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High-dimensional variable clustering based on sub-asymptotic maxima of a weakly dependent random process

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Abstract.

We propose a new class of models for variable clustering called Asymptotic Independent block (AI-block) models, which defines population-level clusters based on the independence of the maxima of a multivariate stationary mixing random process among clusters. This class of models is identifiable, meaning that there exists a maximal element with a partial order between partitions, allowing for statistical inference. We also present an algorithm for recovering the clusters of variables without specifying the number of clusters *a priori*. Our work provides some theoretical insights into the consistency of our algorithm, demonstrating that under certain conditions it can effectively identify clusters in the data with a computational complexity that is polynomial in the dimension. This implies that groups can be learned nonparametrically in which block maxima of a dependent process are only sub-asymptotic.

Keywords: Consistent estimation, Extreme value theory, High dimensional models, Variable clustering.

MSC Codes: 60G70; 62H05; 62M99

1. Introduction

Multivariate extremes arise when two or more extreme events occur simultaneously. These events are of prime interest to assess natural hazard, stemming from heavy rainfall, wind storms and earthquakes since they are driven by joint extremes of several of meteorological variables. It is well known from Sklar's theory (Sklar 1959) that multivariate distributions can be decomposed into two distinct parts: the analysis of marginal distributions and the analysis of the dependence structure described by the copula function. Results from multivariate extreme value theory show that the possible dependence structure of extremes satisfy certain constraints. Indeed, the dependence structure may be described in various equivalent ways (Beirlant et al. 2004; De Haan and Ferreira 2006; Resnick 2008): by the exponent measure (Balkema and Resnick 1977), by the Pickands dependence function (Pickands 1981), by the stable tail dependence function (Huang 1992), by the madogram (Naveau et al. 2009; Boulin et al. 2022), and by the extreme value copula (Gudendorf and Segers 2010).

While the modeling of univariate and low-dimensional extreme events has been well-studied, it remains a challenge to model multivariate extremes, particularly when multiple rare events may occur simultaneously. Recent research in this area has focused on connecting the study of multivariate extremes to modern statistical and machine learning techniques. This has involved the development of new methods for characterizing complex dependence structures between extreme observations, such as sparsity-based approaches (Goix, Sabourin, Cléménçon, et al. 2015; Meyer and Wintenberger 2021; Simpson, Wadsworth, and Tawn 2020), conditional independence and graphical models (Engelke and Hitz 2020; Gissibl and Klüppelberg 2018; Segers 2020), dimensionality reduction (Chautru 2015; Drees and Sabourin 2021), and clustering methods (Cooley and Thibaud 2019; Fomichov and Ivanovs 2022; Janßen and Wan 2020). Our work is aligned with this direction of

research as we propose a clustering algorithm for learning the dependence structure of multivariate extremes and, withal, to bridge important ideas from modern statistics and machine learning to the framework of extreme-value theory. Our approach is remotely related to extremal graphical models. The probabilistic framework of this paper can effectively be seen as a disconnected extremal graph where the connected components are mutually independent of each other (see for example section 8 in Engelke, Ivanovs, and Strokorb 2022).

It is possible to perform clustering on $\mathbf{X}_1, \dots, \mathbf{X}_n$, where n is the number of observations of a random vector $\mathbf{X} \in \mathbb{R}^d$, through two different approaches: by partitioning the set of row indices $\{1, \dots, n\}$ or by partitioning the set of column indices $\{1, \dots, d\}$. The first problem is known as the data clustering problem, while the second is called the variable clustering problem, which is the focus of this paper. In data clustering, observations are drawn from a mixture distribution, and clusters correspond to different realizations of the mixing distribution, which is a distribution over all of \mathbb{R}^d . In the framework of independent and identically distributed (i.i.d.) replications, (Pollard 1981) showed that k -means clustering is strongly consistent, and this result was replicated in the context of extremes by (Janßen and Wan 2020) for spherical k -means.

The problem of variable clustering (see, e.g., Bunea et al. 2020; Eisenach et al. 2020) involves grouping similar components of a random vector $\mathbf{X} = (X^{(1)}, \dots, X^{(d)})$ into clusters. The goal is to recover these clusters from observations $\mathbf{X}_1, \dots, \mathbf{X}_n$. Instead of clustering similar observations based on a dissimilarity measure, the focus is on defining cluster models that correspond to subsets of the components $X^{(j)}$ of $\mathbf{X} \in \mathbb{R}^d$. The goal is to cluster similar variables such that variables within the same cluster are more similar to each other than they are to variables in other clusters. Variable clustering is of particular interest in the study of weather extremes, with examples in the literature on regionalization (Bador et al. 2015; Bernard et al. 2013; Saunders, Stephenson, and Karoly 2021), where spatial phenomena are observed at a limited number of sites. A specific case of interest is clustering these sites according to their extremal dependencies. This can be done using techniques such as k -means or hierarchical clustering with a dissimilarity measure designed for extremes. However, the statistical properties of these procedures have not been extensively studied, and it is not currently known which probabilistic models on \mathbf{X} can be estimated using these techniques. In this paper, we consider model-based clustering, where the population-level clusters are well-defined, offering interpretability and a benchmark to evaluate the performance of a specific clustering algorithm.

The assumption that data are realizations of independent and identically distributed (i.i.d.) random variables is a fundamental assumption in statistical theory and modeling. However, this assumption is often unrealistic for modern datasets or the study of time series. Developing methods and theory to handle departures from this assumption is an important area of research in statistics. One common approach is to assume that the data are drawn from a multivariate stationary and mixing random process, which implies that the dependence between observations weakens over the trajectory. This assumption is widely used in the study of non-i.i.d. processes.

Our contribution is twofold. First, we develop a probabilistic setting for Asymptotic Independent block (AI-block) models to address the problem of clustering extreme values of the target vector. These models are based on the assumption that clusters of components of a multivariate random process are independent relative to their extremes. This approach has the added benefit of being amenable to theoretical analysis, and we show that these models are identifiable (see Theorem 1). Second, we motivate and derive an algorithm specifically designed for these models (see Algorithm (ECO)). We analyze its performance in terms of exact cluster recovery for minimally separated clusters, using a cluster separation metric (see Theorem 4). The issue is investigated in the context of nonparametric estimation using the block maxima method, where the block length is a tuning parameter.

Notations All bold letters \mathbf{x} correspond to vector in \mathbb{R}^d . By considering $B \subseteq \{1, \dots, d\}$, we denote the $|B|$ -subvector of \mathbf{x} by $\mathbf{x}^{(B)} = (X^{(j)})_{j \in B}$. We define by $\mathbf{X} \in \mathbb{R}^d$ a random vector with law H and $\mathbf{X}^{(B)}$ a random subvector of \mathbf{X} with law

$$H^{(B)}(\mathbf{x}^{(B)}) = H(\mathbf{1}, \mathbf{x}^{(B)}, \mathbf{1}), \quad (X^{(j)})_{j \in B} \in [0, 1]^{|B|},$$

where $(\mathbf{1}, \mathbf{x}^{(B)}, \mathbf{1})$ has its j th component equal to $x^{(j)}\mathbb{1}_{\{j \in B\}} + \mathbb{1}_{\{j \notin B\}}$. In a similar way, we note $(\mathbf{0}, \mathbf{x}^{(B)}, \mathbf{0})$ the vector in \mathbb{R}^d which equals $x^{(j)}$ if $j \in B$ and 0 otherwise. When $B = \{1, \dots, d\}$, we will write H instead of $H^{\{\{1, \dots, d\}\}}$. Classical inequalities of vectors such as $\mathbf{x} > \mathbf{0}$ should be understood componentwise. Weak convergence of processes are denoted by \rightsquigarrow . The notation δ_x corresponds to the Dirac measure at x . Let $O = \{O_g\}_{g=1, \dots, G}$ be a partition of $\{1, \dots, d\}$ into G groups and let $s : \{1, \dots, d\} \rightarrow \{1, \dots, G\}$ be a variable index assignment function, thus $O_g = \{a \in \{1, \dots, d\} : s(a) = g\} = \{i_{g,1}, \dots, i_{g,d_g}\}$ with $d_1 + \dots + d_G = d$. Using these notations, the variable $X^{(i_{g,\ell})}$ should be read as the ℓ th element from the g th cluster. Let $\mathbf{X}^{(O_g)}$, $g \in \{1, \dots, G\}$ be extreme value random vectors with $\mathbf{X} = (\mathbf{X}^{(O_1)}, \dots, \mathbf{X}^{(O_G)})$, we say that $\mathbf{X}^{(O_1)}, \dots, \mathbf{X}^{(O_G)}$ are independent if and only if

$$H(\mathbf{x}) = \prod_{g=1}^G H^{(O_g)}(\mathbf{x}^{(O_g)}), \quad \mathbf{x} \in \mathbb{R}^d.$$

The structure of this paper is as follows. In Section 2, we provide background on extreme-value theory and weakly dependent random processes, and describe the probabilistic framework of AI-block models. We show that these models are identifiable and provide a series of equivalent characterizations. In Section 3, we develop a new clustering algorithm for AI-block models and prove that it can recover the target partition with high probability. We provide a process that satisfies our probabilistic and statistical assumptions in Section 4, and compare our approach to existing state-of-the-art methods in Section 5. We illustrate the finite sample performance of our approach on simulated datasets in Section 6. The proofs of our main, auxiliary, and supplementary results are provided in Appendix A, Appendix B, and Appendix C of the supplementary material, respectively. Additional figures and numerical results are presented in Appendix C.3. Throughout the paper, readers will be directed to appendices Appendix B or Appendix C for additional materials when necessary. Otherwise, all the necessary materials can be found in Appendix A.

2. A model for variable clustering

2.1 Background setting

Consider $\mathbf{Z} = (Z^{(1)}, \dots, Z^{(d)})$ and $\mathbf{Z}_t = (Z_t^{(1)}, \dots, Z_t^{(d)})$, where $t \in \mathbb{Z}$ be respectively a d -dimensional random vector with law F and a strictly stationary multivariate random process distributed according to \mathbf{Z} . For the process $(\mathbf{Z}_t, t \in \mathbb{Z})$, let

$$\mathcal{F}_k = \sigma(\mathbf{Z}_t, t \leq k), \quad \text{and} \quad \mathcal{G}_k = \sigma(\mathbf{Z}_t, t \geq k),$$

be respectively the natural filtration and ‘‘reverse’’ filtration of $(\mathbf{Z}_t, t \in \mathbb{Z})$. Many types of mixing conditions exist in the literature. The weakest among those most commonly used is called strong or α -mixing. Specifically, for two σ -fields \mathcal{A}_1 and \mathcal{A}_2 of a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ the α -mixing coefficient of a multivariate random process is defined for $\ell \geq 1$

$$\alpha(\ell) = \sup_{t \in \mathbb{Z}} \alpha(\mathcal{F}_t, \mathcal{G}_{t+\ell}), \tag{1}$$

where

$$\alpha(\mathcal{A}_1, \mathcal{A}_2) = \sup_{A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2} |\mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1)\mathbb{P}(A_2)|.$$

For any process $(\mathbf{Z}_t, t \in \mathbb{Z})$, let

$$\beta(\mathcal{A}_1, \mathcal{A}_2) = \sup \frac{1}{2} \sum_{i,j \in I \times J} |\mathbb{P}(A_i \cap B_j) - \mathbb{P}(A_i)\mathbb{P}(B_j)|,$$

where the sup is taken over all finite partitions $(A_i)_{i \in I}$ and $(B_j)_{j \in J}$ of Ω with the sets A_i in \mathcal{A}_1 and the sets B_j in \mathcal{A}_2 . The β -mixing (or completely regular) coefficient is defined as follows

$$\beta(\ell) = \sup_{t \in \mathbb{Z}} \beta(\mathcal{F}_t, \mathcal{G}_{t+\ell}). \quad (2)$$

By considering

$$\varphi(\mathcal{A}_1, \mathcal{A}_2) = \sup_{A_1, A_2 \in \mathcal{A}_1 \times \mathcal{A}_2, \mathbb{P}(A_1) \neq 0} |\mathbb{P}(A_2|A_1) - \mathbb{P}(A_2)|,$$

the φ -mixing coefficient is defined by

$$\varphi(\ell) = \sup_{t \in \mathbb{Z}} \varphi(\mathcal{F}_t, \mathcal{G}_{t+\ell}) \quad (3)$$

It should be noted that if the original process $(\mathbf{Z}_t, t \in \mathbb{Z})$ satisfies an α - or β - or φ -mixing condition, then the stationary process $(f(\mathbf{Z}_t), t \in \mathbb{Z})$ for a measurable function f also satisfies the same mixing condition. The α -mixing rate, β -mixing rate, and φ -mixing rate of the stationary process are all bounded by the corresponding rate of the original process. In terms of their order, the three mixing coefficients are related as follows:

$$\alpha(\ell) \leq \beta(\ell) \leq \varphi(\ell). \quad (4)$$

This means that the α -mixing coefficient is the weakest, followed by the β -mixing coefficient, and finally the φ -mixing coefficient is the strongest.

Let $\mathbf{M}_m = (M_m^{(1)}, \dots, M_m^{(d)})$ be the vector of component-wise maxima, where we denote by $M_{m,j} = \max_{i=1, \dots, m} Z_i^{(j)}$. Consider a random vector $\mathbf{X} = (X^{(1)}, \dots, X^{(d)})$ with distribution H . A normalizing function a on \mathbb{R} is a non-decreasing, right continuous function that goes to $\pm\infty$ as $x \rightarrow \pm\infty$. In extreme value theory, a fundamental problem is to characterize the limit distribution H in the following limit:

$$\lim_{m \rightarrow \infty} \mathbb{P}\{\mathbf{M}_m \leq \mathbf{a}_m(\mathbf{x})\} = H(\mathbf{x}), \quad (5)$$

where $\mathbf{a}_m = (a_m^{(1)}, \dots, a_m^{(d)})$ with $a_m^{(j)}, 1 \leq j \leq d$ are normalizing functions and H is a non-degenerate distribution. Typically, H is an extreme value distribution, and \mathbf{X} is a max-stable random vector with generalized extreme value margins. In this case, we can write:

$$\mathbb{P}\{\mathbf{X} \leq \mathbf{x}\} = \exp\{-\Lambda(E \setminus [0, \mathbf{x}])\},$$

where Λ is a Radon measure on the cone $E = [0, \infty)^d \setminus \{\mathbf{0}\}$. When (5) holds with H an extreme value distribution, the vector \mathbf{Z} is said to be in max-domain of attraction of the random vector \mathbf{X} with law H , denoted as $F \in \mathcal{D}(H)$. In our context of a dependent process $(\mathbf{Z}_t, t \in \mathbb{Z})$, the limit in (5) will in general be different from a multivariate extremal types distribution and further conditions over the regularity (or mixing conditions) are thus needed to obtain an extremal distribution. In particular, if the random process $(\mathbf{Z}_t, t \in \mathbb{Z})$ is α -mixing that is $\alpha(\ell) \rightarrow 0$ as $\ell \rightarrow \infty$, then a Fisher-Tippett-Gnedenko's type theorem holds for multivariate stationary random processes (see Theorem 4.2 of Hsing 1989).

The max-domain of attraction can be translated into terms of copulae. Let C_m be the copula of \mathbf{M}_m . In this context, the domain of attraction condition can be written as:

$$\lim_{m \rightarrow \infty} C_m(\mathbf{u}) = C(\mathbf{u}).$$

Under the same mixing condition, i.e., $\alpha(\ell) \rightarrow 0$ as $\ell \rightarrow \infty$, C is an extreme value copula and it can be expressed as follows for $\mathbf{u} \in [0, 1]^d$:

$$C(\mathbf{u}) = \exp \left\{ -L \left(-\ln(u^{(1)}), \dots, -\ln(u^{(d)}) \right) \right\},$$

where L is known as the stable tail dependence function (see Gudendorf and Segers 2010 for an overview of extreme value copulae). As it is a homogeneous function of order 1, i.e., $L(a\mathbf{z}) = aL(\mathbf{z})$ for all $a > 0$, we have, for all $\mathbf{z} \in [0, \infty)^d$,

$$L(\mathbf{z}) = (z^{(1)} + \dots + z^{(d)})A(\mathbf{t}),$$

with $t^{(j)} = z^{(j)} / (z^{(1)} + \dots + z^{(d)})$ for $j \in \{2, \dots, d\}$, $t^{(1)} = 1 - (t^{(2)} + \dots + t^{(d)})$, and A is the restriction of L into the d -dimensional unit simplex, viz.

$$\Delta_{d-1} = \{(v^{(1)}, \dots, v^{(d)}) \in [0, 1]^d : v^{(1)} + \dots + v^{(d)} = 1\}.$$

The function A is known as the Pickands dependence function and is often used to quantify the extremal dependence among the elements of \mathbf{X} . Indeed, A satisfies the constraints $1/d \leq \max(t^{(1)}, \dots, t^{(d)}) \leq A(\mathbf{t}) \leq 1$ for all $\mathbf{t} \in \Delta_{d-1}$, with lower and upper bounds corresponding to the complete dependence and independence among maxima. For the latter, it is commonly said that the stationary random process $(\mathbf{Z}_t, t \in \mathbb{Z})$ exhibits asymptotic independence, i.e., the multivariate extreme value distribution H in the max-domain of attraction is equal to the product of its marginal extreme value distributions.

2.2 Proposed AI-block models

In this paper, we focus on the concept of asymptotic independence, which has been observed in various applications, such as the analysis of spatial precipitation patterns (see Lalancette, Engelke, and Volgushev 2021; Le et al. 2018) and water discharges in river networks (Fomichov and Ivanovs 2022). Motivated by these applications, we introduce a new class of models for variable clustering, AI-block models, in which population-level clusters are defined as groups of variables that are dependent within clusters and independent from other clusters relative to their extremes. Formally, the variables of the distribution in the domain of attraction of observed processes can be partitioned into G unknown clusters $O = \{O_1, \dots, O_G\}$, such that variables within the same cluster are dependent, and the clusters are asymptotically independent. In this section, we focus on the identifiability of the model, specifically the existence of a unique maximal element with respect to a certain partial order on partition. We explicitly construct this maximal element, which corresponds to the thinnest partition where this property holds and serves as a target for statistical inference.

Let us consider $\mathbf{X}^{(O_1)}, \dots, \mathbf{X}^{(O_G)}$ to be extreme value random vectors with extreme value copulae $C^{(O_1)}, \dots, C^{(O_G)}$ respectively. Under condition of independence between $\mathbf{X}^{(O_1)}, \dots, \mathbf{X}^{(O_G)}$, the random vector $\mathbf{X} = (\mathbf{X}^{(O_1)}, \dots, \mathbf{X}^{(O_G)})$ is again extreme and one can detail the expression of its extreme value copula. The formal statement of this result is stated in the next proposition.

Proposition 1. *Let $\mathbf{X}^{(O_1)}, \dots, \mathbf{X}^{(O_G)}$ be independent extreme value random vectors with extreme value copulae $C^{(O_1)}, \dots, C^{(O_G)}$. Then the function C defined as*

$$\begin{aligned} C : [0, 1]^d &\longrightarrow [0, 1] \\ \mathbf{u} &\longmapsto \prod_{g=1}^G C^{(O_g)}(u^{(i_{g,1})}, \dots, u^{(i_{g,d_g})}), \end{aligned}$$

is an extreme value copula associated to the random vector $\mathbf{X} = (\mathbf{X}^{(O_1)}, \dots, \mathbf{X}^{(O_G)})$.

As a result, a random vector \mathbf{X} that exhibits asymptotic independence between extreme-valued subvectors therefore inherits this extreme-valued property. Using the definitions and notations so far introduced in this work, we now present the definition of our model.

Definition 1 (Asymptotic Independent-block model). Let $(\mathbf{Z}_t, t \in \mathbb{Z})$ be a d -variate stationary random process with law F and \mathbf{X} a random vector with extreme value distribution H . The random process \mathbf{Z}_t is said to follow an AI-block model if $F \in D(H)$ and for every $g \in \{1, \dots, G\}$, $\mathbf{X}^{(O_g)} = (X^{(i_{g,1})}, \dots, X^{(i_{g,d_g})})$ are extreme value random vectors and $\mathbf{X}^{(O_1)}, \dots, \mathbf{X}^{(O_G)}$ are independent, that is $H = \prod_{g=1}^G H^{(O_g)}$.

Notice that, when $G = 1$, the definition of AI-block models thus reduces to the process $(\mathbf{Z}_t, t \in \mathbb{Z})$ is in the domain of attraction of an extreme value distribution H .

Following Bunea et al. 2020, we introduce the following notation in our framework. We say that \mathbf{Z} follows an AI-block model with a partition O , denoted $\mathbf{Z} \sim O$. We define the set $\mathcal{L}(\mathbf{Z}) = \{O : O \text{ is a partition of } \{1, \dots, d\} \text{ and } \mathbf{Z} \sim O\}$, which is nonempty and finite, and therefore has maximal elements. We introduce a partial order on partitions as follows: let $O = \{O_g\}_g$ and $\{S_{g'}\}_{g'}$ be two partitions of $\{1, \dots, d\}$. We say that S is a sub-partition of O if, for each g' , there exists g such that $S_{g'} \subseteq O_g$. We define the partial order \leq between two partitions O and S of $\{1, \dots, d\}$ as follows:

$$O \leq S, \text{ if } S \text{ is a sub-partition of } O. \quad (6)$$

For any partition $O = \{O_g\}_{1 \leq g \leq G}$, we write $a \stackrel{O}{\sim} b$ where $a, b \in \{1, \dots, d\}$ if there exists $g \in \{1, \dots, G\}$ such that $a, b \in O_g$.

Definition 2. For any two partitions O, S of $\{1, \dots, d\}$, we define $O \cap S$ as the partition induced by the equivalence relation $a \stackrel{O \cap S}{\sim} b$ if and only if $a \stackrel{O}{\sim} b$ and $a \stackrel{S}{\sim} b$.

Checking that $a \stackrel{O \cap S}{\sim} b$ is an equivalence relation is straightforward. With this definition, we have the following interesting properties that lead to the desired result, the identifiability of AI-block models.

Theorem 1. *Let $(\mathbf{Z}_t, t \in \mathbb{Z})$ be a stationary random process, then the following properties hold:*

- (i) *Consider $O \leq S$. Then $\mathbf{Z} \sim S$ implies $\mathbf{Z} \sim O$,*
- (ii) *$O \leq O \cap S$ and $S \leq O \cap S$,*
- (iii) *$\mathbf{Z} \sim O$ and $\mathbf{Z} \sim S$ is equivalent to $\mathbf{Z} \sim O \cap S$,*
- (iv) *The set $\mathcal{L}(\mathbf{Z})$ has a unique maximum $\bar{O}(\mathbf{Z})$, with respect to the partition partial order \leq in (6).*

The proof demonstrates that for any partition such that \mathbf{Z} follows an AI-block model, there exists a maximal partition, denoted by \bar{O} , and its structure is intrinsic of the definition of the extreme value random vector \mathbf{X} . This partition, which represents the thinnest partition of \mathbf{Z} where \mathbf{X} is independent per block, matches our expectations for a reasonable clustering target in these models. With a slight abuse of notation, we will refer to $\bar{O}(\mathbf{Z})$ as \bar{O} throughout the rest of this paper.

2.3 Extremal dependence structure for AI-block models

Under the conditions stated in Proposition 1, \mathbf{X} is an extreme value random vector with a stable tail dependence function L . This function can be expressed in the following form:

$$L(z^{(1)}, \dots, z^{(d)}) = \sum_{g=1}^G L^{(O_g)}(z^{(O_g)}), \quad \mathbf{z} \in [0, \infty)^d, \quad (7)$$

where $L^{(O_1)}, \dots, L^{(O_G)}$ are the stable tail dependence functions with copulae $C^{(O_1)}, \dots, C^{(O_G)}$, respectively. This model is a specific form of the nested extreme value copula, as mentioned in the remark below and discussed in further detail in Hofert, Huser, and Prasad 2018.

Remark 1. Equation (7) can be rewritten as

$$L(\mathbf{z}) = L_{\Pi} \left(L^{(O_1)} \left(z^{(O_1)} \right), \dots, L^{(O_G)} \left(z^{(O_G)} \right) \right),$$

where $L_{\Pi}(z^{(1)}, \dots, z^{(G)}) = \sum_{g=1}^G z^{(g)}$ is a stable tail dependence function corresponding to asymptotic independence. According to Proposition 1, C is an extreme value copula. Therefore, it follows that C , which has the representation

$$C(\mathbf{u}) = C_{\Pi} \left(C^{(O_1)}(\mathbf{u}^{(O_1)}), \dots, C^{(O_G)}(\mathbf{u}^{(O_G)}) \right), \quad C_{\Pi} = \Pi_{g=1}^G u^{(g)},$$

is also a nested extreme value copula, as defined in Hofert, Huser, and Prasad 2018.

Equation (7) can be restricted to the simplex, allowing us to express the stable tail dependence function in terms of the Pickands dependence function. Specifically, the Pickands dependence function A can be written as a convex combination of the Pickands dependence functions $A^{(O_1)}, \dots, A^{(O_G)}$ as follows:

$$\begin{aligned} A(t^{(1)}, \dots, t^{(d)}) &= \frac{1}{z^{(1)} + \dots + z^{(d)}} \left[\sum_{g=1}^G (z^{(i_{g,1})} + \dots + z^{(i_{g,d_g})}) A^{(O_g)}(\mathbf{t}^{(O_g)}) \right] \\ &= \sum_{g=1}^G w^{(O_g)}(\mathbf{t}) A^{(O_g)}(\mathbf{t}^{(O_g)}) =: A^{(O)}(t^{(1)}, \dots, t^{(d)}), \end{aligned} \quad (8)$$

with $t^{(j)} = z^{(j)} / (z^{(1)} + \dots + z^{(d)})$ for $j \in \{2, \dots, d\}$ and $t^{(1)} = 1 - (t^{(2)} + \dots + t^{(d)})$, $w^{(O_g)}(\mathbf{t}) = (z^{(i_{g,1})} + \dots + z^{(i_{g,d_g})}) / (z^{(1)} + \dots + z^{(d)})$ for $g \in \{2, \dots, G\}$ and $w^{(O_1)}(\mathbf{t}) = 1 - (w^{(O_2)}(\mathbf{t}) + \dots + w^{(O_G)}(\mathbf{t}))$, $\mathbf{t}^{(O_g)} = (t^{(i_{g,1})}, \dots, t^{(i_{g,d_g})})$ where $t^{(i_{g,\ell})} = z^{(i_{g,\ell})} / (z^{(i_{g,1})} + \dots + z^{(i_{g,d_g})})$ and $(i_{g,\ell})$ designates the ℓ th variable in the g th cluster for $\ell \in \{1, \dots, d_g\}$ and $g \in \{1, \dots, G\}$. As a convex combination of Pickands dependence functions, A is itself a Pickands dependence function (see page 123 of Falk, Hüsler, and Reiss 2010).

In the context of independence between extreme random variables, it is well-known that the inequality $A(\mathbf{t}) \leq 1$ holds for $\mathbf{t} \in \Delta_{d-1}$, where A is the Pickands dependence function and equality stands if and only if the random variables are independent. This result extends to the case of random vectors, with the former case being a special case where $d_1 = \dots = d_G = 1$.

Proposition 2. Consider an extreme value random vector $\mathbf{X} \in \mathbb{R}^d$ with Pickands dependence function A . Let $A^{(O)}$ be as defined in (8). For all $\mathbf{t} \in \Delta_{d-1}$, we have:

$$\left(A^{(O)} - A \right) (\mathbf{t}) \geq 0,$$

with equality if and only if $\mathbf{X}^{(O_1)}, \dots, \mathbf{X}^{(O_G)}$ are independent.

This proposition states that the difference between the convex combinations of the Pickands dependence functions denoted by $A^{(O)}$ and the Pickands dependence function A is always nonnegative. Equality holds if and only if the subvectors $\mathbf{X}^{(O_1)}, \dots, \mathbf{X}^{(O_G)}$ are independent. The next proposition gives the form of the exponent measure when the random vectors $\mathbf{X}^{(O_1)}, \dots, \mathbf{X}^{(O_G)}$ are independent.

Proposition 3. *Suppose \mathbf{X} is an extreme-value random vector with exponent measure Λ concentrating on $E \setminus [0, \mathbf{x}]$ where $E = [0, \infty]^d \setminus \{\mathbf{0}\}$. The following properties are equivalent:*

- (i) *The vectors $\mathbf{X}^{(O_1)}, \dots, \mathbf{X}^{(O_G)}$ are independent.*
- (ii) *The vectors are blockwise independent: for every $1 \leq g < h \leq G$*

$\mathbf{X}^{(O_g)}$ and $\mathbf{X}^{(O_h)}$, are independent random vectors.

- (iii) *The exponent measure Λ concentrates on*

$$\bigcup_{g=1}^G \{\mathbf{0}\}^{d_1} \times \dots \times]0, \infty[^{d_g} \times \dots \times \{\mathbf{0}\}^{d_G}, \quad (9)$$

so that for $\mathbf{y} > \mathbf{0}$,

$$\Lambda \left(\bigcup_{1 \leq g < h \leq G} \left\{ \mathbf{x} \in E, \exists a \in O_g, x^{(a)} > y^{(a)}, \exists b \in O_h, x^{(b)} > y^{(b)} \right\} \right) = 0$$

These conditions generalize straightforwardly those stated in Proposition 5.24 of Resnick 2008 (see Exercise 5.5.1 of the book aforementioned or the Lemma in Strokorb 2020), we refer to Appendix B.1 for a proof. Furthermore, Equation (9) is geometrically involving: the exponent measure concentrates only the positive orthants where maxima are dependent, we refer to Fig. 3 in Appendix C.3.1 to clarify this statement.

In higher dimensions, the translation of asymptotic independence for random vectors can be computationally expensive and may require parametric assumptions to be tractable. Ideally, we would like a summary statistic that can be estimated empirically and that accurately reflects the underlying clusters. In extreme value theory, independence between the components $X^{(1)}, \dots, X^{(d)}$ of an extreme-value random vector $\mathbf{X} \in \mathbb{R}^d$ can be characterized in a useful way: according to Takahashi 1987, 1994, total independence of \mathbf{X} is equivalent to the existence of a vector $\mathbf{x} = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{R}^d$ such that $H(\mathbf{x}) = H^{(1)}(x^{(1)}) \dots H^{(d)}(x^{(d)})$. In the following, we find conditions on F such that $F \in D(H)$ and $H = \prod_{g=1}^G H^{(O_g)}$ where $H^{(O_g)}$ are extreme value distributions for $g \in \{1, \dots, G\}$ to obtain a similar statement of Takahashi 1987, 1994 translated to our framework.

Condition A. There exist sequences r_m, ℓ_m such that the following statements hold:

- (i) $r_m \rightarrow \infty$ and $r_m = o(m)$,
- (ii) $\ell_m \rightarrow \infty$ and $\ell_m = o(r_m)$,
- (iii) $(r_m/\ell_m)\alpha(\ell_m) = o(1)$, where the coefficient α is given in (1).

Theorem 2. *Suppose that H and $H^{(O_g)}$ are respectively d and d_g continuous extreme value distributions, for $g \in \{1, \dots, G\}$. Suppose that Conditions A holds for $(\mathbf{Z}_t, t \in \mathbf{Z})$, then*

$$\mathbb{P}\{\mathbf{M}_m \leq \mathbf{a}_m(\mathbf{x})\} \xrightarrow{m \rightarrow \infty} \prod_{g=1}^G H^{(O_g)}(\mathbf{x}^{(O_g)}), \quad \forall \mathbf{x} \in \mathbb{R}^d,$$

if and only if

$$\mathbb{P}\{\mathbf{M}_m \leq \mathbf{a}_m(\mathbf{x})\} \xrightarrow{m \rightarrow \infty} H(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^d \quad (10)$$

and there exists a $\mathbf{p} = (\mathbf{p}^{(O_1)}, \dots, \mathbf{p}^{(O_G)}) \in \mathbb{R}^d$ such that $0 < H^{(O_g)}(\mathbf{p}^{(O_g)}) < 1$ and

$$\mathbb{P}\{\mathbf{M}_m \leq \mathbf{a}_m(\mathbf{p})\} \xrightarrow{m \rightarrow \infty} \prod_{g=1}^G H^{(O_g)}(\mathbf{p}^{(O_g)}). \quad (11)$$

Notice that condition in (10) is natural in the study of the multivariate dependence of extreme. Indeed, this condition in AI-block model is directly obtained when subvectors $\mathbf{Z}^{(O_g)}$, $g \in \{1, \dots, G\}$ are in the max-domain of attraction of an extreme value distribution as asked in Definition 1. The random vector \mathbf{Z} is thus in the max-domain of attraction of a multivariate extreme value distribution which is the product of the corresponding subvectors as precisely written in Proposition 1.

The interested reader may find the proof in Appendix B.2. One direct application of this result in AI-block models is that $\mathbf{X}^{(O_1)}, \dots, \mathbf{X}^{(O_G)}$ are independent if and only if there exists an extreme value distribution H such that the process $(\mathbf{Z}_t, t \in \mathbb{Z})$ is in the max-domain of attraction and the following holds:

$$A\left(\frac{1}{d}, \dots, \frac{1}{d}\right) = \sum_{g=1}^G \frac{d_g}{d} A^{(O_g)}\left(\frac{1}{d_g}, \dots, \frac{1}{d_g}\right).$$

Definition 3 (Sum of Extremal COefficients (SECO)). The extremal coefficient of an extreme value random vector \mathbf{X} is defined as (see Smith 1990):

$$\theta := \theta^{\{\{1, \dots, d\}\}} = d A(d^{-1}, \dots, d^{-1}), \quad (12)$$

where A is the Pickands dependence function. For a partition $O = \{O_1, \dots, O_G\}$ of $\{1, \dots, d\}$, we also define the extremal coefficient of the subvectors $\mathbf{X}^{(O_g)}$ as $\theta^{(O_g)} = d_g A^{(O_g)}(d_g^{-1}, \dots, d_g^{-1})$, where $d_g = |O_g|$ is the size of the set O_g and $A^{(O_g)}$ is the Pickands dependence function of $\mathbf{X}^{(O_g)}$. Using these coefficients, we define the following quantity SECO as

$$\text{SECO}(O) = \sum_{g=1}^G \theta^{(O_g)} - \theta. \quad (13)$$

The SECO is a measure that quantifies the deviation of the sum of extremal coefficient of the subvectors $\mathbf{X}^{(O_g)}$ from the extremal coefficient of the full vector \mathbf{X} . When this measure is 0, it indicates that the subvectors $\mathbf{X}^{(O_1)}, \dots, \mathbf{X}^{(O_G)}$ constitute an independent partition. This means that the SECO in (13) captures the asymptotic independent block structure of the random vector \mathbf{X} , regardless of any distributional assumptions.

To perform statistical inference, we state a condition based on the extremal dependence of each cluster. This condition allows to present a simple, yet powerful, algorithm which compare the pairwise extreme dependence between components of the vector, enabling us to make informed conclusions about the dependence structures present in the data.

Condition B. For every $g \in \{1, \dots, G\}$, the extreme value random vector $\mathbf{X}^{(\bar{O}_g)}$, where \bar{O}_g is the maximal element of $\mathcal{L}(\mathbf{Z})$, exhibits dependence between all components.

One sufficient condition to satisfy Condition B is to suppose that exponent measures of the extreme value random vectors $\mathbf{X}^{(\bar{O}_g)}$ have nonnegative Lebesgue densities on the nonnegative orthant $[0, \infty)^{d_g} \setminus \{\mathbf{0}^{(\bar{O}_g)}\}$ for every $g \in \{1, \dots, G\}$. The domination of the exponent measure by the Lebesgue measure is a prerequisite assumption to define conditional independence between nodes of an extremal graphical models, (see, e.g., Engelke and Hitz 2020; Engelke and Volgushev 2022) and Section 7 for a more detailed discussion on this condition). Various classes of tractable extreme value distributions satisfy Condition B. These popular models, commonly used for statistical inference, include the asymmetric logistic model (Tawn 1990), the asymmetric Dirichlet model (Coles and Tawn 1991), the pairwise Beta model (Cooley, Davis, and Naveau 2010) or the Hüsler Reiss model (Hüsler and Reiss 1989).

3. Consistent estimation of minimally separated clusters

3.1 Multivariate tail coefficient

Throughout this section assume that we observe copies $\mathbf{Z}_1 \dots, \mathbf{Z}_n$ of the d -dimensional stationary random process $(\mathbf{Z}_t, t \in \mathbb{Z})$, which is in the max-domain of attraction of \mathbf{X} , an AI-block model as in Definition 1. The sample of size n of $(\mathbf{Z}_t, t \in \mathbb{Z})$ is divided into k blocks of length m , so that $k = \lfloor n/m \rfloor$, the integer part of n/m and there may be a remaining block of length $n - km$. For the i -th block, the maximum value in the j -th component is denoted by

$$M_{m,i}^{(j)} = \max \left\{ Z_t^{(j)} : t \in (im - m, im] \cap \mathbb{Z} \right\}.$$

Let us denote by $\mathbf{M}_{m,i} = (M_{m,i}^{(1)}, \dots, M_{m,i}^{(d)})$ the vector of the componentwise maxima in the i -th block. For a fixed block length m , the sequence of block maxima $(\mathbf{M}_{m,i})_i$ forms a stationary process that exhibits the same regularity of the process $(\mathbf{Z}_t, t \in \mathbb{Z})$. The distribution functions of block maxima are denoted by

$$F_m(\mathbf{x}) = \mathbb{P} \{ \mathbf{M}_{m,1} \leq \mathbf{x} \}, \quad F_m^{(j)}(X^{(j)}) = \mathbb{P} \left\{ M_{m,i}^{(j)} \leq X^{(j)} \right\},$$

with $\mathbf{x} \in \mathbb{R}^d$ and $j \in \{1, \dots, d\}$. Denote by $U_{m,1}^{(j)} = F_m^{(j)}(M_{m,1}^{(j)})$ the unknown uniform margin of $M_{m,1}^{(j)}$ with $j \in \{1, \dots, d\}$. Let C_m be the unique (as the margins of $\mathbf{M}_{m,1}$ are continuous) copula of F_m . In the present context of serial dependence, the domain of attraction condition reads as follows.

Condition C. There exists a copula C such that

$$\lim_{m \rightarrow \infty} C_m(\mathbf{u}) = C(\mathbf{u}), \quad \mathbf{u} \in [0, 1]^d.$$

One way to measure tail dependence for a d -dimensional extreme value random vector is through the use of the extremal coefficient, as defined in Equation (12). According to Schlather and Tawn 2002, the coefficient θ can be interpreted as the number of independent variables that are involved in the given random vector. Let $x \in \mathbb{R}$ be the coefficient $\theta_m(x)$ for the vector of maxima $\mathbf{M}_{m,1}$, which is defined by the following relation:

$$\mathbb{P} \left\{ \bigvee_{j=1}^d U_{m,1}^{(j)} \leq x \right\} = \mathbb{P} \{ U_{m,1}^{(1)} \leq x \}^{\theta_m(x)}.$$

Under Condition C, the coefficient $\theta_m(x)$ of the componentwise maxima $\mathbf{M}_{m,1}$ converges to the extremal coefficient θ of the random vector \mathbf{X} , that is:

$$\theta_m(x) \xrightarrow{m \rightarrow \infty} \theta, \quad \forall x \in \mathbb{R}.$$

It is worth noting that θ is a constant since \mathbf{X} is a multivariate extreme value distribution. To generalize the bivariate madogram for the random vectors $\mathbf{M}_{m,1}$ we follow the same approach as in Marcon et al. 2017; Boulin et al. 2022 and define:

$$\nu_m = \mathbb{E} \left[\bigvee_{j=1}^d U_{m,1}^{(j)} - \frac{1}{d} \sum_{j=1}^d U_{m,1}^{(j)} \right], \quad \nu = \mathbb{E} \left[\bigvee_{j=1}^d H^{(j)}(X^{(j)}) - \frac{1}{d} \sum_{j=1}^d H^{(j)}(X^{(j)}) \right]. \quad (14)$$

Condition C implies that the distribution of $\mathbf{M}_{m,1}$ is sub-asymptotically extreme valued. A common approach for estimating the extremal coefficient in this scenario consists of supposing that the

sample follows exactly the extreme value distribution and to consider $\theta_m(x) := \theta_m$ a sub-asymptotic extremal coefficient which is constant for every x . Thus, we have

$$\theta_m = \frac{1/2 + \nu_m}{1/2 - \nu_m}, \quad 1 \leq \theta_m \leq d.$$

One issue with the sub-asymptotic extremal coefficient is that it is misspecified, as extreme value distributions only arise in the limit as the block size m tends to infinity, while in practice we must use a finite sample size. We study this misspecification error in Section 3.3. A plug-in estimation process can be obtained using:

$$\hat{\theta}_{n,m} = \frac{1/2 + \hat{\nu}_{n,m}}{1/2 - \hat{\nu}_{n,m}}, \quad (15)$$

where $\hat{\nu}_{n,m}$ is an estimate of ν_m obtained using:

$$\hat{\nu}_{n,m} = \frac{1}{k} \sum_{i=1}^k \left[\bigvee_{j=1}^d \hat{U}_{n,m,i}^{(j)} - \frac{1}{d} \sum_{j=1}^d \hat{U}_{n,m,i}^{(j)} \right], \quad (16)$$

and $(\hat{U}_{n,m,1}^{(j)}, \dots, \hat{U}_{n,m,k}^{(j)})$ are the empirical counterparts of $(U_{m,1}^{(j)}, \dots, U_{m,k}^{(j)})$ or, equivalently, scaled ranks of the sample. To establish the strong consistency of this estimator, certain conditions on the mixing coefficients must be satisfied.

Condition D. Let $m_n = o(n)$. The series $\sum_{n \geq 1} \beta(m_n)$ is convergent, where β is defined in (2).

For the sake of notational simplicity, we will write $m = m_n$, $k = k_n$. The convergence of the series of β -mixing coefficients in Condition D is necessary to obtain the strong consistency of $\hat{\nu}_{n,m}$, and it can be achieved through the sufficiency condition of the Glivencko-Cantelli lemma for almost sure convergence.

Proposition 4. Let $(\mathbf{Z}_t, t \in \mathbb{Z})$ be a stationary multivariate random process. Under Conditions C and D, the madogram estimator in (16) is strongly consistent, i.e.,

$$|\hat{\nu}_{n,m} - \nu| \xrightarrow[n \rightarrow \infty]{a.s.} 0,$$

with ν the theoretical madogram of the extreme value random vector \mathbf{X} given in (14).

The consistency stated above is derived for data of fixed dimension d with the sample size n increases to infinity. In the following, we provide non-asymptotic bounds for the error $|\hat{\nu}_{n,m} - \nu_m|$.

Proposition 5. Let $(\mathbf{Z}_t, t \in \mathbb{Z})$ be a stationary process with algebraic φ -mixing distribution, $\varphi(n) \leq \lambda n^{-\zeta}$ where $\lambda > 0$, and $\zeta > 1$ and φ defined in Equation (3). Then the following concentration bound holds

$$\mathbb{P} \left\{ |\hat{\nu}_{n,m} - \nu_m| \geq C_1 k^{-1/2} + C_2 k^{-1} + t \right\} \leq (d + 2\sqrt{e}) \exp \left\{ -\frac{t^2 k}{C_3} \right\},$$

where k is the number of block maxima and C_1 , C_2 and C_3 are constants depending only on ζ and λ .

In Proposition 4, we require $(\mathbf{Z}_t, t \in \mathbb{Z})$ to be absolutely regular, or β -mixing, in order to apply the coupling lemma of Berbee 1979, which is sufficient for the asymptotic analysis. The non-asymptotic analysis in Proposition 5 is more stringent and requires the use of φ -mixing in order to apply Hoeffding and McDiarmid inequalities in a dependent setting, as described in Mohri and Rostamizadeh 2010; Rio 2017. By using the chain of inequalities in (4), the conditions in Proposition 5 therefore imply Conditions A and D.

3.2 Inference in AI-block models

In this section, we present an adapted version of the algorithm developed in Bunea *et al.* 2020 for clustering variables based on a metric on their covariances, named as CORD. Our adaptation involves the use of the extremal correlation as a measure of dependence between the extremes of two variables.

The SECO in (13) can be written in the bivariate setting as

$$\text{SECO}(\{a, b\}) = 2 - \theta(a, b),$$

where for notational convenience, $\theta(a, b) := \theta^{\{a, b\}}$ is the bivariate extremal coefficient between $X^{(a)}$ and $X^{(b)}$ as defined in (12). This metric has a range between 0 and 1, with the boundary cases representing asymptotic independence and comonotonic extremal dependence, respectively. In fact, the bivariate SECO is exactly equal to the extremal correlation χ defined in Coles, Heffernan, and Tawn 1999 as

$$\chi(a, b) = \lim_{q \rightarrow 0} \chi_q(a, b), \text{ where } \chi_q(a, b) = \mathbb{P} \left\{ H^{(a)}(X^{(a)}) > 1 - q | H^{(b)}(X^{(b)}) > 1 - q \right\},$$

whenever the limit exists. In particular, if \mathbf{X} is a multivariate extreme-value distribution, then $\chi(a, b) = \chi_q(a, b)$ for $q \in (0, 1)$. In an AI-block model, the statement

$$\mathbf{X}^{(O_g)} \perp\!\!\!\perp \mathbf{X}^{(O_h)}, \quad g \neq h,$$

is equivalent to

$$\chi(a, b) = \chi(b, a) = 0, \quad \forall a \in O_g, \forall b \in O_h, \quad g \neq h. \quad (17)$$

Thus using Proposition 3, Condition \mathcal{B} and Equation (17), the extremal correlation is a sufficient statistic to recover clusters in an AI-block model. Indeed, by Condition \mathcal{B} and Equation (17), two variables $X^{(a)}$ and $X^{(b)}$ belong to the same cluster of an AI-block model if and only if $\chi(a, b) > 0$. For the estimation procedure, using tools introduced in the previous section, we give a sample version of the extremal correlation associated to $M_{m,1}^{(a)}$ and $M_{m,1}^{(b)}$ by

$$\hat{\chi}_{n,m}(a, b) = 2 - \hat{\theta}_{n,m}(a, b), \quad a, b \in \{1, \dots, d\},$$

where $\hat{\theta}_{n,m}(a, b)$ is the sampling version defined in (15) of $\theta(a, b)$. The strong consistency of this estimate follows directly from Proposition 4.

Let us denote by $\mathcal{X} = (\chi(a, b))_{a, b \in \{1, \dots, d\}}$ be the matrix of all extremal correlations and $\hat{\mathcal{X}} = (\hat{\chi}_{n,m}(a, b))_{a, b \in \{1, \dots, d\}}$ be its sampling version. We present an algorithm, named ECO (Extremal COrrrelation), for estimating the partition \bar{O} using a dissimilarity metric based on the extremal correlation. This algorithm, outlined in Algorithm (ECO), does not require the specification of the number of groups G , as it is automatically estimated by the procedure. The algorithm complexity for computing the k vectors $\hat{\mathbf{U}}_{n,m,i} = (\hat{U}_{n,m,i}^{(1)}, \dots, \hat{U}_{n,m,i}^{(d)})$ for $i \in \{1, \dots, k\}$ is of order $O(dk \ln(k))$ (Cormen *et al.* 2022, Section 2). Given the empirical ranks, computing $\hat{\mathcal{X}}$ and performing the algorithm require $O(d^2 \vee dn \ln(k))$ and $O(d^3)$ computations, respectively. So the overall complexity of the estimation procedure is $O(d^2(d \vee k \ln(k)))$.

In the following, we provide conditions ensuring that our algorithm is consistent.

Theorem 3. *Consider the AI-block model as defined in Definition 1 under Condition \mathcal{B} and $(\mathbf{Z}_t, t \in \mathbb{Z})$ be a stationary multivariate random process. For a given \mathcal{X} and its corresponding estimator $\hat{\mathcal{X}}$, if Conditions \mathcal{C} and \mathcal{D} holds, then for any $\tau > 0$*

$$\mathbb{P} \left\{ \hat{O} = \bar{O} \right\} = 1, \quad \text{as } n \rightarrow \infty.$$

Algorithm (ECO) Clustering procedure for AI-block models

```

1: procedure ECO( $S, \tau, \hat{\mathcal{X}}$ )
2:   Initialize:  $S = \{1, \dots, d\}$ ,  $\hat{\chi}_{n,m}(a, b)$  for  $a, b \in \{1, \dots, d\}$  and  $l = 0$ 
3:   while  $S \neq \emptyset$  do
4:      $l = l + 1$ 
5:     if  $|S| = 1$  then
6:        $\hat{O}_l = S$ 
7:     if  $|S| > 1$  then
8:        $(a_l, b_l) = \arg \max_{a, b \in S} \hat{\chi}_{n,m}(a, b)$ 
9:       if  $\hat{\chi}_{n,m}(a_l, b_l) \leq \tau$  then
10:         $\hat{O}_l = \{a_l\}$ 
11:       if  $\hat{\chi}_{n,m}(a_l, b_l) > \tau$  then
12:         $\hat{O}_l = \{s \in S : \hat{\chi}_{n,m}(a_l, s) \wedge \hat{\chi}_{n,m}(b_l, s) \geq \tau\}$ 
13:        $S = S \setminus \hat{O}_l$ 
14:   return  $\hat{O} = (\hat{O}_l)_l$ 
    
```

A key consideration is the choice of the threshold τ . If $\tau \approx 0$, the algorithm is likely to return the unique cluster $\{1, \dots, d\}$, while if $\tau \approx 1$, it is likely to return the greatest cluster $\{\{1\}, \dots, \{d\}\}$. However, this issue can be addressed through a non-asymptotic analysis of the algorithm, which we present in the next section.

3.3 Estimation in growing dimensions

We provide an extension of Theorem 3, allowing estimation in the case of growing dimensions, by adding non asymptotic bounds on the probability of consistently estimating the maximal element \bar{O} of an AI-block model. Furthermore, this result provides an answer for how to leverage τ in Algorithm (ECO). The difficulty of clustering in AI-block models can be assessed via the size of the Minimal Extremal COrrrelation (MECO) separation between two variables in a same cluster:

$$\text{MECO}(\mathcal{X}) := \min_{a \overset{\bar{O}}{\sim} b} \chi(a, b).$$

In AI-block models, with Condition \mathcal{B} , we always have $\text{MECO}(\mathcal{X}) > \eta$ with $\eta = 0$. However, a large value of η will be needed for retrieving consistently the partition \bar{O} from identical and independent observations.

We are now ready to state the main result of this section.

Theorem 4. *We consider the AI-block model as defined in Definition 1 under Condition \mathcal{B} , and $(\mathbf{Z}_t, t \in \mathbb{Z})$ be a d -multivariate stationary process with algebraic φ -mixing distribution, $\varphi(n) \leq \lambda n^{-\zeta}$ where $\lambda > 0$ and $\zeta > 1$ and φ defined in Equation (3). Define*

$$d_m = \max_{a \neq b} |\chi_m(a, b) - \chi(a, b)|.$$

Let (τ, η) be parameters fulfilling

$$\begin{aligned} \tau &\geq d_m + C_1 k^{-1/2} + C_2 k^{-1} + C_3 \sqrt{\frac{(1 + \gamma) \ln(d)}{k}}, \\ \eta &\geq d_m + C_1 k^{-1/2} + C_2 k^{-1} + C_3 \sqrt{\frac{(1 + \gamma) \ln(d)}{k}} + \tau, \end{aligned}$$

where C_1, C_2, C_3 are universal constants depending only on λ and ζ , k is the number of block maxima, and $\gamma > 0$. For a given \mathcal{X} and its corresponding estimator $\hat{\mathcal{X}}$, if $\text{MECO}(\mathcal{X}) > \eta$, then the output of Algorithm (ECO) is consistent, i.e.,

$$\mathbb{P}\{\hat{O} = \bar{O}\} \geq 1 - 2(1 + \sqrt{e})d^{-2\gamma}.$$

Unsurprising, as Theorem 4 is not concerned with asymptotics, we did not actually assume Condition C. A link between \mathbf{Z} and \mathbf{X} is implicitly provided through the bias term d_m which measures the distance between $\chi_m(a, b)$ and $\chi(a, b)$. This quantity vanishes when Condition C holds as $m \rightarrow \infty$.

Some comments on the implications of Theorem 4 are in order. On a high level, larger dimension d and bias d_m lead to a higher threshold τ . The effects of the dimension d and the bias d_m are intuitive: larger dimension or more bias make the partition recovery problem more difficult. It is clear that the partition recovery problem becomes more difficult as the dimension or bias increases. This is reflected in the bound of the MECO value below which distinguish between noise and asymptotic independence is impossible by our algorithm. Thus, whereas the dimension d increases, the dependence between each component should be stronger in order to distinguish between the two. In other words, for alternatives that are sufficiently separated from the asymptotic independence case, the algorithm will be able to distinguish between asymptotic independence and noise at the $\sqrt{\ln(d)k^{-1}}$ scale. For a more quantitative discussion, our algorithm is able to recover clusters when the data dimension scales at a polynomial rate, i.e. $d = o(n^p)$, with $p > 1$ as η in Theorem 4 decreases with increasing n .

Remark 2. According to Theorem 4, when the bias term is zero ($d_m = 0$), the threshold τ is of order $\sqrt{\ln(d)k^{-1}}$. Thresholding $\hat{\chi}_{n,m}(a, b)$ at this level guarantees exact recovery if the separation MECO is at least 2τ . These results are similar to those for the hard thresholding estimator in Gaussian sequence models, as demonstrated in Section 4.1 of Tsybakov 2014.

3.4 Data-driven selection of the threshold parameter

The performance of Algorithm (ECO) depends crucially on the value of the threshold parameter τ . This threshold involves known quantities such as d and k and a unknown parameter d_m (see Theorem 4). For the latter, there is no simple manner to choose optimally this parameter, as there is no simple way to determine how fast is the convergence to the asymptotic extreme behavior, or how far into the tail the asymptotic block dependence structure appears. Second order conditions, which are commonly used in the literature to ensure convergence to the stable tail dependence function at a certain rate, are theoretically relevant (see Dombry and Ferreira 2019; Einmahl, Krajina, and Segers 2012; Fougères, De Haan, and Mercadier 2015 for examples). However, finding the optimal value for the block length parameter remains a challenging task. In practice, it is advisable to use a data-driven procedure to select the threshold in Algorithm (ECO). We propose to use the following type of cross-validation for this purpose. The idea is to use the SECO criteria presented in Equation (13). Let $\mathbf{Z} \sim O$, given a partition $\hat{O} = \{\hat{O}_g\}_g$, we know from Theorem 2 that the SECO similarity given by

$$\text{SECO}(\hat{O}) = \sum_g \theta^{(\hat{O}_g)} - \theta \tag{18}$$

is equal to 0 if and only if $\hat{O} \leq \bar{O}$. We thus construct a loss function given by the SECO where we evaluate its value over a grid of the τ values. The value of τ which the SECO similarity has minimum values is also the value of τ for which we have consistent recovery of our communities. Our procedure ask to split the data in three subsamples: on one subset, we construct a set of candidate

partitions. The other two subsamples are used to estimate the extremal dependence coefficient for the sub-vector of the candidate partition $\mathbf{X}^{(\hat{O}_g)}$ and for the whole distribution \mathbf{X} . We denote $\hat{\theta}_{(1)}^{(\hat{O}_g)}$ and $\hat{\theta}_{(2)}$ the two madogram-based estimators in the spirit of (15) of $\theta^{(\hat{O}_g)}$ and θ in (18) based on the two independent samples of size n . The cross-validation based estimator of the SECO in (18) is thus defined as

$$\widehat{\text{SECO}}(\hat{O}) = \sum_g \hat{\theta}_{(1)}^{(\hat{O}_g)} - \hat{\theta}_{(2)}. \quad (19)$$

Let \hat{O} be a collection of partitions computed with Algorithm (ECO) from one subsample, by varying τ around its theoretical optimal value, of order $(d_m + \sqrt{\ln(d)k^{-1}})$, on a fine grid. For any $\hat{O} \in \bar{O}$, we evaluate our cross-validation SECO in (19). Proposition 6 offers theoretical support for this procedure, for large n . It shows that, in expectation, the minimum of the proposed criterion is asymptotically attained for subpartitions of \bar{O} .

Proposition 6. *We consider an AI-block model as in Definition 1 and the partial order \leq between two partitions in (6). If Assumptions \mathcal{C} and \mathcal{D} hold, thus*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\widehat{\text{SECO}}(\bar{O}) \right] < \lim_{n \rightarrow \infty} \mathbb{E} \left[\widehat{\text{SECO}}(\hat{O}) \right], \quad \hat{O} \not\leq \bar{O}.$$

Moreover, we give the weak convergence of an estimator of $\text{SECO}(O)$ where $\mathbf{Z} \sim O$ (see Appendix C.2, for details).

4. Hypotheses discussion for a multivariate random persistent process

A trivial example of an AI-block model is given by a partition O such that $\mathbf{Z}^{(O_g)}$ is in domain of attraction of an extreme value random vector $H^{(O_g)}$, $g \in \{1, \dots, G\}$ such that $\mathbf{Z}^{(O_1)}, \dots, \mathbf{Z}^{(O_G)}$ are independent. In this simple model, block independent clusters are sub-asymptotic hence asymptotic and the peculiar dependence structure under study is not inherent of the tail behaviour of the random vector.

More interestingly, in this section we will focus on a process where the dependence between clusters disappears in the distribution tails. To this aim, we recall here a φ -algebraically mixing process. The interested reader is referred for instance to Bücher and Segers 2014. This process satisfies Conditions \mathcal{A} and \mathcal{D} . We show that Conditions \mathcal{B} and \mathcal{C} hold with a bit more work.

Consider i.i.d d -dimensional random vectors $\mathbf{Z}_0, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots$ and independent Bernoulli random variables I_1, I_2, \dots i.i.d. with $\mathbb{P}\{I_t = 1\} = p \in (0, 1]$. For $t = 1, 2, \dots$, define the stationary random process $(\mathbf{Z}_t, t \in \mathbb{Z})$ by

$$\mathbf{Z}_t = \boldsymbol{\xi}_t \delta_1(I_t) + \mathbf{Z}_{t-1} \delta_0(I_t), \quad (20)$$

where we suppose without loss of generality that the process is defined for all $t \in \mathbb{Z}$ using stationarity. The persistence of the process $(\mathbf{Z}_t, t \in \mathbb{Z})$ arises from repeatable values in (20). From this persistence, $(\mathbf{Z}_t, t \in \mathbb{Z})$ is φ -mixing with coefficient of order $O((1-p)^n)$ (see Lemma B.1 of Bücher and Segers 2014), hence algebraically mixing. Using Proposition 4.2 of Bücher and Segers 2014, if C_1 , i.e. the copula of Z_1 , is in the copula domain of attraction of an extreme value copula C , i.e.,

$$\left\{ C_1 \left(\{u^{(1)}\}^{1/m}, \dots, \{u^{(d)}\}^{1/m} \right) \right\}^m \xrightarrow{m \rightarrow \infty} C(u^{(1)}, \dots, u^{(d)}), \quad \mathbf{u} \in [0, 1]^d,$$

then also C_m as $m \rightarrow \infty$ with C_m the copula of the maxima of $\mathbf{Z}_1, \dots, \mathbf{Z}_m$. For $\theta > 0$ and $\beta \geq 1$, let us consider the multivariate outer power transform of a Clayton copula defined such as

$$C_{\theta, \beta}(\mathbf{u}) = \left[1 + \left\{ \sum_{j=1}^d (\{u^{(j)}\}^{-\theta} - 1)^\beta \right\}^{1/\beta} \right]^{-1/\theta}, \quad \mathbf{u} \in [0, 1]^d.$$

The copula of multivariate componentwise maxima of an i.i.d. sample of size m from a continuous distribution with copula $C_{\theta,\beta}$ is equal to

$$\left\{ C_{\theta,\beta}(\{u^{(1)}\}^{1/m}, \dots, \{u^{(d)}\}^{1/m}) \right\}^m = C_{\theta/m,\beta}(u^{(1)}, \dots, u^{(d)}). \quad (21)$$

As $m \rightarrow \infty$, this copula converges to the Gumbel copula with shape parameter $\beta \geq 1$

$$C_{0,\beta}(u^{(1)}, \dots, u^{(d)}) := \lim_{m \rightarrow \infty} C_{\theta/m,\beta}(u^{(1)}, \dots, u^{(d)}) = \exp \left[- \left\{ \sum_{j=1}^d (-\ln u^{(j)})^\beta \right\}^{1/\beta} \right],$$

uniformly in $\mathbf{u} \in [0, 1]^d$. This result is initially stated in Proposition 4.3 in Bücher and Segers 2014 for the bivariate case. The extension to an arbitrary dimension implies no further arguments, the proof is thus omitted. Now let us consider the nested Archimedean copula given by

$$C_{\theta,\beta_0} \left(C_{\theta,\beta_1}^{(O_1)}(\mathbf{u}^{(O_1)}), \dots, C_{\theta,\beta_G}^{(O_G)}(\mathbf{u}^{(O_G)}) \right). \quad (22)$$

We aim to show that this copula is in the domain of attraction of an AI-block model. That is the purpose of the proposition stated below.

Proposition 7. *Consider $1 \leq \beta_0 \leq \min\{\beta_1, \dots, \beta_G\}$, then the nested Archimedean copula given in (22) is in the copula domain of attraction of an extreme value copula given by*

$$C_{0,\beta_0} \left(C_{0,\beta_1}^{(O_1)}(\mathbf{u}^{(O_1)}), \dots, C_{0,\beta_G}^{(O_G)}(\mathbf{u}^{(O_G)}) \right).$$

In particular, taking $\beta_0 = 1$ gives an AI-block model where each extreme value random vector $\mathbf{X}^{(O_g)}$ dependencies correspond to a Gumbel copula with parameter shape β_g .

From the last conclusion of Proposition 7, we obtain Condition \mathcal{C} , that is $(\mathbf{Z}_t, t \in \mathbb{Z})$ in (20) is in max-domain of attraction of an AI-block model. Noticing that the exponent measure of each cluster is absolutely continuous with respect to the Lebesgue measure, Condition \mathcal{B} is thus valid.

Remark 3. Notice that, using results from Bücher and Segers 2014, one can show that the bias d_m in Theorem 4 is of order $1/m$ in the i.i.d. case, i.e. when $p = 1$, see Appendix A.3 for details.

5. Competitor clustering algorithms for extremes

In this section, we present some competitor algorithms: the spherical k -means (Chautru 2015; Fomichov and Ivanovs 2022; Janßen and Wan 2020), k -means and hierarchical clustering using madogram as a dissimilarity (Bador et al. 2015; Bernard et al. 2013; Saunders, Stephenson, and Karoly 2021). The performance of the spherical k -means and hierarchical clustering will be compared with our Algorithm (ECO) in Section 6.

The k -means procedure is a way to identify distinct groups within a population. This procedure involves partitioning a set of data into G groups (to be consistent with our notation). To do this, we first choose cluster centers ψ_1, \dots, ψ_G for the points $\mathbf{Z}_1, \dots, \mathbf{Z}_n \in \mathbb{R}^d$ in order to minimize

$$W_n := \frac{1}{n} \sum_{i=1}^n \min_{g \in \{1, \dots, G\}} d(\mathbf{Z}_i, \psi_g),$$

where $d: \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty)$ is a distance function or, more generally, a dissimilarity function in \mathbb{R}^d . The motivation is to identify cluster centers such that distances of the observations to their nearest

cluster center are minimized. Accordingly, all observations which are closest to the same cluster center are viewed as belonging to the same group.

While the original version of k -means uses the Euclidean distance, several alternatives choices of d have been suggested. As the extremal dependence structure can be described with the angular measure S (see Resnick 2008, section 5 for details), a natural way to measure the distance between two points is by their angle. This corresponds to the spherical k -means clustering which is described as follow: for a given integer G , solve the following optimization problem

$$\frac{1}{n} \sum_{i=1}^n \min_{g \in \{1, \dots, G\}} d(\mathbf{Y}_i, \psi_g),$$

with \mathbf{Y}_i , i.i.d. observations from \mathbf{Y} , a random variable living on the unit sphere with law S . Consistency results with i.i.d. observations and for sufficiently many large observations had been proved for this algorithm in Janßen and Wan 2020. The consistency result gives that the centroids obtained by minimizing the program above are close to the true centroids of the angular distribution.

In the framework of Bador et al. 2015; Bernard et al. 2013; Saunders, Stephenson, and Karoly 2021, the madogram is considered as a dissimilarity measure. This criterion can be read in the present context of block maxima method as

$$W_n = \frac{1}{k} \sum_{i=1}^k \min_{g \in \{1, \dots, G\}} \frac{1}{2} |\hat{\mathbf{U}}_{n,m,i} - \psi_g| = \int_{[0,1]^d} \min_{g \in \{1, \dots, G\}} \frac{1}{2} |\mathbf{u} - \psi_g| d\hat{C}_{n,m}(\mathbf{u}),$$

where $\hat{C}_{n,m}$ is the empirical copula defined as

$$\hat{C}_{n,m}(\mathbf{u}) = \frac{1}{k} \sum_{i=1}^k \mathbf{1}_{\{\hat{\mathbf{U}}_{n,m,i} \leq \mathbf{u}\}}, \quad \mathbf{u} \in [0, 1]^d. \quad (23)$$

For a copula C_m in the domain of attraction of an extreme value copula C , let $\Psi = \{\psi_1, \dots, \psi_G\}$, be a set of cluster centers with $\psi_g \in \mathbb{R}^d$, $g \in \{1, \dots, G\}$ and consider the averaged distance from any observation to the closest element of Ψ as

$$W(\Psi, C) = \int_{[0,1]^d} \min_{\psi \in \Psi} \frac{1}{2} |\mathbf{u} - \psi| dC(\mathbf{u}).$$

To the best of our knowledge, consistency results for k -means procedure using the madogram have not yet been established. The following proposition tries to bridge this gap.

Proposition 8. *Let $(\mathbf{Z}_t, t \in \mathbb{Z})$ be a stationary multivariate random process with continuous univariate margins such that Conditions \mathcal{C} and \mathcal{D} hold. For each $\hat{C}_{n,m}$ in (23) and a given value $G \in \mathbb{N}$, denote by Ψ_G^n a random set which minimizes*

$$W(\Psi, \hat{C}_{n,m}) = \int_{[0,1]^d} \min_{\psi \in \Psi} \frac{1}{2} |\mathbf{u} - \psi| d\hat{C}_{n,m}(\mathbf{u}),$$

among all sets $\Psi \subset [0, 1]^d$ with at most G elements. Accordingly, let us define Ψ_G the optimal set when we replace $\hat{C}_{n,m}$ by C and assume that for a given value of G , the set Ψ_G is uniquely determined. Thus Ψ_G^n converges almost surely to Ψ_G as $n \rightarrow \infty$.

From Proposition 8, the madogram seems to be a relevant dissimilarity to estimate the set of theoretical cluster centers with respect to the extreme value copula of \mathbf{X} . As far as we know, the madogram was used for clustering using the partitioning around medoids algorithm (Bador et al. 2015; Bernard et al. 2013) and the hierarchical clustering (Saunders, Stephenson, and Karoly 2021). For computational convenience, only the hierarchical clustering and spherical k -means are considered in the next Section 6.

6. Numerical results

In this section, we investigate the finite-sample performance of our algorithm to retrieve clusters in AI-block models. We consider a number of AI-block models of increasing complexity where we compare the performance of our algorithm with state-of-the-art methods in literature, the Hierarchical Clustering (HC) using the madogram as dissimilarity and the spherical k -means (SKmeans) algorithms. We design three resulting partitions in the limit model:

- E1 \mathbf{X} is composed of two blocks O_1 and O_2 , of equal lengths where $\mathbf{X}^{(O_1)}$ and $\mathbf{X}^{(O_2)}$ are extreme-valued random vectors with a Logistic distribution and $\beta_1 = \beta_2 = 10/7$.
- E2 \mathbf{X} is composed of $G = 5$ blocks of random sample sizes d_1, \dots, d_5 from a multinomial distribution with parameter $q_g = 0.5^g$ for $g \in \{1, \dots, 4\}$ and $q_5 = 1 - \sum_{g=1}^4 q_g$. Each random vector is distributed according to a Logistic distribution where parameters $\beta_g = 10/7$ for $g \in \{1, \dots, 5\}$.
- E3 We consider the same model as E2 where we add 5 singletons. Then we have 10 resulting clusters. Model with singletons are known to be the hardest model to recover in the clustering literature.

We consider observations from the model in (20) in Section 4, where we simulate them with a nested Archimedean copula as in (22) using the method proposed by the copula R package (Marius Hofert and Martin Mächler 2011). The goal of our algorithm is to cluster d variables in \mathbb{R}^n . Thus, to make comparisons, we transpose the dataset for the k -means algorithm in order to obtain centroids in \mathbb{R}^d . In contrast to our “blindfolded” algorithm that automatically infers the number of clusters, we need to specify it for SKmeans and HC. These procedures with this wisely chosen parameter are called “oracles”. Several simulation frameworks are considered and detailed in the following.

- F1 We first investigate the choice of the intermediate sequence m of the block length used for estimation. We let $m \in \{3, 6, \dots, 30\}$ with a fixed sample size $n = 10000$ and $k = \lfloor n/m \rfloor$.
- F2 We compute the performance of the structure learning method for varying sample size n . Since the value of m which is required for consistent estimation is unknown in practice we choose $m = 20$.
- F3 We show the relationship between the average SECO and exact recovery rate of the method presented in Section 3.4. We use the case $n = 16000$, $k = 800$ and $d = 1600$ to study the “large k , large d ” of our cross-validation approach.

In the simulation study, we use the fixed threshold $\alpha = 2 \times (1/m + \sqrt{\ln(d)/k})$ for F1 and F2 since our theoretical results given in Theorem 4 suggest the usage of a threshold proportional $d_m + \sqrt{\ln(d)/k}$ and we can show, in the i.i.d. settings (where $p = 1$) that $d_m = O(1/m)$ (see details in Section Appendix A.3). For F3, we vary α around its theoretical optimal value, on a fine grid.

Figure 1 states all the results we obtain from each experiment and framework considered in this numerical section. We plot exact recovery rate for Algorithm (ECO) with dimensions $d = 200$ and $d = 1600$. In the “large d ” setting with $d = 1600$, we consider the performance of the HC algorithm using the madogram as a dissimilarity measure and the spherical k -means in the first two frameworks. Each experiment is performed using $p = 0.9$. We refer the reader to Appendix C.3.2 for numerical results using $p \in \{0.5, 0.7, 1.0\}$. As expected, the performance of our algorithm in F1 (see Figure 1, first row) is initially increasing in m , reaches a peak, and then decreases. This phenomenon depicts a trade-off between the sub-asymptotically regime and the accuracy of inference. Indeed, a large block’s length m induces a lesser bias as we reach the domain of attraction. However, the number of block k is consequently decreasing and implies a high variance for the inference process. These joint phenomena explain the parabolic form of the exact recovery rate for our algorithms

for $d \in \{200, 1600\}$. Considering the framework F2 the performance of our algorithm is better as the number of block-maxima increases (see Figure 1, second row).

A classical pitfall for learning algorithms is high dimensional settings. Here, when the dimension increases from 200 to 1600, our algorithm consistently reports the maximal element \bar{O} with a reasonable number of blocks. This is in accordance with our theoretical findings, as the difficulty of clustering in AI-block models, as quantified by η in Theorem 4, scales at a rate of $\sqrt{\ln(d)k^{-1}}$. This rate has a moderate impact on the dimension d . In the framework F3, the numerical studies in Figure 1 (third row) shows that the optimal ranges of τ values, for high exact recovery percentages, are also associated with low average SECO losses. This supports our data-driven choices of τ provided in Section 3.4.

We notice that the HC algorithm using the madogram as dissimilarity performs very well in each configuration even when the inference is strongly biased, i.e., the block length m is small hence we are far from the domain of attraction. This can be explained by the fact that madograms are lower when $a \stackrel{\bar{O}}{\sim} b$ and higher when $a \not\stackrel{\bar{O}}{\sim} b$. This is effectively true by construction of the madogram in the domain of attraction of \mathbf{X} but it is even true in our considered sub-asymptotic framework. Hence, by construction of the HC, i.e., by merging the two closest centroids in terms of madogram, we obtain the correct partitioning of \mathbf{X} even when the domain of attraction is not reached. To compare, our algorithm gives one and unique cluster, i.e., the vector is completely dependent, when the block's length m is too small and we are not yet in the domain of attraction of \mathbf{X} . This behavior is desirable as it corresponds to what it is effectively observed, the whole vector is dependent. This is a leading argument for model-based clustering which are designed for a specific model and where the inference remains coherent with the constructed target. One drawback of using HC with the madogram, as previously described, is the need to specify the number of groups G beforehand, which is not always straightforward. Despite this limitation, the HC procedure with the madogram performs well in retrieving clusters in AI-block models when the true number of clusters is known. Further researches can be lead in order to adapt our algorithm with a hierarchical design as proposed by Lee, Deng, and Ning 2021 for the algorithm of Bunea et al. 2020.

For the same reasons as for the HC case, the SKmeans performs well for Experiment (E1) and Experiment (E2) for all considered values of m . However, when we consider Experiment (E3), its performance drastically decreases. Furthermore, the exact recovery rate decreases as m increases, which is not desirable in extreme settings. However, a rigorous method for choosing G is currently lacking and it remains an hyperparameter that must be chosen by the statistician. When the hyperparameter is known and equal to the true value, clusters are correctly inferred for Experiments E1 and E2 for the HC algorithm and the SKmeans, but not for Experiment E3 for the SKmeans. Our algorithm, with the threshold specified in Theorem 4, can reach this level of performance depicted by the HC with madogram without specifying the number of clusters.

7. Conclusions

Our main focus in this work was to develop and analyze an algorithm for recovering clusters in AI-block models, and to understand how the dependence structure of maxima impacts the difficulty of clustering in these models. This is particularly challenging when we are dealing with high-dimensional data and weakly dependent observations that are sub-asymptotically distributed. In order to better understand these phenomena, we ask stronger assumptions about the extremal dependence structure in our theoretical analysis. Specifically, we assume the asymptotic independence between blocks, which is the central assumption of AI-block models. This assumption allows us to study the effects of the dependence structure and to develop and analyze an efficient algorithm for recovering clusters in these AI-block models. This procedure can recover the clusters with high probability by setting

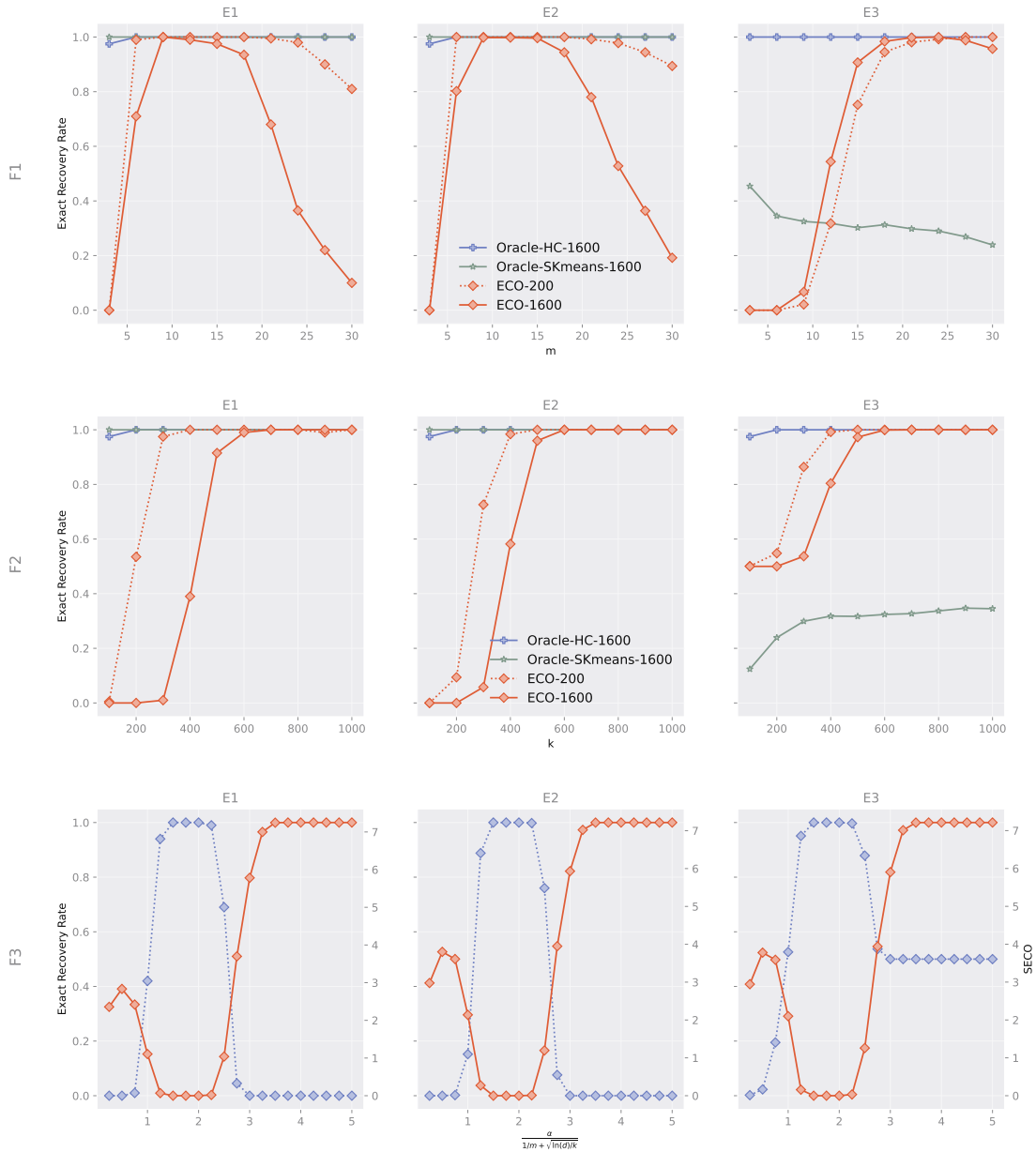


Figure 1. Simulation results with $p = 0.9$. From top to bottom: Framework F1, Framework F2, Framework F3. From left to right: Experiment E1, Experiment E2, Experiment E3. Exact recovery rate for our algorithm (red, diamond points), for the HC (blue, plus points) and the SKmeans (green, star points) for F1 and F2 across 100 runs. Dotted lines correspond to $d = 200$, solid lines to $d = 1600$. The threshold τ is taken as $2 \times (1/m + \sqrt{\ln(d)/k})$. For F3, average SECO losses (red solid lines, circle points) and exact recovery percentages (blue dotted lines, diamond points) across 100 simulations. For better illustration, the SECO losses are standardized first by subtracting the minimal SECO loss in each figure, and the standardized SECO losses plus 1 are then plotted on the logarithmic scale.

a threshold that is only logarithmic in the dimension d . However, it is still of interest to relax assumptions to cover a wider range of scenarios and problems that might be encountered in practice. We outline some potential directions for further researches below.

In this paper, we find a bound for the minimal extremal correlation separation $\eta > 0$. A further goal is to find the minimum value η^* below which it is impossible, with high probability, to exactly

recover \bar{O} by any method. This question can be formally expressed using Le Cam's theory as follows:

$$\inf_{\hat{O}} \sup_{\mathcal{X} \in \mathbb{X}(\eta)} \mathbb{P}_{\mathcal{X}}(\hat{O} \neq \bar{O}) \geq \text{constant} > 0, \quad \forall \eta < \eta^*,$$

with $\mathbb{X}(\eta) = \{\mathcal{X}, \text{MECO}(\mathcal{X}) > \eta\}$ and the infimum is taken over all possible estimators. One possible direction to obtain such a result is to follow methods introduced by Drees 2001 for risk bounds of extreme value index. An interesting consequence of this result is to determine whether our procedure is optimal (in a minimax sense), i.e., whether the order of η^* and the one found in Theorem 4 are the same.

In practice, the dependence structure can be much more complicated and Condition \mathcal{B} may not hold. In its seminal work Ryabko 2017 proposed a conditional independence test to determine whether an element j belongs to a cluster \hat{O}_1 without considering pairwise dependence between all components of the cluster. Specifically, it is asked whether

$$\mathbf{X}^{(\hat{O}_1)} \perp\!\!\!\perp X^{(j)} \mid \left(\{1, \dots, d\} \setminus (\hat{O}_1 \cup \{j\}) \right).$$

Defining a suitable conditional independence for extreme modeling is involving, as noted in Papathopoulos and Strokorb 2016, because the classical definition of conditional independence leads to trivial structure for max-stable distributions. To overcome this hindrance, conditional independence for extremes often involves the distribution of a random vector

$$(\mathbf{X}, Y) = (X^{(1)}, \dots, X^{(d)}, Y),$$

where the conditional limit of \mathbf{X} given Y is large (see, e.g., Heffernan and Tawn 2004; Heffernan and Resnick 2007 or Aghbalou et al. 2021 for a different approach designed for supervised learning). A notion of multivariate conditional independence for extremes, which is used for graph inference, is relatively new. Engelke and Hitz 2020 define such a notion for multivariate Pareto and requires absolute continuity with respect to the Lebesgue measure of the exponent measure, hence our Condition \mathcal{B} . The graphical model introduced in Segers 2020 only requires the existence of a density in the vector tree \mathbf{X} and this can exhibit asymptotic independence. However, when applied in practice (see e.g. Asenova, Mazo, and Segers 2021), an Hüsler-Reiss density is asked, which implies Condition \mathcal{B} . This leads to a trivial maximal element, $\bar{O} = \{1, \dots, d\}$. In Gissibl and Klüppelberg 2018, a causal-max linear Bayesian network is proposed and leads to graphical models with discrete dependence structure on a directed acyclic graph. Considering our framework in directed acyclic graph points out new research directions and possible future works. We may also drop Condition \mathcal{B} by taking advantage of the specific geometry of the angular measure in AI-block models by inferring clusters with methods from topological data analysis.

A more general version of the extremal conditional independence notion in Engelke and Hitz 2020 is given in Definition 3.1 in Engelke, Ivanovs, and Strokorb 2022, which naturally translates to a forest of asymptotically independent trees in the context of graphs. Motivated by modern applications in which extremal graphs are constructed based on a pre-clustering method such as k -medoids (see, e.g., Hentschel, Engelke, and Segers 2022), we see our work as a dedicated tool for this purpose, as the target clusters of our model have an interpretation in terms of extremal graphs. Therefore, combining our work on threshold exceedances (specifically the result of Theorem 4) and extending Theorem 3 of Engelke and Volgushev 2022 to handle mixing observations can lead to a strongly consistent method for learning extremal forests based on the maxima of a multivariate weakly dependent random process.

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Appendix A. Proofs of main results

Appendix A.1 Proofs of Section 2

In Proposition 1, we prove that the function introduced in Section 2.2 is an extreme-value copula. We do this by showing that its margins are distributed uniformly on the unit interval $[0,1]$ and that it is max-stable, which is a defining characteristic of extreme-value copulae.

Proof of Proposition 1 We first show that C is a copula function. It is clear that $C(\mathbf{u}) \in [0, 1]$ for every $\mathbf{u} \in [0, 1]^d$. We check that its univariate margins are uniformly distributed on $[0, 1]$. Without loss of generality, take $u^{(i_1,1)} \in [0, 1]$ and let us compute

$$C(1, \dots, u^{(i_1,1)}, \dots, 1) = C^{(O_1)}(u^{(i_1,1)}, 1, \dots, 1) = u^{(i_1,1)}.$$

So C is a copula function. We now have to prove that C is an extreme-value copula. We recall that C is an extreme-value copula if and only if C is max-stable, that is for every $m \geq 1$

$$C(u^{(1)}, \dots, u^{(d)}) = C(\{u^{(1)}\}^{1/m}, \dots, \{u^{(d)}\}^{1/m})^m.$$

By definition, we have

$$C(\{u^{(1)}\}^{1/m}, \dots, \{u^{(d)}\}^{1/m})^m = \prod_{g=1}^G \left\{ C^{(O_g)} \left(\{u^{(i_{g,1})}\}^{1/m}, \dots, \{u^{(i_{g,d_g})}\}^{1/m} \right) \right\}^m.$$

Using that $C^{(O_1)}, \dots, C^{(O_G)}$ are extreme-value copulae, thus max stable, we obtain

$$C(\{u^{(1)}\}^{1/m}, \dots, \{u^{(d)}\}^{1/m})^m = \prod_{g=1}^G C^{(O_g)} \left(u^{(i_{g,1})}, \dots, u^{(i_{g,d_g})} \right) = C(u^{(1)}, \dots, u^{(d)}).$$

Thus C is an extreme-value copula. Finally, we prove that C is associated to the random vector $\mathbf{X} = (\mathbf{X}^{(O_1)}, \dots, \mathbf{X}^{(O_G)})$, that is

$$\mathbb{P}\{\mathbf{X} \leq \mathbf{x}\} = C(H^{(1)}(x^{(1)}), \dots, H^{(d)}(x^{(d)})), \quad \mathbf{x} \in \mathbb{R}^d.$$

Using mutual independence between random vectors, we have

$$\begin{aligned} \mathbb{P}\{\mathbf{X} \leq \mathbf{x}\} &= \prod_{g=1}^G \mathbb{P}\left\{ X^{(i_{g,1})} \leq x^{(i_{g,1})}, \dots, X^{(i_{g,d_g})} \leq x^{(i_{g,d_g})} \right\} \\ &= \prod_{g=1}^G C^{(O_g)} \left(H^{(i_{g,1})}(x^{(i_{g,1})}), \dots, H^{(i_{g,d_g})}(x^{(i_{g,d_g})}) \right) \\ &= C(H^{(1)}(x^{(1)}), \dots, H^{(d)}(x^{(d)})). \end{aligned}$$

Hence the result. \square

Theorem 1, proved below, establishes several fundamental properties of the set $\mathcal{L}(\mathbf{Z})$, including the fact that subpartitions of an element $O \in \mathcal{L}(\mathbf{Z})$ also belong to $\mathcal{L}(\mathbf{Z})$ (item (i)), the ordering of partitions and their intersections (item (ii)) and the stability of the intersection of two elements $O, S \in \mathcal{L}(\mathbf{Z})$ (item (iii)). Using these results, the theorem also provides an explicit construction of the unique maximal element $\bar{O}(\mathbf{Z})$ of $\mathcal{L}(\mathbf{Z})$ (see item (iv)).

Proof of Theorem 1 For (i), if $\mathbf{Z} \sim S$, then there exist an extreme value random vector with distribution H such that $F \in D(H)$ and a partition $S = \{S_1, \dots, S_G\}$ of $\{1, \dots, d\}$ which induces mutually independent random vectors $\mathbf{X}^{(S_1)}, \dots, \mathbf{X}^{(S_G)}$. As S is a sub-partition of O , it also generates a partition where vectors are mutually independent.

Now let us prove (ii), take $g \in \{1, \dots, G\}$ and $a, b \in (O \cap S)_g$, in particular $a \stackrel{O}{\sim} b$, thus there exists $g' \in \{1, \dots, G'\}$ such that $a, b \in O_{g'}$. The following inclusion $(O \cap S)_g \subseteq O_{g'}$ is hence obtained and the second statement follows.

The third result (iii) comes down from the definition for the direct sense and by (i) and (ii) for the reverse one. We now go to the last item of the theorem, i.e. item (iv). The set $\mathcal{L}(\mathbf{Z})$ is non-empty since the trivial partition $O = \{1, \dots, d\}$ belongs to $\mathcal{L}(\mathbf{Z})$. It is also a finite set, and we can enumerate it $\mathcal{L}(\mathbf{Z}) = \{O_1, \dots, O_M\}$. Define the sequence O'_1, \dots, O'_M recursively according to

- $O'_1 = O_1$,
- $O'_g = O_g \cap O'_{g-1}$ for $g = 2, \dots, M$.

According to (iii), we have that by induction $O'_1, \dots, O'_M \in \mathcal{L}(\mathbf{Z})$. In addition, we have both $O'_{g-1} \leq O'_g$ and $O_g \leq O'_g$, so by induction $O_1, \dots, O_g \leq O'_g$. Hence the partition $\bar{O}(\mathbf{Z}) := O'_M = O_1 \cap \dots \cap O_{M-1}$ is the maximum of $\mathcal{L}(\mathbf{Z})$. \square

We extend the well-known inequality $A \leq 1$, which holds if and only if X_1, \dots, X_d are independent random variables, to the case of random vectors. Specifically, $\mathbf{X}^{(O_1)}, \dots, \mathbf{X}^{(O_G)}$ are independent if and only if $A = A^{(O)}$ (see Equation (8)). We provide two methods for establishing this result: the first leverages the convexity and homogeneity of order one of the stable tail dependence function, while the second takes advantage of the associativity of extreme-value random vectors.

Proof of Proposition 2 For the first method, the stable tail dependence function L is subadditive as an homogeneous convex function under a cone, i.e.,

$$L(\mathbf{x} + \mathbf{y}) \leq L(\mathbf{x}) + L(\mathbf{y}),$$

for every $\mathbf{x}, \mathbf{y} \in [0, \infty)^d$. In particular, we obtain by induction on G

$$L\left(\sum_{g=1}^G \mathbf{x}^{(g)}\right) \leq \sum_{g=1}^G L(\mathbf{x}^{(g)}),$$

where $\mathbf{x}^{(g)} \in [0, \infty)^d$ and $g \in \{1, \dots, G\}$. Consider now $\mathbf{z}^{(O_g)} = (\mathbf{0}, z^{(i_{g,1})}, \dots, z^{(i_{g,d_g})}, \mathbf{0})$, we directly obtain using the equation above

$$L(\mathbf{z}) = L\left(\sum_{g=1}^G \mathbf{z}^{(O_g)}\right) \leq \sum_{g=1}^G L(\mathbf{z}^{(O_g)}) = \sum_{g=1}^G L^{(O_g)}(z^{(i_{g,1})}, \dots, z^{(i_{g,d_g})}).$$

Translating the above inequality in terms of Pickands dependence function results on

$$\begin{aligned} A(\mathbf{t}) &\leq \sum_{g=1}^G \frac{1}{z^{(1)} + \dots + z^{(d)}} L^{(O_g)}(z^{(i_{g,1})}, \dots, z^{(i_{g,d_g})}) \\ &= \sum_{g=1}^G \frac{z^{(i_{g,1})} + \dots + z^{(i_{g,d_g})}}{z^{(1)} + \dots + z^{(d)}} A^{(O_g)}(t^{(i_{g,1})}, \dots, t^{(i_{g,d_g})}), \end{aligned}$$

where $t^{(i)} = z^{(i)}/(z^{(1)} + \dots + z^{(d)})$. Hence the result.

We can also prove this result by using the associativity of extreme-value distributions (see Proposition 5.1 of Marshall and Olkin 1983 or Section 5.4.1 of Resnick 2008), i.e.,

$$\mathbb{E}[f(\mathbf{X})g(\mathbf{X})] \geq \mathbb{E}[f(\mathbf{X})] \mathbb{E}[g(\mathbf{X})],$$

for every increasing (or decreasing) functions f, g . By induction on $G \in \mathbb{N}_*$,

$$\mathbb{E}\left[\prod_{g=1}^G f^{(g)}(\mathbf{X})\right] \geq \prod_{g=1}^G \mathbb{E}\left[f^{(g)}(\mathbf{X})\right]. \quad (24)$$

Take $f^{(g)}(\mathbf{x}) = \mathbb{1}_{\{\lceil -\infty, \mathbf{x}^{(O_g)} \rceil\}}$ for each $g \in \{1, \dots, G\}$, thus Equation (24) gives

$$C(H^{(1)}(x^{(1)}), \dots, H^{(d)}(x^{(d)})) \geq \prod_{g=1}^G C^{(O_g)}\left(H^{(O_g)}\left(\mathbf{x}^{(O_g)}\right)\right),$$

which can be restated in terms of stable tail dependence function as

$$L(\mathbf{z}) \leq \sum_{g=1}^G L^{(O_g)}(\mathbf{z}^{(O_g)}).$$

We obtain the statement expressing this inequality with Pickands dependence function. Finally, notice that (24) with $f^{(g)}(\mathbf{x}) = \mathbb{1}_{\{\lceil -\infty, \mathbf{x}^{(O_g)} \rceil\}}$ for each $g \in \{1, \dots, G\}$ holds as an equality if and only if $\mathbf{X}^{(O_1)}, \dots, \mathbf{X}^{(O_G)}$ are independent random vectors. \square

Appendix A.2 Proofs of Section 3

In analogy to Equation (23) in the main text, let $C_{n,m}^o$ be the empirical estimator of the copula C_m based on the (unobservable) sample $(U_{m,1}^{(j)}, \dots, U_{m,k}^{(j)})$ for $j \in \{1, \dots, d\}$. The proof of Proposition 4 will use twice Lemma 4, which shows that $\|C_{n,m}^o - C\|_\infty$ converges almost surely to 0. The proof of this lemma is postponed to Section Appendix C.1 of supplementary results.

Proof of Proposition 4 We aim to show the following convergence

$$|\hat{\nu}_{n,m} - \nu| \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0,$$

where ν is the theoretical madogram of the extreme value random vector \mathbf{X} given in (14) and $\hat{\nu}_{n,m}$ the madogram estimator in (16). Let us define the following quantity

$$\hat{\nu}_{n,m}^o = \frac{1}{k} \sum_{i=1}^k \left[\bigvee_{j=1}^d U_{m,i}^{(j)} - \frac{1}{d} \sum_{j=1}^d U_{m,i}^{(j)} \right], \quad (25)$$

that is the madogram estimated through the sample $\mathbf{U}_{m,1}, \dots, \mathbf{U}_{m,k}$. Following Lemma A.1 of Marcon *et al.* 2017, we can show that

$$\hat{\nu}_{n,m}^o - \nu = \phi(C_{n,m}^o - C),$$

with $\phi : \ell^\infty([0,1]^d) \rightarrow \ell^\infty(\Delta_{d-1})$, $f \mapsto \phi(f)$ defined by

$$\phi(f) = \frac{1}{d} \sum_{j=1}^d \int_{[0,1]} f(1, \dots, 1, \underbrace{u}_{j\text{-th component}}, 1, \dots, 1) du - \int_{[0,1]} f(u, \dots, u) du.$$

Using Conditions \mathcal{C} and \mathcal{D} , by Lemma 4 in Section Appendix C.1, as $\|C_{n,m}^o - C\|_\infty$ converges almost surely to 0, we obtain that

$$|\hat{\nu}_{n,m}^o - \nu| \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0. \quad (26)$$

Furthermore, using the chain of inequalities and again Lemma 4 in Section Appendix C.1,

$$\begin{aligned} |\hat{\nu}_{n,m} - \hat{\nu}_{n,m}^o| &\leq 2 \sup_{j \in \{1, \dots, d\}} \sup_{x \in \mathbb{R}} \left| \hat{F}_{n,m}^{(j)}(x) - F_m^{(j)}(x) \right| \\ &\leq 2 \sup_{j \in \{1, \dots, d\}} \sup_{u \in [0,1]} \left| \frac{1}{k} \sum_{i=1}^k \mathbf{1}_{\{U_{m,i}^{(j)} \leq u\}} - u \right|. \end{aligned}$$

Then we obtain that

$$|\hat{\nu}_{n,m} - \hat{\nu}_{n,m}^o| \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0. \quad (27)$$

Now, write

$$|\hat{\nu}_{n,m} - \nu| \leq |\hat{\nu}_{n,m} - \hat{\nu}_{n,m}^o| + |\hat{\nu}_{n,m}^o - \nu|,$$

and use Equations (26) and (27) to obtain the statement. \square

In Proposition 5 we state a concentration inequality for the madogram estimator. This inequality is obtained through two main steps, that are using classical concentration inequalities, such as Hoeffding and McDiarmid inequalities and chaining arguments in our specific framework of multivariate mixing random process.

Proof of Proposition 5 As in the proof of Proposition 4, we consider $\hat{\nu}_{n,m}^o$ (see Equation (25)) be the madogram using the sample $\mathbf{U}_{m,1}, \dots, \mathbf{U}_{m,k}$. By the triangle inequality, we have

$$|\hat{\nu}_{n,m} - \nu_m| \leq |\hat{\nu}_{n,m} - \hat{\nu}_{n,m}^o| + |\hat{\nu}_{n,m}^o - \nu_m|.$$

For the second term, using the triangle inequality, we obtain

$$\begin{aligned} |\hat{\nu}_{n,m}^o - \nu_m| &\leq \frac{1}{k} \sum_{i=1}^k \left\{ \left| \bigvee_{j=1}^d U_{n,m,i}^{(j)} - \mathbb{E} \left[\bigvee_{j=1}^d U_{n,m,i}^{(j)} \right] \right| + \left| \frac{1}{d} \sum_{j=1}^d U_{n,m,i}^{(j)} - \mathbb{E} \left[\frac{1}{d} \sum_{j=1}^d U_{n,m,i}^{(j)} \right] \right| \right\} \\ &\triangleq E_1 + E_2, \end{aligned}$$

and for the first term,

$$|\hat{\nu}_{n,m} - \hat{\nu}_{n,m}^o| \leq 2 \sup_{j \in \{1, \dots, d\}} \sup_{x \in \mathbb{R}} \left| \hat{F}_{n,m}^{(j)}(x) - F_m^{(j)}(x) \right| \triangleq E_3.$$

The rest of this proof is devoted to control each term: E_1 , E_2 and E_3 . Notice that the sequences $(\bigvee_{j=1}^d U_{n,m,i}^{(j)})_{i=1}^k$, $(\frac{1}{d} \sum_{j=1}^d U_{n,m,i}^{(j)})_{i=1}^k$ and $(\mathbf{1}_{\{M_{n,m,i}^{(j)} \leq x\}})_{i=1}^k$ share the same mixing regularity as $(\mathbf{Z}_t)_{t \in \mathbb{Z}}$ as measurable transformation of this process. Thus, they are in particular algebraically φ -mixing.

Control of the term E_1 . For every $i \in \{1, \dots, k\}$, we have that $\|\bigvee_{j=1}^d U_{n,m,i}^{(j)}\|_\infty \leq 1$, by applying the Hoeffding's inequality for algebraically φ -mixing sequences (see Corollary 2.1 of Rio 2017) we can control the following event, for $t > 0$,

$$\mathbb{P}\{E_1 \geq t\} \leq \sqrt{e} \exp\left\{-\frac{t^2 k}{2\|\Delta_k\|_\infty^2}\right\},$$

with $\|\Delta_k\|_\infty^2 = (1 + 4 \sum_{i=1}^k \varphi(i))^2 \geq 1 + 4 \sum_{i=1}^{k-1} \varphi(i)$. The term in the numerator can be bounded as

$$1 + 4 \sum_{i=1}^k \varphi(i) \leq 1 + 4 \sum_{i=1}^k \lambda i^{-\zeta} \leq 1 + 4\lambda \left(1 + \int_1^k x^{-\zeta} dx\right) = 1 + 4\lambda \left(1 + \frac{k^{1-\zeta} - 1}{1-\zeta}\right).$$

Using the assumption $\zeta > 1$, we can upper bound $k^{1-\zeta}$ by 1 and obtain

$$1 + 4\lambda \left(1 + \frac{k^{1-\zeta} - 1}{1-\zeta}\right) \leq 1 + 4\lambda \left(1 + \frac{1}{\zeta - 1}\right) = 1 + \frac{4\lambda\zeta}{\zeta - 1}.$$

We thus obtain

$$\mathbb{P}\left\{E_1 \geq \frac{t}{3}\right\} \leq \sqrt{e} \exp\left\{-\frac{t^2 k}{C_3}\right\},$$

where $C_3 > 0$ is a constant depending on ζ and λ .

Control of the term E_2 . This control is obtained with the same arguments used for E_1 . Thus, we obtain, for $t > 0$,

$$\mathbb{P}\left\{E_2 \geq \frac{t}{3}\right\} \leq \sqrt{e} \exp\left\{-\frac{t^2 k}{C_3}\right\}.$$

Control of the term E_3 . This bound is more technical. Before proceeding, we introduce some notations. For every $j \in \{1, \dots, d\}$, we define

$$\alpha_{n,m}^{(j)} = \left(\mathbb{P}_{n,m}^{(j)} - \mathbb{P}_m^{(j)}\right), \quad \beta_{n,m}^{(j)}(x) = \alpha_{n,m}^{(j)}(]-\infty, x]), \quad x \in \mathbb{R},$$

where $\mathbb{P}_{n,m}^{(j)}$ corresponds to the empirical measure for the sample $(M_{m,1}^{(j)}, \dots, M_{m,k}^{(j)})$ and $\mathbb{P}_m^{(j)}$ is the law of the random variable $M_m^{(j)}$. To control the term E_3 , we introduce chaining arguments as used in the proof of Proposition 7.1 of Rio 2017. Let be $j \in \{1, \dots, d\}$ fixed and N be some positive integer to be chosen later. For any real x such that $F_m^{(j)}(x) \neq 0$ and $F_m^{(j)}(x) \neq 1$, let us write $F_m^{(j)}(x)$ in base 2 :

$$F_m^{(j)}(x) = \sum_{l=1}^N b_l(x) 2^{-l} + r_N(x), \quad \text{with } r_N(x) \in [0, 2^{-N}[$$

where $b_l = 0$ or $b_l = 1$. For any L in $[1, \dots, N]$, set

$$\Pi_L(x) = \sum_{l=1}^L b_l(x) 2^{-l} \quad \text{and } i_L = \Pi_L(x) 2^L.$$

Let the reals $(x_L)_L$ be chosen in such a way that $F_m^{(j)}(x_L) = \Pi_L(x)$. With these notations

$$\begin{aligned} \beta_{n,m}^{(j)}(x) &= \beta_{n,m}^{(j)}(\Pi_1(x)) + \beta_{n,m}^{(j)}(x) - \beta_{n,m}^{(j)}(\Pi_N(x)) \\ &\quad + \sum_{L=2}^N \left[\beta_{n,m}^{(j)}(\Pi_L(x)) - \beta_{n,m}^{(j)}(\Pi_{L-1}(x)) \right]. \end{aligned}$$

Let the reals $x_{L,i}$ be defined by $F_m^{(j)}(x_{L,i}) = i2^{-L}$. Using the above equality, we get that

$$\sup_{x \in \mathbb{R}} \left| \beta_{n,m}^{(j)}(x) \right| \leq \sum_{L=1}^N \Delta_L + \Delta_N^*,$$

with

$$\Delta_L = \sup_{i \in [1, 2^L]} \left| \alpha_{n,m}^{(j)}(]x_{L,i-1}, x_{L,i}] \right| \text{ and } \Delta_N^* = \sup_{x \in \mathbb{R}} \left| \alpha_{n,m}^{(j)}(] \Pi_N(x), x]) \right|.$$

From the inequalities

$$-2^{-N} \leq \alpha_{n,m}^{(j)}(] \Pi_N(x), x]) \leq \alpha_{n,m}^{(j)}(] \Pi_N(x), \Pi_N(x) + 2^{-N}]) + 2^{-N},$$

we get that

$$\Delta_N^* \leq \Delta_N + 2^{-N} \text{ and } \mathbb{E} \left[\sup_{x \in \mathbb{R}} \left| \beta_{n,m}^{(j)}(x) \right| \right] \leq 2 \sum_{L=1}^N \|\Delta_L\|_1 + 2^{-N},$$

where $\|\Delta_L\|_1$ is the L^1 -norm of Δ_L . Let N be the natural number such that $2^{N-1} < k \leq 2^N$. For this choice of N , we obtain

$$\mathbb{E} \left[\sup_{x \in \mathbb{R}} \left| \beta_{n,m}^{(j)}(x) \right| \right] \leq 2 \sum_{L=1}^N \|\Delta_L\|_1 + k^{-1}.$$

Hence, using Lemma 7.1 of Rio 2017 (where we divide by \sqrt{k} the considering inequality in the lemma), we obtain that

$$\begin{aligned} \mathbb{E} \left[\sup_{x \in \mathbb{R}} \left| \beta_{n,m}^{(j)}(x) \right| \right] &\leq 2 \frac{C_0}{\sqrt{k}} \sum_{L=1}^N \left(2^{-\frac{(\zeta-1)^2}{(4\zeta)^2}} \right)^L + k^{-1} \\ &\leq \frac{2}{\sqrt{k}} \frac{C_0}{1 - 2^{-\frac{(\zeta-1)^2}{(4\zeta)^2}}} + k^{-1} \triangleq C_1 k^{-1/2} + k^{-1}, \end{aligned}$$

where C_0 and C_1 are constants depending on ζ and λ .

Now, fix $x \in \mathbb{R}$ and denote by $\Phi : \mathbb{R}^k \mapsto [0, 1]$, the function defined by

$$\Phi(x_1, \dots, x_k) = \sup_{x \in \mathbb{R}} \left| \frac{1}{k} \sum_{i=1}^k \mathbb{1}_{\{x_i \leq x\}} - F_m^{(j)}(x) \right|.$$

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, we obtain with some calculations:

$$|\Phi(\mathbf{x}) - \Phi(\mathbf{y})| \leq \sup_{x \in \mathbb{R}} \frac{1}{k} \sum_{i=1}^k \left| \mathbb{1}_{\{x_i \leq x\}} - \mathbb{1}_{\{y_i \leq x\}} \right| \leq \frac{1}{k} \sum_{i=1}^k \mathbb{1}_{\{x_i \neq y_i\}}.$$

Thus, Φ is k^{-1} -Lipschitz with respect to the Hamming distance. Under algebraically φ -mixing process, we may apply Theorem 8 of Mohri and Rostamizadeh 2010 with $(M_{m,1}^{(j)}, \dots, M_{m,k}^{(j)})$, we obtain with probability at least $1 - \exp\{-t^2 k / C_3\}$

$$\sup_{x \in \mathbb{R}} \left| \hat{F}_{n,m}^{(j)}(x) - F_m^{(j)}(x) \right| \leq \mathbb{E} \left[\sup_{x \in \mathbb{R}} \left| \beta_{n,m}^{(j)}(x) \right| \right] + \frac{t}{3} \leq C_1 k^{-1/2} + C_2 k^{-1} + \frac{t}{3}.$$

Thus, with probability at most $\exp\{-t^2k/C_3\}$

$$\sup_{x \in \mathbb{R}} \left| \hat{F}_{n,m}^{(j)}(x) - F_m^{(j)}(x) \right| \geq C_1 k^{-1/2} + k^{-1} + \frac{t}{3}.$$

Using Bonferroni inequality

$$\mathbb{P} \left\{ E_3 \geq \frac{t}{3} \right\} \leq d \mathbb{P} \left\{ \sup_{x \in \mathbb{R}} \left| \hat{F}_{n,m}^{(j)}(x) - F_m^{(j)}(x) \right| \geq t \right\},$$

we thus obtain a control bound for E_3 . Assembling all the controls obtained for E_1 , E_2 and E_3 , we obtain the desired result. \square

We present here the strong consistency of our procedure when the dimension d is fixed the sample size n grows at infinity. The main technicality of the proof has already been tackled in Proposition 4.

Proof of Theorem 3 If a and b are not in the same cluster according to \bar{O} , i.e. $a \not\sim^{\bar{O}} b$, then with Condition \mathcal{B} satisfied, $\chi(a, b) = 0$. Therefore, using Proposition 4 along with Conditions \mathcal{C} and \mathcal{D} , we can conclude that almost surely

$$\lim_{n \rightarrow \infty} \hat{\chi}_{n,m}(a, b) = 0 \leq \tau.$$

Now, if $a \overset{\bar{O}}{\sim} b$, then $\chi(a, b) > 0$ and again by Propositions 4 and Conditions \mathcal{C} and \mathcal{D} , we the almost sure convergence

$$\lim_{n \rightarrow \infty} \hat{\chi}_{n,m}(a, b) = \chi(a, b) > 0,$$

hence

$$a \overset{\bar{O}}{\sim} b \iff \lim_{n \rightarrow \infty} \hat{\chi}_{n,m}(a, b) > 0.$$

Let us prove Theorem 3 by induction on the algorithm step l . We consider the algorithm at some step $l - 1$ and assume that the algorithm was consistent up to this step, i.e. $\hat{O}_j = \bar{O}_j$ for $j = 1, \dots, l - 1$.

If $\lim_{n \rightarrow \infty} \hat{\chi}_{n,m}(a_l, b_l) = 0$, then no $b \in S$ is in the same group of a_l . Since the algorithm has been consistent up to this step l , it means that a_l is a singleton and $\hat{O}_l = \{a_l\}$.

If $\lim_{n \rightarrow \infty} \hat{\chi}_{n,m}(a_l, b_l) > 0$, then $a_l \overset{\bar{O}}{\sim} b$. The equivalence above implies that $\hat{O}_l = S \cap \bar{O}_l$. Since the algorithm has been consistent up until this step, we know that $\hat{O}_l = \bar{O}_l$. Therefore, the algorithm remains consistent at step l with probability tending to one as $n \rightarrow \infty$, and Theorem 3 follows by induction. \square

The proof of Theorem 4 needs the following results : (1) an upper bound over the quantity $|\hat{\theta}_{n,m}(a, b) - \theta_m(a, b)|$ with respect to $|\hat{\nu}_{n,m}(a, b) - \nu_m(a, b)|$ to use the concentration inequality introduced in Proposition 5, (2) exhibit an event such that $\{\hat{O} = \bar{O}\}$. Lemmas 1 and 2 below address these two questions. Then, taking benefits of these results, we show that the probability of the exhibited event such that $\{\hat{O} = \bar{O}\}$ holds with high probability, as stated in Theorem 4.

Lemma 1. Consider a pair $(a, b) \in \{1, \dots, d\}^2$, the following inequality holds:

$$|\hat{\theta}_{n,m}(a, b) - \theta_m(a, b)| \leq 9 |\hat{\nu}_{n,m}(a, b) - \nu_m(a, b)|.$$

Proof of Lemma 1 We may write the respective quantities as $\theta = f(\nu(a, b))$ and $\hat{\theta}_{n,m} = f(\hat{\nu}_{n,m}(a, b))$ where f is a function defined as follows,

$$f : [0, 1/6] \rightarrow [1, 2]$$

$$x \mapsto \frac{1/2+x}{1/2-x},$$

with $f(x) \in [1, 2]$ by definition of the sub-asymptotic extremal coefficient θ_m . The domain of this function is restricted to the interval $[0, 1/6]$ because we have $f(x) \leq 2$, or

$$x + \frac{1}{2} \leq 1 - 2x,$$

which holds if $x \leq 1/6$. The inequality $f(x) \geq 1$ gives the positivity of the domain. In particular, $x < 1/2$ and thus $2^{-1} - x \geq 3^{-1} > 0$. Taking derivative of f , we find that

$$|f'(x)| = \frac{1}{(1/2 - x)^2} \leq 3^2, \quad x \in [0, 1/6].$$

Therefore, f is 9-Lipschitz continuous and we have

$$|\hat{\theta}_{n,m}(a, b) - \theta_m(a, b)| = |f(\hat{\nu}_{n,m}(a, b)) - f(\nu_m(a, b))| \leq 9|\hat{\nu}_{n,m}(a, b) - \nu_m(a, b)|.$$

This completes the proof. \square

Lemma 2. Consider the AI-block model in Definition 1 under Condition \mathcal{B} . Define

$$\kappa = \sup_{a, b \in \{1, \dots, d\}} |\hat{\chi}_{n,m}(a, b) - \chi(a, b)|.$$

Consider parameters (τ, η) fulfilling

$$\tau \geq \kappa, \quad \eta \geq \kappa + \tau. \quad (28)$$

If $\text{MECO}(\mathcal{X}) > \eta$, then Algorithm (ECO) yields $\hat{O} = \bar{O}$.

Proof of Lemma 2 If $a \not\sim^{\bar{O}} b$, then $\chi(a, b) = 0$ by Condition \mathcal{B} and

$$\hat{\chi}_{n,m}(a, b) = \hat{\chi}_{n,m}(a, b) - \chi(a, b) \leq \kappa \leq \tau.$$

Now, if $a \sim^{\bar{O}} b$, if $\mathcal{X} \in \mathbb{X}(\eta)$ then $\chi(a, b) > \kappa + \alpha$ and

$$\kappa + \tau < \chi(a, b) - \hat{\chi}_{n,m}(a, b) + \hat{\chi}_{n,m}(a, b),$$

and thus $\hat{\chi}_{n,m}(a, b) > \tau$. In particular, under (28) and the separation condition $\text{MECO}(\mathcal{X}) > \eta$, we have

$$a \sim^{\bar{O}} b \iff \hat{\chi}_{n,m}(a, b) > \tau. \quad (29)$$

Let us prove the lemma by induction on the algorithm step l . We consider the algorithm at some step $l - 1$ and assume that the algorithm was consistent up to this step, i.e. $\hat{O}_j = \bar{O}_j$ for $j = 1, \dots, l - 1$.

If $\hat{\chi}_{n,m}(a_l, b_l) \leq \tau$, then according to (29), no $b \in S$ is in the same group of a_l . Since the algorithm has been consistent up to this step l , it means that a_l is a singleton and $\hat{O}_l = \{a_l\}$.

If $\hat{\chi}_{n,m}(a_l, b_l) > \tau$, then $a_l \sim^{\bar{O}} b$ according to (29). Furthermore, the equivalence implies that $\hat{O}_l = S \cap \bar{O}_l$. Since the algorithm has been consistent up to this step, we have $\hat{O}_l = \bar{O}_l$. To conclude, the algorithm remains consistent at the step l and the result follows by induction. \square

Proof of Theorem 4 We have that for $t > 0$:

$$\mathbb{P} \left\{ \sup_{a,b \in \{1, \dots, d\}} |\hat{\theta}_{n,m}(a, b) - \theta_m(a, b)| \geq t \right\} \leq d^2 \mathbb{P} \left\{ |\hat{\theta}_{n,m}(a, b) - \theta_m(a, b)| \geq t \right\}.$$

With probability at least $1 - 2(1 + \sqrt{e})d^2 \exp\{-t^2k/C_3\}$, using Proposition 5 and Lemma 1, one has

$$\sup_{a,b \in \{1, \dots, d\}} \left| \hat{\theta}_{n,m}(a, b) - \theta(a, b) \right| \leq d_m + C_1 k^{-1/2} + C_2 k^{-1} + t,$$

By considering $\delta \in]0, 1[$ and solve the following equation

$$\frac{\delta}{d^2} = 2(1 + \sqrt{e}) \exp \left\{ -\frac{kt^2}{C_3} \right\},$$

with respect to t gives that the event

$$\sup_{a,b \in \{1, \dots, d\}} \left| \hat{\theta}_{n,m}(a, b) - \theta(a, b) \right| \geq d_m + C_1 k^{-1/2} + C_2 k^{-1} + C_3 \sqrt{\frac{1}{k} \ln \left(\frac{2(1 + \sqrt{e})d^2}{\delta} \right)},$$

is of probability at most δ . Now, taking $\delta = 2(1 + \sqrt{e})d^{-2\gamma}$, with $\gamma > 0$, we have

$$\sup_{a,b \in \{1, \dots, d\}} \left| \hat{\theta}_{n,m}(a, b) - \theta(a, b) \right| \leq d_m + C_1 k^{-1/2} + C_2 k^{-1} + C_3 \sqrt{\frac{(1 + \gamma) \ln(d)}{k}},$$

with probability at least $1 - 2(1 + \sqrt{e})d^{-2\gamma}$ for C_3 sufficiently large. The result then follows from Lemma 2 along with Condition \mathcal{B} and algebraically φ -mixing random process, since

$$\mathbb{P} \left\{ \kappa \leq d_m + C_1 k^{-1/2} + C_2 k^{-1} + C_3 \sqrt{\frac{(1 + \gamma) \ln(d)}{k}} \right\} \geq 1 - 2(1 + \sqrt{e})d^{-2\gamma},$$

and $\text{MECO}(\mathcal{X}) > \eta$ by assumption. \square

Now, we prove the theoretical result giving support to our cross validation process.

Proof of Proposition 6 Notice that, by using Proposition 4 along with Conditions \mathcal{C} and \mathcal{D} , $\hat{\theta}_{(1)}^{(O_g)}$ for $g \in \{1, \dots, G\}$ and $\hat{\theta}_{(2)}$ are strongly consistent. Then it holds that

$$\widehat{\text{SECO}}(\bar{O}) \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \text{SECO}(\bar{O}) = 0$$

Using Lebesgue's theorem, we obtain

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\widehat{\text{SECO}}(\bar{O}) \right] = 0.$$

Furthermore, applying again Lebesgue theorem

$$0 < \mathbb{E} \left[\text{SECO}(\hat{O}) \right] = \lim_{n \rightarrow \infty} \mathbb{E} \left[\widehat{\text{SECO}}(\hat{O}) \right].$$

To conclude,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\widehat{\text{SECO}}(\bar{O}) \right] < \lim_{n \rightarrow \infty} \mathbb{E} \left[\widehat{\text{SECO}}(\hat{O}) \right].$$

Hence the result. \square

Appendix A.3 Proofs of Section 4

In the following we prove that the model introduced in Section 4 is in the domain of attraction of an AI-block model. This comes down from some elementary algebra where the fundamental argument is given by Proposition 4.2 in Bücher and Segers 2014, from which the inspiration for the model was drawn thereof.

Proof of Proposition 7 We aim to show that the following quantity

$$\left| C_{\theta, \beta_0} \left(C_{\theta, \beta_1}^{(O_1)}(\{\mathbf{u}^{(O_1)}\}^{1/m}), \dots, C_{\theta, \beta_G}^{(O_G)}(\{\mathbf{u}^{(O_G)}\}^{1/m}) \right)^m - C_{0, \beta_0} \left(C_{0, \beta_1}^{(O_1)}(\mathbf{u}^{(O_1)}), \dots, C_{0, \beta_G}^{(O_G)}(\mathbf{u}^{(O_G)}) \right) \right|,$$

converges to 0 uniformly in $\mathbf{u} \in [0, 1]^d$. Using Equation (21) in the main article, the latter term is equal to

$$E_{0,m} := \left| C_{\theta, \beta_0} \left(C_{\theta/m, \beta_1}^{(O_1)}(\mathbf{u}^{(O_1)})^{1/m}, \dots, C_{\theta/m, \beta_G}^{(O_G)}(\mathbf{u}^{(O_G)})^{1/m} \right)^m - C_{0, \beta_0} \left(C_{0, \beta_1}^{(O_1)}(\mathbf{u}^{(O_1)}), \dots, C_{0, \beta_G}^{(O_G)}(\mathbf{u}^{(O_G)}) \right) \right|.$$

Thus, again with Equation (21) and triangle inequality

$$\begin{aligned} E_{0,m} &\leq \left| C_{\theta/m, \beta_0} \left(C_{\theta/m, \beta_1}^{(O_1)}(\mathbf{u}^{(O_1)}), \dots, C_{\theta/m, \beta_G}^{(O_G)}(\mathbf{u}^{(O_G)}) \right) - C_{0, \beta_0} \left(C_{\theta/m, \beta_1}^{(O_1)}(\mathbf{u}^{(O_1)}), \dots, C_{\theta/m, \beta_G}^{(O_G)}(\mathbf{u}^{(O_G)}) \right) \right| \\ &\quad + \left| C_{0, \beta_0} \left(C_{\theta/m, \beta_1}^{(O_1)}(\mathbf{u}^{(O_1)}), \dots, C_{\theta/m, \beta_G}^{(O_G)}(\mathbf{u}^{(O_G)}) \right) - C_{0, \beta_0} \left(C_{0, \beta_1}^{(O_1)}(\mathbf{u}^{(O_1)}), \dots, C_{0, \beta_G}^{(O_G)}(\mathbf{u}^{(O_G)}) \right) \right| \\ &=: E_{1,m} + E_{2,m}. \end{aligned}$$

As $C_{\theta/m, \beta_0}$ converges uniformly to C_{0, β_0} , then, uniformly in $\mathbf{u} \in [0, 1]^d$, $E_{1,m} \xrightarrow{m \rightarrow \infty} 0$. Now, using Lipschitz property of the copula function, one has

$$E_{2,m} \leq \sum_{g=1}^G \left| C_{\theta/m, \beta_g}^{(O_g)}(\mathbf{u}^{(O_g)}) - C_{0, \beta_g}^{(O_g)}(\mathbf{u}^{(O_g)}) \right|,$$

which converges almost surely to 0 as $m \rightarrow \infty$. The limiting copula is an extreme value copula by $\beta_0 \leq \min\{\beta_1, \dots, \beta_G\}$, see Example 3.8 of Hofert, Huser, and Prasad 2018. Hence the result. \square

Proof of Remark 3 Take $a \neq b$ fixed, we have, using Lemma 1

$$|\chi_m(a, b) - \chi(a, b)| = |\theta_m(a, b) - \theta(a, b)| \leq 9 |\nu_m(a, b) - \nu(a, b)|,$$

where $\nu_m(a, b)$ (resp. $\nu(a, b)$) is the madogram computed between $M_m^{(a)}$ and $M_m^{(b)}$ (resp. between $X^{(a)}$ and $X^{(b)}$) and we use Lemma 1 to obtain the inequality. Using the results of Lemma 1 of

Marcon et al. 2017, we have

$$\begin{aligned} \nu_m(a, b) - \nu(a, b) &= \frac{1}{2} \left(\int_{[0,1]} (C_m - C)(\mathbf{1}, x^{(a)}, \mathbf{1}) dx^{(a)} + \int_{[0,1]} (C_m - C)(\mathbf{1}, x^{(b)}, \mathbf{1}) dx^{(b)} \right) \\ &\quad - \int_{[0,1]} (C_m - C)(1, \dots, \underbrace{x}_{a\text{th index}}, 1, \dots, 1, \underbrace{x}_{b\text{th index}}, \dots, 1) dx, \end{aligned}$$

where the integration is taken respectively for the a -th, b -th and a, b -th components. Hence

$$\begin{aligned} |\nu_m(a, b) - \nu(a, b)| &\leq \frac{1}{2} \left(\int_{[0,1]} |(C_m - C)(\mathbf{1}, x^{(a)}, \mathbf{1})| dx^{(a)} + \int_{[0,1]} |(C_m - C)(\mathbf{1}, x^{(b)}, \mathbf{1})| dx^{(b)} \right) \\ &\quad + \int_{[0,1]} |(C_m - C)(1, \dots, \underbrace{x}_{a\text{th index}}, 1, \dots, 1, \underbrace{x}_{b\text{th index}}, \dots, 1)| dx. \end{aligned}$$

Using Proposition 4.3 in Bücher and Segers 2014 we obtain that $|C_m - C|(\mathbf{u}) = O(1/m)$, uniformly in $\mathbf{u} \in [0, 1]^d$. Hence the statement. \square

Appendix A.4 Proof of Proposition 8 in Section 5

Proofs of consistency theorems for k -means clustering oftenly needs a uniform strong law of large numbers (SLLN) stated as

$$\sup_{g \in \mathcal{G}} \left| \int gd(\mathbb{P}_n - \mathbb{P}) \right| \xrightarrow[n \rightarrow \infty]{a.s.} 0, \tag{30}$$

(see Section 4 of Pollard 1981), where \mathbb{P} is a probability measure on $\mathcal{B}(\mathbb{R}^d)$, \mathbb{P}_n the empirical measure and \mathcal{G} a class of functions. Equation (30) is also stated as the class of functions \mathcal{G} is Glivenko-Cantelli (see Chapter 2 of Vaart and Wellner 1996). In Proposition 3.3 of Janßen and Wan 2020, where \mathbb{P} and \mathbb{P}_n are replaced respectively by the angular measure S and its empirical counterpart S_n , it is shown that condition (30) holds. In our framework, we consider the extreme value copula C , the copula of block maxima C_m , its empirical counterpart $\hat{C}_{n,m}$ where the second is in the copula domain of attraction of C . The consistency of k -means clustering using madogram directly comes down from arguments given in Janßen and Wan 2020 if we are able to state Equation (30) for a specific class of function \mathcal{G} where the madogram belongs to.

For this purpose, the notion of bounded variation of functions and in particular the integration by part formula for Lebesgue-Stieltjes integral are of prime interest (see, for example, Theorem 6 of Fermanian, Radulovic, and Wegkamp 2004). We will say that a function f is of bounded variation in the sense of Hardy-Krause if $V_{HK}(f) < \infty$ (see Section 2 of Radulović, Wegkamp, and Zhao 2017 for a definition). Let us consider \mathcal{G} as the class of functions which are continuous and $V_{HK}(g) < \infty$.

Proof of Proposition 8 We want to prove that for every $g \in \mathcal{G}$

$$\int gd\hat{C}_{n,m} \xrightarrow[n \rightarrow \infty]{a.s.} \int gdC. \tag{31}$$

Using integration by parts, we have that

$$\int gd\hat{C}_{n,m} = \Gamma(\hat{C}_{n,m}, g),$$

where $\Gamma(\cdot, g)$ is a linear and Lipschitz function, hence continuous. Now, if we can state that

$$\|\hat{C}_{n,m} - C\|_\infty \xrightarrow[n \rightarrow \infty]{a.s.} 0, \quad (32)$$

we obtain that $\Gamma(\hat{C}_{n,m}, g) \xrightarrow[n \rightarrow \infty]{a.s.} \Gamma(C, g)$, using continuity, hence (31). Let us prove Equation (32). Using triangle inequality, we have

$$\|\hat{C}_{n,m} - C\|_\infty \leq \|\hat{C}_{n,m} - C_{n,m}^o\|_\infty + \|C_{n,m}^o - C\|_\infty.$$

The second term in the right hand side converges almost surely to 0 by Lemma 4 (see Section Appendix C.1) with Conditions \mathcal{C} and \mathcal{D} . It remains to work on the first term. Now, we have

$$\begin{aligned} \|\hat{C}_{n,m} - C_{n,m}^o\|_\infty &\leq \frac{1}{k} \sum_{i=1}^k \left| \mathbb{1}_{\{\hat{U}_{n,m,i}^{(j)} \leq u^{(j)}, 1 \leq j \leq d\}} - \mathbb{1}_{\{U_{m,i}^{(j)} \leq u^{(j)}, 1 \leq j \leq d\}} \right| \\ &\leq \frac{1}{k} \sum_{i=1}^k \prod_{j=1}^d \mathbb{1}_{\{\hat{U}_{n,m,i}^{(j)} \neq U_{m,i}^{(j)}\}}. \end{aligned}$$

Notice that, for every $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, d\}$

$$\begin{aligned} \mathbb{1}_{\{\hat{U}_{n,m,i}^{(j)} \neq U_{m,i}^{(j)}\}} &= \mathbb{1}_{\{|\hat{U}_{n,m,i}^{(j)} - U_{m,i}^{(j)}| > 0\}} \\ &\leq \mathbb{1}_{\{\sup_{x \in \mathbb{R}} |\hat{F}_{n,m}^{(j)}(x) - F_m^{(j)}(x)| > 0\}} = \mathbb{1}_{\{\sup_{u \in [0,1]} \left| \frac{1}{k} \sum_{i=1}^k \mathbb{1}_{\{U_{m,i}^{(j)} \leq u\}} - u \right| > 0\}}. \end{aligned}$$

Thus

$$\|\hat{C}_{n,m} - C_{n,m}^o\|_\infty \leq \mathbb{1}_{\{\sup_{u \in [0,1]} \left| \frac{1}{k} \sum_{i=1}^k \mathbb{1}_{\{U_{m,i}^{(j)} \leq u\}} - u \right| > 0\}}.$$

Denote by

$$\begin{aligned} A_n &= \left\{ \omega \in \Omega : \sup_{u \in [0,1]} \left| \frac{1}{k} \sum_{i=1}^k \mathbb{1}_{\{U_{m,i}^{(j)} \leq u\}} - u \right|(\omega) > 0 \right\}, \\ A_{n,\epsilon} &= \left\{ \omega \in \Omega : \sup_{u \in [0,1]} \left| \frac{1}{k} \sum_{i=1}^k \mathbb{1}_{\{U_{m,i}^{(j)} \leq u\}} - u \right|(\omega) > \epsilon \right\}, \end{aligned}$$

where $\epsilon > 0$. Using Lemma 4 in Section Appendix C.1, we have that

$$\mathbb{P} \left\{ \limsup_{n \rightarrow \infty} A_{n,\epsilon} \right\} = 0.$$

Now, remark that $A_n = \bigcup_{N \geq 1} A_{n,1/N}$. We thus have

$$\mathbb{P} \left\{ \limsup_{n \rightarrow \infty} A_n \right\} = \mathbb{P} \left\{ \limsup_{n \rightarrow \infty} \bigcup_{N \geq 1} A_{n,1/N} \right\} = \mathbb{P} \left\{ \bigcup_{N \geq 1} \limsup_{n \rightarrow \infty} A_{n,1/N} \right\}.$$

Using σ -subadditivity of measures, we have

$$\mathbb{P} \left\{ \limsup_{n \rightarrow \infty} A_n \right\} \leq \sum_{N \geq 1} \mathbb{P} \left\{ \limsup_{n \rightarrow \infty} A_{n,1/N} \right\} = 0.$$

Hence $\mathbf{1}_{A_n} = 0$ almost surely as $n \rightarrow \infty$ which results on $\|\hat{C}_{n,m} - C_{n,m}^o\|_\infty$ converges almost surely to 0. We thus obtain (32) hence (31).

In order to prove Proposition 8, we prove in the following that the function

$$g(\mathbf{u}) = \min_{g \in \{1, \dots, G\}} \frac{1}{2} \sum_{j=1}^d |u^{(j)} - \psi_g^{(j)}|, \quad \psi_g \in [0, 1]^d, \quad g \in \{1, \dots, G\}.$$

is of bounded variations in the sens of Hardy Krause (BHKV). Indeed, for every $j \in \{1, \dots, d\}$, the functions $(\mathbf{u}, \mathbf{v}) \mapsto u^{(j)}$ and $(\mathbf{u}, \mathbf{v}) \mapsto v^{(j)}$ are BHKV on $[0, 1]^d \times [0, 1]^d$ since it depends only on one variable and is monotone in this variable. As the difference between two BHKV functions are BHKV, it follows $\forall j \in \{1, \dots, d\}$ the function $(\mathbf{u}, \mathbf{v}) \mapsto u^{(j)} - v^{(j)}$ is BHKV on $[0, 1]^d \times [0, 1]^d$. Taking the absolute value preserves the BHKV property on $[0, 1]^d$. It follows that $(\mathbf{u}, \mathbf{v}) \mapsto \sum_{j=1}^d |u^{(j)} - v^{(j)}|$ is BHKV as a sum of functions that are BHKV. Multiplying by a constant preserves the bounded variation property of the function. Finally, taking the min over a finite set of values in $[0, 1]^d$ maintains the BHKV property as it holds for every $(\mathbf{u}, \mathbf{v}) \in [0, 1]^d \times [0, 1]^d$. Hence g is BHKV and clearly a continuous function. Thus using Equation (31) along with arguments from Theorem 3.1 in Janßen and Wan 2020 we obtain the consistency the result. \square

Appendix B. Proofs of additional results from Section 2

Appendix B.1 Proof of Proposition 3

We will establish the result proceeding as (iii) \implies (i) \implies (ii) \implies (iii) where we directly have (i) \implies (ii). Now for (iii) \implies (i), suppose Λ concentrates on the set (9). Then for $\mathbf{x} > 0$, noting $A_g(\mathbf{x}) = \{\mathbf{u} \in E, \exists a \in O_g, u_a > x_a\}$ for $g \in \{1, \dots, G\}$, we obtain

$$\begin{aligned} -\ln H(\mathbf{x}) &= \Lambda(E \setminus [0, \mathbf{x}]) = \Lambda\left(\bigcup_{g=1}^G A_g(\mathbf{x})\right) \\ &= \sum_{g=1}^G \Lambda(A_g(\mathbf{x})) + \sum_{g=2}^G (-1)^{g+1} \sum_{1 \leq i_1 < i_2 < \dots < i_l \leq G} \Lambda(A_{i_1}(\mathbf{x}) \cap \dots \cap A_{i_l}(\mathbf{x})), \end{aligned}$$

so that because of Equation (9) in the main paper,

$$-\ln H(\mathbf{x}) = \sum_{g=1}^G \Lambda(A_g(\mathbf{x})),$$

and we have $H(\mathbf{x}) = \prod_{g=1}^G \exp\{-\Lambda(\{u \in E, \exists a \in O_g, u_a > x_a\})\} = \prod_{g=1}^G H^{(O_g)}(\mathbf{x}^{(O_g)})$.

Thus H is written as a product of the G distributions corresponding to random vectors $\mathbf{X}^{(O_1)}, \dots, \mathbf{X}^{(O_G)}$, as desired.

It remains to show (ii) \implies (iii). Set $Q^{(O_k)}(\mathbf{y}^{(O_k)}) = -\ln \mathbb{P}\{\mathbf{X}^{(O_k)} \leq \mathbf{y}^{(O_k)}\}$ for $k \in \{1, \dots, G\}$. We have for $\mathbf{y} > 0$ that blockwise independence implies, with $k \neq l$,

$$Q^{(O_k)}(\mathbf{y}^{(O_k)}) + Q^{(O_l)}(\mathbf{y}^{(O_l)}) = -\ln \mathbb{P}\{\mathbf{X}^{(O_k)} \leq \mathbf{y}^{(O_k)}, \mathbf{X}^{(O_l)} \leq \mathbf{y}^{(O_l)}\}.$$

Since $H(\mathbf{x}) = \exp\{-\Lambda(E \setminus [\mathbf{0}, \mathbf{x}])\}$ for $\mathbf{x} > 0$, we have

$$\begin{aligned} Q^{(O_k)}(\mathbf{y}^{(O_k)}) + Q^{(O_l)}(\mathbf{y}^{(O_l)}) &= \Lambda(\{\mathbf{x}, \exists a \in O_k, x_a > y_a\} \cup \{\mathbf{x}, \exists b \in O_l, x_b > y_b\}) \\ &= \Lambda(\{\mathbf{x}, \exists a \in O_k, x_a > y_a\}) + \Lambda(\{\mathbf{x}, \exists b \in O_l, x_b > y_b\}) \\ &\quad - \Lambda(\{\mathbf{x}, \exists a \in O_k, x_a > y_a, \exists b \in O_l, x_b > y_b\}) \\ &= Q^{(O_k)}(\mathbf{y}^{(O_k)}) + Q^{(O_l)}(\mathbf{y}^{(O_l)}) \\ &\quad - \Lambda(\{\mathbf{x}, \exists a \in O_k, x_a > y_a, \exists b \in O_l, x_b > y_b\}), \end{aligned}$$

and thus

$$\Lambda(\{\mathbf{x}, \exists a \in O_k, x_a > y_a, \exists b \in O_l, x_b > y_b\}) = 0,$$

so that (iii) holds. This is equivalent to Λ concentrates on the set in Equation (9) in the main paper.

Appendix B.2 Proof of Theorem 2

In this section, we will exhibit conditions under which subvectors $\mathbf{X}^{(O_1)}, \dots, \mathbf{X}^{(O_G)}$ of an extreme value random vector \mathbf{X} with law H are independent whence $(\mathbf{Z}_t, t \in \mathbb{Z})$ is a multivariate random process in the domain of attraction of \mathbf{X} , i.e., $\mathbf{Z} \in D(F)$. It is known, by Remark 3 that independence of $\mathbf{X}^{(O_1)}, \dots, \mathbf{X}^{(O_G)}$ is equivalent to blockwise independence. So we first consider the case for $G = 2$. As we consider a stationary mixing random process, the main technical issue is to approximate the quantity $\mathbb{P}\{\mathbf{M}_m \leq \mathbf{a}_m(\mathbf{x})\}$ with the quantity $\mathbb{P}\{\mathbf{M}_{r_m} \leq \mathbf{a}_m(\mathbf{x})\}^k$ for a fixed integers k and an integer sequence $r_m \rightarrow \infty$, which might be expected intuitively. We state below conditions to obtain such an approximation. Specifically, let k be a fixed integer, we consider the maximum of $2k$ sub-blocks as follows

$$M_{r_m,1}^{\text{odd},(j)}, M_{\ell_m,1}^{\text{even},(j)}, \dots, M_{r_m,k}^{\text{odd},(j)}, M_{\ell_m,k}^{\text{even},(j)}, \quad j \in \{1, \dots, d\}$$

such that $M_{r_m,i}^{\text{odd},(j)}, M_{\ell_m,i}^{\text{even},(j)}, i \in \{1, \dots, k\}$ alternatively have length r_m and ℓ_m . We so have the following equality $m = (r_m + \ell_m)k$.

Lemma 3. *Suppose Condition A holds for the stationary process $(\mathbf{Z}_t, t \in \mathbb{Z})$. Fix $\mathbf{x} \in \mathbb{R}^d$, then the following statements are equivalent:*

- (i) $\mathbb{P}\{\mathbf{M}_m \leq \mathbf{a}_m(\mathbf{x})\} \rightarrow c$, as $m \rightarrow \infty$,
- (ii) $\mathbb{P}\{\mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_m(\mathbf{x})\}^k \rightarrow c$, as $m \rightarrow \infty$,
- (iii) $k(1 - \mathbb{P}\{\mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_m(\mathbf{x})\}) \rightarrow -\ln(c)$, as $m \rightarrow \infty$.

Proof of Lemma 3 Let $1 \leq j \leq d$, we are going to show that

$$\left| \mathbb{P}\{\mathbf{M}_m \leq \mathbf{a}_m(\mathbf{x})\} - \mathbb{P}\left\{\bigvee_{i=1}^k \mathbf{M}_{r_m,i}^{\text{odd}} \leq \mathbf{a}_m(\mathbf{x})\right\} \right| \rightarrow 0.$$

As $\bigvee_{i=1}^k M_{r_m,i}^{\text{odd},(j)} \leq M_m^{(j)}$, we have

$$\begin{aligned} &\left| \mathbb{1}_{\{\mathbf{M}_m \leq \mathbf{a}_m(\mathbf{x})\}} - \mathbb{1}_{\{\bigvee_{i=1}^k \mathbf{M}_{r_m,i}^{\text{odd}} \leq \mathbf{a}_m(\mathbf{x})\}} \right| \\ &\leq \sum_{j=1}^d \mathbb{1}_{\{\bigvee_{i=1}^k M_{r_m,i}^{\text{odd},(j)} \leq a_m^{(j)}(x^{(j)}) < \bigvee_{i=1}^k M_{\ell_m,i}^{\text{even},(j)}\}}. \end{aligned}$$

Taking the expectation and using stationarity, we obtain

$$\begin{aligned} & \left| \mathbb{P}\{\mathbf{M}_m \leq \mathbf{a}_m(\mathbf{x})\} - \mathbb{P}\left\{\bigvee_{i=1}^k \mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_m(\mathbf{x})\right\} \right| \\ & \leq \sum_{j=1}^d k \mathbb{P}\{M_{r_m,1}^{\text{odd},(j)} \leq a_m^{(j)}(x^{(j)}) < M_{\ell_m,1}^{\text{even},(j)}\}. \end{aligned}$$

Now, if m is sufficiently large we may choose $\lfloor r_m/\ell_m \rfloor$ sub-blocks of size ℓ_m contained within the first odd block of length r_m

$$M_{\ell_m,1,j}, \dots, M_{\ell_m, \lfloor r_m/\ell_m \rfloor, j}, \quad j \in \{1, \dots, d\}$$

when we, again, keep the ones with odd index so that they are separated from each other and from $M_{\ell_m,1}^{\text{even},(j)}$ by at least ℓ_m observations (if $\lfloor r_m/\ell_m \rfloor$ is odd, we can drop the last block in order that this condition holds). Then for every $j \in \{1, \dots, d\}$

$$\begin{aligned} \mathbb{P}\{M_{r_m,1}^{\text{odd},(j)} \leq a_m^{(j)}(x^{(j)}) < M_{\ell_m,1}^{\text{even},(j)}\} & \leq \mathbb{P}\left\{\bigcap_{s=1}^{\lfloor r_m/\ell_m \rfloor} M_{\ell_m,s,j}^{\text{odd}} \leq a_m^{(j)}(x^{(j)}) < M_{\ell_m,1}^{\text{even},(j)}\right\} \\ & = \mathbb{P}\left\{\bigcap_{s=1}^{\lfloor r_m/\ell_m \rfloor} M_{\ell_m,s,j}^{\text{odd}} \leq a_m^{(j)}(x^{(j)})\right\} \\ & \quad - \mathbb{P}\left\{\bigcap_{s=1}^{\lfloor r_m/\ell_m \rfloor} M_{\ell_m,s,j}^{\text{odd}} \leq a_m^{(j)}(x^{(j)}), M_{\ell_m,1}^{\text{even},(j)} \leq a_m^{(j)}(x^{(j)})\right\}. \end{aligned}$$

By stationarity, $\mathbb{P}\{M_{\ell_m,1}^{\text{odd},(j)} \leq a_m^{(j)}(x^{(j)})\} = \mathbb{P}\{M_{\ell_m,1}^{\text{even},(j)} \leq a_m^{(j)}(x^{(j)})\} =: p_j$, and by tools given in the proof of Lemma 2.5 in Leadbetter 1974, the two terms of the right hand side differ from $p_j^{\ell_m}$ and $p_j^{\ell_m+1}$ by no more than $(\lfloor r_m/\ell_m \rfloor - 1)\alpha(\ell_m)$, $(\lfloor r_m/\ell_m \rfloor)\alpha(\ell_m)$ respectively. Hence

$$\mathbb{P}\{M_{r_m,i}^{\text{odd},(j)} \leq a_m^{(j)}(x^{(j)}) < M_{r_m,i,j}^{\text{even}}\} \leq \frac{1}{\ell_m + 1} + 2\frac{r_m}{\ell_m}\alpha(\ell_m).$$

Thus, by Condition \mathcal{A} as $m \rightarrow \infty$

$$k\mathbb{P}\{M_{r_m,1}^{\text{odd},(j)} \leq a_m^{(j)}(x^{(j)}) < M_{\ell_m,1}^{\text{even},(j)}\} \rightarrow 0.$$

Hence the first result. Now we will prove that

$$\left| \mathbb{P}\left\{\bigvee_{i=1}^k \mathbf{M}_{r_m,i}^{\text{odd}} \leq \mathbf{a}_m(\mathbf{x})\right\} - \prod_{i=1}^k \mathbb{P}\left\{\mathbf{M}_{r_m,i}^{\text{odd}} \leq \mathbf{a}_m(\mathbf{x})\right\} \right| \rightarrow 0,$$

as $m \rightarrow \infty$. Write $\psi_{i,m} = \mathbb{1}_{\{\mathbf{M}_{r_m,i}^{\text{odd}} \leq \mathbf{a}_m(\mathbf{x})\}}$ and note that

$$\mathbb{P}\left\{\bigvee_{i=1}^k \mathbf{M}_{r_m,i}^{\text{odd}} \leq \mathbf{a}_m(\mathbf{x})\right\} = \mathbb{E}\left[\prod_{i=1}^k \psi_{i,m}\right].$$

Proceeding as in proof of Theorem 3.1 in Bücher and Segers 2014, one can show that

$$\left| \mathbb{E} \left[\Pi_{i=1}^k \psi_{i,m} \right] - \Pi_{i=1}^k \mathbb{E}[\psi_{i,m}] \right| \leq 4 \max \alpha \left(\sigma(\psi_{i,m}), \sigma(\Pi_{i'=i+1}^k \psi_{i',m}) \right),$$

where $(\sigma(\psi_{i,m}), \sigma(\Pi_{i'=i+1}^k \psi_{i',m}))$ denotes respectively the sigma-field induced by the random variables $\psi_{i,m}, \Pi_{i'=i+1}^k \psi_{i',m}$. Since the maxima $M_{r_m, i}^{\text{odd}, (j)}$ over different blocks $i \neq i'$ are based on observations that are at least ℓ_m observations apart, the right hand size of the last display is of the order $k \alpha(\ell_m)$. The latter converges to 0 by Condition **A**. Using that, we obtain by stationarity that

$$\mathbb{P} \left\{ \mathbf{M}_{r_m, 1}^{\text{odd}} \leq \mathbf{a}_m(\mathbf{x}) \right\}^k \rightarrow c, \quad \text{as } m \rightarrow \infty.$$

The last equivalence follows straightforwardly with the last expression. \square

Proof of Theorem 2 As announced at the beginning of this section, we prove the result for $G = 2$. Then Proposition 3 and the algebra involved in the proof of Theorem 2.2 of Takahashi 1994 give the statement of Theorem 2 for $G > 2$.

For any $\mathbf{x} \in \mathbb{R}^d$, $0 < H^{(O_1)}(\mathbf{x}^{(O_1)}), H^{(O_2)}(\mathbf{x}^{(O_2)}) < 1$, there exists a finite $c_{\mathbf{x}} > 0$ such that $\{H^{(O_1)}(\mathbf{x}^{(O_1)})\}^{1/s} > H^{(O_1)}(\mathbf{p}^{(O_1)})$, $\{H^{(O_2)}(\mathbf{x}^{(O_2)})\}^{1/s} > H^{(O_2)}(\mathbf{p}^{(O_2)})$, for every $s \geq c_{\mathbf{x}}$, where \mathbf{p} is an element of \mathbb{R}^d such that Equation (11) in the main text holds. Such a $c_{\mathbf{x}}$ can be taken as:

$$c_{\mathbf{x}} = \inf \left\{ c > 0, \left\{ H^{(O_i)}(\mathbf{x}^{(O_i)}) \right\}^{1/c} > H^{(O_i)}(\mathbf{p}^{(O_i)}), i = 1, 2 \right\}$$

Recall that by the decided subdivision of maxima stated at the begining of this section we have $m = (r_m + \ell_m)k$. In particular, by Equation (10) in the main text with Lemma 3 gives

$$\mathbb{P} \left\{ \mathbf{M}_{r_m, 1}^{\text{odd}, (O_i)} \leq \mathbf{a}_{(r_m + \ell_m) \lceil sk \rceil}^{(O_i)}(\mathbf{x}^{(O_i)}) \right\}^{sk} \xrightarrow{m \rightarrow \infty} H^{(O_i)}(\mathbf{x}^{(O_i)})$$

where $\lceil sk \rceil$ is the greatest integer less than or equal to sk . Thus

$$\mathbb{P} \left\{ \mathbf{M}_{r_m, 1}^{\text{odd}, (O_i)} \leq \mathbf{a}_{(r_m + \ell_m) \lceil sk \rceil}^{(O_i)}(\mathbf{x}^{(O_i)}) \right\}^k \xrightarrow{m \rightarrow \infty} \left\{ H^{(O_i)}(\mathbf{x}^{(O_i)}) \right\}^{1/s}, \quad i = 1, 2. \quad (33)$$

From the continuity of $H^{(O_i)}$, $\forall s \geq c_{\mathbf{x}}$, we can find a $\mathbf{q}^{(O_i)} \geq \mathbf{p}^{(O_i)}$ such that

$$H^{(O_i)}(\mathbf{q}^{(O_i)}) = \left\{ H^{(O_i)}(\mathbf{x}^{(O_i)}) \right\}^{1/s}, \quad i = 1, 2$$

Denoting by $\mathbf{q} = (\mathbf{q}^{(O_1)}, \mathbf{q}^{(O_2)})$ with, by construction, $\mathbf{q} \geq \mathbf{p}$.

From Equation (10) in the main text and Lemma 3, we obtain

$$\mathbf{P} \left\{ \mathbf{M}_{r_m, 1}^{\text{odd}, (O_i)} \leq \mathbf{a}_m^{(O_i)}(\mathbf{q}^{(O_i)}) \right\}^k \rightarrow H^{(O_i)}(\mathbf{q}^{(O_i)}) = \left\{ H^{(O_i)}(\mathbf{x}^{(O_i)}) \right\}^{1/s}, \quad i = 1, 2. \quad (34)$$

Denoting by

$$A_1(\mathbf{x}) = \left\{ \mathbf{M}_{r_m, 1}^{\text{odd}, (O_1)} \leq \mathbf{a}_m^{(O_1)}(\mathbf{x}^{(O_1)}) \right\}, \quad A_2(\mathbf{x}) = \left\{ \mathbf{M}_{r_m, 1}^{\text{odd}, (O_2)} \leq \mathbf{a}_m^{(O_2)}(\mathbf{x}^{(O_2)}) \right\},$$

one has for a given \mathbf{x}

$$\begin{aligned} \mathbb{P} \left\{ [A_1(\mathbf{x}) \cap A_2(\mathbf{x})]^c \right\} &= \mathbb{P} \left\{ A_1(\mathbf{x})^c \cup A_2(\mathbf{x})^c \right\} \\ &= \mathbb{P} \left\{ A_1(\mathbf{x})^c \right\} + \mathbb{P} \left\{ A_2(\mathbf{x})^c \right\} - \mathbb{P} \left\{ A_1(\mathbf{x})^c \cap A_2(\mathbf{x})^c \right\}. \end{aligned}$$

We thus obtain

$$\begin{aligned} \mathbb{P}\{A_1(\mathbf{x})^c \cap A_2(\mathbf{x})^c\} &= [1 - \mathbb{P}\{A_1(\mathbf{x})\}] + [1 - \mathbb{P}\{A_2(\mathbf{x})\}] \\ &\quad - [1 - \mathbb{P}\{A_1(\mathbf{x}) \cap A_2(\mathbf{x})\}]. \end{aligned} \quad (35)$$

Then by hypothesis we have in both hands with Lemma 3

$$\begin{aligned} k[1 - \mathbb{P}\{A_i(\mathbf{p})\}] &\xrightarrow{m \rightarrow \infty} -\ln H^{(O_i)}(\mathbf{p}^{(O_i)}), \quad i \in \{1, 2\} \\ k[1 - \mathbb{P}\{A_1(\mathbf{p}) \cap A_2(\mathbf{p})\}] &\xrightarrow{m \rightarrow \infty} -\ln H^{(O_1)}(\mathbf{p}^{(O_1)})H^{(O_2)}(\mathbf{p}^{(O_2)}). \end{aligned}$$

Thus using the identity in (35) with $\mathbf{x} = \mathbf{p}$, we obtain.

$$k \mathbb{P}\{A_1(\mathbf{p})^c \cap A_2(\mathbf{p})^c\} \xrightarrow{m \rightarrow \infty} 0.$$

By $\mathbf{q} \geq \mathbf{p}$ and using that $A_1(\mathbf{q})^c \cap A_2(\mathbf{q})^c \subset A_1(\mathbf{p})^c \cap A_2(\mathbf{p})^c$ as a is non-decreasing, we have

$$k \mathbb{P}\{A_1(\mathbf{q})^c \cap A_2(\mathbf{q})^c\} \xrightarrow{m \rightarrow \infty} 0.$$

Thus we have along with equality (35) (taking $\mathbf{x} = \mathbf{q}$) and Lemma 3

$$\mathbb{P}\left\{\mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_m(\mathbf{q})\right\}^k \xrightarrow{m \rightarrow \infty} H^{(O_1)}(\mathbf{q}^{(O_1)})H^{(O_2)}(\mathbf{q}^{(O_2)}). \quad (36)$$

The RHS of the equation above is equal to

$$\left\{H^{(O_1)}(\mathbf{x}^{(O_1)})\right\}^{1/s} \left\{H^{(O_2)}(\mathbf{x}^{(O_2)})\right\}^{1/s}.$$

We now want to prove the following fact: $\forall \mathbf{x} \in \mathbb{R}^d, \exists s \geq c_{\mathbf{x}}$ such that

$$\left[\exists \mathbf{p} \geq \mathbf{q}, H^{(O_i)}(\mathbf{q}^{(O_i)}) = \left\{H^{(O_i)}(\mathbf{x}^{(O_i)})\right\}^{1/s} \right]$$

implies

$$\left[\lim_{m \rightarrow \infty} \left| \mathbb{P}\left\{\mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_{(r_m + \ell_m) \lceil sk \rceil}(\mathbf{x})\right\}^k - \mathbb{P}\left\{\mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_m(\mathbf{q})\right\}^k \right| = 0 \right].$$

Reducing the proposition ad absurdum gives that $\exists \mathbf{x} \in \mathbb{R}^d, \forall s \geq c_{\mathbf{x}}$ such that

$$\left[\exists \mathbf{p} \geq \mathbf{q}, H^{(O_i)}(\mathbf{q}^{(O_i)}) = \left\{H^{(O_i)}(\mathbf{x}^{(O_i)})\right\}^{1/s} \right]$$

and

$$\left[\lim_{m \rightarrow \infty} \left| \mathbb{P}\left\{\mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_{(r_m + \ell_m) \lceil sk \rceil}(\mathbf{x})\right\}^k - \mathbb{P}\left\{\mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_m(\mathbf{q})\right\}^k \right| \neq 0 \right].$$

Under the conditions of the theorem and all the work done above, we know that the first statement under bracket is true and gives important convergence results given in Equations (33), (36). It remains to show that the second statement under bracket cannot be true. Using Equations (10) in the main text, Equation (36), one has

$$\begin{aligned} \lim_{m \rightarrow \infty} \mathbb{P}\left\{\mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_{(r_m + \ell_m) \lceil sk \rceil}(\mathbf{x})\right\}^k - \mathbb{P}\left\{\mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_m(\mathbf{q})\right\}^k &= \\ \left\{H(\mathbf{x})\right\}^{1/s} - \left\{H^{(O_1)}(\mathbf{x}^{(O_1)})\right\}^{1/s} \left\{H^{(O_2)}(\mathbf{x}^{(O_2)})\right\}^{1/s}. \end{aligned}$$

As H is an extreme value distribution, hence associated (see the proof of Proposition 1 for more details), we have

$$H(\mathbf{x}) \geq H^{(O_1)}(\mathbf{x}^{(O_1)})H^{(O_2)}(\mathbf{x}^{(O_2)}).$$

Then

$$\lim_{m \rightarrow \infty} \mathbb{P} \left\{ \mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_{(r_m+\ell_m)\lceil sk \rceil}(\mathbf{x}) \right\}^k - \mathbb{P} \left\{ \mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_m(\mathbf{q}) \right\}^k \geq 0.$$

Thus if the above limit is different from 0, then

$$\lim_{m \rightarrow \infty} \mathbb{P} \left\{ \mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_{(r_m+\ell_m)\lceil sk \rceil}(\mathbf{x}) \right\}^k - \mathbb{P} \left\{ \mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_m(\mathbf{q}) \right\}^k > \epsilon_1 > 0,$$

with ϵ_1 a positive constant. Of course, for every m , we have

$$\mathbb{P} \left\{ \mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_{(r_m+\ell_m)\lceil sk \rceil}(\mathbf{x}) \right\} \leq \mathbb{P} \left\{ \mathbf{M}_{r_m,1}^{\text{odd},(O_1)} \leq \mathbf{a}_{(r_m+\ell_m)\lceil sk \rceil}^{(O_1)}(\mathbf{x}^{(O_1)}) \right\}.$$

Hence

$$\begin{aligned} \lim_{m \rightarrow \infty} \mathbb{P} \left\{ \mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_{(r_m+\ell_m)\lceil sk \rceil}(\mathbf{x}) \right\}^k - \mathbb{P} \left\{ \mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_m(\mathbf{q}) \right\}^k &\leq \\ \lim_{m \rightarrow \infty} \mathbb{P} \left\{ \mathbf{M}_{r_m,1}^{\text{odd},(O_1)} \leq \mathbf{a}_{(r_m+\ell_m)\lceil sk \rceil}^{(O_1)}(\mathbf{x}^{(O_1)}) \right\}^k - \mathbb{P} \left\{ \mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_m(\mathbf{q}) \right\}^k &. \end{aligned}$$

The RHS of the inequality is equal to by Equations 33 and 36

$$\left\{ H^{(O_1)}(\mathbf{x}^{(O_1)}) \right\}^{1/s} \left(1 - \left\{ H^{(O_2)}(\mathbf{x}^{(O_2)}) \right\}^{1/s} \right).$$

Since $0 < H^{(O_1)}(\mathbf{x}^{(O_1)}), H^{(O_2)}(\mathbf{x}^{(O_2)}) < 1$, the above quantity goes to 0 as $s \rightarrow \infty$, we can thus pick an $\epsilon_2 < \epsilon_1$ and we know that there exists a $C > c_{\mathbf{x}}$ such that if $s \geq C$, then

$$\epsilon_1 < \lim_{m \rightarrow \infty} \mathbb{P} \left\{ \mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_{(r_m+\ell_m)\lceil sk \rceil}(\mathbf{x}) \right\}^k - \mathbb{P} \left\{ \mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_m(\mathbf{q}) \right\}^k \leq \epsilon_2 < \epsilon_1$$

which reduces to an absurdity. So the desired fact is true and we have $\forall \mathbf{x} \in \mathbb{R}^d, \exists s \geq c_{\mathbf{x}}$ and the following convergence result holds

$$\lim_{m \rightarrow \infty} \left| \mathbb{P} \left\{ \mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_{(r_m+\ell_m)\lceil sk \rceil}(\mathbf{x}) \right\}^k - \mathbb{P} \left\{ \mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_m(\mathbf{q}) \right\}^k \right| = 0.$$

So by (36),

$$\mathbb{P} \left\{ \mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_{(r_m+\ell_m)\lceil sk \rceil}(\mathbf{x}) \right\}^k \xrightarrow{m \rightarrow \infty} H^{(O_1)}(\mathbf{q}^{(O_1)})H^{(O_2)}(\mathbf{q}^{(O_2)}),$$

where the RHS is equal to

$$\left\{ H^{(O_1)}(\mathbf{x}^{(O_1)}) \right\}^{1/s} \left\{ H^{(O_2)}(\mathbf{x}^{(O_2)}) \right\}^{1/s}.$$

Hence

$$\mathbb{P} \left\{ \mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_{(r_m+\ell_m)\lceil sk \rceil}(\mathbf{x}) \right\}^{sk} \xrightarrow{m \rightarrow \infty} H^{(O_1)}(\mathbf{x}^{(O_1)})H^{(O_2)}(\mathbf{x}^{(O_2)}).$$

Applying Lemma 1.3 of Takahashi 1994 gives

$$\mathbb{P} \left\{ \mathbf{M}_{r_m,1}^{\text{odd}} \leq \mathbf{a}_m(\mathbf{x}) \right\}^k \xrightarrow{m \rightarrow \infty} H^{(O_1)}(\mathbf{x}^{(O_1)})H^{(O_2)}(\mathbf{x}^{(O_2)}).$$

The proof is completed by applying Lemma 3. □

Appendix C. Further results

Appendix C.1 A usefull Glivenko-Cantelli result for the copula with known margins in a weakly dependent setting

In this section, we will prove an important auxiliary result: the empirical copula estimator $\hat{C}_{n,m}^o$ based on the weakly dependent sample $\mathbf{U}_{m,1}, \dots, \mathbf{U}_{m,k}$ is uniformly strongly consistent towards the extreme value copula C . This result is a main tool to obtain important results in the paper such as Proposition 4, Theorem 3 and Proposition 8. For that purpose, the Berbee's coupling lemma is of prime interest (see, e.g., Chapter 5 in Rio 2017) which gives an approximation of the original process by conveniently defined independent random variables.

Lemma 4. *Under conditions of Proposition 4, we have*

$$\|C_{n,m}^o - C\|_\infty \xrightarrow[n \rightarrow \infty]{a.s.} 0.$$

Proof of Lemma 4 Using triangle inequality, one obtain the following bound

$$\|C_{n,m}^o - C\|_\infty \leq \|C_{n,m}^o - C_m\|_\infty + \|C_m - C\|_\infty. \quad (37)$$

As $\{C_m, m \in \mathbb{N}\}$ is an equicontinuous class of functions (for every m , C_m is a copula hence a 1-Lipschitz function), defined on the compact set $[0, 1]^d$ (by Tychonov's theorem) which converges pointwise to C by Condition C. Then the convergence is uniform over $[0, 1]^d$. Thus the second term of the RHS of Equation (37) converges to 0 almost surely.

Now, let us prove that $\|C_{n,m}^o - C_m\|_\infty$ converges almost surely to 0. By Berbee's coupling lemma (see Theorem 6.1 in Rio 2017 or Theorem 3.1 in Bücher and Segers 2014 for similar applications), we can construct inductively a sequence $(\bar{\mathbf{Z}}_{im+1}, \dots, \bar{\mathbf{Z}}_{im+m})_{i \geq 0}$ such that the following three properties hold:

- (i) $(\bar{\mathbf{Z}}_{im+1}, \dots, \bar{\mathbf{Z}}_{im+m}) \stackrel{d}{=} (\mathbf{Z}_{im+1}, \dots, \mathbf{Z}_{im+m})$ for any $i \geq 0$;
- (ii) both $(\mathbf{Z}_{2im+1}, \dots, \mathbf{Z}_{2im+m})_{i \geq 0}$ and $(\bar{\mathbf{Z}}_{(2i+1)m+1}, \dots, \bar{\mathbf{Z}}_{(2i+1)m+m})_{i \geq 0}$ sequences are independent and identically distributed;
- (iii) $\mathbb{P}\{(\bar{\mathbf{Z}}_{im+1}, \dots, \bar{\mathbf{Z}}_{im+m}) \neq (\mathbf{Z}_{im+1}, \dots, \mathbf{Z}_{im+m})\} \leq \beta(m)$.

Let $\bar{C}_{n,m}^o$ and $\bar{\mathbf{U}}_{m,i}$ be defined analogously to $C_{n,m}^o$ and $\mathbf{U}_{m,i}$ respectively but with $\mathbf{Z}_1, \dots, \mathbf{Z}_n$ replaced with $\bar{\mathbf{Z}}_1, \dots, \bar{\mathbf{Z}}_n$. Now write

$$C_{n,m}^o(\mathbf{u}) = \bar{C}_{n,m}^o(\mathbf{u}) + \{C_{n,m}^o(\mathbf{u}) - \bar{C}_{n,m}^o(\mathbf{u})\}. \quad (38)$$

We will show below that the term under brackets converges uniformly to 0 almost surely. Write $\bar{C}_{n,m}^o(\mathbf{u}) = \bar{C}_{n,m}^{o,\text{odd}}(\mathbf{u}) + \bar{C}_{n,m}^{o,\text{even}}(\mathbf{u})$ where $\bar{C}_{n,m}^{o,\text{odd}}(\mathbf{u})$ and $\bar{C}_{n,m}^{o,\text{even}}(\mathbf{u})$ are defined as sums over the odd and even summands of $\bar{C}_{n,m}^o(\mathbf{u})$, respectively. Since both of these sums are based on i.i.d. summands by properties (i) and (ii), we have $\|\bar{C}_{n,m}^o - C_m\|_\infty \xrightarrow[n \rightarrow \infty]{a.s.} 0$ using Glivenko-Cantelli (see Chapter 2.5 of Vaart and Wellner 1996).

It remains to control the term under brackets on the right hand side of Equation (38), we have that

$$\begin{aligned} |C_{n,m}^o(\mathbf{u}) - \bar{C}_{n,m}^o(\mathbf{u})| &\leq \frac{1}{k} \sum_{i=1}^k \left| \mathbb{1}_{\{\bar{\mathbf{U}}_{m,i} \leq \mathbf{u}\}} - \mathbb{1}_{\{\mathbf{U}_{m,i} \leq \mathbf{u}\}} \right| \\ &\leq \frac{1}{k} \sum_{i=1}^k \mathbb{1}_{\{(\bar{X}_{im+1}, \dots, \bar{X}_{im+m}) \neq (X_{im+1}, \dots, X_{im+m})\}}. \end{aligned}$$

Hence, using Markov's inequality and property (iii), we have

$$\mathbb{P} \left\{ \sup_{u \in [0,1]^d} |\bar{C}_{n,m}^o(\mathbf{u}) - C_{n,m}^o(\mathbf{u})| > \epsilon \right\} \leq \frac{\beta(m)}{\epsilon}.$$

Thus by Condition \mathcal{D} ,

$$\sum_{n \geq 1} \mathbb{P} \left\{ \sup_{u \in [0,1]^d} |\bar{C}_{n,m}^o(\mathbf{u}) - C_{n,m}^o(\mathbf{u})| > \epsilon \right\} < \infty.$$

Applying Borel-Cantelli gives the desired convergence to 0 almost surely of the term under bracket in Equation (38). Gathering all results gives that the term $\|C_{n,m}^o - C_m\|_\infty$ converges almost surely to 0. Hence the statement using Equation (37). \square

Appendix C.2 Weak convergence of an estimator of $A^{(O)} - A$

We now state conditions on the block size m and the number of blocks k , as in Bücher and Segers 2014, to demonstrate the weak convergence of the empirical copula process based on the (unobservable) sample $(U_{n,m,1}^{(j)}, \dots, U_{n,m,k}^{(j)})$ for every $j \in \{1, \dots, d\}$ under mixing conditions. An additional condition will be required within the theorem to establish the weak convergence of the rank-based copula estimator under the same mixing conditions.

Condition \mathcal{F} . There exists a positive integer sequence ℓ_n such that the following statement holds:

- (i) $m_n \rightarrow \infty$ and $m_n = o(n)$
- (ii) $\ell_n \rightarrow \infty$ and $\ell_n = o(m_n)$
- (iii) $k_n \alpha(\ell_n) = o(1)$ and $(m_n/\ell_n) \alpha(\ell_n) = o(1)$
- (iv) $\sqrt{k_n} \beta(m_n) = o(1)$

For notational conveniency, we will write in the following $m_n = m$, $k_n = k$, $\ell_n = \ell$. Note that Condition \mathcal{F} (iii) guarantees that the limit C is an extreme value copula by Theorem 4.2 of Hsing 1989. As usual, the weak convergence of the empirical copula process stems down from the finite dimensional convergence and the asymptotic tightness of the process which then hold from Condition \mathcal{F} (iii) and (iv) respectively. In order to apply Hadamard's differentiability to obtain the weak convergence of the empirical copula based on the sample's scaled ranks, we need a classical condition over the derivatives of the limit copula stated as follows.

Condition \mathcal{G} . For any $j \in \{1, \dots, d\}$, the j th first order partial derivative $\dot{C}^{(j)} = \partial C / \partial u^{(j)}$ exists and is continuous on $\{\mathbf{u} \in [0, 1]^d, u^{(j)} \in (0, 1)\}$.

The estimator of the Pickands dependence function that we present is based on the madogram concept (Cooley, Naveau, and Poncet 2006; Marcon et al. 2017), a notion borrowed from geostatistics in order to capture the spatial dependence structure. Our estimator is defined as

$$\hat{A}_{n,m}(\mathbf{t}) = \frac{\hat{\nu}_{n,m}(\mathbf{t}) + c(\mathbf{t})}{1 - \hat{\nu}_{n,m}(\mathbf{t}) - c(\mathbf{t})},$$

where

$$\hat{\nu}_{n,m}(\mathbf{t}) = \frac{1}{k} \sum_{i=1}^k \left[\bigvee_{j=1}^d \left\{ \hat{U}_{n,m,j}^{(j)} \right\}^{1/t^{(j)}} - \frac{1}{d} \sum_{j=1}^d \left\{ \hat{U}_{n,m,i}^{(j)} \right\}^{1/t^{(j)}} \right], \quad c(\mathbf{t}) = \frac{1}{d} \sum_{j=1}^d \frac{t^{(j)}}{1 + t^{(j)}},$$

and $\hat{U}_{n,m,i}^{(j)} = \hat{F}_{n,m}^{(j)}(M_{m,i}^{(j)})$ corresponds to ranks scaled by k^{-1} . By convention, here $u^{1/0} = 0$ for $u \in (0, 1)$. Let $g \in \{1, \dots, G\}$ and define

$$\hat{A}_{n,m}^{(O_g)}(\mathbf{t}^{(O_g)}) = \hat{A}_{n,m}(\mathbf{0}, \mathbf{t}^{(O_g)}, \mathbf{0})$$

the empirical Pickands dependence function associated to the k -th subvector of \mathbf{X}_p . We consider the empirical process of the difference between estimates of the Pickands dependence functions of subvectors $\mathbf{X}^{(O_g)}$, $g \in \{1, \dots, G\}$, and the estimator of the Pickands dependence function of \mathbf{X} :

$$\mathcal{E}_{nG}(\mathbf{t}) = \sqrt{k} \left(\hat{A}_{n,m}^{(O)}(\mathbf{t}) - \hat{A}_{n,m}(\mathbf{t}) \right),$$

where $\hat{A}_{n,m}^{(O)}(\mathbf{t}) = \sum_{g=1}^G w^{(O_g)}(\mathbf{t}) \hat{A}_{n,m}^{(O_g)}(\mathbf{t}^{(O_g)})$. Noticing that multiplying the above process by d and taking $\mathbf{t} = (d^{-1}, \dots, d^{-1})$ gives

$$\sqrt{k} \widehat{SECO}(O) = \sqrt{k} \left(\sum_{g=1}^G \hat{\theta}_{n,m}^{(O_g)} - \hat{\theta}_{n,m} \right).$$

Hence, the weak convergence of the above empirical process will immediately comes down from the one of the empirical process in \mathcal{E}_{nG} , as stated in the theorem below.

Theorem 5. *Consider the AI-block model in Definition 1 with a given partition O , i.e., $A = A^{(O)}$. Under Conditions \mathcal{C} , \mathcal{F} , \mathcal{G} and $\sqrt{k}(C_m - C) \rightsquigarrow \Gamma$, the empirical process \mathcal{E}_{nG} converges weakly in $\ell^\infty(\Delta_{d-1})$ to a tight Gaussian process having representation*

$$\begin{aligned} \mathcal{E}_G(\mathbf{t}) &= (1 + A(\mathbf{t}))^2 \int_{[0,1]} (N_C + \Gamma)(u^{t^{(1)}}, \dots, u^{t^{(d)}}) du \\ &\quad - \sum_{g=1}^G w^{(O_g)}(\mathbf{t}) \left(1 + A^{(O_g)}(\mathbf{t}^{(O_g)}) \right)^2 \int_{[0,1]} (N_C + \Gamma)(\mathbf{1}, u^{t^{(i_g,1)}}, \dots, u^{t^{(i_g,d_g)}}, \mathbf{1}) du, \end{aligned}$$

where N_C is a continuous tight Gaussian process with representation

$$N_C(u^{(1)}, \dots, u^{(d)}) = B_C(u^{(1)}, \dots, u^{(d)}) - \sum_{j=1}^d \dot{C}_j(u^{(1)}, \dots, u^{(d)}) B_C(\mathbf{1}, u^{(j)}, \mathbf{1}),$$

and B_C is a continuous tight Gaussian process with covariance function

$$\text{Cov}(B_C(\mathbf{u}), B_C(\mathbf{v})) = C(\mathbf{u} \wedge \mathbf{v}) - C(\mathbf{u})C(\mathbf{v}) \stackrel{\mathcal{H}_0}{=} C_{\Pi}(\mathbf{u} \wedge \mathbf{v}) - C_{\Pi}(\mathbf{u})C_{\Pi}(\mathbf{v}).$$

Proof of Theorem 5 The proof is straightforward, notice that by the triangle diagram in Figure 2

$$\mathcal{E}_{nG} = \psi \circ \phi \left(\sqrt{k}(\hat{A}_{n,m} - A) \right),$$

where ϕ is detailed as

$$\begin{aligned} \phi : \ell^\infty(\Delta_{d-1}) &\rightarrow \ell^\infty(\Delta_{d-1}) \otimes (\ell^\infty(\Delta_{d-1}), \dots, \ell^\infty(\Delta_{d-1})) \\ x &\mapsto (x, \phi_1(x), \dots, \phi_G(x)), \end{aligned}$$

with for every $g \in \{1, \dots, G\}$

$$\begin{aligned} \phi_g : \ell^\infty(\Delta_{d-1}) &\rightarrow \ell^\infty(S_d) \\ x &\mapsto w^{(O_g)}(t^{(1)}, \dots, t^{(G)}) x(\mathbf{0}, t^{(i_g,1)}, \dots, t^{(i_g,d_g)}, \mathbf{0}), \end{aligned}$$

$$\begin{array}{ccc}
\sqrt{k} \left(\hat{A}_{n,m} - A \right) & \longrightarrow & \mathcal{E}_{nG} \\
& \searrow \phi & \uparrow \psi \\
\left(\sqrt{k} \left(\hat{A}_{n,m} - A \right); w^{(O_1)} \sqrt{k} \left(\hat{A}_{n,m}^{(O_1)} - A^{(O_1)} \right), \dots, w^{(O_G)} \sqrt{k} \left(\hat{A}_{n,m}^{(O_G)} - A^{(O_G)} \right) \right) & &
\end{array}$$

Figure 2. Commutative diagram of composition of function.

and also

$$\begin{aligned}
\psi : \ell^\infty(\Delta_{d-1}) \otimes (\ell^\infty(\Delta_{d-1}), \dots, \ell^\infty(\Delta_{d-1})) &\rightarrow \ell^\infty(\Delta_{d-1}) \\
(x, \phi_1(x), \dots, \phi_G(x)) &\mapsto \sum_{g=1}^G \phi_g(x) - x.
\end{aligned}$$

The function ϕ_g is a linear and bounded function hence continuous for every g , it follows that ϕ is continuous since each coordinate functions is continuous. As a linear and bounded function, ψ is also a continuous function. Noticing that,

$$(C_m - C)(\mathbf{1}, u, \mathbf{1}) = 0, \quad \forall n \in \mathbb{N},$$

where $m = m_n$ is the block length for a sample size n . We thus have

$$\sqrt{k}(C_m - C)(\mathbf{1}, u, \mathbf{1}) \xrightarrow[n \rightarrow \infty]{} 0,$$

with $k = k_n$ the number of blocks. Therefore $\Gamma(\mathbf{1}, u, \mathbf{1}) = 0$. Combining this equality with Corollary 3.6 of Bücher and Segers 2014 and the same techniques as in the proof of Theorem 2.4 in Marcon *et al.* 2017, we obtain along with Conditions \mathcal{C} , \mathcal{F} , \mathcal{G}

$$\sqrt{k}(\hat{A}_{n,m}(\mathbf{t}) - A(\mathbf{t})) \rightsquigarrow - (1 + A(\mathbf{t}))^2 \int_{[0,1]} (N_C + \Gamma)(u^{t^{(1)}}, \dots, u^{t^{(d)}}) du.$$

Applying the continuous mapping theorem for the weak convergence in $\ell^\infty(\Delta_{d-1})$ (Theorem 1.3.6 of Vaart and Wellner 1996) leads the result. \square

Appendix C.3 Supplementary figures

In this section are gathered supplementary figures to intuit or enrich our findings. Section [Appendix C.3.1](#) contains an illustration of the main idea of Proposition 3. Section [Appendix C.3.2](#) completes numerical results of Section 6 by taking $p \in \{0.5, 0.7, 1.0\}$. Notice that $p = 1.0$ corresponds to the serially independent case.

Appendix C.3.1 Additional figure for Section 2

When asymptotic independence between subvectors of a multivariate extreme value distribution occurs, the exponent measure charges disjoint sets of $E \setminus [0, \mathbf{x}]$, as stated in Equation (9) and Proposition 3, (iii) in the main text. The form of the support of the exponent measure can be represented by the following union of sets:

$$\bigcup_{g=1}^G \mathbf{0}^{d_1} \times \dots \times]0, \infty[^{d_g} \times \dots \times \mathbf{0}^{d_G}.$$

To better understand this concept, we have provided an example in \mathbb{R}^3 in Figure 3.



Figure 3. Support of the exponent measure of an AI-block model with partition $O = \{\{1, 2\}, \{3\}\}$. The measure charges the positive orthant draws by the axes x_1, x_2 (in grey) and the positive axis x_3 (in black), the intersection of those two sets is empty.

Appendix C.3.2 Additional numerical results for Section 6

In Section 6 of the main text, we demonstrate that a specific random process is in the max domain of attraction of an AI-block model. This model is characterized by the persistence parameter, p , which corresponds to the probability that the current observation is equal to the previous one. We use observations drawn from this process to evaluate the performance of our algorithm and state-of-the-art algorithms for clustering extremes in terms of their ability to recover the maximal element of an AI-block model of a weakly dependent multivariate random process.

The specific parameter setting $p = 0.9$ used in these experiments are detailed in Section 6 of the main text. In this auxiliary section, we present the results of these evaluations in Figures 4, 5, 6, where we display the exact recovery rate for each algorithm for varying values of $p \in \{0.5, 0.7, 1.0\}$, respectively. We recall that observations are serially independent when $p = 1.0$.

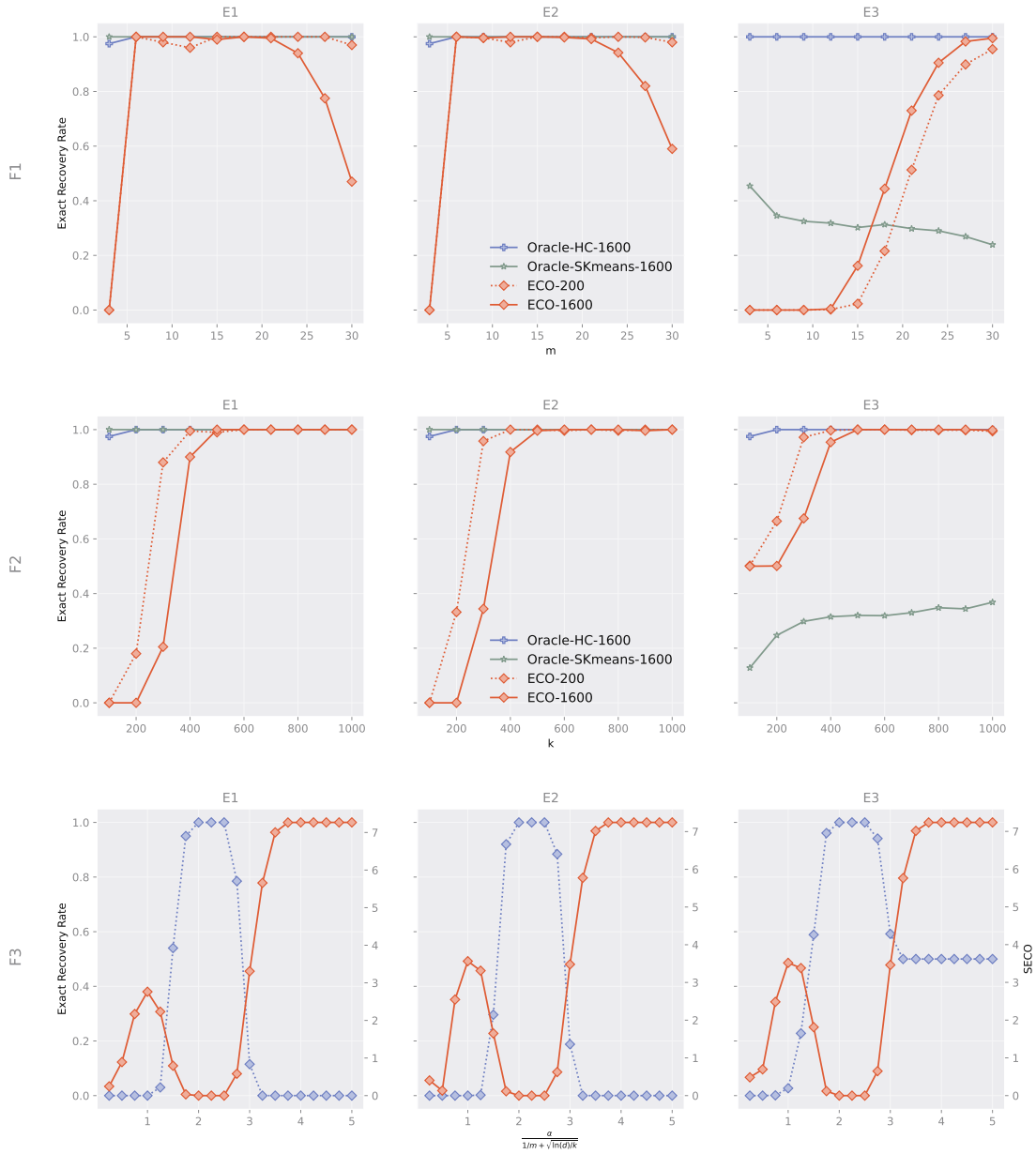


Figure 4. Simulation results for $p = 0.5$. From top to bottom: Framework F1, Framework F2, Framework F3. From left to right: Experiment E1, Experiment E2, Experiment E3. Exact recovery rate for our algorithm (red, diamond points), for the HC k -means (blue, plus points) and the SKmeans (green, star points) for F1 and F2 across 100 runs. Dotted lines correspond to $d = 200$, solid lines to $d = 1600$. The threshold τ is taken as $2 \times (1/m + \sqrt{\ln(d)/k})$. For F3, average SECO losses (red solid lines, circle points) and exact recovery percentages (blue dotted lines, diamond points) across 100 simulations. For better illustration, the SECO losses are standardized first by subtracting the minimal SECO loss in each figure, and the standardized SECO losses plus 1 are then plotted on the logarithmic scale.

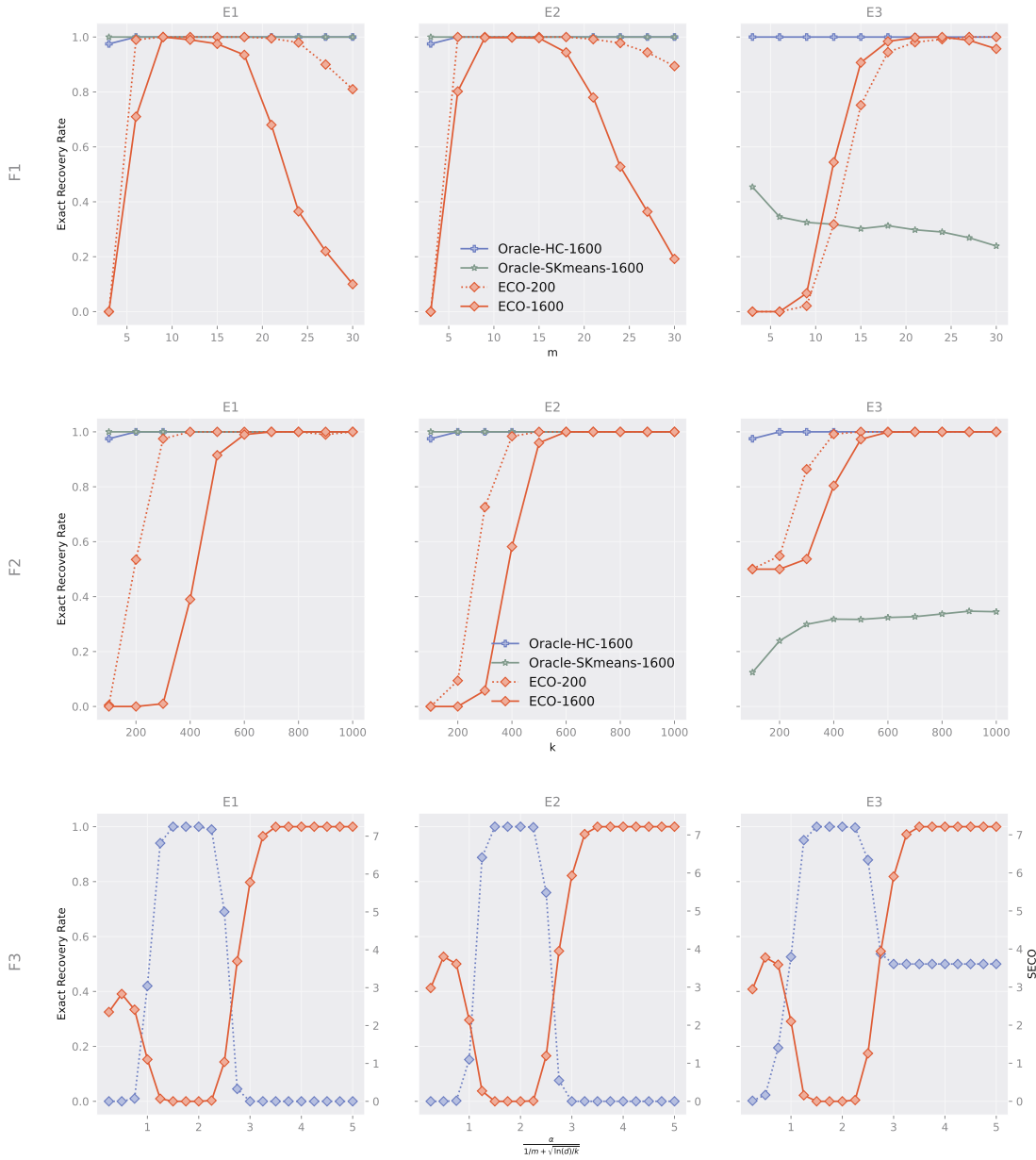


Figure 5. Simulation results for $p = 0.7$. From top to bottom: Framework F1, Framework F2, Framework F3. From left to right: Experiment E1, Experiment E2, Experiment E3. Exact recovery rate for our algorithm (red, diamond points), for the HC k -means (blue, plus points) and the SKmeans (green, star points) for F1 and F2 across 100 runs. Dotted lines correspond to $d = 200$, solid lines to $d = 1600$. The threshold τ is taken as $2 \times (1/m + \sqrt{\ln(d)/k})$. For F3, average SECO losses (red solid lines, circle points) and exact recovery percentages (blue dotted lines, diamond points) across 100 simulations. For better illustration, the SECO losses are standardized first by subtracting the minimal SECO loss in each figure, and the standardized SECO losses plus 1 are then plotted on the logarithmic scale.

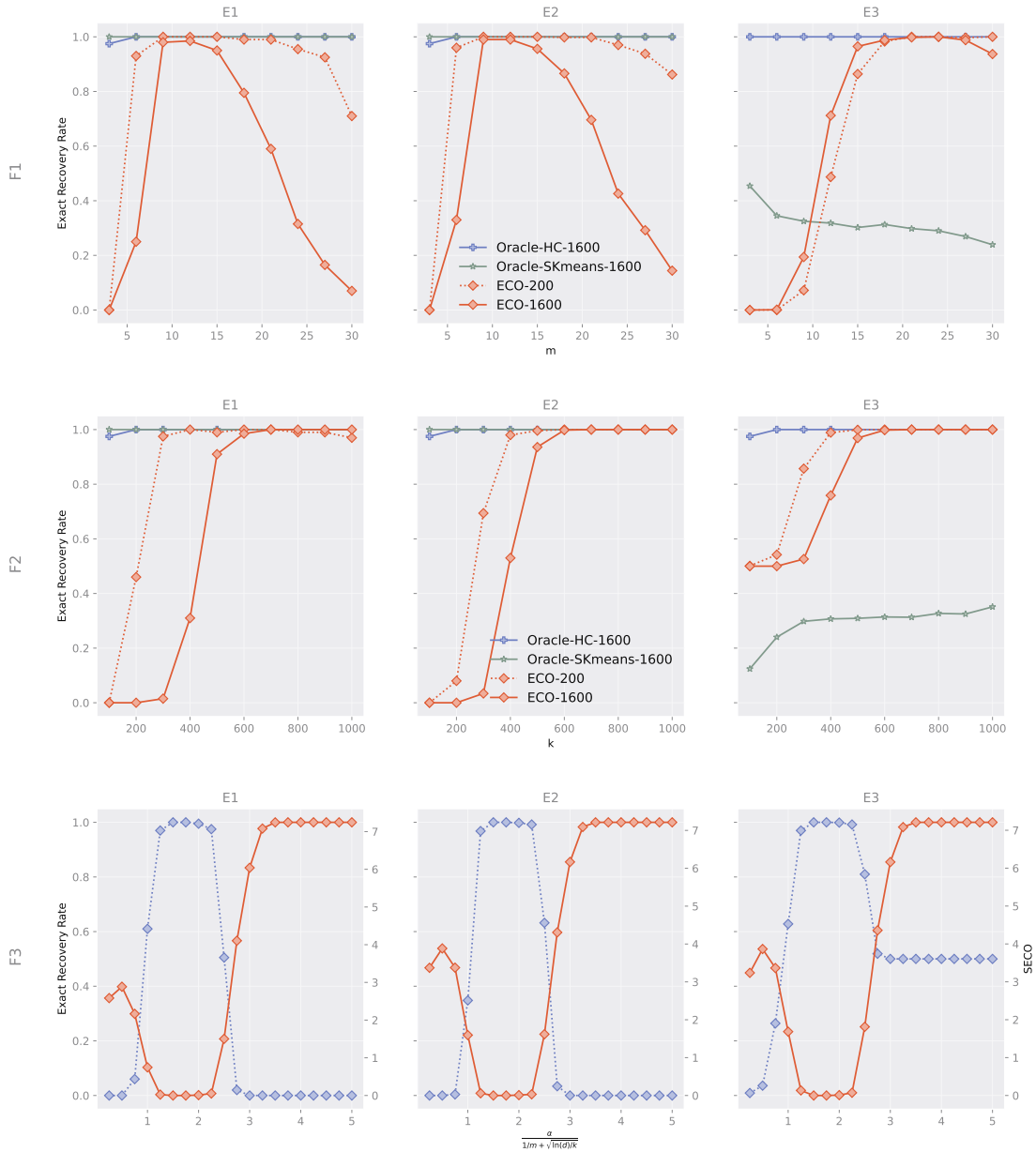


Figure 6. Simulation results for $p = 1.0$ (serially independent case). From top to bottom: Framework **F1**, Framework **F2**, Framework **F3**. From left to right: Experiment **E1**, Experiment **E2**, Experiment **E3**. Exact recovery rate for our algorithm (red, diamond points), for the HC k -means (blue, plus points) and the SKmeans (green, star points) for **F1** and **F2** across 100 runs. Dotted lines correspond to $d = 200$, solid lines to $d = 1600$. The threshold τ is taken as $2 \times (1/m + \sqrt{\ln(d)/k})$. For **F3**, average SECO losses (red solid lines, circle points) and exact recovery percentages (blue dotted lines, diamond points) across 100 simulations. For better illustration, the SECO losses are standardized first by subtracting the minimal SECO loss in each figure, and the standardized SECO losses plus 1 are then plotted on the logarithmic scale.