



HAL
open science

Poincaré inequalities and integrated curvature-dimension criterion for generalised Cauchy and convex measures

Baptiste Nicolas Huguet

► **To cite this version:**

Baptiste Nicolas Huguet. Poincaré inequalities and integrated curvature-dimension criterion for generalised Cauchy and convex measures. 2023. hal-03973129v1

HAL Id: hal-03973129

<https://hal.science/hal-03973129v1>

Preprint submitted on 3 Feb 2023 (v1), last revised 12 Jan 2024 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

POINCARÉ INEQUALITIES AND INTEGRATED CURVATURE-DIMENSION CRITERION FOR GENERALISED CAUCHY AND CONVEX MEASURES

BAPTISTE HUGUET

ABSTRACT. We obtain new sharp weighted Poincaré inequalities for convex measures on Riemannian manifolds. When specified to generalised Cauchy measures, it gives a unified and simple proof of the weighted Poincaré inequality for the whole range of parameters, with the optimal spectral gap, the error term and the extremal functions.

CONTENTS

1. Introduction	1
2. A variance representation formula	4
3. Weighted Poincaré inequalities	6
4. Generalised Cauchy measures	11
5. Eigenfunctions and optimality	16
6. The one-dimensional case	19
Acknowledgement	21
References	21

1. INTRODUCTION

The family of generalised Cauchy distributions is a family of probability measures defined on \mathbb{R}^n by $d\mu_\beta \sim (1 + |x|^2)^{-\beta} dx$, for $\beta > n/2$. This family plays an important role in different mathematical problem. In [6], S. Bobkov and M. Ledoux pointed out that these measures approximate the Gaussian measure as β diverges, up to rescaling. This inspired them to study the property of these measures in terms of functional inequalities and more especially, Poincaré inequalities. Due to their heavy-tail, these measures can not satisfy the classical Poincaré inequality. Anyway, they proved a weighted version of it, with the radial weight $\omega(x) = 1 + |x|^2$. For all $n \geq 1$ and all $\beta \geq n$ (or $\beta > 1$ if $n = 1$), we have

$$\frac{2(\beta - 1)}{\left(\sqrt{1 + \frac{2}{\beta-1}} + \sqrt{\frac{2}{\beta-1}}\right)^2} \text{Var}_{\mu_\beta}(f) \leq \int_{\mathbb{R}^n} |df|^2 (1 + |x|^2) d\mu_\beta,$$

Date: February 3, 2023 *File:* cauchy'poincare.tex.

for all smooth bounded function f . This result allows to recover the classical Poincaré inequality for Gaussian measure in \mathbb{R}^n . However, it does not cover the entire range $n/2 < \beta$ and the constant is not optimal.

Simultaneously, this question has been studied in the perspective of convergence rate of solutions to fast diffusion equations. Indeed, generalised Cauchy measures are the twins of Gaussian measures in the study of heat equation. In [5], A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo and J.-L. Vázquez prove Hardy-Poincaré inequalities for $\beta \in \mathbb{R} \setminus \{n/2\}$. When $\beta > n/2$, i.e when the measure is a probability, these inequalities are equivalent to weighted Poincaré inequalities. Furthermore, in [7], they obtain a complete description of the continuous spectrum and the eigenvalues of an adequate operator for $\beta \geq n/2$. The bottom of this spectrum is then related to the optimal constant of the Poincaré inequality. Their proof relies on deep and involved spectral theory arguments. Besides, it is specific to the generalised Cauchy measures.

These articles paved the way to numerous works on weighted Poincaré inequalities. On the one hand, these works address the question of general convex measures. They establish criteria under which there are weighted Poincaré inequalities satisfied with explicit bound on the optimal constant. On the other hand, they question the relevance of probabilistic arguments in comparison to spectral theory arguments, in particular, Markov semigroups and curvature-dimension theory. These works often lead to more general functional inequalities as logarithmic-Sobolev or other Φ -entropies.

In [15], V.H. Nguyen, proves sharp weighted Poincaré inequalities for convex measures, i.e $d\mu \sim \omega^{-\beta} dx$ on \mathbb{R}^n , where ω is smooth and $\text{Hess}(\omega)$ uniformly lower bounded by some constant $\rho > 0$. His proof is inspired by Hörmander L^2 -method. His result is valid for $\beta \geq n + 1$. In the case of generalised Cauchy measures, his bound is proved to be optimal and associated to an eigenfunction.

In [8], M. Bonnefont, A. Joulin and Y. Ma complete the proof in one-dimension using the intertwining approach. They make a great step in a second article [9] for the n -dimensional case. They obtain a general comparison between the spectral gap of a radially log-concave measure and the spectral gap of its one-dimensional radial part. Applied to generalised Cauchy measures, it gives a weighted Poincaré inequality for the whole range $\beta > n/2$. Nonetheless, they can not reach the optimal constant for $n = 2$ and $(3 + \sqrt{5})/2 \leq \beta \leq 3$ and for $n \geq 3$ and $n + n/(n + 1) \leq \beta \leq n + 1$.

Besides these major articles, several papers recover partial results and enlighten new perspectives. For example, in [17], A. Saumard, uses Stein kernel to recover the optimal bound in dimension 1 for $\beta > 3/2$. In [1], M. Arnaudon, M. Bonnefont and A. Joulin obtain a new bound for $\beta > n$ using intertwining, unfortunately, their bound is not optimal. In [13], the author improves the previous method by mixing it with manifold embedding. He obtains a better lower bound in dimension $n = 2$, even optimal for $1 < \beta \leq 1 + \sqrt{2}$. In [4], D. Bakry, I. Gentil and G. Scheffer recover

the optimal bound for $\beta \geq n + 1$ using harmonicity and curvature-dimension argument. Their result generalised the work of [15] to Riemannian manifold. In [12], I. Gentil and S. Zougmeier recover this Riemannian result using the curvature-dimension criterion on the Laplace operator only.

Each one of these articles uses different perspectives and different techniques. Together, they do not allow to understand the link between curvature-dimension criterion and the weighted Poincaré inequality, especially for generalised Cauchy measures. The goal of this article is to obtain a simple, self-contained and intrinsic proof of sharp Poincaré inequalities for convex measures using curvature-dimension arguments only. When specified to generalised Cauchy measure, we recover the optimal constant, $\lambda_1(-L)$ such that

$$\lambda_1(-L) \operatorname{Var}_{\mu_\beta}(f) \leq \int_{\mathbb{R}^n} |df|^2 (1 + |x|^2) d\mu_\beta,$$

for the whole range of parameters. This optimal constant will be interpreted as a spectral gap. Our main result is the following theorem.

Theorem 1.1. *For $n = 1$, we have*

$$\lambda_1(-L) = \begin{cases} (\beta - 1/2)^2 & \text{if } 1/2 < \beta \leq 3/2 \\ 2(\beta - 1) & \text{if } 3/2 \leq \beta \end{cases} .$$

For $n \geq 2$, we have

$$\lambda_1(-L) = \begin{cases} (\beta - n/2)^2 & \text{if } n/2 < \beta \leq n/2 + 2 \\ 4(\beta - n/2 - 1) & \text{if } n/2 + 2 \leq \beta \leq n + 1 \\ 2(\beta - 1) & \text{if } n + 1 \leq \beta \end{cases} .$$

We also obtain an estimation of the error term in the inequality and the associated extremal functions, when they exist. Our method is inspired by the integrated curvature-dimension criterion. This criterion has been firstly introduced by M. Obata in [16], for the Laplace operator on compact Riemannian manifolds, and developed by M. Ledoux in [14]. An improvement of this criterion is proposed in [11], with an explicit expression of the error term in Poincaré inequality. This improvement allows to characterise extremal functions. With an appropriate choice of diffusion, we prove weighted Poincaré inequalities for convex measures, with explicit sharp lower bound of the optimal constant, extending the classical result to $\beta \leq n + 1$ and to Riemannian manifolds.

Let us describe the structure of this paper. In Section 2 we present the integrated curvature-dimension criterion and its link to weighted Poincaré inequalities. In section 3, we prove weighted Poincaré inequalities for the measures $d\mu \sim \omega^{-\beta}$, where ω is smooth and positive. These inequalities will be satisfied under assumptions which need to be discussed. Section 4 is dedicated to the generalised Cauchy measures. We apply the previous results and obtain a lower bound on the spectral gap. In Section 5, we prove the optimality of our lower bound and we discuss

the existence of extremal functions. Finally, in Section 6, we return to the one-dimensional case, as a toy application of our method. This section can be read at the beginning as it exposes most of our arguments in a simpler case.

2. A VARIANCE REPRESENTATION FORMULA

Let M be a complete Riemannian manifold of dimension n . Let μ be a probability measure on M and L a reversible diffusion operator associated to μ . We denote by \mathbf{P} the \mathbb{L}^2 -semigroup generated by L . We assume that the semigroup \mathbf{P} is ergodic. We denote by Γ and Γ_2 the carré du champ and iterated carré du champ operators, respectively defined on $f \in \mathcal{C}_c^\infty(M)$ by

$$2\Gamma(f) = L(f^2) - 2fLf \quad \text{and} \quad 2\Gamma_2(f) = L\Gamma(f) - 2\Gamma(f, Lf).$$

Under suitable conditions, discussed in [3], these operators can be uniquely extended to the domain $\mathcal{D}(L) \subset \mathbb{L}^2(\mu)$. The link between these operators and functional inequalities satisfied by the measure μ have been studied since the work of D. Bakry and M. Émery [2]. The operator L satisfies the curvature-dimension criterion $CD(\rho, \infty)$, also called Bakry-Émery criterion, with $\rho \in \mathbb{R}$, if

$$\Gamma_2 \geq \rho\Gamma. \quad (1)$$

Moreover, if $\rho > 0$, then under the $CD(\rho, \infty)$ criterion, the associated measure μ satisfies a Poincaré-type inequality with constant ρ

$$\rho \operatorname{Var}_\mu(f) \leq \int_M \Gamma(f) d\mu. \quad (2)$$

For the operator $L = \Delta - \nabla V \cdot \nabla$ associated to $d\mu = e^{-V} dx$, the carré du champ operator is $\Gamma(f) = |df|^2$ and (2) is the classical Poincaré inequality. However, for a general L , this inequality is a weighted Poincaré inequality. Actually, both Poincaré inequality and weighted Poincaré inequality are equivalent to a spectral gap property on the generator L . We denote by $\lambda_1(-L)$ this spectral gap

$$\lambda_1(-L) = \inf \left\{ \frac{\int_M -fLf d\mu}{\operatorname{Var}_\mu(f)}, f \in \mathcal{D}(-L) \right\}.$$

Actually, curvature-dimension criterion implies stronger functional inequalities than Poincaré-type inequalities, such as logarithmic-Sobolev inequalities. In [14], M. Ledoux, proves in \mathbb{R}^n that the classical Poincaré inequality is equivalent to an integrated version of the Bakry-Émery criterion. This result is very general and have been adapted to a larger class of diffusion and space. See [3] or [10] for more details. The version we recall here, comes from [11], where the deficit term is explicit.

Proposition 2.1 ([11]). *For all $f \in \mathcal{C}_c^\infty(M)$ and for all $\rho \neq 0$ we have*

$$\operatorname{Var}_\mu(f) = \frac{1}{\rho} \int_M \Gamma(f) d\mu - \frac{2}{\rho} \int_0^{+\infty} \int_M (\Gamma_2 - \rho\Gamma)(\mathbf{P}_t f) d\mu dt.$$

In particular, $\lambda_1(-L)$ is the largest $\rho > 0$ such that $\int_M (\Gamma_2 - \rho\Gamma) d\mu \geq 0$.

Proof. Let us define $h(t) = \int_M (\mathbf{P}_t f)^2 d\mu$. We have

$$\text{Var}_\mu(f) = -[h_t]_0^{+\infty} = \frac{-1}{2\rho} h'(0) - \frac{1}{\rho} \int_0^{+\infty} \frac{1}{2} h''(t) + \rho h'(t) dt.$$

Then, we compute the first derivatives of h .

$$h'(t) = \int_M \partial_t (\mathbf{P}_t f)^2 d\mu = \int_M 2\mathbf{P}_t f L\mathbf{P}_t f d\mu = -2 \int_M \Gamma(\mathbf{P}_t f) d\mu$$

$$h''(t) = -2 \int_M \partial_t \Gamma(\mathbf{P}_t f) d\mu = -4 \int_M \Gamma(\mathbf{P}_t f, L\mathbf{P}_t f) d\mu = 4 \int_M \Gamma_2(\mathbf{P}_t f) d\mu$$

This proves the formula. Besides, if L satisfies an integrated Bakry-Émery criterion, with constant $\rho > 0$, then it is clear that $\lambda_1(-L) \geq \rho$. Conversely, using Cauchy-Schwarz inequality, we have

$$\int_M \Gamma(f) d\mu = \int_M \left(f - \int_M f d\mu \right) (-L)f d\mu \leq \text{Var}_\mu(f)^{1/2} \left(\int_M \Gamma_2(f) d\mu \right)^{1/2}.$$

Hence, we have

$$\lambda_1(-L) \leq \inf_{f \in \mathcal{C}_c^\infty(M)} \frac{\int_M \Gamma_2(f) d\mu}{\int_M \Gamma(f) d\mu}.$$

□

Let us remark that this formula can be extended to other Φ -entropy inequalities, such as logarithmic-Sobolev inequality, using

$$h(t) = \int_M \Phi(\mathbf{P}_t f) d\mu.$$

Thanks to the explicit expression of the error term this proposition provides a necessary condition for extremal function, up to regularity considerations, or at least, an heuristic for finding them. If L satisfies an integrated curvature-dimension criterion and if $f \in \mathcal{D}(L)$ is extremal, then we have

$$\int_M (\Gamma_2 - \rho\Gamma)(f) d\mu = 0.$$

It has been used in [11] on the Ornstein-Uhlenbeck operator to recover the extremal functions in Poincaré and logarithmic-Sobolev inequalities for the Gaussian measures. The regularity arguments involved will be further discussed in Section 5.

In the following, we use Proposition 2.1 to determine weighted Poincaré inequalities for convex measures. In the case of generalised Cauchy measures, we will also use Proposition 2.1 to find the extremal functions and prove the optimality of our results.

3. WEIGHTED POINCARÉ INEQUALITIES

Let ω be a smooth positive function on M . We assume that there exists $\beta > 0$ such that the measure with density $\omega^{-\beta}$ with respect to the Riemannian volume measure, denoted μ_β , is a probability measure. Up to additional assumptions on ω we will be able to determine on which range of β this measure assumption is satisfied. The goal of this section is to obtain general weighted Poincaré inequalities for the probability μ_β . These inequality will be sharp in the sense that for generalised Cauchy measures, the weight and the constant are optimal. The main innovation of this section is the generalisation of Poincaré inequalities for convex measures to small $\beta \leq n + 1$.

The measure μ is associated to the symmetric diffusion operator on $\mathcal{C}_c^\infty(M)$

$$Lf = \omega \Delta f - (\beta - 1) \langle d\omega, df \rangle.$$

This special operator is chosen such that its carré du champ gives the weight ω in the Poincaré-type inequalities. Indeed, we have the following result.

Proposition 3.1. *For all $f \in \mathcal{C}_c^\infty(M)$, we have*

$$\begin{aligned} \Gamma(f) &= \omega |df|^2, \\ \Gamma_2(f) &= \|\omega \text{Hess}(f)\|_{HS}^2 + \omega^2 \text{Ric}(\nabla f, \nabla f) + \frac{1}{2} [\omega \Delta \omega - (\beta - 1) |d\omega|^2] |df|^2 \\ &\quad + \langle d|df|^2, \omega d\omega \rangle - \langle \Delta f df, \omega d\omega \rangle + (\beta - 1) \omega \text{Hess}(\omega)(\nabla f, \nabla f). \end{aligned}$$

Proof. The computation for Γ does not present any difficulties. We will only develop the computation of Γ_2 . For $f \in \mathcal{C}_c^\infty(M)$, we have

$$\begin{aligned} 2\Gamma_2(f) &= L\Gamma(f) - 2\Gamma(f, Lf) \\ &= \omega \Delta(\omega |df|^2) - (\beta - 1) \langle d\omega, d(\omega |df|^2) \rangle \\ &\quad - 2\omega \langle df, d(\omega \Delta f - (\beta - 1) \langle d\omega, df \rangle) \rangle \\ &= \omega \Delta \omega |df|^2 + 2\omega \langle d\omega, d|df|^2 \rangle + \omega^2 \Delta |df|^2 \\ &\quad - (\beta - 1) |d\omega|^2 |df|^2 - (\beta - 1) \langle d|df|^2, \omega d\omega \rangle \\ &\quad - 2 \langle \Delta f df, \omega d\omega \rangle - 2\omega^2 \langle df, d|df|^2 \rangle + 2(\beta - 1) \omega \langle df, d \langle df, d\omega \rangle \rangle \end{aligned}$$

Then, we use the Bochner formula for the Laplacian

$$\Delta |df|^2 - 2 \langle df, d\Delta f \rangle = 2 \|\text{Hess}(f)\|_{HS}^2 + 2 \text{Ric}(\nabla f, \nabla f). \quad (3)$$

Together with the remark that $\text{Hess}(f)(\nabla f, \cdot) = \frac{1}{2} d|df|^2$, it ends the proof. \square

Remarks that, as ω is positive, then the carré du champ cancels only on constant functions. This means that the diffusion is ergodic and Proposition 2.1 can be applied.

It remains unclear whereas L satisfies or not a curvature-dimension criterion. In Section 4, we will give a positive answer for generalised Cauchy measures but with

a non-positive ρ . This is a reason why we are interested to integrated curvature-dimension criterion. Before going further, we establish the following integration by parts formulae.

Lemme 3.2. *For all $f \in \mathcal{C}_c^\infty(M)$, we have*

$$\int_M \langle d|df|^2, \omega d\omega \rangle d\mu_\beta = \int_M [-\omega \Delta \omega + (\beta - 1)|d\omega|^2] |df|^2 d\mu_\beta, \quad (4)$$

$$\begin{aligned} \int_M \langle \Delta f df, \omega d\omega \rangle d\mu_\beta &= \int_M -\frac{1}{2} \langle d|df|^2, \omega d\omega \rangle - \omega \text{Hess}(\omega)(\nabla f, \nabla f) \\ &\quad + (\beta - 1) \langle df, d\omega \rangle^2 d\mu_\beta. \end{aligned} \quad (5)$$

Furthermore, if $\beta \neq 2$, we have

$$\int_M \langle d|df|^2, \omega d\omega \rangle d\mu_\beta = \frac{1}{\beta - 2} \int_M \omega^2 \Delta |df|^2 d\mu_\beta, \quad (6)$$

$$\int_M \langle \Delta f df, \omega d\omega \rangle d\mu_\beta = \frac{1}{\beta - 2} \int_M \omega^2 \langle df, d\Delta f \rangle + (\omega \Delta f)^2 d\mu_\beta. \quad (7)$$

3.1. Strong convexity. Firstly, we recover a general weighted Poincaré inequality proved in the Euclidean space \mathbb{R}^n in [15, Theorem 12] and Riemannian manifolds with non-negative Ricci curvature in [12]. We also provide a criterion for general Riemannian manifold. As the computation will show, this result will be valid only for $\beta \geq n + 1$ and under strong convexity property. The core step of our method is to rearrange the terms in Γ_2 thanks to a good choice of integration by parts.

Proposition 3.3. *For all $\beta \neq 2$ and $f \in \mathcal{C}_c^\infty(M)$, we have*

$$\begin{aligned} \int_M \Gamma_2(f) d\mu_\beta &= \int_M \frac{\beta - (n + 1)}{\beta - 2} \|\omega \text{Hess}(f)\|_{HS}^2 \\ &\quad + \frac{n}{\beta - 2} \left[\|\omega \text{Hess}(f)\|_{HS}^2 - \frac{1}{n} (\omega \Delta f)^2 \right] \\ &\quad + (\beta - 1) \omega \left[\frac{1}{\beta - 2} \omega \text{Ric} + \text{Hess}(\omega) \right] (\nabla f, \nabla f) d\mu_\beta. \end{aligned}$$

Proof. From Proposition 3.1, we have

$$\begin{aligned} \int_M \Gamma_2(f) d\mu_\beta &= \int_M \|\omega \text{Hess}(f)\|_{HS}^2 + \omega^2 \text{Ric}(\nabla f, \nabla f) \\ &\quad + \frac{1}{2} [\omega \Delta \omega - (\beta - 1)|d\omega|^2] |df|^2 + \frac{1}{2} \underbrace{\langle d|df|^2, \omega d\omega \rangle}_{(i)} \\ &\quad + \frac{1}{2} \underbrace{\langle d|df|^2, \omega d\omega \rangle}_{(ii)} - \underbrace{\langle \Delta f df, \omega d\omega \rangle}_{(iii)} + (\beta - 1) \omega \text{Hess}(\omega)(\nabla f, \nabla f) \end{aligned}$$

$$\begin{aligned}
&= \int_M \|\omega \operatorname{Hess}(f)\|_{HS}^2 + \omega^2 \operatorname{Ric}(\nabla f, \nabla f) + (\beta - 1)\omega \operatorname{Hess}(\omega)(\nabla f, \nabla f) \\
&\quad + \frac{\omega^2}{2(\beta - 2)}(\Delta|df|^2 - 2\langle df, d\Delta|df|^2 \rangle) - \frac{\omega^2}{\beta - 2}(\Delta f)^2
\end{aligned}$$

where we use the equalities (4), (6) and (7) on (i), (ii) and (iii) respectively. Using again the Bochner formula (3), we obtain the result. \square

Remark that this expression of $\int_M \Gamma_2(f) d\mu$ can be found in [15], without the use of the carré du champ formalism.

The previous formula suggests some conditions under which we can obtain a spectral gap inequality. Let us assume that $\beta \neq 2$ and that there exists a $c > 0$ such that

$$\frac{\omega}{\beta - 2} \operatorname{Ric} + \operatorname{Hess}(\omega) \geq c \operatorname{id}. \quad (\text{H1})$$

This assumption is not very convenient because it depends on β and because this dependence is not explicit. Yet, in the Euclidean space, the assumption is equivalent to strong convexity on ω : it exists $\rho_- > 0$ such that

$$\operatorname{Hess}(\omega) \geq \rho_- \operatorname{id}. \quad (8)$$

Remarks that under this convexity assumption, the measure $d\mu_\beta$ is a probability measure for every $\beta > \frac{n}{2}$. The hypothesis (8) is also sufficient in a Riemannian manifold with non-negative Ricci curvature. We can now state our first result.

Theorem 3.4. *For $n \geq 2$ and $\beta \geq n + 1$, under assumption (H1), we have $\lambda_1(-L) \geq c(\beta - 1)$. Moreover, we have the explicit upper bound of the error term*

$$\begin{aligned}
c(\beta - 1) \operatorname{Var}_{\mu_\beta}(f) &\quad - \int_M \Gamma(f) d\mu_\beta \\
&\leq -2 \int_0^{+\infty} \int_M \frac{\beta - (n + 1)}{\beta - 2} \|\omega \operatorname{Hess}(\mathbf{P}_t f)\|_{HS}^2 \\
&\quad + \frac{n}{\beta - 2} \left[\|\omega \operatorname{Hess}(\mathbf{P}_t f)\|_{HS}^2 - \frac{1}{n} (\omega \Delta \mathbf{P}_t f)^2 \right] d\mu_\beta dt.
\end{aligned}$$

For $n = 1$ and $\beta > 1$, under assumption (H1), we have $\lambda_1(-L) \geq c(\beta - 1)$. Moreover, we have the explicit upper bound of the error term

$$c(\beta - 1) \operatorname{Var}_{\mu_\beta}(f) - \int_M \Gamma(f) d\mu_\beta \leq -2 \int_0^{+\infty} \int_M \|\omega \operatorname{Hess}(\mathbf{P}_t f)\|_{HS}^2 d\mu_\beta dt.$$

Proof. Let us begin with the general case $n \geq 2$. We must remark that for $n \geq 2$ and $\beta \geq n + 1$ the assumption $\beta \neq 2$ is redundant as we have $\beta > 2$. Then, under the assumption (H1) we have proved that

$$\int_M \Gamma_2(f) - c(\beta - 1)\Gamma(f) d\mu_\beta \geq \int_M \frac{\beta - (n + 1)}{\beta - 2} \|\omega \operatorname{Hess}(f)\|_{HS}^2$$

$$+ \frac{n}{\beta - 2} \left[\|\omega \text{Hess}(f)\|_{HS}^2 - \frac{1}{n}(\omega \Delta f)^2 \right] d\mu_\beta.$$

The numerical coefficients are non-negative and the Cauchy-Schwarz inequality implies that for all linear operator A acting on the tangent bundle TM , we have

$$\|A\|_{HS}^2 \geq \frac{1}{n}(\text{Tr } A)^2.$$

Applying it to $A = \omega \text{Hess}(f)$, we obtain an average curvature-dimension criterion and we conclude with Proposition 2.1.

In the particular case $n = 1$, if $\beta \neq 2$, after simplifications, we have shown

$$\int_M \Gamma_2(f) - c(\beta - 1)\Gamma(f) d\mu_\beta \geq \int_M \|\omega \text{Hess}(f)\|_{HS}^2 d\mu_\beta.$$

Then, the case $\beta = 2$ can be deduced by dominated convergence. \square

Actually, in the one dimensional case, the result can be directly obtain, without any dominated convergence as we can avoid the integration by parts (6) and (7). This will be illustrated by the example of the generalised Cauchy measures in Section 6.

3.2. Upper bounded convexity. To obtained Proposition 3.3, we have made the choice to keep Hessian terms and only them. Now, in order to obtain Poincaré inequalities for smaller β , we use the integration by parts formulae to make them disappear. This choice of terms is specific to the dimension $n \geq 2$. These Poincaré inequalities for $\beta \leq n + 1$ is an innovation of our work. Nonetheless, it requires a stronger assumption on ω .

Proposition 3.5. *For all $n \geq 2$, $\beta \neq 2$ and $f \in \mathcal{C}_c^\infty(M)$, we have*

$$\begin{aligned} \int_M \Gamma_2(f) d\mu_\beta &= \int_M \frac{n}{n-1} \left[\|\omega \text{Hess}(f)\|_{HS}^2 - \frac{1}{n}(\omega \Delta f)^2 \right] \\ &\quad + \frac{(n+1-\beta)(\beta-1)}{n-1} [|df|^2 |d\omega|^2 - \langle df, d\omega \rangle^2] \\ &\quad + \omega \left[\frac{n}{n-1} \omega \text{Ric} + (\beta-1) \text{Hess}(\omega) \right] (\nabla f, \nabla f) \\ &\quad + \frac{n+1-\beta}{n-1} \omega [\text{Hess}(\omega) - \Delta \omega \text{id}] (\nabla f, \nabla f) d\mu_\beta. \end{aligned}$$

Proof. Let $\lambda \in \mathbb{R}$ be a parameter to be optimised later. We have

$$\begin{aligned} \int_M \Gamma_2(f) d\mu_\beta &= \int_M \|\omega \text{Hess}(f)\|_{HS}^2 + \omega^2 \text{Ric}(\nabla f, \nabla f) \\ &\quad + \frac{1}{2} [\omega \Delta \omega - (\beta-1)|d\omega|^2] |df|^2 + \frac{1}{2} \underbrace{\langle d|df|^2, \omega d\omega \rangle}_{(i)} \end{aligned}$$

$$\begin{aligned}
& + \frac{\lambda}{2} \langle d|df|^2, \omega d\omega \rangle - \lambda \langle \Delta f df, \omega d\omega \rangle \\
& + \frac{1-\lambda}{2} \underbrace{\langle d|df|^2, \omega d\omega \rangle}_{(i)} - (1-\lambda) \underbrace{\langle \Delta f df, \omega d\omega \rangle}_{(ii)} \\
& + (\beta - 1) \omega \text{Hess}(\omega)(\nabla f, \nabla f) d\mu_\beta \\
& = \int_M \|\omega \text{Hess}(f)\|_{HS}^2 + \frac{1-\lambda}{\beta-2} [\|\omega \text{Hess}(f)\|_{HS}^2 - (\omega \Delta f)^2] \\
& + \frac{\lambda}{2} \underbrace{\langle d|df|^2, \omega d\omega \rangle}_{(iv)} - \lambda \underbrace{\langle \Delta f df, \omega d\omega \rangle}_{(v)} + (\beta - 1) \omega \text{Hess}(\omega)(\nabla f, \nabla f) \\
& + \left(1 + \frac{1-\lambda}{\beta-2}\right) \omega^2 \text{Ric}(\nabla f, \nabla f) d\mu_\beta
\end{aligned}$$

Here we have used the formulae (4), (6) and (7) for (i), (ii) and (iii) respectively and Bochner formula. Then we choose λ in order to exactly compensate the terms $(\Delta f)^2$ with the term $\|\text{Hess}(f)\|_{HS}^2$. It results that

$$\lambda = \frac{n+1-\beta}{n-1}.$$

To conclude, we use (4) and (5) on (iv) and (v) respectively. It gives the claimed result. \square

In order to derive some spectral gap inequality, for $\beta < n+1$, the assumption (H1) is not sufficient. In this case, we assume that it exist $\tilde{c} > 0$ such that

$$\frac{n}{n-1} \omega \text{Ric} + (\beta - 1) \text{Hess}(\omega) + \frac{n+1-\beta}{n-1} \omega [\text{Hess}(\omega) - \Delta \omega \text{id}] \geq \tilde{c} \text{id}. \quad (\text{H2})$$

In a general Riemannian manifold, it seems difficult to obtain an handy condition on ω . However, in the Euclidean space \mathbb{R}^n , this condition is equivalent to boundedness of the Hessian operator : it exists $0 < \rho_- \leq \rho_+$ such that

$$\rho_- \text{id} \leq \text{Hess}(\omega) \leq \rho_+ \text{id}. \quad (9)$$

In this case, we will denote by κ the condition number associated :

$$\kappa = \frac{\rho_+}{\rho_-} \geq 1.$$

As previously, in a Riemannian manifold with non-negative Ricci curvature, we can also use this simpler boundedness assumption (9). Even if it is more restrictive than (H2), it is more explicit.

Let us remark that if the Hessian is uniformly bounded, then μ is a probability on \mathbb{R}^n if and only if $\beta > \frac{n}{2}$.

Theorem 3.6. *For $n \geq 2$ and $\beta \leq n+1$, under assumption (H2), we have*

$$\lambda_1(-L) \geq \tilde{c}$$

and for all $f \in \mathcal{C}_c^\infty(M)$

$$\begin{aligned} & \tilde{c} \text{Var}_{\mu_\beta}(f) - \int_M \Gamma(f) d\mu_\beta \\ & \leq -2 \int_0^{+\infty} \int_M \frac{n}{n-1} \left[\|\omega \text{Hess}(\mathbf{P}_t f)\|_{HS}^2 - \frac{1}{n} (\omega \Delta \mathbf{P}_t f)^2 \right] \\ & \quad + \frac{(n+1-\beta)(\beta-1)}{n-1} [|d\mathbf{P}_t f|^2 |d\omega|^2 - \langle d\mathbf{P}_t f, d\omega \rangle^2] d\mu_\beta dt \end{aligned}$$

Furthermore, if $\text{Ric} \geq 0$, then, under assumption (9), we have

$$\lambda_1(-L) \geq \rho_- \left(\beta - 1 - \frac{n+1-\beta}{n-1} (n\kappa - 1) \right), \quad \forall \beta \in \left[\frac{n(n+1)\kappa - 2}{n(\kappa+1) - 2}, n+1 \right].$$

Proof. The first general result is clear using Cauchy-Schwarz inequality. Then, we remark that $\beta - 1 + \lambda \geq 0$ and $\lambda \geq 0$. Under assumption (9), we have

$$\begin{aligned} (\beta - 1 + \lambda) \text{Hess}(\omega) - \lambda \Delta \omega \text{id} & \geq [\rho_-(\beta - 1 + \lambda) - \lambda n \rho_+] \text{id} \\ & \geq \rho_- [\beta - 1 - \lambda(n\kappa - 1)] \text{id} \end{aligned}$$

The term $\rho_- [\beta - 1 - \lambda(n\kappa - 1)]$ is positive if and only if $\frac{n(n+1)\kappa - 2}{n(\kappa+1) - 2} < \beta$. \square

The assumption (9) implies the assumption (8), then it is interesting to remark that the associated spectral gap bounds are continuous at $\beta = n + 1$. As the condition number κ is always greater than 1, we have

$$\frac{n(n+1)\kappa - 2}{n(\kappa+1) - 2} \geq \frac{n}{2} + 1.$$

Hence, our two spectral gap lower bound do not cover the entire range $]n/2, +\infty[$ admissible under assumption (8). Actually, it seems difficult to obtain any result for the small β in this general case but in the particular case of generalised Cauchy measure, we will be able to conclude.

4. GENERALISED CAUCHY MEASURES

In this section we apply our method to the generalised Cauchy measures on \mathbb{R}^n to get some lower bounds of the spectral gap $\lambda_1(-L)$ together with error term estimates. The optimality of these bounds and the extremal functions will be discussed in Section 5. In this section, we assume that $n \geq 2$. The one-dimensional case has already been proved in [8]. We will recover this result in Section 6, with our method, as it is the simplest illustration of it and as it gives nice heuristic for the n -dimensional computations.

We denote by ω the function defined on \mathbb{R}^n by $\omega(x) = 1 + |x|^2$. The generalised Cauchy measure μ_β is the probability measure defined on \mathbb{R}^n , for $\beta > n/2$, by

$$d\mu_\beta = \frac{1}{Z_{n,\beta}} \omega^{-\beta} dx, \quad \text{with} \quad Z_{n,\beta} = \frac{\pi^{n/2} \Gamma(\beta - n/2)}{\Gamma(\beta)}.$$

It is associated to the symmetric operator defined on $C_c^\infty(\mathbb{R}^n)$ by

$$Lf = \omega \Delta f - 2(\beta - 1) \langle x, df \rangle.$$

The main result of this article is the spectral gap of the operator $-L$ for all $n \geq 1$ and all $\beta > n/2$. We recall the major theorem.

Theorem 4.1. *For $n = 1$, we have*

$$\lambda_1(-L) = \begin{cases} (\beta - 1/2)^2 & \text{if } 1/2 < \beta \leq 3/2 \\ 2(\beta - 1) & \text{if } 3/2 \leq \beta \end{cases}.$$

For $n \geq 2$, we have

$$\lambda_1(-L) = \begin{cases} (\beta - n/2)^2 & \text{if } n/2 < \beta \leq n/2 + 2 \\ 4(\beta - n/2 - 1) & \text{if } n/2 + 2 \leq \beta \leq n + 1 \\ 2(\beta - 1) & \text{if } n + 1 \leq \beta \end{cases}.$$

It is noticeable that the expressions obtained are continuous in β . Remark that in dimension $n = 2$ the intermediate range $n/2 + 2 \leq \beta \leq n + 1$ is reduced to $\beta = 3$. This result shows that curvature-dimension argument are sufficient to obtain the spectral gap for generalised Cauchy measures. It is a real improvement. Besides, we will complete this result with the study of error term in the associated weighted Poincaré inequality and the extremal functions.

Firstly, we recover the carré du champ operators. According to Proposition 3.1, they are given by the following formulae.

Proposition 4.2. *For all $f \in C_c^\infty(\mathbb{R}^n)$ we have*

$$\Gamma(f) = \omega |df|^2,$$

$$\Gamma_2(f) = \|\omega \text{Hess}(f)\|_{HS}^2 + [n\omega + 2(\beta - 1)] |df|^2 + 2 \langle d|df|^2, \omega x \rangle - 2 \langle \Delta f df, \omega x \rangle.$$

As we remarked in the general case, these formulae do not clearly establish that the Cauchy measure does not fulfil a $CD(\rho, \infty)$ criterion for a positive ρ . Nonetheless, in this particular case, we can prove a curvature-dimension result.

Proposition 4.3. *For $n \geq 2$, L satisfies $CD(-(n + 8), \infty)$.*

Proof. For all $\alpha, \varepsilon > 0$, we have

$$\begin{aligned} \Gamma_2(f) &= \|\omega \text{Hess}(f)\|_{HS}^2 + [n\omega + 2(\beta - 1)] |df|^2 + 2 \langle d|df|^2, \omega x \rangle - 2 \langle \Delta f df, \omega x \rangle \\ &\geq \|\omega \text{Hess}(f)\|_{HS}^2 + [n\omega + 2(\beta - 1)] |df|^2 + 4 \text{Hess}(f)(\nabla f, \omega x) \\ &\quad - \alpha (\omega \Delta f)^2 - \frac{1}{\alpha} |df|^2 |x|^2 \end{aligned}$$

$$\begin{aligned}
&\geq (1 - n\alpha)\|\omega \text{Hess}(f)\|_{HS}^2 + \left[n\omega + 2(\beta - 1) - \frac{1}{\alpha}|x|^2 \right] |df|^2 \\
&\quad + 4 \text{Hess}(f)(\nabla f, \omega x) \\
&\geq (1 - n\alpha)\|\omega \text{Hess}(f) + \varepsilon \nabla f \otimes x\|_{HS}^2 + (4 - 2(1 - n\alpha)\varepsilon) \text{Hess}(f)(\nabla f, \omega x) \\
&\quad + \left[n\omega + 2(\beta - 1) - \left(\frac{1}{\alpha} + (1 - n\alpha)\varepsilon^2 \right) |x|^2 \right] |df|^2
\end{aligned}$$

We get the claimed result choosing $\alpha = 1/2n$ and $\varepsilon = 4$. \square

Actually, this result is not optimal. In dimension $n = 1$, we prove a $CD(0, \infty)$ criterion in Section 6. In [13], a $CD(0, \infty)$ criterion is proved in dimension $n = 2$. We can make the conjecture that this is general, although, we do not need it, neither for the spectral gap nor for the extremal functions.

First of all, we remark that

$$\text{Hess}(\omega) = 2 \text{id},$$

then ω satisfies the assumption (9) (and so (8)) with $\rho_- = 2$ and $\kappa = 1$. This allows us to use the results from the previous section. In the general case, we have obtained two different spectral gap's lower bounds on two different ranges of β . In the Cauchy particular case, we can reach a lower third range and span all $]n/2, +\infty[$. Those range are the lower range, $n/2 < \beta \leq n/2 + 2$, the intermediate range, $n/2 + 2 \leq \beta \leq n + 1$, and the upper range, $n + 1 \leq \beta$.

4.1. Upper range. In the subsection, we assume that $\beta \geq n + 1$. From Proposition 3.3, the integrations by parts prove the following expression.

$$\begin{aligned}
\int_{\mathbb{R}^n} \Gamma_2(f) d\mu_\beta &= \int_{\mathbb{R}^n} \frac{\beta - (n + 1)}{\beta - 2} \|\omega \text{Hess}(f)\|_{HS}^2 \\
&\quad + \frac{n}{\beta - 2} \left[\|\omega \text{Hess}(f)\|_{HS}^2 - \frac{1}{n} (\omega \Delta f)^2 \right] + 2(\beta - 1) \Gamma(f) d\mu_\beta
\end{aligned}$$

Then, following Theorem 3.4, we obtain the first lower bound.

Corollary 4.4. *For all $n \geq 2$ and $\beta \geq n + 1$, we have the spectral gap lower bound*

$$\lambda_1(-L) \geq 2(\beta - 1),$$

and for all $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}
&2(\beta - 1) \text{Var}_{\mu_\beta}(f) - \int_{\mathbb{R}^n} |df|^2 (1 + |x|^2) d\mu_\beta \\
&= -2 \int_0^{+\infty} \int_{\mathbb{R}^n} \frac{\beta - (n + 1)}{\beta - 2} \|\omega \text{Hess}(\mathbf{P}_t f)\|_{HS}^2 \\
&\quad + \frac{n}{\beta - 2} \left[\|\text{Hess}(\omega \mathbf{P}_t f)\|_{HS}^2 - \frac{1}{n} (\omega \Delta \mathbf{P}_t f)^2 \right] d\mu_\beta dt.
\end{aligned}$$

4.2. Intermediate range. We will now use Proposition 3.5 and Theorem 3.6 to obtain a lower bound for $\beta < n + 1$. As $\text{Hess}(\omega) = 2 \text{id}$, these propositions allow to obtain a bound on the range $n/2 + 1 < \beta \leq n + 1$. However, this bound will only be optimal for $n/2 + 2 < \beta \leq n + 1$. In the literature, in this range, the spectral gap was not fully known for $n \geq 3$ with curvature-dimension arguments.

From Proposition 3.5, we obtain the following formula.

$$\begin{aligned} \int_{\mathbb{R}^n} \Gamma_2(f) d\mu_\beta &= \int_{\mathbb{R}^n} \frac{n}{n-1} \left[\|\omega \text{Hess}(f)\|_{HS}^2 - \frac{1}{n} (\omega \Delta f)^2 \right] \\ &\quad + 4(\beta - 1) \frac{n+1-\beta}{n-1} [|df|^2 |x|^2 - \langle df, x \rangle^2] \\ &\quad + 4\left(\beta - \frac{n}{2} - 1\right) \Gamma(f) d\mu_\beta. \end{aligned} \quad (10)$$

Finally, from Theorem 3.6, we get the following lower bound.

Corollary 4.5. *For all $n \geq 2$ and $n/2 + 1 < \beta \leq n + 1$, we have the spectral gap lower bound*

$$\lambda_1(-L) \geq 4(\beta - n/2 - 1),$$

and for all $f \in C_c^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned} &4(\beta - n/2 - 1) \text{Var}_{\mu_\beta}(f) - \int_{\mathbb{R}^n} |df|^2 (1 + |x|^2) d\mu_\beta \\ &= -2 \int_0^{+\infty} \int_{\mathbb{R}^n} \frac{n}{n-1} \left[\|\omega \text{Hess}(\mathbf{P}_t f)\|_{HS}^2 - \frac{1}{n} (\omega \Delta \mathbf{P}_t f)^2 \right] \\ &\quad + 4(\beta - 1) \frac{n+1-\beta}{n-1} [|d\mathbf{P}_t f|^2 |x|^2 - \langle d\mathbf{P}_t f, x \rangle^2] d\mu_\beta dt. \end{aligned}$$

This result for mid-range β improves the previous results from [9] in which there was still a gap for $\frac{n(n+2)}{n+1} \leq \beta \leq n + 1$ without optimal constant.

4.3. Lower range. As for the one dimensional case, the appropriate formula for the lower range is obtained by combining order one and order two terms, in the way we proved the curvature-dimension criterion. In this range, we must take care of the value $\beta = 2$, reachable in dimensions $n = 2$ or $n = 3$, where Proposition 3.5 is not licit.

Let ε be an optimisation parameter. We have

$$\begin{aligned} \left\| \omega \text{Hess} f + \varepsilon \frac{\nabla f \otimes x + x \otimes \nabla f}{2} \right\|_{HS}^2 &= \|\omega \text{Hess} f\|_{HS}^2 + \varepsilon \langle d|df|^2, x\omega \rangle \\ &\quad + \frac{\varepsilon^2}{2} [|df|^2 |x|^2 + \langle df, x \rangle^2] \\ (\omega \Delta f + \varepsilon \langle df, x \rangle)^2 &= (\omega \Delta f)^2 + 2\varepsilon \langle \Delta f df, x\omega \rangle + \varepsilon^2 \langle df, x \rangle^2 \end{aligned}$$

We inject these twisted terms in equation (10) and then, we use the integration by parts (5) and (4). We obtain the following formula.

$$\begin{aligned} & \int_{\mathbb{R}^n} \Gamma_2(f) d\mu_\beta \\ &= \int_{\mathbb{R}^n} \frac{n}{n-1} \left[\left\| \omega \text{Hess } f + \varepsilon \frac{\nabla f \otimes x + x \otimes \nabla f}{2} \right\|^2 - \frac{1}{n} (\omega \Delta f + \varepsilon \langle df, x \rangle)^2 \right] \\ & \quad + \mathcal{B}_\varepsilon [|df|^2 |x|^2 - \langle df, x \rangle^2] + [\mathcal{C}_\varepsilon + \mathcal{D}_\varepsilon |x|^2] |df|^2 d\mu_\beta, \end{aligned}$$

with

$$\begin{aligned} \mathcal{B}_\varepsilon &= \frac{(n-2)\varepsilon^2 - 8(\beta-1)\varepsilon + 8(\beta-1)(n+1-\beta)}{2(n-1)} \\ \mathcal{C}_\varepsilon &= (n+2)\varepsilon + 4\left(\beta - \frac{n}{2} - 1\right) \\ \mathcal{D}_\varepsilon &= -\varepsilon^2 + (n+2-2(\beta-1))\varepsilon + 4\left(\beta - \frac{n}{2} - 1\right) \end{aligned}$$

We need to reconstruct a factor ω from the term $\mathcal{C}_\varepsilon + \mathcal{D}_\varepsilon |x|^2$. For all $\varepsilon > 0$, we have $\mathcal{C}_\varepsilon \geq \mathcal{D}_\varepsilon$. The maximum of \mathcal{D} is reached for $\varepsilon_0 = n/2 + 2 - \beta$, for which we have

$$\begin{aligned} \mathcal{B}_{\varepsilon_0} &= \frac{2(\beta-1)(3n-6) + n^2 - 4}{4(n-1)} \geq 0, \\ \mathcal{D}_{\varepsilon_0} &= \left(\beta - \frac{n}{2}\right)^2. \end{aligned}$$

Then, for all $n \geq 2$, for all $\beta \geq n/2$ such that $\beta \neq 2$, we have the following formula.

$$\begin{aligned} & \int_{\mathbb{R}^n} \Gamma_2(f) d\mu_\beta \\ &= \int_{\mathbb{R}^n} \frac{n}{n-1} \left[\left\| \omega \text{Hess } f + \varepsilon_0 \frac{\nabla f \otimes x + x \otimes \nabla f}{2} \right\|^2 - \frac{1}{n} (\omega \Delta f + \varepsilon_0 \langle df, x \rangle)^2 \right] \quad (11) \\ & \quad + \frac{2(\beta-1)(3n-6) + n^2 - 4}{4(n-1)} [|df|^2 |x|^2 - \langle df, x \rangle^2] \\ & \quad + \left(\frac{n}{2} + 2 - \beta\right) \left(\beta + \frac{n}{2}\right) |df|^2 + \left(\beta - \frac{n}{2}\right)^2 \Gamma(f) d\mu_\beta. \end{aligned}$$

By a dominated convergence argument, this formula is still valid for $\beta = 2$ when $n = 2$ or $n = 3$. On the range $n/2 < \beta \leq n/2 + 2$, the coefficients are non-negative. Hence, we have proved the following weighted Poincaré inequality on the lower range.

Corollary 4.6. *For all $n \geq 2$ and $n/2 < \beta \leq n/2 + 2$, we have the spectral gap lower bound*

$$\lambda_1(-L) \geq \left(\beta - \frac{n}{2}\right)^2$$

and for all $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned} & \left(\beta - \frac{n}{2}\right)^2 \text{Var}_{\mu_\beta}(f) - \int_{\mathbb{R}^n} |df|^2 (1 + |x|^2) d\mu_\beta \\ &= -2 \int_0^{+\infty} \int_{\mathbb{R}^n} \frac{n}{n-1} \left\| \omega \text{Hess } \mathbf{P}_t f + \varepsilon_0 \frac{\nabla \mathbf{P}_t f \otimes x + x \otimes \nabla \mathbf{P}_t f}{2} \right\|^2 \\ & - \frac{1}{n-1} (\omega \Delta \mathbf{P}_t f + \varepsilon_0 \langle d\mathbf{P}_t f, x \rangle)^2 \\ & + \frac{2(\beta-1)(3n-6) + n^2 - 4}{4(n-1)} [|d\mathbf{P}_t f|^2 |x|^2 - \langle d\mathbf{P}_t f, x \rangle^2] \\ & + \left(\frac{n}{2} + 2 - \beta\right) \left(\beta + \frac{n}{2}\right) |d\mathbf{P}_t f|^2 d\mu_\beta dt. \end{aligned}$$

where $\varepsilon_0 = \beta - \frac{n}{2} - 2$.

The two last corollary gives two lower bounds of $\lambda_1(-1)$, on two intersecting ranges of β , $]n/2 + 1, n + 1]$ and $]n/2, n/2 + 2]$ respectively. To conclude the first part of Theorem 1.1 proof, we remark that on $[n/2 + 1, n/2 + 2]$, we have

$$4\left(\beta - \frac{n}{2} - 1\right) \leq \left(\beta - \frac{n}{2}\right)^2.$$

5. EIGENFUNCTIONS AND OPTIMALITY

The goal of this section is to prove the optimality of the spectral gap bounds obtained in Sections 4 and to describe, if they exist, the extremal functions. Actually, upper bounds have already been obtained in [9] on the whole range of β but the optimality has not been proved for all parameters. Nevertheless, our error estimate brings a new perspective upon this question. The existence of extremal functions is sufficient to prove the optimality of the spectral gap bound. Those extremal functions can be interpreted as eigenfunctions of the operator $-L$. In the lower range $]n/2, n/2 + 2]$, we show that another strategy is needed.

As explained in Section 2, Proposition 2.1 characterises extremal functions, up to regularity arguments. There are two subtleties. The first one is to extend our error estimate formulae to $f \in \mathcal{D}(-L)$. When the operator L is essentially self-adjoint and satisfies a curvature-dimension criterion for some $\rho \in \mathbb{R}$, both the carré du champ and carré du champ iterated operators are uniquely defined on $\mathcal{D}(-L)$ (see [3]). This is satisfied for generalised Cauchy measure. However, in our case, the problem is more critical as we used integrations by parts. The second subtlety is the convergence of $\mathbf{P}_t f$ as t tends to 0. Despite these regularity considerations, Proposition 2.1 remains a good heuristic to find extremal functions.

Looking at the error estimates in Corollary 4.4, 4.5 and 4.6, we see that up to regularity, any extremal function satisfies one of these conditions, depending on the considered ranges of β

$$\begin{aligned} \text{either } & \|\text{Hess}(f)\|_{HS}^2 = 0 & \text{and } & \|\text{Hess}(f)\|_{HS}^2 = \frac{1}{n}(\Delta f)^2, \\ \text{or } & \|\text{Hess}(f)\|_{HS}^2 = \frac{1}{n}(\Delta f)^2 & \text{and } & |df||x| = \langle df, x \rangle, \\ \text{or } & |df| = 0. \end{aligned}$$

This leads to consider affine functions on the upper range and quadratic ones for the intermediate range. Reciprocally, we will show that these functions are extremal. This also shows that there is no extremal function on the lower range.

5.1. Eigenfunctions. The extremal functions exhibited previously can be interpreted as eigenfunctions providing that they are square-integrable. This implies that the spectral gap bounds $2(\beta - 1)$ and $4(\beta - \frac{n}{2} - 1)$ are eigenvalues. This proves their optimality.

Proposition 5.1. *Let be $n \geq 1$ and $\beta > n/2 + 1$. For all $v \in \mathbb{R}^n$, the linear function $x \mapsto \langle v, x \rangle$ belongs to $\mathbb{L}^2(\mu_\beta)$ and is an eigenfunction of $-L$ associated to the eigenvalue $2(\beta - 1)$. In particular, for all $\beta > n/2 + 1$, we have*

$$\inf_{f \in \mathcal{D}(L)} \frac{\int_{\mathbb{R}} \Gamma(f) d\mu_\beta}{\text{Var}_{\mu_\beta}(f)} \leq 2(\beta - 1).$$

For $\beta > n/2 + 2$, the function $x \mapsto |x|^2 - \frac{n}{2\beta - n - 2}$ belongs to $\mathbb{L}^2(\mu_\beta)$ and is an eigenfunction associated to the eigenvalue $4(\beta - \frac{n}{2} - 1)$. In particular, for all $\beta > n/2 + 2$, we have

$$\inf_{f \in \mathcal{D}(L)} \frac{\int_{\mathbb{R}} \Gamma(f) d\mu_\beta}{\text{Var}_{\mu_\beta}(f)} \leq 4(\beta - \frac{n}{2} - 1).$$

In particular, up to regularity, the extremal functions on the upper range are of type $x \in \mathbb{R}^n \mapsto \langle v, x \rangle + c$ and the extremal functions on the intermediate range are of the type $x \in \mathbb{R}^n \mapsto a|x|^2 + c$, with $v \in \mathbb{R}^n$ and $a, c \in \mathbb{R}$. We remark that for $\beta = n + 1$, the space of extremal functions is of dimension 3 whereas it is of dimension 2 on the interior of the upper and intermediate ranges.

5.2. The special case of the lower range. In the lower range, we have shown that, up to regularity, the error term is positive. Hence, it does not admit any extremal functions. In this case, the spectral gap is not an eigenvalue. In [5], it is interpreted as the bottom of continuous spectrum. Then our bound is still optimal. We give a new proof of this optimality based on the one-dimensional argument from [8], recalled in Proposition 6.4. Their proof needs some adaptation to the n -dimensional case. The following heuristic gives an idea of the adequate sequence. Firstly, the optimal ε is $n/2 + 2 - \beta$, which suggests a sequence f_ε

whose \mathbb{L}^2 -norm blows up as $\varepsilon \rightarrow (2\beta - n - 4)/4$. Secondly, the terms to cancel are not related to Hessian but to quadratic functions which suggest a factor $|x|^2$. It leads to the following result.

Proposition 5.2. *For $n \geq 2$, $n/2 < \beta \leq n/2 + 2$ and $\varepsilon < (2\beta - n - 4)/4$ the functions $f_\varepsilon : x \mapsto |x|^2 \omega^\varepsilon(x)$ belongs to $\mathbb{L}^2(\mu_\beta)$ and we have*

$$\lim_{\varepsilon \uparrow (2\beta - n - 4)/4} \frac{\int_{\mathbb{R}} \Gamma(f_\varepsilon) d\mu_\beta}{\text{Var}_{\mu_\beta}(f_\varepsilon)} = \left(\beta - \frac{n}{2}\right)^2.$$

In particular, we have

$$\inf_{f \in \mathcal{D}(L)} \frac{\int_{\mathbb{R}} \Gamma(f) d\mu_\beta}{\text{Var}_{\mu_\beta}(f)} \leq \left(\beta - \frac{n}{2}\right)^2.$$

Proof. Firstly, we remark that $f_\varepsilon \in \mathbb{L}^2(\mu_\beta)$ if and only if $\varepsilon < (2\beta - n - 4)/4$ but at the limit we still have $f_{(2\beta - n - 4)/4} \in \mathbb{L}^1(\mu_\beta)$. So, we have

$$\frac{\text{Var}_{\mu_\beta}(f_\varepsilon)}{\int_{\mathbb{R}^n} f_\varepsilon^2 d\mu_\beta} = 1 - \frac{\left(\int_{\mathbb{R}^n} f_\varepsilon d\mu_\beta\right)^2}{\int_{\mathbb{R}^n} f_\varepsilon^2 d\mu_\beta} \xrightarrow{\varepsilon \rightarrow (2\beta - n - 4)/4} 1.$$

Then, for all $\varepsilon < (2\beta - n - 4)/4$, we have

$$\begin{aligned} \nabla f_\varepsilon(x) &= 2x\omega^\varepsilon(x) + 2\varepsilon x|x|^2\omega^{\varepsilon-1}(x) \\ \text{Hess}(f_\varepsilon(x)) &= 2n\omega^\varepsilon(x) \text{id} + 2n\varepsilon|x|^2\omega^{\varepsilon-1}(x) \text{id} + 8\varepsilon\omega^{\varepsilon-1}(x)x \otimes x \\ &\quad + 4\varepsilon(\varepsilon - 1)|x|^2\omega^{\varepsilon-2}(x)x \otimes x \\ -Lf_\varepsilon(x) &= [4\beta - 2n - 4 - 2\varepsilon(-2\beta + n + 4 - 2\varepsilon)] f_\varepsilon(x) - 2n \frac{f_\varepsilon(x)}{|x|^2} \\ &\quad - 4\varepsilon(\beta - \varepsilon) \frac{f_\varepsilon(x)}{\omega(x)}. \end{aligned}$$

So, we have

$$\begin{aligned} \frac{\int_{\mathbb{R}^n} -f_\varepsilon Lf_\varepsilon d\mu_\beta}{\int_{\mathbb{R}^n} f_\varepsilon^2 d\mu_\beta} &= 4\beta - 2n - 4 - 2\varepsilon(-2\beta + n + 4 + 2\varepsilon) - 2n \frac{\int_{\mathbb{R}^n} f_\varepsilon^2(x)|x|^{-2} d\mu_\beta}{\int_{\mathbb{R}^n} f_\varepsilon^2 d\mu_\beta} \\ &\quad - 4\varepsilon(\beta - \varepsilon) \frac{\int_{\mathbb{R}^n} f_\varepsilon^2 \omega^{-1} d\mu_\beta}{\int_{\mathbb{R}^n} f_\varepsilon^2 d\mu_\beta}. \end{aligned}$$

We remark that, as $\varepsilon \rightarrow (2\beta - n - 4)/4$, $f_\varepsilon^2(x)|x|^{-2}$ and $f_\varepsilon^2\omega^{-1}$ are integrable. So, we have

$$\lim_{\varepsilon \uparrow (2\beta - n - 4)/4} \frac{\int_{\mathbb{R}} \Gamma(f_\varepsilon) d\mu_\beta}{\text{Var}_{\mu_\beta}(f_\varepsilon)} = \lim_{\varepsilon \uparrow (2\beta - n - 4)/4} \frac{\int_{\mathbb{R}} -f_\varepsilon Lf_\varepsilon d\mu_\beta}{\int_{\mathbb{R}^n} f_\varepsilon^2 d\mu_\beta} = \left(\beta - \frac{n}{2}\right)^2.$$

□

6. THE ONE-DIMENSIONAL CASE

In this section, we end the proof of Theorem 1.1 with the dimension $n = 1$. For the sake of completeness, we recover this result with our method and make some further remarks, especially on the error term and on the extremal functions.

We still denote by ω the function

$$\omega : x \in \mathbb{R} \mapsto 1 + x^2,$$

and as shown in Section 4, we have

$$\begin{aligned} Lf &= \omega f'' - 2(\beta - 1)xf'\Gamma(f) = \omega f'^2, \\ \Gamma_2(f) &= \omega^2 f''^2 + [\omega + 2(\beta - 1)]f'^2 + 2f''f'x\omega. \end{aligned}$$

Firstly, we remark that, the operator L satisfies a nice curvature-dimension criterion.

Proposition 6.1. *For all $1/2 < \beta$, L satisfies the $CD(0, \infty)$ criterion but does not satisfy any $CD(\rho, \infty)$ criterion for $\rho > 0$.*

Proof. For all $f \in \mathcal{C}^\infty(\mathbb{R})$, we have

$$\begin{aligned} \Gamma_2(f) &= \omega^2 f''^2 + [\omega + 2(\beta - 1)]f'^2 + 2f''f'x\omega \\ &= (\omega f'' + xf')^2 + (2\beta - 1)f'^2 \\ &= (\omega f'' + xf')^2 + \frac{2\beta - 1}{\omega}\Gamma(f) \end{aligned}$$

So $\Gamma_2 > 0$ but $\frac{2\beta-1}{\omega}$ is not uniformly lower bounded by a positive constant. \square

This elementary proof enlightens one of the two options we have. The first one is the integration by parts, as used in our general method. The second one, which appears here, is the factorisation. A mix between these two approaches allows to recover the spectral gap for the whole range $\beta \in]1/2, +\infty[$.

Let $\varepsilon \in \mathbb{R}$ be an optimisation parameter. We have

$$\Gamma_2(f) = (\omega f'' + \varepsilon x f')^2 + 2(1 - \varepsilon)f''f'x\omega + [2(\beta - 1) + \omega - \varepsilon^2 x^2]f'^2$$

Now, using the integration by parts formula (4), we get

$$\int_{\mathbb{R}} \Gamma_2(f) d\mu_\beta = \int_{\mathbb{R}} (\omega f'' + \varepsilon x f')^2 + [A_\varepsilon + B_\varepsilon |x|^2] f'^2 d\mu_\beta$$

with $A_\varepsilon = 2(\beta - 1) + \varepsilon$ and $B_\varepsilon = 2(\beta - 1) + \varepsilon - \varepsilon(2(\beta - 1) + \varepsilon)$. We optimise to obtain the maximum over ε of $\min\{A_\varepsilon, B_\varepsilon\}$. Two cases appear : small β and large β .

If $\beta \geq 3/2$ then the optimum is reached for $\varepsilon = 0$, as a direct application of Theorem 3.4. We obtain

$$\int_{\mathbb{R}} \Gamma_2(f) d\mu_\beta = \int_{\mathbb{R}} \omega^2 f''^2 + 2(\beta - 1)f'^2 d\mu_\beta.$$

Then, according to Proposition 2.1, we have the following result.

Corollary 6.2. *If $\beta \geq 3/2$, then $\lambda(-L) \geq 2(\beta - 1)$ and for all $f \in \mathcal{C}_c^\infty(\mathbb{R})$, we have*

$$2(\beta - 1) \operatorname{Var}_{\mu_\beta}(f) = \int_{\mathbb{R}} \Gamma(f) d\mu_\beta - 2 \int_0^{+\infty} \int_{\mathbb{R}} \omega^2(\mathbf{P}_t f)^{\prime 2} d\mu_\beta dt.$$

Furthermore, if $\beta > 3/2$ then $\lambda(-L) = 2(\beta - 1)$ as linear functions are eigenfunctions associated to $2(\beta - 1)$.

Actually, affine functions are the only extremal functions, up to regularity, as any extremal function, f , satisfies $f^{\prime 2} = 0$. Remark that we can not conclude yet when $\beta = 3/2$ as affine functions do not belong to $\mathbb{L}^2(\mu_{3/2})$.

If $1/2 < \beta \leq 3/2$, the maximum over ε is obtain for $\varepsilon = 3/2 - \beta$ and we have

$$\begin{aligned} \int_{\mathbb{R}} \Gamma_2(f) d\mu_\beta &= \int_{\mathbb{R}} \left(\omega f'' + \left(\frac{3}{2} - \beta \right) f' \right)^2 + \left(\beta - \frac{1}{2} \right)^2 \omega f'^2 \\ &\quad + \left(\beta - \frac{1}{2} \right) \left(\frac{3}{2} - \beta \right) f'^2 d\mu_\beta. \end{aligned}$$

Then, we obtain the final result.

Corollary 6.3. *If $1/2 < \beta \leq 3/2$, then $\lambda(-L) = (\beta - 1/2)^2$ and for all*

$$\begin{aligned} \left(\beta - \frac{1}{2} \right)^2 \operatorname{Var}_{\mu_\beta}(f) - \int_{\mathbb{R}} \Gamma(f) d\mu_\beta &= -2 \int_0^{+\infty} \int_{\mathbb{R}} \left(\omega \mathbf{P}_t f'' + \left(\frac{3}{2} - \beta \right) \mathbf{P}_t f' \right)^2 \\ &\quad + \left(\frac{3}{2} - \beta \right) \left(\beta - \frac{1}{2} \right) \mathbf{P}_t f'^2 d\mu_\beta dt. \end{aligned}$$

Our calculus only provides a lower bound of the spectral gap. On this range, there is no eigenfunction, as in the n -dimensional case. The article [8] prove the optimality using a sequence of functions.

Proposition 6.4 ([8]). *For $n = 1$, $1/2 < \beta \leq 3/2$ and $\varepsilon < (2\beta - 3)/4$, the function $f_\varepsilon : x \mapsto x\omega^\varepsilon(x)$ belongs to $\mathbb{L}^2(\mu)$ and we have*

$$\lim_{\varepsilon \uparrow (2\beta - 3)/4} \frac{\int_{\mathbb{R}} \Gamma(f_\varepsilon) d\mu_\beta}{\operatorname{Var}_{\mu_\beta}(f_\varepsilon)} = \left(\beta - \frac{1}{2} \right)^2.$$

While looking for an extremal function f associated to $(\beta - 1/2)^2$, we saw that, up to regularity, if it exists, this function would satisfy the equation

$$\omega f'' + \left(\frac{3}{2} - \beta \right) f' = 0.$$

The solutions of this equation are primitives from $\omega^{(2\beta - 3)/4}$. This explains the choice of f_ε . Heuristically, the sequence f_ε converges toward a function close to cancels the Hessian part but who is not in $\mathbb{L}^2(\mu_\beta)$.

ACKNOWLEDGEMENT

I would like to thanks François Bolley for his encouragements and for our many useful discussions. This research is partially supported by the Centre Henri Lebesgue (ANR-11-LABX-0020-0) and the ANR project RAGE "Analyse Réelle et Géométrie" (ANR-18-CE40-0012).

REFERENCES

- [1] Marc Arnaudon, Michel Bonnefont, and Aldéric Joulin. Intertwinings and generalized Brascamp-Lieb inequalities. *Rev. Mat. Iberoam.*, 34:1021–1054, 2018.
- [2] Dominique Bakry and Michel Émery. Diffusions hypercontractives. In *Séminaire de probabilités de Strasbourg XIX*, volume 1123 of *Lecture Notes in Mathematics*, pages 177–206. Springer, 1985.
- [3] Dominique Bakry, Ivan Gentil, and Michel Ledoux. *Analysis and Geometry of Markov Diffusion operators*, volume 348 of *Grundlehren der mathematischen Wissenschaften*. Springer, 2014.
- [4] Dominique Bakry, Ivan Gentil, and Gregory Scheffer. Sharp Beckner-type inequalities for Cauchy and spherical distributions. *Studia Math.*, 251(3):219–245, 2020.
- [5] Adrien Blanchet, Matteo Bonforte, Jean Dolbeault, Gabriele Grillo, and Juan Luis Vázquez. Hardy-Poincaré inequalities and applications to nonlinear diffusions. *C.R. Acad. Sci. Paris*, 334(7):431–436, 2007.
- [6] Sergey G. Bobkov and Michel Ledoux. Weighted Poincaré-type inequalities for Cauchy and other convex measures. *Ann. Probab.*, 37(2):403–427, 2009.
- [7] Matteo Bonforte, Jean Dolbeault, Gabriele Grillo, and Juan Luis Vázquez. Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities. *PNAS*, 107(38):16459–16464, 2010.
- [8] Michel Bonnefont, Aldéric Joulin, and Yutao Ma. A note on spectral gap and weighted Poincaré inequalities for some one-dimensional diffusions. *ESAIM Probability & Statistics*, 20:18–29, 2016.
- [9] Michel Bonnefont, Aldéric Joulin, and Yutao Ma. Spectral gap for spherically symmetric log-concave probability measures, and beyond. *J. Funct. Anal.*, 270(7):2456 – 2482, 2016.
- [10] Patrick Cattiaux and Arnaud Guillin. A journey with the integrated Γ_2 criterion and its weak forms. to be published in *Geometric Aspects of Functional Analysis: Israel Seminar (GAFA) 2020-2022*. Volume 2327 of *Lecture Notes in Mathematics*.
- [11] Djalil Chafaï and Joseph Lehec. On Poincaré and logarithmic Sobolev inequalities for a class of singular Gibbs measures. In *Geometric Aspects of Functional Analysis: Israel Seminar (GAFA) 2017-2019 Volume I*, volume 2256 of *Lecture Notes in Mathematics*, pages 219–246. Springer, 2020.
- [12] Ivan Gentil and Simon Zugmeyer. A family of Beckner inequalities under various curvature-dimension conditions. *Bernoulli*, 27(2):751–771, 2021.
- [13] Baptiste Huguet. Intertwining relations for diffusions in manifolds and applications to functional inequalities. *Stoch. Proc. and their App.*, 145:1–28, 2022.
- [14] Michel Ledoux. On an integral criterion for hypercontractivity of diffusion semigroups and extremal functions. *J. Funct. Anal.*, 105(2):444–465, 1992.
- [15] Van Hoang Nguyen. Dimensional variance inequalities of Brascamp-Lieb type and a local approach to dimensional Prékopa’s theorem. *J. Funct. Anal.*, 266(2):931 – 955, 2014.
- [16] Morio Obata. Certain conditions for a Riemannian manifold to be isometric with a sphere. *J. Math. Soc. Japan*, 14(3):333–340, 1962.

- [17] Adrien Saumard. Weighted Poincaré inequalities, concentration inequalities and tail bounds related to Stein kernels in dimension one. *Bernoulli*, 25(4B):3978 – 4006, 2019.

IRMAR (ENS RENNES), UMR CNRS 6625, UNIV. RENNES, FRANCE

Email address: `baptiste.huguet@math.cnrs.fr`

URL: <https://www.math.cnrs.fr/~bhuguet/>