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# Controllability results for a cross diffusion system with a free boundary by a flatness approach

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**Abstract**

We study a one-dimensional cross diffusion system with a free boundary modeling the Physical Vapor Deposition. Using the flatness approach, we show several results of boundary controllability for this system in spaces of Gevrey class functions. One of the main difficulties consists in the physical constraints on the state and on the control. More precisely, the state corresponds to volume fractions of the $n+1$ chemical species and to the thickness of the film produced in the process, whereas the controls are the fluxes of the chemical species. We obtain the local controllability in the case where we apply $n+1$ nonnegative controls and a controllability result for large time in the case where we apply $n$ controls without any sign constraints. We also show in this last case that the controllability may fail for small times. We illustrate these results with some numerical simulations.
*Keywords:* Control, flatness, cross diffusion, free boundary.

## 1 Introduction and statement of our results

We consider controllability properties of a cross diffusion system with moving boundaries that can model the Physical Vapor Deposition (PVD) process involved in the fabrication of solar panels. Over a substrate, $n+1$ chemical species are introduced in gaseous state and solidify to generate a film occupying a domain of the form $(0, e(t))$ at time $t \geqslant 0$. We follow here the model considered in [1, 3]. For more information about the physical process, we refer to [15].

We denote by $u_k$ the volume fraction of the species $k \in \{0, \dots, n\}$ and by $\varphi_k$ its flux. The physical conditions on these quantities are

$$\sum_{k=0}^{n} u_k = 1, \qquad u_k \geqslant 0 \quad (0 \leqslant k \leqslant n) \tag{1.1}$$

and

$$\varphi_k \geqslant 0 \quad (0 \leqslant k \leqslant n). \tag{1.2}$$

We have denoted by $e(t) > 0$ the thickness of the film at time $t$ and its relation with the fluxes is

$$e(t) = e^0 + \int_0^t \sum_{k=0}^{n} \varphi_k(s)\,\mathrm{d}s \qquad (t \geqslant 0). \tag{1.3}$$

In what follows, the state variables of the system are $e$ and

$$u := (u_1, \dots, u_n)^\top \in \mathbb{R}^n,$$

since $u_0$ can be obtained from $u$ with the relation (1.1). We also set

$$\varphi := (\varphi_1, \dots, \varphi_n)^\top \in \mathbb{R}^n.$$

Then the model considered here is (see [3])

$$\begin{cases} \partial_t u - A\partial_x^2 u = 0 & t > 0,\ x \in (0, e(t)), \\ A\partial_x u(t,0) = 0 & t > 0, \\ A\partial_x u(t,e(t)) + e'(t)u(t,e(t)) = \varphi(t) & t > 0, \\ u(0,x) = u^0(x) & x \in (0, e^0). \end{cases} \tag{1.4}$$

To be more precise in the model considered in [1], the matrix $A \in \mathcal{M}_n(\mathbb{R})$ can depend on $u$. Here we assume that $A$ is independent of $u$, and invertible:

$$A \in \mathrm{GL}_n(\mathbb{R}). \tag{1.5}$$

The existence of solutions to the above system has been proven under some particular conditions on the function $A(u)$ that give a gradient flow structure to (1.4) and permit to use the boundedness by entropy technique (see [2, 6]). For an introduction to this method and to general cross diffusion systems, one can refer to the book [7]. One can see (1.3)–(1.4) as a controlled system where the state is $(e, u)$ and the controls are the fluxes $\varphi_k$. To our knowledge, the only work dealing with the controllability of the above system is [3] where the authors show the finite-time stabilization of the linearized system around some equilibrium states. Their result is based on the backstepping method.

In our work, we use a different approach to control the system: the differential flatness. This method, which comes from the finite-dimensional framework, has been successfully applied to several partial differential equations: the heat equation [11, 13], general parabolic equations [12], the Schrödinger equation [14], the linear Korteweg–de Vries equation [10] and a class of semilinear heat equations [8]. The authors of the present paper have used this method to show some controllability properties of another free boundary model, the Stefan problem, in [4].

As for the Stefan problem, the difficulties to handle (1.3)–(1.4) are the free-boundary and the physical restrictions that impose bounds on the solutions or on the controls. Here we want to use the controls $\varphi_k$ to pass from a stationary state to another with the restrictions (1.1) and (1.2). With these two constraints, we obtain a local controllability result (Theorem 1.1). To go further, we also investigate the problem where (1.1) is satisfied and where there is no sign constraints on the controls (Theorems 1.2 and 1.5). The set of "physically reasonable" stationary states of (1.3)–(1.4) is

$$\mathcal{S} := \left\{ (\overline{e}, \overline{u}) \in \mathbb{R}^{n+1}\ ;\ \overline{e} > 0, \quad \overline{u}_k > 0\ (1 \leqslant k \leqslant n), \quad \sum_{k=1}^n \overline{u}_k \in (0,1) \right\}. \tag{1.6}$$

In order to state our mains results, we first recall the notion of Gevrey functions. A function $f : \mathbb{R} \to \mathbb{R}$ is a Gevrey function of order $\sigma \geqslant 1$ if it is $C^\infty$ and if for any compact subset $K$ of $\mathbb{R}$, there exist $M, R \in \mathbb{R}_+^*$ such that its derivatives satisfy

$$\left| f^{(\ell)}(t) \right| \leqslant M \frac{\ell!^\sigma}{R^\ell} \qquad (\ell \in \mathbb{N},\ t \in K).$$

Similarly, $g : \mathbb{R}^2 \to \mathbb{R}$ is a Gevrey function of order $(\sigma_1, \sigma_2)$ if it is $C^\infty$ and if for any compact subset $K$ of $\mathbb{R}^2$, there exist $M, R_1, R_2 \in \mathbb{R}_+^*$ such that

$$\left| \partial_t^\ell \partial_x^k g(t,x) \right| \leqslant M \frac{\ell!^{\sigma_1}}{R_1^\ell} \frac{k!^{\sigma_2}}{R_2^k} \qquad (\ell, k \in \mathbb{N},\ (t,x) \in K).$$

These definitions can be extended to vector-valued functions. The interest for such functions is due to the fact that for any $\sigma > 1$, the function

$$\theta_\sigma(t) := \begin{cases} 1 & \text{if } t \leqslant 0, \\ \dfrac{e^{-\left((1-t)^{-c_\sigma}\right)}}{e^{-((1-t)^{-c_\sigma})} + e^{-(t^{-c_\sigma})}} & \text{if } t \in (0,1), \\ 0 & \text{if } t \geqslant 1, \end{cases} \tag{1.7}$$

with $c_\sigma = (\sigma-1)^{-1}$ is a Gevrey function of order $\sigma$ (see, for instance [5, Appendix B]). More precisely one can show the existence of $M, R > 0$ such that

$$\left|\theta_\sigma^{(i)}(t)\right| \leqslant M\frac{(i!)^\sigma}{R^i} \qquad (i \in \mathbb{N},\ t \in \mathbb{R}). \tag{1.8}$$

For more information on the Gevrey functions, we can refer for instance to [16–18].

We are now in position to give the main results of this paper. The first main result states as follows:

**Theorem 1.1.** *Assume* (1.5), $\sigma \in (1,2)$, $T > 0$ *and* $(e^0, u^0) \in \mathcal{S}$. *Let us consider* $(e^1, u^1) \in \mathcal{S}$ *and assume that* $e^1 > e^0 > 0$. *There exists* $\varepsilon > 0$ *such that if*

$$\left|u^0 - u^1\right| \leqslant \varepsilon, \tag{1.9}$$

*then there exist a control* $(\varphi_k)_{k\in\{0,\dots,n\}}$ *Gevrey of order* $\sigma$ *satisfying* (1.2) *in* $(0,T)$, *and a solution* $(e,u)$ *of* (1.3)–(1.4) *such that* $e$ *is Gevrey of order* $\sigma$, $u$ *is Gevrey of order* $\sigma$ *in time and* 1 *in space,* $(e(t), u(t,x)) \in \mathcal{S}$, *for every* $t \in [0,T]$ *and* $x \in (0, e(t))$ *and*

$$e(T) = e^1, \quad u(T,\cdot) = u^1.$$

In the case where we do not consider the condition (1.2), we can control globally the system (1.3)–(1.4). First we consider the case where $\varphi_0 \equiv 0$, that is we have only $n$ scalar controls $\varphi_1, \dots, \varphi_n$ acting on (1.3)–(1.4). Our second main result is stated below:

**Theorem 1.2.** *Assume* (1.5) *and* $\sigma \in (1,2)$. *Let us consider* $(e^0, u^0), (e^1, u^1) \in \mathcal{S}$ *with*

$$e^0\left(1 - \sum_{k=1}^n u_k^0\right) = e^1\left(1 - \sum_{k=1}^n u_k^1\right). \tag{1.10}$$

*Then there exists* $T_0 > 0$ *such that for any* $T \geqslant T_0$, *there exist a control* $(\varphi_k)_k$ *Gevrey of order* $\sigma$ *with* $\varphi_0 \equiv 0$ *and a solution* $(e,u)$ *of* (1.3)–(1.4) *such that* $e$ *is Gevrey of order* $\sigma$, $u$ *is Gevrey of order* $\sigma$ *in time and* 1 *in space,* $(e(t), u(t,x)) \in \mathcal{S}$, *for every* $t \in [0,T]$ *and* $x \in (0, e(t))$, *and*

$$e(T) = e^1, \quad u(T,\cdot) = u^1.$$

**Remark 1.3.** *The compatibility condition* (1.10) *is necessary if* $\varphi_0 \equiv 0$ *and it will be justified by the Corollary 2.2.*

To complete Theorem 1.2, we show that for $\varphi_0 \equiv 0$, there are cases where the problem is not controllable for small times $T > 0$:

**Proposition 1.4.** *Assume* (1.5) *and that* $A$ *has a real eigenvalue. Then, there exist* $(e^0, u^0), (e^1, u^1) \in \mathcal{S}$ *satisfying* (1.10) *and* $T_0 > 0$ *such that if there exist a control* $(\varphi_k)_k$ *with* $\varphi_0 \equiv 0$ *and a solution* $(e,u)$ *of* (1.3)–(1.4) *such that* $(e(t), u(t,x)) \in \mathcal{S}$, *for every* $t \in [0,T]$ *and* $x \in (0, e(t))$ *and*

$$e(T) = e^1, \quad u(T,\cdot) = u^1,$$

*then*

$$T \geqslant T_0.$$

In the case where we use the $n+1$ controls and where we do not consider the condition (1.2), then the system is controllable for any time $T > 0$:

**Theorem 1.5.** *Assume* (1.5), $\sigma \in (1,2)$ *and* $T > 0$. *Let us consider* $(e^0, u^0), (e^1, u^1) \in \mathcal{S}$. *Then there exist a control* $(\varphi_k)_{k\in\{0,\dots,n\}}$ *Gevrey of order* $\sigma$ *and a solution* $(e,u)$ *of* (1.3)–(1.4) *such that* $e$ *is Gevrey of order* $\sigma$, $u$ *is Gevrey of order* $\sigma$ *in time and* 1 *in space,* $(e(t), u(t,x)) \in \mathcal{S}$, $t \in [0,T]$ *and* $x \in (0, e(t))$ *and*

$$e(T) = e^1, \quad u(T,\cdot) = u^1.$$

The outline of the paper is the following: in Section 2, we give some preliminary results and recall the flatness method that we use in our main results. Then we show these main results in Section 3. Finally, we present some numerical illustrations of our results in Section 4.

# 2 Preliminaries and the flatness method

## 2.1 Property of the system (1.4)

One of the difficulties to deal with our system is the boundary condition in (1.4):

$$A\partial_x u(t,e(t)) + e'(t)u(t,e(t)) = \varphi(t) \qquad (t>0). \tag{2.1}$$

We show that condition (2.1) can be written in different and more convenient ways:

**Lemma 2.1.** *Assume $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+; \mathbb{R}^n)$ satisfies*

$$\begin{cases} \partial_t u - \partial_x^2(Au) = 0 & (t>0,\ x>0), \\ A\partial_x u(t,0) = 0 & (t>0), \end{cases} \tag{2.2}$$

*and let set $u_0 := 1 - \sum_{k=1}^n u_k$. Then (1.3), (2.1) are equivalent to*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{e(t)} u_k(t,x)\,\mathrm{d}x = \varphi_k(t) \quad (0 \leqslant k \leqslant n,\ t>0), \qquad e(0) = e^0. \tag{2.3}$$

*Proof.* Using (2.2), we have that for $k \in \{1,\dots,n\}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{e(t)} u_k(t,x)\,\mathrm{d}x = \int_0^{e(t)} \partial_t u_k(t,x)\,\mathrm{d}x + e'(t)u_k(t,e(t)) = \partial_x[Au(t,e(t))]_k + e'(t)u_k(t,e(t)). \tag{2.4}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{e(t)} u_0(t,x)\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{e(t)} \left(1 - \sum_{k=1}^n u_k(t,x)\right)\mathrm{d}x = e'(t) - \sum_{k=1}^n \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{e(t)} u_k(t,x)\,\mathrm{d}x. \tag{2.5}$$

We can now show the equivalence: if (2.3) holds, then (2.4) yields (2.1) and (2.5) yields

$$e'(t) = \sum_{k=0}^n \varphi_k(t) \tag{2.6}$$

which, combined with $e(0) = e^0$, implies (1.3). Conversely, (1.3) implies $e(0) = e^0$ and (2.6). Then (2.4) and (2.1) yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{e(t)} u_k(t,x)\,\mathrm{d}x = \varphi_k(t)$$

for $k \in \{1,\dots,n\}$ and the case $k=0$ is obtained from (2.6) and from (2.5). □

We deduce the following compatibility conditions for the reachable states if $\varphi_0 \equiv 0$:

**Corollary 2.2.** *Assume $(e,u)$ is a smooth solution of (1.3)–(1.4) with $\varphi_0 \equiv 0$. Then for any $t>0$,*

$$\int_0^{e(t)} \left(1 - \sum_{k=1}^n u_k(t,x)\right)\mathrm{d}x = \int_0^{e^0} \left(1 - \sum_{k=1}^n u_k^0(x)\right)\mathrm{d}x. \tag{2.7}$$

**Remark 2.3.** *This result justifies the condition (1.10) between the initial and the final conditions in Theorem 1.2. Moreover, it can be used to obtain a numerical algorithm in the case where $\varphi_0 \equiv 0$. This corresponds to the result below.*

**Corollary 2.4.** *Assume $(e,u)$ is a smooth solution of (1.3)–(1.4) in $[0,T]$, with $\varphi_0 \equiv 0$ and such that $(e(t),u(t,x)) \in \mathcal{S}$, for every $t \in [0,T]$ and $x \in [0,e(t)]$. Then for any $t > 0$, $e$ is the solution of the Cauchy problem*

$$e'(t) = \frac{\sum_{k=1}^n (A\partial_x u)_k\,(t,e(t))}{1-\sum_{k=1}^n u_k(t,e(t))} \quad (t \in [0,T]), \qquad e(0) = e^0. \tag{2.8}$$

*Proof.* Differentiating (2.7) with respect to time, we obtain that

$$e'(t)\left(1-\sum_{k=1}^n u_k(t,e(t))\right) = \int_0^{e(t)} \sum_{k=1}^n \partial_t u_k(t,x)\,\mathrm{d}x.$$

Then using the first two equations of (1.4), we deduce that

$$e'(t)\left(1-\sum_{k=1}^n u_k(t,e(t))\right) = \sum_{k=1}^n (A\partial_x u)_k(t,e(t)).$$

Using that

$$\sum_{k=1}^n u_k(t,e(t)) \in (0,1),$$

we deduce the result. □

**Remark 2.5.** *The above result allows us to solve numerically (1.3)–(1.4) if $\varphi_0 \equiv 0$. By the flatness method described in the next section, we obtain a smooth solution of (2.2) that can be approximated through a truncation. Then, we can solve (2.8) by a standard numerical method, provided that $\sum_{k=1}^n u_k < 1$. Note that, with this condition, the Cauchy problem (2.8) admits a unique local solution thanks to the Cauchy-Lipschitz theorem. In the proof of Theorem 1.2, we use directly Corollary 2.2 instead of Corollary 2.4 to obtain the existence and uniqueness of $e$.*

## 2.2 The flatness approach

Theorems 1.1, 1.2 and 1.5 are based on the flatness approach. This strategy was introduced in [11] for the heat equation, and we follow here the same strategy by searching the solution $u$ of (1.4) in the form

$$u(t,x) := \sum_{i=0}^{\infty} \frac{x^{2i}}{(2i)!} A^{-i}\alpha^{(i)}(t) \qquad ((t,x) \in \mathbb{R}^2), \tag{2.9}$$

where $\alpha : \mathbb{R} \to \mathbb{R}^n$ is Gevrey of order $\sigma \in (1,2)$, i.e., there exist $M_1, R_1 > 0$ such that

$$\left|\alpha^{(i)}(t)\right| \leqslant M_1 \frac{(i!)^\sigma}{R_1^i} \qquad (i \in \mathbb{N},\ t \in \mathbb{R}). \tag{2.10}$$

Here and in what follows, we use the notation $|\cdot|$ for the maximum norm of $\mathbb{R}^n$. Formally, we see that for all $(t,x)$, we have the following relations

$$\partial_t u(t,x) - A\partial_x^2 u(t,x) = \sum_{i=0}^{\infty} \frac{x^{2i}}{(2i)!} A^{-i}\alpha^{(i+1)}(t) - \sum_{i=1}^{\infty} \frac{x^{2i-2}}{(2i-2)!} A^{-(i-1)}\alpha^{(i)}(t) = 0,$$
$$\partial_x u(t,0) = 0.$$

To justify the above property, we need to show that $u$ given by (2.9), with $\alpha$ given later by (3.17), is well-defined and is a Gevrey function. First, we set

$$f_i(t,x) := \frac{x^{2i}}{(2i)!} A^{-i}\alpha^{(i)}(t) \qquad ((t,x) \in \mathbb{R}^2,\ i \in \mathbb{N}).$$

If $2i < k$, then $\partial_t^\ell \partial_x^k f_i \equiv 0$, and if $2i \geqslant k$, then (2.10) yields

$$\left|\partial_t^\ell \partial_x^k f_i(t,x)\right| \leqslant M_1 \frac{((\ell+i)!)^\sigma}{R_1^{\ell+i}} \|A^{-1}\|^i \frac{|x|^{2i-k}}{(2i-k)!}, \tag{2.11}$$

where here $\|\cdot\|$ denotes the matrix norm induced by the maximum norm in $\mathbb{R}^n$. Using that $(\ell+i)! \leqslant 2^{\ell+i}\ell! i!$ and the Stirling formula, we deduce that $u$ given by (2.9), with $\alpha$ satisfying (2.10), is well-defined. Furthermore, for every $r > 0$, there exist $M > 0$ such that

$$\left|\partial_t^\ell \partial_x^k u(t,x)\right| \leqslant M \frac{(\ell!)^\sigma}{(R_1/2^\sigma)^\ell} \frac{k!}{r^k} \qquad (\ell, k \in \mathbb{N},\ t > 0,\ |x| \leqslant r).$$

We have thus obtained the following result:

**Proposition 2.6.** *Assume $\alpha$ satisfies* (2.10)*. Then $u$ given by* (2.9) *is Gevrey of class $\sigma$ in time and* 1 *in space and is a solution of* (2.2)*.*

# 3 Proof of the main results

## 3.1 Proof of Theorem 1.1

We consider $u$ given by (2.9) with $\alpha$ given by

$$\alpha(t) := \theta_\sigma\left(\frac{t}{T}\right) u^0 + \left(1 - \theta_\sigma\left(\frac{t}{T}\right)\right) u^1 \in \mathbb{R}^n \qquad (t \in \mathbb{R}), \tag{3.1}$$

where $\theta_\sigma$ is defined by (1.7). We also set

$$e(t) := \frac{e^1 - e^0}{T} t + e^0 \qquad (t \in [0,T]). \tag{3.2}$$

In particular,

$$e'(t) = \frac{e^1 - e^0}{T} > 0 \quad \text{and} \quad e^0 \leqslant e(t) \leqslant e^1 \qquad (t \in [0,T]).$$

Applying Proposition 2.6, we deduce the regularity of $(e,u)$ and that (1.3)–(1.4) hold with the controls given by

$$\varphi(t) := A\partial_x u(t,e(t)) + e'(t)u(t,e(t)) \quad \text{and} \quad \varphi_0(t) := e'(t) - \sum_{k=1}^n \varphi_k(t) \quad (t > 0). \tag{3.3}$$

Moreover, using the properties (1.7) of $\theta_\sigma$, we also have

$$u(t,x) = u^0 \quad \text{if } t \leqslant 0, \qquad u(t,x) = u^1 \quad \text{if } t \geqslant T. \tag{3.4}$$

There exists $d \in (0,1)$ such that

$$d \leqslant \min_{k \in \{1,\dots,n\}} u_k^0 \quad \text{and} \quad \sum_{k=1}^n u_k^0 \leqslant 1 - d. \tag{3.5}$$

Taking $\varepsilon \leqslant d/2$ in (1.9), we deduce from the above estimates and from (3.1) that

$$\frac{d}{2} \leqslant \min_{k \in \{1,\dots,n\}} \alpha_k \quad \text{and} \quad \sum_{k=1}^n \alpha_k \leqslant 1 - \frac{d}{2} \quad \text{in } (0,T). \tag{3.6}$$

Now, we can write

$$\sum_{k=1}^{n} u_k(t,x) = \sum_{k=1}^{n} \alpha_k(t) + \sum_{i=1}^{\infty} \frac{x^{2i}}{(2i)!} \sum_{k=1}^{n} \left[A^{-i}\alpha^{(i)}(t)\right]_k \tag{3.7}$$

and

$$u_k(t,x) = \alpha_k(t) + \sum_{i=1}^{\infty} \frac{x^{2i}}{(2i)!} \left[A^{-i}\alpha^{(i)}(t)\right]_k \qquad (1 \leqslant k \leqslant n). \tag{3.8}$$

Using (1.8) and (1.9), we have

$$\left|\alpha^{(i)}(t)\right| \leqslant \varepsilon M \frac{(i!)^{\sigma}}{(RT)^i} \qquad (i \in \mathbb{N}^*,\ t \in \mathbb{R}) \tag{3.9}$$

and thus

$$\left|\sum_{i=1}^{\infty} \frac{x^{2i}}{(2i)!} \left[A^{-i}\alpha^{(i)}(t)\right]_k\right| \leqslant M\varepsilon \sum_{i=1}^{\infty} \left(\frac{\|A^{-1}\|(e^1)^2}{RT}\right)^i \frac{(i!)^{\sigma}}{(2i)!} \qquad (t \in [0,T],\ x \in [0,e(t)]). \tag{3.10}$$

Taking $\varepsilon$ small enough, we deduce that

$$\frac{d}{4} \leqslant \min_{k\in\{1,\dots,n\}} u_k(t,x) \quad \text{and} \quad \sum_{k=1}^{n} u_k(t,x) \leqslant 1 - \frac{d}{4} \qquad (t \in [0,T],\ x \in [0,e(t)])\,. \tag{3.11}$$

Finally, from (3.3) and (3.8), we have

$$\varphi_k(t) = e'(t)\alpha_k(t) + \sum_{i=1}^{\infty} \frac{e(t)^{2i-1}}{(2i-1)!} \left[A^{-(i-1)}\alpha^{(i)}(t)\right]_k + e'(t) \sum_{i=1}^{\infty} \frac{e(t)^{2i}}{(2i)!} \left[A^{-i}\alpha^{(i)}(t)\right]_k \qquad (t > 0,\ 1 \leqslant k \leqslant n) \tag{3.12}$$

and

$$\varphi_0(t) = e'(t)\left(1 - \sum_{k=1}^{n} \alpha_k(t)\right) - \sum_{i=1}^{\infty} \frac{e(t)^{2i-1}}{(2i-1)!} \sum_{k=1}^{n} \left[A^{-(i-1)}\alpha^{(i)}(t)\right]_k - e'(t) \sum_{i=1}^{\infty} \frac{e(t)^{2i}}{(2i)!} \sum_{k=1}^{n} \left[A^{-i}\alpha^{(i)}(t)\right]_k \quad (t > 0). \tag{3.13}$$

The relations in (3.2) and (3.6) yield

$$e'(t)\alpha_k(t) \geqslant \frac{e^1 - e^0}{T}\frac{d}{2} > 0, \quad e'(t)\left(1 - \sum_{k=1}^{n} \alpha_k(t)\right) \geqslant \frac{e^1 - e^0}{T}\frac{d}{2} > 0 \qquad (t \in [0,T]). \tag{3.14}$$

On the other hand, for any $k \in \{1,\dots,n\}$,

$$\left|\sum_{i=1}^{\infty} \frac{e(t)^{2i-1}}{(2i-1)!} \left[A^{-(i-1)}\alpha^{(i)}(t)\right]_k\right| \leqslant M\varepsilon \sum_{i=1}^{\infty} \frac{\|A^{-1}\|^{i-1}\left(e^1\right)^{2i-1}}{(RT)^i} \frac{(i!)^{\sigma}}{(2i-1)!} \qquad (t \in [0,T]) \tag{3.15}$$

and

$$\left|\sum_{i=1}^{\infty} \frac{e(t)^{2i}}{(2i)!} \left[A^{-i}\alpha^{(i)}\right]_k (t)\right| \leqslant M\varepsilon \sum_{i=1}^{\infty} \left(\frac{\|A^{-1}\|(e^1)^2}{RT}\right)^i \frac{(i!)^{\sigma}}{(2i)!} \qquad (t \in [0,T]). \tag{3.16}$$

Taking $\varepsilon$ small enough, and combining (3.12)–(3.16), we deduce (1.2) in $(0,T)$.

This concludes the proof of Theorem 1.1. □

### 3.2 Proof of Theorem 1.2

We consider $u$ given by (2.9) with $\alpha$ defined through the function $\theta_\sigma$ given in (1.7):

$$\alpha(t) := \theta_\sigma\left(\frac{t}{T}\right)u^0 + \left(1-\theta_\sigma\left(\frac{t}{T}\right)\right)u^1 \in \mathbb{R}^n \quad (t\in\mathbb{R}). \tag{3.17}$$

Using (1.8), we have

$$\left|\alpha^{(i)}(t)\right| \leqslant M_1\frac{(i!)^\sigma}{(RT)^i} \qquad (i\in\mathbb{N}^*,\ t\in\mathbb{R}) \tag{3.18}$$

Applying Proposition 2.6, we deduce the regularity of $u$, and that it satisfies (2.2), and

$$u(0,x)=u^0, \quad u(T,x)=u^1 \qquad (x\in\mathbb{R}).$$

Note that we have for $(t,x)\in\mathbb{R}^2$,

$$\sum_{k=1}^n u_k(t,x) = \sum_{k=1}^n \alpha_k(t) + \sum_{i=1}^\infty \frac{x^{2i}}{(2i)!}\sum_{k=1}^n\left[A^{-i}\alpha^{(i)}(t)\right]_k \tag{3.19}$$

and

$$u_k(t,x) = \alpha_k(t) + \sum_{i=1}^\infty \frac{x^{2i}}{(2i)!}\left[A^{-i}\alpha^{(i)}(t)\right]_k \qquad (1\leqslant k\leqslant n). \tag{3.20}$$

From (3.18), we deduce that if $|x|\leqslant r$ and $T\geqslant 1$, then

$$\left|\sum_{i=1}^\infty \frac{x^{2i}}{(2i)!}\left[A^{-i}\alpha^{(i)}(t)\right]_k\right| \leqslant \frac{f(r)}{T}, \qquad \left|\sum_{i=1}^\infty \frac{x^{2i}}{(2i)!}\sum_{k=1}^n\left[A^{-i}\alpha^{(i)}(t)\right]_k\right| \leqslant \frac{nf(r)}{T}, \tag{3.21}$$

where

$$f(r) := M_1\sum_{i=1}^\infty\left(\frac{r^2\|A^{-1}\|}{R}\right)^i\frac{(i!)^\sigma}{(2i)!} < \infty. \tag{3.22}$$

Now, let us consider $d\in(0,1)$ such that

$$d\leqslant \min_{k\in\{1,\dots,n\}} u_k^0, \quad \sum_{k=1}^n u_k^0 \leqslant 1-d, \quad d\leqslant \min_{k\in\{1,\dots,n\}} u_k^1, \quad \text{and} \quad \sum_{k=1}^n u_k^1\leqslant 1-d, \tag{3.23}$$

Then, the above relations and (3.17) yield

$$d\leqslant \min_{k\in\{1,\dots,n\}}\alpha_k \quad \text{and} \quad \sum_{k=1}^n \alpha_k \leqslant 1-d. \tag{3.24}$$

We then set

$$r := \frac{2}{d}\,e^0\left(1-\sum_{k=1}^n u_k^0\right) > 0 \tag{3.25}$$

and we consider $T\geqslant 1$ such that

$$\frac{nf(r)}{T}\leqslant\frac{d}{2}.$$

Combining (3.19)–(3.21), (3.24) and the above relation, we deduce

$$\frac{d}{2}\leqslant \min_{k\in\{1,\dots,n\}} u_k(t,x) \quad \text{and} \quad \sum_{k=1}^n u_k(t,x)\leqslant 1-\frac{d}{2} \qquad (t\geqslant 0,\ |x|\leqslant r). \tag{3.26}$$

In particular, the function

$$U(t,x) := \int_0^x \left(1 - \sum_{k=1}^n u_k(t,y)\right) \mathrm{d}y$$

is increasing in $x \in [0,r]$, and satisfies $U(t,0) = 0$, and, with (3.25),

$$U(t,r) \geqslant r\frac{d}{2} = e^0 \left(1 - \sum_{k=1}^n u_k^0\right).$$

Therefore, there exists a unique $e(t) \in [0,r]$ such that

$$\int_0^{e(t)} \left(1 - \sum_{k=1}^n u_k(t,x)\right) \mathrm{d}x = e^0 \left(1 - \sum_{k=1}^n u_k^0\right).$$

One can check that $U$ is Gevrey of order $\sigma$ in time and 1 in space. Thus using the Implicit Gevrey map theorem (see [18, Theorem 8.1]), we deduce that $e$ is Gevrey of order $\sigma$. We have $e(0) = e^0$ and if we set

$$\varphi_k(t) := \frac{\mathrm{d}}{\mathrm{d}t} \int_0^{e(t)} u_k(t,x)\,\mathrm{d}x \quad (1 \leqslant k \leqslant n,\ t > 0)$$

then we see that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^{e(t)} u_0(t,x)\,\mathrm{d}x = 0$$

and thus (2.3) holds and from Lemma 2.1, we deduce that $(e,u)$ is a solution of (1.3)–(1.4). From (3.26) and $e(t) \leqslant r$, we deduce that $(e(t),u(t)) \in \mathcal{S}$ for all $t \in [0,T]$.
This concludes the proof of Theorem 1.2. □

## 3.3 Proof of Proposition 1.4

We now prove that the controllability of the system can fail for small times.

We denote by $A^\top$ the transpose of $A$ and by hypothesis, there exist $a \in \mathbb{R}^n$, $a \neq 0$ and $\lambda \neq 0$ such that

$$A^\top a = \lambda a.$$

Since $a \neq 0$, there exist $u^0, u^1 \in \mathbb{R}^n$ such that

$$u^0 \cdot a \neq u^1 \cdot a,$$

together with

$$\min_{k\in\{1,\dots,n\}} u_k^0 > 0, \quad \min_{k\in\{1,\dots,n\}} u_k^1 > 0, \quad \sum_{k=1}^n u_k^0 < 1, \quad \sum_{k=1}^n u_k^1 < 1.$$

We also choose $e^0 > 0$ and $e^1 > 0$ such that (1.10) holds. Let us consider a control $(\varphi_k)_k$ with $\varphi_0 \equiv 0$ and a solution $(e,u)$ of (1.3)–(1.4) satisfying $(e(t),u(t,x)) \in \mathcal{S}$, for every $t \in [0,T]$ and $x \in (0,e(t))$, and

$$e(T) = e^1, \quad u(T,x) = u^1 \quad (x \in (0,e^1)).$$

We set

$$v := u \cdot a + \sum_{k=1}^n |a_k|, \quad v^0 := u^0 \cdot a + \sum_{k=1}^n |a_k|, \quad v^1 := u^1 \cdot a + \sum_{k=1}^n |a_k|$$

and we have $v^0 \neq v^1$ and $v^0, v^1 > 0$. Since $(e(t), u(t,x)) \in \mathcal{S}$ for all $t \in [0,T]$, we have $v(t,x) > 0$, for all $(t,x)$. From Corollary 2.2, we deduce

$$e(t) > L := \int_0^{e^0} \left(1 - \sum_{k=1}^n u_k^0(x)\right) \mathrm{d}x > 0.$$

Taking the inner product of the first equation of (1.4) with $a$, and using the above results, we deduce that

$$\begin{cases} \partial_t v - \lambda \partial_x^2 v = 0 & \text{in } (0,T) \times (0,L), \\ v(t,0) > 0, \ v(t,L) > 0 & t \in (0,T), \\ v(0,x) = v^0 > 0, \quad v(T,x) = v^1 > 0 & x \in (0,L). \end{cases}$$

If $\lambda > 0$, we can apply [9, Theorem 2.1] to deduce the existence of $T_0(v^0, v^1, L, \lambda) > 0$ such that

$$T \geqslant T_0.$$

If $\lambda < 0$, we can perform the change of variable $t \mapsto T - t$ and apply [9, Theorem 2.1] to deduce the result. This concludes the proof of Proposition 1.4. □

### 3.4 Proof of Theorem 1.5

We end this section by the proof of the controllability result in arbitrarily small time, using $n+1$ controls which are not subject to any sign constraint.

Using $\varphi_0$, we can control $e$ independently of $u$. More precisely, let us set

$$e(t) := e^0 \theta_\sigma\left(\frac{3t}{T}\right) + \varepsilon T^{1/2}\left(\theta_\sigma\left(1 - \frac{3t}{T}\right) - \theta_\sigma\left(3 - \frac{3t}{T}\right)\right) + e^1 \theta_\sigma\left(3 - \frac{3t}{T}\right), \tag{3.27}$$

with $\varepsilon \in (0,1)$ small enough that we specify later. We define $u$ with (2.9) where $\alpha$ is defined through the function $\theta_\sigma$ given in (1.7):

$$\alpha(t) := \theta_\sigma\left(\frac{3t}{T} - 1\right) u^0 + \left(1 - \theta_\sigma\left(\frac{3t}{T} - 1\right)\right) u^1 \in \mathbb{R}^n \qquad (t \in \mathbb{R}). \tag{3.28}$$

We have that $\alpha$ satisfies (3.18). Then, applying Proposition 2.6, we deduce the regularity of $(e,u)$ and that it satisfies (1.3)–(1.4) with the controls given by

$$\varphi(t) := A\partial_x u(t, e(t)) + e'(t) u(t, e(t)) \quad \text{and} \quad \varphi_0(t) := e'(t) - \sum_{k=1}^n \varphi_k(t) \quad (t > 0). \tag{3.29}$$

Using the properties (1.7) of $\theta_\sigma$, we also have

$$u(t,x) = u^0 \quad \text{if } t \leqslant \frac{T}{3}, \qquad u(t,x) = u^1 \quad \text{if } t \geqslant \frac{2T}{3}, \tag{3.30}$$

and

$$e(t) = e^0 \quad \text{if } t \leqslant 0, \qquad e(t) = \varepsilon T^{1/2} \quad \text{if } t \in \left(\frac{T}{3}, \frac{2T}{3}\right), \qquad e(t) = e^1 \quad \text{if } t \geqslant T \tag{3.31}$$

and $e(t) \geqslant \min(e^0, e^1, \varepsilon T^{1/2}) > 0$ in the other intervals. It only remains to show that

$$0 < \min_{k \in \{1,\dots,n\}} u_k(t,x) \quad \text{and} \quad \sum_{k=1}^n u_k(t,x) < 1 \qquad (t \in (0,T), \ x \in (0, e(t))). \tag{3.32}$$

Using (3.30), we deduce the above relations for $t \notin \left(\frac{T}{3}, \frac{2T}{3}\right)$. If $t \in \left(\frac{T}{3}, \frac{2T}{3}\right)$, we use (2.9) to write (3.19), with

$$\sum_{k=1}^{n} \alpha_k(t) = \theta_\sigma\left(\frac{3t}{T} - 1\right)\sum_{k=1}^{n} u_k^0 + \left(1 - \theta_\sigma\left(\frac{3t}{T} - 1\right)\right)\sum_{k=1}^{n} u_k^1 \ \in (0,1) \qquad (t \in (0,T)).$$

As in the proof of Theorem 1.2, we only need to estimate the sum in the right-hand side of (3.19): using (3.31), $e(t) = \varepsilon T^{1/2}$ and combining it with (3.18), we obtain

$$\left|\sum_{i=1}^{\infty} \frac{x^{2i}}{(2i)!}\sum_{k=1}^{n}\left[A^{-i}\alpha^{(i)}(t)\right]_k\right| \leqslant M_1 n\varepsilon^2 \sum_{i=1}^{\infty}\left(\frac{\|A^{-1}\|}{R}\right)^i \frac{(i!)^\sigma}{(2i)!} \qquad (x \in (0, \varepsilon T^{1/2})). \tag{3.33}$$

In particular, taking $\varepsilon$ small enough, we conclude that

$$\sum_{k=1}^{n} u_k \in (0,1) \qquad \left(t \in \left(\frac{T}{3}, \frac{2T}{3}\right),\ x \in (0, e(t))\right).$$

The other relation in (3.32) is obtained similarly by taking $\varepsilon$ possibly smaller.
This concludes the proof of Theorem 1.5. □

# 4 Numerical illustrations

Within the flatness method, one can deduce naturally a numerical algorithm to approximate the solution of (1.4). Assuming that the solution $u$ is given by (2.9), where $\alpha : \mathbb{R} \to \mathbb{R}^n$ is a Gevrey function of order $\sigma \in (1,2)$ satisfying (2.10), one can approach $u$ by its partial sums:

$$u_N(t,x) := \sum_{i=0}^{N} \frac{x^{2i}}{(2i)!} A^{-i}\alpha^{(i)}(t) \quad ((t,x) \in \mathbb{R}^2). \tag{4.1}$$

The following result gives an estimate of the error between $u$ and $u_N$. The proof is standard, but we recall it for completeness.

**Lemma 4.1.** *Assume $\sigma \in (1,2)$, and let $L > 0$. There exists a constant $C > 0$ such that for any Gevrey function $\alpha$ satisfying* (2.10)*, there exists $N_0 \in \mathbb{N}^*$ such that for any $N \geqslant N_0$, $u$ and $u_N$, given respectively by* (2.9) *and* (4.1)*, satisfy*

$$\|u - u_N\|_{L^\infty(\mathbb{R}\times[-L,L])} \leqslant CM_1 \frac{\sqrt{N}}{\ln N}\left(\frac{L^2\left\|A^{-1}\right\|}{4R_1}\right)^N \frac{1}{(N!)^{2-\sigma}}. \tag{4.2}$$

*Proof.* From (2.10) and the Stirling formula, there exist a constant $C > 0$ and $N_0 \in \mathbb{N}^*$ such that for any $L > 0$, $x \in [-L, L]$ and $i \geqslant N_0$,

$$\left|\frac{x^{2i}}{(2i)!} A^{-i}\alpha^{(i)}(t)\right| \leqslant CM_1\left(\frac{L^2\left\|A^{-1}\right\| e^{2-\sigma}}{4R_1}\right)^i i^{i(\sigma-2)+\frac{1}{2}(\sigma-1)}. \tag{4.3}$$

We thus set

$$K := \frac{L^2\left\|A^{-1}\right\| e^{2-\sigma}}{4R_1}, \quad g(y) := (2-\sigma)y\ln y - \frac{\sigma-1}{2}\ln y - y\ln K$$

so that (4.3) can be written as

$$\left|\frac{x^{2i}}{(2i)!} A^{-i}\alpha^{(i)}(t)\right| \leqslant CM_1 e^{-g(i)}. \tag{4.4}$$

Using that $\sigma \in (1,2)$, we deduce that taking $N_0$ large enough, we have

$$g'(y) \geqslant g'(N) \geqslant \frac{2-\sigma}{2}\ln(N) > 0 \quad (y \geqslant N \geqslant N_0).$$

Therefore, we deduce from the above relation, together with (2.9), (4.1) and (4.4) that

$$\|u - u_N\|_{L^\infty(\mathbb{R}\times[-L,L])} \leqslant CM_1 \int_N^\infty e^{-g(y)}\, dy \leqslant \frac{CM_1}{\ln(N)} e^{-g(N)}.$$

We conclude the proof of this lemma by using the Stirling formula. □

We will now give different illustrations of our results. All the calculations are performed using `Matlab`. For $(t,x) \in \mathbb{R}^2$, $u(t,x)$ from (2.9) is approximated by (4.1). Thanks to the previous proposition, one does not need to take $N$ to be very large (most of the time, we use $N = 100$ nonetheless) one can observe in Figure 3a that the convergence is fast. The derivatives of $\alpha$ are computed inductively as in the appendix of [14]. As $\theta_\sigma$ from (1.7) converges pointwise to a Heaviside function when $\sigma$ goes to 1 in order for the numerical computation to go well and to avoid an explosion (see Figure 3b) one will benefit from taking $\sigma$ close to 2 therefore, we chose to work with $\sigma = 2 - \frac{1}{32}$.

**Remark 4.2.** *According to [3] (see also [1]) a matrix form which is of physical interest for this system is given by $A(u) = (A_{i,j}(u))_{i,j\in\{1,\dots,n\}}$ with*

$$\begin{cases} A_{i,i}(u) = \displaystyle\sum_{1\leqslant k\neq i\leqslant n} (K_{i,k} - K_{k,0})u_k + K_{k,0}, & \text{for } 1 \leqslant i \leqslant n, \\ A_{i,j}(u) = -(K_{i,j} - K_{i,0})u_i, & \text{for } 1 \leqslant i \neq j \leqslant n, \end{cases}$$

*where for every $0 \leqslant i \neq j \leqslant n$, $K_{i,j} = K_{j,i} > 0$.*

All the computations are done with $n = 4$, and $A = A(u_0)$ constructed as in the previous remark with:

$$(K_{i,j})_{i,j\in\{1,\dots,n\}} = \begin{pmatrix} * & 2 & 2 & 5 \\ 2 & * & 5 & 7 \\ 2 & 5 & * & 3 \\ 5 & 7 & 3 & * \end{pmatrix} \quad \text{and} \quad (K_{i,0})_{i\in\{1,\dots,n\}} = \begin{pmatrix} 5 \\ 8 \\ 7 \\ 9 \end{pmatrix}.$$

## 4.1 Illustrations for Theorem 1.2.

For this part, we take,

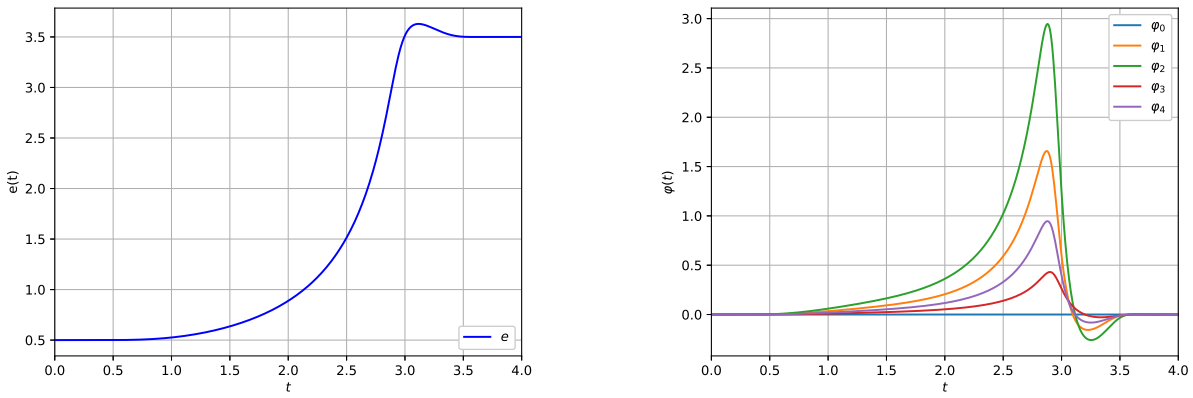
$$e^0 = 1/2, \qquad u^0 = \frac{1}{10}\begin{pmatrix} 1 \\ 2 \\ 2.5 \\ 1 \end{pmatrix} \quad \text{and} \quad u^1 = \frac{1}{10}\begin{pmatrix} 2.5 \\ 4.5 \\ 1 \\ 1.5 \end{pmatrix}. \tag{4.5}$$

Recall that when $\varphi_0$ is set to 0, $u^0$, $u^1$, $e^0$ and $e^1$ satisfy the compatibility condition (2.4). With the values given in (4.5), this leads to

$$e^1 = 7/2. \tag{4.6}$$

Here, we compute $e$ as the solution of the Cauchy problem given in Corollary 2.4 using the `ode45` function from `Matlab` by approximating $u$ and $\partial_x u$ with truncated series as described before, $\varphi$ and $\varphi_0$ are then computed from $u_N$ and $e$ using (3.3). The result can be seen in Figures 1 and 2. In Figure 1a we can see that the condition (1.10) is numerically verified (as $e_1$ isn't used in the algorithm that computes $e$), and in Figure 1b, the fact that $\varphi_0 \equiv 0$ is also numerically verified. In Figure 2 we give the graphs of $u(t,\cdot)$ for different values of $t$ which allows us to see that the transition from $u^0$ to $u^1$ is quite smooth.

In Figure 3a one can observe the fast convergence of $\varphi_1$ with respect to $N$. In fact the control has *numerically converged* for $N = 4$. Finally, in Figure 3b we illustrate the dependence of the control with respect to $\sigma$. We see in particular the fact that for $\sigma$ close to 1, the control becomes singular. In fact, at a given time $T > 0$, taking $\sigma$ close to 1 leads to a violation of the state constraints. Roughly speaking, the more $\sigma$ is small the larger $T$ has to be. As example, the state constraint is still violated for $\sigma = 1.3$ and $T = 100$.




(a) Graph of $e$.

(b) Graphs of the controls.

Figure 1: Numerical test with $T = 4$, $N = 100$ and $\sigma = 2 - \frac{1}{32}$. The values of $e^0$, $u^0$ and $u^1$ are given in (4.5).

### 4.2 Illustration of Theorem 1.5

To numerically illustrate Theorem 1.5, we take $e^0$, $u^0$, $e^1$ and $u^1$ as in the previous paragraph, i.e., they are given by (4.5) and (4.6). Here, $e$ is given by (3.27) and $\varphi_0, \dots, \varphi_n$ are obtained through (3.3). In (3.27), we chose $\varepsilon = 0.1$, recall that this value has to be chosen small enough to ensure the state constraints on $u$.

Using $\varphi_0$ we are able to control our system from the same steady states as in the previous case and with the same parameters in arbitrarily small time (we take $T = 0.25$ for the numerical illustration). Recall that in the $n$ controls case a minimal time is required to ensure that $\sum_{i=1}^{n} u_i < 1$, and this constraint is violated even for $T = 3$ (when $\varphi_0 \equiv 0$). On Figure 4, we have displayed the trajectory of $e$ and the controls $\varphi_0, \dots, \varphi_4$.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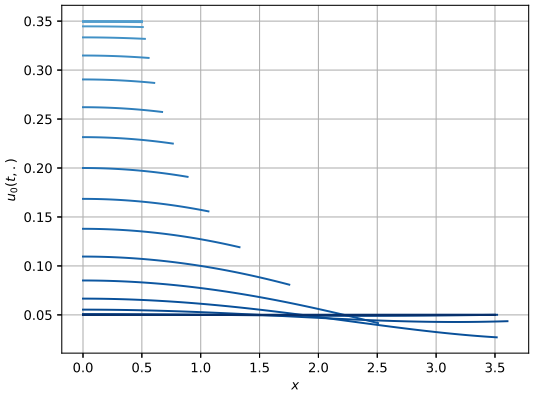
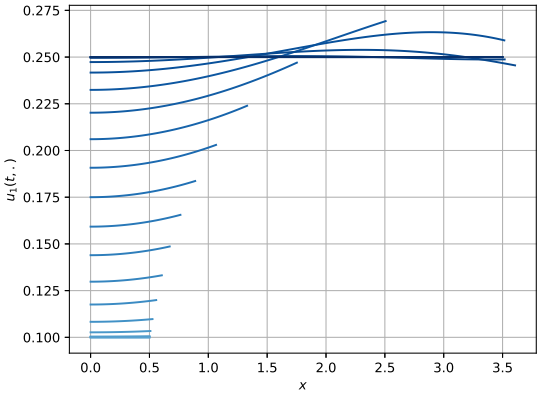
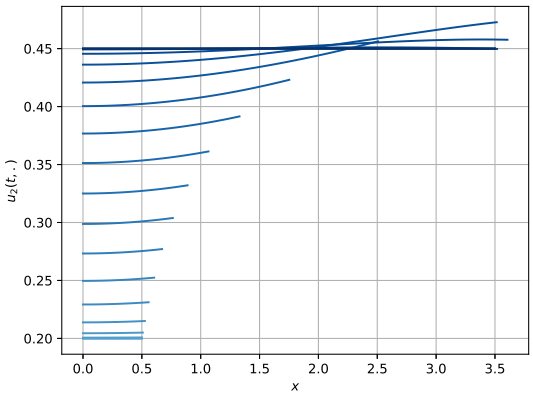


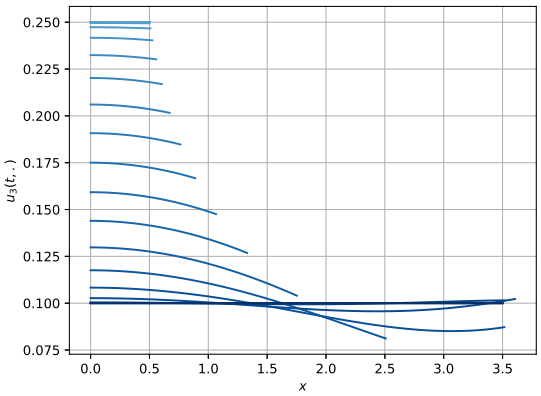

(a) Representation of $u_0(t, \cdot)$.



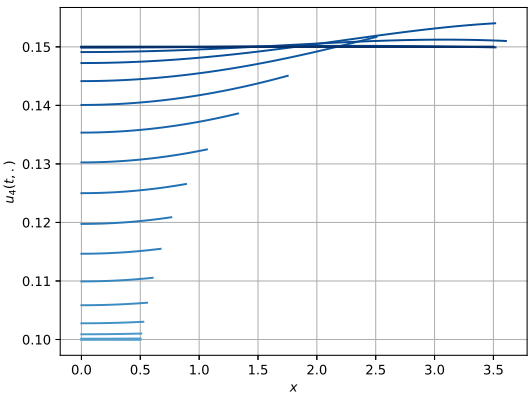

(b) Representation of $u_1(t, \cdot)$.



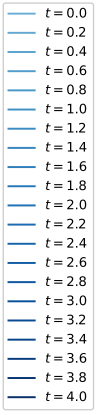

(c) Representation of $u_2(t, \cdot)$.




(d) Representation of $u_3(t, \cdot)$.

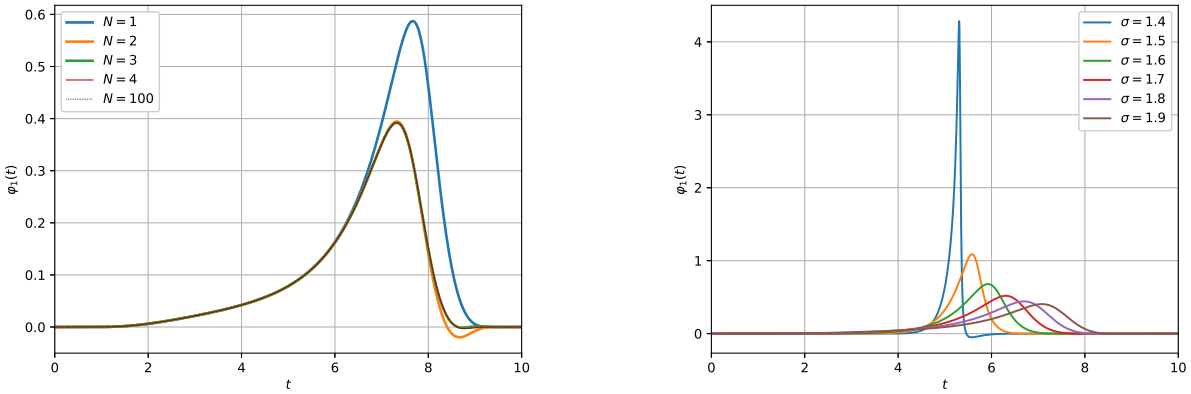



(e) Representation of $u_4(t, \cdot)$.




(f) Legend of the previous graphs.

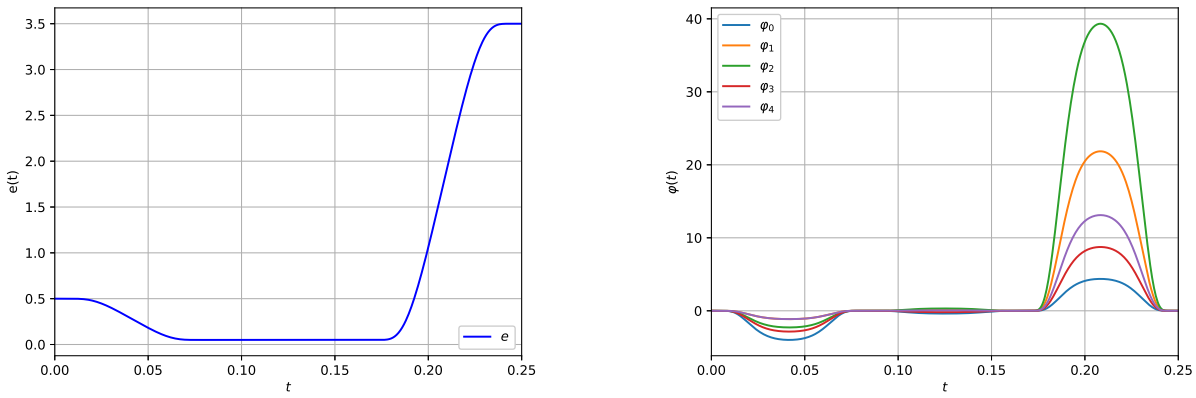
Figure 2: Representation of $u(t, \cdot)$ for different values of $t$. The parameters are given in Figure 1.




(a) Convergence of $\varphi_1$ with respect to $N$ with $T = 10$ and $\sigma = 2 - \frac{1}{32}$.

(b) Behaviour $\varphi_1$ with respect to $\sigma$ with $T = 10$ and $N = 100$.

Figure 3: Dependency of $\varphi_1$ with respect to $N$ or $\sigma$. The values of $e^0$, $u^0$ and $u^1$ are given in (4.5).




(a) Graph of $e$.

(b) Graphs of the controls.

Figure 4: Numerical test with $T = 0.25$, $N = 100$, $\varepsilon = 0.1$ (see (3.27)) and $\sigma = 2 - \frac{1}{32}$. The values of $e^0$, $u^0$, $e^1$ and $u^1$ are given in (4.5) and (4.6).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