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Matrix and Spectral Similarity Networks

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Abstract

Matrices are fundamentally important in science and technology thanks to their versatility for representing and modeling a wide range of natural and abstract structures and dynamics. A great deal of the properties of matrices stem from their spectral structure, namely the involved eigenvalues and eigenvectors, to the point that the systematic study of eigenvalues distribution (the matrix spectrum) has constituted a subject of great attention. In the present work, we develop an approach to visualizing and characterizing the spectral structure of square, real-valued random matrices in terms of complex networks obtained by comparing the coincidence similarity between the eigenvalues and/or eigenvectors of a given ensemble of matrices. An index to quantify how much a matrix is symmetric, corresponding to coincidence similarity index between the upper and lower triangular parts of a given matrix, is also proposed and employed to complement the study of the considered random matrices. The potential of the approach is illustrated respectively to three types of random matrix ensembles, namely gaussian (Ginibre) ensemble, uniform random matrices, and quasi-symmetric Gaussian random matrices. In all cases, the respective eigenvalues networks revealed an impressive structured heterogeneity of interrelationships between the obtained eigenvalues. Additional results of potential interest are also presented and discussed.

1 Introduction

Thanks to their flexibility for representing several systems and dynamics, matrices (e.g. [1, 2, 3, 4, 5]) are among the most important mathematical structures in science and technology. At the same time, the properties of matrices are to a great extent determined by their respective spectral structures, more specifically their *eigenvalues* and *eigenvectors* (e.g. [6, 7, 8]). For instance, Euclidean spaces, as well as rotation and reflection transformations, are underlain by symmetric matrices, whose eigenvalues can be expressed in terms of real values, while their respective eigenvectors are orthogonal. In addition, the spectral structure of matrices defines the dynamics of respectively associated systems of linear differential equations.

Though much has been learnt about the spectral properties of several types of matrices, the intrinsically complex respective structures implies that much can still be respectively investigated, especially in cases where analytical approaches are not particularly straightforward or even viable, as well as respectively to specific types of deterministic and random matrices. In addition, the particularly ample scope of matrices in science and technology also implied that these structures can be approached from

a wide variety of perspectives. Yet another motivation for complementary approaches to the representation and characterization of matrix properties concerns the possibility to obtain preliminary insights about some problem or aspect of particular interest, which can then be eventually further developed in analytical terms.

Being a generalization of matrices, graphs and networks constitute mathematical structures that are even more general than matrices for representing a wide range of structures and dynamics. One of the main benefits of employing network science for modeling natural and abstract problems include the possibility to effectively represent properties of particular interest in terms of respective *relationships*, expressed as links. It thus becomes possible not only to visualize the studied structures in terms of a representation that emphasized inter-relationships, but also to perform several characterization in terms of respective topological measurements, as well as developing models and simulations of the studied structures and systems.

The present work aims at representing, visualizing, and characterizing the properties of square real-valued matrices from the perspective of *similarities* between their spectral properties, namely eigenvalues and eigenvectors. For instance, given an ensemble of matrices, it would be

possible to represent each matrix as a network node, while the pairwise relationships between the matrices would be defined in terms of some quantification of the similarity between their respective eigenvalues or eigenvectors. It therefore becomes an issue of particular relevance to resource to some similarity comparison approach that can provide enhanced selectivity and sensitivity to structural differences between the supplied set of matrices.

Suggested recently [9, 10, 11], the *coincidence similarity index* provides a particularly strict, in the sense of its enhanced selectivity and sensitivity, means for comparing two non-zero vectors having the same dimensions. In addition, these properties are tunable through a parameter D so that higher values yield more selective and sensitive comparisons. As a consequence of these characteristics, the coincidence similarity index results particularly interesting for situations in which markedly similar vectors need to be compared, which is often found in practice.

The coincidence similarity index also possesses other interesting features such as being naturally normalized within $[0, 1]$ (for comparison of vectors with non-negative entries) or $[-1, 1]$ (for comparison of vectors also with negative entries), while being robust to perturbations of the vector components.

One particularly interesting application of the coincidence similarity index is as a means to transform datasets, where each data element is characterized in terms of a set of respective properties or *features*, into respective graphs or complex networks [12]. Once a dataset is so transformed, the relationships between typically high dimensional data elements can not only be effectively visualized, but also characterized, e.g. in terms of measurements of the topology of the obtained network, and/or modeled.

Thanks to its interesting properties, the coincidence similarity index and the respective networks have been successfully applied to several situations, including non-linear filtering of 3D scalar fields [13], by template matching [10], image segmentation [11], relationships between city structures [14], defining the autorrelation and cross-relation of networks [15], automatic motifs identification [16], enzymes networks [17], neuromorphological networks [18], as well as finding patterns in bipartite networks [19].

More recently [20], coincidence similarity complex networks have been applied as a means to characterize inter-relationships between Jacobian matrices and metric tensors underlying nonlinear vector fields. In a sense, the present work can be understood as a spin off of that previous work, in the sense that here we also resource to coincidence complex networks for matrices characterization, but in terms of the respective spectral structure.

2 Basic Concepts

There are several ways in which a matrix can be understood. One of them is simply as a set of entries organized as a table. In this work, we focus on real-valued square matrices, with dimension $N \times N$. The distribution of the values of the entries of a matrix can be defined in several manners, be it as a consequence of the representation of a given structure or dynamics, or being completely random though adhering to some stochastic model (e.g. uniformly random, or normally distributed).

Another interesting approach to matrix is as the linear combination of several other matrices with the same dimension, so that the properties of the resulting matrix becomes potentially related to the properties of the respective components. One particularly relevant situation related to the case in which each of the component terms of the linear combination consists of a matrix determined by the *external* or *tensor* product of specific vectors.

Yet another manner in which matrix are defined is as having the columns (or lines) corresponding to vectors of a basis underlying some vector space. This type of matrix is then of particular importance in basis transformation and coordinates changes.

Whatever the origin of a matrix, much of its properties stem from the two following aspects: (a) its *diagonal* structure, of which the *identity matrix* can be understood as prototypes; and (b) the types of values taken by the matrix entries (e.g. positive real values, generic real values, complex values, etc.).

Identity matrices can then be understood as being a particular case of diagonal matrices of the type:

$$A = \begin{bmatrix} \beta & 0 & \dots & 0 \\ 0 & \beta & \dots & 0 \\ \dots & & & \dots \\ 0 & 0 & \dots & \beta \end{bmatrix} = \beta \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & & & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix} \quad (1)$$

where $\beta \in \mathbb{R}$. Identity matrices are obtained when $\beta = 1$. Observe that this type of matrix is inherently *symmetric*, i.e. $A = A^T$. Recall that the product of a matrix by its transpose (or vice-versa) necessarily yields a symmetric matrix. In addition, the product of two (or more) symmetric matrices results a symmetric matrix.

An interesting approach that can be used to inter-related matrices consists of the concept of *matrix similarity*. More specifically, two square matrices A and B are said to be *similar* provided a third matrix C can be found so

that:

$$\begin{aligned} B &= C^{-1} A C \\ \text{or} \\ B &= C A C^{-1} \end{aligned}$$

Therefore, matrix B can be understood as corresponding to the linear coordinate change by matrix C of the linear map established by matrix A in the original coordinate system, i.e. the latter matrix acted in the original space S as:

$$\mathbf{y} = A \mathbf{x} \quad (2)$$

When S is linearly transformed to new coordinates \tilde{S} by means of a transformation matrix C , we can write:

$$C \tilde{\mathbf{y}} = A C \tilde{\mathbf{x}} \implies \tilde{\mathbf{y}} = C^{-1} A C \tilde{\mathbf{x}} = B \tilde{\mathbf{x}} \quad (3)$$

Thus, matrix B can be understood to implement, in the new space \tilde{S} , the counterpart of the original linear transformation in space S .

In addition to the above interpretation of similar matrices, they are also of particular interest because A and B will share several of their properties, including their spectral structure.

Given an $N \times N$ matrix A , we can write:

$$A \mathbf{x} = \lambda \mathbf{x} \iff [A - I] \mathbf{x} = 0 \quad (4)$$

where $\lambda \in \mathbb{C}$.

In case a vector \mathbf{x} and a complex scalar λ can be found that satisfies the above equation, the scalar λ and the vector \mathbf{x} are said to constitute an *eigenvalue* and corresponding *eigenvector* of A . A maximum of N pairs of distinct eigenvalues and eigenvectors can be found for an $N \times N$ matrix A . The set of eigenvalues and eigenvectors of a matrix is often said to constitute its *spectral structure*. More specifically, the set of eigenvalues is said to correspond to the *spectrum* of the matrix, constituting a subject of great interest in the characterization and analysis of matrices (e.g. [21, 22, 23, 24, 25, 26, 27]).

The spectral properties of a given $N \times N$ matrix A is strongly dependent on the type of numeric entries and symmetry (diagonal structure) of A . In particular, symmetric matrices have distinct real eigenvalues associated to orthogonal eigenvectors. Thus, in case matrix A has N distinct real eigenvalues, associated to N respective eigenvectors that are mutually orthonormal, it is possible to consider the respective canonical basis and write:

$$\begin{aligned} \Lambda &= \lambda_1 [\mathbf{x}_1]^T \mathbf{x}_1 + \lambda_2 [\mathbf{x}_2]^T \mathbf{x}_2 + \dots + \lambda_N [\mathbf{x}_M]^T \mathbf{x}_N = \\ &= \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & & \dots & \\ 0 & 0 & \dots & \lambda_N \end{bmatrix} \end{aligned} \quad (5)$$

Given a generic invertible matrix C , we can apply the similarity principle to get a new matrix B as:

$$B = C^{-1} \Lambda C \quad (6)$$

which will have the same eigenvalues as matrix A . In the particular case in which C is orthonormal, we have that $C^{-1} = C$, so that the previous equation can be rewritten as:

$$B = C^T \Lambda C \quad (7)$$

Therefore, we have that B is necessarily symmetric. The *spectral theorem* states that a symmetric matrix A can be diagonalized as in the above equation, with Λ corresponding to the eigenvalues, while the matrix C is obtained by taking each respective orthonormalized eigenvector for its rows. In other words, given a symmetric matrix A , which we will understand as performing a linear transformation in a given vector space S , it is possible to find a linear basis change to a new space in which the matrix A becomes simply a diagonal matrix \tilde{A} , which is the same as saying that A and \tilde{A} are similar.

Because of the particular importance of matrix symmetry on the respective spectral properties, it is interesting to consider some measurement capable of quantifying how much a given square matrix with real values is symmetric. Though there are other possible approaches more commonly adopted for this purpose, such as those based on comparing norms, in the present work we resource to a symmetry index that compares the coincidence similarity index (to be described in Section 3) of the two triangular parts of a given matrix A . We will illustrate this method in terms of the following example.

Let the matrix A to have its symmetry index calculated be as follows:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 7 \\ -2 & 5 & 8 & 9 \\ 4 & 8 & 8 & 9 \end{bmatrix}$$

The respective upper and lower triangular parts can be expressed in terms of the following vector:

$$U = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 4 \\ 7 \\ 9 \end{bmatrix}; \quad L = \begin{bmatrix} 2 \\ -2 \\ 5 \\ 4 \\ 8 \\ 8 \end{bmatrix}$$

where special care has been taken in order to implement respective indexing preserving the original symmetry. Now, the coincidence similarity index can be readily obtained as corresponding to:

$$C(U, L) = 0.7241379$$

The maximum coincidence similarity value $\mathcal{C} = 1$ will be obtained if and only the upper and lower triangular parts of the matrix are identical, while the minimum value $\mathcal{C} = -1$ will result whenever one of the triangular parts has the opposite sign of the other.

In the present work, we shall focus on *random matrices* (e.g. [28, 29, 30, 31, 32]), namely matrices that are composed of random entries. Random matrices have been extensively studied and also applied to several interesting problems, including the characterization of complex networks (e.g. [33, 34, 35, 36]).

In particular, we shall consider the three following types of random matrix ensembles: (a) Gaussian or Ginibre) ensemble; (b) uniform random matrices, with uniform probability from the intervals $[0, 1]$ or $[-1, 1]$; and (c) quasi-Gaussian random matrices, which are suggested as a means to take into account the particularly interesting situation in which the matrices present strong, but not perfect, symmetry.

Random matrices have been the subject of several interesting studies, many of which being relatively recent. One of the more systematic approaches was developed by Jean Ginibre (1938–2020) in the 60s (e.g. [37]), focusing on Gaussian random matrices, which is also the *Ginibre ensemble*.

More recently, the *circular law*, was developed (e.g. [38, 39, 40, 41, 42]), which states that the eigenvalues of Gaussian random matrices of dimension $N \times N$ with mean 0 and variance 1 converge, when $N \rightarrow \infty$, to a circular distribution with radius 1, centered at the origin of the Argand plane. Figure 1 illustrates the eigenvalue distributions in the Argand plane respectively to increasing values of N .

Additional results regarding ensembles of random matrices have been reported, including but not limited to [43, 44, 45, 46, 47, 48].

As expected, the densities present a perfect bilateral symmetry respectively to the real axis, which is a direct consequent of the fact that, in the adopted case of real-valued matrices, the complex eigenvalues always take place as conjugated pairs.

Of particular importance for the present work, this convergence takes progressively, with most of the eigenvalues being distributed along the real axis, especially for the smaller values of N . As the same time, the region immediately surrounding the real axis resulted less dense for each N .

In order to further illustrate the variation of eigenvalue density, Figure 2 presents hexagonal histograms of eigenvalue densities for $N = 12, 20$, and 26 . Not only the less dense region around the real axis can be evidently perceived, but also the two halves of the plot also present growing density as one moves from their border toward

their centers.

3 Coincidence Similarity Networks

The coincidence similarity index allows particularly strict quantification of the similarity between two non-zero vectors \mathbf{v}_i and \mathbf{v}_j .

Among additional interesting properties of the coincidence similarity index [10, 49], we have: (i) enhanced selectivity and sensitivity; (ii) normalization in the intervals $[0, 1]$ or $[-1, 1]$; (iii) noticeable tolerance to perturbations of some of the features, as well as invariance to affine and other types of transformations.

In case all features are non-negative, the *Jaccard similarity* between two non-zero feature vectors \mathbf{v}_i and \mathbf{v}_j can be expressed as:

$$\mathcal{J}(\mathbf{v}_i, \mathbf{v}_j) = \frac{\sum_{k=1}^F \min(v_{i,k}, v_{j,k})}{\sum_{k=1}^F \max(v_{i,k}, v_{j,k})} \quad (8)$$

with $0 \leq \mathcal{J}(\mathbf{v}_i, \mathbf{v}_j) \leq 1$.

The *interiority index* (also known as overlap, e.g. [50]) can be written as:

$$\mathcal{I}(\mathbf{v}_i, \mathbf{v}_j) = \frac{\sum_{k=1}^F \min(v_{i,k}, v_{j,k})}{\sum_{k=1}^F \min(v_{i,k}, v_{j,k})} \quad (9)$$

with $0 \leq \mathcal{I}(\mathbf{v}_i, \mathbf{v}_j) \leq 1$.

The *coincidence similarity* between two non-zero feature vectors can now be defined as:

$$\mathcal{C}(\mathbf{v}_i, \mathbf{v}_j) = [\mathcal{J}(\mathbf{v}_i, \mathbf{v}_j)]^D [\mathcal{I}(\mathbf{v}_i, \mathbf{v}_j)] \quad (10)$$

with $0 \leq \mathcal{C}(\mathbf{v}_i, \mathbf{v}_j) \leq 1$, and $D \in \mathbb{R}$.

Higher values of D yield more strict (selective and sensitive) comparisons of the similarity between the compared vectors (or matrices).

Multiset-based generalizations of the above indices to be used in case the feature vectors can have negative, can be expressed as ([10, 11, 9, 12]):

$$\mathcal{J}(\mathbf{v}_i, \mathbf{v}_j) = \frac{\sum_{k=1}^F s_{xy} \min(|v_{i,k}|, |v_{j,k}|)}{\sum_{k=1}^F \max(|v_{i,k}|, |v_{j,k}|)} \quad (11)$$

$$\mathcal{I}(\mathbf{v}_i, \mathbf{v}_j) = \frac{\sum_{k=1}^F \min(|v_{i,k}|, |v_{j,k}|)}{\sum_{k=1}^F \min(|v_{i,k}|, |v_{j,k}|)} \quad (12)$$

$$\text{where: } s_{xy} = \text{sign}(v_{i,k}) \text{sign}(v_{j,k}) \quad (13)$$

with $-1 \leq \mathcal{J}(\mathbf{v}_i, \mathbf{v}_j), \mathcal{I}(\mathbf{v}_i, \mathbf{v}_j), \mathcal{C}(\mathbf{v}_i, \mathbf{v}_j) \leq 1$. Related the Jaccard for negative values, in the context of L1 norm, have also been reported [51, 52].

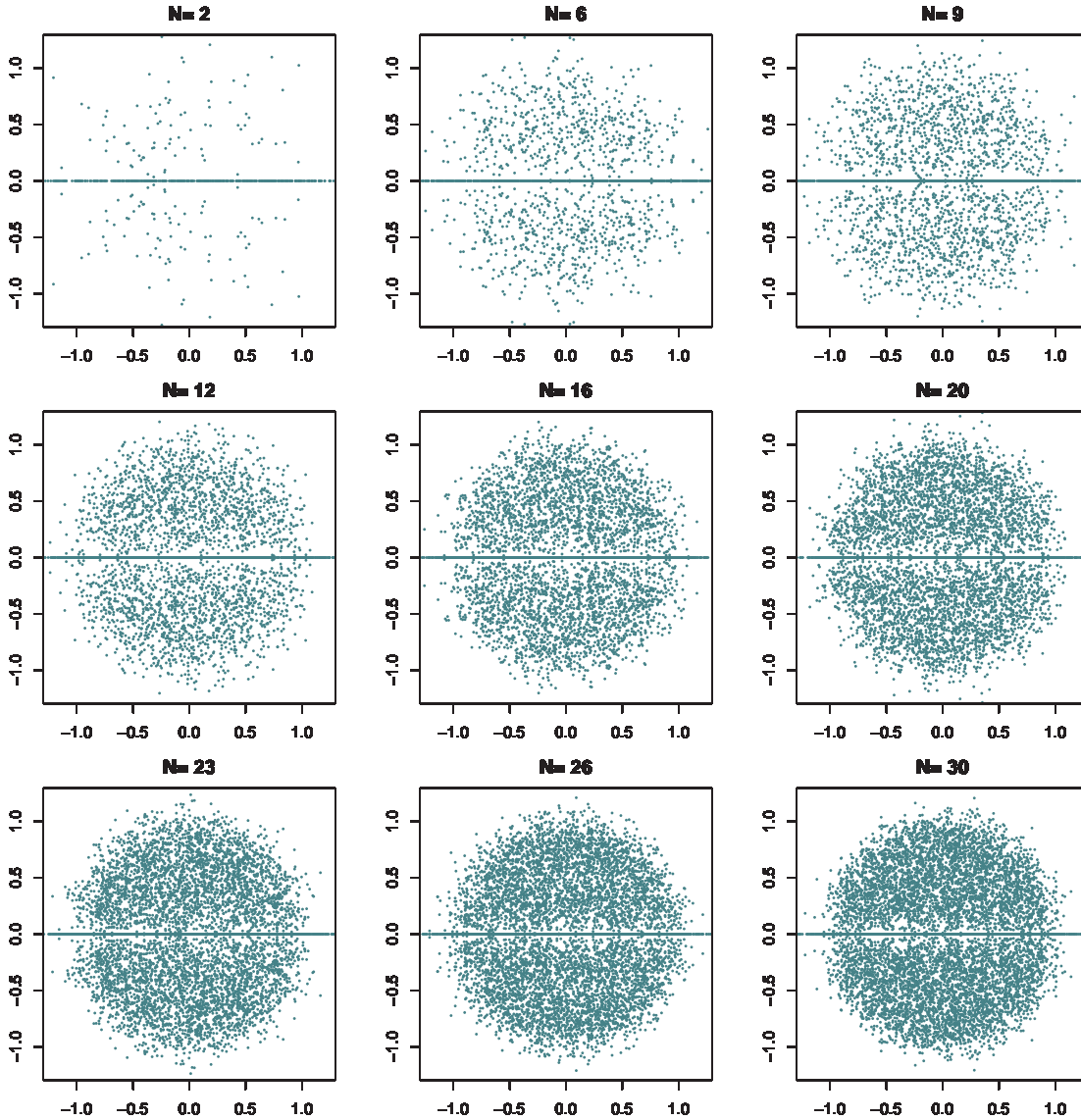


Figure 1: Illustration of the *circular law* of matrix eigenvalues respectively to increasing values of N , considering 50 realizations for each configuration. The number of eigenvalues in each case is thus equal to $50N$. The following three aspects are of particular important to the present work, as they related to symmetry and heterogeneity of the eigenvalues distributions for the considered values of N : (i) the eigenvalues tend to concentrate along the real axis; (ii) for the smaller values of N , the eigenvalues tend to concentrate along the real axis, while the other eigenvalues are bilaterally distributed as a consequence of being complex conjugate; and (iii) lower distribution local density tends to result at the regions that are most adjacent to the real axis, giving rise to “voids”.

4 Matrix Networks

The first type of coincidence network considered in the present work relates to similarity relationships between the entry of square matrices. More specifically, given a set of N_M matrices with the same dimension $N \times N$, each matrix is mapped into a respective node, while the weight of the links between each pair of matrices is taken as corresponding to the respective coincidence similarity between the entries of those matrices.

For instance, consider the three following matrices:

$$\begin{aligned}
 A &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}; & \vec{B} &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 10 & 6 \\ 7 & 8 & 9 \end{bmatrix}; \\
 C &= \begin{bmatrix} 1 & 10 & 3 \\ 10 & 5 & 10 \\ 7 & 10 & 9 \end{bmatrix} & & (14)
 \end{aligned}$$

In order to calculate the respective coincidence similar-

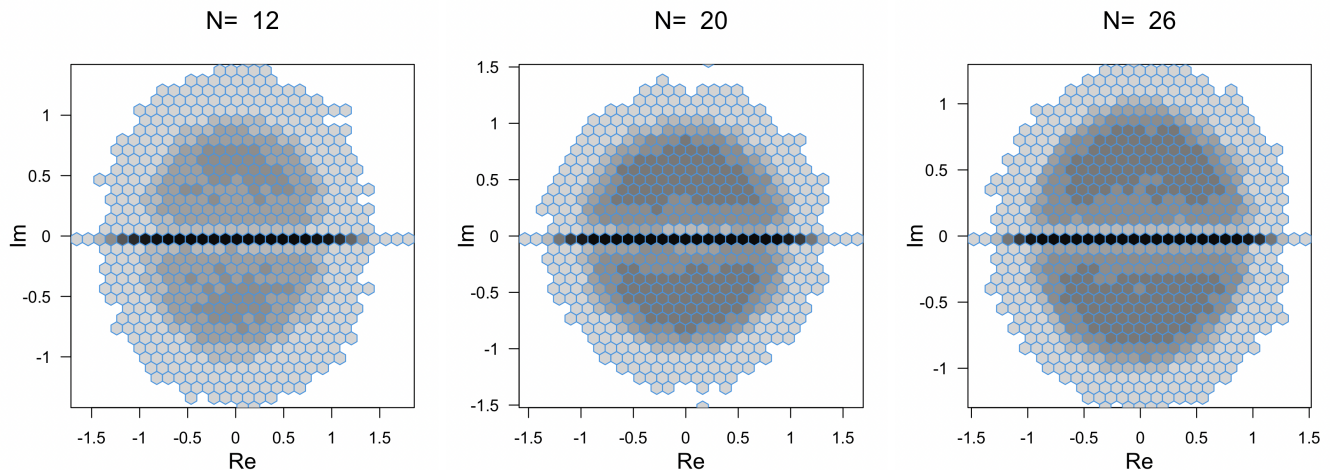


Figure 2: Hexagonal histograms of eigenvalues density obtained for the gaussian random matrix ensemble for $N = 12, 20,$ and 26 . A total of 20000 matrices was taken into account in each case. Not only the less dense region around the real axis can be observed, but each of the two halves (up and down) also present a gradient of density progressing from their borders to their respective centers.

ity index, we first vectorize the two matrices as:

$$\begin{aligned}\vec{A} &= [1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9]; \\ \vec{B} &= [1 \ 2 \ 3 \ 4 \ 10 \ 6 \ 7 \ 8 \ 9]; \\ \vec{C} &= [1 \ 10 \ 3 \ 10 \ 5 \ 10 \ 7 \ 10 \ 9]\end{aligned}$$

and then apply the coincidence similarity index. In the case of this particular example, for $D = 3$, we get:

$$\begin{aligned}\mathcal{C}(\vec{A}, \vec{B}) &= 0.729; \\ \mathcal{C}(\vec{A}, \vec{C}) &= 0.3318161; \\ \mathcal{C}(\vec{B}, \vec{C}) &= 0.2391035\end{aligned}$$

which effectively reflect the similarity between the pairwise comparisons between these three matrices.

Given that in this work we will be restricted to random matrices having elements independent and uniformly drawn in the intervals $[0, 1]$ or $[-1, 1]$, the respectively obtained networks will be mostly regular, not incorporating any specific modularity or patterns in their structure, unless as a consequence of statistical fluctuations.

Figure 3 illustrates a *matrix network* considering 400 random matrices with dimension 4×4 , with entries drawn with uniform probability from the interval $[-1, 1]$. The coincidence similarities were obtained for $D = 3$. As could be expected, a mostly regular network has been obtained, with no particular structure other than a few outlier nodes at the borders of the obtained structure.

5 Eigenvectors Networks

Given an ensemble of matrices, in addition to obtaining coincidence similarity networks between their respective samples, it is also of interest to consider coincidence

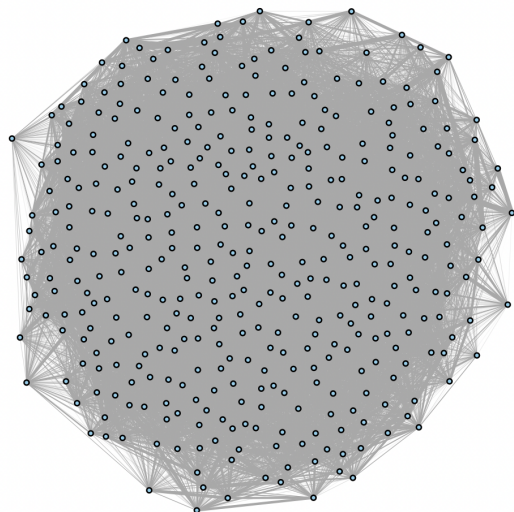


Figure 3: Matrix network considering 400 matrices with dimension 4×4 , whose entries were drawn with uniform probability from the interval $[-1, 1]$, and adopting $D = 3$. As expected, no particular modularity or interconnectivity patterns can be observed, other than a few outlier nodes, along the network border, resulting from statistical fluctuations. As all other network visualizations shown in this work, the result shown in this figure was obtained by using the Fruchterman-Reingold visualization methodology (e.g. [53]).

similarity networks between the respective eigenvectors. Henceforth, we take all eigenvectors normalized to unit magnitude.

In order to provide subsidies for better understanding the obtained networks, we will consider two measurements applicable to each eigenvalue. The first consists of the norm of the eigenvalue. The second corresponds to a quantification of how much an eigenvalue λ is imaginary,

which is calculated as:

$$\xi(\lambda) = \frac{|\text{Im}(\lambda)|}{|\text{Re}(\lambda)| + |\text{Im}(\lambda)|} \quad (15)$$

This measurement is henceforth referred to as the *imaginary index* of a given eigenvalue.

The feature vectors used to obtain the eigenvector networks of a given $N \times N$ matrix consists of the $1 \times 2N^2$ vectors containing the real and imaginary parts of the obtained eigenvectors. The order of these entries is immaterial for the coincidence similarity calculation, provided that it is consistently kept. The zero entries (e.g. in case a matrix has only real eigenvectors) also do not influence the coincidence calculation, provided the compared matrices are not completely zero. The eigenvectors will always be taken after being normalized to unit magnitude.

As an example, let the matrix A and respective eigenvectors as follows:

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix};$$

$$\mathbf{x}_1 = \begin{bmatrix} 0 - \frac{\sqrt{2}i}{2} \\ -\frac{\sqrt{2}}{2} + 0i \end{bmatrix}; \quad \mathbf{x}_2 = \begin{bmatrix} 0 + \frac{\sqrt{2}i}{2} \\ -\frac{\sqrt{2}}{2} + 0i \end{bmatrix}$$

The respective feature vector will be:

$$\mathbf{v} = \left[0 \quad \frac{\sqrt{2}}{2} \quad -\frac{\sqrt{2}}{2} \quad 0 \quad 0 \quad \frac{\sqrt{2}}{2} \quad -\frac{\sqrt{2}}{2} \quad 0 \right]^T$$

Figure 4 depicts the eigenvector network obtained for Gaussian random matrices with $N = 5$. The obtained network is mostly regular, with absence of well-defined modularity of other types of interconnection patterns. Therefore, these matrices will not be further studied in the present work.

6 Eigenvalues Networks: Gaussian Ensembles

The concept of coincidence similarity networks can be readily extended to represent, characterize and model the eigenvalues of specific matrix ensembles. The methodology respectively adopted in the present work consists of obtaining the eigenvalues of the given matrices, and then defining respective feature vectors containing the real part of each eigenvalue followed by the respective imaginary part. Given that all matrices considered in here are real-valued, complex eigenvalues will always occur in conjugated pairs. In addition, we will henceforth take the eigenvalues in decreasing order of magnitude. In case of complex eigenvalues with the same magnitude, they are taken in increasing order of signs along the respective coordinates.

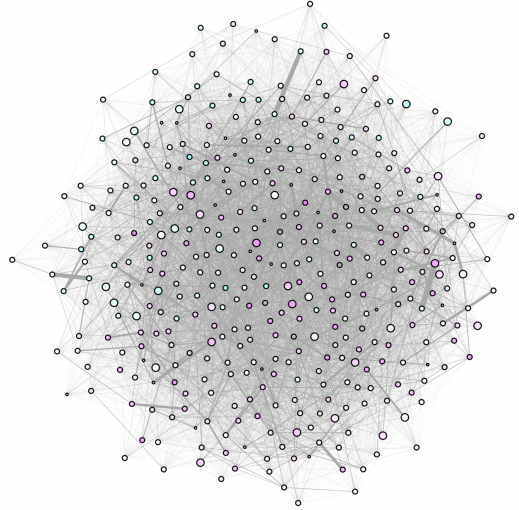


Figure 4: *Eigenvector network* obtained for Gaussian random matrices with $N = 5$. Except for some more intense pairwise interconnections, Nt particular patterns of interconnectivity can be discerned in this mostly regular network. The size and color of nodes correspond, respectively, to the average magnitude and imaginary index of the respectively associated eigenvalues.

Given matrices with dimension $N \times N$, each respective feature vector will have $2N$ entries. In all cases, a total of 400 matrices has been considered for each simulation, therefore yielding $400N$ eigenvalues in each case. Thus, the obtained networks do not reflect the individual eigenvalues, as is typically the case with Argand plane representations, but the properties of the the whole set of respective eigenvalues.

For instance, let the matrix A and respective eigenvalues as follows:

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix};$$

$$\lambda_1 = 1 + 2i$$

$$\lambda_2 = 1 - 2i$$

This matrix will be represented in terms of the following feature vector:

$$\mathbf{v} = \left[1 \quad 2 \quad 1 \quad -2 \right]^T$$

The set of eigenvalues of each matrix are then associated to nodes of the eigenvalues network, while the respective link weights are defined by the respective pairwise coincidence similarity indices. In other words, each considered matrix is represented as a respective node, with feature vectors corresponding to its spectral structure.

Before proceeding to the eigenvalues networks, it is interesting to take into account the fact the relative number R_{im} of matrices with at least one eigenvalue with non-zero imaginary parts increases quick and steadily with N , as

illustrated in Figure 5(a). This ratio can be verified to increase quick and steadily with N .

The ratio r_{im} of eigenvalues with with non-zero imaginary parts is shown in Figure 5(b), presenting an almost linear increase.

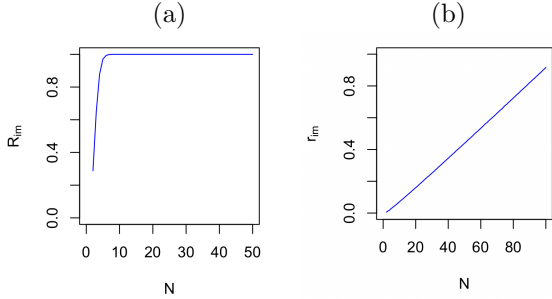


Figure 5: (a) Ratios R_{im} of matrices with at least one eigenvalue with non-zero imaginary part in terms of N , for (b) Ratios r_{im} of eigenvalues with non-zero imaginary parts. The ratios were obtained while considering 1000 matrices.

Figure 6 depicts the eigenvalues networks obtained for $N = 2, 3, 4$, and 5. It should be observed that here, as well as in the remainder of the present work, the values of D have been chosen to allow enhanced visualization of the networks.

A number of interesting results can be readily observed from Figure 6. First, we have that the obtained networks are far from being regular or uniform. In all cases, the networks present a strong bilateral symmetry that, surprisingly, does not reflect the fact that in real-valued matrices all complex eigenvalues are conjugated pairs. Actually, it has been verified that this overall bilateral symmetry is determined by the sign of the real part of the second eigenvalue. This can be taken as an indication that at least some of the other obtained modules are also determined by sign relationships between the real and imaginary portions of the involved eigenvalues.

In addition to the overall bilateral organization, the obtained networks present modules, or clusters, of more interconnected nodes giving rise to an intricate interconnection patterns that can be observed to increase steadily with N . To a good extent, these modules can be verified to involve eigenvalues with similar magnitudes (size of the nodes) and/or imaginary index values (colors, progressing from cyan to magenta/white in the adopted heatmap).

For instance, in the case $N = 2$, the obtained network can be understood to be composed of three main modules, those at the two respective extremities plus a central module. The former two modules can be verified to be almost purely real, so that they are located near the real axis in the respective Argand plot. As one moves from these two external modules into the center of the network, the imaginary index decreases progressively through magenta

and cyan colors, up to the central point of the network. A relatively small change of eigenvalues magnitude can be observed in this case.

A slightly more intricate network can be observed for $N = 3$, which contains a larger number of eigenvalues with non-zero imaginary parts. Here, the eigenvalues with high imaginary index have been mostly split into four groups, those at the extremities of the network, plus another two respectively adjacent smaller modules composed almost exclusively by eigenvalues with high imaginary index values.

A substantially more complex eigenvalue network can be discerned for $N = 4$, with both extremal modules being split into two submodules, while the central module also becomes thus separated. The obtained modules can be observed to involve eigenvalues sharing imaginary index as well as magnitude values.

An even more intricate network has been obtained for $N = 5$, in which case the above observed modules have become further separated.

7 Eigenvalues Networks: Uniform Ensembles

In this section, eigenvalues coincidence networks are obtained and discussed respectively to uniform random matrix ensembles. More specifically, we consider random matrices with all entries draw with uniformly random probability from intervals $[0, 1]$ and $[-1, 1]$.

As with Gaussian matrices, we start by characterizing how the ratio R_{im} of matrices having at least one eigenvalue with non-zero imaginary parts and the ration r_{im} of number of eigenvalues with non-zero imaginary parts. Figures 7 and 8 show these two ratios respectively to the two considered intervals, with results that are closely similar to those obtained respectively to Gaussian random matrices.

In order to complement the analysis above, Figure 8 presents the ratio of *total* eigenvalues with non-zero imaginary parts. Interestingly, almost identical and nearly linear increases can be observed in both cases.

Figure 9 presents the eigenvalues coincidence networks obtained for the two considered types of uniform random matrices respectively to $N = 2$. The network obtained for matrices entries drawn from the interval $[0, 1]$ resulted markedly distinct from that obtained while considering the interval $[-1, 1]$, with the former network involving a single module that is, however, characterized by an axis along which eigenvalues with decreasing magnitudes (network sizes) can be observed. Observe that no eigenvalues with non-zero imaginary parts result for $N = 2$ when the matrices entries are drawn from the strictly non-negative

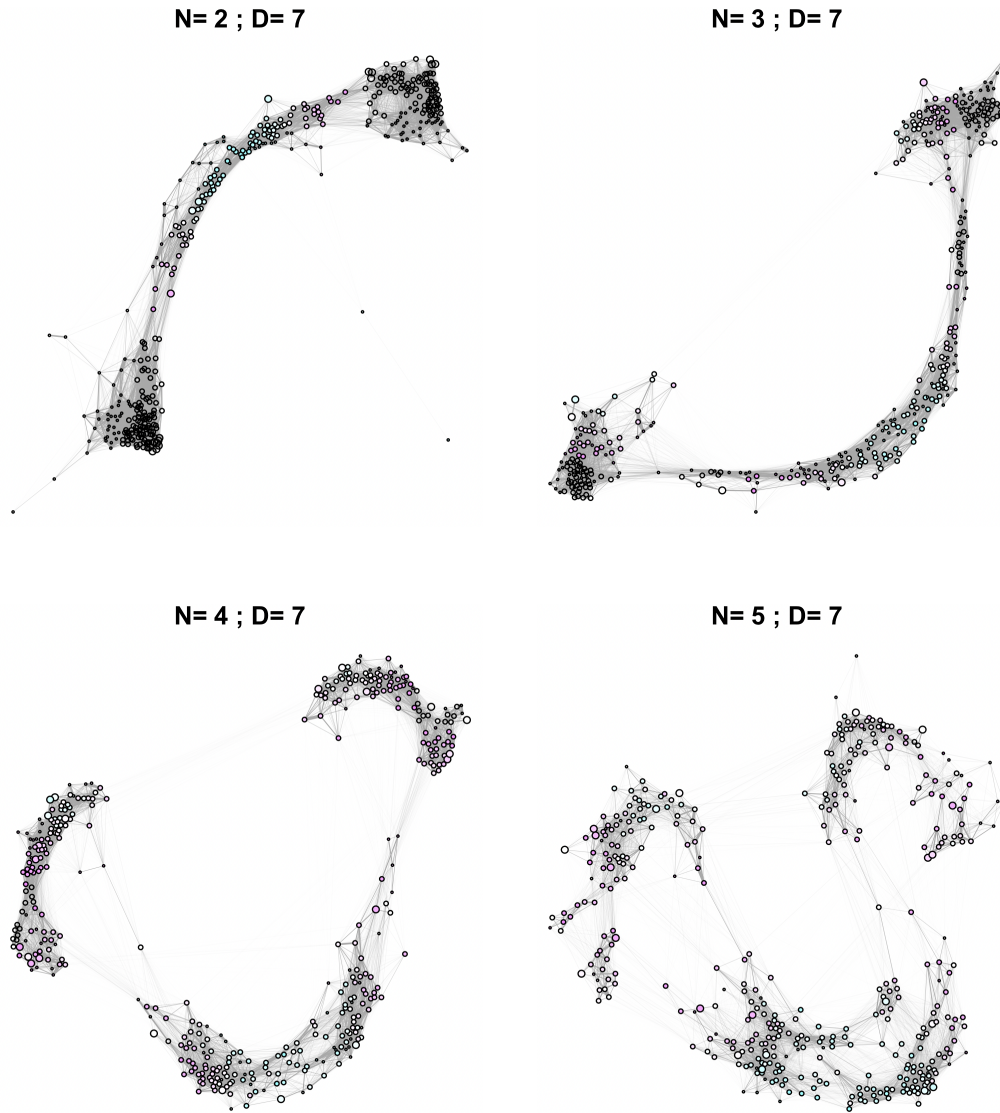


Figure 6: *Eigenvalues networks* obtained for Gaussian random matrices respectively to several values of N . The coincidence similarity index was applied with $D = 7$, and 400 matrices were considered in each case. Each node corresponds to the real and imaginary parts of all eigenvalues respective matrix, and the pairwise links indicate the similarity between these sets of eigenvalues. The size of the nodes reflects the average magnitude of respective eigenvalues, and the color of each node indicates the values of the imaginary indices. The dominant bilateral symmetry obtained for each N is a direct consequence of the eigenvalues appearing as complex conjugates. At the same time, intricate patterns of interconnectivity can be observed that are related to the specific properties of the eigenvalues, especially the sign of the real part of the second eigenvalue.

interval $[0, 1]$. Contrariwise, a network that closely resembles the respective Gaussian counterpart can be observed for the interval $[-1, 1]$. The increasingly more modular network obtained in that case is a direct consequence of the fact that a larger percentage of eigenvalues with non-zero imaginary parts result for that interval.

The interrelationships between the eigenvalues obtained for uniform random matrices for $N = 3$ are shown in terms of the respective eigenvalue coincidence network depicted in Figure 10, respectively to the intervals $[0, 1]$ and $[-1, 1]$. Though still constituting a single module with relatively uniform interconnections, the eigenvalue

coincidence network obtained for the interval $[0, 1]$ resulted more structured than in the previous case respective to $N = 2$. A bilateral symmetry of interconnections can now be noticed, while nodes with similar average magnitude and imaginary index values tended to interconnect more intensely. The eigenvalue coincidence network obtained for the interval $[-1, 1]$ is mostly similar to the respective Gaussian counterpart.

The eigenvalue coincidence networks obtained for $N = 4$ and 5 are presented, respectively, in Figure 11 and 12. The networks obtained considering the interval $[0, 1]$ are progressively more modular, with interconnections taking

8 Eigenvalues Networks: Quasi-Symmetric Gaussian Ensembles

The third and last type of random matrix ensemble to be studied from the perspective of eigenvalues networks consists of what we shall call *quasi-symmetric matrices*, meaning matrices with respectively strong symmetry, which influences markedly the respective spectral properties. This type of matrices, which has special importance in several theoretical and applied problems, are rarely obtained in the two types of ensembles considered in the previous sections because the ample stochasticity of those models will rarely lead to symmetric entries. Therefore, it becomes of special interest to consider eigenvalue coincidence networks of quasi-symmetric ensembles, which constitutes the subject to be developed in the present section.

The specific model to be adopted in this section for generating this type of matrix consists of starting with a symmetric gaussian random matrix and then adding some relatively small perturbation to the off-diagonal elements. In particular, we shall consider these perturbations to be random values uniformly distributed in the interval $[-1, 1]$. More specifically, let A be a symmetric gaussian random matrix with dimension $N \times N$. The respective quasi-symmetric version, for perturbation intensity α , can be obtained as:

$$A_\alpha = A + \alpha R(N, N) \quad (16)$$

where $R(N, N)$ is an $N \times N$ matrix with entries corresponding to random values uniformly distributed in the interval $[-1, 1]$.

Quasi-symmetric ensembles of other types of random matrices (e.g. randomly uniform) can also be obtained from the above approach.

Figure 13 illustrates the distributions in the Argand plane of the eigenvalues distributions obtained for this type of matrix respectively to several values of N .

As could be expected, the overall symmetry of these quasi-gaussian matrices, as gauged by using the symmetry index suggested in the present work, decrease steadily as N increases, which is illustrated in Figure 14.

Figure 15 depicts the eigenvalue coincidence networks obtained for quasi-Gaussian random matrices for increasing values of N , obtained for $\alpha = 0.2$. Interestingly, except for the marked overall bilateral symmetry, all obtained networks are substantially distinct from the respective counterparts obtained in the case of the Gaussian and uniform ensembles.

More specifically, in all the four considered cases, two “satellite” dense communities of nodes can be identified. Interestingly, the separation between these two modules

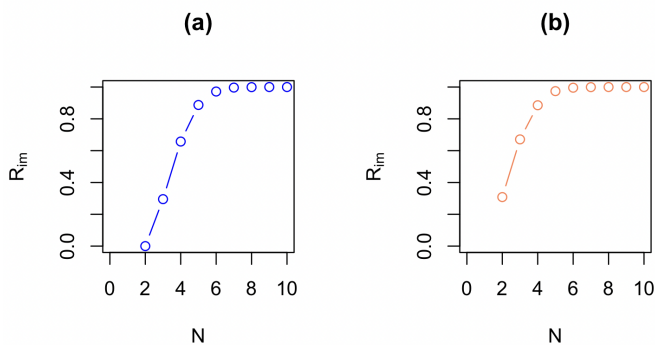


Figure 7: Ratios R_{im} of matrices with at least one eigenvalue with non-zero imaginary parts in terms of N , for matrices of dimension $N \times N$: (a) matrix elements drawn from interval $[0, 1]$; (b) elements drawn from $[-1, 1]$. The ratios were obtained while considering 5000 matrices.

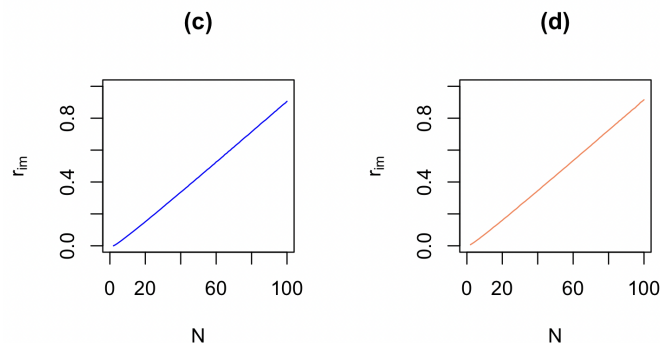


Figure 8: Ratios r_{im} of total eigenvalues with non-zero imaginary parts in terms of N , for matrices of dimension $N \times N$: (a) matrix elements drawn from interval $[0, 1]$; (b) elements drawn from $[-1, 1]$. The ratios were obtained while considering 5000 matrices. The ratios were obtained while considering 1000 matrices.

into account the average magnitude and imaginary index values of the respective matrices. At the same time, the eigenvalue coincidence networks obtained for the interval $[-1, 1]$ are mostly similar to the respective Gaussian counterparts discussed in the previous section.

Though the eigenvalue coincidence networks obtained for entries drawn from the intervals $[0, 1]$ and $[-1, 1]$ presented modularity and interconnection intricacy that increased with N , the networks obtained for the second interval are characterized by a higher level of detailed interconnections and modularity, which is a direct consequence of the wider range of eigenvalues with non-zero imaginary parts, which contribute to the specificity of the network connectivity.

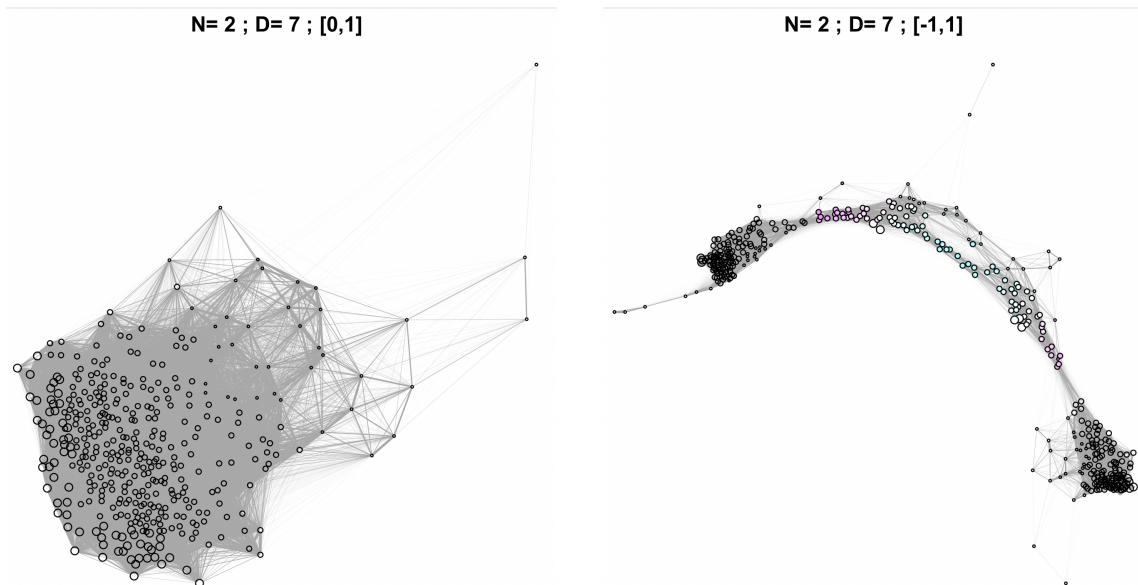


Figure 9: Matrix network considering 400 matrices with dimension 2×2 , whose entries were drawn with uniform probability from the intervals $[0, 1]$ and $[-1, 1]$, and adopting $D = 7$. Two completely distinct eigenvalue coincidence networks have been respectively obtained. Please refer to the text for a discussion.

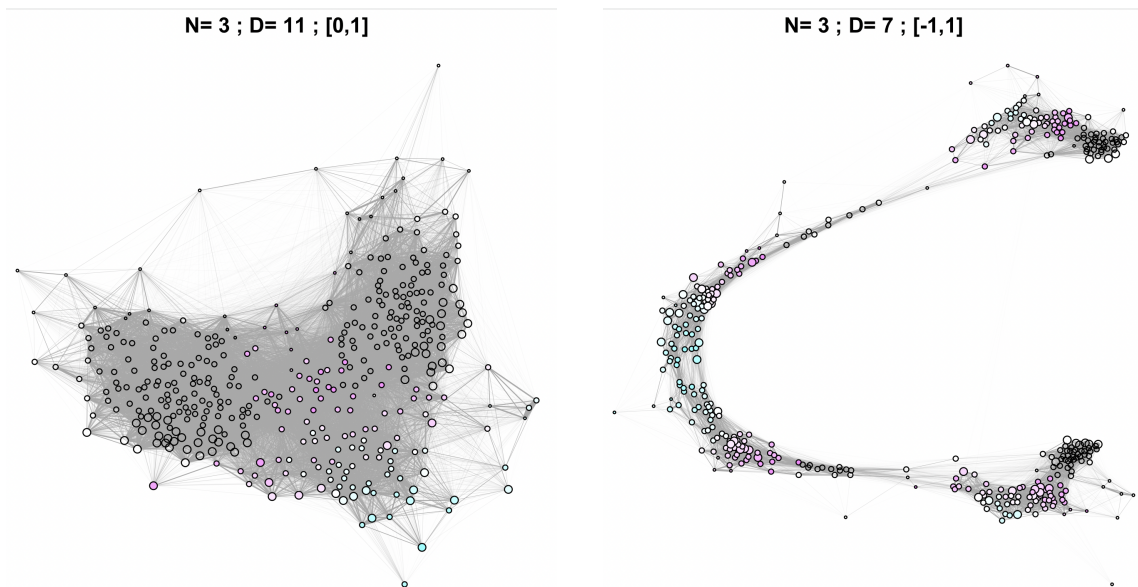


Figure 10: Matrix network considering 400 matrices with dimension 3×3 , whose entries were drawn with uniform probability from the intervals $[0, 1]$ and $[-1, 1]$, and adopting $D = 11$ and $D = 7$. Completely distinct eigenvalue coincidence networks have been respectively obtained. Please refer to the text for a discussion.

tend to become less intense as N increases. At the same time, secondary modules start appear which are adjacent to the two main satellites. Also of interest is the fact that the average magnitude of the eigenvalues, as well as their imaginary index values, seem to play a less important role in defining the local interconnectivity between the considered matrices.

9 Concluding Remarks

Matrices and their spectral structure plays a fundamental role in science and technology. In particular, the eigenvalues and eigenvectors of a given matrix determines to a great extent, and often completely, the structure and/or dynamics of the respectively modeled entities.

While several interesting approaches have employed random matrices to better understand complex networks,

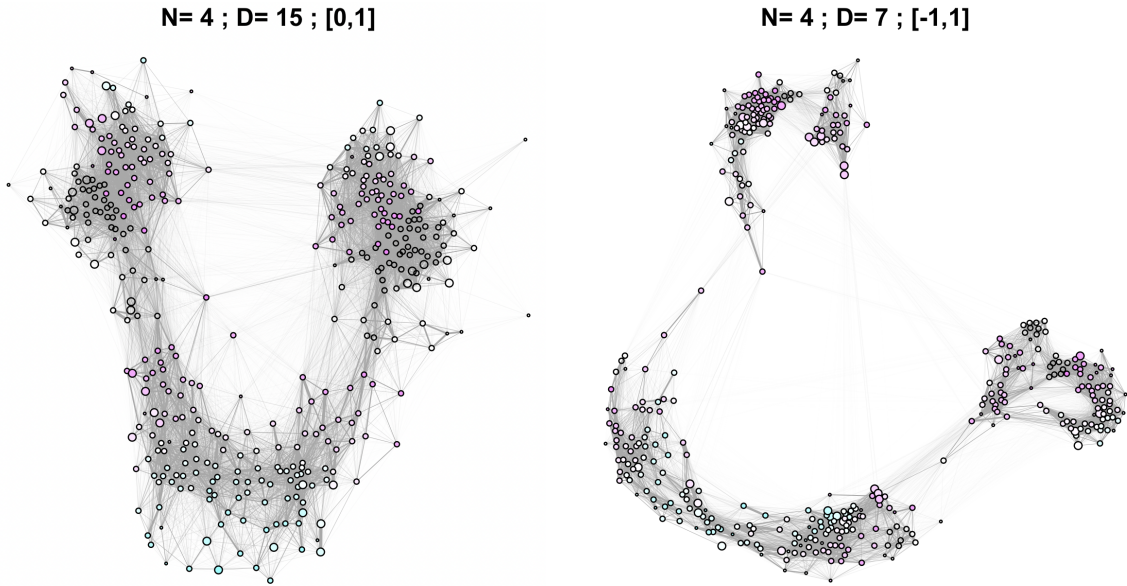


Figure 11: Matrix network considering 400 matrices with dimension 4×4 , whose entries were drawn with uniform probability from the intervals $[0, 1]$ and $[-1, 1]$, and adopting $D = 15$ and $D = 7$. Two completely distinct eigenvalue coincidence networks have been respectively obtained. Please refer to the text for a discussion.

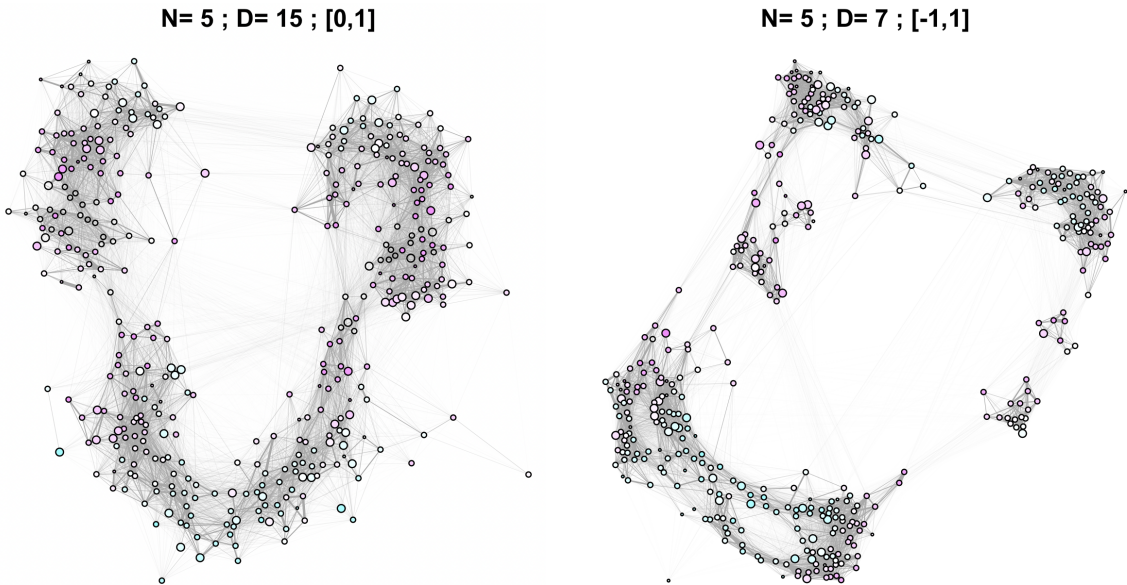


Figure 12: Matrix network considering 400 matrices with dimension 5×5 , whose entries were drawn with uniform probability from the intervals $[0, 1]$ and $[-1, 1]$, and adopting $D = 15$ and $D = 7$. Two completely distinct eigenvalue coincidence networks have been respectively obtained. Please refer to the text for a discussion.

the present work has considered the possibility of using coincidence similarity complex networks for better understanding random matrices. Analogously to matrices, complex networks are also characterized by great versatility for representing, characterizing, and modeling real-world and abstract structures and dynamics. In a sense, matrices are particular cases of graphs, being associated to the concept of lattices.

More specifically, the main principle underlying the present work consisted in representing interrelationships between real-value, square matrices, in terms of complex networks where each matrix, or its spectral properties, were mapped into a respective network, while the interconnections were defined by the respective pairwise coincidence similarity index values. Introduced recently, coincidence similarity networks are characterized by being

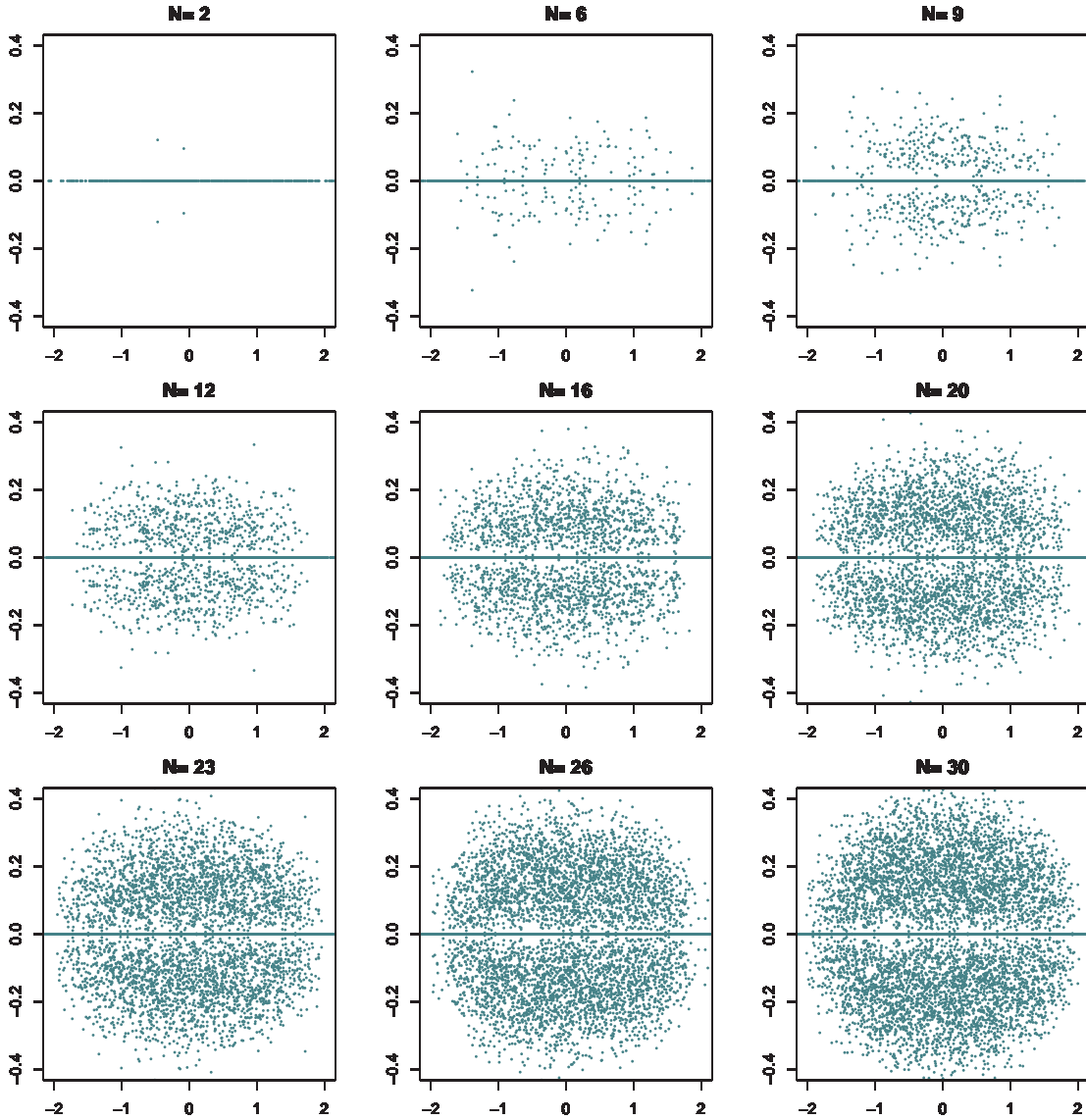


Figure 13: The distribution of eigenvalues of quasi-Gaussian matrices in the Argand plane. As with gaussian and uniform random matrices, the region adjacent to the real-axis is again characterized by smaller local density of eigenvalues. Observe the different scales in the real and imaginary axes.

capable of implementing particularly strict quantifications of the similarity between two given feature vectors, in the sense of being strongly selective and sensitive to the respective feature differences. This capability, which can be controlled by the parameter D , allows the coincidence similarity index to be effectively employed to derive respective coincidence similarity networks characterized by enhanced levels of details and modularity.

Three types of coincidence similarity networks have been considered in the present work respectively to the characterization of the interrelationship between matrices, namely the Gaussian (or Ginibre), uniform, and quasi-symmetric Gaussian ensembles. Several results have been reported and discussed, including:

- The number of matrices with at least one eigenvalue with non-zero imaginary part increases abruptly with N in all considered cases;
- The number of eigenvalues with eigenvalue with non-zero imaginary part increases almost linearly with N in all considered cases;
- The distribution of eigenvalues in the Argand plane confirmed the existence of two major clusters presenting perfect bilateral symmetry around the real axis, both of which presenting increasing eigenvalue densities from their borders to their centers;
- An index for quantifying the symmetry of a given matrix has been described, based on the coincidence

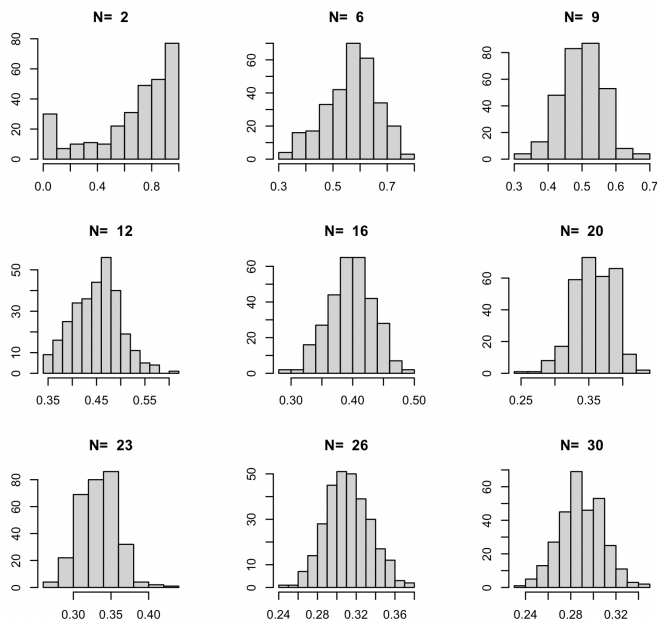


Figure 14: Histograms of values of the symmetry index, described in Section 2, obtained for quasi-Gaussian matrices respectively to several values of N . Observe that, as could be expected, the symmetry of the matrices decreases steadily with N , as larger matrices can be more strongly influenced by the random perturbations.

similarity index taken between the two tridiagonal parts of the matrix;

- A quasi-symmetric Gaussian ensemble of random matrix has been considered that involves almost symmetric matrices with uniformly random fluctuations added to their off-diagonal elements with intensity α . These matrices are of particular interest given that several situations often involve nearly symmetric matrices, which tend to have a higher concentration of purely real eigenvalues, or eigenvalues with relatively small imaginary parts;
- Thanks to its enhanced selectivity and sensitivity, coincidence similarity networks could reveal intricate patterns of interrelationships between the eigenvalues of matrices of three types of random matrices ensembles;
- At least for the considered configurations and features, the matrix and eigenvector networks resulted mostly uniform, with little or no interconnectivity specificities;
- Similar eigenvalues coincidence networks were observed, for the considered matrix dimensions, for both Gaussian and uniform random matrices with entries drawn from the interval $[-1, 1]$;
- In all the three considered random matrix ensembles, a dominant bilateral symmetry has been observed

which is not a consequence of the complex eigenvalues appearing as complex conjugates, reflecting instead the sign of the real part of the second eigenvalue, which have been taken in decreasing order of respective magnitude;

- The modularity and intricacy of the interconnectivity patterns resulting in the eigenvalue networks were observed to increase with N , since larger matrices account for larger types of symmetries and combinations of correlated eigenvector features;
- In the case of uniform random matrices, the eigenvalue coincidence networks resulted markedly less modular and intricate when the respective matrix entries were drawn from the interval $[0, 1]$ than when considering the interval $[-1, 1]$, which is a consequence of the former type of networks having less eigenvalues with non-zero imaginary parts;
- The eigenvalue coincidence networks obtained for the quasi-symmetric Gaussian ensemble resulted markedly distinct from their respective Gaussian counterparts, being relatively less intricate and modular;
- The observed interconnectivity patterns of the eigenvalue networks could only be obtained by considering the whole set of eigenvalues for each considered matrices, which cannot be discerned in the more traditional Argand plots of the eigenvalues. It is the specific structure of the sequence of eigenvalues that determine much of the observed network modularity and intricate interconnecting patterns;

The described concepts, methods and results pave the way to several subsequent works. Regarding the suggested symmetry index, it would be interesting to complement its characterization by taking into account also the structure of the main diagonal, which also plays an important role in determining the eigenvalues structure. Of particular interest concerning the eigenvalues networks would be to further study the obtained patterns in the light of several topological measurements, as well as by employing community finding methods, so that each community of eigenvalues could be separately characterized. In addition, It would be interesting to consider other ensembles of random networks, including cases involving modularity and non-homogeneous interconnections. It would also be interesting to consider applications of the reported concepts and methods to problems involving the characterization of interrelationships between sets of experimental matrices. Yet another possibility would be to apply the concepts of autorrelation and cross-relation [15] as a

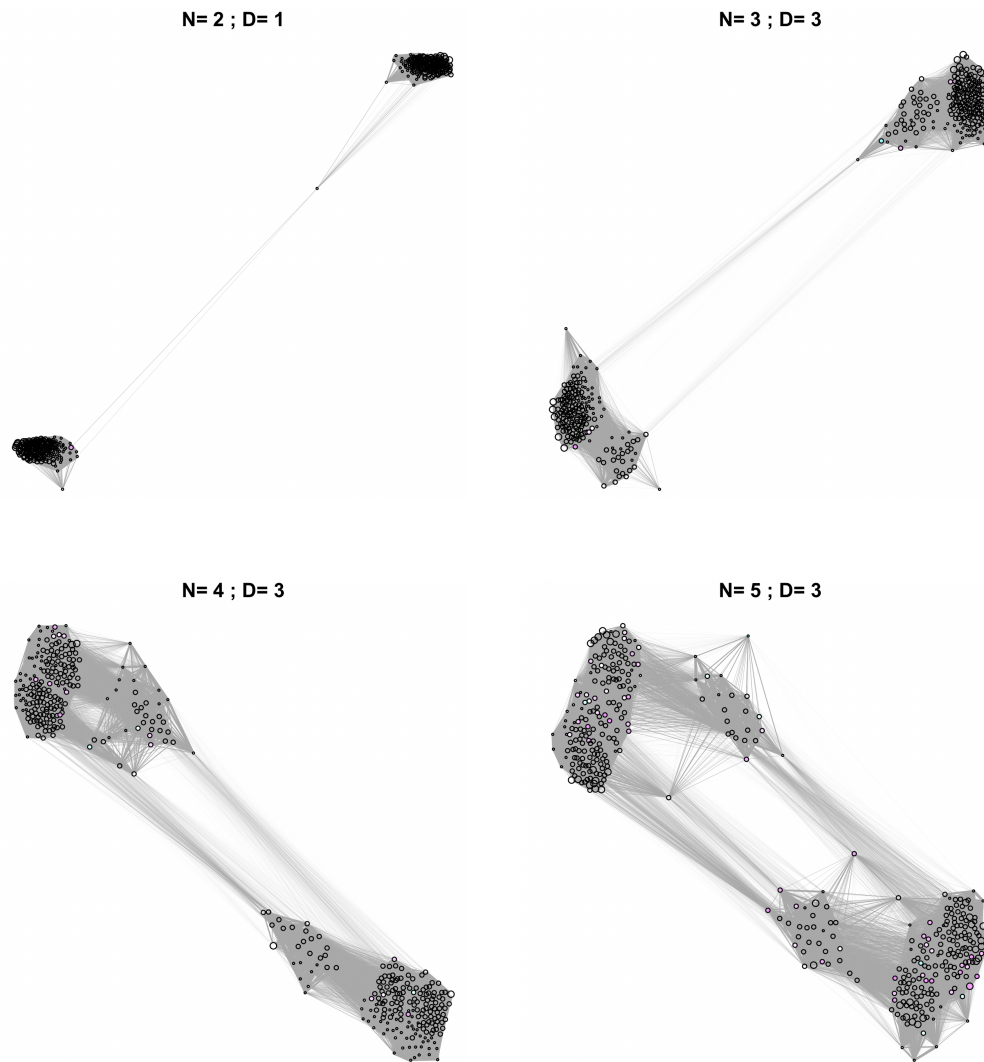


Figure 15: *Eigenvalues networks* obtained for quasi-Gaussian random matrices respectively to several values of N . The coincidence similarity index was applied with $D = 1$ or $D = 3$, and 400 matrices were considered in each case. Please refer to the text for more information and respective discussion.

means of obtaining even more detailed eigenvalues networks.

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Observations

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