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To cite this version:

Shu Shen, Jianqing Yu. Flat vector bundles and analytic torsion on orbifolds. Communications in Analysis and Geometry, 2022, 30 (3), pp.575-656. 10.4310/CAG.2022.v30.n3.a3 . hal-03974197

HAL Id: hal-03974197

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Submitted on 5 Feb 2023

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Communications in
Analysis and Geometry
Volume 00, Number 0, 1–85, 0000

Flat vector bundles and analytic torsion on orbifolds

Shu Shen and Jianqing Yu

arXiv:1704.08369v3 [math.DG] 9 Sep 2020

This article is devoted to a study of flat orbifold vector bundles. We construct a bijection between the isomorphic classes of proper flat orbifold vector bundles and the equivalence classes of representations of the orbifold fundamental groups of base orbifolds. We establish a Bismut-Zhang like anomaly formula for the Ray-Singer metric on the determinant line of the cohomology of a compact orbifold with coefficients in an orbifold flat vector bundle. We show that the analytic torsion of an acyclic unitary flat orbifold vector bundle is equal to the value at zero of a dynamical zeta function when the underlying orbifold is a compact locally symmetric space of reductive type, which extends one of the results obtained by the first author for compact locally symmetric manifolds.

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Introduction

Orbifolds were introduced by Satake [43] under name of V-manifold as manifolds with quotient singularities. They appear naturally, for example, in the geometry of 3-manifolds, in the symplectic reduction, in the problems on moduli spaces, and in string theory, etc.

It is natural to consider the index theoretic problem and the associated secondary invariants on orbifolds. Satake [44] and Kawasaki [27, 28] extended the classical Gauss-Bonnet-Chern Theorem, the Hirzebruch signature Theorem and the Riemann-Roch-Hirzebruch Theorem. For the secondary invariants, Ma [31] studied the holomorphic torsions and Quillen metrics associated with holomorphic orbifold vector bundles, and Farsi [18] introduced an orbifold version eta invariant and extended the Atiyah-Patodi-Singer Theorem. In this article, we study flat orbifold vector bundles and the associated secondary invariants, i.e., analytic torsions or more precisely Ray-Singer metrics.

Let us recall some results on flat vector bundles on manifolds. Let Z be a connected smooth manifold, and let F be a complex flat vector bundle on Z. Equivalently, F can be obtained via a complex representation of the fundamental group $\pi_1(Z)$ of Z, which is called the holonomy representation. Denote by $H^{\cdot}(Z, F)$ the cohomology of the sheaf of locally constant sections of F.

Assume that Z is compact. Given metrics g^{TZ} and g^F on TZ and F, the Ray-Singer metric [42] on the determinant line λ of $H^{\cdot}(Z, F)$ is defined by the product of the analytic torsion with an L^2-metric on λ.

If g^F is flat, or equivalently if the holonomy representation is unitary, then the celebrated Cheeger-Müller Theorem [13, 38] tells us that in this

case the Ray-Singer metric coincides with the so-called Reidemeister metric [41], which is a topological invariant of the unitarily flat vector bundles constructed with the help of a triangulation on Z. Bismut-Zhang [9] and Müller [39] simultaneously considered generalizations of this result. In [39], Müller extended it to the case where g^F is unimodular or equivalently the holonomy representation is unimodular. In [9], Bismut and Zhang studied the dependence of the Ray-Singer metric on g^{TZ} and g^F. They gave an anomaly formula [9, Theorem 0.1] for the variation of the logarithm of the Ray-Singer metric on g^{TZ} and g^F as an integral of a locally calculable Chern-Simons form on Z. They generalized the original Cheeger-Müller Theorem to arbitrary flat vector bundles with arbitrary Hermitian metrics [9, Theorem 0.2]. In [10], Bismut and Zhang also considered the extensions to the equivariant case. Note that both in [9, 10], the existence of a Morse function whose gradient satisfies the Smale transversality condition [48, 49] plays an important role.

From the dynamical side, motivated by a remarkable similarity [35, Section 3] between the analytic torsion and Weil's zeta function, Fried [22] showed that, when the underlying manifold is hyperbolic, the analytic torsion of an acyclic unitarily flat vector bundle is equal to the value at zero of the Ruelle dynamical zeta function. In [23, p.66 Conjecture], he conjectured similar results hold true for more general spaces.

In [47], following the early contribution of Fried [22] and Moscovici-Stanton [37], the author showed the Fried conjecture on closed locally symmetric manifolds of the reductive type. The proof is based on Bismut's explicit semisimple orbital integral formula [5, Theorem 6.1.1]. (See Ma's talk [32] at Séminaire Bourbaki for an introduction.)

In this article, we extend most of the above results to orbifolds. Now, we will describe our results in more details and explain the techniques used in the proof.

0.1. Orbifold fundamental group and holonomy representation

Let Z be a connected orbifold with the associated groupoid $\mathcal{G}$. Following Thurston [50], let X be the universal covering orbifold of Z with the deck transformation group Γ, which is called orbifold fundamental group of Z. Then, $Z = \Gamma\backslash X$. In an analogous way as in the classical homotopy theory of ordinary paths on topological spaces, Haefliger [26] introduced the $\mathcal{G}$-paths and their homotopy theory. He gave an explicit construction of X and Γ following the classical methods.

If F is a complex proper flat orbifold vector bundle of rank r, in Section 2, we constructed a parallel transport along a $\mathcal{G}$-path. In this way, we obtain a representation $\rho : \Gamma \to \mathrm{GL}_r(\mathbf{C})$ of Γ, which is called the holonomy representation of F. Denote by $\mathcal{M}_r^{\mathrm{pr}}(Z)$ the isomorphic classes of complex proper flat orbifold vector bundles of rank r on Z, and denote by $\mathrm{Hom}(\Gamma, \mathrm{GL}_r(\mathbf{C}))/_\sim$ the equivalence classes of complex representations of Γ of dimension r. We show the following theorem.

Theorem 0.1. *The above construction descends to a well-defined bijection*

$$\mathcal{M}_r^{\mathrm{pr}}(Z) \simeq \mathrm{Hom}\big(\Gamma, \mathrm{GL}_r(\mathbf{C})\big)/_\sim. \tag{0.1}$$

The difficulty of the proof lies in the injectivity, which consists in showing that F is isomorphic to the quotient of $X \times \mathbf{C}^r$ by the Γ-action induced by the deck transformation on X and by the holonomy representation on $\mathbf{C}^r$. Indeed, applying Haefliger's construction, in subsection 2.5, we show directly that the universal covering orbifold of the total space of F is $X \times \mathbf{C}^r$. Moreover, its deck transformation group is isomorphic to Γ with the desired action on $X \times \mathbf{C}^r$.

We remark that on the universal covering orbifold there exist non trivial and non proper flat orbifold vector bundles. Thus, Theorem 0.1 no longer holds true for non proper orbifold vector bundles.

On the other hand, for a general orbifold vector bundle E which is not necessarily proper, there exists a proper subbundle E^{pr} of E such that

$$C^\infty(Z, E) = C^\infty\left(Z, E^{\mathrm{pr}}\right). \tag{0.2}$$

Moreover, if E is flat, E^{pr} is also flat. For a Γ-space V, we denote by V^Γ the set of fixed points in V. By Theorem 0.1 and (0.2), we get:

Corollary 0.2. *For any (possibly non proper) flat orbifold vector bundle F on a connected orbifold Z, there exists a representation of the orbifold fundamental group $\rho : \Gamma \to \mathrm{GL}_r(\mathbf{C})$ such that*

$$C^\infty(Z, F) = C^\infty\left(X, \mathbf{C}^r\right)^\Gamma. \tag{0.3}$$

By abuse of notation, in this case, although ρ is not unique, we still call ρ a holonomy representation of F.

Waldron informed us that in his PhD thesis [52] he proved Theorem 0.1 in a more abstract setting using differentiable stacks.

0.2. Analytic torsion on orbifolds

Assume that Z is a compact orbifold of dimension m. Let ΣZ be the strata of Z, which has a natural orbifold structure. Write $Z \coprod \Sigma Z = \coprod_{i=0}^{l_0} Z_i$ as a disjoint union of connected components. We denote $m_i \in \mathbf{N}$ the multiplicity of Z_i (see (2.18)). Let F be a complex flat orbifold vector bundle on Z. Let λ be the determinant line of the cohomology $H^{\cdot}(Z, F)$ (see (4.16)).

Let g^{TZ} and g^F be metrics on TZ and F. Denote by $\square^Z$ the associated Hodge Laplacian acting on the space $\Omega^{\cdot}(Z, F)$ of smooth forms with values in F. By the orbifold Hodge Theorem [14, Proposition 2.1], we have the canonical isomorphism

$$H^{\cdot}(Z, F) \simeq \ker \square^Z. \tag{0.4}$$

As in the case of smooth manifolds, by [27] (or by the short time asymptotic expansions of the heat trace [31, Proposition 2.1]), the analytic torsion $T(F)$ is still well-defined. It is a real positive number defined by the following weighted product of the zeta regularized determinants

$$T(F) = \prod_{i=1}^{m} \det \left(\square^Z|_{\Omega^i(Z,F)} \right)^{(-1)^i i/2}. \tag{0.5}$$

Let $|\cdot|_\lambda^{\mathrm{RS},2}$ be the L^2-metric on λ induced by g^{TZ}, g^F via (0.4). The Ray-Singer metric on λ is then given by

$$\|\cdot\|_\lambda^{\mathrm{RS}} = T(F)|\cdot|_\lambda^{\mathrm{RS}}. \tag{0.6}$$

We remark that as in the smooth case, if Z is of even dimension and orientable, if F is unitarily flat, in Proposition 4.6, we show that $T(F) = 1$.

In Section 4, we study the dependence of $\|\cdot\|_\lambda^{\mathrm{RS},2}$ on g^{TZ} and g^F. To state our result, let us introduce some notation. Let (g'^{TZ}, g'^F) be another pair of metrics. Let $\|\cdot\|_\lambda'^{\mathrm{RS},2}$ be the Ray-Singer metric for (g'^{TZ}, g'^F). Let ∇^{TZ} and ∇'^{TZ} be the respective Levi-Civita connections on TZ for g^{TZ} and g'^{TZ}. Denote by $o(TZ)$ the orientation line of Z. Consider the Euler form $e(TZ, \nabla^{TZ}) \in \Omega^m(Z, o(TZ))$ and the first odd Chern form $\frac{1}{2}\theta(\nabla^F, g^F) = \frac{1}{2}\operatorname{Tr}[(g^F)^{-1}\nabla^F g^F] \in \Omega^1(Z)$. Denote by

$$e\left(Z_i, \nabla^{TZ_i}\right) \in \Omega^{\dim Z_i}(Z_i, o(TZ_i)), \qquad \theta_i\left(\nabla^F, g^F\right) \in \Omega^1(Z_i) \tag{0.7}$$

the canonical extensions of $e(TZ, \nabla^{TZ})$ and $\theta(\nabla^F, g^F)$ to Z_i (see subsection 3.4). Let

$$\widetilde{e}(TZ_i, \nabla^{TZ_i}, \nabla'^{TZ_i}) \in \Omega^{\dim Z_i - 1}(Z_i, o(TZ_i))/d\Omega^{\dim Z_i - 2}(Z_i, o(TZ_i)) \tag{0.8}$$

and $\widetilde{\theta}_i(\nabla^F, g^F, g'^F) \in C^\infty(Z_i)$ be the associated Chern-Simons forms such that

$$\begin{aligned} d\,\widetilde{e}\left(TZ_i, \nabla^{TZ_i}, \nabla'^{TZ_i}\right) &= e\left(Z_i, \nabla'^{TZ_i}\right) - e\left(Z_i, \nabla^{TZ_i}\right), \\ d\,\widetilde{\theta}_i\left(\nabla^F, g^F, g'^F\right) &= \theta_i\left(\nabla^F, g'^F\right) - \theta_i\left(\nabla^F, g^F\right). \end{aligned} \tag{0.9}$$

In Section 4, we show:

Theorem 0.3. *The following identity holds:*

$$\begin{aligned} \log\left(\frac{\|\cdot\|_\lambda^{\prime \mathrm{RS},2}}{\|\cdot\|_\lambda^{\mathrm{RS},2}}\right) = \sum_{i=0}^{l_0} \frac{1}{m_i}\Bigg(&\int_{Z_i} \widetilde{\theta}_i\left(\nabla^F, g^F, g'^F\right) e\left(Z_i, \nabla^{TZ_i}\right) \\ &- \int_{Z_i} \theta_i\left(\nabla^F, g'^F\right) \widetilde{e}\left(TZ_i, \nabla^{TZ_i}, \nabla'^{TZ_i}\right)\Bigg). \end{aligned} \tag{0.10}$$

The arguments in Section 4 are inspired by Bismut-Lott [8, Theorem 3.24], who gave a unified proof for the family local index theorem and the anomaly formula [9, Theorem 0.1]. Conceptually, their proof is simpler and more natural than the original proof given by Bismut-Zhang [9, Section IV]. Also, our proof relies on the finite propagation speeds for the solutions of hyperbolic equations on orbifolds, which is originally due to Ma [31].

If Z is of odd dimension and orientable, then all the Z_i, for $0 \leq i \leq l_0$, is of odd dimension. By Theorem 0.3, the Ray-Singer metric $\|\cdot\|_\lambda^{\mathrm{RS},2}$ does not depend on the metrics g^{TZ}, g^F; it becomes a topological invariant.

0.3. A solution of Fried conjecture on locally symmetric orbifolds

In [22, p. 537], Fried raised the question of extending his result [22, Theorem 1] to hyperbolic orbifolds on the equality between the analytic torsion and the zero value of the Ruelle dynamical zeta function associated to a unitarily flat acyclic vector bundle on hyperbolic manifolds. In Section 5, we extend Fried's result to more general compact odd dimensional [1] locally symmetric orbifolds of the reductive type.

[1] The even dimensional case is trivial.

Let G be a linear connected real reductive group with Cartan involution $\theta \in \mathrm{Aut}(G)$. Let $K \subset G$ be the set of fixed points of θ in G, so that K is a maximal compact subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K. Let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be the Cartan decomposition. Let B be an $\mathrm{Ad}(G)$-invariant and θ-invariant non degenerate bilinear form on $\mathfrak{g}$ such that $B|_{\mathfrak{p}} > 0$ and $B|_{\mathfrak{k}} < 0$. Recall that an element $\gamma \in G$ is said to be semisimple if and only if γ can be conjugated to $e^a k^{-1}$ with $a \in \mathfrak{p}$, $k \in K$, $\mathrm{Ad}(k)a = a$. And γ is said to be elliptic if and only if γ can be conjugated into K. Note that if γ is semisimple, its centralizer $Z(\gamma)$ in G is still reductive with maximal compact subgroup $K(\gamma)$.

Take $X = G/K$ to be the associated symmetric space. Then, $B|_{\mathfrak{p}}$ induces a G-invariant Riemannian metric g^{TX} on X such that (X, g^{TX}) is of non positive sectional curvature. Let d_X be the Riemannian distance on X.

Let $\Gamma \subset G$ be a cocompact discrete subgroup of G. Set $Z = \Gamma \backslash G/K$. Then Z is a compact orbifold with universal covering orbifold X. To simplify the notation in Introduction, we assume that Γ acts effectively on X. Then Γ is the orbifold fundamental group of Z. Clearly, Γ contains only semisimple elements. Let Γ_+ be the subset of Γ consisting of non elliptic elements. Take $[\Gamma]$ to be the set of conjugacy classes of Γ. Denote by $[\Gamma_+] \subset [\Gamma]$ the set of non elliptic conjugacy classes.

Proceeding as in the proof for the manifold case [15, Proposition 5.15], the set of closed geodesics of positive lengths consists of a disjoint union of smooth connected compact orbifolds $\coprod_{[\gamma]\in[\Gamma_+]} B_{[\gamma]}$. Moreover, $B_{[\gamma]}$ is diffeomorphic to $\Gamma \cap Z(\gamma)\backslash Z(\gamma)/K(\gamma)$. Also, all the elements in $B_{[\gamma]}$ have the same length $l_{[\gamma]} > 0$. Clearly, the geodesic flow induces a locally free $\mathbb{S}^1$-action on $B_{[\gamma]}$. By an analogy to the multiplicity m_i of Z_i in $Z \coprod \Sigma Z$, we can define the multiplicity $m_{[\gamma]}$ of the quotient orbifold $\mathbb{S}^1 \backslash B_{[\gamma]}$ (see (5.60)). Denote by $\chi_{\mathrm{orb}}\big(\mathbb{S}^1 \backslash B_{[\gamma]}\big) \in \mathbf{Q}$ the orbifold Euler characteristic number [44, Section 3.3] (see also (3.21)) of $\mathbb{S}^1 \backslash B_{[\gamma]}$. In Section 5, we show:

Theorem 0.4. *If* $\dim Z$ *is odd, and if* F *is a unitarily flat orbifold vector bundle on* Z *with holonomy* $\rho : \Gamma \to \mathrm{U}(r)$*, then the dynamical zeta function*

$$R_\rho(\sigma) = \exp\left(\sum_{[\gamma]\in[\Gamma_+]} \mathrm{Tr}[\rho(\gamma)] \frac{\chi_{\mathrm{orb}}\left(\mathbb{S}^1 \backslash B_{[\gamma]}\right)}{m_{[\gamma]}} e^{-\sigma l_{[\gamma]}} \right) \tag{0.11}$$

is well-defined and holomorphic on $\mathrm{Re}\,(\sigma) \gg 1$*, and extends meromorphically to* $\mathbf{C}$*. There exist explicit constants* $C_\rho \in \mathbf{R}$ *with* $C_\rho \neq 0$ *and* $r_\rho \in \mathbf{Z}$ *(see*

(5.117)*) such that as* $\sigma \to 0$*,*

$$R_\rho(\sigma) = C_\rho T(F)^2 \sigma^{r_\rho} + \mathcal{O}(\sigma^{r_\rho+1}). \tag{0.12}$$

Moreover, if $H^{\cdot}(Z, F) = 0$*, we have*

$$C_\rho = 1, \qquad r_\rho = 0, \tag{0.13}$$

so that

$$R_\rho(0) = T(F)^2. \tag{0.14}$$

The proof of Theorem 0.4 is similar to the one given in [47], except that in the current case, we also need to take account of the contribution of elliptic orbital integrals in the analytic torsion. On the other hand, let us note that a priori elliptic elements do not contribute to the dynamical zeta function. This seemingly contradictory phenomenon has already appeared in the smooth case. In fact, in the current case, the elliptic and non elliptic orbital integrals are related via functional equations of certain Selberg zeta functions.

We refer the readers to the papers of Giulietti-Liverani-Pollicott [24] and Dyatlov-Zworski [16, 17] for other points of view on the dynamical zeta function on negatively curved manifolds.

Let us also mention Fedosova's recent work [19–21] on the Selberg zeta function and the asymptotic behavior of the analytic torsion of unimodular flat orbifold vector bundles on hyperbolic orbifolds.

0.4. Organisation of the article

This article is organized as follows. In Section 1, we introduce some basic notation on the determinant line and characteristic forms. Also we recall some standard terminology on group actions on topological spaces.

In Section 2, we recall the definition of orbifolds, orbifold vector bundles, and the $\mathcal{G}$-path theory of Haefliger [26]. We show Theorem 0.1.

In Section 3, we explain how to extend the usual differential calculus and Chern-Weil theory on manifolds to orbifolds.

In Section 4, we study the analytic torsion and Ray-Singer metric on orbifolds. Following [8], we show in a unified way an orbifold version of Gauss-Bonnet-Chern Theorem and Theorem 0.3. Some estimates on heat kernels are postponed to Section 4.4.

In Section 5, we study the analytic torsion on locally symmetric orbifold using the Selberg trace formula and Bismut's semisimple orbital integral formula. We show Theorem 0.4.

0.5. Notation

In the whole paper, we use the superconnection formalism of Quillen [40] (see also [4, Section 1.3]). Here we just briefly recall that if A is a $\mathbf{Z}_2$-graded algebra, if $a, b \in A$, the supercommutator $[a, b]$ is given by $[a, b] = ab - (-1)^{\deg a \deg b} ba$. If B is another $\mathbf{Z}_2$-graded algebra, we denote by $A \widehat{\otimes} B$ the super tensor algebra. If $E = E^+ \oplus E^-$ is a $\mathbf{Z}_2$-graded vector space, the algebra $\mathrm{End}(E)$ is $\mathbf{Z}_2$-graded. If $\tau = \pm 1$ on $E^{\pm}$, if $a \in \mathrm{End}(E)$, the supertrace $\mathrm{Tr_s}[a]$ is defined by $\mathrm{Tr_s}[a] = \mathrm{Tr}[\tau a]$. We make the convention that $\mathbf{N} = \{0, 1, 2, \cdots\}$ and $\mathbf{N}^* = \{1, 2, \cdots\}$. If A is a finite set, we denote by $|A|$ its cardinality.

Acknowledgement

The work started while S.S. was visiting the University of Science and Technology of China in July, 2016. He would like to thank the School of Mathematical Sciences for hospitality. S.S. was supported by a grant from the European Research Council (E.R.C.) under European Union's Seventh Framework Program (FP7/2007-2013) /ERC grant agreement (No. 291060) and by Collaborative Research Centre "Space-Time-Matte" (SFB 647) founded by the German Research Foundation (DFG). J.Y. was partially supported by NSFC (No. 11401552, No. 11771411).

1. Preliminary

The purpose of this section is to recall some basic definitions and constructions. This section is organized as follows. In subsection 1.1, we introduce the basic conventions on determinant lines.

In subsection 1.2, we recall some standard terminology of group actions on topological spaces, which will be used in the whole paper.

In subsection 1.3, we recall the Chern-Weil construction on characteristic forms and the associated secondary classes of Chern-Simons forms on manifolds.

1.1. Determinants

Let V be a complex vector space of finite dimension. We denote by V^* the dual space of V, and by $\Lambda^{\cdot}V$ the exterior algebra of V. Set $\det V = \Lambda^{\dim V}V$. Clearly, $\det V$ is a line. We use the convention that $\det 0 = \mathbf{C}$. If λ is a line, we denote by $\lambda^{-1} = \lambda^*$ the dual line.

1.2. Group actions

Let L be a topological group acting continuously on a topological space S. The action of L is said to be free if for any $g \in L$ and $g \neq 1$, the set of fixed points of g in S is empty. The action of L is said to be effective if the morphism of groups $L \to \mathrm{Homeo}(S)$ is injective, where $\mathrm{Homeo}(S)$ is the group of homeomorphisms of S. The action of L is said to be properly discontinuous if for any $x \in S$ there is a neighborhood U of x such that the set

$$\{g \in L : gU \cap U \neq \varnothing\} \tag{1.1}$$

is finite.

If L acts on the right (resp. left) on the topological space S_0 (resp. S_1), denote by S_0/L (resp. $L\backslash S_1$) the quotient space, and by $S_0 \times_L S_1$ the quotient of $S_0 \times S_1$ by the left action defined by

$$g(x_0, x_1) = (x_0 g^{-1}, g x_1), \qquad \text{for } g \in L \text{ and } (x_0, x_1) \in S_0 \times S_1. \tag{1.2}$$

If S_2 is another left L-space, denote by $S_1 \ {}_L\!\times S_2$ the quotient of $S_1 \times S_2$ by the evident left action of L.

1.3. Characteristic forms on manifolds

Let S be a manifold. Denote by $\big(\Omega^{\cdot}(S), d^S\big)$ the de Rham complex of S, and by $H^{\cdot}(S)$ its de Rham cohomology.

Let E be a real vector bundle of rank r equipped with a Euclidean metric g^E. Let ∇^E be a metric connection, and let $R^E = (\nabla^E)^2$ be the curvature of ∇^E. Then R^E is a 2-form on S with values in antisymmetric endomorphisms of E.

If A is an antisymmetric matrix, denote by $\mathrm{Pf}[A]$ the Pfaffian [9, (3.3)] of A. Then $\mathrm{Pf}[A]$ is a polynomial function of A, which is a square root of $\det[A]$. Let $o(E)$ be the orientation line of E. The Euler form of $\big(E, \nabla^E\big)$ is

given by

$$e\left(E,\nabla^E\right)=\mathrm{Pf}\left[\frac{R^E}{2\pi}\right]\in\Omega^r\big(S,o(E)\big). \tag{1.3}$$

The cohomology class $e(E)\in H^r(S,o(E))$ of $e(E,\nabla^E)$ does not depend on the choice of (g^E,∇^E). More precisely, if g'^E is another metric on E, and if ∇'^E is another connection on E which preserves g'^E, we can define a class of Chern-Simons $(r-1)$-form

$$\widetilde{e}\left(E,\nabla^E,\nabla'^E\right)\in\Omega^{r-1}\big(S,o(E)\big)/d\Omega^{r-2}\big(S,o(E)\big) \tag{1.4}$$

such that

$$d^S\widetilde{e}\left(E,\nabla^E,\nabla'^E\right)=e\left(E,\nabla'^E\right)-e\left(E,\nabla^E\right). \tag{1.5}$$

Note that if r is odd, then $e(E,\nabla^E)=0$ and $\widetilde{e}(E,\nabla^E,\nabla'^E)=0$.

Let us describe the construction of $\widetilde{e}(E,\nabla^E,\nabla'^E)$. Take a smooth family $(g_s^E,\nabla_s^E)_{s\in\mathbf{R}}$ of metrics and metric connections such that

$$\left(g_0^E,\nabla_0^E\right)=\left(g^E,\nabla^E\right),\qquad \left(g_1^E,\nabla_1^E\right)=\left(g'^E,\nabla'^E\right). \tag{1.6}$$

Set

$$\pi:\mathbf{R}\times S\to S. \tag{1.7}$$

We equip π^*E with a Euclidean metric g^{π^*E} and with a metric connection ∇^{π^*E} defined by

$$g^{\pi^*E}|_{\{s\}\times S}=g_s^E,\qquad \nabla^{\pi^*E}=ds\wedge\left(\frac{d}{ds}+\frac{1}{2}g_s^{E,-1}\frac{d}{ds}g_s^E\right)+\nabla_s^E. \tag{1.8}$$

Write

$$e\left(\pi^*E,\nabla^{\pi^*E}\right)=e\left(E,\nabla_s^E\right)+ds\wedge\alpha_s\in\Omega^r\big(\mathbf{R}\times S,\pi^*o(E)\big). \tag{1.9}$$

Since $e\left(\pi^*E,\nabla^{\pi^*E}\right)$ is closed, by (1.9), for $s\in\mathbf{R}$, we have

$$\frac{\partial}{\partial s}e\left(E,\nabla_s^E\right)=d^S\alpha_s. \tag{1.10}$$

Then, $\widetilde{e}(E,\nabla^E,\nabla'^E)\in\Omega^{r-1}\big(S,o(E)\big)/d\Omega^{r-2}\big(S,o(E)\big)$ is defined by the class of

$$\int_0^1 \alpha_s ds\in\Omega^{r-1}\big(S,o(E)\big). \tag{1.11}$$

Note that $\widetilde{e}(E,\nabla^E,\nabla'^E)$ does not depend on the choice of smooth family $(g_s^E,\nabla_s^E)_{s\in\mathbf{R}}$ (c.f. [12, Proposition 2.7]). Also, (1.5) is a consequence of (1.6) and (1.10).

Let us recall the definition of the $\widehat{A}$-form of (E,∇^E). For $x\in\mathbf{C}$, set

$$\widehat{A}(x)=\frac{x/2}{\sinh(x/2)}. \tag{1.12}$$

The $\widehat{A}$-form of (E,∇^E) is given by

$$\widehat{A}\left(E,\nabla^E\right)=\left[\det\left(\widehat{A}\left(-\frac{R^E}{2i\pi}\right)\right)\right]^{1/2}\in\Omega^{\cdot}(S). \tag{1.13}$$

Let L be a compact Lie group. Assume that L acts fiberwisely and linearly on the vector bundle E over S, which preserves (g^E,∇^E). Take $g\in L$. Assume that g preserves the orientation of E. Let $E(g)$ be the subbundle of E defined by the fixed points of g. Let $\pm\theta_1,\cdots,\pm\theta_{s_0}$, $0<\theta_i\leq\pi$ be the district nonzero angles of the action of g on E. Let E_{θ_i} be the subbundle of E on which g acts by a rotation of angle θ_i. The subbundles $E(g)$ and E_{θ_i} are canonically equipped with Euclidean metrics and metric connections $\nabla^{E(g)},\nabla^{E_{\theta_i}}$.

For $\theta\in\mathbf{R}-2\pi\mathbf{Z}$, set

$$\widehat{A}_\theta(x)=\frac{1}{2\sinh\left(\frac{x+i\theta}{2}\right)}. \tag{1.14}$$

Given θ_i, let $\widehat{A}_{\theta_i}(E_{\theta_i},\nabla^{E_{\theta_i}})$ be the corresponding multiplicative genus. The equivariant $\widehat{A}$-form of (E,∇^E) is given by

$$\widehat{A}_g\left(E,\nabla^E\right)=\widehat{A}\left(E(g),\nabla^{E(g)}\right)\prod_{i=1}^{s_0}\widehat{A}_{\theta_i}\left(E_{\theta_i},\nabla^{E_{\theta_i}}\right)\in\Omega^{\cdot}(S). \tag{1.15}$$

Let E' be a complex vector bundle carrying a connection $\nabla^{E'}$ with curvature $R^{E'}$. Assume that E' is equipped with a fiberwise linear action of

L, which preserves $\nabla^{E'}$. For $g \in L$, the equivariant Chern character form of $(E', \nabla^{E'})$ is given by

$$\mathrm{ch}_g\left(E', \nabla^{E'}\right) = \mathrm{Tr}\left[g \exp\left(-\frac{R^{E'}}{2i\pi}\right)\right] \in \Omega^{\mathrm{even}}(S). \tag{1.16}$$

As before, $\widehat{A}_g(E, \nabla^E)$, $\mathrm{ch}_g(E', \nabla^{E'})$ are closed. Their cohomology classes do not depend on the choice of connections. The closed forms in (1.15) and (1.16) on S are exactly the ones that appear in the Lefschetz fixed point formula of Atiyah-Bott [2, 3]. Note that there are questions of signs to be taken care of, because of the need to distinguish between θ_i and $-\theta_i$. We refer to the above references for more detail.

Let F be a flat vector bundle on S with flat connection ∇^F. Let g^F be a Hermitian metric on F. Assume that F is equipped with a fiberwise and linear action of L which preserves ∇^F and g^F. Following [9, Definition 4.1], put

$$\omega\left(\nabla^F, g^F\right) = \left(g^F\right)^{-1} \nabla^F g^F. \tag{1.17}$$

Then, $\omega(\nabla^F, g^F)$ is a 1-form on S with values in symmetric endomorphisms of F. For $x \in \mathbf{C}$, set

$$h(x) = xe^{x^2}. \tag{1.18}$$

Following [8, Defintion 1.7] and [6, Definition 1.7], for $g \in L$, the equivariant odd Chern character form of (F, ∇^F) is given by

$$h_g\left(\nabla^F, g^F\right) = \sqrt{2i\pi}\,\mathrm{Tr}\left[gh\left(\frac{\omega\left(\nabla^F, g^F\right)/2}{\sqrt{2i\pi}}\right)\right] \in \Omega^{\mathrm{odd}}(S). \tag{1.19}$$

When $g = 1$, we denote by $h\left(\nabla^F, g^F\right) = h_1\left(\nabla^F, g^F\right)$.

By [8, Theorem 1.11] and [6, Theorem 1.8], we know that the cohomology class $h_g(\nabla^F) \in H^{\mathrm{odd}}(S)$ of $h_g(\nabla^F, g^F)$ does not depend on g^F. If g'^F is another L-invariant Hermitian metric on F, we can define the class of Chern-Simons form $\widetilde{h}_g(\nabla^F, g^F, g'^F) \in \Omega^{\mathrm{even}}(S)/d\Omega^{\mathrm{odd}}(S)$ such that

$$d^S\widetilde{h}_g(\nabla^F, g^F, g'^F) = h_g\left(\nabla^F, g'^F\right) - h_g\left(\nabla^F, g^F\right). \tag{1.20}$$

More precisely, choose a smooth family of L-invariant metrics $(g_s^F)_{s\in\mathbf{R}}$ such that

$$g_0^F = g^F, \qquad g_1^F = g'^F. \tag{1.21}$$

Consider the projection π defined in (1.7). Equip π^*F with the following flat connection and Hermitian metric

$$\nabla^{\pi^*F} = d^{\mathbf{R}} + \nabla^F, \qquad g^{\pi^*F}\Big|_{\{s\}\times S} = g_s^F. \tag{1.22}$$

Write

$$h_g\left(\nabla^{\pi^*F}, g^{\pi^*F}\right) = h_g\left(\nabla^F, g_s^F\right) + ds \wedge \beta_s \in \Omega^{\mathrm{odd}}(\mathbf{R}\times S). \tag{1.23}$$

As (1.11), $\widetilde{h}_g\left(\nabla^F, g^F, g'^F\right) \in \Omega^{\mathrm{even}}(S)/d\Omega^{\mathrm{odd}}(S)$ is defined by the class of

$$\int_0^1 \beta_s ds \in \Omega^{\mathrm{even}}(S). \tag{1.24}$$

By [8, Theorem 1.11] and [6, Theorem 1.11], $\widetilde{h}_g\left(\nabla^F, g^F, g'^F\right)$ does not depend on the choice of the smooth family of metrics $(g_s^F)_{s\in\mathbf{R}}$. Also, $\widetilde{h}_g\left(\nabla^F, g^F, g'^F\right)$ satisfies (1.20).

2. Topology of orbifolds

The purpose of this section is to introduce some basic definitions and related constructions for orbifolds. We show Theorem 0.1, which claims a bijection between the isomorphism classes of proper flat orbifold vector bundles and the equivalent classes of representations of the orbifold fundamental group.

This section is organized as follows. In subsection 2.1, we recall the definition of orbifolds and the associated groupoid $\mathcal{G}$.

In subsection 2.2, we introduce the resolution for the singular set of an orbifold.

In subsection 2.3, we recall the definition of orbifold vector bundles.

In subsection 2.4, the orbifold fundamental group and the universal covering orbifold are constructed using the $\mathcal{G}$-path theory of Haefliger [11, 26].

Finally, in subsection 2.5, we define the holonomy representation for a proper flat orbifold vector bundle. We restate and show Theorem 0.1.

2.1. Definition of orbifolds

In this subsection, we recall the definition of orbifolds following [44, Section 1] and [1, Section 1.1]. Let Z be a topological space, and let $U \subset Z$ be a connected open subset of Z. Take $m \in \mathbf{N}$.

Definition 2.1. An m-dimensional orbifold chart for U is given by a triple $(\widetilde{U}, G_U, \pi_U)$, where

- $\widetilde{U} \subset \mathbf{R}^m$ is a connected open subset of $\mathbf{R}^m$;
- G_U is a finite group acting smoothly and effectively on the left on $\widetilde{U}$;
- $\pi_U : \widetilde{U} \to U$ is a G_U-invariant continuous map which induces a homeomorphism of topological spaces

$$G_U \backslash \widetilde{U} \simeq U. \tag{2.1}$$

Remark 2.2. In [44, Section 1], it is assumed that the codimension of the fixed point set of G_U in $\widetilde{U}$ is bigger than or equal to 2. In this article, we do not make this assumption.

Let $U \hookrightarrow V$ be an embedding of connected open subsets of Z, and let $(\widetilde{U}, G_U, \pi_U)$ and $(\widetilde{V}, G_V, \pi_V)$ be respectively orbifold charts for U and V.

Definition 2.3. An embedding of orbifold charts is a smooth embedding $\phi_{VU} : \widetilde{U} \to \widetilde{V}$ such that the diagram

$$\begin{tikzcd} \widetilde{U} \arrow[r, "\phi_{VU}"] \arrow[d, "\pi_U"] & \widetilde{V} \arrow[d, "\pi_V"] \\ U \arrow[r, hook] & V \end{tikzcd} \tag{2.2}$$

commutes.

We recall the following proposition. The proof was given by Satake [44, Lemmas 1.1, 1.2] under the assumption that the codimension of the fixed point set is bigger than or equal to 2. For general cases, see [36, Appendix] for example.

Proposition 2.4. *Let $\phi_{VU} : (\widetilde{U}, G_U, \pi_U) \hookrightarrow (\widetilde{V}, G_V, \pi_V)$ be an embedding of orbifold charts. The following statements hold:*

1) *if $g \in G_V$, then $x \in \widetilde{U} \to g\phi_{VU}(x) \in \widetilde{V}$ is another embedding of orbifold charts. Conversely, any embedding of orbifold charts $(\widetilde{U}, G_U, \pi_U) \hookrightarrow (\widetilde{V}, G_V, \pi_V)$ is of such form;*
2) *there exists a unique injective morphism $\lambda_{VU} : G_U \to G_V$ of groups such that ϕ_{VU} is λ_{VU}-equivariant;*

3) if $g \in G_V$ such that $\phi_{VU}(\widetilde{U}) \cap g\phi_{VU}(\widetilde{U}) \neq \varnothing$, then g is in the image of λ_{VU}, and so $\phi_{VU}(\widetilde{U}) = g\phi_{VU}(\widetilde{U})$.

Let $U_1, U_2 \subset Z$ be two connected open subsets of Z with orbifold charts $(\widetilde{U}_1, G_{U_1}, \pi_{U_1})$ and $(\widetilde{U}_2, G_{U_2}, \pi_{U_2})$.

Definition 2.5. The orbifold charts $(\widetilde{U}_1, G_{U_1}, \pi_{U_1})$ and $(\widetilde{U}_2, G_{U_2}, \pi_{U_2})$ are said to be compatible if for any $z \in U_1 \cap U_2$, there is an open connected neighborhood $U_0 \subset U_1 \cap U_2$ of z with orbifold chart $(\widetilde{U}_0, G_{U_0}, \pi_{U_0})$ such that there exist two embeddings of orbifold charts

$$\phi_{U_iU_0} : (\widetilde{U}_0, G_{U_0}, \pi_{U_0}) \hookrightarrow (\widetilde{U}_i, G_{U_i}, \pi_{U_i}), \qquad \text{for } i = 1, 2. \tag{2.3}$$

The diffeomorphism $\phi_{U_2U_0}\phi^{-1}_{U_1U_0} : \phi_{U_1U_0}(\widetilde{U}_0) \to \phi_{U_2U_0}(\widetilde{U}_0)$ is called a coordinate transformation.

Definition 2.6. An orbifold atlas on Z consists of an open connected cover $\mathcal{U} = \{U\}$ of Z and compatible orbifold charts $\widetilde{\mathcal{U}} = \{(\widetilde{U}, G_U, \pi_U)\}_{U\in\mathcal{U}}$.

An orbifold atlas $(\mathcal{V}, \widetilde{\mathcal{V}})$ is called a refinement of $(\mathcal{U}, \widetilde{\mathcal{U}})$, if $\mathcal{V}$ is a refinement of $\mathcal{U}$ and if every orbifold chart in $\widetilde{\mathcal{V}}$ has an embedding into some orbifold chart in $\widetilde{\mathcal{U}}$.

Two orbifold atlases are said to be equivalent if they have a common refinement.

The equivalent class of an orbifold atlas is called an orbifold structure on Z.

Definition 2.7. An orbifold is a second countable Hausdorff space equipped with an orbifold structure. It said to have dimension m, if all the orbifold charts which define the orbifold structure are of dimension m.

Remark 2.8. Let U, V be two connected open subsets of an orbifold with respectively orbifold charts $(\widetilde{U}, G_U, \pi_U)$ and $(\widetilde{V}, G_V, \pi_V)$, which are compatible with the orbifold structure. If $U \subset V$, and if $\widetilde{U}$ is simply connected, then there exists an embedding of orbifold charts $(\widetilde{U}, G_U, \pi_U) \hookrightarrow (\widetilde{V}, G_V, \pi_V)$.

Remark 2.9. For any point z of an orbifold, there exists an open connected neighborhood $U_z \subset Z$ of z with a compatible orbifold chart $(\widetilde{U}_z, G_z, \pi_z)$ such that $\pi_z^{-1}(z)$ contains only one point $x \in \widetilde{U}_z$. Such a chart is called to be centered at x. Clearly, x is a fixed point of G_z. The isomorphism class of the group G_z does not depend on the different choices of centered orbifold charts, and is called the local group at z.

Moreover, we can choose $(\widetilde{U}_z, G_z, \pi_z)$ to be a linear chart centered at 0, which means

$$\widetilde{U}_z = \mathbf{R}^m, \qquad x = 0 \in \mathbf{R}^m, \qquad G_z \subset \mathrm{O}(m). \tag{2.4}$$

In the sequel, let Z be an orbifold with orbifold atlas $(\mathcal{U}, \widetilde{\mathcal{U}})$. We assume that $\mathcal{U}$ is countable and that each $\widetilde{U} \in \widetilde{\mathcal{U}}$ is simply connected. When we talk of an orbifold chart, we mean the one which is compatible with $\widetilde{\mathcal{U}}$.

Let us introduce a groupoid $\mathcal{G}$ associated to the orbifold Z with orbifold atlas $(\mathcal{U}, \widetilde{\mathcal{U}})$. Recall that a groupoid is a category whose morphisms, which are called arrows, are isomorphisms. We define $\mathcal{G}_0$, the objects of $\mathcal{G}$, to be the countable disjoint union of smooth manifold

$$\mathcal{G}_0 = \coprod_{U \in \mathcal{U}} \widetilde{U}. \tag{2.5}$$

An arrow g from $x_1 \in \mathcal{G}_0$ to $x_2 \in \mathcal{G}_0$, denoted by $g : x_1 \to x_2$, is a germ of coordinate transformation g defined near x_1 such that $g(x_1) = x_2$. We denote by $\mathcal{G}_1$ the set of arrows. This way defines a groupoid $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$. By [1, Section 1.4], $\mathcal{G}_1$ is equipped with a topology such that $\mathcal{G}$ is a proper, effective, étale Lie groupoid.

For $x_1, x_2 \in \mathcal{G}_0$, we call x_1 and x_2 in the same orbit if there is an arrow $g \in \mathcal{G}_1$ from x_1 to x_2. We denote by $\mathcal{G}_0/\mathcal{G}_1$ the orbit space equipped with the quotient topology. The projection $\pi_U : \widetilde{U} \to U$ induces a homeomorphism of topological spaces

$$\mathcal{G}_0/\mathcal{G}_1 \simeq Z. \tag{2.6}$$

Let Y and Z be two orbifolds. Following [43, p. 361], we introduce:

Definition 2.10. A continuous map $f : Y \to Z$ between orbifolds is called smooth if for any $y \in Y$, there exist

- an open connected neighborhood $U \subset Y$ of y, an open connected neighborhood $V \subset Z$ of $f(y)$ such that $f(U) \subset V$,
- orbifold charts $(\widetilde{U}, G_U, \pi_U)$ and $(\widetilde{V}, G_V, \pi_V)$ for U and V,

- a smooth map $\widetilde{f}_U : \widetilde{U} \to \widetilde{V}$ such that the following diagram

$$\begin{tikzcd} \widetilde{U} \arrow[r, "\widetilde{f}_U"] \arrow[d, "\pi_U"] & \widetilde{V} \arrow[d, "\pi_V"] \\ U \arrow[r, "f|_U"] & V \end{tikzcd} \tag{2.7}$$

  commutes.

We denote by $C^\infty(Y, Z)$ the space of smooth maps from Y to Z.

Two orbifolds Y and Z are called isomorphic if there are smooth maps $f : Y \to Z$ and $f' : Z \to Y$ such that $ff' = \mathrm{id}$ and $f'f = \mathrm{id}$. Clearly, this is the case if $f : Y \to Z$ is a smooth homeomorphism such that each lifting $\widetilde{f}_U$ is a diffeomorphism. Moreover, in this case, by Proposition 2.4, there is an isomorphism of group $\rho_U : G_U \to G_V$ such that $\widetilde{f}_U$ is ρ_U-equivariant. Also, any possible lifting has the form $g\widetilde{f}_U$, $g \in G_U$.

Definition 2.11. An action of Lie group L on Z is said to be smooth, if the action $L \times Z \to Z$ is smooth.

The following proposition is an extension of [50, Proposition 13.2.1]. We include a detailed proof since some constructions in the proof will be useful to show Theorem 0.1.

Proposition 2.12. *Let Γ be a discrete group acting smoothly and properly discontinuously on the left on an orbifold X. Then $\Gamma\backslash X$ has a canonical orbifold structure induced from X.*

Proof. Let $p : X \to \Gamma\backslash X$ be the natural projection. We equip $\Gamma\backslash X$ with the quotient topology. Since X is Hausdorff and second countable, and since the Γ-action is properly discontinuous, then $\Gamma\backslash X$ is also Hausdorff and second countable.

Take $z \in \Gamma\backslash X$. We choose $x \in X$ such that $p(x) = z$. Set

$$\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}. \tag{2.8}$$

As the Γ-action is properly discontinuous, Γ_x is a finite group, and there exists an open connected Γ_x-invariant neighborhood $V_x \subset X$ of x such that

for $\gamma \in \Gamma - \Gamma_x$,

$$\gamma V_x \cap V_x = \varnothing. \tag{2.9}$$

Then, $p(V_x) \subset \Gamma\backslash X$ is an open connected neighborhood of z. Also, we have

$$\Gamma_x\backslash V_x \simeq \Gamma\backslash\Gamma V_x = p(V_x). \tag{2.10}$$

By taking V_x small enough, there is an orbifold chart $(\widetilde{V}_x, H_x, \pi_x)$ for V_x centered at $\widetilde{x} \in \widetilde{V}_x$ (see Remark 2.9). As Γ acts smoothly on X, we can assume that homeomorphism of V_x defined by $\gamma_x \in \Gamma_x$ lifts to a local diffeomorphism $\widetilde{\gamma}_x$ defined near $\widetilde{x}$ such that $\pi_x\widetilde{\gamma}_x = \gamma_x\pi_x$ holds near $\widetilde{x}$.

By Proposition 2.4, the lifting $\widetilde{\gamma}_x$ is not unique, and all possible liftings can be written as $h_x\widetilde{\gamma}_x$ for some $h_x \in H_x$. Let G_x be the group of local diffeomorphism defined near x generated by $\{\widetilde{\gamma}_x\}_{\gamma_x\in\Gamma_x}$ and H_x. Then G_x is a finite group. By choosing V_x small enough and by (2.10), G_x acts on $\widetilde{V}_x$ such that

$$G_x\backslash\widetilde{V}_x \simeq p(V_x). \tag{2.11}$$

Since the G_x-action on $\widetilde{V}_x$ is effective, $(\widetilde{V}_x, G_x, p\circ\pi_x)$ is an orbifold chart of Z for $p(V_x)$.

The family of open sets $\{p(V_x)\}$ covers $\Gamma\backslash X$. It remains to show that two such orbifold charts $(\widetilde{V}_{x_1}, G_{x_1}, p\circ\pi_{x_1})$ and $(\widetilde{V}_{x_2}, G_{x_2}, p\circ\pi_{x_2})$ are compatible. Its proof consists of two steps.

In the fist step, we consider the case $x_2 = \gamma x_1$ for some $\gamma \in \Gamma$. We can assume that $V_{x_2} = \gamma V_{x_1}$, and that $\gamma|_{V_{x_1}}$ lifts to $\widetilde{\gamma}_{x_1} : \widetilde{V}_{x_1} \to \widetilde{V}_{x_2}$. Then $\widetilde{\gamma}_{x_1}$ defines an isomorphism between the orbifold charts $(\widetilde{V}_{x_1}, G_{x_1}, p\circ\pi_{x_1})$ and $(\widetilde{V}_{x_2}, G_{x_2}, p\circ\pi_{x_2})$.

In the second step, we consider general $x_1, x_2 \in X$ such that $p(V_{x_1}) \cap p(V_{x_2}) \neq \varnothing$. Because of the first step, we can assume that $V_{x_1} \cap V_{x_2} \neq \varnothing$. For $x_0 \in V_{x_1} \cap V_{x_2}$, take an open connected neighborhood $V_{x_0} \subset V_{x_1} \cap V_{x_2}$ of x_0 and an orbifold chart $(\widetilde{V}_{x_0}, H_{x_0}, \pi_{x_0})$ of X as before. We can assume that there exist two embeddings $\phi_{V_{x_i}V_{x_0}} : (\widetilde{V}_{x_0}, H_{x_0}, \pi_{x_0}) \hookrightarrow (\widetilde{V}_{x_i}, H_{x_i}, \pi_{x_i})$, for $i = 1, 2$, of orbifold charts of X. Then, $\phi_{V_{x_i}V_{x_0}}$ also define two embeddings of orbifold charts of Z,

$$(\widetilde{V}_{x_0}, G_{x_0}, p\circ\pi_{x_0}) \hookrightarrow (\widetilde{V}_{x_i}, G_{x_i}, p\circ\pi_{x_i}). \tag{2.12}$$

The proof our proposition is completed. □

Remark 2.13. By the construction, H_x is a normal subgroup of G_x, and $\gamma_x \to \widetilde{\gamma}_x$ induces a surjective morphism of groups

$$\Gamma_x \to G_x/H_x. \tag{2.13}$$

If the action of Γ on X is effective, then (2.13) is an isomorphism of groups. Thus, the following sequence of groups

$$1 \to H_x \to G_x \to \Gamma_x \to 1 \tag{2.14}$$

is exact.

2.2. Singular set of orbifolds

Let Z be an orbifold with orbifold atlas $(\mathcal{U},\widetilde{\mathcal{U}})$. Put

$$Z_{\text{reg}} = \{z \in Z : G_z = \{1\}\}, \qquad Z_{\text{sing}} = \{z \in Z : G_z \neq \{1\}\}. \tag{2.15}$$

Then $Z = Z_{\text{reg}} \cup Z_{\text{sing}}$. Clearly, Z_{reg} is a smooth manifold. However, Z_{sing} is not necessarily an orbifold. Following [27, Section 1], we will introduce the orbifold resolution ΣZ for Z_{sing}.

Let $[G_z]$ be the set of conjugacy classes of G_z. Set

$$\Sigma Z = \{(z, [g]) : z \in Z, [g] \in [G_z] - \{1\}\}. \tag{2.16}$$

By [27, Section 1], ΣZ possess a natural orbifold structure. Indeed, take $U \in \mathcal{U}$ and $(\widetilde{U}, \pi_U, G_U) \in \widetilde{\mathcal{U}}$. For $g \in G_U$, denote by $\widetilde{U}^g \subset \widetilde{U}$ the set of fixed points of g in $\widetilde{U}$, and by $Z_{G_U}(g) \subset G_U$ the centralizer of g in G_U. Clearly, $Z_{G_U}(g)$ acts on $\widetilde{U}^g$, and the quotient $Z_{G_U}(g)\backslash\widetilde{U}^g$ depends only on the conjugacy class $[g] \in [G_U]$. The map $x \in \widetilde{U}^g \to (\pi_U(x), [g]) \in \Sigma U$ induces an identification

$$\coprod_{[g]\in[G_U]\backslash\{1\}} Z_{G_U}(g)\backslash\widetilde{U}^g \simeq \Sigma U. \tag{2.17}$$

By (2.17), we equip ΣU with the induced topology and orbifold structure. The topology and the orbifold structure on ΣZ is obtained by gluing ΣU. We omit the detail.

We decompose $\Sigma Z = \coprod_{i=1}^{l_0} Z_i$ following its connected components. If $(z, [g]) \in Z_i$, set

$$m_i = \left|\ker\left(Z_{G_U}(g) \to \mathrm{Diffeo}(\widetilde{U}^g)\right)\right| \in \mathbf{N}^*. \tag{2.18}$$

By definition, m_i is locally constant, and is called the multiplicity of Z_i. In the sequel, we write

$$Z_0 = Z, \qquad m_0 = 1. \tag{2.19}$$

2.3. Orbifold vector bundle

We recall the definition of orbifold vector bundles.

Definition 2.14. A complex orbifold vector bundle E of rank r on Z consists of an orbifold $\mathcal{E}$, called the total space, and a smooth map $\pi : \mathcal{E} \to Z$, such that

1) there is an orbifold atlas $(\mathcal{U}, \widetilde{\mathcal{U}})$ of Z such that for any $U \in \mathcal{U}$ and $(\widetilde{U}, G_U, \pi_U) \in \widetilde{\mathcal{U}}$, there exist a finite group G_U^E acting smoothly on $\widetilde{U}$ which induces a surjective morphism of groups $G_U^E \to G_U$, a representation $\rho_U^E : G_U^E \to \mathrm{GL}_r(\mathbf{C})$, and a G_U^E-invariant continuous map $\pi_U^E : \widetilde{U} \times \mathbf{C}^r \to \pi^{-1}(U)$ which induces a homomorphism of topological spaces

$$\widetilde{U} \times_{G_U^E} \mathbf{C}^r \simeq \pi^{-1}(U); \tag{2.20}$$

2) the triple $(\widetilde{U} \times \mathbf{C}^r, G_U^E, \pi_U^E)$ is a (compatible) orbifold chart on $\mathcal{E}$;
3) if $U_1, U_2 \in \mathcal{U}$ such that $U_1 \cap U_2 \neq \varnothing$, and for any $z \in U_1 \cap U_2$, there exist a connected open neighborhood $U_0 \subset U_1 \cap U_2$ of z with a simply connected orbifold chart $(\widetilde{U}_0, G_{U_0}, \pi_{U_0})$ and the triple $(G_{U_0}^E, \rho_{U_0}^E, \pi_{U_0}^E)$ such that (1) and (2) hold, and that the embeddings of orbifold charts of $\mathcal{E}$

$$\phi_{U_iU_0}^E : \left(\widetilde{U}_0 \times \mathbf{C}^r, G_{U_0}^E, \pi_{U_0}^E\right) \hookrightarrow \left(\widetilde{U}_i \times \mathbf{C}^r, G_{U_i}^E, \pi_{U_i}^E\right), \quad \text{for } i = 1, 2, \tag{2.21}$$

have the following form

$$\phi_{U_iU_0}^E(x, v) = \left(\phi_{U_iU_0}(x), g_{U_iU_0}^E(x)v\right), \qquad \text{for } (x, v) \in \widetilde{U}_0 \times \mathbf{C}^r, \tag{2.22}$$

where $\phi_{U_iU_0} : (\widetilde{U}_0, G_{U_0}, \pi_{U_0}) \hookrightarrow (\widetilde{U}_i, G_{U_i}, \pi_{U_i})$ is an embedding of orbifold charts of Z, and $g^E_{U_iU_0} \in C^\infty\big(\widetilde{U}_0, \mathrm{GL}_r(\mathbf{C})\big)$.

The vector bundle E is called proper if the surjective morphism $G^E_U \to G_U$ is an isomorphism, and is called flat if $g^E_{U_iU_0}$ can be chosen to be constant.

Remark 2.15. The embedding $\phi^E_{U_iU_0}$ exists since $\widetilde{U}_0 \times \mathbf{C}^r$ is simply connected. By Proposition 2.4, it is uniquely determined by the first component $\phi_{U_iU_0}$ when E is proper.

Remark 2.16. We can define the real orbifold vector bundle in an obvious way.

In the sequel, for $U \in \mathcal{U}$, we will denote by $\widetilde{E}_U$ the trivial vector bundle of rank r on $\widetilde{U}$, and by E_U the restriction of E to U. Their total spaces are given respectively by

$$\widetilde{\mathcal{E}}_U = \widetilde{U} \times \mathbf{C}^r, \qquad \mathcal{E}_U = \widetilde{U} \times_{G_U} \mathbf{C}^r. \tag{2.23}$$

Let us identify the associated groupoid $\mathcal{G}^E = (\mathcal{G}^E_0, \mathcal{G}^E_1)$ for the total space $\mathcal{E}$ of a proper orbifold vector bundle E. By (2.5), the object of $\mathcal{G}^E$ is given by

$$\mathcal{G}^E_0 = \coprod_{U \in \mathcal{U}} \widetilde{\mathcal{E}}_U = \mathcal{G}_0 \times \mathbf{C}^r. \tag{2.24}$$

If $g \in \mathcal{G}_1$ is represented by the germ of the transformation $\phi_{U_2U_0}\phi^{-1}_{U_1U_0}$, denote by g^E_* the germ of transformation $g^E_{U_2U_0} g^{E,-1}_{U_1U_0}$. By Remark 2.15, g^E_* is uniquely determined by g. Thus, if g is an arrow form x, and if $v \in \mathbf{C}^r$, (g, v) defines an arrow from (x, v) to $(gx, g^E_* v)$. This way gives an identification

$$\mathcal{G}^E_1 = \mathcal{G}_1 \times \mathbf{C}^r. \tag{2.25}$$

We give some examples of orbifold vector bundles.

Example 2.17. The tangent bundle TZ of an orbifold Z is a real proper orbifold vector bundle locally defined by $\{(T\widetilde{U}, G_U)\}_{U \in \mathcal{U}}$.

Example 2.18. Assume Z is covered by linear charts $\{(\widetilde{U}, G_U, \pi_U)\}_{U \in \mathcal{U}}$ (see Remark 2.9). The orientation line $o(TZ)$ is a real proper orbifold line

bundle on Z, locally defined by $(\widetilde{U} \times \mathbf{R}, G_U)$ where the action of $g \in G_U$ is given by

$$g : (x, v) \in \widetilde{U} \times \mathbf{R} \to (gx, (\operatorname{sign}\det(g))v) \in \widetilde{U} \times \mathbf{R}. \tag{2.26}$$

Clearly, $o(TZ)$ is flat. If $o(TZ)$ is trivial, Z is called orientable.

Example 2.19. If E, F are orbifold vector bundles on Z, then E^*, $\overline{E}$, $\Lambda^{\cdot}(E)$, $\mathscr{T}(E) = \oplus_{k\in\mathbf{N}} E^{\otimes k}$ and $E \otimes F$ are defined in an obvious way.

Example 2.20. Let E be an orbifold vector bundle on Z. For $U \in \mathcal{U}$, let $V_U \subset \mathbf{C}^r$ be subspace of $\mathbf{C}^r$ of the fixed points of $\ker(G_U^E \to G_U)$. Then G_U acts on V_U, and $\{(\widetilde{U} \times V_U, G_U)\}_{U\in\mathcal{U}}$ defines a proper orbifold vector bundle E^{pr} on Z. Clearly, if E is flat, then E^{pr} is also flat.

A smooth section of E is defined by a smooth map $s : Z \to E$ in the sense of Definition 2.10 such that $\pi \circ s = \mathrm{id}$ and that each local lift $\widetilde{s}_U$ of $s|_U$ is G_U^E-invariant. The space of smooth sections of E is denote by $C^\infty(Z, E)$. The space of differential forms with values in E is defined by $\Omega^{\cdot}(Z, E) = C^\infty(Z, \Lambda^{\cdot}(T^*Z) \otimes_{\mathbf{R}} E)$. For $k \in \mathbf{N}$, we define $C^k(Z, E)$ in a similar way. Also, the space of distributions $\mathcal{D}'(Z, E)$ of E is defined by the topological dual of $C^\infty(Z, E^*)$. By definition, we have

$$C^\infty(Z, E) = C^\infty(Z, E^{\mathrm{pr}}). \tag{2.27}$$

For this reason, most of results in this paper can be extended to non proper flat vector bundles.

Assume now E is proper. By (2.6), $s \in C^\infty(Z, E)$ can be represented by $\{s_U \in C^\infty(\widetilde{U}, \widetilde{E}_U)^{G_U}\}_{U\in\mathcal{U}}$ a family of G_U-invariant sections such that for any $x_1 \in U_1, x_2 \in U_2$ and $g \in \mathcal{G}_1$ from x_1 to x_2, near x_1 we have

$$g^* s_{U_2} = s_{U_1}. \tag{2.28}$$

We have the similar description for elements of $C^\infty(Z, E)$ and $D'(Z, E)$.

We call g^E a Hermitian metric on E, if g^E is a section in $C^\infty\big(Z, E^* \otimes \overline{E}^*\big)$ such that g^E is represented by a family $\{g^{\widetilde{E}_U}\}_{U\in\widetilde{U}}$ of G_U-invariant metrics on $\widetilde{E}_U$ such that (2.28) holds. If E is the real orbifold vector bundle TZ, g^{TZ} is called a Riemannian metric on Z.

Two orbifold vector bundles E and F are called isomorphic if there is $f \in C^\infty(Z, E^* \otimes F)$ and $g \in C^\infty(Z, F^* \otimes E)$ such that $fg = \mathrm{id}$ and $gf = \mathrm{id}$.

Let Γ be a discrete group acting smoothly and properly discontinuously on an orbifold X. Let $\rho : \Gamma \to \mathrm{GL}_r(\mathbf{C})$ be a representation of Γ. By Proposition 2.12, $\Gamma\backslash X$ and

$$\mathcal{F} = X \times_{\Gamma} \mathbf{C}^r \tag{2.29}$$

have canonical orbifold structures. The projection $X \times \mathbf{C}^r \to X$ descends to a smooth map of orbifolds

$$\pi : \mathcal{F} \to \Gamma\backslash X. \tag{2.30}$$

Proposition 2.21. *Assume that the action of Γ on X is smooth, properly discontinuous and effective. Then* (2.30) *defines canonically a proper flat vector bundle F on $\Gamma\backslash X$.*

Proof. Recall that $p : X \to \Gamma\backslash X$ is the projection. For $x \in X$, we use the same notations Γ_x, V_x, $(\widetilde{V}_x, H_x)$ and G_x as in the proof of Proposition 2.12. Then, $\Gamma\backslash X$ is covered by

$$p(V_x) \simeq \Gamma_x\backslash V_x \simeq G_x\backslash \widetilde{V}_x. \tag{2.31}$$

The stabilizer subgroup of Γ at $(x, 0) \in X \times \mathbf{C}^r$ is Γ_x. By (2.9), if $\gamma \in \Gamma - \Gamma_x$,

$$\gamma(V_x \times \mathbf{C}^r) \cap (V_x \times \mathbf{C}^r) = \varnothing. \tag{2.32}$$

As in (2.10), we have

$$\pi^{-1}\big(p(V_x)\big) = \Gamma\backslash\Gamma(V_x \times \mathbf{C}^r) \simeq V_x \times_{\Gamma_x} \mathbf{C}^r. \tag{2.33}$$

Since the action of Γ on X is effective, by Remark 2.13, we have a morphism of groups $G_x \to \Gamma_x$. The group G_x acts on $\mathbf{C}^r$ via the composition of $G_x \to \Gamma_x$ and $\rho|_{\Gamma_x} : \Gamma_x \to \mathrm{GL}_r(\mathbf{C})$. Thus, G_x acts on $\widetilde{V}_x \times \mathbf{C}^r$ effectively such that

$$V_x \times_{\Gamma_x} \mathbf{C}^r \simeq \widetilde{V}_x \times_{G_x} \mathbf{C}^r. \tag{2.34}$$

By Proposition 2.12, $(\widetilde{V}_x \times \mathbf{C}^r, G_x)$ is an orbifold chart of $\mathcal{F}$ for $\pi^{-1}(p(V_x))$.

Take two $(\widetilde{V}_{x_1} \times \mathbf{C}^r, G_{x_1})$ and $(\widetilde{V}_{x_2} \times \mathbf{C}^r, G_{x_2})$ orbifold charts of $\mathcal{F}$. It remains to show the compatibility condition (2.22). We proceed as in the proof of Proposition 2.12. If $x_2 = \gamma x_1$, then

$$(x, v) \in \widetilde{V}_{x_1} \times \mathbf{C}^r \to (\widetilde{\gamma}_{x_1} x, \rho(\gamma)v) \in \widetilde{V}_{x_2} \times \mathbf{C}^r \tag{2.35}$$

defines an isomorphism of orbifold charts on $\mathcal{F}$.

For general $x_1, x_2 \in X$, we can assume that $V_{x_1} \cap V_{x_2} \neq \varnothing$. For $x_0 \in V_{x_1} \cap V_{x_2}$, take V_{x_0}, $\widetilde{V}_{x_0}$ and $\phi_{V_{x_i} V_{x_0}}$ as in the proof of Proposition 2.12. Then

$$(x, v) \in \widetilde{V}_{x_0} \times \mathbf{C}^r \to (\phi_{V_{x_i} V_{x_0}}(x), v) \in \widetilde{V}_{x_i} \times \mathbf{C}^r \tag{2.36}$$

define two embeddings of orbifold charts of $\mathcal{F}$. From (2.35) and (2.36), we deduce that (2.30) defines a flat orbifold vector bundle on $\Gamma \backslash X$. The properness is clear from the construction. The proof of our proposition is completed. □

Remark 2.22. Take $A \in \mathrm{GL}_r(\mathbf{C})$. Let $\rho_A : \gamma \in \Gamma \to A\rho(\gamma)A^{-1} \in \mathrm{GL}_r(\mathbf{C})$ be another representation of Γ. Then

$$(x, v) \in X \times \mathbf{C}^r \to (x, Av) \in X \times \mathbf{C}^r \tag{2.37}$$

descends to an isomorphism between flat orbifold vector bundles $X \times_{\rho} \mathbf{C}^r$ and $X \times_{\rho_A} \mathbf{C}^r$.

2.4. Orbifold fundamental groups and universal covering orbifold

In this subsection, following [26], [11, Section III.$\mathcal{G}$.3], we recall the constructions of the orbifold fundamental group and the universal covering orbifold. We assume that the orbifold Z is connected. Let $\mathcal{G}$ be the groupoid associated with some orbifold atlas $(\mathcal{U}, \widetilde{\mathcal{U}})$.

Definition 2.23. A continuous $\mathcal{G}$-path $c = (b_1, \ldots, b_k; g_0, \ldots, g_k)$ starting at $x \in \mathcal{G}_0$ and ending at $y \in \mathcal{G}_0$ parametrized by $[0, 1]$ is given by

1) a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ of $[0, 1]$;
2) continuous paths $b_i : [t_{i-1}, t_i] \to \mathcal{G}_0$, for $1 \leq i \leq k$;
3) arrows $g_i \in \mathcal{G}_1$ such that $g_0 : x \to b_1(0)$, $g_i : b_i(t_i) \to b_{i+1}(t_i)$, for $1 \leq i \leq k-1$, and $g_k : b_k(1) \to y$.

If $x = y$, we call that c is a $\mathcal{G}$-loop based at x.

Two $\mathcal{G}$-paths

$$c = (b_1, \cdots, b_k; g_0, \cdots, g_k), \qquad c' = (b'_1, \cdots, b'_{k'}; g'_0, \cdots, g'_{k'}), \tag{2.38}$$

such that c ending at y and c' starting at y can be composed into a $\mathcal{G}$-path (with a suitable reparametrization [11, Section III.$\mathcal{G}$.3.4]),

$$cc' = (b_1, \cdots, b_k, b'_1, \cdots, b'_{k'}; g_0, \cdots, g'_0 g_k, \cdots, g'_{k'}). \tag{2.39}$$

Also, we can define the inverse of a $\mathcal{G}$-path in an obvious way.

Definition 2.24. We define an equivalence relation on $\mathcal{G}$-paths generated by

1) subdivision of the partition and adjunction by identity elements of $\mathcal{G}_1$ on new partition points.
2) for some $1 \leq i_0 \leq k$, replacement of the triple $(b_{i_0}, g_{i_0-1}, g_{i_0})$ by the triple $(hb_{i_0}, hg_{i_0-1}, g_{i_0}h^{-1})$, where $h \in \mathcal{G}_1$ is well-defined near the path $b_{i_0}([t_{i_0-1}, t_{i_0}])$.

The equivalent class of $\mathcal{G}$-paths is called the path on the orbifold Z.

Remark 2.25. If Z is equipped with a Riemannian metric, then the length of a path on Z represented by the $\mathcal{G}$-path $c = (b_1, \ldots, b_k; g_0, \ldots, g_k)$ is defined by the sum of the lengths of b_i. Clearly, this definition does not depend on the choice of the representative c. The set of paths on Z with length 0 is just the orbifold $Z \coprod \Sigma Z$.

Remark 2.26. Following [25, Section 2.4.2], if Z is equipped with a Riemannian metric, a $\mathcal{G}$-path $c = (b_1, \ldots, b_k; g_0, \ldots, g_k)$ is called a $\mathcal{G}$-geodesic, if for all $1 \leq i \leq k$, b_i is a geodesic and if for all $1 \leq i \leq k-1$, $g_{i,*}\dot{b}_i(t_i) = \dot{b}_{i+1}(t_i)$. The geodesic on Z is defined by the equivalence class of the $\mathcal{G}$-geodesics. Similarly, we can define the closed geodesic on Z by the equivalent class of the closed geodesic $\mathcal{G}$-paths, i.e., a $\mathcal{G}$-geodesic starting and ending at the same point such that $g_{0,*}g_{k,*}\dot{b}_k(1) = \dot{b}_0(0)$.

Definition 2.27. An elementary homotopy between two $\mathcal{G}$-paths c and c' is a family, parametrized by $s \in [0, 1]$, of $\mathcal{G}$-paths $c^s = (b^s_1, \cdots, b^s_k; g^s_0, \cdots, g^s_k)$, over the subdivisions $0 = t^s_0 \leq t^s_1 \leq \cdots \leq t^s_k = 1$, where t^s_i, b^s_i and g^s_i depend continuously on the parameter s, the elements g^s_0 and g^s_k are independent of s and $c^0 = c, c^1 = c'$.

Definition 2.28. Two $\mathcal{G}$-paths are said to be homotopic (with fixed extremities) if one can pass from the first to the second by equivalences of $\mathcal{G}$-paths and elementary homotopies. The homotopy class of a $\mathcal{G}$-path c will be denoted by $[c]$.

As ordinary paths in topological spaces, the composition and inverse operations of $\mathcal{G}$-paths are well-defined for their homotopy classes.

Definition 2.29. Take $x_0 \in \mathcal{G}_0$. With the operations of composition and inverse of $\mathcal{G}$-paths, the homotopy classes of $\mathcal{G}$-loops based at x_0 form a group $\pi_1^{\mathrm{orb}}(Z, x_0)$ called the orbifold fundamental group.

As Z is connected, any two points of $\mathcal{G}_0$ can be connected by a $\mathcal{G}$-path. Thus, the isomorphic class of the group $\pi_1^{\mathrm{orb}}(Z, x_0)$ does not depend on the choice of x_0. Also, it depends only on the orbifold structure of Z. In the sequel, for simplicity, we denote by $\Gamma = \pi_1^{\mathrm{orb}}(Z, x_0)$.

Remark 2.30. As a fundamental group of a manifold, Γ is countable.

In the rest of this subsection, following [11, Section III.$\mathcal{G}$.3.20], we will construct the universal covering orbifold X of Z. Let us begin with introducing a groupoid $\widehat{\mathcal{G}}$. Assume that $\mathcal{U} = \{U_z\}$ and $\widetilde{\mathcal{U}} = \{(\widetilde{U}_z, G_z, \pi_z)\}$ where all the $\widetilde{U}_z$ are simply connected and are centered at $x \in \widetilde{U}_z$ as in Remark 2.9. Fix $x_0 \in \mathcal{G}_0$ as before.

Let $\widehat{\mathcal{G}}_0$ be the space of homotopy classes of $\mathcal{G}$-paths starting at x_0. The group Γ acts naturally on $\widehat{\mathcal{G}}_0$ by composition at the starting point x_0. We denote by

$$\widehat{p} : \widehat{\mathcal{G}}_0 \to \mathcal{G}_0 \tag{2.40}$$

the projection sending $[c] \in \widehat{\mathcal{G}}_0$ to its ending point. Clearly, $\widehat{p}$ is Γ-invariant.

Define a topology and manifold structure on $\widehat{\mathcal{G}}_0$ as follows. For $x_1, x_2 \in \widetilde{U}_z$, we denote by $c_{x_1x_2} = (b_{x_1x_2}; \mathrm{id}, \mathrm{id})$ a $\mathcal{G}$-path starting at x_1 and ending at x_2, where $b_{x_1x_2}$ is a path in $\widetilde{U}_z$ connecting x_1 and x_2. Note that since $\widetilde{U}_z$ is simply connected, the homotopy class $[c_{x_1x_2}]$ does not depend on the choice of $b_{x_1x_2}$. For each $U_z \in \mathcal{U}$, we fix a $\mathcal{G}$-path c_z starting at x_0 and ending at $x \in \widetilde{U}_z$. For $a \in \Gamma$, set

$$\widetilde{V}_{z,a} = \left\{ [c] \in \widehat{p}^{-1}(\widetilde{U}_z) : \left[c c_{\widehat{p}([c])x} c_z^{-1} \right] = a \right\}. \tag{2.41}$$

By (2.40) and (2.41), we have

$$\widehat{p}^{-1}(\widetilde{U}_z) = \coprod_{a \in \Gamma} \widetilde{V}_{z,a}, \qquad \widehat{\mathcal{G}}_0 = \coprod_{U_z \in \mathcal{U}, a \in \Gamma} \widetilde{V}_{z,a}. \tag{2.42}$$

Also,

$$\widehat{p} : \widetilde{V}_{z,a} \to \widetilde{U}_z \tag{2.43}$$

is a bijection. We equip $\widetilde{V}_{z,a}$ with a topology and a manifold structure via (2.43). Clearly, the choice of c_z is irrelevant. By (2.42), $\widehat{\mathcal{G}}_0$ is a countable disjoint union of smooth manifolds such that (2.40) is a Galois covering with deck transformation group Γ.

If $y \in \mathcal{G}_0$ and if $g \in \mathcal{G}_1$ is defined near y, we denote by

$$c_{y,g} = (b_y; \mathrm{id}, g) \tag{2.44}$$

the $\mathcal{G}$-path, where b_y is the constant path at y. Set

$$\widehat{\mathcal{G}}_1 = \big\{([c], g) \in \widehat{\mathcal{G}}_0 \times \mathcal{G}_1 : g \text{ is defined near } \widehat{p}([c]) \in \mathcal{G}_0\big\}. \tag{2.45}$$

Then, $([c], g) \in \widehat{\mathcal{G}}_1$ represents an arrow from $[c]$ to $[c][c_{\widehat{p}[c],g}]$. This defines a groupoid $\widehat{\mathcal{G}} = (\widehat{\mathcal{G}}_0, \widehat{\mathcal{G}}_1)$.

Let

$$X = \widehat{\mathcal{G}}_0 / \widehat{\mathcal{G}}_1 \tag{2.46}$$

be the orbit space of $\widehat{\mathcal{G}}$ equipped with the quotient topology. The action of Γ on $\widehat{\mathcal{G}}_0$ descends to an effective and continuous action on X. The projection $\widehat{p}$ descends to a Γ-invariant continuous map

$$p : X \to Z. \tag{2.47}$$

Theorem 2.31. *Assume that Z is a connected orbifold. Then, the topological space X defined in* (2.46) *is connected and has a canonical orbifold structure such that Γ acts smoothly, effectively and properly discontinuously on X. Moreover,* (2.47) *induces an isomorphism of orbifolds*

$$\Gamma \backslash X \to Z. \tag{2.48}$$

Proof. Let us begin with showing that the topological space X is connected. Take any $\mathcal{G}$-path $c = (b_1, \cdots, b_k; g_0, \cdots, g_k)$ starting at x_0. By our construction of $\widehat{\mathcal{G}_1}$, the images in X of the $\mathcal{G}$-paths c and $(b_1, \cdots, b_k; g_0, \cdots, g_{k-1}, \mathrm{id})$ are in the same connected component of X. The same holds true for the $\mathcal{G}$-paths $(b_1, \cdots, b_{k-1}; g_0, \cdots, g_{k-1})$ and $(b_1, \cdots, b_k; g_0, \cdots, g_{k-1}, \mathrm{id})$. By induction argument, the images in X of c and the constant $\mathcal{G}$-path at x_0 are in the same connected component of X. So X is connected.

Let us construct an orbifold atlas on X. For $a \in \Gamma$, let $\pi_{z,a}$ be the composition of continuous maps $\widetilde{V}_{z,a} \hookrightarrow \widehat{\mathcal{G}}_0 \to X$. Set

$$V_{z,a} = \pi_{z,a}(\widetilde{V}_{z,a}) \subset X. \tag{2.49}$$

By (2.42),

$$p^{-1}(U_z) = \bigcup_{a \in \Gamma} V_{z,a}. \tag{2.50}$$

Recall that $x \in \widetilde{U}_z$ and $\pi_z(x) = z$. By (2.44), for $g \in G_z$, $c_{x,g}$ is a $\mathcal{G}$-loop based at x. Set

$$r_z : g \in G_z \to [c_z c_{x,g} c_z^{-1}] \in \Gamma. \tag{2.51}$$

Then, r_z is a morphism of groups. By (2.46) and (2.50),

$$p^{-1}(U_z) = \coprod_{[a] \in \Gamma / \operatorname{Im}(r_z)} V_{z,a}. \tag{2.52}$$

Using the fact that $p^{-1}(U_z)$ is open in X, we can deduce that $V_{z,a}$ is open in X.

Put

$$H_z = \ker r_z. \tag{2.53}$$

By (2.51) and (2.53), H_z acts on $\widetilde{V}_{z,a}$ by

$$(g, [c]) \in H_z \times \widetilde{V}_{z,a} \to [c][c_{\widehat{p}([c]), g^{-1}}] \in \widetilde{V}_{z,a}. \tag{2.54}$$

Then $\pi_{z,a}$ induces a homeomorphism of topological spaces

$$H_z \backslash \widetilde{V}_{z,a} \simeq V_{z,a}. \tag{2.55}$$

As the H_z-action on $\widetilde{V}_{z,a}$ is effective, $(\widetilde{V}_{z,a}, H_z, \pi_{z,a})$ is an orbifold chart for $V_{z,a}$. Moreover, the compatibility of each charts is a consequence of (2.43) and the compatibility of charts in $\widetilde{\mathcal{U}}$. Hence, $\{(\widetilde{V}_{z,a}, H_z, \pi_{z,a})\}_{U_z \in \mathcal{U}, a \in \Gamma}$ forms an orbifold atlas on X.

As Γ is countable, X is second countable. We will show that X is Hausdorff. Indeed, take $y_1, y_2 \in X$ and $y_1 \neq y_2$. If $p(y_1) \neq p(y_2)$, as Z is Hausdorff, take respectively open neighborhoods U_1 and U_2 of $p(y_1)$ and of $p(y_2)$

such that $U_1 \cap U_2 = \varnothing$. Then $p^{-1}(U_1) \cap p^{-1}(U_2) = \varnothing$. Assume $p(y_1) = p(y_2)$. By adding charts in $\mathcal{U}$, we can assume that there is $U_z \in \mathcal{U}$ such that $p(y_1) = p(y_2) = z$ with orbifold charts $(\widetilde{U}_z, G_z, \pi_z)$ centered at x. Assume y_1, y_2 are represented by $\mathcal{G}$-paths c_1 and c_2 starting at x_0 and ending at x. For $i = 1, 2$, set

$$a_i = [c_i][c_{z_0}^{-1}] \in \Gamma. \tag{2.56}$$

As $y_1 \neq y_2$, then $[a_1] \neq [a_2] \in \Gamma / \operatorname{Im}(r_z)$. Thus,

$$y_1 \in V_{z,a_1}, \qquad y_2 \in V_{z,a_2}, \qquad V_{z,a_1} \cap V_{z,a_2} = \varnothing. \tag{2.57}$$

In summary, we have shown that X is an orbifold.

Note that $\gamma V_{z,a} = V_{z,\gamma a}$. By (2.52), the set

$$\{\gamma \in \Gamma : \gamma V_{z,a} \cap V_{z,a} \neq \varnothing\} = a \operatorname{Im}(r_z) a^{-1} \subset \Gamma \tag{2.58}$$

is finite. Then the Γ-action on X is properly discontinuous. As Γ acts on $\widehat{\mathcal{G}}_0$, the Γ-action is smooth.

We claim that (2.48) is homeomorphism of topological space. Indeed, by the construction, (2.48) is injective. It is surjective as Z is connected. The continuity of the inverse (2.48) is a consequence of (2.52).

The isomorphism of orbifolds between $\Gamma \backslash X$ and Z is a consequence of Proposition 2.12 and (2.43), (2.55) and (2.58). The proof our theorem is completed. □

Remark 2.32. By (2.52), (2.55), and by the covering orbifold theory of Thurston [50, Definition 13.2.2], $p : X \to Z$ is a covering orbifold of Z. Moreover, we can show that for any connected covering orbifold $p' : Y \to Z$, there exists a covering orbifold $p'' : X \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[dr, "p''"] \arrow[dd, "p"'] & \\ & Y \arrow[dl, "p'"] \\ Z & \end{tikzcd} \tag{2.59}$$

commutes. For this reason, X is called a universal covering orbifold of Z. As in the case of the classical covering theory of topological spaces, the universal

covering orbifold is unique up to covering isomorphism. Also, Γ is isomorphic to the orbifold deck transformation group of X.

Remark 2.33. If a connected covering orbifold X' of Z has a trivial orbifold fundamental group, then X' is a universal covering orbifold of Z. In particular, if Z has a covering orbifold X', which is a connected simply connected manifold, then X' is a universal covering orbifold of Z.

Example 2.34. The teardrop Z_n with $n \geqslant 2$ (see Figure 2.1) is an example of an orbifold with a trivial orbifold fundamental group which is not a manifold (c.f. [50, p. 304]). Its underlying topological space is a 2-sphere $\mathbb{S}^2$, and its singular set consists of a single point, whose neighbourhood is modelled on $\mathbf{R}^2/(\mathbf{Z}/n\mathbf{Z})$, where the cyclic group $\mathbf{Z}/n\mathbf{Z}$ acts by rotations.

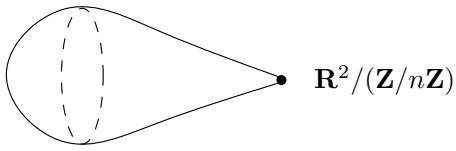

Figure 2.1: Teardrop Z_n.

2.5. Flat vector bundles and holonomy

In this subsection, we still assume that Z is a connected orbifold. Let F be a proper flat orbifold vector bundle on Z. Let $(\mathcal{U}, \widetilde{\mathcal{U}})$ be an orbifold atlas as in Definition 2.14. Let $\mathcal{G}$ be the associated groupoid. We fix $x_0 \in \mathcal{G}$.

For a $\mathcal{G}$-path $c = (b_1, \cdots, b_k; g_0, \cdots, g_k)$, the parallel transport τ_c of F along c is defined by

$$\tau_c = g^F_{k,*} \cdots g^F_{0,*} \in \mathrm{GL}_r(\mathbf{C}). \tag{2.60}$$

It depends only on the homotopy class of c. In particular, it defines a representation, called holonomy representation of F,

$$\rho : \Gamma \to \mathrm{GL}_r(\mathbf{C}). \tag{2.61}$$

The isomorphic class of the representation ρ is independent of the choice of orbifold atlas on Z, of the local trivialization of F, and of the choice of x_0. Moreover, it does not depend on the isomorphic class of F.

Let $\mathrm{Hom}(\Gamma, \mathrm{GL}_r(\mathbf{C}))/_\sim$ be the set of equivalent classes of complex representations of Γ of dimension r, and let $\mathcal{M}_r^{\mathrm{pr}}(Z)$ be the set of isomorphic classes of proper complex flat orbifold vector bundles of rank r on Z. By Proposition 2.21 and Remark 2.22, the map

$$\rho \in \mathrm{Hom}(\Gamma, \mathrm{GL}_r(\mathbf{C}))/_\sim \to X_\rho \times \mathbf{C}^r \in \mathcal{M}_r^{\mathrm{pr}}(Z) \tag{2.62}$$

is well-defined.

Theorem 2.35. *The map* (2.62) *is one-one and onto, whose inverse is given by the holonomy representation* (2.61).

Proof. Step 1. The holonomy representation of $X_\rho \times \mathbf{C}^r$ is isomorphic to ρ.

Assume that the orbifold Z is covered by $\{U_z\}$ with simply connected orbifold charts $\{\widetilde{U}_z\}$ centered at $x \in \widetilde{U}_z$ and X is covered by $\{V_{z,a}\}$ as (2.52) such that $p(V_{z,a}) = U_z$.

Take $\gamma \in \Gamma$. Let $c = (b_1, \cdots, b_k; g_0, \cdots, g_k)$ be a $\mathcal{G}$-loop based at x_0 which represents γ. It is enough to show the parallel transport along c is $\rho(\gamma)$. For $1 \leq i \leq k$, take

$$x_i = b_i(t_{i-1}). \tag{2.63}$$

Up to equivalence relation of c and up to adding charts into the orbifold atlas of Z, we can assume that there are orbifold charts $\widetilde{U}_{z_i}$ of Z centered at x_i such that $b_i : [t_{i-1}, t_i] \to \widetilde{U}_{z_i}$. Also, we assume that $\widetilde{U}_{z_0}$ is an orbifold chart of Z centered at x_0.

Let $c_{z_1} = c_{x_0,g_0}$ as in (2.44). For $2 \leq i \leq k$, set

$$c_{z_i} = (b_1, \cdots, b_{i-1}; g_0, \cdots, g_{i-1}). \tag{2.64}$$

By (2.41), for $1 \leq i \leq k$, c_{z_i} is a $\mathcal{G}$-path starting at x_0 and ending at x_i such that

$$[c_{z_i}] \in \widetilde{V}_{z_i,1}. \tag{2.65}$$

We claim that for $1 \leq i \leq k$,

$$V_{z_{i-1},1} \cap V_{z_i,1} \neq \varnothing, \qquad V_{z_k,1} \cap V_{z_0,\gamma} \neq \varnothing. \tag{2.66}$$

Indeed, $c'_{z_i} = (b_1, \cdots, b_{i-1}; g_0, \cdots, g_{i-2}, \mathrm{id})$ projects to the same element of X as c_{z_i}, and $[c'_{z_i}] \in \widetilde{V}_{z_{i-1},1}$. Also, $c'_{z_{k+1}}$ projects to the same element of X as c.

Recall that $\gamma V_{z_0,1} = V_{z_0,\gamma}$. By (2.35), (2.36) and (2.66), for $1 \leq i \leq k-1$, we have

$$g_{i,*} = 1, \qquad g_{k,*} = \rho(\gamma). \tag{2.67}$$

By (2.60), the parallel transport along c is $\rho(\gamma)$.

Step 2. If F has holonomy ρ, then F is isomorphic to $X \times_\rho \mathbf{C}^r$.

We will construct the bundle isomorphism. By (2.24) and (2.25), the groupoid of the total space $\mathcal{F}$ is given by $\mathcal{G}^F = (\mathcal{G}_0 \times \mathbf{C}^r, \mathcal{G}_1 \times \mathbf{C}^r)$. Let us construct a universal covering orbifold of $\mathcal{F}$ by determining its groupoid $\widehat{\mathcal{G}}^F$.

Take $(x_0, 0), (x_1, u) \in \mathcal{G}_0^F$. Let (c, v) be a $\mathcal{G}^F$-path starting at $(x_0, 0)$ and ending at (x_1, u). Then there is a partition of $[0, 1]$ given by $0 = t_0 < \cdots < t_k = 1$ such that $c = (b_1, \cdots, b_k; g_0, \cdots, g_k)$ as in Definition 2.23. Also, $v = (v_1, \cdots, v_k)$, where $v_i : [t_{i-1}, t_i] \to \mathbf{C}^r$ is a continuous path such that $v_1(0) = 0$, $g_{k,*}^F v_k(1) = u$ and for $1 \leq i \leq k-1$,

$$g_{i,*}^F v_i(t_i) = v_{i+1}(t_i). \tag{2.68}$$

Put $w : [0, 1] \to \mathbf{C}^r$ a continuous path such that for $t \in [t_{i-1}, t_i]$,

$$w(t) = g_{0,*}^{F,-1} \cdots g_{i-1,*}^{F,-1} v_i(t). \tag{2.69}$$

We identify (c, v) with (c, w) via (2.69). Then, (c, v) is homotopic to (c', v') if and only if c, c' are homotopic as $\mathcal{G}$-path and w, w' are homotopic as ordinary continuous paths in $\mathbf{C}^r$. Since any continuous path $w : [0, 1] \to \mathbf{C}^r$ such that $w(0) = 0$ is homotopic to the path $t \in [0, 1] \to tw(1)$, we have the identification

$$[c, v] \in \widehat{\mathcal{G}}_0^F \to ([c], w(1)) = ([c], \tau_c^{-1} u) \in \widehat{\mathcal{G}}_0 \times \mathbf{C}^r. \tag{2.70}$$

In particular, we have an isomorphism of groups

$$[c] \in \Gamma \to [(c, 0)] \in \pi_1^{\mathrm{orb}}(\mathcal{F}, (x_0, 0)), \tag{2.71}$$

where 0 is the constant loop at $0 \in \mathbf{C}^r$.

In the same way, we identify

$$([c, v], g) \in \widehat{\mathcal{G}}_1^F \to \big(([c], g), w(1)\big) \in \widehat{\mathcal{G}}_1 \times \mathbf{C}^r. \tag{2.72}$$

We deduce that $\big(([c], g), w(1)\big)$ represents an arrow from $\big([c], w(1)\big) \in \widehat{\mathcal{G}}_0^F$ to $\big([cc_g], w(1)\big) \in \widehat{\mathcal{G}}_0^F$. Therefore, the orbit space of $\widehat{\mathcal{G}}^F$, which is also the universal covering orbifold of $\mathcal{F}$, is given by $X \times \mathbf{C}^r$.

By the identification (2.70), the projection (2.40) is given by

$$\widehat{p}_\rho : \big([c], w(1)\big) \in \widehat{\mathcal{G}}_0 \times \mathbf{C}^r \to \big(\widehat{p}([c]), w(1)\big) \in \mathcal{G}_0 \times \mathbf{C}^r. \tag{2.73}$$

The group Γ acts on the left on $\widehat{\mathcal{G}}_0$, and on the left on $\mathbf{C}^r$ by ρ. As in (2.40), the projection (2.73) is a Galois covering with deck transformation group Γ. And $\widehat{p}_\rho$ descends to a Γ-invariant continuous map

$$p_\rho : X \times \mathbf{C}^r \to \mathcal{F}. \tag{2.74}$$

By Theorem 2.31 , p_ρ induces an isomorphism of orbifolds

$$X \times_\rho \mathbf{C}^r \simeq \mathcal{F}. \tag{2.75}$$

Using the fact that (2.73) is linear on the $\mathbf{C}^r$, we can deduce that (2.75) is an isomorphism of orbifold vector bundles. The proof of our theorem is completed. □

Remark 2.36. The properness condition is necessary. Indeed, Theorem 2.35 implies that the proper flat vector bundle is trivial on the universal cover. Consider a non trivial finite group H acting effectively on $\mathbf{C}^r$. Then $H\backslash\mathbf{C}^r$ is a non proper orbifold vector bundle over a point. Clearly, it is not trivial.

Remark 2.37. By (2.27) and Theorem 2.35, we get Corollary 0.2.

3. Differential calculus on orbifolds

The purpose of this section is to explain briefly how to extend the usual differential calculus to orbifolds. To simplify our presentation, we assume that the underlying orbifold is compact. We assume also that the orbifold vector bundles are proper. By (2.27), all the constructions in this section extend trivially to non proper orbifold vector bundles.

This section is organized as follows. In subsections 3.1-3.3, we introduce differential operators, integration of differential forms, integral operators and Sobolev space on orbifolds.

In subsection 3.4, we explain Chern-Weil theory for the orbifold vector bundles. The Euler form, odd Chern character form, their Chern-Simons classes, and their canonical extensions to $Z \coprod \Sigma Z$ are constructed in detail.

3.1. Differential operators on orbifolds

Let Z be a compact orbifold with atlas $(\mathcal{U},\widetilde{\mathcal{U}})$. Let E be a proper orbifold vector bundle on Z such that (2.20) holds.

A differential operator D of order p is a family $\{\widetilde{D}_U : C^\infty(\widetilde{U},\widetilde{E}_U) \to C^\infty(\widetilde{U},\widetilde{E}_U)\}_{U\in\mathcal{U}}$ of G_U-invariant differential operators of order p such that if $g \in \mathcal{G}_1$ is an arrow from $x_1 \in \widetilde{U}_1$ to $x_2 \in \widetilde{U}_2$, then near x_1, we have

$$g^*\widetilde{D}_{U_2} = \widetilde{D}_{U_1}. \tag{3.1}$$

If each $\widetilde{D}_U$ is elliptic, then D is called elliptic.

If $s \in C^\infty(Z,E)$ is represented by the family $\{s_U \in C^\infty(\widetilde{U},\widetilde{E}_U)^{G_U}\}$ such that (2.28) holds. By (2.28) and (3.1), $\{\widetilde{D}_U\widetilde{s}_U \in C^\infty(\widetilde{U},\widetilde{E}_U)^{G_U}\}_{U\in\mathcal{U}}$ defines a section of E, which is denoted by Ds. Clearly, $D : C^\infty(Z,E) \to C^\infty(Z,E)$ is a linear operator such that

$$\mathrm{Supp}(Ds) \subset \mathrm{Supp}(s). \tag{3.2}$$

As in the manifold case, the differential operator acts naturally on distributions.

Example 3.1. A connection ∇^E on E is a first order differential operator from $C^\infty(Z,E)$ to $\Omega^1(Z,E)$ such that ∇^E is represented by a family $\{\nabla^{\widetilde{E}_U}\}_{U\in\widetilde{U}}$ of G_U-invariant connections on $\widetilde{E}_U$ such that (3.1) holds. The curvature $R^E = (\nabla^E)^2$ is defined as usual. It is a section of $\Lambda^2(T^*Z)\otimes_{\mathbf{R}} \mathrm{End}(E)$. As usual, ∇^E is called metric with respect to a Hermitian metric g^E if $\nabla^E g^E = 0$.

Example 3.2. Let (Z, g^{TZ}) be a Riemannian orbifold. If g^{TZ} is defined by the family $\{g^{T\widetilde{U}}\}_{U\in\mathcal{U}}$ of Riemannian metrics, then the family of Levi-civita connections on $(\widetilde{U}, g^{T\widetilde{U}})$ defines the Levi-civita connection ∇^{TZ} on (Z, g^{TZ}).

Example 3.3. Let F be a flat orbifold vector bundle on Z. The de Rham operator $d^Z : \Omega^\cdot(Z,F) \to \Omega^{\cdot+1}(Z,F)$ is a first order differential operator represented by the family of de Rham operators

$$\left\{d^{\widetilde{U}} : \Omega^\cdot(\widetilde{U},\mathbf{C}^r) \to \Omega^{\cdot+1}(\widetilde{U},\mathbf{C}^r)\right\}_{U\in\mathcal{U}}. \tag{3.3}$$

Clearly, $(d^Z)^2 = 0$. The complex $\left(\Omega^\cdot(Z,F), d^Z\right)$ is called the orbifold de Rham complex with values in F. Denote by $H^\cdot(Z,F)$ the corresponding

cohomology. When $F = \mathbf{R}$ is the trivial bundle, we denote simply by $\Omega^{\cdot}(Z)$ and $H^{\cdot}(Z)$.[2] Clearly, $\nabla^F = d^Z|_{C^\infty(Z,F)}$ defines a connection on F with vanishing curvature. As in manifold case, such a connection will be called flat. We say F is unitarily flat, if there exists a Hermitian metric g^F on F such that $\nabla^F g^F = 0$. Clearly, this is equivalent to say the holonomy representation ρ is unitary.

3.2. Integral operators on orbifolds

Since Z is Hausdorff and compact, there exists a (finite) partition of unity subordinate to $\mathcal{U}$. That means there is a (finite) family of smooth functions $\{\phi_i \in C_c^\infty(Z,[0,1])\}_{i\in I}$ on Z such that the support $\operatorname{Supp}\phi_i$ is contained in some $U_i \in \mathcal{U}$, and that

$$\sum_{i\in I} \phi_i = 1. \tag{3.4}$$

Denote by

$$\widetilde{\phi}_i = \pi_{U_i}^*\big(\phi_i\big) \in C_c^\infty(\widetilde{U}_i)^{G_{U_i}}. \tag{3.5}$$

Following [44, p. 474], for $\alpha \in \Omega^{\cdot}\big(Z, o(TZ)\big)$ which is represented by the invariant forms $\{\alpha_{U_i} \in \Omega^{\cdot}(\widetilde{U}_i, o(T\widetilde{U}_i))^{G_{U_i}}\}_{i\in I}$, define

$$\int_Z \alpha = \sum_{i\in I} \frac{1}{|G_{U_i}|} \int_{\widetilde{U}_i} \widetilde{\phi}_i \alpha_{U_i}. \tag{3.6}$$

By (3.4) and (3.6), we get:

Proposition 3.4. *If $\alpha \in \Omega^{\cdot}\big(Z, o(TZ)\big)$, then α is integrable on Z_{reg} such that*

$$\int_Z \alpha = \int_{Z_{\mathrm{reg}}} \alpha. \tag{3.7}$$

From (3.7), we see that the definition (3.6) does not depend on the choice of orbifold atlas and the partition of unity. Also, we have the orbifold Stokes formula.

[2] By Satake [43], $H^{\cdot}(Z)$ coincides with the singular cohomology of the underlying topological space Z. In general, $H^{\cdot}(Z,F)$ coincides with the cohomology of the sheaf of locally constant sections of F.

Theorem 3.5. *The following identity holds: for* $\alpha \in \Omega^{\cdot}\big(Z, o(TZ)\big)$,

$$\int_Z d^Z \alpha = 0. \tag{3.8}$$

Let us introduce integral operators. Let (E, g^E) be a Euclidean orbifold vector bundle on Z. Fix a volume form $dv_Z \in \Omega^{\cdot}(Z, o(TZ))$ of Z. Then, we can define the space $L^2(Z, E)$ of L^2-sections in an obviously way. By (3.7), we have

$$L^2(Z, E) = L^2(Z_{\mathrm{reg}}, E_{\mathrm{reg}}). \tag{3.9}$$

As in manifold case, with the help of dv_Z, we have the natural embedding $C^\infty(Z, E) \to \mathcal{D}'(Z, E)$.

By our local description of smooth sections and distributions (2.28), the Schwartz kernel theorem still holds for orbifolds. That means for any continuous linear map $A : C^\infty(Z, E) \to \mathcal{D}'(Z, E)$, there exists a unique $p \in \mathcal{D}'(Z \times Z, E \boxtimes E^*)$ such that for $s_1 \in C^\infty(Z, E)$ and $s_2 \in C^\infty(Z, E^*)$, we have

$$\langle As_1, s_2\rangle = \langle p, s_2 \otimes s_1\rangle. \tag{3.10}$$

Assume that p is of class C^k for some $k \in \mathbf{N}$. Then A is called integral operator. The restriction of p to the regular part defines a bounded section $p_{\mathrm{reg}} \in C^k(Z_{\mathrm{reg}} \times Z_{\mathrm{reg}}, E_{\mathrm{reg}} \boxtimes E^*_{\mathrm{reg}})$ such that for $s \in C^\infty(Z, E)$ and $z \in Z_{\mathrm{reg}}$,

$$As(z) = \int_{z' \in Z_{\mathrm{reg}}} p_{\mathrm{reg}}(z, z') s(z') dv_{Z_{\mathrm{reg}}}. \tag{3.11}$$

Using (3.9), A extends uniquely to a bounded operator on $L^2(Z, E)$. Moreover, since $p_{\mathrm{reg}}(z, z')$ is bounded,

$$\int_{(z,z') \in Z_{\mathrm{reg}} \times Z_{\mathrm{reg}}} |p_{\mathrm{reg}}(z, z')|^2 dv_{Z_{\mathrm{reg}} \times Z_{\mathrm{reg}}} < \infty. \tag{3.12}$$

Then A is in the Hilbert-Schmidt class. If A is in the trace class, then

$$\mathrm{Tr}\,[A] = \int_{z \in Z_{\mathrm{reg}}} \mathrm{Tr}^E\,[p_{\mathrm{reg}}(z, z)]\, dv_{Z_{\mathrm{reg}}} = \int_{z \in Z} \mathrm{Tr}^E\,[p(z, z)]\, dv_Z. \tag{3.13}$$

Now we give another description of integral operators. For any local chart $\widetilde{U}$, there is a G_U-invariant integral operator $\widetilde{A}_U : C^\infty(\widetilde{U}, \widetilde{E}_U) \to C^\infty(\widetilde{U}, \widetilde{E}_U)$

with integral kernel $\widetilde{p}_U \in C^k(\widetilde{U} \times \widetilde{U}, \widetilde{E}_U \boxtimes \widetilde{E}_U^*)$ such that if $s \in C_c^\infty(U, E|_U)$, then $As|_U$ is defined by the invariant section

$$\widetilde{A}_U s_U(x) = \int_{x' \in \widetilde{U}} \widetilde{p}_U(x, x') s_U(x') dv_{\widetilde{U}}. \tag{3.14}$$

Then the restriction of the integral kernel p on $U \times U$ is represented by the invariant section (see [31, (2.2)])

$$\sum_{g \in G_U} g\widetilde{p}_U(g^{-1}x, x') \in C^k\big(\widetilde{U} \times \widetilde{U}, \widetilde{E}_U \boxtimes \widetilde{E}_U^*\big)^{G_U \times G_U}. \tag{3.15}$$

If A is in trace class, we have

$$\mathrm{Tr}[A] = \sum_{i \in I} \frac{1}{|G_{U_i}|} \sum_{g \in G_{U_i}} \int_{x \in \widetilde{U}_i} \widetilde{\phi}_i(x) \,\mathrm{Tr}\left[g\widetilde{p}_{U_i}(g^{-1}x, x)\right] dv_{\widetilde{U}_i}. \tag{3.16}$$

3.3. Sobolev space on orbifolds

Let (Z, g^{TZ}) be a compact Riemannian orbifold of dimension m. Let ∇^{TZ} be the Levi-civita connection on TZ, and let R^{TZ} be the corresponding curvature. Let (E, g^E) be a proper Hermitian orbifold vector bundle with connection ∇^E. When necessary, we identity E with $\overline{E}^*$ via g^E.

Denote still by $\nabla^{\mathscr{T}(T^*Z) \otimes_{\mathbf{R}} E}$ the connection $\mathscr{T}(T^*Z) \otimes_{\mathbf{R}} E$ induced by ∇^{TZ} and ∇^E. For $q \in \mathbf{N}$, take $\mathcal{H}^q(Z, E)$ to be the Hilbert completion of $C^\infty(Z, E)$ under the norm defined by

$$\|s\|_q^2 = \sum_{j=0}^{q} \int_Z \left| \left(\nabla^{\mathscr{T}(T^*Z) \otimes_{\mathbf{R}} E}\right)^j s(z) \right|^2 dv_Z. \tag{3.17}$$

Let $\mathcal{H}^{-q}(Z, E)$ be the dual of $\mathcal{H}^q(Z, E)$. If $q \in \mathbf{R}$, we can define $\mathcal{H}^q(Z, E)$ by interpolation. As in the case of smooth sections, $s \in \mathcal{H}^q(Z, E)$ can be represented by the family $\{s_U \in \mathcal{H}^q(\widetilde{U}, \widetilde{E}_U)^{G_U}\}_{U \in \mathcal{U}}$ of G_U-invariant sections such that (2.28) holds.

Using these local descriptions, we have

$$\bigcap_{q \in \mathbf{R}} \mathcal{H}^q(Z, E) = C^\infty(Z, E), \qquad \bigcup_{q \in \mathbf{R}} \mathcal{H}^q(Z, E) = \mathcal{D}'(Z, E). \tag{3.18}$$

Moreover, if $q > q'$, we have the compact embedding

$$\mathcal{H}^q(Z, E) \hookrightarrow \mathcal{H}^{q'}(Z, E), \tag{3.19}$$

and if $q \in \mathbf{N}$ and $q > m/2$, we have the continuous embedding

$$\mathcal{H}^q(Z,E) \hookrightarrow C^{q-[m/2]}(Z,E). \tag{3.20}$$

3.4. Characteristic forms on orbifolds

Assume now (E, g^E) is a real Euclidean proper orbifold vector bundle of rank r with a metric connection ∇^E. The Euler form $e(E,\nabla^E) \in \Omega^r\big(Z, o(E)\big)$ is defined by the family of closed forms $\Big\{e\big(\widetilde{E}_U, \nabla^{\widetilde{E}_U}\big)\Big\}_{U\in\mathcal{U}}$. Following [44, Section 3.3], the orbifold Euler characteristic number is defined by

$$\chi_{\mathrm{orb}}(Z) = \int_Z e\left(TZ, \nabla^{TZ}\right). \tag{3.21}$$

If $\nabla^{E'}$ is another metric connection, the class of Chern-Simons form

$$\widetilde{e}(E,\nabla^E,\nabla'^E) \in \Omega^{r-1}\big(Z, o(E)\big)/d\Omega^{r-2}\big(Z, o(E)\big) \tag{3.22}$$

is defined by the family $\Big\{\widetilde{e}\big(\widetilde{E}_U, \nabla^{\widetilde{E}_U}, \nabla'^{\widetilde{E}_U}\big)\Big\}_{U\in\mathcal{U}}$. Clearly, (1.5) still holds true.

Let (F,∇^F) be a proper orbifold flat vector bundle on Z with a Hermitian metric g^F. The odd Chern character $h\left(\nabla^F, g^F\right) \in \Omega^{\mathrm{odd}}(Z)$ of (F,∇^F) is defined by the family of closed odd forms $\Big\{h\big(\nabla^{\widetilde{F}_U}, g^{\widetilde{F}_U}\big)\Big\}_{U\in\mathcal{U}}$. If g'^F is another Hermitian metric on F, the class of Chern-Simons form

$$\widetilde{h}(\nabla^F, g^F, g'^F) \in \Omega^{\mathrm{even}}(Z)/d\Omega^{\mathrm{odd}}(Z) \tag{3.23}$$

is defined by the family $\Big\{h\big(\nabla^{\widetilde{F}_U}, g^{\widetilde{F}_U}, g'^{\widetilde{F}_U}\big)\Big\}_{U\in\mathcal{U}}$. As before, (1.20) still holds true.

The degree 1-part of $h(\nabla^F, g^F)$ and the degree 0-part of $\widetilde{h}(\nabla^F, g^F, g'^F)$ will be especially important in the formulation of Theorem 0.3. We denote by

$$\begin{aligned} \theta\left(\nabla^F, g^F\right) &= 2h\left(\nabla^F, g^F\right)^{[1]}, \\ \widetilde{\theta}\left(\nabla^F, g^F, g'^F\right) &= 2\widetilde{h}\left(\nabla^F, g^F, g'^F\right)^{[0]}. \end{aligned} \tag{3.24}$$

By (1.17)-(1.20) and (3.24), we have

$$\begin{aligned} &\theta\left(\nabla^F, g^F\right) = \mathrm{Tr}\left[\left(g^F\right)^{-1}\nabla^F g^F\right], \\ &d^Z\widetilde{\theta}\left(\nabla^F, g^F, g'^F\right) = \theta\left(\nabla^F, g'^F\right) - \theta\left(\nabla^F, g^F\right). \end{aligned} \tag{3.25}$$

Let $\|\cdot\|_{\det F}$ and $\|\cdot\|'_{\det F}$ be the metrics on the line bundle $\det F$ induced by the metrics g^F and g'^F. By [9, (4.12)], we have

$$\widetilde{\theta}\left(\nabla^F, g^F, g'^F\right) = \log\left(\frac{\|\cdot\|'_{\det F}}{\|\cdot\|_{\det F}}\right)^2. \tag{3.26}$$

The odd Chern character form $h(\nabla^F, g^F)$ and the Chern-Simons class $\widetilde{h}(\nabla^F, g^F, g'^F)$ can be extended to $Z\coprod\Sigma Z$. Recall that for $U\in\mathcal{U}$ and $g\in G_U$, $\widetilde{U}^g$ is an orbifold chart of $Z\coprod\Sigma Z$. The restriction of $(\widetilde{F}_U, \nabla^{\widetilde{F}_U})$ to $\widetilde{U}^g$ is a flat vector bundle. The element g acts fiberwisely on $\widetilde{F}_U|_{\widetilde{U}^g}$ and preserves $\nabla^{\widetilde{F}_U}$ and $g^{\widetilde{F}_U}$. The family

$$\left\{h_g\left(\nabla^{\widetilde{F}_U}, g^{\widetilde{F}_U}\right)\in\Omega^{\mathrm{odd}}(\widetilde{U}^g)\right\}_{U\in\mathcal{U}, g\in G_U} \tag{3.27}$$

defines a closed differential form $h_\Sigma(\nabla^F, g^F)\in\Omega^{\mathrm{odd}}\left(Z\coprod\Sigma Z\right)$. Denote by $h_i(\nabla^F, g^F)$ the restriction of $h_\Sigma(\nabla^F, g^F)$ to $Z_i\subset Z\coprod\Sigma Z$. Similarly, we can define

$$\begin{aligned} \widetilde{h}_i(\nabla^F, g^F, g'^F) &\in \Omega^{\mathrm{even}}(Z_i)/\Omega^{\mathrm{odd}}(Z_i), \\ \theta_i(\nabla^F, g^F) &\in \Omega^1(Z_i), \\ \widetilde{\theta}_i(\nabla^F, g^F, g'^F) &\in C^\infty(Z_i). \end{aligned} \tag{3.28}$$

The rank of F can be extended to a locally constant function on $Z\coprod\Sigma Z$ in a similar way. Indeed, the family $\{\mathrm{Tr}[\rho_U^F(g)]\in C^\infty(\widetilde{U}^g)\}_{U\in\mathcal{U}, g\in G_U}$ of constant functions defines a locally constant function ρ on $Z\coprod\Sigma Z$. Denote by ρ_i its value at Z_i. Clearly,

$$\rho_0 = \mathrm{rk}[F]. \tag{3.29}$$

4. Ray-Singer metric of orbifolds

In this section, given metrics g^{TZ} and g^F on TZ and F, we introduce the Ray-Singer metric on the determinant of the de Rham cohomology $H^{\cdot}(Z, F)$. We establish the anomaly formula for the Ray-Singer metric. In particular, when

Z is of odd dimension and orientable, the Ray-Singer metric is a topological invariant.

In subsection 4.1, we introduce the Hodge Laplacian associated to the metrics (g^{TZ}, g^F). We state Gauss-Bonnet-Chern Theorem for compact orbifolds.

In subsection 4.2, we construct the analytic torsion and the Ray-Singer metric. We restate the anomaly formula.

In subsection 4.3, following [8], we interpret the analytic torsion as a transgression of odd Chern forms. We state Theorem 4.13, which extends the main result of Bismut-Lott, and from which the anomaly formula follows.

In subsection 4.4, we prove Gauss-Bonnet-Chern Theorem and Theorem 4.13 in a unified way. Using an argument due to [31, p. 2230], which is based on the finite propagation speeds for the solutions of hyperbolic equations [33, Theorem D.2.1], we can turn our problem into a local one. Since the orbifold locally is a quotient of a manifold by some finite group, we can then rely on the results of Bismut-Goette [6], where the authors there consider some similar problems in the equivariant setting.

4.1. Hodge Laplacian

Let Z be a compact orbifold of dimension m, and let F be a proper flat orbifold vector bundle of rank r with flat connection ∇^F. Put

$$\begin{aligned} \chi_{\mathrm{top}}(Z,F) &= \sum_{i=0}^{m}(-1)^i \dim_{\mathbf{C}} H^i(Z,F), \\ \chi'_{\mathrm{top}}(Z,F) &= \sum_{i=1}^{m}(-1)^i i \dim_{\mathbf{C}} H^i(Z,F). \end{aligned} \tag{4.1}$$

Take a Riemannian metric g^{TZ} and a Hermitian metric g^F on F. We apply the construction of subsection 3.3 to the Hermitian orbifold vector bundle $E = \Lambda^{\cdot}(T^*Z) \otimes_{\mathbf{R}} F$ with the Hermitian metric induced by g^{TZ} and g^F, and with the connection $\nabla^{\Lambda^{\cdot}(T^*Z)\otimes_{\mathbf{R}}F}$ induced by the Levi-Civita connection ∇^{TZ} and the flat connection ∇^F. Let $d^{Z,*}$ be the formal adjoint of d^Z. Put

$$D^Z = d^Z + d^{Z,*}, \qquad \square^Z = D^{Z,2} = \left[d^Z, d^{Z,*}\right]. \tag{4.2}$$

Then $d^{Z,*}$ is a first order differential operator, represented by the family of the formal adjoint $d^{\widetilde{U},*}$ of $d^{\widetilde{U}}$ with respect to the L^2-metric defined by $g^{T\widetilde{U}}$ and $g^{\widetilde{F}_U}$. Also, $\square^Z$ is a formally self-adjoint second order elliptic operator

acting on $\Omega^{\cdot}(Z,F)$, which is represented by the family of Hodge Laplacian $\square^{\widetilde{U}}$ acting on $\Omega^{\cdot}(\widetilde{U},\widetilde{F}_U)$ associated with $g^{T\widetilde{U}}$ and $g^{\widetilde{F}_U}$. Also, the operator $\left(\square^Z,\Omega^{\cdot}(Z,F)\right)$ is essentially self-adjoint. And the domain of the self-adjoint extension is $\mathcal{H}^2(Z,\Lambda^{\cdot}(T^*Z)\otimes_{\mathbf{R}}F)$. The following theorem is well-known (e.g., [31, Proposition 2.2], [14, Proposition 2.1]).

Theorem 4.1. *The following orthogonal decomposition holds:*

$$\Omega^{\cdot}(Z,F)=\ker\square^Z\oplus\operatorname{Im}\left(d^Z|_{\Omega^{\cdot}(Z,F)}\right)\oplus\operatorname{Im}\left(d^{Z,*}|_{\Omega^{\cdot}(Z,F)}\right). \tag{4.3}$$

In particular, we have the canonical identification of the vector spaces

$$\ker\square^Z\simeq H^{\cdot}(Z,F). \tag{4.4}$$

In the sequel, we still denote by $\square^Z$ the self-adjoint extension of the operator $\left(\square^Z,\Omega^{\cdot}(Z,F)\right)$. By (3.20), for $k\gg 1$, the operator $(1+\square^Z)^{-k}$ has a continuous kernel. In particular, $(1+\square^Z)^{-k}$ is in the Hilbert-Schmidt class, and $(1+\square^Z)^{-2k}$ is in the trace class. By the above argument, if f lies in the Schwartz space $\mathcal{S}(\mathbf{R})$, then $f(\square^Z)$ has a smooth kernel, and is in the trace class. For $t>0$, the same statement holds true for the heat operator $\exp(-t\square^Z)$ of $\square^Z$. In this way, most of results on compact manifolds, which have been obtained by the functional calculus of the Hodge Laplacian, still hold true for compact orbifolds.

Let $N^{\Lambda^{\cdot}(T^*Z)}$ be the number operator on $\Lambda^{\cdot}(T^*Z)$. We write $\operatorname{Tr_s}[\cdot]=\operatorname{Tr}\left[(-1)^{N^{\Lambda^{\cdot}(T^*Z)}}\cdot\right]$ for the supertrace. By the classical argument of Mckean-Singer formula [34], we get:

Proposition 4.2. *For $t>0$, the following identity holds:*

$$\chi_{\mathrm{top}}(Z,F)=\operatorname{Tr_s}\left[\exp\left(-t\square^Z\right)\right]. \tag{4.5}$$

Recall that $\chi_{\mathrm{orb}}(Z)$ and ρ_i are defined in (3.21) and (3.29).

Theorem 4.3. *When $t\to 0$, we have*

$$\operatorname{Tr_s}\left[\exp\left(-t\square^Z\right)\right]\to\sum_{i=0}^{l_0}\rho_i\frac{\chi_{\mathrm{orb}}(Z_i)}{m_i}. \tag{4.6}$$

In particular,

$$\chi_{\mathrm{top}}(Z,F)=\sum_{i=0}^{l_0}\rho_i\frac{\chi_{\mathrm{orb}}(Z_i)}{m_i}. \tag{4.7}$$

Proof. Equation (4.7) is a consequence of (4.5) and (4.6). The proof of (4.6) will be given in subsection 4.4.1. □

4.2. Analytic torsion and its anomaly

By (4.3), let P^Z be the orthogonal projection onto $\ker \square^Z$. By the short time asymptotic expansions of the heat trace [31, Proposition 2.1], proceeding as in [4, Proposition 9.35], the function

$$\theta(s) = -\frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr_s} \left[N^{\Lambda^\cdot(T^*Z)} \exp\left(-t\,\square^Z\right) \left(1 - P^Z\right) \right] t^{s-1} dt \tag{4.8}$$

defined on the region $\{s \in \mathbf{C} : \mathrm{Re}\,(s) > m/2\}$ is holomorphic, and has a meromorphic extension to $\mathbf{C}$ which is holomorphic at $s = 0$.

Definition 4.4. The analytic torsion of F is defined by

$$T\left(F, g^{TZ}, g^F\right) = \exp\left(\theta'(0)/2\right) > 0. \tag{4.9}$$

Remark 4.5. The formalism of Voros [51] on the regularized determinant of the resolvent of Laplacian extends to orbifolds trivially, as the proof relies only on the short time asymptotic expansions of the heat trace and on the functional calculus. Thus the weighted product of zeta regularized determinants

$$\sigma \to \prod_{i=1}^m \det\left(\sigma + \square^Z|_{\Omega^i(Z,F)}\right)^{(-1)^i i} \tag{4.10}$$

is a meromorphic function on $\mathbf{C}$ such that when $\sigma \to 0$, we have

$$\begin{aligned} \prod_{i=1}^m \det\left(\sigma + \square^Z|_{\Omega^i(Z,F)}\right)^{(-1)^i i} \\ = T\left(Z, g^{TZ}, g^F\right)^2 \sigma^{\chi'_{\mathrm{top}}(Z,F)} + \mathcal{O}(\sigma^{\chi'_{\mathrm{top}}(Z,F)+1}). \end{aligned} \tag{4.11}$$

We have a generalization of [42, Theorem 2.3].

Proposition 4.6. *If Z is an orientable even dimensional compact orbifold and if F is a unitarily flat orbifold vector bundle, then for any Riemannian*

metric g^{TZ} and any flat Hermitian metric g^F,

$$\mathrm{Tr_s}\left[\left(N^{\Lambda^\cdot(T^*Z)}-\frac{m}{2}\right)\exp\left(-t\square^Z\right)\right]=0. \tag{4.12}$$

In particular,

$$T\left(F,g^{TZ},g^F\right)=1. \tag{4.13}$$

Proof. Let $*^Z:\Lambda^\cdot(T^*Z)\to\Lambda^{m-\cdot}(T^*Z)\otimes o(TZ)$ be the Hodge star operator associated to g^{TZ}, which is locally defined by the family $\{*^{\widetilde{U}}:\Lambda^\cdot(T^*\widetilde{U})\to\Lambda^{m-\cdot}(T^*\widetilde{U})\otimes_{\mathbf{R}}o(T\widetilde{U})\}_{U\in\mathcal{U}}$. Note that Z is orientable, so $o(TZ)$ is trivial. Write $\star^Z=*^Z\otimes_{\mathbf{R}}\mathrm{id}_F:\Lambda^\cdot(T^*Z)\otimes_{\mathbf{R}}F\to\Lambda^{m-\cdot}(T^*Z)\otimes_{\mathbf{R}}F$. Clearly, we have

$$\star^Z\left(N^{\Lambda^\cdot(T^*Z)}-\frac{m}{2}\right)\star^{Z,-1}=-\left(N^{\Lambda^\cdot(T^*Z)}-\frac{m}{2}\right). \tag{4.14}$$

Since g^F is flat, we have

$$\star^Z\square^Z\star^{Z,-1}=\square^Z. \tag{4.15}$$

Note that when m is even, $\star^Z$ is an even isomorphism of $\Omega^\cdot(Z,F)$. Hence, by (4.14) and (4.15), we get (4.12). Equation (4.13) is a consequence of (4.12). $\square$

Set

$$\lambda=\bigotimes_{i=0}^{m}\left(\det H^i(Z,F)\right)^{(-1)^i}. \tag{4.16}$$

Then λ is a complex line. Let $|\cdot|_\lambda^{\mathrm{RS},2}$ be the L^2-metric on λ induced via (4.4).

Definition 4.7. The Ray-Singer metric on λ is defined by

$$\|\cdot\|_\lambda^{\mathrm{RS}}=T\left(F,g^{TZ},g^F\right)|\cdot|_\lambda^{\mathrm{RS}}. \tag{4.17}$$

Let (g^{TX},g^F) and (g'^{TX},g'^F) be two pairs of metrics on TX and F. Let $\|\cdot\|_\lambda^{\mathrm{RS},2}$ and $\|\cdot\|_\lambda'^{\mathrm{RS},2}$ be the corresponding Ray-Singer metrics on λ. We restate Theorem 0.3.

Theorem 4.8. *The following identity holds:*

$$\log\left(\frac{\|\cdot\|_{\lambda}^{\prime\mathrm{RS},2}}{\|\cdot\|_{\lambda}^{\mathrm{RS},2}}\right)=\sum_{i=0}^{l_0}\frac{1}{m_i}\int_{Z_i}\Big(\widetilde{\theta}_i\left(\nabla^F,g^F,g'^F\right)e\left(TZ_i,\nabla^{TZ_i}\right)-\theta_i\left(\nabla^F,g^F\right)\widetilde{e}\left(TZ_i,\nabla^{TZ_i},\nabla'^{TZ_i}\right)\Big). \tag{4.18}$$

Proof. The proof of our theorem will be given in Remark 4.15. □

Corollary 4.9. *If all the Z_i's are of odd dimension, then $\|\cdot\|_{\lambda}^{\mathrm{RS},2}$ does not depend on g^{TZ} or g^F. In particular, this is the case if Z is an orientable odd dimensional orbifold.*

Proof. When $\dim Z_i$ is odd, $e\left(TZ_i,\nabla^{TZ_i}\right)=0$ and $\widetilde{e}\left(TZ_i,\nabla^{TZ_i},\nabla'^{TZ_i}\right)=0$. By Theorem 4.8, we get Corollary 4.9. □

Corollary 4.10. *If for $0\leq i\leq l_0$,*

$$\chi_{\mathrm{orb}}(Z_i)=0, \tag{4.19}$$

and if F is unitarily flat, then $\|\cdot\|_{\lambda}^{RS,2}$ does not depend on g^{TZ} or on the flat Hermitian metric g^F.

Proof. Take (g^{TX},g^F) and (g'^{TX},g'^F) two pairs of metrics on TX and F such that $\nabla^F g^F=0$ and $\nabla^F g'^F=0$. By (3.27), for $0\leq i\leq l_0$,

$$\theta_i\left(\nabla^F,g^F\right)=\theta_i\left(\nabla^F,g'^F\right)=0. \tag{4.20}$$

By (3.25) and (4.20), $\widetilde{\theta}_i(\nabla^F,g^F,g'^F)$ is closed. It becomes a constant $c_i\in\mathbf{C}$ as Z_i is connected. Using (4.19), we get

$$\int_{Z_i}\widetilde{\theta}_i\left(F,g^F,g'^F\right)e\left(TZ_i,\nabla^{TZ_i}\right)=c_i\int_{Z_i}e\left(TZ_i,\nabla^{TZ_i}\right)=0. \tag{4.21}$$

By (4.18), (4.20), and (4.21), we get $\|\cdot\|_{\lambda}^{\mathrm{RS},2}=\|\cdot\|_{\lambda}^{\prime\mathrm{RS},2}$. □

Remark 4.11. If F is not proper, we can define the analytic torsion and Ray-Singer metric in the same way. Indeed, we have

$$H^{\cdot}(X,F)=H^{\cdot}(X,F^{\mathrm{pr}}),\qquad T\left(F,g^{TZ},g^F\right)=T\left(F^{\mathrm{pr}},g^{TZ},g^{F^{\mathrm{pr}}}\right). \tag{4.22}$$

Also, the Ray-Singer metrics of F and F^{pr} coincides. For this reason, all the result in this section holds true for non proper flat orbifold vector bundle.

4.3. Analytic torsion as a transgression

Let $(g_s^{TZ}, g_s^F)_{s\in\mathbf{R}}$ be a smooth family of metrics on TZ and F such that

$$(g_s^{TZ}, g_s^F)|_{s=0} = (g^{TZ}, g^F), \qquad (g_s^{TZ}, g_s^F)|_{s=1} = (g'^{TZ}, g'^F). \tag{4.23}$$

Then, $s \in \mathbf{R} \to \log T(F, g_s^{TZ}, g_s^F)$ is a smooth function. Following [8, Section III], we will interpret the analytic torsion function $\log T(F, g_\cdot^{TZ}, g_\cdot^F)$ as a transgression for some odd Chern forms associated to certain flat superconnections on $\mathbf{R}$.

Recall that π is defined in (1.7), and $g^{\pi^*(TZ)}$, g^{π^*F} are defined in (1.8) with $E = TZ$ or F. Consider now a trivial infinite dimensional vector bundle W on $\mathbf{R}$ defined by

$$\mathbf{R} \times \Omega^\cdot(Z, F) \to \mathbf{R}. \tag{4.24}$$

Let g^W be a Hermitian metric on W such that g_s^W is the L^2-metric on $\Omega^\cdot(Z, F)$ induced by (g_s^{TZ}, g_s^F). Put

$$A' = d^{\mathbf{R}} + d^Z. \tag{4.25}$$

Then A' is a flat superconnection on W. Let A'' be the adjoint of A' with respect to g^W. For $s \in \mathbf{R}$, denote by $d_s^{Z,*}, \square_s^Z$ the corresponding objects for (g_s^{TZ}, g_s^F), and by $*_s^Z$ the Hodge star operator with respect to g_s^{TZ}. Thus,

$$A'' = d^{\mathbf{R}} + d_s^{Z,*} + ds \wedge \left(g_s^{F,-1} \frac{\partial g_s^F}{\partial s} + *_s^{Z,-1} \frac{\partial *_s^Z}{\partial s} \right). \tag{4.26}$$

Set

$$A = \frac{1}{2}(A'' + A'), \qquad B = \frac{1}{2}(A'' - A'). \tag{4.27}$$

Then A is a superconnection on W, and B is a fibrewise first order elliptic differential operator. The curvature of A is given by

$$A^2 = -B^2. \tag{4.28}$$

It is a fibrewise second order elliptic differential operator.

Following [8, Definition 2.7], we introduce a deformation of g^W. For $t > 0$, set

$$g_t^W = t^{N^{\Lambda^\cdot(T^*Z)}} g^W. \tag{4.29}$$

Let A_t'' be the adjoint of A' with respect to g_t^W. Clearly,

$$A_t'' = t^{-N^{\Lambda^\cdot(T^*Z)}} A'' t^{N^{\Lambda^\cdot(T^*Z)}}. \tag{4.30}$$

We define A_t and B_t as in (4.27), i.e.,

$$A_t = \frac{1}{2}(A_t'' + A'), \qquad B_t = \frac{1}{2}(A_t'' - A'). \tag{4.31}$$

Theorem 4.12. *For $t > 0$, we have*

$$\mathrm{Tr_s}\left[\exp\left(B_t^2\right)\right] = \chi_{\mathrm{top}}(Z, F). \tag{4.32}$$

Proof. Theorem 4.12 can be proved using the technique of the local family index theory as in [8, Theorem 3.15]. Since here the parameter space $\mathbf{R}$ is of dimension 1, we give a short proof. By construction, $\mathrm{Tr_s}\left[\exp\left(B_t^2\right)\right]$ is an even form on $\mathbf{R}$, thus it is a function. By (4.25), (4.26), (4.30) and (4.31), we have

$$\mathrm{Tr_s}\left[\exp\left(B_t^2\right)\right] = \mathrm{Tr_s}\left[\exp\left(-t\Box_s^Z/4\right)\right]. \tag{4.33}$$

By (4.5) and (4.33), we get (4.32). □

Recall that h is defined in (1.18). Following [8, (2.22) and (2.23)], for $t > 0$, set

$$u_t = \mathrm{Tr_s}\left[h(B_t)\right] \in \Omega^1(\mathbf{R}), \quad v_t = \mathrm{Tr_s}\left[\frac{N^{\Lambda^\cdot(T^*Z)}}{2} h'(B_t)\right] \in C^\infty(\mathbf{R}). \tag{4.34}$$

The subspace $\ker(\Box_s^Z) \subset \Omega^\cdot(Z, F)$ defines a finite dimensional subbundle $W_0 \subset W$ on $\mathbf{R}$. By Theorem 4.1, the fiber of W_0 is $H^\cdot(Z, F)$. As in [8, Section III.f], we equip W_0 with the restricted metric g^{W_0} and the induced

connection ∇^{W_0}. Put

$$
\begin{aligned}
u_0 &= \sum_{i=0}^{l_0} \frac{1}{m_i} \int_{Z_i} e\left(\pi^*(TZ_i), \nabla^{\pi^*(TZ_i)}\right) h_i\left(\nabla^{\pi^*F}, g^{\pi^*F}\right), \\
u_\infty &= h\left(\nabla^{W_0}, g^{W_0}\right),
\end{aligned}
\tag{4.35}
$$

and

$$
v_0 = \frac{m}{4}\chi_{\mathrm{top}}(Z,F), \qquad v_\infty = \frac{1}{2}\chi'_{\mathrm{top}}(Z,F). \tag{4.36}
$$

For a smooth family $\{\alpha_t\}_{t>0}$ of differential forms on $\mathbf{R}$, we say $\alpha_t = \mathcal{O}(t)$ if for all the compact $K \subset \mathbf{R}$, and for all $k \in \mathbf{N}$, there is $C > 0$ such that $\|\alpha_t\|_{C^k(K)} \leq Ct$.

Theorem 4.13. *For $t > 0$, u_t is a closed 1-form on $\mathbf{R}$ such that its cohomology class does not depend on $t > 0$ and that the following identity of 1-forms holds:*

$$
\frac{\partial}{\partial t} u_t = d^{\mathbf{R}}\left(\frac{v_t}{t}\right). \tag{4.37}
$$

As $t \to 0$, we have

$$
u_t = u_0 + \mathcal{O}\left(\sqrt{t}\right), \qquad v_t = v_0 + \mathcal{O}\left(\sqrt{t}\right), \tag{4.38}
$$

as $t \to \infty$,

$$
u_t = u_\infty + \mathcal{O}(1/\sqrt{t}), \qquad v_t = v_\infty + \mathcal{O}(1/\sqrt{t}). \tag{4.39}
$$

Proof. By [8, Theorems 1.8 and 3.20], u_t is closed and (4.37) holds. The first equation of (4.38) will be proved in subsection 4.4.2. The first equation of (4.39) can be proved as [8, Theorem 3.16], whose proof is based on functional calculus. Proceeding as [8, Theorem 3.21], we can show the second equations of (4.38) and (4.39) as consequence of the first equations[3] of (4.38) and (4.39). □

[3]More precisely, we need the corresponding results on a larger parametrized space $\mathbf{R} \times (0, \infty)$. We leave the details to readers.

Corollary 4.14. *The following identities in $C^\infty(\mathbf{R})$ and $\Omega^1(\mathbf{R})$ hold:*

(4.40)
$$\log T\left(F, g_\cdot^{TZ}, g_\cdot^F\right) = -\int_0^\infty \left\{v_t - v_\infty h'(0) - (v_0 - v_\infty)\, h'\left(\frac{i\sqrt{t}}{2}\right)\right\}\frac{dt}{t},$$

and

$$u_0 - u_\infty = d^{\mathbf{R}} \log T\left(F, g_\cdot^{TZ}, g_\cdot^F\right). \tag{4.41}$$

Proof. By Theorem 4.13, proceeding as [8, Theorems 3.23 and 3.29], we get Corollary 4.14. □

Remark 4.15. Theorem 4.8 is just (4.41). Indeed, if $\|\cdot\|_{\lambda,s}^{\mathrm{RS},2}$ denotes the Ray-Singer metric on λ associated to (g_s^{TZ}, g_s^F). By (4.17), (4.41), and using the fact that our base manifold $\mathbf{R}$ is of dimension 1, we have

$$\begin{aligned} ds \wedge \frac{\partial}{\partial s}\left\{\log\left(\|\cdot\|_{\lambda,s}^{\mathrm{RS},2}\right)\right\} \\ = \sum_{i=0}^{l_0} \frac{1}{m_i}\int_{Z_i} e\left(\pi^*(TZ_i), \nabla^{\pi^*(TZ_i)}\right)\theta_i\left(\nabla^{\pi^*F}, g^{\pi^*F}\right). \end{aligned} \tag{4.42}$$

By (1.9) and (1.23) with $\alpha_{i,s} \in \Omega^{\dim Z_i - 1}(Z_i, o(TZ_i))$, $\beta_{i,s}^{[0]} \in C^\infty(Z_i)$ defined in an obvious way, we have

$$\begin{aligned} e\left(\pi^*(TZ_i), \nabla^{\pi^*(TZ_i)}\right) &= e\left(TZ_i, \nabla^{TZ_i}\right) + d^Z \int_0^s \alpha_{i,s} ds + ds \wedge \alpha_{i,s}, \\ \theta_i\left(\nabla^{\pi^*F}, g^{\pi^*F}\right) &= \theta_i\left(\nabla^F, g^F\right) + 2d^Z \int_0^s \beta_{i,s}^{[0]} ds + 2ds \wedge \beta_{i,s}^{[0]}. \end{aligned} \tag{4.43}$$

By (4.43), we have

$$\begin{aligned} \int_{Z_i} e\left(\pi^*(TZ_i), \nabla^{\pi^*(TZ_i)}\right)\theta_i\left(\nabla^{\pi^*F}, g^{\pi^*F}\right) \\ = ds \int_{Z_i}\left(2\beta_{i,s}^{[0]}\, e\left(TZ_i, \nabla^{\pi^*(TZ)}\right) - \theta_i\left(\nabla^F, g^F\right) \wedge \alpha_{i,s}\right). \end{aligned} \tag{4.44}$$

By integrating (4.44) with respect to the variable s form 0 to 1, and by (4.42), we get (4.18).

4.4. Estimates on heat kernels

4.4.1. Proof of Theorem 4.3. We follow [6, Section 13.2]. Take $\alpha_0 > 0$. Let $f, g : \mathbf{R} \to [0,1]$ be smooth even functions such that

$$f(s) = \begin{cases} 1, & |s| \leq \alpha_0/2; \\ 0, & |s| \geq \alpha_0, \end{cases} \qquad g(s) = 1 - f(s). \tag{4.45}$$

Definition 4.16. For $t > 0$ and $a \in \mathbf{C}$, set

$$F_t(a) = \int_{\mathbf{R}} e^{2isa} e^{-s^2} f\left(\sqrt{t}s\right) \frac{ds}{\sqrt{\pi}}, \quad G_t(a) = \int_{\mathbf{R}} e^{2isa} e^{-s^2} g\left(\sqrt{t}s\right) \frac{ds}{\sqrt{\pi}}. \tag{4.46}$$

By (4.45) and (4.46), we get

$$\exp\left(-a^2\right) = F_t(a) + G_t(a). \tag{4.47}$$

Moreover, F_t and G_t are even holomorphic functions, whose restriction to $\mathbf{R}$ lies in $\mathcal{S}(\mathbf{R})$. By (4.46), we find that given $m, m' \in \mathbf{N}$, $c > 0$, there exist $C > 0, C' > 0$ such that if $t \in (0,1]$, $a \in \mathbf{C}$, $|\operatorname{Im}(a)| \leq c$,

$$|a|^m \left| G_t^{(m')}(a) \right| \leq C \exp(-C'/t). \tag{4.48}$$

There exist uniquely well-defined holomorphic functions $\mathscr{F}_t(a)$ and $\mathscr{G}_t(a)$ such that

$$F_t(a) = \mathscr{F}_t(a^2), \qquad G_t(a) = \mathscr{G}_t(a^2). \tag{4.49}$$

By (4.47) and (4.49), we have

$$\exp(-a) = \mathscr{F}_t(a) + \mathscr{G}_t(a). \tag{4.50}$$

By (4.50), we get

$$\exp\left(-t\square^Z\right) = \mathscr{F}_t\left(t\square^Z\right) + \mathscr{G}_t\left(t\square^Z\right). \tag{4.51}$$

If A is a bounded operator, let $\|A\|$ be its operator norm. If A is in the trace class, let $\|A\|_1 = \operatorname{Tr}\left[\sqrt{A^*A}\right]$ be its trace norm.

Proposition 4.17. *There exist $c > 0$ and $C > 0$ such that for $t \in (0, 1]$,*

$$\left\|\mathscr{G}_t\left(t\square^Z\right)\right\|_1 \leq Ce^{-c/t}. \tag{4.52}$$

In particular, as $t \to 0$, we have

$$\mathrm{Tr_s}\left[\exp\left(-t\square^Z\right)\right] = \mathrm{Tr_s}\left[\mathscr{F}_t\left(t\square^Z\right)\right] + \mathcal{O}(e^{-c/t}). \tag{4.53}$$

Proof. By (4.48) and (4.49), for any $k \in \mathbf{N}$, the operator $\left(1+\square^Z\right)^k \mathscr{G}_t\left(t\square^Z\right)$ is a bounded operator such that there exist $C > 0$ and $C' > 0$,

$$\left\|\left(1+\square^Z\right)^k \mathscr{G}_t\left(t\square^Z\right)\right\| \leq C\exp\left(-C'/t\right). \tag{4.54}$$

Take $k \in \mathbf{N}$ big enough such that $(1+\square^Z)^{-k}$ is of trace class. Then

$$\left\|\mathscr{G}_t\left(t\square^Z\right)\right\|_1 \leq \left\|(1+\square^Z)^{-k}\right\|_1 \left\|(1+\square^Z)^k \mathscr{G}_t\left(t\square^Z\right)\right\|. \tag{4.55}$$

By (4.54) and (4.55), we get (4.52). By (4.51) and (4.52), we get (4.53). □

Assume that Z is covered by a finite family $\mathcal{U} = \{U_i\}_{i\in I}$ of connected open sets with orbifold atlas $\widetilde{\mathcal{U}} = \{(\widetilde{U}_i, G_{U_i}, \pi_{U_i})\}_{i\in I}$. Let $\{\phi_i\}_{i\in I}$ be a partition of unity subordinate to $\{U_i\}_{i\in I}$. Let $\widetilde{\mathscr{F}}_t\left(t\square^Z\right)_{U_i}(z, z')$ be the smooth kernel in the sense (3.14). By (3.16), we have

$$\begin{aligned}\mathrm{Tr_s}&\left[\mathscr{F}_t\left(t\square^Z\right)\right] \\ &= \sum_{i\in I}\frac{1}{|G_{U_i}|}\sum_{g\in G_{U_i}}\int_{\widetilde{U}_i}\widetilde{\phi}_i(x)\,\mathrm{Tr_s}\left[g\widetilde{\mathscr{F}}_t\left(t\square^Z\right)_{U_i}(g^{-1}x, x)\right]dv_{\widetilde{U}_i}.\end{aligned} \tag{4.56}$$

For $i \in I$, and $g \in G_{U_i}$, let $N_{\widetilde{U}_i^g/\widetilde{U}_i}$ be the normal bundle of $\widetilde{U}_i^g$ in $\widetilde{U}_i$. We identity $N_{\widetilde{U}_i^g/\widetilde{U}_i}$ with the orthogonal bundle of $T\widetilde{U}_i^g$ in $T\widetilde{U}_i|_{\widetilde{U}_i^g}$. For $\epsilon_0 > 0$, set

$$N_{\widetilde{U}_i^g/\widetilde{U}_i,\epsilon_0} = \left\{(y, Y) \in N_{\widetilde{U}_i^g/\widetilde{U}_i} : \mathrm{dist}\left(y, \mathrm{Supp}(\widetilde{\phi}_i)\right) < \epsilon_0, |Y| < \epsilon_0\right\}. \tag{4.57}$$

Take $\epsilon_0 > 0$ small enough such that for all $i \in I$,

$$\overline{\{x \in \widetilde{U}_i : \mathrm{dist}(x, \mathrm{Supp}(\widetilde{\phi}_i)) < \epsilon_0\}} \subset \widetilde{U}_i, \tag{4.58}$$

and such that all $i \in I$, $g \in G_{U_i}$, the exponential map $(y, Y) \in N_{\widetilde{U}_i^g/\widetilde{U}_i,\epsilon_0} \to \exp_y(Y) \in \widetilde{U}_i$ defines a diffeomorphism from $N_{\widetilde{U}_i^g/\widetilde{U}_i,\epsilon_0}$ onto its image $\widetilde{U}_{i,\epsilon_0,g} \subset$

$\widetilde{U}_i$. Also, there exists $\delta_0 > 0$ such that for all $i \in I$, $g \in G_{U_i}$ if $x \in \mathrm{Supp}(\widetilde{\phi}_i)$ and $\mathrm{dist}(g^{-1}x, x) < \delta_0$, then

$$x \in \widetilde{U}_{i,\epsilon_0,g}. \tag{4.59}$$

Let $dv_{\widetilde{U}^g}$ be the induced Riemannian volume of $\widetilde{U}^g$, and let dY be the induced Lebesgue volume on the fiber of $N_{\widetilde{U}_i^g/\widetilde{U}_i}$. Let $k_i : N_{\widetilde{U}_i^g/\widetilde{U}_i,\epsilon_0} \to \mathbf{R}_+^*$ be a smooth function such that on $\widetilde{U}_{i,\epsilon_0,g}$ we have

$$dv_{\widetilde{U}_i} = k_i(y, Y) dv_{\widetilde{U}^g} dY. \tag{4.60}$$

Clearly, $k_i(y, 0) = 1$.

For $x \in \mathrm{Supp}(\widetilde{\phi}_i)$ and $r \in (0, \epsilon_0)$, let $B_x^{\widetilde{U}_i}(r)$ be the geodesic ball of center x and radius r. Using the result of the finite propagation speeds for the solutions of hyperbolic equations on orbifolds [31, Section 2.3] (see also [33, Theorem D.2.1]), by taking $\alpha_0 < \frac{1}{4}\min\{\delta_0, \epsilon_0\}$, for $x \in \mathrm{Supp}(\widetilde{\phi}_i)$, we find the support of $\widetilde{\mathscr{F}_t}\left(t\square^Z\right)_{U_i}(x, \cdot)$ in $B_x^{\widetilde{U}_i}(4\alpha_0)$. Moreover, $\widetilde{\mathscr{F}_t}\left(t\square^Z\right)_{U_i}(x, \cdot)$ depends only on the Hodge Laplacian $\square^{\widetilde{U}_i}$ acting on $\Omega^{\cdot}(\widetilde{U}_i, \widetilde{F}_{U_i})$. Using (4.59) and (4.60), we get

$$\begin{aligned} \int_{\widetilde{U}_i} \widetilde{\phi}_i(x)\, \mathrm{Tr_s}\left[g\widetilde{\mathscr{F}_t}\left(t\square^Z\right)_{U_i}(g^{-1}x, x)\right] dv_Z \\ = \int_{y\in\widetilde{U}_i^g} dv_{\widetilde{U}_i^g} \int_{Y\in N_{\widetilde{U}_i^g/\widetilde{U}_i,y},|Y|<\epsilon_0} \widetilde{\phi}_i(y, Y) \\ \mathrm{Tr_s}\left[g\widetilde{\mathscr{F}_t}\left(t\square^Z\right)_{U_i}\left(g^{-1}(y, Y), (y, Y)\right)\right] k_i(y, Y) dY. \end{aligned} \tag{4.61}$$

Consider an isometric embedding of $(\widetilde{U}_i, g^{T\widetilde{U}_i})$ into a compact manifold (X_i, g^{TX_i}). We extend the trivial Hermitian vector bundle $(\widetilde{F}_{U_i}, g^{\widetilde{F}_{U_i}})$ to a trivial Hermitian vector bundle (F_i, g^{F_i}) on X_i. Thus, when restricted on $\widetilde{U}_i$,

$$\square^{\widetilde{U}_i} = \square^{X_i}. \tag{4.62}$$

Using the results of the finite propagation speeds for the solutions of hyperbolic equations on orbifolds [31, Section 2.3] and on manifolds [33, Theorem D.2.1], for $x, y \in \widetilde{U}_i$, we have

$$\widetilde{\phi}_i(x)\widetilde{\mathscr{F}_t}\left(t\square^Z\right)_{U_i}(x, y) = \widetilde{\phi}_i(x)\mathscr{F}_t\left(t\square^{X_i}\right)(x, y). \tag{4.63}$$

Recall that for $x \in \widetilde{U}_i$ and $g \in G_{U_i}$, $g : F_{g^{-1}x} \to F_x$ is a linear map. In particular, $g\mathscr{F}_t\left(t\square^{X_i}\right)(g^{-1}x, x)$ is well-defined on $\widetilde{U}_i$. By (4.63), for $y \in \widetilde{U}_i^g$, we have

$$\begin{aligned}
&\int_{Y \in N_{\widetilde{U}_i^g/\widetilde{U}_i,y}, |Y|<\epsilon_0} \widetilde{\phi}_i(y,Y)\,\mathrm{Tr_s}\Big[g\widetilde{\mathscr{F}_t}\left(t\square^Z\right)_{U_i}\left(g^{-1}(y,Y),(y,Y)\right)\Big]k_i(y,Y)dY \\
&= \int_{Y \in N_{\widetilde{U}_i^g/\widetilde{U}_i,y}, |Y|<\epsilon_0} \widetilde{\phi}_i(y,Y)\,\mathrm{Tr_s}\left[g\mathscr{F}_t\left(t\square^{X_i}\right)\left(g^{-1}(y,Y),(y,Y)\right)\right]k_i(y,Y)dY \\
&\qquad = t^{\frac{1}{2}\dim N_{\widetilde{U}_i^g/\widetilde{U}_i}} \int_{Y \in N_{\widetilde{U}_i^g/\widetilde{U}_i,y}, \sqrt{t}|Y|<\epsilon_0} \phi_i\left(y,\sqrt{t}Y\right) \\
&\qquad\qquad \mathrm{Tr_s}\left[g\mathscr{F}_t\left(t\square^{X_i}\right)\left(g^{-1}(y,\sqrt{t}Y),(y,\sqrt{t}Y)\right)\right]k_i\left(y,\sqrt{t}Y\right)dY.
\end{aligned} \tag{4.64}$$

Let $\left[e\left(T\widetilde{U}^g, \nabla^{T\widetilde{U}^g}\right)\right]^{\max}$ be the function defined on $\widetilde{U}^g$ such that

$$e\left(T\widetilde{U}^g, \nabla^{T\widetilde{U}^g}\right) = \left[e\left(T\widetilde{U}^g, \nabla^{T\widetilde{U}^g}\right)\right]^{\max} dv_{\widetilde{U}^g}. \tag{4.65}$$

Theorem 4.18. *There exist $c > 0$ and $C > 0$ such that for any $i \in I$, $g \in G_{U_i}$ and $(y, \sqrt{t}Y) \in N_{\widetilde{U}_i^g/\widetilde{U}_i,\epsilon_0}$, we have*

$$\begin{aligned}
t^{\frac{1}{2}\dim N_{\widetilde{U}_i^g/\widetilde{U}_i}} \Big| &\widetilde{\phi}_i\left(y,\sqrt{t}Y\right)k_i\left(y,\sqrt{t}Y\right) \\
&\mathrm{Tr_s}\left[g\mathscr{F}_t\left(t\square^{X_i}\right)\left(g^{-1}(y,\sqrt{t}Y),(y,\sqrt{t}Y)\right)\right]\Big| \le C\exp(-c|Y|^2).
\end{aligned} \tag{4.66}$$

As $t \to 0$, we have

$$\begin{aligned}
t^{\frac{1}{2}\dim N_{\widetilde{U}_i^g/\widetilde{U}_i}} &\int_{Y \in N_{\widetilde{U}_i^g/\widetilde{U}_i,y}, \sqrt{t}|Y|<\epsilon_0} \Big\{\widetilde{\phi}_i(y,\sqrt{t}Y)k_i\left(y,\sqrt{t}Y\right) \\
&\mathrm{Tr_s}\left[g\mathscr{F}_t\left(t\square^{X_i}\right)\left(g^{-1}(y,\sqrt{t}Y),(y,\sqrt{t}Y)\right)\right]dY\Big\} \\
&\to \widetilde{\phi}_i(y,0)\,\mathrm{Tr}[\rho^F_{U_i}(g)]\left[e\left(T\widetilde{U}^g, \nabla^{T\widetilde{U}^g}\right)\right]^{\max}.
\end{aligned} \tag{4.67}$$

Proof. Theorem 4.18 is a consequence of [6, Theorems 13.13 and 13.15]. □

The end of the proof of Theorem 4.3. By (4.53), (4.56), (4.61), (4.64)-(4.67), and the dominated convergence Theorem, we get

$$\begin{aligned}(4.68)\quad &\lim_{t\to 0}\mathrm{Tr_s}\left[\exp\left(-t\square^Z\right)\right]\\ &=\sum_{i\in I}\frac{1}{|G_{U_i}|}\sum_{g\in G_{U_i}}\mathrm{Tr}[\rho^F_{U_i}(g)]\int_{\widetilde{U}_i^g}\widetilde{\phi}_i(y,0)e\left(T\widetilde{U}_i^g,\nabla^{T\widetilde{U}_i^g}\right).\end{aligned}$$

As $\widetilde{\phi}_i$ is G_{U_i}-invariant, the integral on the right-hand side of (4.68) depends only on the conjugation class of G_{U_i}. Thus,

$$\begin{aligned}(4.69)\quad &\sum_{i\in I}\frac{1}{|G_{U_i}|}\sum_{g\in G_{U_i}}\mathrm{Tr}[\rho^F_{U_i}(g)]\int_{\widetilde{U}_i^g}\widetilde{\phi}_i(y,0)e\left(T\widetilde{U}_i^g,\nabla^{T\widetilde{U}_i^g}\right)\\ &=\sum_{i\in I}\sum_{[g]\in[G_{U_i}]}\frac{\mathrm{Tr}[\rho^F_{U_i}(g)]}{|Z_{G_{U_i}}(g)|}\int_{\widetilde{U}_i^g}\widetilde{\phi}_i(y,0)e\left(T\widetilde{U}_i^g,\nabla^{T\widetilde{U}_i^g}\right)=\sum_{i=0}^{l_0}\rho_i\frac{\chi_{\mathrm{orb}}(Z_i)}{m_i}.\end{aligned}$$

By (4.68) and (4.69), we get (4.6). □

4.4.2. The end of the proof of Theorem 4.13. It remains to show the first identity of (4.38). Following [6, p. 68], we introduce a new Grassmann variable $\mathbf{z}$ which is anticommute with ds. For two operators P, Q in the trace class, set

$$\mathrm{Tr}^{\mathbf{z}}[P+\mathbf{z}Q]=\mathrm{Tr}[Q]. \tag{4.70}$$

By (1.18), (4.34), (4.50) and (4.70), we have

$$\begin{aligned}(4.71)\quad u_t&=\mathrm{Tr_s^{\mathbf{z}}}\left[\exp\left(-A_t^2+\mathbf{z}B_t\right)\right]\\ &=\mathrm{Tr_s^{\mathbf{z}}}\left[\mathscr{F}_t\left(A_t^2-\mathbf{z}B_t\right)\right]+\mathrm{Tr_s^{\mathbf{z}}}\left[\mathscr{G}_t\left(A_t^2-\mathbf{z}B_t\right)\right].\end{aligned}$$

We follow the same strategy used in the proof of Theorem 4.3. As in (4.52), proceeding as in [6, Theorem 13.6], when $t\to 0$, we have

$$\mathrm{Tr_s^{\mathbf{z}}}\left[\mathscr{G}_t\left(A_t^2-\mathbf{z}B_t\right)\right]=\mathcal{O}(e^{-c/t}). \tag{4.72}$$

Moreover, the principal symbol of the lifting of $A_t^2-\mathbf{z}B_t$ on $\widetilde{U}_i$ is scalar, and is equal to $t|\xi|^2/4$ for $\xi\in T^*\widetilde{U}_i$. Take $\alpha_0<\min\{\delta_0,\epsilon_0\}$. As in the case of $\widetilde{\mathscr{F}_t}\left(t\square^Z\right)$, for $x\in\widetilde{U}_i$ and $x\in\mathrm{Supp}(\widetilde{\phi}_i)$, the support of $\widetilde{\mathscr{F}_t}\left(A_t^2-\mathbf{z}B_t\right)_{U_i}(x,\cdot)$

is $B_x^{\widetilde{U}_i}(\alpha_0)$ and its value depends only on the restriction of $A_t^2 - \mathbf{z}B_t$ on U_i. Also,

(4.73)
$$\mathrm{Tr}_{\mathrm{s}}^{\mathbf{z}}\left[\mathscr{F}_t\left(A_t^2-\mathbf{z}B_t\right)\right]=\sum_{i\in I}\frac{1}{|G_{U_i}|}\sum_{g\in G_{U_i}}\int_{y\in\widetilde{U}_i^g}\Bigg\{\int_{Y\in N_{\widetilde{U}^g/\widetilde{U}_i,y},|Y|<\epsilon_0}\widetilde{\phi}_i(y,Y)$$
$$\mathrm{Tr}_{\mathrm{s}}^{\mathbf{z}}\left[g\widetilde{\mathscr{F}_t}\left(A_t^2-\mathbf{z}B_t\right)_{U_i}\left(g^{-1}(y,Y),(y,Y)\right)\right]k_i(y,Y)dY\Bigg\}dv_{\widetilde{U}_i^g}$$

As in (4.62), we can replace $A_t^2 - \mathbf{z}B_t$ by the corresponding operator on manifolds. Recall that $h_g\left(\nabla^{\pi^*\widetilde{F}_{U_i}}, g^{\pi^*\widetilde{F}_{U_i}}\right) \in \Omega^{\mathrm{odd}}(\widetilde{U}_i^g)$ is defined in (3.27). Proceeding as [6, Theorem 3.24], as $t \to 0$, we have

$$\begin{aligned}(4.74)\quad &\int_{y\in\widetilde{U}_i^g}\Bigg\{\int_{Y\in N_{\widetilde{U}^g/\widetilde{U}_i,y},|Y|<\epsilon_0}\widetilde{\phi}_i(y,Y)k_i(y,Y)\\ &\qquad\mathrm{Tr}_{\mathrm{s}}^{\mathbf{z}}\left[g\widetilde{\mathscr{F}_t}\left(A_t^2-\mathbf{z}B_t\right)_{U_i}\left(g^{-1}(y,Y),(y,Y)\right)\right]dY\Bigg\}dv_{\widetilde{U}_i^g}\\ &=\int_{\widetilde{U}_i^g}\widetilde{\phi}_i(y,0)e\left(\pi^*(T\widetilde{U}_i^g),\nabla^{\pi^*(T\widetilde{U}_i^g)}\right)h_g\left(\nabla^{\pi^*\widetilde{F}_{U_i}},g^{\pi^*\widetilde{F}_{U_i}}\right)+\mathcal{O}(\sqrt{t}).\end{aligned}$$

Proceeding now as in (4.69), by (4.72)-(4.74), we get the first identity of (4.38). □

5. Analytic torsion on compact locally symmetric space

Let G be a linear connected real reductive group with maximal compact subgroup $K \subset G$, and let $\Gamma \subset G$ be a discrete cocompact subgroup of G. The corresponding locally symmetric space $Z = \Gamma\backslash G/K$ is a compact orientable orbifold. The purpose of this section is to show Theorem 0.4 which claims an equality between the analytic torsion of an acyclic unitarily flat orbifold vector bundle F on Z and the zero value of the dynamical zeta function associated to the holonomy of F.

This section is organized as follows. In subsections 5.1 and 5.2, we recall some facts on reductive groups and the associated symmetric spaces.

In subsections 5.3 and 5.4, we recall the definition of semisimple elements and the semisimple orbital integrals. We recall the Bismut formula for semisimple orbital integrals [5, Theorem 6.1.1].

In subsection 5.5, we introduce the discrete cocompact subgroup Γ and the associated locally symmetric spaces. We recall the Selberg trace formula.

In subsection 5.6, we introduce a Ruelle dynamical zeta function associated to the holonomy of a unitarily flat orbifold vector bundle on Z. We restate Theorem 0.4. When the fundamental rank $\delta(G) \in \mathbf{N}$ of G does not equal to 1 or when G has noncompact center, we show Theorem 0.4.

Subsections 5.7-5.11 are devoted to the case where G has compact center and $\delta(G) = 1$. In subsection 5.7, we recall some notation and results proved in [47, Sections 6A and 6B]. In subsection 5.8, we introduce a class of representations of K. In subsection 5.9, using the Bismut formula, we evaluate the orbital integrals for the heat operators of the Casimir associated to the K-representations constructed in subsection 5.8. In subsection 5.10, we introduce the Selberg zeta functions, which are shown to be meromorphic on $\mathbf{C}$ and satisfy certain functional equations. In subsection 5.11, we show that the dynamical zeta function equals an alternating product of certain Selberg zeta functions. We show Theorem 0.4.

5.1. Reductive groups

Let G be a linear connected real reductive group [29, p. 3], let $\theta \in \mathrm{Aut}(G)$ be the Cartan involution. That means G is a closed connected group of real matrices that is stable under transpose, and θ is the composition of transpose and inverse of matrices. Let $K \subset G$ be the subgroup of G fixed by θ, so that K is a maximal compact subgroup of G.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K. The Cartan involution θ acts naturally as Lie algebra automorphism of $\mathfrak{g}$. Then $\mathfrak{k}$ is the eigenspace of θ associated with the eigenvalue 1. Let $\mathfrak{p} \subset \mathfrak{g}$ be the eigenspace with the eigenvalue -1, so that

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}. \tag{5.1}$$

By [29, Proposition 1.2], we have the diffeomorphism

$$(Y, k) \in \mathfrak{p} \times K \to e^{Y} k \in G. \tag{5.2}$$

Set

$$m = \dim \mathfrak{p}, \qquad n = \dim \mathfrak{k}. \tag{5.3}$$

Let B be a real-valued non degenerate bilinear symmetric form on $\mathfrak{g}$ which is invariant under the adjoint action of G, and also under θ. Then (5.1) is an orthogonal splitting with respect to B. We assume B to be positive on $\mathfrak{p}$, and negative on $\mathfrak{k}$. The form $\langle \cdot, , \cdot \rangle = -B(\cdot, \theta \cdot)$ defines an $\mathrm{Ad}(K)$-invariant

scalar product on $\mathfrak{g}$ such that the splitting (5.1) is still orthogonal. We denote by $|\cdot|$ the corresponding norm.

Let $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$ be the complexification of $\mathfrak{g}$ and let $\mathfrak{u} = \sqrt{-1}\mathfrak{p} \oplus \mathfrak{k}$ be the compact form of $\mathfrak{g}$. Let $G_{\mathbf{C}}$ and U be the connected group of complex matrices associated to the Lie algebras $\mathfrak{g}_{\mathbf{C}}$ and $\mathfrak{u}$. By [29, Propositions 5.3 and 5.6], if G has a compact center, $G_{\mathbf{C}}$ is a linear connected complex reductive group with maximal compact subgroup U.

Let $\mathscr{U}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. We identify $\mathscr{U}(\mathfrak{g})$ with the algebra of left-invariant differential operators on G. Let $C^{\mathfrak{g}} \in \mathscr{U}(\mathfrak{g})$ be the Casimir element. If $e_1, \cdots, e_m$ is an orthonormal basis of $\mathfrak{p}$, if $e_{m+1}, \cdots, e_{m+n}$ is an orthonormal basis of $\mathfrak{k}$, then

$$C^{\mathfrak{g}} = -\sum_{i=1}^{m} e_i^2 + \sum_{i=m+1}^{n+m} e_i^2. \tag{5.4}$$

Classically, $C^{\mathfrak{g}}$ is in the center of $\mathscr{U}(\mathfrak{g})$.

We define $C^{\mathfrak{k}}$ similarly. Let τ be a finite dimensional representation of K on V. We denote by $C^{\mathfrak{k},V}$ or $C^{\mathfrak{k},\tau} \in \mathrm{End}(V)$ the corresponding Casimir operator acting on V, so that

$$C^{\mathfrak{k},V} = C^{k,\tau} = \sum_{i=m+1}^{m+n} \tau(e_i)^2. \tag{5.5}$$

Let $\delta(G) \in \mathbf{N}$ be the fundamental rank of G, that is defined by the difference between the complex ranks of G and K. If $T \subset K$ is a maximal torus of K with Lie algebra of $\mathfrak{t} \subset \mathfrak{k}$, set

$$\mathfrak{b} = \{Y \in \mathfrak{p} : [Y, \mathfrak{t}] = 0\}. \tag{5.6}$$

Put

$$\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{t}, \qquad H = \exp(\mathfrak{b})T. \tag{5.7}$$

By [29, Theorem 5.22], $\mathfrak{h} \subset \mathfrak{g}$ (resp. $H \subset G$) is a θ-invariant Cartan subalgebra (resp. subgroup). Therefore,

$$\delta(G) = \dim \mathfrak{b}. \tag{5.8}$$

Moreover, up to conjugation, $\mathfrak{h} \subset \mathfrak{g}$ (resp. $H \subset G$) is the unique Cartan subalgebra (resp. subgroup) with minimal noncompact dimension.

5.2. Symmetric space

Let $\omega^{\mathfrak{g}}$ be the canonical left-invariant 1-form on G with values in $\mathfrak{g}$, and let $\omega^{\mathfrak{p}}, \omega^{\mathfrak{k}}$ be its components in $\mathfrak{p}, \mathfrak{k}$, so that

$$\omega^{\mathfrak{g}} = \omega^{\mathfrak{p}} + \omega^{\mathfrak{k}}. \tag{5.9}$$

Set $X = G/K$. Then $p : G \to X = G/K$ is a K-principle bundle equipped with the connection form $\omega^{\mathfrak{k}}$.

Let τ be a finite dimensional orthogonal representation of K on the real Euclidean space E_τ. Let $\mathcal{E}_\tau$ be the associated Euclidean vector bundle with total space $G \times_K E_\tau$. It is equipped a Euclidean connection $\nabla^{\mathcal{E}_\tau}$ induced by $\omega^{\mathfrak{k}}$. We identify $C^\infty(X, \mathcal{E}_\tau)$ with the K-invariant subspace $C^\infty(G, E_\tau)^K$ of smooth E_τ-valued functions on G. Let $C^{\mathfrak{g},X,\tau}$ be the Casimir element of G acting on $C^\infty(X, \mathcal{E}_\tau)$.

Observe that K acts isometrically on $\mathfrak{p}$ by adjoint action. Using the above construction, the total space of the tangent bundle TX is given by

$$G \times_K \mathfrak{p}. \tag{5.10}$$

It is equipped with a Euclidean metric g^{TX} and a Euclidean connection ∇^{TX}, which coincides with the Levi-Civita connection of the Riemannian manifold (X, g^{TX}). Classically, (X, g^{TX}) has non positive sectional curvature.

If $E_\tau = \Lambda^\cdot(\mathfrak{p}^*)$ is equipped with the K-action induced by the adjoint action, then $C^\infty(X, \mathcal{E}_\tau) = \Omega^\cdot(X)$. In this case, we write $C^{\mathfrak{g},X} = C^{\mathfrak{g},X,\tau}$. By [5, Proposition 7.8.1], $C^{\mathfrak{g},X}$ coincides with the Hodge Laplacian acting on $\Omega^\cdot(X)$.

Let dv_X be the Riemannian volume of (X, g^{TX}). Define $[e(TX, \nabla^{TX})]^{\max}$ as in (4.65). Since both dv_X and $e(TX, \nabla^{TX})$ are G-invariant, we see that $[e(TX, \nabla^{TX})]^{\max} \in \mathbf{R}$ is a constant. Note that $\delta(G)$ and $\dim X$ have the same parity. By [47, Proposition 4.1], if $\delta(G) \neq 0$, then

$$\left[e\left(TX, \nabla^{TX}\right)\right]^{\max} = 0. \tag{5.11}$$

If $\delta(G) = 0$, G has a compact center. Then U is a compact group with maximal torus T. Denote by $W(T, U)$ (resp. $W(T, K)$) the Weyl group of U (resp. K) with respect to T, and by $\mathrm{vol}(U/K)$ the volume of U/K induced by $-B$. Then, [47, Proposition 4.1] asserts

$$\left[e\left(TX, \nabla^{TX}\right)\right]^{\max} = (-1)^{m/2} \frac{|W(T, U)|/|W(T, K)|}{\mathrm{vol}(U/K)}. \tag{5.12}$$

5.3. Semisimple elements

If $\gamma \in G$, we denote by $Z(\gamma) \subset G$ the centralizer of γ in G, and by $\mathfrak{z}(\gamma) \subset \mathfrak{g}$ its Lie algebra. If $a \in \mathfrak{g}$, let $Z(a) \subset G$ be the stabilizer of a in G, and let $\mathfrak{z}(a) \subset \mathfrak{g}$ be its Lie algebra.

Following [5, Section 3.1], $\gamma \in G$ is said to be semisimple if and only if there is $g_\gamma \in G$, such that $\gamma = g_\gamma e^a k^{-1} g_\gamma^{-1}$ with

$$a \in \mathfrak{p}, \qquad k \in K, \qquad \mathrm{Ad}(k)a = a. \tag{5.13}$$

Set

$$a_\gamma = \mathrm{Ad}(g_\gamma)a, \qquad k_\gamma = g_\gamma k g_\gamma^{-1}. \tag{5.14}$$

Therefore, $\gamma = e^{a_\gamma} k_\gamma^{-1}$. Moreover, this decomposition does not depend on the choice of g_γ. By [5, (3.3.3)], we have

$$Z(\gamma) = Z(a_\gamma) \cap Z(k_\gamma), \qquad \mathfrak{z}(\gamma) = \mathfrak{z}(a_\gamma) \cap \mathfrak{z}(k_\gamma). \tag{5.15}$$

By [30, Proposition 7.25], $Z(\gamma)$ is reductive. The corresponding Cartan evolution and bilinear form are given by

$$\theta_{g_\gamma} = g_\gamma \theta g_\gamma^{-1}, \qquad B_{g_\gamma}(\cdot,\cdot) = B\left(\mathrm{Ad}(g_\gamma^{-1})\cdot, \mathrm{Ad}(g_\gamma^{-1})\cdot\right). \tag{5.16}$$

Let $K(\gamma) \subset Z(\gamma)$ be the fixed point of θ_{g_γ}, so $K(\gamma)$ is a maximal compact subgroup $Z(\gamma)$. Let $\mathfrak{k}(\gamma) \subset \mathfrak{z}(\gamma)$ be the Lie algebra of $K(\gamma)$. Let

$$\mathfrak{z}(\gamma) = \mathfrak{p}(\gamma) \oplus \mathfrak{k}(\gamma) \tag{5.17}$$

be the Cartan decomposition of $\mathfrak{z}(\gamma)$. Let

$$X(\gamma) = Z(\gamma)/K(\gamma) \tag{5.18}$$

be the associated symmetric space.

Let $Z^0(\gamma)$ be the connected component of the identity in $Z(\gamma)$. Similarly, $Z^0(\gamma)$ is reductive with maximal compact subgroup $Z^0(\gamma) \cap K(\gamma)$. Also, $Z^0(\gamma) \cap K(\gamma)$ coincides with $K^0(\gamma)$, the connected component of the identity in $K(\gamma)$. Clearly, we have

$$X(\gamma) = Z^0(\gamma)/K^0(\gamma). \tag{5.19}$$

The semisimple element γ is called elliptic if $a_\gamma = 0$. Assume now γ is semisimple and nonelliptic. Then $a_\gamma \neq 0$. Let $\mathfrak{z}^{a,\perp}(\gamma)$ (resp. $\mathfrak{p}^{a,\perp}(\gamma)$) be the

orthogonal spaces to a_γ in $\mathfrak{z}(\gamma)$ (resp. $\mathfrak{p}(\gamma)$) with respect to B_{g_γ}. Thus,

$$\mathfrak{z}^{a,\perp}(\gamma) = \mathfrak{p}^{a,\perp}(\gamma) \oplus \mathfrak{k}(\gamma). \tag{5.20}$$

Moreover, $\mathfrak{z}^{a,\perp}(\gamma)$ is a Lie algebra. Let $Z^{a,\perp,0}(\gamma)$ be the connected subgroup of $Z^0(\gamma)$ that is associated with the Lie algebra $\mathfrak{z}^{a,\perp}(\gamma)$. By [5, (3.3.11)], $Z^{a,\perp,0}(\gamma)$ is reductive with maximal compact subgroup $K^0(\gamma)$ with Cartan decomposition (5.20), and

$$Z^0(\gamma) = \mathbf{R} \times Z^{a,\perp,0}(\gamma), \tag{5.21}$$

so that $e^{ta_\gamma} \in Z^0(\gamma)$ maps into $t|a| \in \mathbf{R}$. Set

$$X^{a,\perp}(\gamma) = Z^{a,\perp,0}(\gamma)/K^0(\gamma). \tag{5.22}$$

By (5.19), (5.21), and (5.22), we have

$$X(\gamma) = \mathbf{R} \times X^{a,\perp}(\gamma), \tag{5.23}$$

so that the action e^{ta_γ} on $X(\gamma)$ is just the translation by $t|a|$ on $\mathbf{R}$.

5.4. Semisimple orbital integral

Recall that τ is a finite dimensional orthogonal representation of K on the real Euclidean space E_τ. Let $p_t^{X,\tau}(x,x')$ be the smooth kernel of the heat operator $\exp(-tC^{\mathfrak{g},X,\tau}/2)$ with respect to the Riemannian volume dv_X. As in [5, (4.1.6)], let $p_t^{X,\tau}(g)$ be the equivariant representation of the section $p_t^{X,\tau}(p1,\cdot)$. Then $p_t^{X,\tau}(g)$ is a $K \times K$-invariant function in $C^\infty(G, \mathrm{End}(E_\tau))$.

Let dv_G be the left-invariant Riemannian volume on G induced by the metric $-B(\cdot,\theta\cdot)$. For a semisimple element $\gamma \in G$, denote by $dv_{Z^0(\gamma)}$ the left-invariant Riemannian volume on $Z^0(\gamma)$ induced by $-B_{g_\gamma}(\cdot,\theta_{g_\gamma}\cdot)$. Clearly, the choice of g_γ is irrelevant. Let $dv_{Z^0(\gamma)\backslash G}$ be the Riemannian volume on $Z^0(\gamma)\backslash G$ such that $dv_G = dv_{Z^0(\gamma)} dv_{Z^0(\gamma)\backslash G}$. By [5, Definition 4.2.2, Proposition 4.4.2], the orbital integral

$$\begin{aligned} &\mathrm{Tr}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\tau}/2\right)\right] \\ &\qquad = \frac{1}{\mathrm{vol}(K^0(\gamma)\backslash K)} \int_{Z^0(\gamma)\backslash G} \mathrm{Tr}^{E_\tau}\left[p_t^{X,\tau}(g)\right] dv_{Z^0(\gamma)\backslash G} \end{aligned} \tag{5.24}$$

is well-defined.

Remark 5.1. In [5, Definition 4.2.2], the volume $\mathrm{vol}(K_0(\gamma)\backslash K)$ are normalized to be 1. By [5, (3.3.18)], in the definition of the orbital integral (5.24), we can replace $K^0(\gamma), Z^0(\gamma)$ by $K(\gamma), Z(\gamma)$.

Remark 5.2. As the notation $\mathrm{Tr}^{[\gamma]}$ indicates, the orbital integral only depends on the conjugacy class of γ in G. However, the notation $[\gamma]$ will be used later for the conjugacy class of a discrete group Γ. Here, we consider $\mathrm{Tr}^{[\gamma]}$ as an abstract symbol.

We will also consider the case where $E_\tau = E_\tau^+ \oplus E_\tau^-$ is a $\mathbf{Z}_2$-graded representation of K. In this case, We will use the notation $\mathrm{Tr_s}^{[\gamma]}\left[\exp(-tC^{\mathfrak{g},X,\tau}/2)\right]$ when the trace on the right-hand side of (5.24) is replaced by the supertrace on E_τ.

In [5, Theorem 6.1.1], for any semisimple element $\gamma \in G$, Bismut gave an explicit formula for $\mathrm{Tr}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\tau}/2\right)\right]$. For the later use, let us recall the formula when γ is elliptic.

Assume now $\gamma \in K$. By (5.14), we can take $g_\gamma = 1$. Then $\mathfrak{p}(\gamma) \subset \mathfrak{p}$, $\mathfrak{k}(\gamma) \subset \mathfrak{k}$. Let $\mathfrak{p}^\perp(\gamma) \subset \mathfrak{p}$, $\mathfrak{k}^\perp(\gamma) \subset \mathfrak{k}$ be the orthogonal space of $\mathfrak{p}(\gamma)$, $\mathfrak{k}(\gamma)$. Take $\mathfrak{z}^\perp(\gamma) = \mathfrak{p}^\perp(\gamma) \oplus \mathfrak{k}^\perp(\gamma)$. Recall that $\widehat{A}$ is defined in (1.12). Following [5, Theorem 5.5.1], for $Y \in \mathfrak{k}(\gamma)$, put

$$J_\gamma(Y) = \frac{\widehat{A}\big(i\,\mathrm{ad}(Y)|_{\mathfrak{p}(\gamma)}\big)}{\widehat{A}\big(i\,\mathrm{ad}(Y)|_{\mathfrak{k}(\gamma)}\big)} \left[\frac{1}{\det\big(1-\mathrm{Ad}(\gamma)\big)|_{\mathfrak{z}^\perp(\gamma)}} \frac{\det\big(1-\exp(-i\,\mathrm{ad}(Y))\,\mathrm{Ad}(\gamma)\big)|_{\mathfrak{k}^\perp(\gamma)}}{\det\big(1-\exp(-i\,\mathrm{ad}(Y))\,\mathrm{Ad}(\gamma)\big)|_{\mathfrak{p}^\perp(\gamma)}}\right]^{1/2}. \tag{5.25}$$

Note that by [5, Section 5.5], the square root in (5.25) is well-defined, and its sign is chosen such that

$$J_\gamma(0) = \Big(\det\big(1-\mathrm{Ad}(\gamma)\big)|_{\mathfrak{p}^\perp(\gamma)}\Big)^{-1}. \tag{5.26}$$

Moreover, J_γ is an $\mathrm{Ad}\left(K^0(\gamma)\right)$-invariant analytic function on $\mathfrak{k}(\gamma)$ such that there exist $c_\gamma > 0, C_\gamma > 0$, for $Y \in \mathfrak{k}(\gamma)$,

$$|J_\gamma(Y)| \leq C_\gamma \exp\left(c_\gamma |Y|\right). \tag{5.27}$$

Denote by dY be the Lebesgue measure on $\mathfrak{k}(\gamma)$ induced by $-B$. Recall that $C^{\mathfrak{k},\mathfrak{p}}, C^{\mathfrak{k},\mathfrak{p}}$ are defined in (5.5). By [5, Theorem 6.1.1], for $t > 0$, we have

$$\begin{aligned}(5.28)\quad & \mathrm{Tr}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\tau}/2\right)\right] \\ &= \frac{1}{(2\pi t)^{\dim \mathfrak{z}(\gamma)/2}} \exp\left(\frac{t}{16}\,\mathrm{Tr}^{\mathfrak{p}}\left[C^{\mathfrak{k},\mathfrak{p}}\right] + \frac{t}{48}\,\mathrm{Tr}^{\mathfrak{k}}\left[C^{\mathfrak{k},\mathfrak{k}}\right]\right) \\ &\quad \int_{Y\in\mathfrak{k}(\gamma)} J_\gamma(Y)\,\mathrm{Tr}^{E_\tau}\left[\tau(\gamma)\exp(-i\tau(Y))\right]\exp\left(-|Y|^2/2t\right)dY.\end{aligned}$$

5.5. Locally symmetric spaces

Let $\Gamma \subset G$ be a discrete cocompact subgroup of G. By [46, Lemma 1], the elements of Γ are semisimple. Let $\Gamma_e \subset \Gamma$ be the subset of elliptic elements in Γ. Set $\Gamma_+ = \Gamma - \Gamma_e$. Let $[\Gamma]$ be the set of conjugacy classes of Γ, and let $[\Gamma_e] \subset [\Gamma]$ and $[\Gamma_+] \subset [\Gamma]$ be respectively the subsets of $[\Gamma]$ formed by the conjugacy classes of elements in Γ_e and Γ_+. Clearly, $[\Gamma_e]$ is a finite set.

The group Γ acts properly discontinuously and isometrically on the left on X. Take $Z = \Gamma\backslash X$ to be the corresponding locally symmetric space. By Proposition 2.12 and Theorem 2.31, Z is a compact orbifold. Note that by (5.2), X is a contractible manifold. By Remark 2.33, X is the universal covering orbifold of Z. The Riemannian metric g^{TX} on X induces a Riemannian metric g^{TZ} on Z. Clearly, (Z, g^{TZ}) has nonpositive curvature.

Let $\Delta_\Gamma \subset \Gamma$ be the subgroup of the elements in Γ that act like the identity on X. Clearly, Δ_Γ is a finite group given by

$$\Delta_\Gamma = \Gamma \cap K \cap Z(\mathfrak{p}), \tag{5.29}$$

where $Z(\mathfrak{p}) \subset G$ is the stabiliser of $\mathfrak{p}$ in G. Thus, the orbifold fundamental group of Z is Γ/Δ_Γ.

Let F be a (possibly non proper) flat vector bundle on Z with holonomy $\rho' : \Gamma/\Delta_\Gamma \to \mathrm{GL}_r(\mathbf{C})$ such that

$$C^\infty(Z, F) = C^\infty(X, \mathbf{C}^r)^{\Gamma/\Delta_\Gamma}. \tag{5.30}$$

Take ρ to be the composition of the projection $\Gamma \to \Gamma/\Delta_\Gamma$ and ρ'. Then

$$C^\infty(Z, F) = C^\infty(X, \mathbf{C}^r)^{\Gamma}. \tag{5.31}$$

By abuse of notation, we still call $\rho : \Gamma \to \mathrm{GL}_r(\mathbf{C})$ the holonomy of F. In the rest of this section, we assume F is unitarily flat, or equivalently ρ is

unitary. Let g^F be the associate flat Hermitian metric on F. Since g^{TZ} and g^F are fixed in the whole section, we write

$$T(F) = T\left(F, g^{TZ}, g^F\right). \tag{5.32}$$

The group Γ acts on the Euclidean vector bundles like $\mathcal{E}_\tau$, and preserves the corresponding connections $\nabla^{\mathcal{E}_\tau}$. The vector bundle $\mathcal{E}_\tau$ descends to a (possibly non proper) orbifold vector bundle $\mathcal{F}_\tau$ on Z. The total space of $\mathcal{F}_\tau$ is given by $\Gamma\backslash G \times_K E_\tau$, and we have the identification of vector spaces

$$C^\infty(Z, \mathcal{F}_\tau) \simeq C^\infty(\Gamma\backslash G, E_\tau)^K. \tag{5.33}$$

By (5.31) and (5.33), we identify $C^\infty(Z, \mathcal{F}_\tau \otimes_{\mathbf{R}} F)$ with the Γ-invariant subspace of $C^\infty(X, \mathcal{E}_\tau \otimes_{\mathbf{R}} \mathbf{C}^r)$. Let $C^{\mathfrak{g},Z,\tau,\rho}$ be the Casimir operator of G acting on $C^\infty(Z, \mathcal{F}_\tau \otimes_{\mathbf{R}} F)$. As we see in subsection 5.2, when $E_\tau = \Lambda^\cdot(\mathfrak{p}^*)$,

$$\Omega^\cdot(Z, F) \simeq C^\infty(Z, \mathcal{F}_\tau \otimes_{\mathbf{R}} F), \tag{5.34}$$

and the Hodge Laplacian acting on $\Omega^\cdot(Z, F)$ is given by

$$\square^Z = C^{\mathfrak{g},Z,\tau,\rho}. \tag{5.35}$$

For $\gamma \in \Gamma$, set $\Gamma(\gamma) = Z(\gamma) \cap \Gamma$. By [46, Lemma 2] (see also [47, Proposition 4.9]), $\Gamma(\gamma)$ is cocompact in $Z(\gamma)$. Then $\Gamma(\gamma)\backslash X(\gamma)$ is a compact locally symmetric orbifold. Clearly, it depends only on the conjugacy class of γ in Γ. Denote by $\mathrm{vol}(\Gamma(\gamma)\backslash X(\gamma))$ the Riemannian volume of $\Gamma(\gamma)\backslash X(\gamma)$ induced by B_{g_γ}.

The group $K(\gamma)$ acts on the right on $\Gamma(\gamma)\backslash Z(\gamma)$. For $h \in \Gamma(\gamma)\backslash Z(\gamma)$, let $K(\gamma)_h$ be the stabilizer of h in $K(\gamma)$. Since $\Gamma(\gamma)\backslash X(\gamma)$ is connected, the cardinal of a generic stabilizer is well defined and depends only on the conjugacy class of γ in Γ. We denote it by $n_{[\gamma]}$. Then, we have

$$\frac{\mathrm{vol}(\Gamma(\gamma)\backslash Z(\gamma))}{\mathrm{vol}(K(\gamma))} = \frac{\mathrm{vol}(\Gamma(\gamma)\backslash X(\gamma))}{n_{[\gamma]}}. \tag{5.36}$$

Let us note that even if $K(\gamma)$ acts effectively on $\Gamma(\gamma)\backslash Z(\gamma)$, $n_{[\gamma]}$ is not necessarily equal to 1.

Proposition 5.3. *For $\gamma \in \Gamma$, we have*

$$n_{[\gamma]} = \big|K \cap \Gamma(\gamma) \cap Z(\mathfrak{p}(\gamma))\big|. \tag{5.37}$$

In particular, if $\gamma = e$, we have

$$n_{[e]} = |\Delta_\Gamma|, \tag{5.38}$$

and if $\gamma \in \Delta_\Gamma$, we have

$$n_{[\gamma]} = |\Gamma(\gamma) \cap \Delta_\Gamma|. \tag{5.39}$$

Proof. For a generic element $g = e^f h$ in $Z(\gamma)$ with $f \in \mathfrak{p}(\gamma)$ and $h \in K(\gamma)$, the stabliser of $\Gamma(\gamma)g \in \Gamma(\gamma)\backslash Z(\gamma)$ in $K(\gamma)$ is given by

$$K(\gamma)_{\Gamma(\gamma)g} = K(\gamma) \cap g^{-1}\Gamma(\gamma)g = g^{-1}\big(e^f K(\gamma) e^{-f} \cap \Gamma(\gamma)\big)g. \tag{5.40}$$

Then,

$$n_{[\gamma]} = \left|e^f K(\gamma) e^{-f} \cap \Gamma(\gamma)\right|. \tag{5.41}$$

Since $e^f K(\gamma) e^{-f}$ is compact, since $\Gamma(\gamma)$ is discrete, and since f can vary in an open dense set, we can deduce that

$$n_{[\gamma]} = |K(\gamma) \cap Z(\mathfrak{p}(\gamma)) \cap \Gamma(\gamma)|, \tag{5.42}$$

from which we get (5.37). By (5.29) and (5.37), we get (5.38). If $\gamma \in \Delta_\Gamma$, by (5.29), we get $\mathfrak{p}(\gamma) = \mathfrak{p}$. Combining the result with (5.37), we get (5.39). □

Theorem 5.4. *There exist $c > 0$ and $C > 0$ such that, for $t > 0$, we have*

$$\sum_{[\gamma]\in[\Gamma_+]} \frac{\mathrm{vol}(\Gamma(\gamma)\backslash X(\gamma))}{n_{[\gamma]}} \left|\mathrm{Tr}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\tau}/2\right)\right]\right| \leq C\exp\left(-\frac{c}{t} + Ct\right). \tag{5.43}$$

For $t > 0$, the following identity holds:

$$\begin{aligned} &\mathrm{Tr}\left[\exp\left(-tC^{\mathfrak{g},Z,\tau,\rho}/2\right)\right] \\ &\quad = \sum_{[\gamma]\in[\Gamma]} \mathrm{Tr}[\rho(\gamma)] \frac{\mathrm{vol}\left(\Gamma(\gamma)\backslash X(\gamma)\right)}{n_{[\gamma]}} \mathrm{Tr}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\tau}/2\right)\right]. \end{aligned} \tag{5.44}$$

Proof. The proof is identical to the one given in [47, Theorem 4.10]. One difference is that we need to show an estimate like [47, (4-26)]. This can be deduced from [46, Lemma 8], which states that Γ possesses a normal torsion free subgroup of finite index.

Let us explain the reason for which the coefficients $n_{[\gamma]}$ appear in (5.44). Indeed, the restriction on the diagonal of the trace of the integral kernel of $\exp(-tC^{\mathfrak{g},Z,\tau,\rho}/2)$ is given by

$$\frac{1}{|\Delta_\Gamma|}\sum_{\gamma\in\Gamma}\mathrm{Tr}[\rho(\gamma)]\,\mathrm{Tr}^{\mathcal{E}_\tau}\left[\gamma_x p_t^{X,\tau}(x,\gamma x)\right], \tag{5.45}$$

where γ_x denotes the obvious element in $\mathrm{Hom}(\mathcal{E}_{\tau,x},\mathcal{E}_{\tau,\gamma x})$ induced by γ (see [5, p. 79]). By (5.36), (5.38), and (5.45), we have

$$\begin{aligned}\mathrm{Tr}\left[\exp\left(-tC^{\mathfrak{g},Z,\tau,\rho}/2\right)\right]&\\ =\frac{1}{\mathrm{vol}(K)}\int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma}\mathrm{Tr}[\rho(\gamma)]\,\mathrm{Tr}^{E_\tau}&\left[p_t^{X,\tau}(g^{-1}\gamma g)\right]dv_{\Gamma\backslash G},\end{aligned} \tag{5.46}$$

where $dv_{\Gamma\backslash G}$ is the volume form on $\Gamma\backslash G$ induced by dv_G. Proceeding as in [5, (4.8.8)-(4.8.12)], by Remark 5.1, (5.24), and (5.46), we get

$$\begin{aligned}\mathrm{Tr}\left[\exp\left(-tC^{\mathfrak{g},Z,\tau,\rho}/2\right)\right]&\\ =\sum_{[\gamma]\in[\Gamma]}\mathrm{Tr}[\rho(\gamma)]\frac{\mathrm{vol}(\Gamma(\gamma)\backslash Z(\gamma))}{\mathrm{vol}(K(\gamma))}&\,\mathrm{Tr}^{[\gamma]}\left[\exp(-tC^{\mathfrak{g},X,\tau}/2)\right].\end{aligned} \tag{5.47}$$

By (5.36) and (5.47), we get (5.44). □

Remark 5.5. By [1, Theorem 3.14, Remark 3.15] and [5, (4.8.22)], when counting with multiplicites, we have the identification of the orbifolds,

$$Z\coprod\Sigma Z\simeq\coprod_{[\gamma]\in[\Gamma_e]}\Gamma(\gamma)\backslash X(\gamma), \tag{5.48}$$

where the multiplicity of each component Z_i of $Z\coprod\Sigma Z$ is m_i (see 2.18) and the multiplicity of $\Gamma(\gamma)\backslash X(\gamma)$ is $n_{[\gamma]}$. Note also that, by Remark 2.25, we can consider $\coprod_{[\gamma]\in[\Gamma_e]}\Gamma(\gamma)\backslash X(\gamma)$ as the space of paths on Z of length 0.

Remark 5.6. Assume that Γ acts effectively on X. Then Z can be represented by the action groupoid $\mathcal{G}$ (see [1, Example 1.32]) whose object is X and whose arrow is $\Gamma\times X$. An arrow $(\gamma,x)\in\Gamma\times X$ maps x to γx. By

Remark 2.26, any closed geodesic on Z can be represented by the closed $\mathcal{G}$-geodesic $c = (b_0; \mathrm{id}, \gamma^{-1})$ where $b_0 : [0,1] \to X$ is a geodesic on X such that $\gamma b_0 = b_1$ and $\gamma_* \dot{b}_0(0) = \dot{b}_0(1)$. By [5, Theorem 3.1.2], the length of b_0 is given by $|a_\gamma|$ (see (5.14)). Moreover, the space of the closed $\mathcal{G}$-geodesics is given by $\cup_{\gamma\in\Gamma} X(\gamma)$. Thus, the space of the closed geodesics on Z is given by $\coprod_{[\gamma]\in[\Gamma]} \Gamma(\gamma)\backslash X(\gamma)$, whose component has the multiplicity $n_{[\gamma]}$. For general Γ, as in Remark 5.5, the same result still holds ture.

We extend [37, Corollary 2.2] and [5, Theorem 7.9.3] to orbifolds (see also [47, Corollary 4.3]).

Corollary 5.7. *Let F be a unitarily flat orbifold vector bundle on Z. If $\dim Z$ is odd and $\delta(G) \neq 1$, then for any $t > 0$, we have*

$$\mathrm{Tr_s}\left[N^{\Lambda^\cdot(T^*Z)} \exp\left(-t\Box^Z/2\right)\right] = 0. \tag{5.49}$$

In particular,

$$T(F) = 1. \tag{5.50}$$

Proof. Since $\dim Z$ is odd, $\delta(G)$ is odd. Since $\delta(G) \neq 1$, $\delta(G) \geqslant 3$. By [47, Theorem 4.12], for any $\gamma \in G$ semisimple, we have

$$\mathrm{Tr_s}^{[\gamma]}\left[N^{\Lambda^\cdot(T^*X)} \exp\left(-tC^{\mathfrak{g},X}/2\right)\right] = 0. \tag{5.51}$$

By (5.35), (5.44), and (5.51), we get (5.49). □

Suppose that $\delta(G) = 1$. Let us recall some notation in [47, (4-49)-(4-52), (6-15), (6-16)]. Up to sign, we fix an element $a_1 \in \mathfrak{b}$ such that $B(a_1, a_1) = 1$. As in subsection 5.3, set

$$M = Z^{a_1,\perp,0}(e^{a_1}), \qquad K_M = K^0(e^{a_1}), \tag{5.52}$$

and

$$\mathfrak{m} = \mathfrak{z}^{a_1,\perp}(e^{a_1}), \qquad \mathfrak{p}_\mathfrak{m} = \mathfrak{p}^{a_1,\perp}(e^{a_1}), \qquad \mathfrak{k}_\mathfrak{m} = \mathfrak{k}(e^{a_1}). \tag{5.53}$$

As in subsection 5.3, M is a connected reductive group such that $\delta(M) = 0$ with Lie algebra $\mathfrak{m}$, with maximal compact subgroup K_M, and with Cartan

decomposition $\mathfrak{m} = \mathfrak{p}_{\mathfrak{m}} \oplus \mathfrak{k}_{\mathfrak{m}}$. Let

$$X_M = M/K_M \tag{5.54}$$

be the corresponding symmetric space. For $k \in T$, we have $\delta(Z^0(k)) = 1$. Denote by $M^0(k)$, $\mathfrak{m}(k)$, $\mathfrak{p}_{\mathfrak{m}}(k)$, $\mathfrak{k}_{\mathfrak{m}}(k)$, $X_M(k)$ the analogies of M, $\mathfrak{m}$, $\mathfrak{p}_{\mathfrak{m}}$, $\mathfrak{k}_{\mathfrak{m}}$, $X_{\mathfrak{m}}$ when G is replaced by $Z^0(k)$.

Assume that $\delta(G) = 1$ and that G has noncompact center. By [47, 4-51], we have

$$G = \mathbf{R} \times M, \qquad K = K_M, \qquad X = \mathbf{R} \times X_M. \tag{5.55}$$

Recall that $H = \exp(\mathfrak{b})T$. Note that if $\gamma = e^a k^{-1} \in H$ with $a \neq 0$, then

$$X^{a,\perp}(\gamma) = X_M(k). \tag{5.56}$$

We have an extension of [47, Proposition 4.14].

Proposition 5.8. *Let $\gamma \in G$ be a semisimple element. If γ can not be conjugated into H by elements of G, then for $t > 0$, we have*

$$\mathrm{Tr_s}^{[\gamma]}\left[N^{\Lambda^\cdot(T^*X)} \exp\left(-tC^{\mathfrak{g},X}/2\right)\right] = 0. \tag{5.57}$$

If $\gamma = e^a k^{-1} \in H$ with $a \in \mathfrak{b}$ and $k \in T$, then for $t > 0$, we have

$$\begin{aligned} \mathrm{Tr}^{[\gamma]}&\left[N^{\Lambda^\cdot(T^*X)} \exp\left(-tC^{\mathfrak{g},X}/2\right)\right] \\ &= -\frac{1}{\sqrt{2\pi t}} e^{-|a|^2/2t}\left[e\left(TX_M(k), \nabla^{TX_M(k)}\right)\right]^{\max}. \end{aligned} \tag{5.58}$$

Proof. Equations (5.57), (5.58) with $\gamma = 1$ or $\gamma = e^a k^{-1}$ with $a \neq 0$ are just [47, (4-45), (4-53), (4-54)]. Equation (5.58) for general $\gamma \in H$ is a consequence of [47, (4-55), (4-56), (4-58)] and (5.56). □

5.6. Ruelle dynamical zeta functions

By Remark 5.6 (c.f. [15, Proposition 5.15] for manifold case), the space of the closed geodesics on Z of positive lengths consists of a disjoint union of

smooth connected compact orbifolds

$$\coprod_{[\gamma]\in[\Gamma_+]} B_{[\gamma]}. \tag{5.59}$$

Moreover, $B_{[\gamma]}$ is diffeomorphic to $\Gamma(\gamma)\backslash X(\gamma)$ with multiplicity $n_{[\gamma]}$. Also, all the elements in $B_{[\gamma]}$ have the same length $l_{[\gamma]} = |a_\gamma| > 0$.

The group $\mathbb{S}^1$ acts locally freely on $B_{[\gamma]}$ by rotation. Then $\mathbb{S}^1\backslash B_{[\gamma]}$ is still an orbifold. Set

$$m_{[\gamma]} = n_{[\gamma]} \left|\ker\left(\mathbb{S}^1 \to \mathrm{Diffeo}\big(B_{[\gamma]}\big)\right)\right| \in \mathbf{N}^*. \tag{5.60}$$

We define $m_{[\gamma]}$ to be the multiplicity of $\mathbb{S}^1\backslash B_{[\gamma]}$.

Proposition 5.9. *For $\gamma \in \Gamma_+$ such that $\gamma = a_\gamma k_\gamma^{-1}$ as in* (5.14)*, we have*

$$\frac{\chi_{\mathrm{orb}}\left(\mathbb{S}^1\backslash B_{[\gamma]}\right)}{m_{[\gamma]}} = \frac{\mathrm{vol}\left(\Gamma(\gamma)\backslash X(\gamma)\right)}{|a_\gamma| n_{[\gamma]}} \left[e\left(TX^{a,\perp}(\gamma), \nabla^{TX^{a,\perp}(\gamma)}\right)\right]^{\max}. \tag{5.61}$$

In particular, if $\delta(G) \geqslant 2$, then for all $[\gamma] \in [\Gamma_+]$, we have

$$\chi_{\mathrm{orb}}\left(\mathbb{S}^1\backslash B_{[\gamma]}\right) = 0. \tag{5.62}$$

Also, if $\delta(G) = 1$ and if γ can not be conjugated into H, then (5.62) *still holds.*

Proof. The proof of our proposition is identical to the one given in [47, Proposition 5.1, Corollary 5.2]. □

Recall that $\rho : \Gamma \to U(r)$ is a unitary representation of Γ.

Definition 5.10. The Ruelle dynamical zeta function R_ρ is said to be well-defined if

- for $\mathrm{Re}\,(\sigma) \gg 1$, the sum

$$\Xi_\rho(\sigma) = \sum_{[\gamma]\in[\Gamma_+]} \mathrm{Tr}[\rho(\gamma)] \frac{\chi_{\mathrm{orb}}\left(\mathbb{S}^1\backslash B_{[\gamma]}\right)}{m_{[\gamma]}} e^{-\sigma l_{[\gamma]}} \tag{5.63}$$

  converges absolutely to a holomorphic function;
- the function $R_\rho(\sigma) = \exp(\Xi_\rho(\sigma))$ has a meromorphic extension to $\sigma \in \mathbf{C}$.

By (5.62), if $\delta(G) \geqslant 2$, the dynamical zeta function R_ρ is well-defined and

$$R_\rho \equiv 1. \tag{5.64}$$

We restate Theorem 0.4, which is the main result of this section.

Theorem 5.11. *If* $\dim Z$ *is odd, then the dynamical zeta function* $R_\rho(\sigma)$ *is well-defined. There exist explicit constants* $C_\rho \in \mathbf{R}$ *with* $C_\rho \neq 0$ *and* $r_\rho \in \mathbf{Z}$ *(see* (5.117)*) such that as* $\sigma \to 0$*,*

$$R_\rho(\sigma) = C_\rho T(F)^2 \sigma^{r_\rho} + \mathcal{O}(\sigma^{r_\rho+1}). \tag{5.65}$$

Moreover, if $H^{\cdot}(Z, F) = 0$*, we have*

$$C_\rho = 1, \qquad r_\rho = 0, \tag{5.66}$$

so that

$$R_\rho(0) = T(F)^2. \tag{5.67}$$

Proof. If $\delta(G) \neq 1$, Theorem 5.11 is a consequence of (5.50) and (5.64). Assume now $\delta(G) = 1$ and G has noncompact center. Proceeding as [47, Theorem 5.6], up to evident modification, we see that the dynamical zeta function $R_\rho(\sigma)$ extends meromorphically to $\sigma \in \mathbf{C}$ such that the following identity of meromorphic function holds,

$$\begin{aligned} R_\rho(\sigma) = {} & \prod_{i=1}^{m} \det\left(\sigma^2 + \square^Z|_{\Omega^i(Z,F)}\right)^{(-1)^i i} \\ & \exp\left(\sigma \sum_{\substack{[\gamma]\in[\Gamma_e] \\ \gamma = g_\gamma k^{-1} g_\gamma^{-1}}} \mathrm{Tr}\left[\rho(\gamma)\right] \frac{\mathrm{vol}\left(\Gamma(\gamma)\backslash X(\gamma)\right)}{n_{[\gamma]}} \left[e\left(TX_M(k), \nabla^{TX_M(k)}\right)\right]^{\max}\right), \end{aligned} \tag{5.68}$$

from which we get (5.65)-(5.67). The proof for the case where $\delta(G) = 1$ and where G has compact center will be given in subsections 5.8-5.11. $\square$

Remark 5.12. By (5.67), we have the formal identity

$$2\log T(F) = \sum_{[\gamma]\in[\Gamma_+]} \mathrm{Tr}[\rho(\gamma)] \frac{\chi_{\mathrm{orb}}\left(\mathbb{S}^1 \backslash B_{[\gamma]}\right)}{m_{[\gamma]}}. \tag{5.69}$$

We note the similarity between (4.7) and (5.69).

Remark 5.13. The formal identity (5.69) can be deduced formally using the path integral argument and Bismut-Goette's V-invariant [7] as in [47, Section 1E]. We leave the details to readers.

5.7. Reductive group with $\delta(G) = 1$ and with compact center

From now on, we assume that $\delta(G) = 1$ and that G has compact center. Let us introduce some notation following [47, Sections 6A and 6B]. We use the notation in (5.52)-(5.54). Let $Z(\mathfrak{b}) \subset G$ be the stabilizer of $\mathfrak{b}$ in G, and let $\mathfrak{z}(\mathfrak{b}) \subset \mathfrak{g}$ be its Lie algebra. We define $\mathfrak{p}(\mathfrak{b})$, $\mathfrak{k}(\mathfrak{b})$, $\mathfrak{p}^{\perp}(\mathfrak{b})$, $\mathfrak{k}^{\perp}(\mathfrak{b})$, $\mathfrak{z}^{\perp}(\mathfrak{b})$ in an obvious way as in subsection 5.3, so that

$$\mathfrak{p}(\mathfrak{b}) = \mathfrak{b} \oplus \mathfrak{p}_{\mathfrak{m}}, \qquad \mathfrak{k}(\mathfrak{b}) = \mathfrak{k}_{\mathfrak{m}}, \tag{5.70}$$

and

$$\mathfrak{p} = \mathfrak{b} \oplus \mathfrak{p}_{\mathfrak{m}} \oplus \mathfrak{p}^{\perp}(\mathfrak{b}), \qquad \mathfrak{k} = \mathfrak{k}_{\mathfrak{m}} \oplus \mathfrak{k}^{\perp}(\mathfrak{b}). \tag{5.71}$$

Let $Z^0(\mathfrak{b})$ be the connected component of the identity in $Z(\mathfrak{b})$. By (5.21), we have

$$Z^0(\mathfrak{b}) = \mathbf{R} \times M. \tag{5.72}$$

Set

$$\mathfrak{z}^{\perp}(\mathfrak{b}) = \mathfrak{p}^{\perp}(\mathfrak{b}) \oplus \mathfrak{k}^{\perp}(\mathfrak{b}). \tag{5.73}$$

Recall that we have fixed $a_1 \in \mathfrak{b}$ such that $B(a_1, a_1) = 1$. The choice of a_1 fixes an orientation of $\mathfrak{b}$. By [47, Proposition 6.2], there exists unique $\alpha \in \mathfrak{b}^*$ such that $\langle \alpha, a_1 \rangle > 0$, and that for any $a \in \mathfrak{b}$, the action of $\mathrm{ad}(a)$ on $\mathfrak{z}^{\perp}(\mathfrak{b})$ has only two eigenvalues $\pm\langle \alpha, a \rangle \in \mathbf{R}$. Take $a_0 = a_1/\langle \alpha, a_1 \rangle \in \mathfrak{b}$. We have

$$\langle \alpha, a_0 \rangle = 1. \tag{5.74}$$

Let $\mathfrak{n} \subset \mathfrak{z}^{\perp}(\mathfrak{b})$ (resp. $\overline{\mathfrak{n}}$) be the $+1$ (resp. -1) eigenspace of $\mathrm{ad}(a_0)$, so that

$$\mathfrak{z}^{\perp}(\mathfrak{b}) = \mathfrak{n} \oplus \overline{\mathfrak{n}}. \tag{5.75}$$

Clearly, $\overline{\mathfrak{n}} = \theta\mathfrak{n}$, and M acts on $\mathfrak{n}$ and $\overline{\mathfrak{n}}$. As explained in [47, Section 5.1], $\dim \mathfrak{n}$ is even. Set

$$l = \frac{1}{2} \dim \mathfrak{n}. \tag{5.76}$$

Let $\mathfrak{u}(\mathfrak{b}) \subset \mathfrak{u}$ and $\mathfrak{u}_{\mathfrak{m}} \subset \mathfrak{u}$ be respectively the compact forms of $\mathfrak{z}(\mathfrak{b})$ and of $\mathfrak{m}$. Then,

$$\mathfrak{u}(\mathfrak{b}) = \sqrt{-1}\mathfrak{b} \oplus \sqrt{-1}\mathfrak{p}_{\mathfrak{m}} \oplus \mathfrak{k}_{\mathfrak{m}}, \qquad \mathfrak{u}_{\mathfrak{m}} = \sqrt{-1}\mathfrak{p}_{\mathfrak{m}} \oplus \mathfrak{k}_{\mathfrak{m}}. \tag{5.77}$$

Let $\mathfrak{u}^{\perp}(\mathfrak{b}) \subset \mathfrak{u}$ be the orthogonal space of $\mathfrak{u}(\mathfrak{b})$, so that

$$\mathfrak{u} = \sqrt{-1}\mathfrak{b} \oplus \mathfrak{u}_{\mathfrak{m}} \oplus \mathfrak{u}^{\perp}(\mathfrak{b}). \tag{5.78}$$

Let $U(\mathfrak{b}) \subset U$ and $U_M \subset U$ be respectively the corresponding connected subgroups of complex matrices of groups associated to the Lie algebras $\mathfrak{u}(\mathfrak{b})$ and $\mathfrak{u}_{\mathfrak{m}}$. By [47, Section 6B], $U(\mathfrak{b})$ and U_M are compact such that

$$U(\mathfrak{b}) = \exp(\sqrt{-1}\mathfrak{b})U_M. \tag{5.79}$$

Clearly, $U(\mathfrak{b})$ acts on $\mathfrak{b}, \mathfrak{u}_{\mathfrak{m}}, \mathfrak{u}^{\perp}(\mathfrak{b})$ and preserves the splitting (5.78).

Put

$$Y_{\mathfrak{b}} = U/U(\mathfrak{b}). \tag{5.80}$$

By [47, Propositions 6.7], $Y_{\mathfrak{b}}$ is a Hermitian symmetric space of the compact type. Let $\omega^{Y_{\mathfrak{b}}} \in \Omega^2(Y_{\mathfrak{b}})$ be the canonical Kälher form on $Y_{\mathfrak{b}}$ induced by B. As in subsection 5.2, $U \to Y_{\mathfrak{b}}$ is a $U(\mathfrak{b})$-principle bundle on $Y_{\mathfrak{b}}$ with canonical connection. Let $(TY_{\mathfrak{b}}, \nabla^{TY_{\mathfrak{b}}})$ and $(N_{\mathfrak{b}}, \nabla^{N_{\mathfrak{b}}})$ the Hermitian vector bundle with Hermitian connection induced by the representation of $U(\mathfrak{b})$ on $\mathfrak{u}_{\mathfrak{m}}$ and $\mathfrak{u}_{\mathfrak{m}}^{\perp}$. For a vector space E, we still denote by E the corresponding trivial bundle on $Y_{\mathfrak{b}}$. By (5.78), we have an analogy of [5, (2.2.1)],

$$\mathfrak{u} = \sqrt{-1}\mathfrak{b} \oplus N_{\mathfrak{b}} \oplus TY_{\mathfrak{b}}. \tag{5.81}$$

Take $k \in T$. Denote by $\mathfrak{n}(k)$, $U^0(k)$, $Y_{\mathfrak{b}}(k)$ and $\omega^{Y_{\mathfrak{b}}(k)}$ the analogies of $\mathfrak{n}$, U, $Y_{\mathfrak{b}}$ and $\omega^{Y_{\mathfrak{b}}}$ when G is replaced by $Z^0(k)$. The embedding $U^0(k) \to U$ induces an embedding $Y_{\mathfrak{b}}(k) \to Y_{\mathfrak{b}}$. Clearly, k acts on the left on $Y_{\mathfrak{b}}$, and $Y_{\mathfrak{b}}(k)$ is fixed by the action of k. Recall that the equivariant $\widehat{A}$-forms $\widehat{A}_{k^{-1}}\left(N_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}, \nabla^{N_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}}\right)$ and $\widehat{A}_{k^{-1}}\left(TY_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}, \nabla^{TY_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}}\right)$ are defined in (1.15). Let $\widehat{A}^{\mathfrak{u}_{\mathfrak{m}}}_{k^{-1}}(0)$ and $\widehat{A}^{\mathfrak{u}^{\perp}(\mathfrak{b})}_{k^{-1}}(0)$ be the components of degree 0 of the form $\widehat{A}_{k^{-1}}\left(N_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}, \nabla^{N_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}}\right)$ and $\widehat{A}_{k^{-1}}\left(TY_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}, \nabla^{TY_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}}\right)$. Following

[5, (7.7.3)], set

$$\widehat{A}_{k^{-1}}(0) = \widehat{A}^{\mathfrak{u}_{\mathfrak{m}}}_{k^{-1}}(0)\widehat{A}^{\mathfrak{u}^{\perp}(\mathfrak{b})}_{k^{-1}}(0). \tag{5.82}$$

By (5.81), as in [5, (7.7.5)], the following identity of closed forms on $Y_{\mathfrak{b}}(k)$ holds:

$$\widehat{A}_{k^{-1}}(0) = \widehat{A}_{k^{-1}}\left(N_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}, \nabla^{N_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}}\right)\widehat{A}_{k^{-1}}\left(TY_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}, \nabla^{TY_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}}\right), \tag{5.83}$$

which generalizes [47, Proposition 6.8].

5.8. Auxiliary virtual representations of K

We follow [47, Sections 6C and 7A]. Denote by $RO(K_M)$ and $RO(K)$ the real representation rings of K_M and K. Since K_M and K have the same maximal torus T, the restriction $RO(K) \to RO(K_M)$ is injective.

By [47, Proposition 6.10], we have the identity in $RO(K_M)$,

$$\left(\sum_{i=1}^{m}(-1)^{i-1}i\Lambda^i(\mathfrak{p}^*)\right)|_{K_M} = \sum_{i=0}^{\dim \mathfrak{p}_{\mathfrak{m}}}\sum_{j=0}^{2l}(-1)^{i+j}\Lambda^i(\mathfrak{p}^*_{\mathfrak{m}})\otimes\Lambda^j(\mathfrak{n}^*). \tag{5.84}$$

By [47, Corollary 6.12], each term on the right hand side of (5.84) has a lift to $RO(K)$. More precisely, let us recall [47, Assumption 7.1].

Assumption 5.14. *Let η be a real finite dimensional representation of M on the vector space E_η such that*

1) *the restriction $\eta|_{K_M}$ to K_M can be lifted into $RO(K)$;*
2) *the action of the Lie algebra $\mathfrak{u}_{\mathfrak{m}} \subset \mathfrak{m}\otimes_{\mathbf{R}}\mathbf{C}$ on $E_\eta\otimes_{\mathbf{R}}\mathbf{C}$, induced by complexification, can be lifted to an action of Lie group U_M;*
3) *the Casimir element $C^{\mathfrak{u}_{\mathfrak{m}}}$ of $\mathfrak{u}_{\mathfrak{m}}$ acts on $E_\eta\otimes_{\mathbf{R}}\mathbf{C}$ as a scalar $C^{\mathfrak{u}_{\mathfrak{m}},\eta}\in\mathbf{R}$.*

By [47, Corollary 6.12], let $\widehat{\eta} = \widehat{\eta}^+ - \widehat{\eta}^- \in RO(K)$ be a real virtual finite dimensional representation of K on $E_{\widehat{\eta}} = E^+_{\widehat{\eta}} - E^-_{\widehat{\eta}}$ such that the following identity in $RO(K_M)$ holds:

$$E_{\widehat{\eta}}|_{K_M} = \sum_{i=0}^{\dim \mathfrak{p}_{\mathfrak{m}}}(-1)^i\Lambda^i(\mathfrak{p}^*_{\mathfrak{m}})\otimes E_\eta|_{K_M}. \tag{5.85}$$

Note that M acts on $\mathfrak{n}$ by adjoint action. By [47, Corollary 6.12 and Proposition 6.13], for $0 \leq j \leq 2l$, the induced representation η_j of K_M on $\Lambda^j(\mathfrak{n}^*)$ satisfies Assumption 5.14, such that the following identity in $RO(K)$ holds,

$$\sum_{i=1}^{m}(-1)^{i-1} i\Lambda^i(\mathfrak{p}^*) = \sum_{j=0}^{2l}(-1)^j E_{\widehat{\eta}_j}. \tag{5.86}$$

5.9. Evaluation of $\mathrm{Tr_s}^{[\gamma]}[\exp(-tC^{\mathfrak{g},X,\widehat{\eta}}/2)]$

In [47, Theorem 7.3], we evaluate the orbital integral $\mathrm{Tr_s}^{[\gamma]}[\exp(-tC^{\mathfrak{g},X,\widehat{\eta}}/2)]$ when $\gamma = 1$ or when γ is a non elliptic semisimple element. In this subsection, we evaluate $\mathrm{Tr_s}^{[\gamma]}[\exp(-tC^{\mathfrak{g},X,\widehat{\eta}}/2)]$ when γ is elliptic. To state the result, let us introduce some notation [47, (7-3)-(7-7)].

Recall that T is a maximal torus of K_M, K and U_M. Denote by $W(T, U_M)$ and $W(T, K)$ the corresponding Weyl groups, and denote by $\mathrm{vol}(K/K_M)$ and $\mathrm{vol}(U_M/K_M)$ the Riemannian volumes induced by $-B$. Set

$$c_G = (-1)^{(\dim \mathfrak{p}-1)/2}\frac{|W(T, U_M)|}{|W(T, K)|}\frac{\mathrm{vol}(K/K_M)}{\mathrm{vol}(U_M/K_M)} \in \mathbf{R}. \tag{5.87}$$

The constant $c_{Z^0(k)}$ is defined in a similar way.

As in [47, (7-3)], by (2) of Assumption 5.14, U_M acts on $E_\eta \otimes_{\mathbf{R}} \mathbf{C}$. We extend this action to $U(\mathfrak{b})$ such that $\exp(\sqrt{-1}\mathfrak{b})$ acts trivially. Denote by $F_{\mathfrak{b},\eta}$ the Hermitian vector bundle on $Y_\mathfrak{b}$ with total space $U \times_{U(\mathfrak{b})} (E_\eta \otimes_{\mathbf{R}} \mathbf{C})$ with Hermitian connection $\nabla^{F_{\mathfrak{b},\eta}}$.

Note that the Kähler form $\omega^{Y_\mathfrak{b}(k)}$ defines a volume form $dv_{Y_\mathfrak{b}(k)}$ on $Y_\mathfrak{b}(k)$. For a $U^0(k)$-invariant differential form β on $Y_\mathfrak{b}(k)$, as in [47, (7-7)], define $[\beta]^{\max} \in \mathbf{R}$ such that

$$\beta - [\beta]^{\max} dv_{Y_\mathfrak{b}(k)} \tag{5.88}$$

has degree smaller than $\dim Y_\mathfrak{b}(k)$. Recall that $a_0 \in \mathfrak{b}$ is defined in (5.74).

Theorem 5.15. *Let $\gamma \in G$ be semisimple. If γ can not be conjugated into H by elements of G, then for $t > 0$, we have*

$$\mathrm{Tr_s}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\widehat{\eta}}/2\right)\right] = 0. \tag{5.89}$$

If $\gamma = k^{-1} \in T$, then for $t > 0$, we have

$$\begin{aligned}(5.90)\quad & \mathrm{Tr_s}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\widehat{\eta}}/2\right)\right] \\ &= \frac{c_{Z^0(k)}}{\sqrt{2\pi t}}\exp\left(\frac{t}{16}\mathrm{Tr}\left[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^{\perp}(\mathfrak{b})}\right]-\frac{t}{2}C^{\mathfrak{u}_{\mathfrak{m}},\eta}\right)\left[\exp\left(-\frac{\omega^{Y_{\mathfrak{b}}(k),2}}{8\pi^2|a_0|^2t}\right)\right. \\ &\qquad \left.\widehat{A}_{k^{-1}}\left(TY_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)},\nabla^{TY_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}}\right)\mathrm{ch}_{k^{-1}}\left(F_{\mathfrak{b},\eta}|_{Y_{\mathfrak{b}}(k)},\nabla^{F_{\mathfrak{b},\eta}|_{Y_{\mathfrak{b}}(k)}}\right)\right]^{\max}.\end{aligned}$$

If $\gamma = e^a k^{-1} \in H$ with $a \neq 0$, then for any $t > 0$, we have

$$\begin{aligned}(5.91)\quad & \mathrm{Tr_s}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\widehat{\eta}}/2\right)\right] = \frac{1}{\sqrt{2\pi t}}\left[e\left(TX_M(k),\nabla^{TX_M(k)}\right)\right]^{\max} \\ & \exp\left(-\frac{|a|^2}{2t}+\frac{t}{16}\mathrm{Tr}\left[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^{\perp}(\mathfrak{b})}\right]-\frac{t}{2}C^{\mathfrak{u}_{\mathfrak{m}},\eta}\right)\frac{\mathrm{Tr}^{E_\eta}\left[\eta(k^{-1})\right]}{\left|\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{z}_0^{\perp}}\right|^{1/2}}.\end{aligned}$$

Proof. Equations (5.89), (5.90) with $\gamma = 1$, and (5.91) are [47, Theorem 5.15]. It remains to show (5.90) for a non trivial $\gamma = k^{-1} \in T$. Set

$$(5.92)\quad \mathfrak{p}_{\mathfrak{m}}^{\perp}(k) = \mathfrak{p}_{\mathfrak{m}} \cap \mathfrak{z}^{\perp}(k), \quad \mathfrak{k}_{\mathfrak{m}}^{\perp}(k) = \mathfrak{k}_{\mathfrak{m}} \cap \mathfrak{z}^{\perp}(k), \quad \mathfrak{m}^{\perp}(k) = \mathfrak{m} \cap \mathfrak{z}^{\perp}(k).$$

By (5.92), we have

$$(5.93)\qquad \mathfrak{p}_{\mathfrak{m}} = \mathfrak{p}_{\mathfrak{m}}(k) \oplus \mathfrak{p}_{\mathfrak{m}}^{\perp}(k), \qquad \mathfrak{k}_{\mathfrak{m}} = \mathfrak{k}_{\mathfrak{m}}(k) \oplus \mathfrak{k}_{\mathfrak{m}}^{\perp}(k).$$

Similarly, k acts on $\mathfrak{p}^{\perp}(\mathfrak{b})$ and $\mathfrak{k}^{\perp}(\mathfrak{b})$. Set

$$(5.94)\quad \begin{aligned}&\mathfrak{p}_1^{\perp}(\mathfrak{b}) = \mathfrak{p}^{\perp}(\mathfrak{b}) \cap \mathfrak{z}(k), && \mathfrak{k}_1^{\perp}(\mathfrak{b}) = \mathfrak{k}^{\perp}(\mathfrak{b}) \cap \mathfrak{z}(k), && \mathfrak{z}_1^{\perp}(k) = \mathfrak{z}^{\perp}(\mathfrak{b}) \cap \mathfrak{z}(k), \\ &\mathfrak{p}_2^{\perp}(\mathfrak{b}) = \mathfrak{p}^{\perp}(\mathfrak{b}) \cap \mathfrak{z}^{\perp}(k), && \mathfrak{k}_2^{\perp}(\mathfrak{b}) = \mathfrak{k}^{\perp}(\mathfrak{b}) \cap \mathfrak{z}^{\perp}(k), && \mathfrak{z}_2^{\perp}(k) = \mathfrak{z}^{\perp}(\mathfrak{b}) \cap \mathfrak{z}^{\perp}(k).\end{aligned}$$

Then

$$(5.95)\qquad \mathfrak{p}^{\perp}(\mathfrak{b}) = \mathfrak{p}_1^{\perp}(\mathfrak{b}) \oplus \mathfrak{p}_2^{\perp}(\mathfrak{b}), \qquad \mathfrak{k}^{\perp}(\mathfrak{b}) = \mathfrak{k}_1^{\perp}(\mathfrak{b}) \oplus \mathfrak{k}_2^{\perp}(\mathfrak{b}).$$

By (5.92) and (5.94), we get

$$(5.96)\quad \begin{aligned}&\mathfrak{p}(k) = \mathfrak{b} \oplus \mathfrak{p}_{\mathfrak{m}}(k) \oplus \mathfrak{p}_1^{\perp}(\mathfrak{b}), && \mathfrak{k}(k) = \mathfrak{k}_{\mathfrak{m}}(k) \oplus \mathfrak{k}_1^{\perp}(\mathfrak{b}), \\ &\mathfrak{p}^{\perp}(k) = \mathfrak{p}_{\mathfrak{m}}^{\perp}(k) \oplus \mathfrak{p}_2^{\perp}(\mathfrak{b}), && \mathfrak{k}^{\perp}(k) = \mathfrak{k}_{\mathfrak{m}}^{\perp}(k) \oplus \mathfrak{k}_2^{\perp}(\mathfrak{b}).\end{aligned}$$

As in the case of [47, (6-5)], we have isomorphisms of representations of T,

$$\mathfrak{p}_1^{\perp}(\mathfrak{b}) \simeq \mathfrak{k}_1^{\perp}(\mathfrak{b}) \simeq \mathfrak{n}(k), \tag{5.97}$$

where the first isomorphism is given by $\mathrm{ad}(a_0)$. Moreover, $\mathrm{ad}(a_0)$ induces an isomorphism of representations of T,

$$\mathfrak{p}_2^{\perp}(\mathfrak{b}) \simeq \mathfrak{k}_2^{\perp}(\mathfrak{b}). \tag{5.98}$$

Set

$$\begin{aligned} &\mathfrak{u}_{\mathfrak{m}}(k) = \sqrt{-1}\mathfrak{p}_{\mathfrak{m}}(k) \oplus \mathfrak{k}_{\mathfrak{m}}(k), \qquad \mathfrak{u}_{\mathfrak{m}}^{\perp}(k) = \sqrt{-1}\mathfrak{p}_{\mathfrak{m}}^{\perp}(k) \oplus \mathfrak{k}_{\mathfrak{m}}^{\perp}(k), \\ &\mathfrak{u}_1^{\perp}(\mathfrak{b}) = \sqrt{-1}\mathfrak{p}_1^{\perp}(\mathfrak{b}) \oplus \mathfrak{k}_1^{\perp}(\mathfrak{b}), \qquad \mathfrak{u}_2^{\perp}(\mathfrak{b}) = \sqrt{-1}\mathfrak{p}_2^{\perp}(\mathfrak{b}) \oplus \mathfrak{k}_2^{\perp}(\mathfrak{b}). \end{aligned} \tag{5.99}$$

Proceeding as [47, 7-18], by (5.28) and by the Weyl integral formula for Lie algebra [47, (7-17)], we have

$$\begin{aligned} &\mathrm{Tr_s}^{[k^{-1}]}\left[\exp\left(-tC^{\mathfrak{g},X,\widehat{\eta}}/2\right)\right] \\ &= \frac{1}{(2\pi t)^{\dim \mathfrak{z}(k)/2}} \exp\left(\frac{t}{16}\,\mathrm{Tr}^{\mathfrak{p}}\left[C^{\mathfrak{k},\mathfrak{p}}\right] + \frac{t}{48}\,\mathrm{Tr}^{\mathfrak{k}}\left[C^{\mathfrak{k},\mathfrak{k}}\right]\right) \\ &\quad \frac{\mathrm{vol}(K^0(k)/T)}{|W(T,K^0(k))|}\int_{Y\in\mathfrak{t}} \det\big(\mathrm{ad}(Y)\big)\big|_{\mathfrak{k}(k)/\mathfrak{t}}\, J_{k^{-1}}(Y) \\ &\qquad \mathrm{Tr_s}^{E_{\widehat{\eta}}}\left[\widehat{\eta}(k^{-1})\exp\big(-i\widehat{\eta}(Y)\big)\right]\exp\left(-|Y|^2/2t\right)dY. \end{aligned} \tag{5.100}$$

As $\mathfrak{t}$ is also the Cartan subalgebra of $\mathfrak{u}_{\mathfrak{m}}(k)$, we will rewrite the integral on the right-hand side as an integral over $\mathfrak{u}_{\mathfrak{m}}(k)$. By (5.25), (5.96)-(5.98), for $Y \in \mathfrak{t}$, we have

$$\begin{aligned} J_{k^{-1}}(Y) &= \frac{\widehat{A}\big(i\,\mathrm{ad}(Y)|_{\mathfrak{p}_{\mathfrak{m}}(k)}\big)}{\widehat{A}\big(i\,\mathrm{ad}(Y)|_{\mathfrak{k}_{\mathfrak{m}}(k)}\big)}\left[\det\big(1-\mathrm{Ad}(k^{-1})\big)|_{\mathfrak{p}_2^{\perp}(\mathfrak{b})\oplus\mathfrak{k}_2^{\perp}(\mathfrak{b})}\right]^{-1/2} \\ &\left[\frac{1}{\det\big(1-\mathrm{Ad}(k^{-1})\big)|_{\mathfrak{p}_{\mathfrak{m}}^{\perp}(k)\oplus\mathfrak{k}_{\mathfrak{m}}^{\perp}(k)}}\frac{\det\big(1-\exp(-i\,\mathrm{ad}(Y))\,\mathrm{Ad}(k^{-1})\big)|_{\mathfrak{k}_{\mathfrak{m}}^{\perp}(k)}}{\det\big(1-\exp(-i\,\mathrm{ad}(Y))\,\mathrm{Ad}(k^{-1})\big)|_{\mathfrak{p}_{\mathfrak{m}}^{\perp}(k)}}\right]^{1/2}. \end{aligned} \tag{5.101}$$

As in [47, (7-22)], by (5.85), (5.97), and (5.101), for $Y \in \mathfrak{t}$, we have

$$
\begin{aligned}
(5.102)\quad & \frac{\det(\mathrm{ad}(Y))|_{\mathfrak{k}(k)/\mathfrak{t}}}{\det(\mathrm{ad}(Y))|_{\mathfrak{u}_{\mathfrak{m}}(k)/\mathfrak{t}}} J_{k^{-1}}(Y)\,\mathrm{Tr_s}^{E_{\widehat{\eta}}}\left[\widehat{\eta}(k^{-1})\exp\big(-i\widehat{\eta}(Y)\big)\right] \\
&= (-1)^{\frac{\dim \mathfrak{p}_{\mathfrak{m}}(k)}{2}} \det\big(\mathrm{ad}(Y)\big)\big|_{\mathfrak{n}(k)} \widehat{A}^{-1}\left(i\,\mathrm{ad}(Y)|_{\mathfrak{u}_{\mathfrak{m}}(k)}\right) \\
&\mathrm{Tr_s}^{E_\eta}\left[\eta(k^{-1})\exp(-i\eta(Y))\right]\left[\det\big(1-\mathrm{Ad}(k^{-1})\big)|_{\mathfrak{u}_2^{\perp}(\mathfrak{b})}\right]^{-1/2} \\
&\qquad \left[\frac{\det\big(1-\exp(-i\,\mathrm{ad}(Y))\,\mathrm{Ad}(k^{-1})\big)|_{\mathfrak{u}_{\mathfrak{m}}^{\perp}(k)}}{\det\big(1-\mathrm{Ad}(k^{-1})\big)|_{\mathfrak{u}_{\mathfrak{m}}^{\perp}(k)}}\right]^{1/2}.
\end{aligned}
$$

Let $U_M(k)$ be the centralizer of k in U_M, and let $U_M^0(k)$ be the connected component of the identity in $U_M(k)$. The right-hand side of (5.102) is $\mathrm{Ad}(U_M^0(k))$-invariant. By (5.87), (5.100) and (5.102), and using again the Weyl integral formula [47, (7-17)], as in [47, (7-24)], we get

$$
\begin{aligned}
(5.103)\quad & \mathrm{Tr_s}^{[k^{-1}}\left[\exp\left(-tC^{\mathfrak{g},X,\widehat{\eta}}/2\right)\right] = \frac{(-1)^{\frac{\dim \mathfrak{n}(k)}{2}}}{(2\pi t)^{\dim \mathfrak{z}(k)/2}} c_{Z^0(k)} \\
& \frac{\exp\left(\frac{t}{16}\,\mathrm{Tr}^{\mathfrak{p}}\left[C^{\mathfrak{k},\mathfrak{p}}\right]+\frac{t}{48}\,\mathrm{Tr}^{\mathfrak{k}}\left[C^{\mathfrak{k},\mathfrak{k}}\right]\right)}{\left[\det\big(1-\mathrm{Ad}(k^{-1})\big)|_{\mathfrak{u}_2^{\perp}(\mathfrak{b})}\right]^{1/2}} \int_{Y\in\mathfrak{u}_{\mathfrak{m}}(k)} \det\big(\mathrm{ad}(Y)\big)|_{\mathfrak{n}(k)} \\
& \qquad \widehat{A}^{-1}\left(i\,\mathrm{ad}(Y)|_{\mathfrak{u}_{\mathfrak{m}}(k)}\right)\mathrm{Tr}^{E_\eta}\left[\eta(k^{-1})\exp(-i\eta(Y))\right] \\
& \qquad \left[\frac{\det\big(1-\exp(-i\,\mathrm{ad}(Y))\,\mathrm{Ad}(k^{-1})\big)|_{\mathfrak{u}_{\mathfrak{m}}^{\perp}(k)}}{\det\big(1-\mathrm{Ad}(k^{-1})\big)|_{\mathfrak{u}_{\mathfrak{m}}^{\perp}(k)}}\right]^{1/2} \exp(-|Y|^2/2t)dY.
\end{aligned}
$$

Proceeding as in [47, (7-25)-(7-44)], by (5.103), we get

$$
\begin{aligned}
(5.104)\quad & \mathrm{Tr_s}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\widehat{\eta}}/2\right)\right] = \frac{c_{Z^0(k)}}{\sqrt{2\pi t}} \\
& \exp\left(\frac{t}{16}\,\mathrm{Tr}\left[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^{\perp}(\mathfrak{b})}\right]-\frac{t}{2}C^{\mathfrak{u}_{\mathfrak{m}},\eta}\right)\left[\exp\left(-\frac{\omega^{Y_{\mathfrak{b}}(k),2}}{8\pi^2|a_0|^2t}\right)\widehat{A}_{k^{-1}}(0)\right. \\
& \qquad \left.\widehat{A}_{k^{-1}}^{-1}\left(N_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)},\nabla^{N_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}}\right)\mathrm{ch}_{k^{-1}}\left(F_{\mathfrak{b},\eta}|_{Y_{\mathfrak{b}}(k)},\nabla^{F_{\mathfrak{b},\eta}|_{Y_{\mathfrak{b}}(k)}}\right)\right]^{\max}.
\end{aligned}
$$

By (5.83) and (5.104), we get (5.90). □

5.10. Selberg zeta functions

We follow [47, Section 7C]. Recall that $\rho : \Gamma \to \mathrm{U}(r)$ is a unitary representation of Γ.

Definition 5.16. For $\sigma \in \mathbf{C}$, we define a formal sum

$$\Xi_{\eta,\rho}(\sigma) = -\sum_{\substack{[\gamma]\in[\Gamma_+]\\ \gamma=g_\gamma e^a k^{-1} g_\gamma^{-1}}} \mathrm{Tr}[\rho(\gamma)] \frac{\chi_{\mathrm{orb}}\big(\mathbb{S}^1\backslash B_{[\gamma]}\big)}{m_{[\gamma]}} \frac{\mathrm{Tr}^{E_\eta}\left[\eta(k^{-1})\right]}{\left|\det\left(1-\mathrm{Ad}(e^a k^{-1})\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2}} e^{-\sigma|a|} \tag{5.105}$$

and a formal Selberg zeta function

$$Z_{\eta,\rho}(\sigma) = \exp\big(\Xi_{\eta,\rho}(\sigma)\big). \tag{5.106}$$

The formal Selberg zeta function is said to be well defined if the same conditions as in Definition 5.10 hold.

Recall that the Casimir operator $C^{g,Z,\widehat{\eta},\rho}$ acting on $C^\infty(Z, \mathcal{F}_{\widehat{\eta}} \otimes_{\mathbf{C}} F)$ is a formally self-adjoint second order elliptic operator, which is bounded from below. Set

$$m_{\eta,\rho}(\lambda) = \dim_{\mathbf{C}} \ker\left(C^{\mathfrak{g},Z,\widehat{\eta}^+,\rho} - \lambda\right) - \dim_{\mathbf{C}} \ker\left(C^{\mathfrak{g},Z,\widehat{\eta}^-,\rho} - \lambda\right). \tag{5.107}$$

As in [47, (7-59)], consider the quotient of zeta regularized determinants

$$\det_{\mathrm{gr}}\left(C^{\mathfrak{g},Z,\widehat{\eta},\rho} + \sigma\right) = \frac{\det\left(C^{\mathfrak{g},Z,\widehat{\eta}^+,\rho} + \sigma\right)}{\det\left(C^{\mathfrak{g},Z,\widehat{\eta}^-,\rho} + \sigma\right)}. \tag{5.108}$$

By Remark 4.5, it is a meromorphic function on $\mathbf{C}$. Its zeros and poles belong to the set $\{-\lambda : \lambda \in \mathrm{Sp}(C^{\mathfrak{g},Z,\widehat{\eta},\rho})\}$. The order of zero at $\sigma = -\lambda$ is $m_{\eta,\rho}(\lambda)$.

Set

$$\sigma_\eta = \frac{1}{8}\,\mathrm{Tr}\left[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})}\right] - C^{\mathfrak{u}_{\mathfrak{m}},\eta}. \tag{5.109}$$

Let $P_{\eta,\rho}(\sigma)$ be the odd polynomial defined by

$$(5.110)\quad P_{\eta,\rho}(\sigma) = \sum_{\substack{[\gamma]\in[\Gamma_e]\\ \gamma=g_\gamma k^{-1}g_\gamma^{-1}}} c_{Z^0(k)}\,\mathrm{Tr}[\rho(\gamma)]\frac{\mathrm{vol}(\Gamma(\gamma)\backslash X(\gamma))}{n_{[\gamma]}} \Bigg(\sum_{j=0}^{\dim \mathfrak{n}(k)/2}(-1)^j\frac{\Gamma(-j-\frac{1}{2})}{j!(4\pi)^{2j+\frac{1}{2}}|a_0|^{2j}}\sigma^{2j+1} \Big[\omega^{Y_{\mathfrak{b}}(k),2j}\widehat{A}_{k^{-1}}\Big(TY_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)},\nabla^{TY_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}}\Big)\mathrm{ch}_{k^{-1}}\Big(\mathcal{F}_{\mathfrak{b},\eta}|_{Y_{\mathfrak{b}}(k)},\nabla^{\mathcal{F}_{\mathfrak{b},\eta}|_{Y_{\mathfrak{b}}(k)}}\Big)\Big]^{\max}\Bigg).$$

Theorem 5.17. *There is $\sigma_0>0$ such that*

$$(5.111)\quad \sum_{\substack{[\gamma]\in[\Gamma_+]\\ \gamma=g_\gamma e^a k^{-1}g_\gamma^{-1}}} \frac{\left|\chi_{\mathrm{orb}}\left(\mathbb{S}^1\backslash B_{[\gamma]}\right)\right|}{m_{[\gamma]}}\frac{e^{-\sigma_0|a|}}{\left|\det\left(1-\mathrm{Ad}(e^a k^{-1})\right)|_{\mathfrak{z}_{\mathfrak{d}}^\perp}\right|^{1/2}}<\infty.$$

The Selberg zeta function $Z_{\eta,\rho}(\sigma)$ has a meromorphic extension to $\sigma\in\mathbf{C}$ such that the following identity of meromorphic functions on $\mathbf{C}$ holds:

$$(5.112)\quad Z_{\eta,\rho}(\sigma)=\det{}_{\mathrm{gr}}\left(C^{\mathfrak{g},Z,\widehat{\eta},\rho}+\sigma_\eta+\sigma^2\right)\exp\left(P_{\eta,\rho}(\sigma)\right).$$

The zeros and poles of $Z_{\eta,\rho}(\sigma)$ belong to $\{\pm i\sqrt{\lambda+\sigma_\eta}:\lambda\in\mathrm{Sp}(C^{\mathfrak{g},Z,\widehat{\eta},\rho})\}$. If $\lambda\in\mathrm{Sp}(C^{\mathfrak{g},Z,\widehat{\eta},\rho})$ and $\lambda\neq-\sigma_\eta$, the order of zero at $\sigma=\pm i\sqrt{\lambda+\sigma_\eta}$ is $m_{\eta,\rho}(\lambda)$. The order of zero at $\sigma=0$ is $2m_{\eta,\rho}(-\sigma_\eta)$. Also,

$$(5.113)\quad Z_{\eta,\rho}(\sigma)=Z_{\eta,\rho}(-\sigma)\exp\left(2P_{\eta,\rho}(\sigma)\right).$$

Proof. Proceeding as in [47, Theorem 7.6], by Theorems 5.4 and 5.15, our theorem follows. □

5.11. The proof of Theorem 5.11 when G has compact center and $\delta(G)=1$

We apply the results of subsection 5.10 to η_j. Recall that $\alpha\in\mathfrak{b}^*$ is defined in (5.74). Proceeding as in [47, Theorem 7.7], by (5.86), we find that $R_\rho(\sigma)$ is well-defined and holomorphic on the domain $\sigma\in\mathbf{C}$ and $\mathrm{Re}\,(\sigma)\gg1$, and

that

$$R_\rho(\sigma) = \prod_{j=0}^{2l} Z_{\eta_j,\rho}\big(\sigma + (j-l)|\alpha|\big)^{(-1)^{j-1}}. \tag{5.114}$$

By Theorem 5.17 and (5.114), $R_\rho(\sigma)$ has a meromorphic extension to $\sigma \in \mathbf{C}$.

For $0 \leq j \leq 2l$, put

$$r_j = m_{\eta_j,\rho}(0). \tag{5.115}$$

By the orbifold Hodge theorem 4.1, as in [47, (7-74)], we have

$$\chi'_{\mathrm{top}}(Z,F) = 2\sum_{j=0}^{l-1}(-1)^{j-1}r_j + (-1)^{l-1}r_l. \tag{5.116}$$

Set

$$C_\rho = \prod_{j=0}^{l-1}\big(-4(l-j)^2|\alpha|^2\big)^{(-1)^{j-1}r_j}, \qquad r_\rho = 2\sum_{j=0}^{l}(-1)^{j-1}r_j. \tag{5.117}$$

Proceeding as in [47, (7-76)-(7-78)], we get (5.65). If $H^{\cdot}(Z,F) = 0$, proceeding as in [47, Corollary 8.18], for all $0 \leq j \leq 2l$, we have

$$r_j = 0. \tag{5.118}$$

By (5.117), we get (5.66), which completes the proof of Theorem 5.11 in the case where G has compact center and $\delta(G) = 1$. □

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