak{U}_{eq}^\dagger$  only in the functional  $\mathcal{S}_{res}$ . Let us note that  $\delta(n_b\Omega_b + n_d\Omega_d) = 1/\Omega_b \delta(n_b + n^*)$  where  $n^* = n_d\Omega_d/\Omega_b$  and  $\Omega_b > 0$ . Moreover it was seen in the section on quasilinear theory that various processes lead to a broadening of the resonance function, i.e. the Kronecker delta function can be replaced by

$$\delta(n_b - n^*) \rightarrow \frac{1}{\pi} \frac{\Delta n_b}{(n_b - n^*)^2 + \Delta n_b^2}$$

If the resonance broadening width  $\Delta n_b$  is large enough against 1, the summation over the index  $n_b$  can be replaced by a continuous integral  $\int_{-\infty}^{+\infty} dn_b$ . One gets

$$\dot{S}_{res} \left( U_{eq}, \partial_{\mathbf{J}} U_{eq}^\dagger \right) = \frac{\pi}{2T_0^2} \int d\gamma \mathfrak{F}_{eq} |H'_{n^*}|^2 \frac{1}{\Omega_b} \left( n^* \frac{\partial \mathfrak{U}_{eq}^\dagger}{\partial J_b} + n_d \frac{\partial \mathfrak{U}_{eq}^\dagger}{\partial J_d} \right) \bigg|_{\mathbf{J}=\mathbf{J}_R}^2$$

Let us remember that

- if  $\sigma \ll 1$ , then  $H'_{n^*} = -\frac{h'}{2}$ , and  $\omega_I = \sqrt{C_\Lambda h'}$  (the island size is equal to  $4\omega_I$ )
- if  $\sigma \gg 1$ , then

$$H'_{n^*} = -\frac{h'}{2} \sqrt{\frac{2}{\pi\sigma\theta_b}} \cos(\omega\tau)$$

and

$$\omega_I = \left[ C_\Lambda \sqrt{\frac{2}{\pi\sigma\theta_b}} h' \right]^{1/2}$$

Combining these various ingredients, the entropy production bears the form

$$\dot{S}_{res} \left( U_{eq}, \partial_{\mathbf{J}} U_{eq}^\dagger \right) = \frac{1}{T_0^2} \frac{\pi}{8} \int d\gamma \mathfrak{F}_{eq} \frac{\omega_I^4}{\Omega_b} \cos^2(\omega\tau) \Xi(\mathbf{J}) \left( \frac{\partial \mathfrak{U}_{eq}^\dagger}{\partial \omega} \right)^2$$

where  $d\gamma = dS d\omega / |C_\Lambda|$ , and  $dS$  is the element of integration along the  $n_b = 0$  resonant surface, and

$$\frac{\partial \mathfrak{U}_{eq}^\dagger}{\partial \omega} = \frac{1}{C_\Lambda} \left( n_b \frac{\partial \mathfrak{U}_{eq}^\dagger}{\partial J_b} + n_d \frac{\partial \mathfrak{U}_{eq}^\dagger}{\partial J_d} \right) \Big|_{\mathbf{J}=\mathbf{J}_R}$$

measures the departure from thermodynamic equilibrium. The latter expression is a consequence of the relation  $\mathbf{J} = \mathbf{J}_R + \Lambda(n_b \hat{\mathbf{e}}_b + n_d \hat{\mathbf{e}}_d)$  and  $\omega = C_\Lambda \Lambda$ . It reflects a diffusion that occurs mainly along the direction  $\omega$ , i.e. across resonant surfaces. The form factor  $\Xi(\mathbf{J})$  accounts for various corrections that are discussed below. In the simplest case,  $\Xi(\mathbf{J}) = 1$ .

A special case occurs when only deeply trapped particles matter, so that  $\Omega_b \rightarrow \omega_b$  does not depend on the action  $J_b$ <sup>37</sup>. Nevertheless it yields a useful order of magnitude. In this limit the curvature reads

$$C_\Lambda \simeq n_d^2 \frac{\partial^2 \Omega_d}{\partial J_d^2}$$

Also, from Eq.(77)  $J_b = \lambda_m \theta_b^2 / 4 \rightarrow 0$ , it can reasonably be assumed that  $\mathfrak{U}_{eq}^\dagger$  does not depend on  $\theta_b$ . In this case

$$\frac{\partial \mathfrak{U}_{eq}^\dagger}{\partial \omega} \simeq \frac{1}{C_\Lambda} n_d \frac{\partial \mathfrak{U}_{eq}^\dagger}{\partial J_d} \Big|_{\mathbf{J}=\mathbf{J}_R}$$

Keeping up with this approximation. The covariant form reads

$$\dot{S}_{res} \left( U_{eq}, \partial_{\mathbf{J}} U_{eq}^\dagger \right) = \frac{1}{T_0^2} \frac{\pi}{8} \int d\gamma F_{eq}(\mathbf{J}) \frac{h'^2}{\Omega_b} \cos^2(\omega\tau) \Xi(\mathbf{J}) \left( (\mathbf{n}' - \sigma \mathbf{n}) \cdot \frac{\partial U_{eq}^\dagger}{\partial \mathbf{J}} \right)^2$$

The expression above bears the advantage of simplicity, and is correct in the limit  $\sigma \ll 1$ , with  $\tau = 0$  and  $\Xi(\mathbf{J}) = 1$ . However it overestimates transport coefficients in the case  $\sigma \gg 1$  since it assumes that quasi-linear theory always apply. The next step is to analyse separately the weak and strong perturbation regimes:

- In the weak perturbation regime  $\tau \omega_I = \frac{\omega_{II}}{\omega_I} \leq 2$ , the usual quasi-linear limit is recovered, i.e.  $\Xi(\mathbf{J}) = 1$ . The diffusion coefficient in the frequency space reads  $D_\omega \sim \pi \omega_I^4 / \Omega_b$ . It can be understood as follows. The equation of motion Eq.(105) is  $d_t \omega = -\omega_I^2 \sin \zeta$ . It implies that  $\omega$  moves by  $\omega_I$  whenever  $\zeta$  changes by  $\pi$ . In presence of two perturbations  $\zeta$  changes sign every half-period  $\pi / \omega_b$ . A simple random walk argument yields  $D_\omega \sim \pi \omega_I^4 / \omega_b$ .
- In the strong perturbation regime  $\omega_I \tau = \omega_{II} / \omega_I \geq S_0$ , the quasi-linear formula cannot be applied. Since  $d_t \zeta$  scales as  $\omega_I^2 \tau = \omega_{II}$ , the phase changes by  $\pi$  after a time  $\tau_{eff} \simeq \pi / \omega_{II}$ . The diffusion coefficient is therefore  $D_\omega \sim \pi \omega_I^2 / \tau$ . The ratio of the diffusion coefficient to the quasilinear reference value is of the order of  $\Omega_b / (\tau \omega_I^2)$ , up to a constant that should be determined numerically. Here we decided to write the form factor as

$$\Xi(\mathbf{J}) = \frac{S_{st}}{2} \frac{\Omega_b}{\tau \omega_I^2} = \frac{S_{st}}{S_{II}}$$

In this formulation the form factor is the inverse ratio of the Chirikov overlap parameter in the strong perturbation regime  $S_{II}$  to some reference value  $S_{st}$ , which for

<sup>37</sup>This assumption is highly questionable. As mentioned above, stochasticity occurs if the bounce frequency  $\Omega_b$  is low enough. This situation actually occurs more easily for barely trapped/passing particles for which  $\Omega_b \rightarrow 0$

consistency must be chosen larger than  $S_c$ . Indeed the transition between the two diffusive regimes occurs when  $S_{II} = S_{st}$ , which obviously must be above the stochasticity threshold, otherwise some contradiction occurs. *Hence the diffusion coefficient in the strong perturbation is always smaller than the quasilinear value.* It seems strange that the strong perturbation regime produces a diffusion coefficient that is smaller than the one found in the weak perturbation regime. But this is fact reminiscent of the discussion on the Dupree time and renormalisation procedure, where the correlation time turns out to decrease with the quasilinear diffusion coefficient. The choice proposed in [31] is  $S_{st} = 2$ .

The reshaped expression of the entropy production rate is therefore

$$\dot{S}_{res} \left( U_{eq}, \partial_{\mathbf{J}} U_{eq}^\dagger \right) = \frac{1}{T_0^2} \frac{\pi}{8} \int d\gamma \mathfrak{F}_{eq} \frac{\omega_I^4}{\Omega_b} \min \left( 1, \frac{S_{st}}{S_{II}} \right) \left( \frac{\partial \mathfrak{U}_{eq}^\dagger}{\partial \omega} \right)^2$$

An alternative to the function  $\min(1, S_{st}/S_{II})$  is a Lorentzian  $\frac{1}{1+S_{II}/S_{st}}$ , which can be justified on the basis of a non linear resonance broadening theory mentioned in the section on the quasi-linear theory.

This expression can now be written in a covariant form

$$\dot{S}_{res} \left( U_{eq}, \partial_{\mathbf{J}} U_{eq}^\dagger \right) = \frac{1}{T_0^2} \frac{\pi}{8} \int d\gamma F_{eq} \frac{h'^2}{\Omega_b} \Lambda(\mathbf{J}) \left( (\mathbf{n}' - \sigma \mathbf{n}) \cdot \frac{\partial U_{eq}^\dagger}{\partial \mathbf{J}} \right)^2$$

where the form factor  $\Lambda(\mathbf{J})$  is such that

- if  $\sigma \ll 1$ , then  $\Lambda(\mathbf{J}) = 1$
- if  $\sigma \gg 1$ , then

$$\Lambda(\mathbf{J}) = \frac{2}{\pi \sigma \theta_b} \min \left( 1, \frac{S_{st}}{S_{II}} \right) \simeq \frac{2}{\pi \sigma \theta_b} \frac{1}{1 + \frac{S_{II}}{S_{st}}}$$

where

$$S_{II} = 2 \sqrt{\frac{2}{\pi \sigma \theta_b}} \frac{K_\Lambda h'}{\Omega_b}$$

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