



# Magnetic coordinates and equilibrium magnetic field

Xavier Garbet

## ► To cite this version:

Xavier Garbet. Magnetic coordinates and equilibrium magnetic field. Doctoral. France. 2023. hal-03974960

**HAL Id: hal-03974960**

**<https://hal.science/hal-03974960>**

Submitted on 6 Feb 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0 International License

# Magnetic coordinates and equilibrium magnetic field

X. Garbet<sup>1,2</sup>

<sup>1</sup>CEA, IRFM, Saint-Paul-lez-Durance, F-13108, France

<sup>2</sup>School of Physical and Mathematical Sciences, Nanyang Technological University, 637371 Singapore

February 6, 2023

## Abstract

This note addresses some properties of confining magnetic configurations. In a first step, it is shown that confinement is ensured if the field lines lie on surfaces, called magnetic (or flux) surfaces. These surfaces have to be tori in accordance with the hairy ball theorem. The existence of magnetic flux surfaces is granted when the configuration is axisymmetric. When it is not, flux surfaces can still exist under some reasonable conditions. A set of curvilinear flux coordinates can then be built, that reflect the structure of the magnetic field. The second step consists in calculating the force exerted on the plasma. The condition of force balance leads to a constraint on the magnetic field called the Grad-Shafranov equation. The equation prescribes the magnetic field, for given pressure and current profiles. Examples of solutions of the Grad-Shafranov are given. It is also shown that a vertical magnetic field is needed to ensure an equilibrium.

**Warning:** hasty readers may skip sections with a star.

## 1 Introduction

The choice of coordinates in a magnetised plasma is essential. The notion of field lines allows understanding this point. Field lines are defined as curves that are tangent to the magnetic field at every point. Conversely the magnetic field is a continuous vector tangent field associated with a set of field lines. When the magnetic field is intense, charged particles follow closely field lines, at least for some time - hence the name magnetic guide field<sup>1</sup>. Magnetic field lines can therefore be seen as geodesics. As in general relativity, it is appealing to define a set of coordinates well adapted to describe these geodesics. A logical next step is to derive an equation that constrains the magnetic topology, here the Grad-Shafranov equation, for given sources which appear to be the thermal pressure and current density profiles. Good confinement properties of a magnetic field are shown to imply that field lines are winded on magnetic flux surfaces, or flux surfaces in short, which are tori. Flux surfaces and field lines offer a powerful way to construct a set of coordinates that spouse the magnetic topology. These are called flux coordinates. Flux coordinates reflect the structure of the magnetic field, and thus embed some basic properties of the field, like magnetic fluxes, hence their name. This simplifies calculations and more importantly allows a deep understanding of the physics at play. This matters for

---

<sup>1</sup>This is a proxy - in fact charged particles drift away from curved field lines. However drift velocities scale as the inverse of the field, hence vanish at high field.

instance to devise equilibrium magnetic configurations with good confinement properties. Differential geometry appears as a powerful mathematical tool to define and handle flux coordinates<sup>2</sup>.

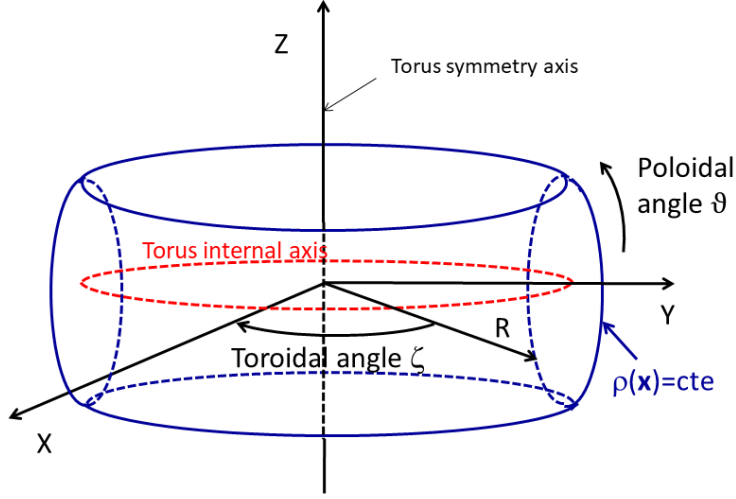


Figure 1: System of toroidal coordinates in an axisymmetric configuration.

## 2 Structure of the equilibrium magnetic field in a tokamak

One way of ensuring charged particle confinement is to enforce field lines to lie on bounded surfaces. These surfaces, when they exist, are called magnetic flux surfaces (or flux surfaces)[1, 2]<sup>3</sup>. Flux surfaces have to be tori thanks to the “hairy ball” theorem [3]. A casual version of this theorem is the following. Field lines can be seen as “hair”. Hair cannot be combed on a sphere without a field singularity (at the poles), whereas they can be combed smoothly on a torus. [3] The ‘hairy ball’ (or “hedgehog”) theorem can be formulated in a slightly more precise fashion. Under reasonable conditions (e.g. magnetic field smoothness on the invariant manifold, no singularity), a compact manifold in a 3D phase space is a 2-dimensional torus. While the “reasonable” conditions of smoothness can be granted, the key aspect is the compactness, which requires that field lines are bounded. This is the hard part of the demonstration, which is addressed in the next sections.

### 2.1 Existence of flux surfaces

A toroidal system of coordinates  $(\rho, \theta, \zeta)$  is chosen, anticipating that field lines are wound on tori in a well suited confining magnetic configuration. The surface  $\rho(\mathbf{x}) = cte$  is a

<sup>2</sup>A reader not familiar with curvilinear coordinates may read first the appendix A.

<sup>3</sup>This condition is sufficient but certainly not necessary. One could think of a finite volume ergodised by a set of field lines, surrounded by flux surfaces. This system could be still called confining while not satisfying the condition of flux surface existence. However a set of flux surfaces is optimal in terms of confinement, admitting that particles follow closely field lines. It is also stressed that the existence of magnetic surfaces does not ensure perfect confinement, because of a slow transport process called Arnold diffusion.

torus, hence a coordinate that characterises the distance to the torus internal axis, while  $\theta$  and  $\zeta$  are poloidal and toroidal angles. A configuration is said axisymmetric if it is left invariant by a rotation around a symmetry axis (labelled by the coordinate  $Z$ ) - see example on Fig.1. However the most general case is not axisymmetric - an emblematic example is the stellarator (see Fig.2)<sup>4</sup>. The angles  $(\theta, \zeta)$  are *not* necessarily geometric angles<sup>5</sup>. The notation  $\varphi$  is used when the toroidal angle is geometric *and* oriented in the conventional trigonometric direction. It must be stressed again that the winding of field lines on toroidal flux surfaces, is by no means granted. Nevertheless nothing prevents the use of an arbitrary set of toroidal coordinates. The vector potential can be written as

$$\mathbf{A} = \chi(\mathbf{x})\nabla\theta - \psi(\mathbf{x})\nabla\zeta + \nabla G_A$$

where  $\chi, \psi$  and  $G_A$  are functions of  $\mathbf{x} = (\rho, \theta, \zeta)$ . The functions  $\chi$  and  $\psi$  get a physical interpretation by using the Gauss flux theorem that relates the contour integral of the vector potential to the flux of the magnetic fields. More precisely if  $\Psi$  is the flux of the magnetic field across a closed curve that lies on the torus  $\rho = cte$  and turns around the vertical axis (contour  $\mathcal{C}$  at constant  $\rho$  and  $\theta$ ), and  $\chi$  the magnetic flux trough a curve that turns around the axis of the torus  $\rho = cte$  in a “poloidal” plane  $\zeta = cte$  (contour  $\mathcal{C}'$  at constant  $\rho$  and  $\zeta$ ), then<sup>6</sup>

$$\Psi(\rho, \theta) = -2\pi \int_{\mathcal{C}} d\zeta \psi(\mathbf{x}) \quad (1)$$

$$\chi(\rho, \zeta) = 2\pi \int_{\mathcal{C}'} d\theta \chi(\mathbf{x}) \quad (2)$$

$\Psi$  and  $\chi$  are called “poloidal” and “toroidal” magnetic fluxes<sup>7</sup>. The magnetic field  $\mathbf{B} = \nabla \times \mathbf{A}$  bears the following Clebsch representation [1]

$$\mathbf{B} = \nabla\chi \times \nabla\theta + \nabla\zeta \times \nabla\psi \quad (3)$$

It is always possible to choose the radial coordinate  $\rho$  as  $\psi$  (or respectively  $\chi$ ). The relation Eq.(1) (resp. Eq.(2)) then implies that  $\rho$  is minus the poloidal flux normalised to  $2\pi$  (resp. the toroidal flux normalised to  $2\pi$ ). In some special cases, a scalar field may be a function of  $\rho$  only, i.e. does not depend on the poloidal and toroidal angles. Such a field is called a “flux function”. By definition, field lines are such that  $\mathbf{B} \times d\ell = 0$ , where  $d\ell$  is a length element along the field line. This condition can be expressed as

$$\frac{d\rho}{\mathbf{B} \cdot \nabla\rho} = \frac{d\theta}{\mathbf{B} \cdot \nabla\theta} = \frac{d\zeta}{\mathbf{B} \cdot \nabla\zeta}$$

Choosing without loss of generality the radial coordinate as the normalised toroidal flux,  $\rho = \chi$ , one gets the two following equations

$$\frac{d\theta}{d\zeta} = \frac{d\psi}{d\chi} \quad (4)$$

$$\frac{d\chi}{d\zeta} = -\frac{d\psi}{d\theta} \quad (5)$$

This set of equations is a set Hamiltonian equations where  $\psi(\chi, \theta, \zeta)$  plays the role of the Hamiltonian and  $\zeta$  the role of time [5, 3]. This corresponds to a dynamical system with

---

<sup>4</sup>Note however that the modulus of the magnetic field can be axisymmetric in a stellarator

<sup>5</sup>A geometric angle  $\theta$ , measured in radian, is defined as the ratio of the length of a circular arc to the radius of the circle. This *does not imply* that the curves  $\rho = cte$  are circles. Any periodic function of the geometric angle is also an angle. A similar definition applies to the angle  $\zeta$ .

<sup>6</sup>The toroidal length element is  $(\nabla\rho \times \nabla\theta) \sqrt{g} d\zeta$ , while a poloidal length element is  $(\nabla\zeta \times \nabla\rho) \sqrt{g} d\theta$ . See Appendix A for details.

<sup>7</sup>Note there is some piece of arbitrariness in these definitions. For instance, if the toroidal geometric angle is taken as the orientation direction for the contour  $\mathcal{C}$ , then  $\Psi$  is the poloidal flux (without a minus).

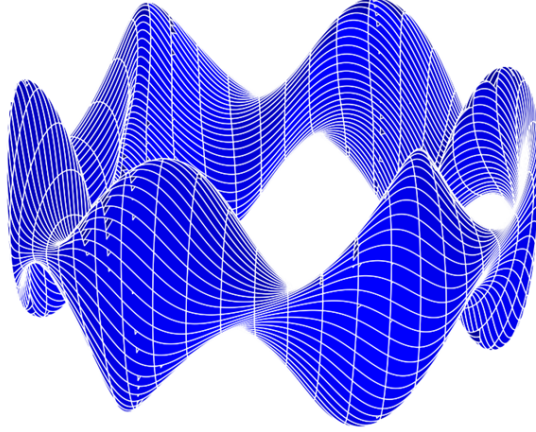


Figure 2: Example of an non-axisymmetric configuration, here a quasi-helically symmetric stellarator (from [4]).

1.5 degree of freedom (a couple of conjugate variables plus time). The solution of these equations is generically characterised by field lines that ergodise a large fraction of the accessible volume. Obviously this means no confinement.

However if the system is axisymmetric, then the function  $\psi$  does not depend on  $\zeta$ .

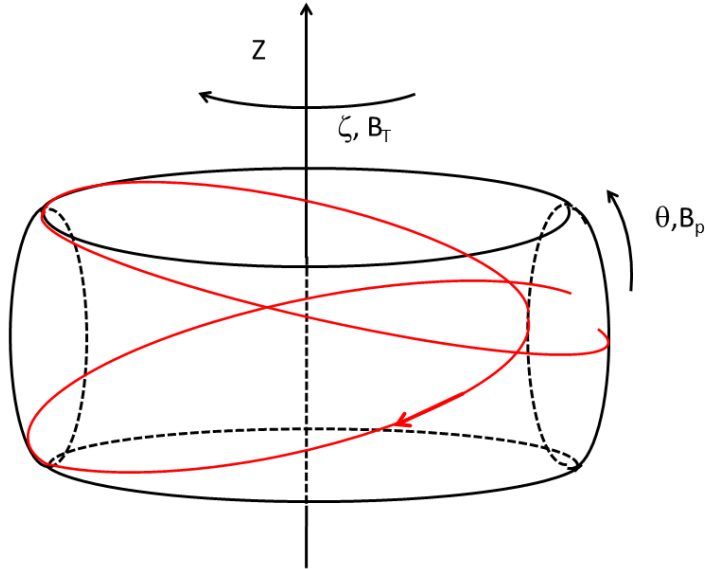


Figure 3: Helical field lines in an axisymmetric configuration.

This is equivalent to a  $2D$  dynamical system where the “Hamiltonian”  $\psi$  is an invariant of motion. In this case field lines (trajectories) lie on surfaces of constant  $\psi$ , and are helices (see Fig.3). These surfaces are magnetic flux surfaces, and are tori in accordance with the hairy ball theorem. The coordinates  $(\rho, \zeta, \theta)$  are called “flux coordinates” as

long as the coordinate  $\rho$  is a function of  $\psi$ , i.e. a label of flux surface. The special case of axisymmetric configurations is detailed in the next section.

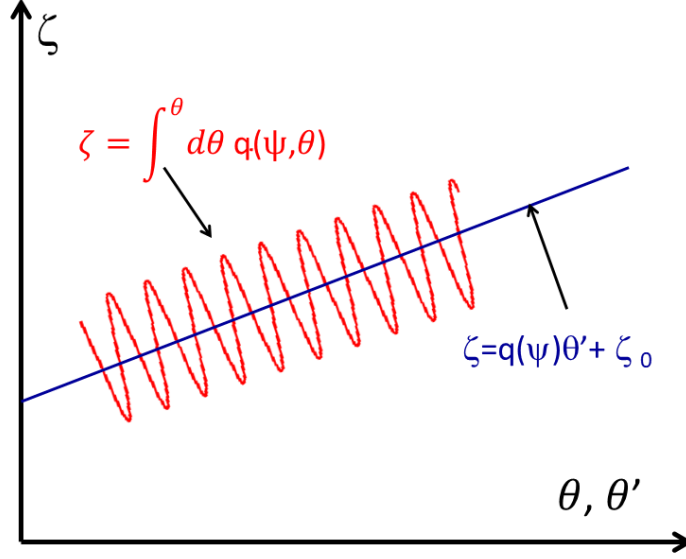


Figure 4: Straight and non straight field line coordinates in an axisymmetric configuration.

## 2.2 Flux coordinates in axisymmetric configurations

The function  $\psi$  is the poloidal magnetic flux normalised to  $-2\pi$ . The radial coordinate  $\rho$  is chosen as  $\rho = \psi$ . The set of angle coordinates  $(\theta, \zeta)$  can be related to any other set of angle variables  $(\theta', \zeta')$  via relations of the form

$$\begin{aligned}\theta' &= \theta + \tilde{\theta}(\psi, \theta, \zeta) \\ \zeta' &= \zeta + \tilde{\zeta}(\psi, \theta, \zeta)\end{aligned}$$

where  $\tilde{\theta}$  and  $\tilde{\zeta}$  are  $2\pi$ -periodic functions in  $(\theta, \zeta)$  with zero mean. In the special axisymmetrical case, the toroidal angle  $\zeta$  can be chosen as minus the toroidal geometric angle  $\varphi$ , and  $\zeta' = \zeta$ . The function  $\tilde{\theta}$  depends on  $(\psi, \theta)$  only. The expression of the magnetic field Eq.(3) imposes that

$$\frac{\mathbf{B} \cdot \nabla \zeta}{\mathbf{B} \cdot \nabla \theta} = \frac{\partial \chi}{\partial \psi}(\psi, \theta) = q_\theta(\psi, \theta)$$

where the function  $q_\theta$  measures the local pitch angle of field lines. The function  $q_\theta$  can be split as

$$q_\theta(\psi, \theta) = q(\psi) + \tilde{q}(\psi, \theta)$$

where

$$q(\psi) = \frac{1}{2\pi} \int_0^{2\pi} d\theta q_\theta(\psi, \theta)$$

and  $\tilde{q}(\psi, \theta)$  is a modulation, i.e. a  $2\pi$ -periodic function with zero mean. Defining the angle  $\theta'(\psi, \theta)$  via the equation

$$\left. \frac{\partial \theta'}{\partial \theta} \right|_\psi = 1 + \frac{\tilde{q}(\psi, \theta)}{q(\psi)}$$

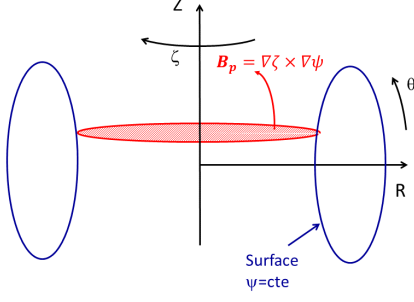


Figure 5: Poloidal magnetic flux in an axisymmetric configuration.

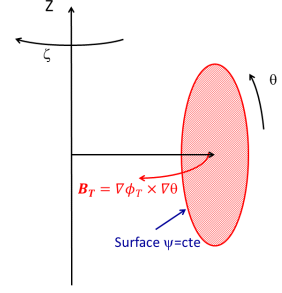


Figure 6: Toroidal magnetic flux in an axisymmetric configuration.

it appears that the new poloidal angle  $\theta'$  is such that

$$\frac{\mathbf{B} \cdot \nabla \zeta}{\mathbf{B} \cdot \nabla \theta'} = q(\psi)$$

Hence the safety factor depends on the poloidal flux only with this choice of angles, so that a field line appears as straight in a plane  $(\zeta, \theta')$  (see Fig.4) - hence the name straight field line coordinates, which are a special choice of flux coordinates. The function  $q(\psi)$  is called safety factor, and its inverse rotational transform (sometimes with a  $2\pi$  prefactor). An immediate consequence is that  $\chi$  depends on  $\psi$  only, and is the toroidal magnetic flux normalised to  $2\pi$  - geometric definitions of the poloidal and toroidal fluxes are shown on Fig.5 and Fig.6. To simplify the notation, this special set of ordinates will still be noted  $(\theta, \zeta)$ , i.e. prime are removed. The expression of the vector potential is particularly agreeable in this set of coordinates

$$\mathbf{A} = \chi(\psi) \nabla \theta - \psi \nabla \zeta$$

up to a gauge function, and the magnetic field is

$$\mathbf{B} = \nabla \chi \times \nabla \theta + \nabla \zeta \times \nabla \psi \quad (6)$$

From now on, the notations  $(\theta, \zeta)$  will correspond to this particular choice of coordinates, called straight field line coordinates because of the constancy of the safety factor on a magnetic flux surface, i.e. the equation of a field line is  $\zeta = q(\psi)\theta + \zeta_0$ , where  $\zeta_0$  is the value of the toroidal angle at  $\theta = 0$ . Usually the poloidal angle is set to zero in the equatorial plane, which fully determines the angle  $\theta$ . It is sometimes convenient to replace  $\theta$  with a coordinate  $\alpha = \zeta - q(\psi)\theta$ <sup>8</sup>. In the system of coordinates  $(\psi, \alpha, \zeta)$ , the equation of a field line is  $\psi = cte$  and  $\alpha = \zeta - q(\psi)\theta = cte$ . In other words, a field line is fully determined by the values of  $\alpha$  and  $\psi$ . Obviously this implies that  $\mathbf{B} \cdot \nabla \psi = 0$ , and  $\mathbf{B} \cdot \nabla \alpha = 0$ . A compact Clebsch formulation of the field is

$$\mathbf{B} = \nabla \alpha \times \nabla \psi$$

### 3 Magnetostatic equilibrium

#### 3.1 Force balance equation

We admit here that flux surfaces exist, labelled by the radial coordinate  $\rho$ . This property was shown for axisymmetric configurations, and it will be seen that it holds for non-axisymmetric configurations under some reasonable assumptions. The full determination

<sup>8</sup>N.B.: the coordinate  $\alpha$  is not an angle unless  $q$  is an integer

of the magnetic field requires the determination of the flux function  $\psi(\rho, \theta, \zeta)$ <sup>9</sup>. The equilibrium field is chosen up-down symmetrical from now on. Some physical considerations other than the Maxwell equations have to be accounted for in order to constrain  $\psi$ . The MHD force balance equation reads [6, 7]

$$\rho_m (\partial_t + \mathbf{V} \cdot \nabla) \mathbf{V} = -\nabla P + \mathbf{J} \times \mathbf{B} + \rho_m \nu \nabla^2 \mathbf{V} \quad (7)$$

where  $\rho_m$  is the mass density,  $\mathbf{V}$  the fluid velocity,  $P$  the pressure,  $\nu$  the kinematic viscosity (supposed constant to simplify the notations). This equation can be understood as a Navier-Stokes equation that incorporates the Lorentz magnetic force  $\mathbf{J} \times \mathbf{B}$ . For a hot plasma immersed in a strong magnetic field, the viscous force is small against the other terms, unless some singularity of the flow appears. If the Mach number is small the inertial force  $(\mathbf{V} \cdot \nabla) \mathbf{V}$  is also small against the pressure gradient. In a steady situation, this means that the plasma equilibrium is ruled by the force balance equation

$$\mathbf{F} = \mathbf{J} \times \mathbf{B} - \nabla P = 0 \quad (8)$$

where  $\mathbf{F}$  is the force exerted on the plasma. The current density  $\mathbf{J}$  is divergence-free  $\nabla \cdot \mathbf{J} = 0$  (a consequence of charge conservation), and is tightly linked to the magnetic field via the Ampère equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad (9)$$

Moreover there is no current normal to the flux surface due to the constraint  $\mathbf{B} \cdot \nabla P = 0$ , which results from the projection of the magnetostatic force balance equation along the magnetic field. Indeed  $\mathbf{B} \cdot \nabla P = 0$  imposes that the pressure is a flux function, i.e. depends on  $\rho$  only. Thus each flux surface is an isobar. Moreover the perpendicular component of the current

$$\mathbf{J}_\perp = \frac{\mathbf{B} \times \nabla P}{B^2} \quad (10)$$

satisfies  $\mathbf{J}_\perp \cdot \nabla \rho = 0$ . Hence the current density  $\mathbf{J}$  is locally tangent to a flux surface. The current  $\mathbf{J}_\perp$ , related to the pressure gradient, is said “diamagnetic” since its poloidal component produces a field that counteracts the toroidal magnetic field, as shown in Fig.7.

### 3.2 Field line bending and curvature

The force exerted on the plasma Eq.(8) can be written in an alternative form using the Ampère equation Eq.(9)

$$\mathbf{F} = -\nabla \left( P + \frac{B^2}{2\mu_0} \right) + \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B}$$

This expression gives some insight in the nature of the electromagnetic forces. The quantity  $\frac{B^2}{2\mu_0}$  acts as the thermal pressure  $P$ , and is accordingly called magnetic pressure - it is also the density of magnetic energy. The force  $(\mathbf{B} \cdot \nabla) \mathbf{B}$  is somewhat more difficult to grasp. It can be written

$$\frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} = \frac{B^2}{\mu_0} \boldsymbol{\kappa} + \left[ (\mathbf{e}_\parallel \cdot \nabla) \left( \frac{B^2}{2\mu_0} \right) \right] \mathbf{e}_\parallel$$

where  $\boldsymbol{\kappa} = (\mathbf{e}_\parallel \nabla \cdot) \mathbf{e}_\parallel$  is the field line curvature. The second term can be regrouped with the gradient of the magnetic pressure, so that the total force reads in fact

$$\mathbf{F} = -\nabla_\perp \left( P + \frac{B^2}{2\mu_0} \right) + \frac{B^2}{\mu_0} \boldsymbol{\kappa}$$

where the perpendicular gradient is defined as  $\nabla_\perp = \nabla - \mathbf{e}_\parallel (\mathbf{e}_\parallel \cdot \nabla)$ , and the property  $(\mathbf{e}_\parallel \cdot \nabla) P = 0$  has been used. Two conclusions can be drawn:

---

<sup>9</sup>It could be  $\chi(\rho, \theta, \zeta)$  as well - it will be seen that the knowledge of one gives the other one - this is already clear for an axisymmetric configuration since  $\chi$  is a function of  $\psi$ .



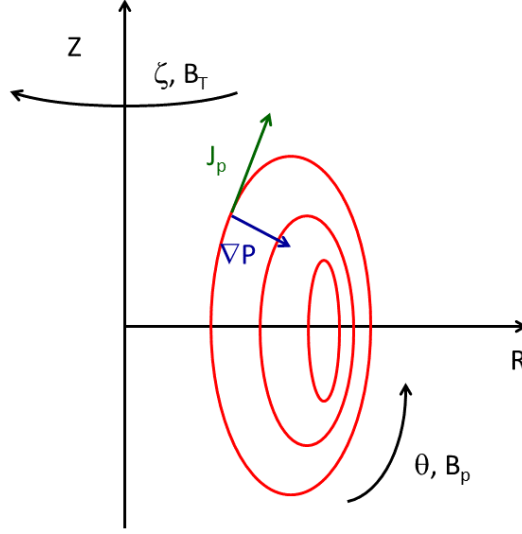


Figure 7: Balance between pressure and Lorentz forces (poloidal projection).

- The steady state condition  $\mathbf{F} = 0$ , gives a useful relationship

$$\kappa = \frac{\mu_0 \nabla_{\perp} P}{B^2} + \frac{\nabla_{\perp} B}{B}$$

that relates the field line curvature with the perpendicular gradient of kinetic and magnetic pressures.

- Let us assume that the plasma is submitted to a small displacement  $\xi$  directed along the normal to the field line, opposite to the curvature  $\kappa$  (see Fig. 8). The projection of the force  $\frac{B^2}{\mu_0} \kappa$  along  $\xi$  is negative, hence opposes to the displacement. This implies that this component of the force *opposes field line bending*.

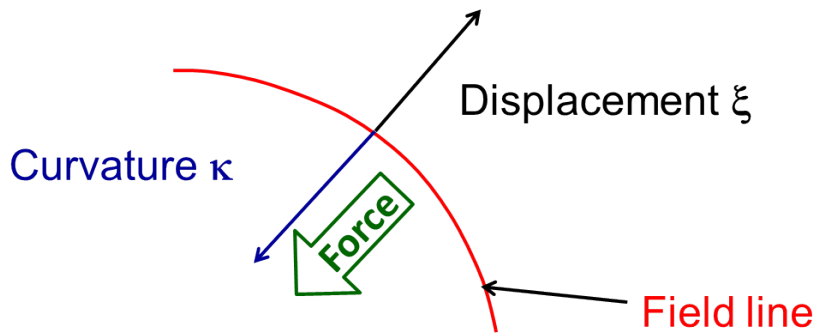


Figure 8: Field line bending and normal component of the Lorentz force.

### 3.3 Tensorial expression of the force balance equation

Since the magnetic field is divergence free, the force can be rewritten in a tensorial form<sup>10</sup>

$$F^k = \frac{\partial T^{ik}}{\partial X^i} = 0 \quad (11)$$

<sup>10</sup>Reminder: the  $X^i$ s are cartesian coordinates

where the symmetrical tensor  $T^{ik}$  is the sum of the Maxwell stress and pressure stress tensors

$$T^{ik} = - \left( P + \frac{B^2}{2\mu_0} \right) \delta^{ik} + \frac{B^i B^k}{\mu_0}$$

The pressure  $P$  has to be replaced by a thermal stress tensor in presence of pressure anisotropies. This expression also provides the matching condition across a surface that separates two parts of the plasma, or the plasma from the vacuum. Indeed the force on the surface that encloses a finite volume is

$$\mathbf{F}_S = - \oint d\mathbf{S} \left( P + \frac{B^2}{2\mu_0} \right) + \oint (d\mathbf{S} \cdot \mathbf{B}) \mathbf{B}$$

Continuity is ensured if  $P + \frac{B^2}{2\mu_0}$  and  $\mathbf{n} \cdot \mathbf{B}$  are continuous across the surface, where  $\mathbf{n}$  is the vector normal to the surface.

## 4 Axisymmetric configurations

Axisymmetry simplifies considerably the determination of the equilibrium magnetic field. As seen in section 2.2, it derives from a vector potential  $\mathbf{A} = \chi \nabla \theta - \psi \nabla \zeta$  where the toroidal flux function  $\chi$  is a function of  $\psi$  only, and  $\theta$  is a straight field line poloidal angle. It also appears that the magnetic field can be related to the poloidal current that flows across a major radius that lies on a flux surface. The force balance equation then leads to a differential equation that rules  $\psi(\mathbf{x})$ , the Grad-Shafranov equation.

### 4.1 Alternative expression of the magnetic field

Let us apply the Gauss flux theorem on the magnetic field

$$\oint_{\mathcal{C}} \mathbf{B} \cdot d\mathbf{x} = \mu_0 \iint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{S}$$

where the contour  $\mathcal{C}$  that bounds the surface  $\mathcal{S}$  is a circle of major radius  $R$  that lies on a flux surface  $\psi = cte$ <sup>11</sup>. The right hand side is the current that flows through the contour  $\mathcal{C}$ , and depends on  $\psi$  only. Moreover the divergence-free condition on the current density imposes that  $\mathbf{J} \cdot d\mathbf{S}$  does not depend on a peculiar choice of the contour  $\mathcal{C}$ , provided it lies on the flux surface. The contour integral  $\oint_{\mathcal{C}} \mathbf{B} \cdot d\mathbf{x}$  is therefore equal to a flux function  $I$  that depends on  $\psi$  only (see Fig. 9). A mixed co- and contravariant expression of the field, frequently found in the literature, is therefore

$$\mathbf{B} = I(\psi) \nabla \zeta + \nabla \zeta \times \nabla \psi \quad (12)$$

The magnetic field is thus the sum of a toroidal component  $\mathbf{B}_T = I(\psi) \nabla \zeta$  and a poloidal field  $\mathbf{B}_p = \nabla \zeta \times \nabla \psi$ . The intensity of the poloidal field is usually smaller than its toroidal counterpart. The total field intensity decreases with major radius since  $|\nabla \zeta| = 1/R$ , hence the denomination “low field side” (by convention the equatorial plane on the low field side is often chosen as the origin of the poloidal angles  $\theta = 0$ ). The current  $I(\psi)$  is related to the Jacobian of the coordinate systems  $(\psi, \theta, \zeta)$  via the relation

$$\sqrt{g} = [(\nabla \zeta \times \nabla \psi) \cdot \nabla \theta]^{-1} = \frac{1}{\mathbf{B} \cdot \nabla \theta} = \frac{qR^2}{I}$$

---

<sup>11</sup>a choice made possible by the assumption of axi-symmetry.

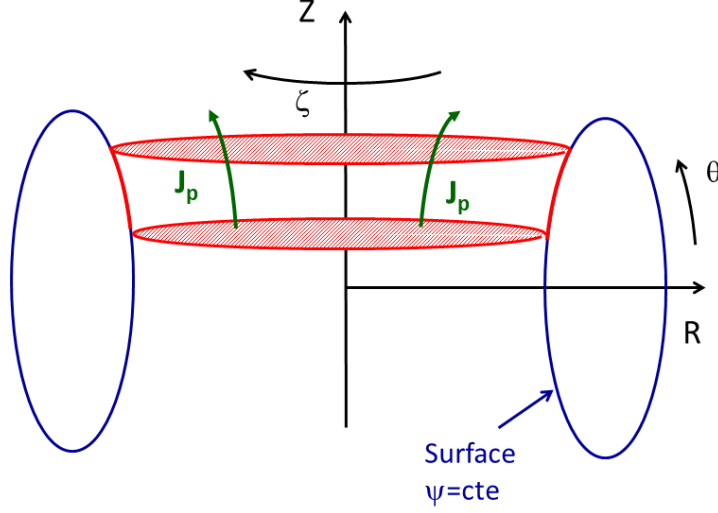


Figure 9: The flux of current across a circle that lies on a flux surface does not depend on the choice of its location. The poloidal current density  $\mathbf{J}_p$  is tangent to the flux surface, so that its flux across the surface vanishes. The current is divergence free, which implies that its flux across one circle has to be equal to its flux across another circle.

## 4.2 Grad-Shafranov equation

The perpendicular diamagnetic current can be recast as

$$\mathbf{J}_\perp = \frac{dP}{d\psi} \frac{\mathbf{B}}{B^2} \times \nabla\psi = \frac{dP}{d\psi} \left( I \frac{\mathbf{B}}{B^2} - R^2 \nabla\zeta \right)$$

so that the current density becomes

$$\mu_0 \mathbf{J} = K \mathbf{B} - \mu_0 \frac{dP}{d\psi} R^2 \nabla\zeta$$

where  $K$  is a function of  $(\psi, \theta)$ . The second part of this expression is divergence-free. Hence the condition  $\nabla \cdot \mathbf{J} = 0$  imposes that  $\mathbf{B} \cdot \nabla K = 0$ , i.e.  $K$  is a function of  $\psi$  only (Fig.10). This function is identified as  $K = -\frac{dI}{d\psi}$  by using the Ampère equation  $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ . Hence the expression

$$\mu_0 \mathbf{J} = -\frac{dI}{d\psi} \mathbf{B} - \mu_0 \frac{dP}{d\psi} R^2 \nabla\zeta \quad (13)$$

This expression of the current calls for some comments. The first part of the current is “paramagnetic” since its poloidal component produces a field in the same direction as the toroidal field. The second part is the toroidal component of the diamagnetic current. Quite clearly, the parallel current density

$$J_\parallel = -\frac{dI}{d\psi} B - \mu_0 \frac{I}{B} \frac{dP}{d\psi}$$

is not constant on a flux surface. The paramagnetic component goes like  $B$  while the diamagnetic piece goes like  $1/B$ . This structure comes from the compressibility of the perpendicular diamagnetic current, which must be balanced by a parallel current, called Pfirsch-Schluter current, which ensures an overall divergence-free current, i.e.

$$\mathbf{B} \cdot \nabla \left( \frac{J_\parallel}{B} \right) + \nabla \cdot \mathbf{J}_\perp = 0$$

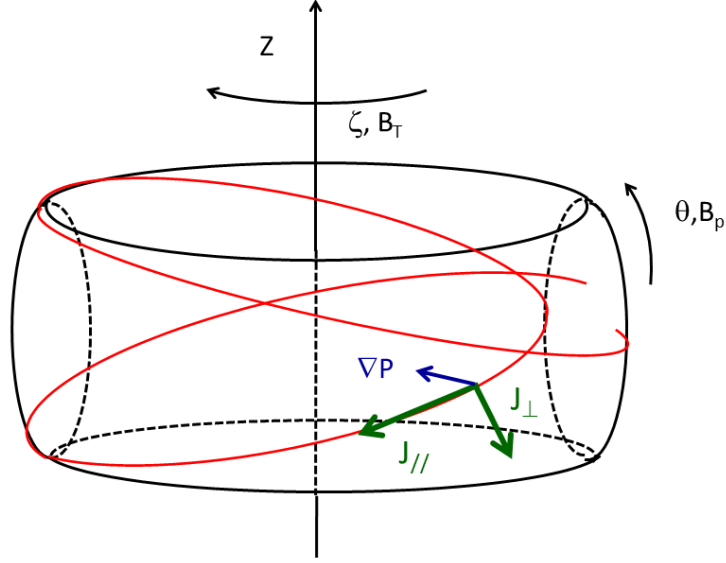


Figure 10: Current density in an axisymmetric configuration.

The projection of Eq.(13) on the toroidal covariant vector  $\mathbf{e}_\zeta = R^2 \nabla \zeta$  yields the toroidal current

$$\mu_0 J_\zeta = \mu_0 R^2 (\mathbf{J} \cdot \nabla \zeta) = -I \frac{dI}{d\psi} - \mu_0 R^2 \frac{dP}{d\psi} \quad (14)$$

Using the identity  $\nabla \zeta \cdot \nabla \times \mathbf{B} = -\nabla \cdot (\nabla \zeta \times \mathbf{B})$ <sup>12</sup>, one finds

$$\nabla_*^2 \psi = R^2 \nabla \cdot \left( \frac{1}{R^2} \nabla \psi \right) = -I \frac{dI}{d\psi} - \mu_0 R^2 \frac{dP}{d\psi} \quad (15)$$

where  $\nabla_*^2 \psi$  is a conventional notation for the modified Laplacian that appears in the l.h.s of this equation. This is the Grad-Shafranov equation, which gives the field  $\psi(\mathbf{x})$  for given pressure  $P(\psi)$  and current  $I(\psi)$  profiles, plus a boundary condition. The boundary condition can be given by the value of  $\psi$  on a flux surface (fixed boundary), or by matching the solution with an external solution consistent with the current that flows in the poloidal coils. If the matching is handled on the separatrix (the last flux surface with closed field lines), and if it is admitted that no current flows in the outer region, it is then convenient to write the field as  $\mathbf{B} = \nabla v$  ( $v$  for “vacuum”), where  $v$  is a scalar field such that  $\nabla^2 v = 0$ . This field is itself matched with the field produced by the coils. Matching conditions on the separatrix are satisfied if  $P + \frac{B^2}{2\mu_0}$  and  $\mathbf{n} \cdot \mathbf{B}$  are continuous across the separatrix, where  $\mathbf{n}$  is the vector normal to the surface.

### 4.3 Grad-Shafranov equation in curvilinear coordinates\*

The Grad-Shafranov equation Eq.(15) is covariant, so that any set of coordinates can be used. The contour lines of  $\psi$  in a poloidal plane give the shape of the flux surfaces. The set of cylindrical coordinates  $(R, \zeta, Z)$  is the easiest one to handle. The effective Laplacian then reads

$$\nabla_*^2 \psi = R \frac{\partial}{\partial R} \left( \frac{1}{R} \frac{\partial \psi}{\partial R} \right) + \frac{\partial^2 \psi}{\partial Z^2}$$

The cylindrical set of coordinates  $(R, \zeta, Z)$  is not always the most suited to a given problem. In some cases the straight field lines set of coordinates  $(\rho, \theta, \zeta)$  is more appropriate.

<sup>12</sup>This comes from the more general identity  $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b}$ .

The subset  $(R, Z)$  must then be replaced by the new set  $(\rho, \theta)$ . The coordinate  $\zeta$  is orthogonal to any set  $(\rho, \theta)$ , and can be treated separately. To keep on handling a *direct* set of coordinates (positive signature), it is convenient to reorder the cylindrical coordinates as  $(R, Z, \zeta)$ , which is direct provided that  $\zeta$  is chosen anti-trigonometric. The subset  $(R, Z)$  behaves as a set of two Cartesian coordinates. A 2D problem is then isolated by determining the metric tensor associated with the change of variables  $(R, Z) \rightarrow (\rho, \theta)$ . The distance element is

$$ds^2 = dR^2 + dZ^2 + R^2 d\zeta^2 = g_{ik} d\xi^i d\xi^k$$

The elements of the reduced metric 2D tensor

$$g_{ik} = \begin{pmatrix} g_{\rho\rho} & g_{\rho\theta} \\ g_{\rho\theta} & g_{\theta\theta} \end{pmatrix}$$

can be calculated by using the expressions of  $R(\rho, \theta)$  and  $Z(\rho, \theta)$  and calculating their partial derivatives. The other coefficients satisfy  $g_{\zeta\varphi} = R^2$ , and  $g_{\zeta\rho} = g_{\zeta\theta} = 0$ . The metric tensor is easily inverted by isolating the (2,2) submatrix to give

$$g^{ik} = \frac{1}{J} \begin{pmatrix} g_{\theta\theta} & -g_{\rho\theta} \\ -g_{\rho\theta} & g_{\rho\rho} \end{pmatrix}$$

where  $J = g_{\rho\rho}g_{\theta\theta} - g_{\rho\theta}^2$  is the determinant of the submatrix. Moreover  $g^{\zeta\zeta} = 1/R^2$ , and  $g^{\zeta\rho} = g^{\zeta\theta} = 0$ . The Jacobian is  $\sqrt{g} = J^{1/2}R$ . Using the covariant expression of divergence and gradient operators (see Appendix A), the Grad-Shafranov equation can then be written as

$$\frac{R^2}{\sqrt{g}} \frac{d\psi}{d\rho} \left[ \frac{\partial}{\partial\rho} \left( \frac{g_{\theta\theta}}{\sqrt{g}} \frac{d\psi}{d\rho} \right) - \frac{\partial}{\partial\theta} \left( \frac{g_{\theta\rho}}{\sqrt{g}} \frac{d\psi}{d\rho} \right) \right] = -I \frac{dI}{d\rho} - R^2 \mu_0 \frac{dP}{d\rho}$$

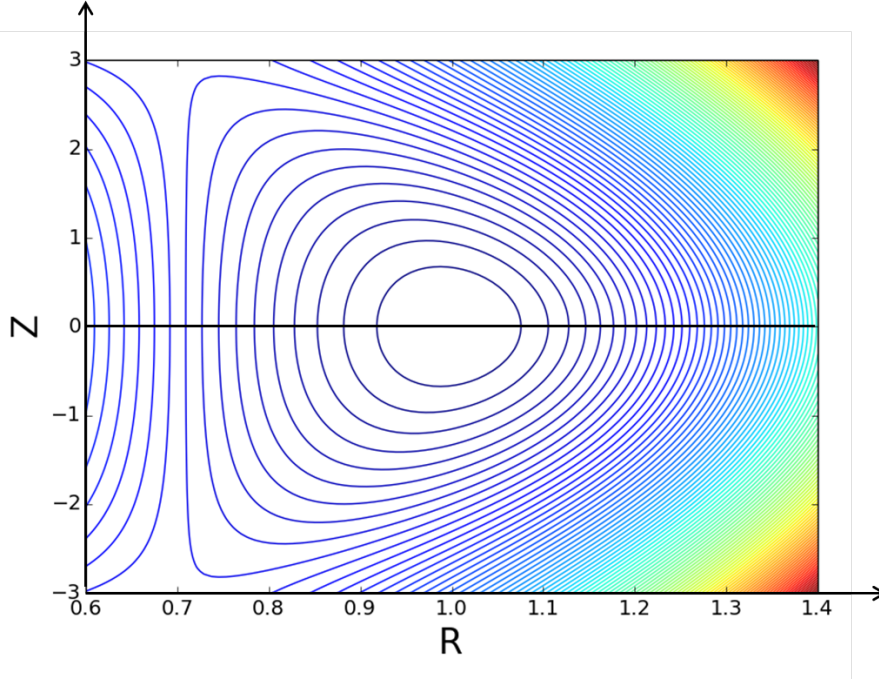


Figure 11: Shape of flux surfaces for a Solov'ev equilibrium. Parameters are  $\alpha = 6$ ,  $\gamma = 0.5$ ,  $R_0 = 1$ .

## 4.4 Solving the Grad-Shafranov equation

The Grad-Shafranov equation is usually solved numerically. Analytical solutions are found in some special cases that deserve being commented. Analytical solutions are useful for many applications, like trajectories computation, stability calculations or transport solvers. Moreover they are mandatory to verify equilibrium numerical solutions. Two main lines can be identified. A first strategy consists in *solving the Grad-Shafranov in Cartesian coordinates*  $(R, Z)$  for a class of pressure and current profiles. This provides the poloidal flux  $\psi(R, Z)$ , and therefore the field. Poloidal cross-sections of the flux surfaces are the contour lines of  $\psi(R, Z)$ . Therefore the shape of the flux surfaces is the outcome of the calculation. The second approach consists in *imposing the geometry of flux surfaces* via a change of coordinates  $(R, Z) \rightarrow (\rho, \theta)$ , where  $\rho$  is a flux label. This change of coordinates is parametrized by a set of flux functions that prescribe the shape. The Grad-Shafranov equation is then solved in curvilinear coordinates by using standard procedures of differential geometry. Solving the Grad-Shafranov equation fixes the shaping functions. In that case, geometry imposes the magnetic field.

Solov'ev equilibria [8, 9] are a first instance of exact global solutions of the Grad-Shafranov equation. If the functions  $I \frac{dI}{d\psi}$  and  $\frac{dP}{d\psi}$  are chosen as suitable functions of  $R$  and  $Z$ , it is possible to construct exact solutions  $\psi(R, Z)$  of Eq.(15). A simple example is given in Appendix B and in Fig.11. The main advantage of the family of Solov'ev equilibria is that it allows a large variety of shapes, and more importantly null point of the poloidal field (X-point). Hence it is possible to calculate equilibria close to those encountered in modern tokamaks.

Another important class of solutions are Miller equilibria [10], after ideas developed by Mercier and Luc [11]. In an earlier work Mercier studied the topology of the magnetic field in non axisymmetric configurations near the magnetic axis, which has deep implications on the various paths to design a stellarator [12]. These are *local* solutions of the Grad-Shafranov equation, i.e. valid in the vicinity of a reference flux surface. Their main advantage is that the pressure gradient and the safety factor can be changed without changing the shape of the reference flux surface. This is quite useful for parametric studies of the effect of the geometry at fixed gradients. This is why Miller equilibria are widely used in turbulence local codes. The basic principles are described in Appendix C. Solutions can be found for D-shape flux surfaces [10]

$$\begin{aligned} R &= R_0 + \Delta(\rho) + \rho \cos \{ \vartheta + [\sin^{-1} \delta(\rho)] \sin \vartheta \} \\ Z &= \rho \kappa(\rho) \sin \vartheta \end{aligned} \quad (16)$$

where the functions  $\kappa(\rho)$  and  $\delta(\rho)$  describe the ellipticity and triangularity of the flux surfaces, and  $\Delta(\rho)$  is a Shafranov shift (Fig.12). This description allows highly shaped flux surfaces. Explicit calculations are available when  $(R, Z) \rightarrow (\rho, \vartheta)$  are written as Fourier series in  $\vartheta$  [13]. The drawback of Miller equilibria is also their advantage: they are not global solutions of the Grad-Shafranov equation.

Finally another way to define an equilibrium is to define the correspondence between the set of coordinates  $(R, Z)$  and the set  $(\rho, \vartheta)$  as an expansion in a small parameter  $\epsilon$ , where  $\epsilon$  is an inverse aspect ration  $a/R_0$  ( $a$  the plasma size). The normalised pressure  $\frac{\mu_0 P}{B^2}$  scales as  $\epsilon^2$  and the current flux  $I$  as  $B_0 R_0 (1 + o(\epsilon^2))$ , where  $B_0$  is a reference field, and  $R_0$  a reference major radius. A convenient expansion, sometimes called the ‘‘Culham’’ equilibrium [14], is

$$\begin{aligned} R &= R_0 + \epsilon \rho \cos \vartheta + \epsilon^2 \Delta(\rho) + \sum_m \epsilon^m S_m(\rho) \cos(m\vartheta) + \epsilon^3 L(\rho) \cos \vartheta \\ Z &= \epsilon \rho \sin \vartheta + \sum_{m \geq 1} \epsilon^m S_m(\rho) \sin(m\vartheta) - \epsilon^3 L(\rho) \sin \vartheta \end{aligned}$$

where  $L(\rho)$  is an adjustment needed to define a proper safety factor, and hence to build a system of straight field line coordinates  $(\theta, \varphi)$ . The function  $\Delta(\rho)$  describes the radial

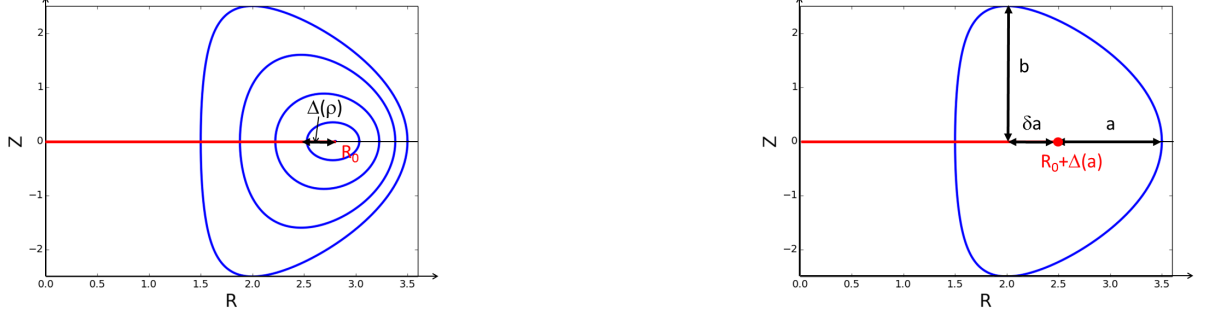


Figure 12: D-shape flux surfaces with Shafranov shift, elongation and triangularity as described by Eq.(16). Left panel: effect of the Shafranov shift  $\Delta(\rho)$ . Right panel: shaping parameters for a single flux surface with minor radius  $a$  -  $\kappa = b/a$  is the elongation,  $\delta$  is the triangularity. Parameters are  $R_0/a = 2.8$ ,  $\Delta(a)/a = 0.3$ ,  $a = 1m$ ,  $\kappa = b/a = 2.5$ ,  $\delta = 0.5$ .

shift of flux surfaces, and is called “Shafranov shift” [15]. The other functions  $S_m(\rho)$  are used to define the geometry of the flux surfaces. The simplest case  $S_m = 0$  corresponds to shifted circular flux surfaces [15] and is treated in Appendix D - see Fig.13. In most cases, the analysis is restricted to  $m = 1$  and  $m = 2$ . The function  $E(\rho) = S_1(\rho)$  describes the ellipticity of the flux surfaces, while  $T(\rho) = S_2(\rho)$  is related to their triangularity. The Grad-Shafranov equation can be solved order by order in  $\epsilon$ . One important property is that  $S_m(\rho)$  is found to vary as  $\rho^m$ . Hence the triangularity and higher order shaping functions fade away when getting close to the magnetic axis. One drawback of this expansion is that it does not allow defining a separatrix with a null point of the poloidal field due to the ordering. On the other hand, the advantage of this expansion is that it provides a global solution of the Grad-Shafranov equation. Hence it has proved quite useful for MHD stability [14] calculation and transport solvers [16]. A set of circular concentric flux surfaces  $\Delta = 0, S_m = 0$  is not a solution of the Grad-Shafranov solution. Nevertheless it is often used as a proxy that allows tractable analytic calculations.

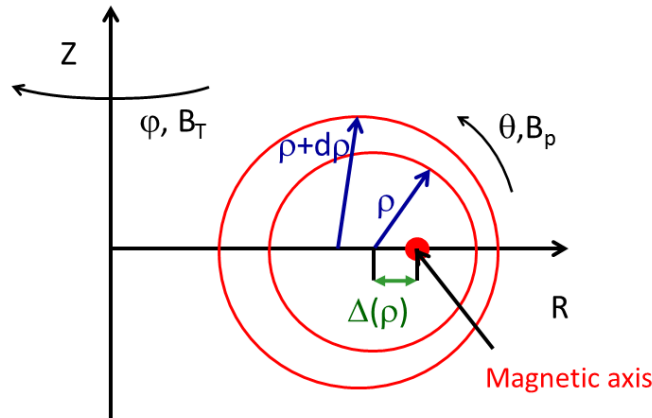


Figure 13: Shifted circular flux surfaces.

## 5 Non-axisymmetric configurations

The existence of flux surfaces is simpler to demonstrate when the configuration is axisymmetric. Axisymmetry also grants a compact form of the magnetic field, leading to a tractable form of the force balance equation. It appears that some of these results still hold in non-axisymmetric configurations under some reasonable assumptions related to the regularity of the pressure field.

### 5.1 Representation of the magnetic field

The equations of field lines Eq.(4,5) exhibit an Hamiltonian structure, where the poloidal flux plays the role of an Hamiltonian, and the toroidal angle replaces the time. In the case of an axisymmetric configuration, the Hamiltonian is obviously an invariant of motion, which suffices to demonstrate the existence of magnetic field lines. The argument is not valid for a non axisymmetric configuration. However an alternative argument, proposed by Boozer [2], and based on the force balance equation, can be used to infer the existence of flux surfaces. If the magnetostatic force balance equation is fulfilled, i.e.  $\mathbf{J} \times \mathbf{B} = \nabla P$ , where  $\mathbf{J}$  is the current density, then the pressure  $P$  is a dynamical invariant since  $(\mathbf{B} \cdot \nabla)P = 0$ . Assuming that the pressure is a smooth function  $P(\mathbf{x})$ , and that isobar surfaces are bounded, this grants the existence of invariant (isobar) tori by virtue of the ‘‘hairy ball’’ theorem, already mentioned<sup>13</sup>. This can be understood as follows. Let us start from the Clebsch expression of the magnetic field Eq.(3). Pressure is chosen as the radial coordinate, i.e.  $\rho = P$ , a label of flux surface since  $(\mathbf{B} \cdot \nabla)P = 0$ . The flux functions  $\chi$  and  $\psi$  are functions of the set of flux coordinates  $(\rho, \vartheta, \zeta)$ . The Clebsch representation Eq.(3) combined with the condition  $(\mathbf{B} \cdot \nabla)\rho = 0$  implies

$$\mathbf{B} = \frac{\partial \chi}{\partial \rho} \nabla \rho \times \nabla \vartheta + \frac{\partial \psi}{\partial \rho} \nabla \zeta \times \nabla \rho$$

and the constraint

$$\frac{\partial \chi}{\partial \zeta} + \frac{\partial \psi}{\partial \vartheta} = 0 \quad (17)$$

The functions  $\chi$  and  $\psi$  can be decomposed as a mean over the angles  $\vartheta$  and  $\zeta$  and a modulation

$$\begin{aligned} \chi(\rho, \vartheta, \zeta) &= \bar{\chi}(\rho) + \tilde{\chi}(\rho, \vartheta, \zeta) \\ \psi(\rho, \vartheta, \zeta) &= \bar{\psi}(\rho) + \tilde{\psi}(\rho, \vartheta, \zeta) \end{aligned}$$

The constraint Eq.(17) can be incorporated in the expressions of fluxes as

$$\begin{aligned} \frac{\partial \chi}{\partial \rho}(\rho, \vartheta, \zeta) &= \frac{d\bar{\chi}}{d\rho}(\rho) + \frac{\partial \lambda}{\partial \vartheta}(\rho, \vartheta, \zeta) \\ \frac{\partial \psi}{\partial \rho}(\rho, \vartheta, \zeta) &= \frac{d\bar{\psi}}{d\rho}(\rho) - \frac{\partial \lambda}{\partial \zeta}(\rho, \vartheta, \zeta) \end{aligned}$$

where  $\lambda$  is a periodic function of  $\vartheta$  and  $\zeta$  with zero mean. The Clebsch representation of the magnetic field Eq.(17) then becomes  $\mathbf{B} = \nabla \bar{\chi} \times \nabla \theta + \nabla \zeta \times \nabla \bar{\psi}$ , where

$$\theta = \vartheta + \frac{1}{\frac{d\bar{\chi}}{d\rho}} \lambda(\rho, \vartheta, \zeta)$$

is a straight field line poloidal angle. The bars on the new toroidal and poloidal fluxes  $\bar{\chi}$  and  $\bar{\psi}$  are now removed to simplify the notations, so that the final structure of the magnetic field is

$$\mathbf{B} = \nabla \chi \times \nabla \theta + \nabla \zeta \times \nabla \psi \quad (18)$$

---

<sup>13</sup>If field lines are spatially bounded, then the invariant manifold associated with the field lines Hamiltonian dynamics Eq.(4,5) is compact. The ‘‘hairy ball’’ or ‘‘hedgehog’’ theorem can then be applied, i.e. a compact manifold in a 3D phase space has to be a 2-dimensional torus.



where  $\chi$  and  $\psi$  are flux functions that depend on  $\rho$  only. Defining the safety factor  $q(\rho)$  (inverse of the rotational transform) as  $q = \frac{d\chi}{d\psi}$ , it appears that the Clebsch structure

$$\mathbf{B} = \nabla(\zeta - q\theta) \times \nabla\psi$$

that was found for axisymmetric configurations also holds for a non axisymmetric field.

The derivation above calls for some comments. It relies on the condition  $(\mathbf{B} \cdot \nabla)P = 0$ , and the implicit assumption that the pressure field exhibits some nice properties of continuity and smoothness (differentiability). Besides isobar surfaces should be bounded. Let us choose a set of straight field line coordinates  $(\rho, \theta, \zeta)$ . The pressure can be decomposed as a Fourier series

$$P = \sum_{mn} P_{mn} e^{i(m\theta + n\zeta)}$$

It is easy to find the expression of the pressure gradient along the field  $(\mathbf{B} \cdot \nabla)P$  by using the expression of the field Eq.(18) and a set of flux coordinates  $(\rho, \theta, \zeta)$

$$(\mathbf{B} \cdot \nabla)P = i(\mathbf{B} \cdot \nabla\zeta) \sum_{m \neq 0, n \neq 0} \left( n + \frac{m}{q(\rho)} \right) P_{mn} e^{i(m\theta + n\zeta)}$$

The condition  $(\mathbf{B} \cdot \nabla)P = 0$  imposes that the pressure  $P$  is a function of  $\rho$ , i.e. a flux function as expected, *up to* an infinite sum of Kronecker delta functions centred on  $n + m/q(\rho)$ . The flux surfaces which satisfy  $n + m/q(\rho) = 0$  are called resonant surfaces<sup>14</sup>. One may expect that any singularity  $\delta(n + m/q(\rho))$  will be regularised after some time by dissipative processes (e.g. viscous dissipation that was neglected in the force balance equation), thus imposing after a relaxation time that the pressure is a flux function. However any force  $\mathbf{F}$ , even small, that will appear in the force balance equation (e.g. thermal noise) will induce singularities in the pressure field via the equation  $(\mathbf{B} \cdot \nabla)P = \mathbf{B} \cdot \mathbf{F}$ . The usual argument that is put forward is that dissipative processes, and/or instabilities localised on resonant surfaces, will wipe out these singularities over a fast time scale. In other words, the plasma should self-organize<sup>15</sup>. We will assume in this section that this is indeed the case.

## 5.2 Structure of the current density\*

As noticed by Boozer [2], the current density behaves with respect to the magnetic field in the same way as the magnetic field with respect to the vector potential. Indeed it is divergence-free  $\nabla \cdot \mathbf{J} = 0$  and is the curl of the magnetic field  $\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$  (Ampère equation). Moreover the force balance equation imposes that the current lies on flux surfaces  $\mathbf{J} \cdot \nabla\rho = 0$  (see section 3.1). Hence the current density can be written in a Clebsch form similar to the magnetic field<sup>16</sup>

$$\mu_0 \mathbf{J} = \nabla \mathcal{G} \times \nabla \theta + \nabla \mathcal{I} \times \nabla \zeta$$

where  $\mathcal{G}$  and  $\mathcal{I}$  are functions of the set of straight field line coordinates  $(\rho, \theta, \zeta)$ . Following closely the methodology described in the previous section, the constraint  $\mathbf{J} \cdot \nabla\rho = 0$  is used to find an equivalent form of the current density

$$\mu_0 \mathbf{J} = \frac{\partial \mathcal{G}}{\partial \rho} \nabla \rho \times \nabla \theta + \frac{\partial \mathcal{I}}{\partial \rho} \nabla \rho \times \nabla \zeta$$

<sup>14</sup>A flux surface does not necessarily exists in the plasma since it requires that the safety factor is equal to a rational number  $-m/n$  that may be out of the range of accessible values of the function  $q(\rho)$ .

<sup>15</sup>Other ways exist that grant the existence of flux surfaces. For instance stellarators are known to exhibit flux surfaces even without plasma. These surfaces are probed with electron beams.

<sup>16</sup>The notations  $\mathcal{G}$  and  $\mathcal{I}$  are inverted compared with Boozer's conventions.

together with the constraint

$$\frac{\partial \mathcal{G}}{\partial \zeta} - \frac{\partial \mathcal{I}}{\partial \theta} = 0$$

The current flux functions  $\mathcal{G}$  and  $\mathcal{I}$  can be decomposed as sums of means  $G(\rho)$  and  $I(\rho)$  and modulations, constrained by Eq.(19). This can be formulated in a convenient form

$$\begin{aligned} \frac{\partial \mathcal{G}}{\partial \rho}(\rho, \theta, \zeta) &= \frac{dG(\rho)}{d\rho} + \frac{\partial \nu}{\partial \vartheta}(\rho, \theta, \zeta) \\ \frac{\partial \mathcal{I}}{\partial \rho}(\rho, \theta, \zeta) &= \frac{dI(\rho)}{d\rho} + \frac{\partial \nu}{\partial \zeta}(\rho, \theta, \zeta) \end{aligned}$$

where  $\nu(\rho, \theta, \zeta)$  is a periodic function in  $\theta$  and  $\zeta$  with zero mean. The current density becomes

$$\mu_0 \mathbf{J} = \nabla G \times \nabla \theta + \nabla I \times \nabla \zeta - \nabla \nu \times \nabla \rho$$

This relation provides the *covariant expression* of the magnetic field

$$\mathbf{B} = G(\rho) \nabla \theta + I(\rho) \nabla \zeta - \nu(\rho, \theta, \zeta) \nabla \rho + \nabla H(\rho, \theta, \zeta) \quad (19)$$

where  $H$  is a gauge function. Of course one would like to manipulate 3 flux functions and not 4. This is done by changing the system of coordinates, i.e. by changing  $(\theta, \zeta)$  in another set  $(\theta', \zeta')$ . As noticed by Boozer, the magnetic field is invariant by any change of the form

$$\begin{aligned} \theta &= \theta' + \iota(\rho) \tilde{\zeta}(\rho, \theta', \zeta') \\ \zeta &= \zeta' + \tilde{\zeta}(\rho, \theta', \zeta') \end{aligned}$$

where  $\iota(\rho) = 1/q(\rho)$ . Since this change of coordinates leaves invariant the combination  $\zeta' - q\theta' = \zeta - q\theta$ . The structure of the field Eq.(19) remains the same with  $(\theta, \zeta)$  replaced by  $(\theta', \zeta')$ , but with new functions  $\nu'$  and  $G'_B$  such that

$$\begin{aligned} \nu' &= \nu + \left( \iota \frac{dG}{d\rho} + \frac{dI}{d\rho} \right) \tilde{\zeta} \\ H' &= H + (\iota G + I) \tilde{\zeta} \end{aligned}$$

Hence  $\nu'$  or  $H'$  can be set to zero via a suitable choice of  $\tilde{\zeta}$ . It must be stressed that even if the toroidal angle  $\zeta$  would be chosen as the geometric angle in the first place, the new toroidal angle is usually not a geometric angle any more. The choice  $\nu' = 0$  corresponds to Hamada coordinates, while  $H' = 0$  determines the Boozer coordinates. Both have their interest for instance in the context of the Hamiltonian formulation of trajectory equations (Boozer coordinates) or neoclassical transport (Hamada coordinates). It appears that the Grad-Shafranov equation can be written in an elegant form when using Boozer/Hamada coordinates. This form is particular compact when choosing Hamada coordinates - moreover the Jacobian then appears to be a flux function. More details can be found in Appendix E or in references [1, 2, 17].

### 5.3 Force exerted on the plasma\*

The force balance equation Eq.(7) reads  $\rho_m (\partial_t + \mathbf{V} \cdot \nabla) \mathbf{V} = \mathbf{F}$ , where  $\mathbf{F} = -\nabla P + \mathbf{J} \times \mathbf{B}$  is the force exerted on the plasma when the viscous force is neglected. The force has no component along the field lines, i.e.  $\mathbf{B} \cdot \mathbf{F} = 0$ . Thus it can be decomposed as [18]

$$\mathbf{F} = F_\rho \nabla \rho + F_\perp \frac{\mathbf{B}}{B^2} \times \nabla \rho$$

Both the magnetic field and the current density can be expressed in a covariant/contravariant form  $\mathbf{J} = J_i \mathbf{e}^i = J^i \mathbf{e}_i$ ,  $\mathbf{B} = B_i \mathbf{e}^i = B^i \mathbf{e}_i$ , with  $B^\rho = \mathbf{B} \cdot \nabla \rho = 0$  since the magnetic field is tangent to a flux surface. The force covariant components then read

$$\begin{aligned} F_\rho &= \sqrt{g} \left( J^\theta B^\zeta - J^\zeta B^\theta \right) - \frac{dP}{d\rho} \\ F_\lambda &= -J^\rho \end{aligned}$$

The field structure Eq.(18) imposes that

$$\begin{aligned} B^\theta &= \frac{1}{\sqrt{g}} \frac{d\psi}{d\rho} \\ B^\zeta &= \frac{1}{\sqrt{g}} \frac{d\chi}{d\rho} \end{aligned}$$

The contravariant components of the current density are computed from the Ampère's equation, using the expression of a curl operator in curvilinear coordinates Eq.(20). After a bit of algebra, the force components can be written in the following form

$$\begin{aligned} F_\rho &= \frac{1}{\mu_0} \left( (\mathbf{B} \cdot \nabla) B_\rho - B^\theta \frac{\partial B_\theta}{\partial \rho} - B^\zeta \frac{\partial B_\zeta}{\partial \rho} \right) - \frac{dP}{d\rho} \\ F_\lambda &= -\frac{1}{\mu_0 \sqrt{g}} \left( \frac{\partial B_\theta}{\partial \zeta} - \frac{\partial B_\zeta}{\partial \theta} \right) \end{aligned}$$

It is reminded that the covariant components of the field are related to the contravariant components via the relation

$$B_i = g_{i\theta} B^\theta + g_{i\zeta} B^\zeta$$

Since the force must vanish,  $F_\lambda = -J^\rho = -\mathbf{J} \cdot \nabla \rho = 0$ , which imposes that the current is tangent to flux surfaces, a result that was already obtained in the special axisymmetric case. At this point, it is useful to express force  $F_\rho$  component in the special axisymmetric case. The condition  $F_\lambda = 0$  imposes that  $B_\zeta = I(\rho)$  depends on  $\rho$  only, a result that was also already found. The component  $F_\rho$  is found to be

$$\begin{aligned} F_\rho &= -\frac{1}{\mu_0 \sqrt{g}} \frac{d\psi}{d\rho} \left[ \frac{\partial}{\partial \rho} \left( \frac{g_{\theta\theta}}{\sqrt{g}} \frac{d\psi}{d\rho} \right) - \frac{\partial}{\partial \theta} \left( \frac{g_{\theta\rho}}{\sqrt{g}} \frac{d\psi}{d\rho} \right) \right] \\ &\quad - \frac{1}{\mu_0 R^2} I \frac{dI}{d\rho} - \frac{dP}{d\rho} \end{aligned}$$

The condition  $F_\rho = 0$  appears to be equivalent to the Grad-Shafranov equation Eq.(15), using the covariant expression of the divergence of a vector  $\nabla \cdot \mathbf{V}$  (see Appendix A), with  $V_\rho = \frac{1}{R^2} \frac{d\psi}{d\rho}$ ,  $V_\theta = V_\zeta = 0$  and  $g^{\rho\rho} = R^2 \frac{g_{\theta\theta}}{g}$ ,  $g^{\theta\rho} = -R^2 \frac{g_{\theta\rho}}{g}$ . Hence the condition of force balance is equivalent to the Grad-Shafranov equation. It provides a powerful method to calculate the equilibrium of a magnetic configuration that is not axisymmetric [18, 19].

## 6 Variational principle and virial theorem

### 6.1 Energy principle and related variational methods

The Grad-Shafranov equation is not easy to solve, and its analogue in non-axisymmetric configurations is even more challenging. This is why it is sometimes replaced by a variational principle, which allows using a projection on a basis of functions and a suitable truncature. One common form is based on the plasma energy

$$\mathcal{W} = \int d^3\mathbf{x} \left( \frac{B^2}{2\mu_0} + \frac{P}{s-1} \right)$$

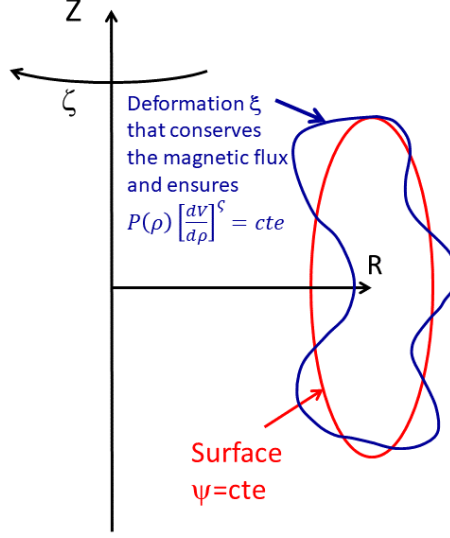


Figure 14: Deformation of a flux surface that conserves the toroidal magnetic flux while keeping  $P(\rho) \left( \frac{dV}{d\rho} \right)^\varsigma$  constant.

where  $\varsigma$  is the adiabatic index<sup>17</sup>. This definition of the plasma energy is not the most general that could be thought of. It does not include the energy density associated with the electric field, and reduces the energy density of the plasma to its thermal energy  $\frac{P}{\varsigma-1}$ , in principle valid when the plasma can be described as a perfect gas. Let us assume that flux surfaces exist, and labelled by a flux coordinate  $\rho$ . The position on a flux surface is a vector  $\mathbf{x}(\rho, \theta, \zeta)$ , where  $(\theta, \zeta)$  are poloidal and toroidal angles. The force balance equation is equivalent to minimizing  $\mathcal{W}$  with respect to any virtual displacement of the magnetic field and pressure that is consistent with the equations of ideal MHD, i.e. that preserves the magnetic fluxes, and under the constraint of an adiabatic transform that keeps  $P(\rho) \left( \frac{dV}{d\rho} \right)^\varsigma$  constant (see Fig. 14), where

$$\frac{dV}{d\rho} = \iint_{\rho} \frac{d\theta d\zeta}{\mathcal{J}}$$

is the differential volume element ( $dV$  is the volume element between the flux surfaces  $\rho$  and  $\rho + d\rho$ ). The transform must be done for a given plasma surface. In other words, if  $\boldsymbol{\xi}$  labels the displacement of flux surfaces, which therefore become  $\mathbf{x}(\rho, \theta, \varphi) + \boldsymbol{\xi}$  after a transformation as described above, the variation of energy is

$$\delta\mathcal{W} = \int d^3\mathbf{x} (\nabla P - \mathbf{J} \times \mathbf{B}) \cdot \boldsymbol{\xi}$$

The demonstration is given in Appendix F. If  $\delta\mathcal{W}$  exhibits an extremum with respect to the displacement  $\boldsymbol{\xi}$  then  $\mathbf{J} \times \mathbf{B} = \nabla P$ . Quite interestingly, this property does not depend on the precise value of the adiabatic index  $\varsigma$ . Hence a “Lagrangian” version of this principle can be devised by choosing  $\varsigma = 0$

$$\mathcal{L} = \int d^3\mathbf{x} \left( \frac{B^2}{2\mu_0} - P \right)$$

<sup>17</sup>The traditional notation  $\gamma$  is avoided here as it is also the usual notation for an instability growth rate.

Variations must then be done at constant pressure  $P(\rho)$  when changing the shape of the flux surfaces. This variational approach has proved efficient to compute equilibria in stellarators [18, 19], and in particular to design optimised non-axisymmetric configurations. The above theorem applies when the last closed flux surface is prescribed. The energy principle can be extended to match with a vacuum field  $\mathbf{B}_v$  by adding  $-\int d^3\mathbf{x} \frac{B_v^2}{2\mu_0}$  to  $\mathcal{W}$ .

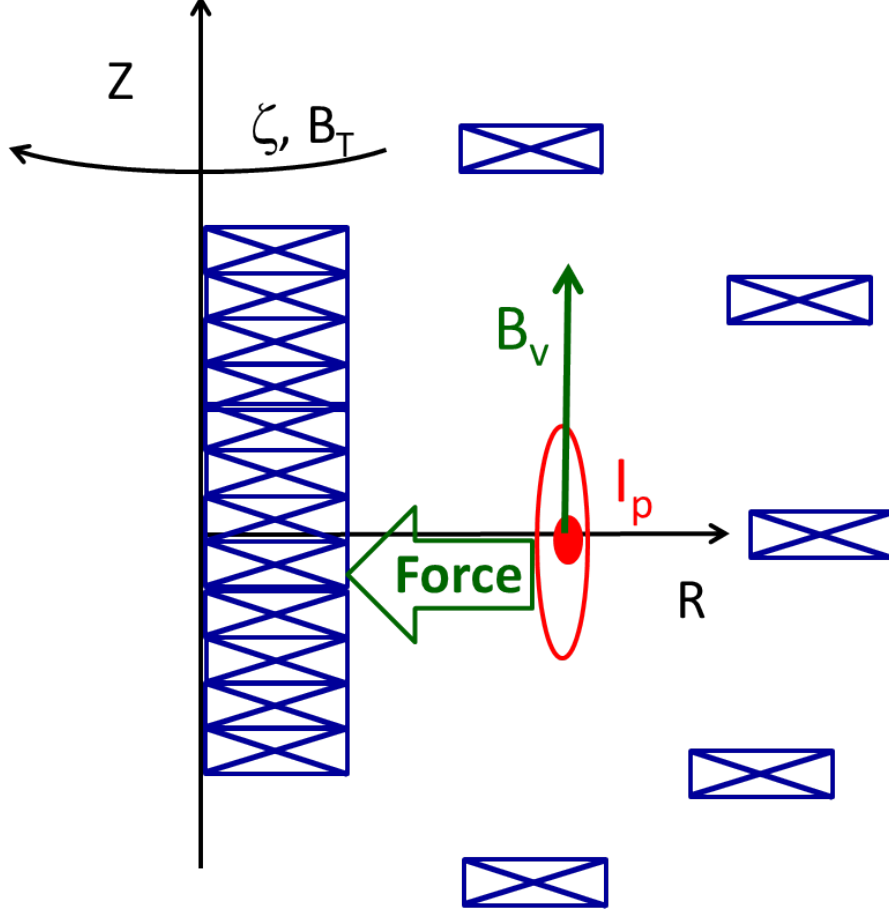


Figure 15: A vertical field combined with the plasma current balances the radial force due to thermal and magnetic forces.

## 6.2 Virial theorem

The virial theorem consists in multiplying the tensorial form of the force balance equation Eq.(11) by a displacement vector  $\boldsymbol{\xi}$  and integrate over a suitable volume  $\mathcal{V}$ . This is the usual procedure to build the weak variational form associated with a differential equation. The resulting functional exhibits an extremum when varying the displacement  $\boldsymbol{\xi}$ . This extremum provides a solution of the force balance equation. It provides a convenient way to resolve numerically the equilibrium magnetic configuration by expanding  $\boldsymbol{\xi}$  on a suitable set of basis functions<sup>18</sup>. The original version of the virial theorem corresponds to a displacement  $\boldsymbol{\xi} = X_i \hat{\mathbf{e}}^i = X^i \hat{\mathbf{e}}_i$ , and  $\hat{\mathbf{e}}^i = \hat{\mathbf{e}}_i$  are the Cartesian unit vectors. Using the relation

$$\frac{\partial}{\partial X^i} (X_k T^{ik}) = T^{kk} + X_k \frac{\partial}{\partial X^i} T^{ik}$$

<sup>18</sup>In this case the solution is an approximate one due to the truncature. The variational aspect grants it is optimal for a given set of basis functions.

and noting that the last term vanishes, an integration over the volume  $\mathcal{V}$  yields

$$\int_{\mathcal{V}} d^3\mathbf{x} \left( 3P + \frac{B^2}{2\mu_0} \right) = \int_{\partial\mathcal{V}} (d\mathbf{S} \cdot \boldsymbol{\xi}) \left( P + \frac{B^2}{2\mu_0} \right) - \frac{1}{\mu_0} \int_{\partial\mathcal{V}} (d\mathbf{S} \cdot \mathbf{B}) (\boldsymbol{\xi} \cdot \mathbf{B})$$

where  $\partial\mathcal{V}$  is the surface that encloses the volume  $\mathcal{V}$ , and  $d\mathbf{S}$  a surface element on  $\partial\mathcal{V}$ . The right hand side vanishes if the surface  $\partial\mathcal{V}$  is located far away from the plasma since the pressure is then null and the magnetic field decreases as the cubic power of the distance (for a dipolar field - multipolar components decay even faster). Hence an external force must be exerted to maintain the plasma. Some geometrical considerations show that this force is horizontal. It comes from both the pressure gradient and electromagnetic forces. A modified version of the virial theorem can be devised to determine this force. The displacement vector is now chosen to lie in a poloidal plane  $\boldsymbol{\xi} = X\hat{\mathbf{e}}_X + Y\hat{\mathbf{e}}_Y$ . This displacement is radial since  $\boldsymbol{\xi} = R\hat{\mathbf{e}}_R$ , where  $R$  is the major radius, and  $\hat{\mathbf{e}}_R = \nabla R$  is the unit vector in the conventional set of cylindrical coordinates  $(R, Z, \varphi)$ . This allows calculating a moment of the radial force

$$\langle RF^R \rangle_{\mathcal{V}} = \int_{\mathcal{V}} d^3\mathbf{x} \xi_k \frac{\partial}{\partial X^i} T^{ik}$$

where the bracket indicates an average over the volume  $\mathcal{V}$ . The integral over the volume of Eq.(20) yields the covariant expression

$$\begin{aligned} \langle RF^R \rangle_{\mathcal{V}} &= \int_{\mathcal{V}} d^3\mathbf{x} \left( 2P + \frac{B_Z^2}{2\mu_0} \right) \\ &\quad - \int_{\partial\mathcal{V}} (d\mathbf{S} \cdot \boldsymbol{\xi}) \left( P + \frac{B^2}{2\mu_0} \right) \end{aligned}$$

where the property  $d\mathbf{S} \cdot \mathbf{B} = 0$  has been used since  $\partial\mathcal{V}$  is a flux surface. Inside the plasma, the flux surfaces shift radially to ensure the force balance equation  $\langle RF^R \rangle = 0$ . This is nothing else than the Shafranov shift. Outside the separatrix, an external radial force is needed to maintain the plasma. This force is due to three causes [20, 21]:

- a force due to pressure. This force is analogous to the one felt by an inflated tire, which expands. It is sometimes called "tire tube force".
- an electromagnetic force due to the plasma current and the poloidal field. A current filament tends to expand under the Lorentz force associated to the current and the magnetic field it produces. This force is called the "hoop force".
- an electromagnetic force due to the Lorentz force that results from the toroidal magnetic field and the poloidal current that flows in the plasma - this current can be diamagnetic or paramagnetic. The toroidal field is inversely proportional to the major radius. A net force results from the imbalance between the jump of magnetic field on high and low field sides. It is called "1/R force".

This radial force is compensated in a tokamak by a vertical magnetic field, such that the Lorentz force due to the vertical field and the toroidal plasma current balances the expanding forces (Fig.15).

## Appendices

# A Curvilinear coordinates: definition and properties

## A.1 Curvilinear coordinates

A position vector is noted  $\mathbf{x}$ . Cartesian coordinates provide the usual framework to characterize a vector, namely  $\mathbf{x} = X^i \hat{\mathbf{e}}_i$ , where  $X^i = X_i$  are the Cartesian coordinates, and the  $\hat{\mathbf{e}}_i = \hat{\mathbf{e}}^i$  is a set of orthonormal basis vectors<sup>19</sup>. The gradient of a differentiable function  $f(X^i)$  is defined as  $\nabla f = \frac{\partial f}{\partial X^i} \hat{\mathbf{e}}^i$ , or put it differently  $df = \nabla f \cdot d\mathbf{x}$ . Curvilinear coordinates are defined as a set  $(\xi^1, \xi^2, \xi^3)$  that can be mapped on the set of Cartesian coordinates  $(X^1, X^2, X^3)$ , i.e. each coordinate  $\xi^i$  is a well defined and invertible function of the  $X^i$ 's - invertible means that the  $X^i$ 's are well behaved functions of the  $\xi^i$ 's. The coordinates  $\xi^i$  are vector fields, noted  $\xi^i(\mathbf{x})$ . The subspace  $\xi^i(\mathbf{x}) = cte$  is a surface. The coordinate gradient  $\nabla \xi^i$  is a vector normal to this surface since  $d\xi^i = \nabla \xi^i \cdot d\mathbf{x}$  (Fig.16)). Hence the set  $(\nabla \xi^1, \nabla \xi^2, \nabla \xi^3)$  form a basis provided the Jacobian is non zero. One can define a dual basis by using the cross-products of the vectors  $\nabla \xi^i$ . Any vector field  $\mathbf{B}(\mathbf{x})$  can then be written as

$$\mathbf{B} = B^i \mathbf{e}_i = B_i \mathbf{e}^i$$

where

$$\mathbf{e}^i = \nabla \xi^i = \frac{\partial \xi^i}{\partial \mathbf{x}} \quad ; \quad \mathbf{e}_i = \frac{\partial \mathbf{x}}{\partial \xi^i} = \epsilon_{ijk} \sqrt{g} (\nabla \xi^j \times \nabla \xi^k)$$

where  $\sqrt{g} = |(\nabla \xi^1 \times \nabla \xi^2) \cdot \nabla \xi^3|^{-1}$  is the Jacobian of the set of coordinates  $(\xi^1, \xi^2, \xi^3)$ . The  $\mathbf{e}_i$  (respectively  $\mathbf{e}^i$ ) are the covariant (resp. contravariant vectors)<sup>20</sup>, and satisfy the property  $\mathbf{e}_i \cdot \mathbf{e}^k = \delta_i^k$ . The latter property reflects the derivative chain rule

$$\frac{\partial \xi_i}{\partial X^j} \cdot \frac{\partial X^j}{\partial \xi^k} = \delta_i^k$$

If the contravariant vectors  $\mathbf{e}^i = \nabla \xi^i$  are known, a matrix inversion provides the covariant vectors  $\mathbf{e}_i = \partial \mathbf{x} / \partial \xi^i$ . Obviously  $B^i = \mathbf{B} \cdot \mathbf{e}^i$  and  $B_i = \mathbf{B} \cdot \mathbf{e}_i$ . The norm of a vector field is  $B = |\mathbf{B}| = (B_i B^i)^{1/2}$ . The vectors  $\mathbf{e}^i$  or  $\mathbf{e}_i$  are usually *not* orthogonal, and not normal. Nevertheless, in some special cases (e.g. cylindrical or spherical coordinates), they do form a set of orthogonal vectors. In this case it is convenient to normalise them and thus build a convenient orthonormal basis. They are then noted with an arrow, to make a distinction with co- or contra-variant vectors and defined as  $\hat{\mathbf{e}}_i = \mathbf{e}_i / |\mathbf{e}_i|$ . An important property is  $\hat{\mathbf{e}}_i = \hat{\mathbf{e}}^i$ , so that co- or contra-variant notations do not matter any more. Obviously Cartesian coordinates is a special case of a set of orthogonal vectors.

## A.2 Metric tensor

An elementary distance element is

$$ds^2 = g_{ik} d\xi^i d\xi^k = g^{ik} d\xi_i d\xi_k$$

where the metric elements are

$$g_{ik} = \mathbf{e}_i \cdot \mathbf{e}_k = \frac{\partial \mathbf{x}}{\partial \xi^i} \cdot \frac{\partial \mathbf{x}}{\partial \xi^k} \quad ; \quad g^{ik} = \mathbf{e}^i \cdot \mathbf{e}^k = \nabla \xi^i \cdot \nabla \xi^k$$

The metric matrix  $g_{ik}$  is in fact a tensor, as it changes from one system of coordinates to another one via usual chain rules for changes of variables. The tensor  $g_{ik}$  is the inverse of the tensor  $g^{ik}$ . With the conventions above,  $\sqrt{g}$  is the determinant of the matrix  $g_{ik}$ . Hence

$$\sqrt{g} = \det(g_{ik}) = \frac{1}{\det(g^{ik})}$$

The change from co- to contra- coordinates read  $B^i = g^{ik} B_k$  and  $B_i = g_{ik} B^k$ .

<sup>19</sup>Doubled indices are summed throughout this note.

<sup>20</sup>The index is upward when the component is contravariant, otherwise covariant

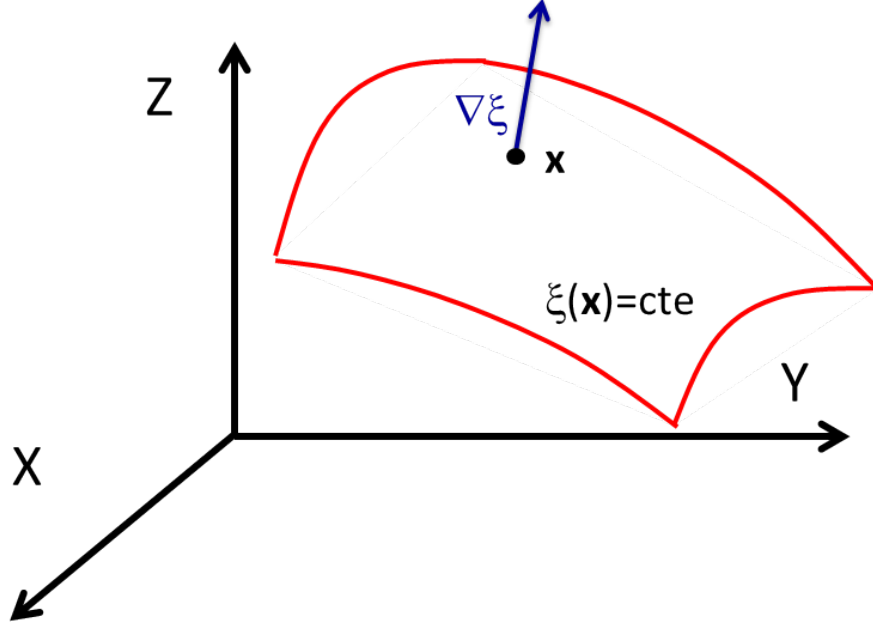


Figure 16: Surface  $\xi(\mathbf{x}) = cte$  and its normal vector  $\nabla \xi$ .

### A.3 Most common expressions of vectorial operators in curvilinear coordinates

The knowledge of the metric tensor gives access to explicit expression of most vectorial operators, which are reminded below. Divergence of a vector  $\mathbf{B}$

$$\nabla \cdot \mathbf{B} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^i} (\sqrt{g} B^i)$$

Curl of a vector  $\mathbf{B}$

$$(\nabla \times \mathbf{B})^i = \frac{1}{\sqrt{g}} \epsilon^{ijk} \frac{\partial B_k}{\partial \xi^j}$$

Components of a scalar gradient

$$(\nabla \phi)^i = g^{ik} \frac{\partial \phi}{\partial \xi^k}$$

Laplacian of a scalar

$$\nabla^2 \phi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^i} \left( \sqrt{g} g^{ik} \frac{\partial \phi}{\partial \xi^k} \right)$$

The element of volume is  $d\mathcal{V} = d^3\mathbf{x} = \sqrt{g} d\xi^1 d\xi^2 d\xi^3$ . The integration element on the surface  $\xi^i = cte$  is  $dS_i = \epsilon_{ijk} \sqrt{g} d\xi^j d\xi^k$ . This is consistent with the Green-Ostrogradsky theorem

$$\int_{\mathcal{V}} d\mathcal{V} \nabla \cdot \mathbf{B} = \oint_{\partial \mathcal{V}} d\mathbf{S} \cdot \mathbf{B}$$

where  $\mathcal{V}$  is a volume and  $\partial \mathcal{V}$  the surface that encloses it. The element of length is given by the relation  $d\mathbf{x} = d\xi^i \mathbf{e}_i$ , which is consistent with the Gauss theorem

$$\int_{\mathcal{S}} d\mathbf{S} \cdot \nabla \times \mathbf{B} = \oint_{\partial \mathcal{S}} d\mathbf{x} \cdot \mathbf{B}$$

where  $\mathcal{S}$  is a surface and  $\partial \mathcal{S}$  the curve that bounds it



## A.4 Flux surface average

Let us choose a set of curvilinear coordinates  $(\rho, \theta, \zeta)$ , periodic in  $\theta$  and  $\zeta$ . A surface average of a function  $f(\rho, \theta, \zeta)$  is defined as the variation of the volume integral of  $f$  between two nearby surfaces  $\rho$  and  $\rho + \delta\rho$

$$\langle f \rangle_\rho = \frac{\delta \int_\rho d^3\mathbf{x} f}{\delta \int_\rho d^3\mathbf{x}} = \frac{\int_\rho d\theta d\zeta \sqrt{g} f}{\int_\rho d\theta d\zeta \sqrt{g}}$$

In the special case of a magnetic field of the form Eq.(6) in an axisymmetric configuration, the magnetic surface average, defined as the average of a function in between two nearby magnetic surfaces  $\psi$  and  $\psi + \delta\psi$ , becomes

$$\langle f \rangle_\psi = \frac{\int_\psi \frac{d\theta}{\mathbf{B} \cdot \nabla \theta} f}{\int_\psi \frac{d\theta}{\mathbf{B} \cdot \nabla \theta}}$$

## B Solov'ev equilibrium

Solov'ev equilibria are exact global solutions of the Grad-Shafranov equation [8]. Let us remind its form

$$\nabla_*^2 \psi = -I \frac{dI}{d\psi} - \mu_0 R^2 \frac{dP}{d\psi}$$

where

$$\nabla_*^2 \psi = R^2 \nabla \cdot \left( \frac{1}{R^2} \nabla \psi \right) = R \frac{\partial}{\partial R} \left( \frac{1}{R} \frac{\partial \psi}{\partial R} \right) + \frac{\partial^2 \psi}{\partial Z^2}$$

and let us assume that

$$\begin{aligned} \mu_0 \frac{dP}{d\psi} &= -A \\ I \frac{dI}{d\psi} &= \frac{\gamma}{1+\alpha^2} R_0^2 A \end{aligned}$$

where  $A$ ,  $\gamma < 1$ ,  $R_0$  and  $\alpha$  are constant parameters. A particular solution of Eq.(20) is then

$$\psi = \frac{A}{2(1+\alpha^2)} \left\{ (R^2 - \gamma R_0^2) Z^2 + \frac{\alpha^2}{4} (R^2 - R_0^2)^2 \right\}$$

The level curves of the function  $\psi(R, Z)$  are cross sections of flux surfaces in a poloidal plane. Near the magnetic axis  $R = R_0, Z = 0$ , i.e. where  $x = \frac{R-R_0}{R_0} \ll 1$  and  $z = \frac{Z}{R_0} \ll 1$ , a Taylor development of Eq.(20) at order 3 in  $x \sim z \sim \epsilon$  yields

$$[x + \Delta(x, z)]^2 + \frac{1-\gamma}{\alpha^2} z^2 = x_\psi^2 + o(\epsilon^4)$$

where  $x_\psi^2 = \frac{2\psi}{AR_0^4} \left( \frac{1}{\alpha^2} + 1 \right)$  is a flux label, and

$$\Delta(x, z) = \frac{1}{2} x^2 + \frac{1}{\alpha^2} z^2$$

Near the axis magnetic axis  $x = z = 0$ , flux surfaces are ellipses with ellipticity  $\frac{\alpha}{\sqrt{1-\gamma}}$ . The horizontal displacement  $\Delta(x, z)$  is a Shafranov shift (see following section). The separatrix is determined by the "X-points" such that  $|\nabla \psi| = 0$  (null poloidal magnetic field), i.e.  $\frac{\partial \psi}{\partial R} = \frac{\partial \psi}{\partial Z} = 0$ . Using the general expression of  $\psi$  Eq.(20) two X-points exist such that  $R_X = \sqrt{\gamma} R_0$  and  $Z_X = \pm \sqrt{\frac{1-\gamma}{2}} \alpha R_0$ . Generalisations of this simple example to more sophisticated equilibria have been developed, which allow a broad variety of surface shapes [22].

## C Miller equilibrium

Finding a global solution of the Grad-Shafranov equation is difficult for arbitrary pressure and current profiles. However it is relatively easy to build a local solution near a flux surface with prescribed shape and poloidal magnetic field [11, 10]. Let us consider a given flux surface with a poloidal flux  $\psi_0$ . The surface cross section, a poloidal curve labelled  $\mathcal{C}$ , can be described by a curvilinear abscissa  $\ell$ . The position  $\mathbf{x}_p$  on this curve is prescribed by a vertical and horizontal positions i.e.  $\mathbf{x}_p(\ell) = R_s(\ell)\hat{\mathbf{e}}_R + Z_s(\ell)\hat{\mathbf{e}}_Z$ . Near the reference flux surface, the vertical and horizontal coordinates can be written in the form

$$\begin{aligned} R(\ell, \rho) &= R_s(\ell) + \rho \cos \vartheta(\ell) \\ Z(\ell, \rho) &= Z_s(\ell) + \rho \sin \vartheta(\ell) \end{aligned}$$

where  $\rho$  is a flux label that measures the distance to the reference flux surface, and  $\vartheta(\ell)$  is a poloidal angle to be defined. Let us define  $\hat{\mathbf{e}}_\ell$  as the unit tangent vector associated with the curve  $\mathcal{C}$ , then

$$\frac{d\mathbf{x}_p}{d\ell} = \hat{\mathbf{e}}_\ell = \frac{dR_s}{d\ell}\hat{\mathbf{e}}_R + \frac{dZ_s}{d\ell}\hat{\mathbf{e}}_Z$$

The condition  $|\hat{\mathbf{e}}_\ell|^2 = 1$  imposes that  $\frac{dR_s}{d\ell}$  and  $\frac{dZ_s}{d\ell}$  must be cosine and sine functions of an angle. There is some flexibility on the choice of the angle origin. Anticipating that derivatives of  $Z$  and  $R$  with respect to  $\rho$  and  $\ell$  will be needed to compute the metric tensor, a suitable choice is

$$\begin{aligned} \frac{dR_s}{d\ell} &= -\sin \vartheta(\ell) \\ \frac{dZ_s}{d\ell} &= \cos \vartheta(\ell) \end{aligned}$$

so that  $\hat{\mathbf{e}}_\ell = -\sin \vartheta(\ell)\hat{\mathbf{e}}_R + \cos \vartheta(\ell)\hat{\mathbf{e}}_Z$ . The derivative of the tangent vector with respect to  $\ell$  is  $\frac{d\hat{\mathbf{e}}_\ell}{d\ell} = -\frac{1}{R_c(\ell)}\hat{\mathbf{e}}_\rho$ , where  $\hat{\mathbf{e}}_\rho = \cos \vartheta(\ell)\hat{\mathbf{e}}_R + \sin \vartheta(\ell)\hat{\mathbf{e}}_Z$  is a unit vector normal to the curve  $\mathcal{C}$  and  $R_c$  is the curvature radius. This imposes that  $\frac{d\vartheta}{d\ell} = \frac{1}{R_c(\ell)}$ . Note that with these conventions the coordinate system  $(\rho, \ell, \zeta)$  is direct. These relations give the needed partial derivatives

$$\begin{pmatrix} dR \\ dZ \end{pmatrix} = \begin{pmatrix} \cos \vartheta & -J \sin \vartheta \\ \sin \vartheta & J \cos \vartheta \end{pmatrix} \begin{pmatrix} d\rho \\ d\ell \end{pmatrix}$$

where  $J(\rho, \ell) = 1 + \frac{\rho}{R_c(\ell)}$  is the matrix determinant. A matrix inversion provide the reverse relationship

$$\begin{pmatrix} d\rho \\ d\ell \end{pmatrix} = \begin{pmatrix} \cos \vartheta & \sin \vartheta \\ -\frac{\sin \vartheta}{J} & \frac{\cos \vartheta}{J} \end{pmatrix} \begin{pmatrix} dR \\ dZ \end{pmatrix}$$

This result can be summarised as  $\nabla \rho = \hat{\mathbf{e}}_\rho$ ,  $\nabla \ell = \frac{1}{J}\hat{\mathbf{e}}_\ell$  and  $\nabla \zeta = \frac{1}{R}\hat{\mathbf{e}}_\zeta$ . The 3D Jacobian is  $\sqrt{g} = JR$ .

We now solve the Grad-Shafranov equation,

$$\nabla_*^2 \psi = R^2 \nabla \cdot \left( \frac{1}{R^2} \nabla \psi \right) = -I \frac{dI}{d\psi} - \mu_0 R^2 \frac{dP}{d\psi}$$

The effective Laplacian  $\nabla_*^2 \psi$  can be reformulated as

$$\nabla_*^2 \psi = \frac{R^2}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \frac{\sqrt{g}}{R^2} g^{ik} \frac{\partial \psi}{\partial x^k} \right)$$

where it is reminded that the metric elements satisfy  $g^{ik} = \nabla \xi^i \cdot \nabla \xi^k$ , so that

$$g^{ik} = \begin{pmatrix} |\nabla \rho|^2 & \nabla \rho \cdot \nabla \ell \\ \nabla \rho \cdot \nabla \ell & |\nabla \ell|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{J^2} \end{pmatrix}$$



choice of coordinates is necessary. A reasonable starting point is the set of cylindrical coordinates  $(R, \zeta, Z)$ . The subset of coordinates  $(R, Z)$  is to be replaced by a new set of flux coordinates  $(\rho, \vartheta)$ , where  $\rho$  is a label of flux surface, and  $\vartheta$  a poloidal angle - here the conventional geometrical angle. The coordinate  $\zeta$  is orthogonal to any set  $(\rho, \vartheta)$ , so that it can be treated separately. To keep on handling a *direct* set of coordinates (positive signature), it is convenient to reorder the cylindrical coordinates as  $(R, Z, \zeta)$ , which is direct if  $\zeta$  is anti-trigonometric, so that the subset  $(R, Z)$  behaves as Cartesian coordinates. This is not mandatory, but makes life easier. The equation of a circle centred on  $R = R_0$  is  $Z = \rho \cos \vartheta$ ,  $R - R_0 = \rho \sin \vartheta$ . However concentric circular flux surfaces are not consistent with the Grad-Shafranov equation, because of the thermal and electromagnetic forces that push the plasma outward. Because of these forces, flux surfaces are shifted horizontally. A minimal update is to prescribe flux surfaces of the form (Fig.17)

$$\begin{aligned} R &= R_0 + \Delta(\rho) + \rho \cos \vartheta \\ Z &= \rho \sin \vartheta \end{aligned}$$

where  $\rho$  is the radius of each circular flux surface ( $\rho = 0$  is the magnetic axis), and  $R_0$  the major radius of the magnetic axis. The length  $\Delta(\rho)$  measures the horizontal displacement of the centres of flux surfaces with respect to the magnetic axis. This displacement is called ‘‘Shafranov’’ shift and is due to forces exerted by kinetic and magnetic pressures. The inverse aspect ratio  $\epsilon = a/R_0$  is assumed to be small parameter  $\epsilon \ll 1$ , where  $a$  is the plasma minor radius. It is anticipated that the Shafranov shift normalised to  $a$  scales as  $\epsilon^2$ . The magnetic field is written as Eq.(12)

$$\mathbf{B} = I(\rho) \nabla \zeta + \nabla \zeta \times \nabla \psi$$

where

$$\frac{d\psi}{d\rho} = b_\theta(\rho) R_0$$

and  $B_0$  is the magnetic field at the magnetic axis. The cylindrical set of coordinates  $(R, Z, \zeta)$  is associated with a well defined set of orthonormal vectors<sup>21</sup>

$$\hat{\mathbf{e}}_R = \nabla R \quad \hat{\mathbf{e}}_Z = \nabla Z \quad \hat{\mathbf{e}}_\zeta = R \nabla \zeta$$

A covariant expression of the field is obtained in the following way. The elements  $(dR, dZ)$  are related to  $(d\rho, d\vartheta)$  via the matrix relation

$$\begin{pmatrix} dR \\ dZ \end{pmatrix} = \begin{pmatrix} \cos \vartheta + \frac{d\Delta}{d\rho} & -\rho \sin \vartheta \\ \sin \vartheta & \rho \cos \vartheta \end{pmatrix} \begin{pmatrix} d\rho \\ d\vartheta \end{pmatrix}$$

Its determinant is  $J(\rho, \vartheta) = \rho \left(1 + \frac{d\Delta}{d\rho} \cos \vartheta\right)$ . This matrix is readily inverted in

$$\begin{pmatrix} d\rho \\ d\vartheta \end{pmatrix} = \frac{1}{J(\rho, \vartheta)} \begin{pmatrix} \rho \cos \vartheta & \rho \sin \vartheta \\ -\sin \vartheta & \cos \vartheta + \frac{d\Delta}{d\rho} \end{pmatrix} \begin{pmatrix} dR \\ dZ \end{pmatrix}$$

The Jacobian of the system of flux coordinates  $(\rho, \vartheta, \zeta)$  is therefore  $\sqrt{g} = [(\nabla \rho \times \nabla \vartheta) \cdot \nabla \zeta]^{-1} = JR$ . These relations yield the vectors  $\nabla \rho$ ,  $\nabla \vartheta$ , and therefore a relationship between  $\nabla \zeta$  and  $\nabla \rho \times \nabla \vartheta$ . An alternative expression of the magnetic field is then derived

$$\mathbf{B} = I(\rho) \frac{\rho}{R_0} \left[ 1 + \left( \frac{d\Delta}{d\rho} - \frac{\rho}{R_0} + o(\epsilon^2) \right) \cos \vartheta \right] \nabla \rho \times \nabla \vartheta + \nabla \zeta \times \nabla \psi$$

The geometrical angle  $\vartheta$  is not a straight field line coordinate since

$$\frac{\mathbf{B} \cdot \nabla \vartheta}{\mathbf{B} \cdot \nabla \zeta} = \frac{b_\theta(\rho) R_0^2}{I(\rho) \rho} \left[ 1 - \left( \frac{d\Delta}{d\rho} - \frac{\rho}{R_0} + o(\epsilon^2) \right) \cos \vartheta \right]$$

---

<sup>21</sup>This set differs generally speaking from the co- or contra-variant basis vectors, which are usually not normal, nor orthogonal.

is a function of  $\rho$  and  $\vartheta$ . It is convenient to replace  $\vartheta$  with a straight field line coordinate  $\theta$  such that  $\frac{\mathbf{B} \cdot \nabla \zeta}{\mathbf{B} \cdot \nabla \theta} = q(\rho)$ , with  $q(\rho)$  the safety factor. The geometrical angle  $\vartheta$  is related to the intrinsic poloidal angle  $\theta$  via the near identity relation

$$\theta = \vartheta - \delta(\rho, \vartheta) + o(\epsilon^2)$$

where the function  $\delta$  is

$$\delta(\rho, \vartheta) = - \left( \frac{d\Delta}{d\rho} - \frac{\rho}{R_0} \right) \sin \vartheta$$

This provides a link between the “poloidal” field  $b_\theta(\rho)$  and the current  $I(\rho)$ , namely

$$b_\theta(\rho) = I(\rho) \frac{\rho}{q R_0^2}$$

The magnetic field can be then written in the expected covariant form

$$\mathbf{B} = \nabla \chi \times \nabla \theta + \nabla \zeta \times \nabla \psi$$

where  $d\chi/d\psi = q(\rho)$ . The Jacobian of the new system of coordinates  $(\rho, \theta, \zeta)$  is  $\sqrt{g} = \rho R^2 / R_0$ .

The Grad-Shafranov equation involves the effective Laplacian

$$\nabla_*^2 \psi = R^2 \nabla \cdot \left( \frac{1}{R^2} \nabla \psi \right) = \frac{R^2}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \frac{\sqrt{g}}{R^2} g^{ik} \frac{\partial \psi}{\partial x^k} \right)$$

The metric tensor associated with the set of variables  $(\rho, \theta, \zeta)$  is formally derived by using the identity  $g_{ik} = \frac{\partial \mathbf{x}}{\partial \xi^i} \cdot \frac{\partial \mathbf{x}}{\partial \xi^k}$  and  $g^{ik} = \nabla \xi^i \cdot \nabla \xi^k$  where  $\xi^i = (\rho, \theta, \zeta)$ . It can also be done by calculating the distance Eq.(16) -  $g^{ik}$  can also be computed by inverting  $g_{ik}$ . The change of variables from  $(R, Z)$  to  $(\rho, \theta)$  is given by Eqs.(20), where  $\vartheta = \theta + \delta(\rho, \theta)$ , and  $\delta(\rho, \theta) = - \left( \frac{d\Delta}{d\rho} - \frac{\rho}{R_0} \right) \sin \theta + o(\epsilon^2)$ . After a little bit of algebra, one gets

$$g_{ik} = \begin{pmatrix} 1 + 2 \frac{d\Delta}{d\rho} \cos \theta & \rho^2 \frac{\partial \delta}{\partial \rho} - \rho \frac{d\Delta}{d\rho} \sin \theta & 0 \\ \rho^2 \frac{\partial \delta}{\partial \rho} - \rho \frac{d\Delta}{d\rho} \sin \theta & \rho^2 \left( 1 + 2 \frac{\partial \delta}{\partial \theta} \right) & 0 \\ 0 & 0 & R^2 \end{pmatrix}$$

where all coefficients are computed up to a  $o(\epsilon^2)$ . The determinant  $g$  matches with the Jacobian  $\sqrt{g} = \rho R^2 / R_0$  that was already calculated. The matrix is easily inverted by isolating the  $(2, 2)$  submatrix to give

$$g^{ik} = \begin{pmatrix} 1 - 2 \frac{d\Delta}{d\rho} \cos \theta & \frac{1}{\rho} \left[ \frac{d}{d\rho} \left( \rho \frac{d\Delta}{d\rho} \right) - \frac{\rho}{R_0} \right] \sin \theta & 0 \\ \frac{1}{\rho} \left[ \frac{d}{d\rho} \left( \rho \frac{d\Delta}{d\rho} \right) - \frac{\rho}{R_0} \right] \sin \theta & \frac{1}{\rho^2} \left[ 1 + 2 \left( \frac{d\Delta}{d\rho} - \frac{\rho}{R_0} \right) \cos \theta \right] & 0 \\ 0 & 0 & \frac{1}{R^2} \end{pmatrix}$$

This provides a new expression of the effective Laplacian

$$\nabla_*^2 \psi = \frac{1}{\rho} \frac{d}{d\rho} \left[ \rho |\nabla \rho|^2 \frac{d\psi}{d\rho} \right] + \frac{\partial}{\partial \theta} \left[ (\nabla \theta \cdot \nabla \rho) \frac{d\psi}{d\rho} \right]$$

where the relations  $g^{\rho\rho} = |\nabla \rho|^2$  and  $g^{\rho\theta} = \nabla \theta \cdot \nabla \rho$  have been used, and

$$\begin{aligned} |\nabla \rho|^2 &= 1 - 2 \frac{d\Delta}{d\rho} \cos \theta \\ \rho \nabla \theta \cdot \nabla \rho &= \left[ \frac{d}{d\rho} \left( \rho \frac{d\Delta}{d\rho} \right) - \frac{\rho}{R_0} \right] \sin \theta \end{aligned}$$

The Grad-Shafranov equation

$$\nabla_*^2 \psi = -I \frac{dI}{d\psi} - \mu_0 R^2 \frac{dP}{d\psi}$$

can then be solved order per order in  $\frac{\rho}{R_0}$ . At lowest order

$$\frac{b_\theta}{\rho} \frac{d}{d\rho} \left( \rho \frac{db_\theta}{d\rho} \right) = -\frac{I}{R_0^2} \frac{dI}{d\rho} - \mu_0 \frac{dP}{d\rho}$$

At order 1 in  $\rho/R_0$ , the follow equation is obtained

$$\frac{d}{d\rho} \left( \rho b_\theta^2 \frac{d\Delta}{d\rho} \right) = -\frac{\rho}{R_0} b_\theta^2 + 2\mu_0 \frac{\rho^2}{R_0} \frac{dP}{d\rho}$$

which provides the radial derivative of the Shafranov shift

$$\begin{aligned} \frac{d\Delta}{d\rho} &= -\frac{\rho}{R_0} \left( \beta_p(\rho) + \frac{l_i(\rho)}{2} \right) \\ \beta_p(\rho) &= -\frac{2\mu_0}{\rho^2 b_\theta^2} \int_0^\rho d\rho \rho^2 \frac{dP}{d\rho} \\ l_i(\rho) &= \frac{2}{\rho^2 b_\theta^2} \int_0^\rho d\rho \rho b_\theta^2 \end{aligned}$$

As expected the Shafranov shift is related to the thermal and magnetic pressures. The quantity  $l_i(\rho)$  is the internal inductance of the flux surface  $r$ . Indeed, noting that the volume element is approximately  $dV = 2\pi R_0 2\pi \rho d\rho$ , the definition above appears to be the volume integral of the density of poloidal magnetic energy  $\frac{b_\theta^2}{2\mu_0}$ , normalised to the magnetic energy at the surface. The quantity  $\beta_p(\rho)$  is called poloidal  $\beta$  and measures the ratio of the thermal pressure to the density of poloidal magnetic energy.

## E Boozer and Hamada coordinates

Some more details are given here on two special choices of coordinates, called Boozer and Hamada coordinates, after their inventors. This section is highly inspired from textbook [1] and review articles [2, 17]. interested readers should consult them for getting more details <sup>22</sup>. The contravariant expression of the magnetic field is <sup>23</sup>

$$\mathbf{B} = G\nabla\theta + I\nabla\zeta + K\nabla\psi + \nabla H \quad (20)$$

where the “currents”  $I$  and  $G$  depend on  $\psi$  only, and  $K$ ,  $H$  depend on  $(\psi, \theta, \zeta)$ . Boozer coordinates correspond to the choice  $H = 0$ , while Hamada coordinates are such that  $K = 0$ . The Lorentz force can be calculated by using the Clebsh form of the magnetic field Eq.(6)

$$\mathbf{B} = \nabla\chi \times \nabla\theta + \nabla\zeta \times \nabla\psi \quad (21)$$

The final result reads

$$\mu_0 \mathbf{J} \times \mathbf{B} = \frac{1}{\sqrt{g}} \left[ q \frac{\partial K}{\partial \zeta} + \frac{\partial K}{\partial \theta} - \frac{dG}{d\psi} - q \frac{dI}{d\psi} \right]$$

<sup>22</sup>It is recalled here that the notations used in tokamak and stellarator communities are usually different. The notation  $\psi$  dubs the poloidal flux in tokamaks, while it often designates the toroidal flux in stellarators. Also the currents  $I$  and  $G$  are permuted. It is also reminded that stellarator experts prefer to use the rotational transform  $\iota$  rather than the safety factor  $q$  - both are related via the relation  $\iota = 1/q$ .

<sup>23</sup>Here we choose  $\psi$ , the poloidal flux, as a radial coordinate  $\rho$ , i.e.  $\rho = \psi$ . To mark the difference, the notation  $-\nu$  is replaced by  $K$ .

where  $1/\sqrt{g} = (\nabla\psi \times \nabla\theta) \cdot \nabla\zeta$ . The force balance equation Eq.(7) then bears a convenient form

$$q \frac{\partial K}{\partial \zeta} + \frac{\partial K}{\partial \theta} - \frac{dG}{d\psi} - q \frac{dI}{d\psi} = \sqrt{g} \mu_0 \frac{dP}{d\psi} \quad (22)$$

Note that this equation does not depend on the gauge function  $H$ . If Hamada coordinates are chosen,  $K = 0$ . It then appear that the Jacobian  $\sqrt{g}$  depends on  $\psi$  only, is a flux function. The Grad-Shafranov equation becomes particularly simple

$$\frac{dG}{d\psi} + q \frac{dI}{d\psi} = -\mu_0 \frac{dP}{d\psi} \frac{d\mathcal{V}}{d\psi} \quad (23)$$

where  $\mathcal{V}$  is the volume enclosed by a magnetic surface normalised to  $4\pi^2$ , i.e.  $\frac{d\mathcal{V}}{d\psi} = \sqrt{g}$ .

Handling the force balance equation Eq.(22) is somewhat less straightforward when using Boozer coordinates  $H = 0, K \neq 0$ . Let us note that an average of Eq.(22) over the angles  $\theta$  and  $\zeta$  provides a solvability constraint Eq.(23) (with a different meaning though since the coordinates are different) that can be used to yield the equation

$$q \frac{\partial K}{\partial \zeta} + \frac{\partial K}{\partial \theta} = \mu_0 \frac{dP}{d\psi} \left( \sqrt{g} - \frac{d\mathcal{V}}{d\psi} \right) \quad (24)$$

The Jacobian is readily obtained by multiplying Eq.(20) by Eq.(21). This procedure yields

$$\sqrt{g} = \frac{Iq + G}{B^2}$$

The equation Eq.(24) hence provides the function  $K$  vs the current  $I, G$  and pressure  $P$ . Let us stress however that this equation is singular on resonant surfaces. Some regularisation is therefore needed either by setting locally a zero pressure gradient, or introducing some dissipation.

## F Variational energy principle

This appendix aims at calculating the variation of the energy

$$\mathcal{W} = \int d^3\mathbf{x} \left( \frac{B^2}{2\mu_0} + \frac{P}{\varsigma - 1} \right)$$

when deforming set for flux surfaces with a virtual displacement  $\boldsymbol{\xi}$  consistent with the equations of ideal MHD. Let us remind here the basic equations of ideal MHD [6, 7, 23, 24]

$$\begin{aligned} \rho_m d_t \mathbf{V} &= -\nabla P + \mathbf{J} \times \mathbf{B} \\ \partial_t \mathbf{B} &= \nabla \times (\mathbf{V} \times \mathbf{B}) \\ d_t P + \varsigma P \nabla \cdot \mathbf{V} &= 0 \\ d_t \rho_m + \rho_m \nabla \cdot \mathbf{V} &= 0 \end{aligned}$$

where  $d_t = \partial_t + \mathbf{V} \cdot \nabla$  is a Lagrangian derivative,  $\mathbf{V}$  the plasma velocity,  $P$  the pressure,  $\mathbf{B}$  the magnetic field and  $\rho_m$  the mass density, and  $\varsigma$  the adiabatic index. The two last equations are consistent with the adiabatic transform constraint  $d_t (P \rho_m^{-\varsigma}) = 0$ . A virtual MHD displacement is such that  $\partial_t \boldsymbol{\xi} = \mathbf{V}$ , where  $\boldsymbol{\xi}$  is supposed infinitesimal. One major property of ideal MHD is the frozen-in law that grants flux conservation under the plasma motion. Hence  $d_t \rho = 0$ , i.e. flux surfaces remain flux surfaces after a displacement. The variation of the magnetic field verifies  $\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$ , so that the change of magnetic energy density is<sup>24</sup>

$$\frac{1}{\mu_0} \mathbf{B} \cdot \delta \mathbf{B} = -\boldsymbol{\xi} \cdot (\mathbf{J} \times \mathbf{B}) + \frac{1}{\mu_0} \nabla \cdot [(\boldsymbol{\xi} \times \mathbf{B}) \times \mathbf{B}]$$

---

<sup>24</sup>This property is most readily demonstrated by using the identity  $\nabla(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$  with  $\mathbf{A} = \boldsymbol{\xi} \times \mathbf{B}$  and the Ampère's equation  $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ .

The variation of pressure reads

$$\delta P = (\varsigma - 1)\boldsymbol{\xi} \cdot \nabla P - \nabla \cdot [\varsigma P \boldsymbol{\xi}]$$

If the transform of flux surfaces is such that the last surface that encloses the plasma is left unchanged, all divergence terms, i.e. the Poynting and thermal energy fluxes, vanish after integration of the plasma volume since  $\boldsymbol{\xi} = 0$  at the boundary. The variation of energy then verifies

$$\delta \mathcal{W} = \int d^3 \mathbf{x} (\nabla P - \mathbf{J} \times \mathbf{B}) \cdot \boldsymbol{\xi}$$

An extremum of  $\delta \mathcal{W}$  is reached when the force balance equation is satisfied. However it must be kept in mind that this is true under some specific conditions: magnetic flux conservation, adiabatic transform and constant plasma boundary.

## G Using the virial theorem

It is quite instructive to apply the virial theorem [15, 23] to a set of shifted circular flux surfaces, as described in the previous section.

$$\begin{aligned} \langle RF^R \rangle \mathcal{V} &= \int_{\mathcal{V}} d^3 \mathbf{x} \left( 2P + \frac{B_Z^2}{2\mu_0} \right) \\ &- \int_{\partial \mathcal{V}} (d\mathbf{S} \cdot \boldsymbol{\xi}) \left( P + \frac{B^2}{2\mu_0} \right) \end{aligned} \quad (25)$$

where  $d^3 \mathbf{x} = 2\pi R dR dZ$  and  $\boldsymbol{\xi} = R \hat{\mathbf{e}}_R$ . Flux surfaces are given by

$$\begin{aligned} R &= R_0 + \Delta(\rho) + \rho \cos \vartheta \\ Z &= \rho \sin \vartheta \end{aligned}$$

where  $\Delta(\rho)$  is the Shafranov shift, and is supposed unknown at this stage. The magnetic field is of the form

$$\mathbf{B} = B_T \hat{\mathbf{e}}_\varphi + B_p \hat{\mathbf{e}}_\vartheta$$

where the “toroidal” and “poloidal” magnetic field intensities are

$$\begin{aligned} B_T(\rho, \vartheta) &= B_0 \left( 1 - \frac{\rho}{R_0} \cos \vartheta + o(\epsilon^2) \right) \\ B_p(\rho, \vartheta) &= b_\theta(\rho) (1 - \Lambda(\rho) \cos \vartheta + o(\epsilon^2)) \end{aligned}$$

with

$$\Lambda(\rho) = \frac{d\Delta}{d\rho} + \frac{\rho}{R_0}$$

The vectors

$$\begin{aligned} \hat{\mathbf{e}}_\rho &= \cos \vartheta \hat{\mathbf{e}}_R + \sin \vartheta \hat{\mathbf{e}}_Z \\ \hat{\mathbf{e}}_\vartheta &= \hat{\mathbf{e}}_\varphi \times \hat{\mathbf{e}}_\rho = -\sin \vartheta \hat{\mathbf{e}}_R + \cos \vartheta \hat{\mathbf{e}}_Z \end{aligned}$$

and  $\hat{\mathbf{e}}_\varphi = R \nabla \varphi$  constitute a set of orthonormal vectors. A quick check of this non-covariant formulation can be done by noting that the distance between 2 flux surfaces of radius  $\rho$  and  $\rho + d\rho$  is  $d\ell(\rho, \theta) = d\rho(1 + \frac{d\Delta}{d\rho} \cos \vartheta)$ . The poloidal magnetic flux is such that  $d\psi = B_p R d\ell$ , and is a function of  $\rho$  only, as expected.

The elements of integration elements that appear in Eq.(25) require some care when moving from the system of coordinates  $(R, Z)$  to  $(\rho, \vartheta)$ . The normal to the surface is



$\mathbf{n} = \mathbf{e}_\rho$ , so that  $d\mathbf{S} = d\vartheta d\varphi \rho R \mathbf{e}_\rho$ . The volume element is  $d^3\mathbf{x} = 2\pi R \rho d\rho d\vartheta$ , so that the following relations hold

$$\begin{aligned} \int_V d_3\mathbf{x} \left\{ 2P + \frac{B_Z^2}{\mu_0} \right\} &= 2V(\rho) \left\{ \langle P \rangle + \frac{1}{2} \left\langle \frac{b_\theta^2}{2\mu_0} \right\rangle \right\} \\ \int_{\partial V} (d\mathbf{S} \cdot \boldsymbol{\xi}) P &= 2V(\rho) P(\rho) \\ \int_{\partial V} (d\mathbf{S} \cdot \boldsymbol{\xi}) \left( P + \frac{B_p^2}{2\mu_0} \right) &= V(\rho) \frac{b_\theta^2}{\mu_0} \frac{R_0}{\rho} \frac{d\Delta}{d\rho} \\ \int_{\partial V} (d\mathbf{S} \cdot \boldsymbol{\xi}) \frac{B_T^2}{2\mu_0} &= 0 \end{aligned}$$

where  $V = 2\pi R_0 \rho^2$  is the volume enclosed by the flux surface and  $\langle \dots \rangle = \frac{2\pi^2 R_0}{V} \int_0^\rho 2\rho' d\rho' \dots$ . The toroidal field does not contribute since  $B_T^2 d\mathbf{S} \cdot \boldsymbol{\xi} = B_0^2/R_0^2 d\vartheta d\varphi \rho \cos \vartheta$ , so that the average over  $\vartheta$  cancels exactly on each flux surface. Hence the force balance equation  $\langle RF_R \rangle = 0$  imposes

$$\frac{d\Delta}{d\rho} = -\frac{\rho}{R_0} \left( \beta_p(\rho) + \frac{l_i(\rho)}{2} \right)$$

The virial theorem can also be used to estimate the vertical field necessary to maintain the plasma equilibrium. If the integration volume  $V$  is chosen to be very large in Eq.(25), all the surface terms can be neglected since the magnetic field decreases as the cubic power of the distance far from the current loop. Hence it is found that  $\langle RF_R \rangle$  is positive, so that no equilibrium can be found. An equilibrium exists if a vertical field is applied. The corresponding average force is

$$\langle RF_R \rangle = V \frac{2R_0}{a} \frac{B_\theta^2(a)}{\mu_0} \frac{B_V}{B_\theta(a)}$$

There exists several ways to calculate this field. One consists in assuming that the toroidal field is discontinuous at the separatrix. Hence the contribution of the toroidal field does no longer vanish. This implies that 3 causes emerge for the onset of the radial force: pressure force, Lorentz force due to the poloidal field and associated toroidal current (this will come in via the plasma inductance), and the Lorentz force due to the toroidal field discontinuity combined with the corresponding poloidal current. The vertical field necessary to maintain the equilibrium is

$$B_V = B_{\theta a} \frac{a}{2R_0} \left[ \ln \left( \frac{8R_0}{a} \right) + \beta_p(a) + \frac{l_i(a)}{2} - \frac{3}{2} \right]$$

## References

- [1] R.D. Hazeltine and J.D. Meiss. *Plasma Confinement*. Dover, 1992.
- [2] A. H. Boozer. Physics of magnetically confined plasmas. *Rev. Mod. Phys.*, 76:1071–1141, 2005.
- [3] V.I. Arnold. *Mathematical Methods of Classical Mechanics*. Springer, 1978.
- [4] J. Nuehrenberg and R. Zille. Quasi-helically symmetric toroidal stellarators. *Physics Letters A*, 129(2):113 – 117, 1988.
- [5] H. Goldstein. *Classical Mechanics*. Addison-Wesley, 1980.
- [6] P.A. Davidson. *An Introduction to Magnetohydrodynamics*. Cambridge University Press, 2001.
- [7] H. Goedbloed. *Magnetohydrodynamics*. Cambridge University Press, 2004.

- [8] L. S. Solov'ev. The theory of hydromagnetic stability of toroidal plasma configurations. *Zh. Eksp. Teor. Fiz.*, 53:626, 1967.
- [9] L. S. Solov'ev. The theory of hydromagnetic stability of toroidal plasma configurations. *Sov. Phys. JETP*, 26:400, 1968.
- [10] R. L. Miller, M. S. Chu, J. M. Greene, Y. R. Lin-Liu, and R. E. Waltz. Noncircular, finite aspect ratio, local equilibrium model. *Physics of Plasmas*, 5(4):973–978, 1998.
- [11] N. Luc C. Mercier. *Report No. EUR-5127e 140*. European Communities, Brussels, 1974.
- [12] C. Mercier. Equilibrium and stability of a toroidal magnetohydrodynamic system in the neighbourhood of a magnetic axis. *Nuclear Fusion*, 4(3):213, 1964.
- [13] J Candy. A unified method for operator evaluation in local grad shafranov plasma equilibria. *Plasma Physics and Controlled Fusion*, 51(10):105009, 2009.
- [14] J. W. Connor and R. J. Hastie. *Report No. CLM-M106*. Culham Laboratory Report, 1985.
- [15] V.D. Shafranov. *Reviews of Plasma Physics*, volume 2. Consultants bureau, New York, 1997.
- [16] R. Fitzpatrick, C. G. Gimblett, and R. J. Hastie. On the '1 1 / 2 -d' evolution of tokamak plasmas in the case of large aspect ratio. *Plasma Physics and Controlled Fusion*, 34(2):161, 1992.
- [17] Per Helander. Theory of plasma confinement in non-axisymmetric magnetic fields. *Reports on Progress in Physics*, 77(8):087001, jul 2014.
- [18] S. P. Hirshman and J. C. Whitson. Steepest descent moment method for three dimensional magnetohydrodynamic equilibria. *The Physics of Fluids*, 26(12):3553–3568, 1983.
- [19] S.P. Hirshman, W.I. van RIJ, and P. Merkel. Three dimensional free boundary calculations using a spectral green's function method. *Computer Physics Communications*, 43(1):143 – 155, 1986.
- [20] J.P. Freidberg. *Plasma Physics and Fusion Energy*. Cambridge University Press, 2007.
- [21] J.P. Freidberg. *Ideal MHD*. Cambridge University Press, 2014.
- [22] A. J. Cerfon and J. P. Freidberg. One size fits all analytic solutions to the grad-shafranov equation. *Physics of Plasmas*, 17(3):032502, 2010.
- [23] R.B. White. *Theory of Tokamak Plasmas*. North Holland, 1989.
- [24] D. Biskamp. *Nonlinear Magnetohydrodynamics*. Cambridge University Press, 1993.