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MOMENTS OF PARTITION FUNCTIONS OF 2D GAUSSIAN POLYMERS IN THE WEAK DISORDER REGIME - I

CLÉMENT COSCO AND OFER ZEITOUNI

ABSTRACT. Let $W_N(\beta) = \mathbb{E}_0 \left[e^{\sum_{n=1}^N \beta \omega(n, S_n) - N\beta^2/2} \right]$ be the partition function of a two-dimensional directed polymer in a random environment, where $\omega(i, x), i \in \mathbb{N}, x \in \mathbb{Z}^2$ are i.i.d. standard normal and $\{S_n\}$ is the path of a random walk. With $\beta = \beta_N = \beta \sqrt{\pi/\log N}$ and $\hat{\beta} \in (0, 1)$ (the subcritical window), $\log W_N(\beta_N)$ is known to converge in distribution to a Gaussian law of mean $-\lambda^2/2$ and variance λ^2 , with $\lambda^2 = \log(1/(1 - \hat{\beta}^2))$ (Caravenna, Sun, Zygouras, *Ann. Appl. Probab.* (2017)). We study in this paper the moments $\mathbb{E}[W_N(\beta_N)^q]$ in the subcritical window, for $q = O(\sqrt{\log N})$. The analysis is based on ruling out triple intersections

1. INTRODUCTION AND STATEMENT OF RESULTS

We consider in this paper the partition function of two dimensional directed polymers in Gaussian environment, and begin by introducing the model. Set

$$(1) \quad W_N(\beta, x) = \mathbb{E}_x \left[e^{\sum_{n=1}^N \beta \omega(n, S_n) - N\beta^2/2} \right], \quad x \in \mathbb{Z}^d.$$

Here, $\{\omega_{n,x}\}_{n \in \mathbb{Z}_+, x \in \mathbb{Z}^d}$ are i.i.d. standard centered Gaussian random variables of law \mathbb{P} , $\{S_n\}_{n \in \mathbb{Z}_+}$ is simple random walk, and \mathbb{E}_x denotes the law of simple random walk started at $x \in \mathbb{Z}^2$. Thus, $W_N(\beta, x)$ is a random variable measurable on the σ -algebra $\mathcal{G}_N := \sigma\{\omega(i, x) : i \leq N, x \in \mathbb{Z}^d\}$. For background, motivation and results on the rich theory surrounding this topic, we refer the reader to [15]. In particular, we mention the relation with the d dimensional stochastic heat equation (SHE).

The random variables $W_N(\beta, 0)$ form a \mathcal{G}_N positive martingale, and therefore converge almost surely to a limit $W_\infty(\beta, 0)$. It is well known that in dimensions $d = 1, 2$, for any $\beta > 0$ we have $W_\infty(\beta, 0) = 0$, a.s., while for $d \geq 3$, there exists $\beta_c > 0$ so that $W_\infty(\beta, 0) > 0$ a.s. for $\beta < \beta_c$ and $W_\infty(\beta, 0) = 0$ for $\beta > \beta_c$. We refer to these as the *weak* and *strong* disorder regimes, respectively. In particular, for $d = 2$, which is our focus in this paper, for any $\beta > 0$, we are in the strong disorder regime.

A meaningful rescaling in dimension 2 was discovered in the context of the SHE by Bertini and Cancrini [2] and was later generalized by Caravenna, Sun and Zygouras [5], in both the SHE and polymer setups, to a wider range of parameters

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for which a phase transition occurs. See also [8, 9, 10, 22, 28]. Introduce the mean intersection local time for random walk

$$(2) \quad R_N = \mathbb{E}_0^{\otimes 2} \left[\sum_{n=1}^N \mathbf{1}_{S_n^1 = S_n^2} \right] \sim \frac{\log N}{\pi}.$$

The asymptotic behavior of R_N follows from the local limit theorem [25, Sec. 1.2]. Further, the Erdős-Taylor theorem [20] states that $\frac{\pi}{\log N} \sum_{n=1}^N \mathbf{1}_{S_n^1 = S_n^2}$ converges in distribution to an exponential random variable of parameter 1.

Set

$$(3) \quad \beta_N = \frac{\hat{\beta}}{\sqrt{R_N}}, \quad \hat{\beta} \geq 0.$$

We will use the short-notation $W_N = W_N(\beta_N, 0)$. With it, see [5], one has

$$(4) \quad \forall \hat{\beta} < 1 : \quad \log W_N \xrightarrow{(d)} \mathcal{N} \left(-\frac{\lambda^2}{2}, \lambda^2 \right), \quad \text{with } \lambda^2(\hat{\beta}) = \log \frac{1}{1 - \hat{\beta}^2}.$$

The convergence in (4) has recently been extended in [26] to the convergence of W_N to the exponential of a Gaussian, in all L^p . (The critical case $\hat{\beta} = 1$, which we will not study in this paper, has received considerable attention, see [2, 7, 9, 23].)

The spatial behavior of $W_N(\beta_N, x)$ is also of interest. Indeed, one has, see [8],

$$(5) \quad G_N(x) := \sqrt{R_N} \left(\log W_N(\beta_N, x\sqrt{N}) - \mathbb{E} \log W_N(\beta_N, x\sqrt{N}) \right) \xrightarrow{(d)} \sqrt{\frac{\hat{\beta}^2}{1 - \hat{\beta}^2}} G(x),$$

with $G(x)$ a log-correlated Gaussian field on \mathbb{R}^2 . The convergence in (5) is in the weak sense, i.e. for any smooth, compactly supported function ϕ , $\int \phi(x) G_N(x) dx$ converges to a centered Gaussian random variable of variance $\hat{\beta}^2 \sigma_\phi^2 / (1 - \hat{\beta}^2)$, where

$$(6) \quad \sigma_\phi^2 = \frac{1}{2\pi} \iint \phi(x) \phi(y) \int_{|x-y|^2/2}^{\infty} z^{-1} e^{-z} dz.$$

One recognizes σ_ϕ^2 in (6) as the variance of the integral of ϕ against the solution of the *Edwards-Wilkinson* equation. For a related result in the KPZ/SHE setup, see [8, 22, 28].

Logarithmically correlated fields, and in particular their extremes and large values, have played an important recent role in the study of various models of probability theory at the critical dimension, ranging from their own study [3, 4, 17, 29], random walk and Brownian motion [1, 16], random matrices [12, 13, 14], Liouville quantum gravity [18, 24], turbulence [11], and more. In particular, exponentiating Gaussian logarithmically correlated fields yields Gaussian multiplicative chaoses, with the ensuing question of convergence towards them.

In the context of polymers, (5) opens the door to the study of such questions. A natural role is played by the random measure

$$\mu_N^\gamma(x) = \frac{e^{\gamma G_N(x)}}{\mathbb{E} e^{\gamma G_N(x)}},$$

and it is natural to ask about its convergence towards a GMC, and about extremes of $G_N(x)$ for x in some compact subsets of \mathbb{R}^2 .

A preliminary step toward any such analysis involves evaluating exponential moments of $G_N(0)$. This is our goal in this paper. In the following, $q = q(N)$ denotes an integer $q \geq 2$ that may depend on N . Our main result is the following.

Theorem 1.1. *There exists $\hat{\beta}_0 \leq 1$ so that if $\hat{\beta} < \hat{\beta}_0$ and*

$$(7) \quad \limsup_{N \rightarrow \infty} \frac{3\hat{\beta}^2}{(1 - \hat{\beta}^2)} \frac{1}{\log N} \binom{q}{2} < 1,$$

then,

$$(8) \quad \mathbb{E}[W_N^q] \leq e^{\binom{q}{2}\lambda^2(1+\varepsilon_N)},$$

where $\varepsilon_N = \varepsilon(N, \hat{\beta}) \searrow 0$ as $N \rightarrow \infty$.

The proof will show that in Theorem 1.1, $\hat{\beta}_0$ can be taken as $1/96$, but we do not expect this to be optimal.

Remark 1.2. *With a similar method, we can also prove that the estimate (8) holds for all $\hat{\beta} < 1$ at the cost of choosing $q^2 = o(\log N / \log \log N)$, see Section 2.4 for details. In particular, we obtain that the partition function possesses all (fixed) moments in the region $\hat{\beta} < 1$:*

$$(9) \quad \forall q \in \mathbb{N}, \quad \sup_N \mathbb{E}[W_N^q] < \infty.$$

As mentioned above, (9) was independently proved in [26]. (See also [27] for further precision and a multivariate generalization of the Erdős-Taylor theorem.) They also observed that together with the convergence in distribution (4), the estimate (9) implies that for all fixed $q \in \mathbb{N}$,

$$(10) \quad \mathbb{E}[W_N^q] e^{-\binom{q}{2}\lambda^2(\hat{\beta})} \xrightarrow{N \rightarrow \infty} 1.$$

Note however that the estimate (8) does not yield (10) when $q \rightarrow \infty$ with $N \rightarrow \infty$.

Remark 1.3. *Theorem 1.1 is of course not enough to prove convergence toward a GMC. For that, one would need to improve the error in the exponent from $O(q^2\varepsilon_N)$ to $O(1)$, to obtain a complementary lower bound and, more important, to derive similar multi-point estimates. We hope to return to these issues in future work.*

The structure of the paper is as follows. In the next Section 2, we use a well-worn argument to reduce the computation of moments to certain estimates concerning the intersection of (many) random walks. After some standard preliminaries, we state there our main technical estimate, Theorem 2.1, which provides intersection estimates under the extra assumptions that all intersections are in pairs, i.e. that no triple (or more) points exist. The rest of the section provides the proof of Theorem 1.1. Section 3 then develops the induction scheme that is used to prove Theorem 2.1. Since we assume that there are no triple (or more) intersections, we may consider particles as matched in pairs at intersection times. The induction is essentially on the number of instances in which “matched particles” break the match and create a different matching. Section 4 provides a discussion of our results, their limitations, and possible extensions. In particular we explain there why the constraint on q in Theorem 1.1 limits our ability to obtain the expected sharp upper bounds on the maximum of $\log W_N(\hat{\beta}_N, x\sqrt{N})$. The appendices collect several auxiliary results and a sharpening of one of our estimates, see Proposition B.1.

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2. INTERSECTION REPRESENTATION, REDUCED MOMENTS, AND PROOF OF THEOREM 1.1

Throughout the rest of the paper, we let $p(n, x) = p_n(x) = P_0(S_n = x)$. There is a nice formula for the q -th moment of the partition function whose importance is apparent in previous work on directed polymers, for example in [6, 7]. Indeed,

$$\mathbb{E}[W_N^q] = \mathbb{E}_0^{\otimes q} \mathbb{E} e^{\sum_{i=1}^q \sum_{n=1}^N (\beta_N \omega(n, S_n^i) - \beta_N^2/2)},$$

where S^1, \dots, S^q are q independent copies of the simple random walk and $\mathbb{E}_X^{\otimes q}$ denotes the expectation for the product measure started at $X = (x^1, \dots, x^q)$. (If the starting point X is not specified, we assume $X = \mathbf{0}$.) Since the $\omega(i, x)$ are Gaussian and the variance of $\sum_{i=1}^q \beta_N \omega(n, S_n^i)$ is equal to $\beta_N^2 \sum_{i=1 \dots q, j=1 \dots q} \mathbf{1}_{S_n^i = S_n^j}$, we have the following formula for the moment in terms of intersections of q independent random walks:

$$(11) \quad \mathbb{E}[W_N^q] = \mathbb{E}^{\otimes q} \left[e^{\beta_N^2 \sum_{1 \leq i < j \leq q} \sum_{n=1}^N \mathbf{1}_{S_n^i = S_n^j}} \right].$$

2.1. No triple estimate. The key step in upper bounding the right hand side of (11) is to restrict the summation to subsets where there are no triple (or more) intersection. More precisely, denote by

$$(12)$$

$$F_n = \{\exists(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) : 1 \leq \bar{\alpha} < \bar{\beta} < \bar{\gamma} \leq q, S_n^{\bar{\alpha}} = S_n^{\bar{\beta}} = S_n^{\bar{\gamma}}\},$$

$$(13)$$

$$K_n =$$

$$\{\exists(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}) : 1 \leq (\bar{\alpha} < \bar{\beta}), (\bar{\gamma} < \bar{\delta}) \leq q, \{\bar{\alpha}, \bar{\beta}\} \cap \{\bar{\gamma}, \bar{\delta}\} = \emptyset, S_n^{\bar{\alpha}} = S_n^{\bar{\beta}}, S_n^{\bar{\gamma}} = S_n^{\bar{\delta}}\}$$

and let

$$G_T = \bigcap_{n \in \llbracket 1, T \rrbracket} (F_n \cup K_n)^c$$

be the event that there is no triple (or more) intersection, i.e. that at each given time no more than a pair of particles are involved in an intersection.

The following theorem is the technically involved part of this paper. Its proof will be presented in Section 3.

Theorem 2.1. *Fix $\hat{\beta} \in (0, 1)$. Assume that either $q(N) = q_0 \in \mathbb{N}$ is constant, or that $q(N) \rightarrow \infty$ as $N \rightarrow \infty$ with the condition (7). Then, uniformly in $T \in \llbracket 1, N \rrbracket$ as $N \rightarrow \infty$,*

$$(14) \quad \sup_{X \in (\mathbb{Z}^2)^q} \mathbb{E}_X^{\otimes q} \left[e^{\beta_N^2 \sum_{n=1}^T \sum_{1 \leq i < j \leq q} \mathbf{1}_{S_n^i = S_n^j}} \mathbf{1}_{G_T} \right] \leq c e^{\lambda_{T,N}^2 \binom{q}{2} (1+o(1))},$$

where $c = c(\hat{\beta}) > 0$ in the case $q \rightarrow \infty$ and $c = c(\hat{\beta}, q_0)$ when $q = q_0$, and $\lambda_{T,N}$ is defined as

$$(15) \quad \lambda_{T,N}^2(\hat{\beta}) = \lambda_{T,N}^2 = \log \frac{1}{1 - \hat{\beta}^2 \frac{\log T}{\log N}}.$$

Note that as soon as $q > 9$, the expression in the left side of (14) trivially vanishes if $X = \mathbf{0}$. The X 's of interest are those that allow for non-existence of triple or more intersections.

Assuming Theorem 2.1, the proof of Theorem 1.1 is relatively straightforward. We will need the preliminary results collected in the next subsection.

2.2. A short time a priori estimate. The following lemma is a variation on Khas'minskii's lemma [30, p.8, Lemma 2.1].

Lemma 2.2. *Let \mathcal{Z} be the set of all nearest-neighbor walks on \mathbb{Z}^2 , that is $Z \in \mathcal{Z}$ if $Z = (Z_i)_{i \in \mathbb{N}}$ where $Z_i \in \mathbb{Z}^2$ and $Z_{i+1} - Z_i \in \{\pm \mathbf{e}_j, j \leq d\}$ where \mathbf{e}_j are the canonical vectors of \mathbb{Z}^2 . If for some $k \in \mathbb{N}$ and $\kappa \in \mathbb{R}$, one has*

$$(16) \quad \eta = \sup_{Z \in \mathcal{Z}} \sup_{x \in \mathbb{Z}^2} \left(e^{\kappa^2} - 1 \right) \mathbb{E}_x \left[\sum_{n=1}^k \mathbf{1}_{S_n=Z_n} \right] < 1,$$

then

$$(17) \quad \sup_{Z \in \mathcal{Z}} \sup_{x \in \mathbb{Z}^2} \mathbb{E}_x \left[e^{\kappa^2 \sum_{n=0}^k \mathbf{1}_{S_n=Z_n}} \right] \leq \frac{1}{1 - \eta}.$$

Proof. Let $\Lambda_2 = (e^{\kappa^2} - 1)$. We have:

$$\begin{aligned} \mathbb{E}_x \left[e^{\kappa^2 \sum_{n=1}^k \mathbf{1}_{S_n=Z_n}} \right] &= \mathbb{E}_x \left[\prod_{n=1}^k (1 + \Lambda_2 \mathbf{1}_{S_n=Z_n}) \right] \\ &= \sum_{p=0}^{\infty} \Lambda_2^p \sum_{1 \leq n_1 < \dots < n_p \leq k} \mathbb{E}_x \left[\prod_{i=1}^p \mathbf{1}_{S_{n_i}=Z_{n_i}} \right] \\ &= \sum_{p=0}^{\infty} \Lambda_2^p \sum_{1 \leq n_1 < \dots < n_{p-1} \leq k} \mathbb{E}_x \left[\prod_{i=1}^{p-1} \mathbf{1}_{S_{n_i}=Z_{n_i}} \mathbb{E}_{S_{n_{p-1}}} \left[\sum_{n=1}^{k-n_{p-1}} \mathbf{1}_{S_n=Z_{n+n_{p-1}}} \right] \right] \\ &\stackrel{(16)}{\leq} \sum_{p=0}^{\infty} \Lambda_2^{p-1} \eta \sum_{1 \leq n_1 < \dots < n_{p-1} \leq k} \mathbb{E}_x \left[\prod_{i=1}^{p-1} \mathbf{1}_{S_{n_i}=Z_{n_i}} \right] \leq \dots \leq \sum_{p=0}^{\infty} \eta^p = \frac{1}{1 - \eta}. \end{aligned}$$

□

The next lemma gives an a-priori rough estimate on the moments of $W_k(\beta_N) = W_k(\beta_N, 0)$ when k is small.

Lemma 2.3. *Let $\hat{\beta} > 0$. Let $b_N > 0$ be a deterministic sequence such that $b_N = o(\sqrt{\log N})$ as $N \rightarrow \infty$. Assume that $q = O(\sqrt{\log N}) > 1$. Then, for all $k \leq e^{b_N}$,*

$$\mathbb{E}[W_k(\beta_N)^q] = \mathbb{E}^{\otimes q} \left[e^{\beta_N^2 \sum_{1 \leq i < j \leq q} \sum_{n=1}^k \mathbf{1}_{S_n^i=S_n^j}} \right] \leq e^{\frac{1}{\pi}(1+\varepsilon_N)q^2 \beta_N^2 \log(k+1)},$$

for $\varepsilon_N = \varepsilon_N(\hat{\beta}) \rightarrow 0$ as $N \rightarrow \infty$.

Proof. Let $N_k^{i,j} = \sum_{n=1}^k \mathbf{1}_{S_n^i = S_n^j}$. By Hölder's inequality, we find that

$$\mathbb{E}[W_k(\beta_N)^q] \leq \mathbb{E}^{\otimes q} \left[e^{\frac{q\beta_N^2}{2} \sum_{1 < j \leq q} N_k^{1,j}} \right]^{q/q} = \mathbb{E}^{\otimes 2} \left[\mathbb{E}^{\otimes 2} \left[e^{\frac{q\beta_N^2}{2} N_k^{1,2}} \middle| S^1 \right]^{q-1} \right],$$

by independence of the $(N_{1,j})_{1 < j}$ conditioned on S^1 . We now estimate the above conditional expectation using Lemma 2.2. Let $\kappa^2 = q\beta_N^2/2 \rightarrow 0$ and η be as in (16). For any $Z \in \mathcal{Z}$ and $y \in \mathbb{Z}^2$,

$$\mathbb{E}_y \left[\sum_{n=1}^k \mathbf{1}_{S_n = Z_n} \right] \leq \sum_{n=1}^k \sup_x p_n(x),$$

where, see Appendix A for an elementary proof,

$$(18) \quad \forall n \geq 1 : \quad \sup_x p_n(x) := p_n^* \leq \frac{2}{\pi n}.$$

Thus, $\eta \leq \frac{1}{\pi}(1 + o(1))q\beta_N^2 \log(k+1) \rightarrow 0$, uniformly for $k \leq e^{bN}$ as $N \rightarrow \infty$. Lemma 2.2 then yields that for such k 's,

$$\mathbb{E}[W_k(\beta_N)^q] \leq \left(\frac{1}{1 - \frac{1}{\pi}(1 + o(1))q\beta_N^2 \log k} \right)^{q-1} = e^{\frac{1}{\pi}(1+o(1))q^2\beta_N^2 \log(k+1)}.$$

□

2.3. Proof of Theorem 1.1. We begin by assuming that $q(N) = q_0$ is constant or that $q(N) \rightarrow \infty$ as $N \rightarrow \infty$ with the condition (7). This allows us to apply Theorem 2.1. As a first step, we will prove that

$$(19) \quad \mathbb{E}[W_N^q] \leq C e^{(q/2)\lambda^2(1+\varepsilon_N)},$$

where $C = 1$ if $q \rightarrow \infty$ and $C = C(\hat{\beta}, q_0)$ when $q = q_0$, and $\varepsilon_N = \varepsilon_N(\hat{\beta}) \rightarrow 0$ as $N \rightarrow \infty$. As a second step, we treat the general $q(N)$ case (i.e. only assuming condition (7)) by a diagonalization argument.

Recall the definitions of $\lambda_{k,N}$ in (15) and that $\lambda = \lambda_{N,N}(\hat{\beta})$. By standard convexity arguments, we note that $x \leq \log(\frac{1}{1-x}) \leq \frac{x}{1-x}$ for all $x \in [0, 1)$; hence for all $a > 1$ and $\hat{\beta} < 1$ such that $a\hat{\beta}^2 < 1$,

$$(20) \quad \forall k \leq N : \quad a\hat{\beta}^2 \frac{\log k}{\log N} \leq \lambda_{k,N}(\sqrt{a}\hat{\beta})^2 \leq \frac{a\hat{\beta}^2}{1 - a\hat{\beta}^2} \frac{\log k}{\log N}.$$

Now, let

$$I_{s,t} = \beta_N^2 \sum_{n=s+1}^t \sum_{i < j \leq q} \mathbf{1}_{S_n^i = S_n^j} \quad \text{and} \quad I_k = I_{0,k},$$

and define

$$(21) \quad M(X) := \mathbb{E}_X^{\otimes q} [e^{I_N}] \quad \text{and} \quad M = \sup_{X \in (\mathbb{Z}^2)^q} M(X).$$

By (11), it is enough to have a bound on $M(0)$. In fact what we will give is a bound on M . To do so, we let $T = T_N > 0$ such that $\log T = o(\sqrt{\log N})$ and introduce the event

$$\tau_T := \inf\{n > T : F_n \cup K_n \text{ occurs}\}.$$

We then decompose $M(X)$ as follows:

$$M(X) = \mathbb{E}_X^{\otimes q} [e^{I_N} \mathbf{1}_{\tau_T \leq N}] + \mathbb{E}_X^{\otimes q} [e^{I_N} \mathbf{1}_{\tau_T > N}] =: A(X) + B(X).$$

We start by bounding $B(X)$ from above. By Markov's property,

$$\begin{aligned} \sup_{X \in (\mathbb{Z}^2)^q} B(X) &\leq \sup_{X \in (\mathbb{Z}^2)^q} \mathbb{E}_X^{\otimes q} [e^{I_T}] \sup_{Y \in (\mathbb{Z}^2)^q} \mathbb{E}_Y^{\otimes q} [e^{I_{N-T}} \mathbf{1}_{\tau_0 > N-T}] \\ &\leq C e^{\frac{1}{\pi}(1+\varepsilon_N)q^2 \beta_N^2 \log T} e^{\binom{q}{2} \lambda_{N-T,N}^2 (1+o(1))} \leq C e^{\binom{q}{2} \lambda^2 (1+o(1))}, \end{aligned}$$

where in the second inequality, we used Lemma 2.3 and Theorem 2.1 and in the last inequality, we used that $\beta_N^2 \log T$ vanishes as $N \rightarrow \infty$ and that $\lambda_{N-T,N}^2 < \lambda^2(\hat{\beta})$. Note that the constant C depends on $\hat{\beta}$ and might depend on q_0 in the constant case $q = q_0$ (because of Theorem 2.1).

We will now deal with $A(X)$ and show that

$$(22) \quad \sup_{X \in (\mathbb{Z}^2)^q} A(X) \leq M \varepsilon_N,$$

with $\varepsilon_N \rightarrow 0$. This, together with the last two displays, implies that

$$M(1 - \varepsilon_N) \leq C e^{\binom{q}{2} \lambda^2 (1+o(1))}.$$

Absorbing the constant C in the $o(1)$ term in the case that $q \rightarrow \infty$, this entails (19).

Toward the proof of (22), we first use Markov's property to obtain that

$$A(X) = \sum_{k=T}^N \mathbb{E}_X^{\otimes q} [e^{I_k + I_{k,N}} \mathbf{1}_{\tau_T = k}] \leq M \sum_{k=T}^N \mathbb{E}_X^{\otimes q} [e^{I_k} \mathbf{1}_{\tau_T = k}].$$

In what follows, we use the phrase "no triple+ at time n " to denote the event that $F_n \cup K_n$ does not hold. Similarly, for $\mathcal{I} \subset \llbracket 1, q \rrbracket$, we use the phrase "no triple+ for particles of \mathcal{I} " to denote the event $(\cup_{n \in \mathcal{I}} (F_n \cup K_n))^c$. We then decompose over which event, F_n or K_n , occurred at τ_T , and then over which particles participated in the event:

$$\begin{aligned} A(X) &\leq M \sum_{\bar{\alpha}, \bar{\beta}, \bar{\gamma} \leq q} \sum_{k=T}^N \mathbb{E}_X^{\otimes q} \left[e^{I_k} \mathbf{1}_{\text{no triple+ in } \llbracket T, k-1 \rrbracket} \mathbf{1}_{S_k^{\bar{\alpha}} = S_k^{\bar{\beta}} = S_k^{\bar{\gamma}}} \right] \\ (23) \quad &+ M \sum_{(\bar{\alpha} < \bar{\beta}) \neq (\bar{\gamma} < \bar{\delta})} \sum_{k=T}^N \mathbb{E}_X^{\otimes q} \left[e^{I_k} \mathbf{1}_{\text{no triple+ in } \llbracket T, k-1 \rrbracket} \mathbf{1}_{S_k^{\bar{\alpha}} = S_k^{\bar{\beta}}, S_k^{\bar{\gamma}} = S_k^{\bar{\delta}}} \right], \\ &=: A_1(X) + A_2(X). \end{aligned}$$

We next handle $A_1(X)$, the argument for $A_2(X)$ is similar. Write

$$I_k = J_k + J_k^{\bar{\alpha}} + J_k^{\bar{\beta}} + J_k^{\bar{\gamma}},$$

where

$$J_k = \beta_N^2 \sum_{n=1}^k \sum_{\substack{i < j \leq q \\ i, j \notin \{\bar{\alpha}, \bar{\beta}, \bar{\gamma}\}}} \mathbf{1}_{S_n^i = S_n^j} \quad \text{and} \quad J_k^{i_0} = \beta_N^2 \sum_{n=1}^k \sum_{j \in \llbracket 1, q \rrbracket \setminus \{i_0\}} \mathbf{1}_{S_n^{i_0} = S_n^j}.$$

If we let $\frac{1}{a} + \frac{3}{b} = 1$ with $1 < a \leq 2$ and $1 < b$, we have

$$\begin{aligned}
& \mathbb{E}_X^{\otimes q} \left[e^{I_k} \mathbf{1}_{\text{no triple+ in } \llbracket T, k-1 \rrbracket} \mathbf{1}_{S_k^{\bar{\alpha}} = S_k^{\bar{\beta}} = S_k^{\bar{\gamma}}} \right] \\
(24) \quad & \leq \mathbb{E}_X^{\otimes q} \left[e^{aJ_k} \mathbf{1}_{\text{no triple+ in } \llbracket T, k-1 \rrbracket \text{ for particles of } \llbracket 1, q \rrbracket \setminus \{\bar{\alpha}, \bar{\beta}, \bar{\gamma}\}} \mathbf{1}_{S_k^{\bar{\alpha}} = S_k^{\bar{\beta}} = S_k^{\bar{\gamma}}} \right]^{1/a} \\
(25) \quad & \times \prod_{i_0 \in \{\bar{\alpha}, \bar{\beta}, \bar{\gamma}\}} \mathbb{E}_X^{\otimes q} \left[e^{bJ_k^{i_0}} \mathbf{1}_{S_k^{\bar{\alpha}} = S_k^{\bar{\beta}} = S_k^{\bar{\gamma}}} \right]^{1/b}.
\end{aligned}$$

We will treat separately the last two factors. Before doing so, we specify our choice of a, b and $\hat{\beta}$. We assume that $\hat{\beta}^2 < 1/72$ and $a < 3/2$, with a close enough to $3/2$ (and so b close to 9) in such a way that

$$\begin{aligned}
(26) \quad & (i) \ 8b\hat{\beta}^2 < 1, \quad (ii) \ \limsup_{N \rightarrow \infty} \frac{1}{\pi} q^2 \beta_N^2 < 1/a \quad \text{and} \\
& (iii) \ \limsup_{N \rightarrow \infty} \frac{\hat{\beta}^2}{1 - a\hat{\beta}^2} \frac{\binom{q}{2}}{\log N} < 1/a.
\end{aligned}$$

Note that (ii) and (iii) are assured to hold for a close enough to $3/2$ thanks to the assumption (7) which implies that $\limsup_N \pi^{-1} q^2 \beta_N^2 \leq \frac{2}{3}$. We chose $\hat{\beta}^2 < 1/72$ to allow (i).

We first bound the factor appearing in (24). If $k \leq e^{(\log N)^{1/3}}$, then it is bounded by

$$\mathbb{E}_X^{\otimes q} [e^{aJ_k}]^{1/a} \mathbb{P}_{(x^{\bar{\alpha}}, x^{\bar{\beta}}, x^{\bar{\gamma}})}^{\otimes 3} \left(S_k^{\bar{\alpha}} = S_k^{\bar{\beta}} = S_k^{\bar{\gamma}} \right)^{1/a} \leq C e^{\frac{1}{\pi} (1+\varepsilon_N) q^2 a \beta_N^2 (\log(k+1))/a} k^{-2/a},$$

for some $c > 0$ and uniformly in $X \in (\mathbb{Z}^2)^q$, where we have used in the inequality Lemma 2.3 and that $\sum_x p_k(x)^3 \leq (p_k^*)^2 \leq k^{-2}$ by (18). For $k \geq e^{(\log N)^{1/3}}$, we rely on (14) to bound the same factor by

$$\begin{aligned}
& \mathbb{P}_{(x^{\bar{\alpha}}, x^{\bar{\beta}}, x^{\bar{\gamma}})}^{\otimes 3} \left(S_k^{\bar{\alpha}} = S_k^{\bar{\beta}} = S_k^{\bar{\gamma}} \right)^{1/a} \\
& \times \mathbb{E}_X^{\otimes q} \left[e^{aJ_k} \mathbf{1}_{\text{no triple+ in } \llbracket T, k-1 \rrbracket \text{ for particles in } \llbracket 1, q \rrbracket \setminus \{\bar{\alpha}, \bar{\beta}, \bar{\gamma}\}} \right]^{1/a} \\
& \leq C k^{-2/a} \left(\mathbb{E}_X^{\otimes q} [e^{aJ_T}] \right) \\
& \sup_Y \mathbb{E}_Y^{\otimes (q-3)} \left[e^{aJ_{k-T-1}} \mathbf{1}_{\text{no triple+ in } \llbracket 1, k-T-1 \rrbracket \text{ for particles in } \llbracket 1, q \rrbracket \setminus \{\bar{\alpha}, \bar{\beta}, \bar{\gamma}\}} \right]^{1/a} \\
& \leq C e^{\frac{1}{\pi} (1+\varepsilon_N) q^2 a \beta_N^2 (\log T)/a} e^{\binom{q}{2} \lambda_{k,N}^2 (\sqrt{a}\hat{\beta})/a} k^{-2/a}.
\end{aligned}$$

For the factor in (25), we apply the Cauchy-Schwarz inequality to find that

$$(27) \quad \mathbb{E}_X^{\otimes q} \left[e^{bJ_k^{i_0}} \mathbf{1}_{S_k^{\bar{\alpha}} = S_k^{\bar{\beta}} = S_k^{\bar{\gamma}}} \right]^{1/b} \leq \mathbb{E}_X^{\otimes q} \left[e^{2bJ_k^{i_0}} \right]^{1/2b} k^{-1/b},$$

where we again used that $\sum_x p_k(x)^3 \leq (p_k^*)^2 \leq k^{-2}$ by (18). Now observe that by conditioning on S^{i_0} , we have

$$\mathbb{E}_X^{\otimes q} \left[e^{2bJ_k^{i_0}} \right] \leq \sup_{y \in \mathbb{Z}^2} \mathbb{E}_y^S \left[\sup_{x \in \mathbb{Z}^2} \mathbb{E}_x^{S^2} \left[e^{2b\beta_N^2 \sum_{n=1}^k \mathbf{1}_{S_n^1 = S_n^2}} \right]^{q-1} \right],$$

where uniformly on all nearest neighbors walks $Z \in \mathcal{Z}$,

$$2b\beta_N^2 \sup_{x \in \mathbb{Z}^2} \mathbb{E}_x \sum_{n=1}^k \mathbf{1}_{S_k=Z_k} \leq 4(1+o(1))b\hat{\beta}^2 \frac{\log(k+1)}{\log N}$$

because of the definition of R_N and $\sup_x p_n(x) \leq 2/(\pi n)$, see (2) and (18). Hence by Lemma 2.2 with (26)-(i),

$$\begin{aligned} \sup_{X \in (\mathbb{Z}^2)^q} \mathbb{E}_X^{\otimes q} \left[e^{bJ_k^{i_0}} \mathbf{1}_{S_k^{\hat{\alpha}}=S_k^{\hat{\beta}}=S_k^{\hat{\gamma}}} \right]^{1/b} &\leq \left(\frac{1}{1-8b\hat{\beta}^2 \frac{\log(k+1)}{\log N}} \right)^{(q-1)/2b} k^{-1/b} \\ &\leq e^{c \frac{\log(k+1)}{\sqrt{\log N}}} k^{-1/b}, \end{aligned}$$

for some universal constant $c > 0$, using (7).

Coming back to (23), we find that

$$\begin{aligned} (28) \quad \sup_{X \in (\mathbb{Z}^2)^q} A_1(X) &\leq Mq^3 \sum_{k=T}^{\lfloor e^{(\log N)^{1/3}} \rfloor} e^{\frac{1}{\pi}(1+\varepsilon_N)q^2\beta_N^2 \log(k+1)} k^{-2/a} e^{3c \frac{\log(k+1)}{\sqrt{\log N}}} k^{-3/b} \\ &+ CMq^3 e^{\frac{1}{\pi}(1+\varepsilon_N)q^2\beta_N^2 \log T} \sum_{k=\lfloor e^{(\log N)^{1/3}} \rfloor}^N e^{\lambda_{k,N}^2(\sqrt{a}\hat{\beta})\frac{q}{2}/a} k^{-2/a} e^{3c \frac{\log(k+1)}{\sqrt{\log N}}} k^{-3/b}. \end{aligned}$$

By (26)-(ii), there exists $\delta > 0$ such that the first sum in the RHS of (28) can be bounded by

$$Mq^3 \sum_{k=T}^{\lfloor e^{(\log N)^{1/3}} \rfloor} k^{-1-\delta} \leq MCq^3 T^{-\delta},$$

for N large enough. Hence, we can set $T = \lfloor e^{(\log N)^{1/4}} \rfloor$ (which satisfies $\log T = o(\sqrt{\log N})$), so that $q^3 T^{-\delta} \rightarrow 0$ as $N \rightarrow \infty$. Relying on (20), the second sum in (28) is bounded by

$$\begin{aligned} CMq^3 e^{c' \log T} \sum_{k=\lfloor e^{(\log N)^{1/3}} \rfloor}^N e^{\frac{\hat{\beta}^2}{1-a\hat{\beta}^2} \frac{q}{2} \log(k+1)} e^{3c \frac{\log(k+1)}{\sqrt{\log N}}} k^{-1-1/a} \\ \leq CMq^3 e^{c' \log T} \sum_{k=\lfloor e^{(\log N)^{1/3}} \rfloor}^N k^{-1-\delta} \leq CMq^3 e^{-\delta(\log N)^{1/3} + c(\log N)^{1/4}}, \end{aligned}$$

for some $\delta, c' > 0$, where we used (26)-(iii). Then again the quantity multiplying M in the last line vanishes as $N \rightarrow \infty$, thus from (28) we obtain (22).

When dealing with A_2 , we have to use Hölder's inequality as in (24), (25) with 4 particles instead of 3, so in this case we can choose $a \sim 3/2$ and $b \sim 12$, and the condition (i) in (26) is satisfied with the restriction $\hat{\beta}^2 < 1/96$. The rest of the argument follows the same line as for A_1 .

As a result, we have shown that (19) holds. When $q = q_0$, although the constant C in (19) might depend on q_0 , it still yields that W_N is bounded in any L^p , $p > 1$. This fact combined with (4) implies the convergence (10) for all fixed q , which in particular implies that (8) holds in the case $q = q_0$ as well.

We now turn to the general case, where we only assume that $q(N)$ satisfies (7). Suppose that (8) does not hold, so that we can find $\varepsilon_0 > 0$ and a subsequence $q'_N = q(\varphi(N))$ such that

$$(29) \quad \forall N \in \mathbb{N}, \quad \mathbb{E}W_N^{q'_N} > e^{\lambda^2(q'_N/2)(1+\varepsilon_0)}.$$

One can distinguish two cases. If q'_N is bounded, then up to extracting a subsequence, we can suppose that q'_N converges to some $q_0 \geq 2$. Then, one can check that by (4), we must have $\mathbb{E}W_N^{q'_N} \rightarrow e^{\lambda^2(q_0/2)}$ (for example, using Skorokhod's representation theorem and Vitali's convergence theorem with the fact that W_N is bounded in any L^p). But this is impossible by (29). On the other hand, if q'_N is not bounded, up to extracting a subsequence we can suppose that $q'_N \rightarrow \infty$. But then (29) cannot be true because (8) holds with $q = q'_N \rightarrow \infty$. Therefore (8) must hold for any sequence $q(N)$ that satisfies (7). \square

2.4. On Remark 1.2. We describe the changes needed for obtaining the claim in Remark 1.2. Recall the definitions of F_n and K_n , see (12) and (13), and (11). Set

$$A_N = \sum_{n=1}^N \mathbf{1}_{(F_n \cup K_n)^c} \sum_{1 \leq i < j \leq q} \mathbf{1}_{S_n^i = S_n^j},$$

$$B_N = \sum_{n=1}^N \mathbf{1}_{F_n} \sum_{1 \leq i < j \leq q} \mathbf{1}_{S_n^i = S_n^j}, \quad C_N = \sum_{n=1}^N \mathbf{1}_{K_n} \sum_{1 \leq i < j \leq q} \mathbf{1}_{S_n^i = S_n^j}$$

Note that for any $u_N \geq 1$, we can check that $\mathbb{E}_X^{\otimes q} [e^{u_N \beta_N^2 A_N}]$ is bounded above by $\Psi_{N,q}(X)$ of (37) with $T = N$ and β_N replaced by $\beta_N u_N$. Using Hölder's inequality it is enough to show (together with the proof of Theorem 2.1, which actually controls $\sup_X \Psi_{N,q}(X)$) that for any $\hat{\beta} < 1$ and $q_N = o(\log N / \log \log N)$, there exist $v_N \rightarrow \infty$ so that

$$(30) \quad \sup_X \mathbb{E}_X^{\otimes q} [e^{v_N \beta_N^2 B_N}]^{1/v_N} \rightarrow_{N \rightarrow \infty} 1, \quad \sup_X \mathbb{E}_X^{\otimes q} [e^{v_N \beta_N^2 C_N}]^{1/v_N} \rightarrow_{N \rightarrow \infty} 1.$$

We sketch the proof of the first limit in (30), the proof of the second is similar. By Corollary C.2 (applied on the space of q -tuples of path, with $f(Y_n) = v_N \beta_N^2 \sum_{1 \leq i < j \leq q} \mathbf{1}_{S_n^i = S_n^j} \mathbf{1}_{F_n}$), it suffices to show that

$$(31) \quad \limsup_{N \in \mathbb{N}} \sup_{X \in \mathbb{Z}^q} \mathbb{E}_X^{\otimes q} [v_N \beta_N^2 B_N] = 0.$$

To see (31), fix $K \in \llbracket 1, N \rrbracket$. By (18), we have that

$$(32) \quad \mathbb{E}_X^{\otimes q} \sum_{n=1}^N \mathbf{1}_{F_n} \sum_{1 \leq i < j \leq q} \mathbf{1}_{S_n^i = S_n^j} \leq \sum_{n=1}^K \frac{q(q-1)}{2} \frac{C}{n} + \sum_{n=K+1}^N \sum_{1 \leq i < j \leq q} \mathbb{E}_X^{\otimes q} \mathbf{1}_{F_n} \mathbf{1}_{S_n^i = S_n^j}.$$

For $i < j \leq q$ and $r \in \{0, 1, 2\}$, further denote

$$F_n^{i,j;r} = \{\exists (\bar{\alpha}, \bar{\beta}, \bar{\gamma}) : \bar{\alpha} < \bar{\beta} < \bar{\gamma} \leq q, S_n^{\bar{\alpha}} = S_n^{\bar{\beta}} = S_n^{\bar{\gamma}}, |\{\bar{\alpha}, \bar{\beta}, \bar{\gamma}\} \cap \{i, j\}| = r\}.$$

We have that

$$\sum_{1 \leq i < j \leq q} \mathbb{E}_X^{\otimes q} \mathbf{1}_{F_n} \mathbf{1}_{S_n^i = S_n^j} = \sum_{1 \leq i < j \leq q} \mathbb{E}_X^{\otimes q} \sum_{r=0}^2 \mathbf{1}_{F_n^{i,j;r}} \mathbf{1}_{S_n^i = S_n^j}.$$

We first focus on the term $r = 0$. By independence, (18) and the union bound,

$$\sum_{1 \leq i < j \leq q} \mathbb{E}_X^{\otimes q} \mathbf{1}_{F_n^{i,j;0}} \mathbf{1}_{S_n^i = S_n^j} \leq \sum_{1 \leq i < j \leq q} \frac{C}{n} \sum_{\bar{\alpha} < \bar{\beta} < \bar{\gamma} \leq q} \sup_{x_i \in \mathbb{Z}^2} \sum_{y \in \mathbb{Z}^2} \prod_{i=1}^3 \mathbb{P}_{x_i}(S_n = y) \leq \frac{Cq^5}{n^3}.$$

When $r = 1$, the condition in the indicator function becomes that there exist $\bar{\alpha} < \bar{\beta} \leq q$ such that $S_n^i = S_n^j = S_n^{\bar{\alpha}} = S_n^{\bar{\beta}}$. Hence, the term for $r = 1$ is bounded by

$$\sum_{1 \leq i < j \leq q} \sum_{\bar{\alpha} < \bar{\beta} \leq q} \sup_{x_i \in \mathbb{Z}^2} \sum_{y \in \mathbb{Z}^2} \prod_{i=1}^4 \mathbb{P}_{x_i}(S_n = y) \leq \frac{Cq^4}{n^3}.$$

Similarly, we can bound the term for $r = 2$ by a constant times q^3/n^2 . Using (32), we find that for all $K \in \llbracket 1, N \rrbracket$,

$$(33) \quad \sup_{X \in \mathbb{Z}^q} \mathbb{E}_X^{\otimes q} [v_N \beta_N^2 B_N] \leq \frac{C v_N \hat{\beta}^2}{\log N} \left(\frac{q(q-1)}{2} \log K + \frac{q^5}{K^2} + \frac{q^4}{K^2} + \frac{q^3}{K} \right).$$

For $K = \lfloor (\log N)^{3/4} \rfloor$, and $q^2 = o(\log N / \log \log N)$, we find that (31) holds with a well-chosen $v_N \rightarrow \infty$.

3. NO TRIPLE INTERSECTIONS - PROOF OF THEOREM 2.1

Recall that $T \in \llbracket 1, N \rrbracket$. For compactness of notation in the rest of the paper, set

$$(34) \quad \sigma_N^2 = \sigma_N^2(\hat{\beta}) = e^{\beta_N^2} - 1.$$

By (2), there exist $\delta_N = \delta(N, \hat{\beta})$ and $\delta'_N = \delta'(N, \hat{\beta})$ that vanish as $N \rightarrow \infty$ such that

$$(35) \quad \sigma_N^2 = \frac{\hat{\beta}^2}{R_N} (1 + \delta_N) = \frac{\pi \hat{\beta}^2}{\log N} (1 + \delta'_N).$$

3.1. Expansion in chaos. In this section, we show that the moment without triple intersections can be bounded by a rather simple expansion. Introduce the following notation: for $\mathbf{n} = (n_0, n_1, \dots, n_k)$ and $\mathbf{x} = (x_0, x_1, \dots, x_k)$, let $p_{\mathbf{n}, \mathbf{x}} = \prod_{i=1}^k p(n_i - n_{i-1}, x_i - x_{i-1})$.

Proposition 3.1. *For all $X = (x_0^1, \dots, x_0^q) \in (\mathbb{Z}^2)^q$, we have*

$$(36) \quad \mathbb{E}_X^{\otimes q} \left[e^{\beta_N^2 \sum_{n=1}^T \sum_{1 \leq i < j \leq q} \mathbf{1}_{S_n^i = S_n^j}} \mathbf{1}_{G_T} \right] \leq \Psi_{N,q}(X),$$

where

$$(37) \quad \Psi_{N,q}(X) = \sum_{k=0}^{\infty} \sigma_N^{2k} \sum_{\substack{1 \leq n_1 < \dots < n_k \leq T, (i_1 < j_1), \dots, (i_k < j_k) \\ \mathbf{x}^1 \in (\mathbb{Z}^2)^k, \dots, \mathbf{x}^q \in (\mathbb{Z}^2)^k}} \prod_{r=1}^k \mathbf{1}_{x_r^{i_r} = x_r^{j_r}} \prod_{i=1}^q p_{(0, n_1, \dots, n_k), (x_0^i, \mathbf{x}^i)},$$

where we recall (34) for the definition of σ_N .

(By convention, here and throughout the paper, the term $k = 0$ in sums as (37) equals 1.)

Proof. We will use here the lexicographical ordering of 3-tuples (n, i, j) and use the shorthand notation $(n_1, i_1, j_1) < \cdots < (n_k, i_k, j_k)$ to denote a collection of k 3-tuples satisfying $(n_1, i_1, j_1) < \cdots < (n_k, i_k, j_k) \in \llbracket 1, T \rrbracket \times \llbracket 1, q \rrbracket^2$ with $i_r < j_r$ for all $r \leq k$. For brevity, we write G for G_T .

For $X = (x_0^1, \dots, x_0^q) \in (\mathbb{Z}^2)^q$, using the identity $e^{\beta_N^2 \mathbf{1}_{S_n^i = S_n^j}} - 1 = \sigma_N^2 \mathbf{1}_{S_n^i = S_n^j}$,

$$\begin{aligned} M_{N,q}^{\text{no triple}}(X) &:= \mathbb{E}_X^{\otimes q} \left[e^{\beta_N^2 \sum_{n=1}^T \sum_{1 \leq i < j \leq q} \mathbf{1}_{S_n^i = S_n^j} \mathbf{1}_G} \right] \\ &= \mathbb{E}_X^{\otimes q} \left[\prod_{n \in \llbracket 1, T \rrbracket, i < j \leq q} \left(1 + \sigma_N^2 \mathbf{1}_{S_n^i = S_n^j} \right) \mathbf{1}_G \right], \end{aligned}$$

Expand the previous product to obtain that:

$$(38) \quad M_{N,q}^{\text{no triple}}(X) = \sum_{k=0}^{\infty} \sigma_N^{2k} \sum_{(n_1, i_1, j_1) < \cdots < (n_k, i_k, j_k)} \mathbb{E}_X^{\otimes q} \left[\prod_{r=1}^k \mathbf{1}_{S_{n_r}^{i_r} = S_{n_r}^{j_r}} \mathbf{1}_G \right].$$

Since there are no triple or more particle intersections on the event G , the above sum can be restricted to 3-tuples $(n_r, i_r, j_r)_{r \leq k}$ such that $n_r < n_{r+1}$ for all $r < k$. Hence,

$$\begin{aligned} M_{N,q}^{\text{no triple}}(X) &= \sum_{k=0}^{\infty} \sigma_N^{2k} \sum_{1 \leq n_1 < \cdots < n_k \leq T, (i_1 < j_1), \dots, (i_k < j_k)} \mathbb{E}_X^{\otimes q} \left[\prod_{r=1}^k \mathbf{1}_{S_{n_r}^{i_r} = S_{n_r}^{j_r}} \mathbf{1}_G \right] \\ &\leq \Psi_{N,q}(X), \end{aligned}$$

where Ψ is defined in (37), and where we have bounded $\mathbf{1}_G$ by 1 in the inequality. \square

3.2. Decomposition in two-particle intersections. In this section, we rewrite $\Psi_{N,q}$ in terms of successive two-particle interactions. We generalize a decomposition used in [7, Section 5.1] that was restricted to a third moment computation ($q = 3$). The following notation is borrowed from their paper. Let

$$(39) \quad U_N(n, x) := \begin{cases} \sigma_N^2 \mathbb{E}_0^{\otimes 2} \left[e^{\beta_N^2 \sum_{l=1}^{n-1} \mathbf{1}_{S_l^1 = S_l^2}} \mathbf{1}_{S_n^1 = S_n^2 = x} \right] & \text{if } n \geq 1, \\ \mathbf{1}_{x=0} & \text{if } n = 0, \end{cases}$$

and

$$(40) \quad U_N(n) := \sum_{z \in \mathbb{Z}^2} U_N(n, z) = \begin{cases} \sigma_N^2 \mathbb{E}_0^{\otimes 2} \left[e^{\beta_N^2 \sum_{l=1}^{n-1} \mathbf{1}_{S_l^1 = S_l^2}} \mathbf{1}_{S_n^1 = S_n^2} \right] & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Observe that, by the identity $e^{\beta_N^2 \mathbf{1}_{S_l^1 = S_l^2}} - 1 = \sigma_N^2 \mathbf{1}_{S_l^1 = S_l^2}$, one has for all $n \geq 1$,

$$\begin{aligned} (41) \quad &\mathbb{E}_0^{\otimes 2} \left[e^{\beta_N^2 \sum_{l=1}^{n-1} \mathbf{1}_{S_l^1 = S_l^2}} \mathbf{1}_{S_n^1 = S_n^2 = x} \right] = \mathbb{E}_0^{\otimes 2} \left[\prod_{l=1}^{n-1} \left(1 + \sigma_N^2 \mathbf{1}_{S_l^1 = S_l^2} \right) \mathbf{1}_{S_n^1 = S_n^2 = x} \right] \\ &= \sum_{k=0}^{\infty} \sigma_N^{2k} \sum_{n_0=0 < n_1 < \cdots < n_k < n} \mathbb{E}_0^{\otimes 2} \left[\prod_{r=1}^k \mathbf{1}_{S_{n_r}^1 = S_{n_r}^2} \mathbf{1}_{S_n^1 = S_n^2 = x} \right]. \end{aligned}$$

Hence for all $n \geq 1$:

$$(42) \quad U_N(n, x) = \sigma_N^2 \sum_{k=0}^{\infty} \sigma_N^{2k} \sum_{\substack{n_0=0 < n_1 < \dots < n_k < n = n_{k+1} \\ x_0=0, x_1, \dots, x_k \in \mathbb{Z}^2, x_{k+1}=x}} \prod_{r=1}^{k+1} p_{n_r - n_{r-1}}(x_r - x_{r-1})^2.$$

Now, in the sum in (37), we observe that (only) two particles interact at given times ($n_1 < \dots < n_k$). So we define $a_1 = n_1$ and $b_1 = n_r$ such that (n_1, n_2, \dots, n_r) are the successive times that verify $(i_1, j_1) = (i_2, j_2) = \dots = (i_r, j_r)$ before a new couple of particles $\{i_{r+1}, j_{r+1}\} \neq \{i_1, j_1\}$ is considered, and we let $k_1 = r$ be the number of times the couple is repeated. Define then $a_2 \leq b_2, a_3 \leq b_3, \dots, a_m \leq b_m$ similarly for the next interacting couples, with m denoting the number of alternating couples and k_1, \dots, k_m the numbers of times the couples are repeated successively.

Further let $\mathbf{X} = (X_1, \dots, X_m)$ and $\mathbf{Y} = (Y_1, \dots, Y_m)$ with $X_r = (x_r^1, \dots, x_r^q)$ and $Y_r = (y_r^1, \dots, y_r^q)$ denote respectively the positions of the particles at time a_r and b_r . We also write $X = (x_0^p)_{p \leq q}$, for the initial positions of the particles at time 0. We call a *diagram* \mathbf{I} of size $m \in \mathbb{N}$ any collection of m couples $\mathbf{I} = ((i_1 < j_1), \dots, (i_m < j_m))$ such that $\{i_r, j_r\} \neq \{i_{r+1}, j_{r+1}\}$. We denote by $\mathcal{D}(m, q)$ the set of all diagrams of size m .

If we re-write $\Psi_{N,q}(X)$ according to the decomposition that we just described, we find that:

$$\begin{aligned} \Psi_{N,q}(X) &= \sum_{m=0}^{\infty} \sum_{\substack{1 \leq a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_m \leq b_m \leq T \\ \mathbf{X}, \mathbf{Y} \in (\mathbb{Z}^2)^{m \times q}, (i_r, j_r)_{r \leq m} \in \mathcal{D}(m, q)}} \sum_{k_1 \in [1, b_1 - a_1 + 1], \dots, k_m \in [1, b_m - a_m + 1]} \sigma_N^{2k_1 + \dots + 2k_m} \\ &\times \prod_{p \leq q} p_{a_1}(x_1^p - x_0^p) \left(\prod_{r=1}^{m-1} \prod_{p \leq q} p_{a_{r+1} - b_r}(x_{r+1}^p - y_r^p) \right) \prod_{r=1}^m \mathbf{1}_{x_r^{i_r} = x_r^{j_r}} \mathbf{1}_{y_r^{i_r} = y_r^{j_r}} \\ &\times \sum_{\substack{a_r < n_1 < \dots < n_{k_r-2} < b_r \\ \mathbf{z}^1 \in (\mathbb{Z}^2)^{k_r-2}, \dots, \mathbf{z}^q \in (\mathbb{Z}^2)^{k_r-2}}} \prod_{s=1}^{k_r-2} \mathbf{1}_{z_s^{i_r} = z_s^{j_r}} \prod_{p \leq q} p_{(a_r, n_1, \dots, n_{k_r-2}, b_r), (x_r^p, z_1^p, \dots, z_{k_r-2}^p, y_r^p)}. \end{aligned}$$

See Figure 1 for a pictorial description of the intersections associated with a diagram.

Summing over all the configurations between time a_r and b_r gives a contribution of $\sigma_N^2 \mathbf{1}_{x_r^{i_r} = y_r^{i_r}}$ when $a_r = b_r$, and

$$\begin{aligned} &\sum_{k=2}^{\infty} \sigma_N^{2k} \sum_{\substack{n_0 = a_r < n_1 < \dots < n_{k-2} < b_r = n_{k-1} \\ x_0 = x_r^{i_r}, x_1, \dots, x_{k-2} \in \mathbb{Z}^2, x_{k-1} = y_r^{i_r}}} \prod_{i=1}^{k-1} p_{n_i - n_{i-1}}(x_i - x_{i-1})^2 \\ &= \sigma_N^2 U_N(b_r - a_r, y_r^{i_r} - x_r^{i_r}), \end{aligned}$$

when $a_r < b_r$ (in this case $k_r \geq 2$ by definition). It directly follows that:

$$(43) \quad \Psi_{N,q}(X) = \sum_{m=0}^{\infty} \sigma_N^{2m} \sum_{\substack{1 \leq a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_m \leq b_m \leq T \\ \mathbf{X}, \mathbf{Y} \in \mathbb{Z}^{m \times q}, \mathbf{I} = (i_r, j_r)_{r \leq m} \in \mathcal{D}(m, q)}} A_{X, \mathbf{a}, \mathbf{b}, \mathbf{X}, \mathbf{Y}, \mathbf{I}},$$

where

$$(44) \quad \begin{aligned} A_{X,\mathbf{a},\mathbf{b},\mathbf{X},\mathbf{Y},\mathbf{I}} &= \prod_{p \leq q} p_{a_1}(x_1^p - x_0^p) \prod_{r=1}^m U_N(b_r - a_r, y_r^{i_r} - x_r^{i_r}) \mathbf{1}_{x_r^{i_r} = x_r^{j_r}} \mathbf{1}_{y_r^{i_r} = y_r^{j_r}} \\ &\times \prod_{p \notin \{i_r, j_r\}} p(b_r - a_r, y_r^p - x_r^p) \prod_{r=1}^{m-1} \prod_{p \leq q} p(a_{r+1} - b_r, x_{r+1}^p - y_r^p). \end{aligned}$$

We can further simplify the expression (43). Let $\mathbf{I} = (i_r, j_r)_{r \leq m} \in \mathcal{D}(m, q)$ be any diagram. For all $r \leq m$, denote by \bar{k}_r^1 the last index $l < r$ such that $i_r \in \{i_l, j_l\}$, i.e. $\bar{k}_r^1 = \sup\{l \in \llbracket 1, r-1 \rrbracket : i_r \in \{i_l, j_l\}\}$. When the set is empty we set $\bar{k}_r^1 = 0$. Define \bar{k}_r^2 similarly for j_r instead of i_r and let $\bar{k}_r = \bar{k}_r^1 \vee \bar{k}_r^2$. See figure 1.

Proposition 3.2. *For all $X \in (\mathbb{Z}^2)^q$,*

$$(45) \quad \Psi_{N,q}(X) = \sum_{m=0}^{\infty} \sigma_N^{2m} \sum_{\substack{1 \leq a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_m \leq b_m \leq T \\ \mathbf{x}, \mathbf{y} \in \mathbb{Z}^m, \mathbf{I} = (i_r, j_r)_{r \leq m} \in \mathcal{D}(m, q)}} \tilde{A}_{X,\mathbf{a},\mathbf{b},\mathbf{x},\mathbf{y},\mathbf{I}},$$

where

$$(46) \quad \begin{aligned} \tilde{A}_{X,\mathbf{a},\mathbf{b},\mathbf{x},\mathbf{y},\mathbf{I}} &= \prod_{p \in \{i_1, j_1\}} p_{a_1}(x_1 - x_0^p) \prod_{r=1}^m U_N(b_r - a_r, y_r - x_r) \\ &\times \prod_{r=1}^{m-1} p(a_{r+1} - b_{\bar{k}_{r+1}^1}, x_{r+1} - y_{\bar{k}_{r+1}^1}) p(a_{r+1} - b_{\bar{k}_{r+1}^2}, x_{r+1} - y_{\bar{k}_{r+1}^2}). \end{aligned}$$

Proof. Denote $x_r = x_r^{i_r}$ and $y_r = y_r^{i_r}$. We obtain (46) from (44) by using the semi group property of the random walk transition probabilities and summing, at intersection times, over the location of particles not involved in the intersection. \square

Proposition 3.3. *We have that*

$$(47) \quad \sup_{X \in (\mathbb{Z}^2)^q} \Psi_{N,q}(X) \leq \sum_{m=0}^{\infty} \sum_{\mathbf{I} \in \mathcal{D}(m, q)} \sigma_N^{2m} A_{m,N,\mathbf{I}},$$

where

$$(48) \quad A_{m,N,\mathbf{I}} = \sum_{\substack{u_i \in \llbracket 1, T \rrbracket, v_i \in \llbracket 0, T \rrbracket, 1 \leq i \leq m \\ \sum_{i=1}^m u_i \leq T}} p_{2u_1}^* U_N(v_m) \prod_{r=1}^{m-1} U_N(v_r) p_{v_r + 2u_{r+1} + 2\tilde{u}_{r+1}}^*.$$

with

$$(49) \quad \tilde{u}_r = \begin{cases} \sum_{i=\bar{k}_r+1}^{r-1} u_i & \text{if } \bar{k}_r < r-1, \\ \frac{u_{r-1}}{2} & \text{if } \bar{k}_r = r-1, \end{cases}$$

and $p_k^* = \sup_{x \in \mathbb{Z}^2} p_k(x)$.

Proof. By (45), it is enough to show that

$$(50) \quad \sup_{X \in (\mathbb{Z}^2)^q} \sum_{\substack{1 \leq a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_m \leq b_m \leq T \\ \mathbf{x}, \mathbf{y} \in \mathbb{Z}^m}} \tilde{A}_{X,\mathbf{a},\mathbf{b},\mathbf{x},\mathbf{y},\mathbf{I}} \leq A_{m,N,\mathbf{I}}.$$

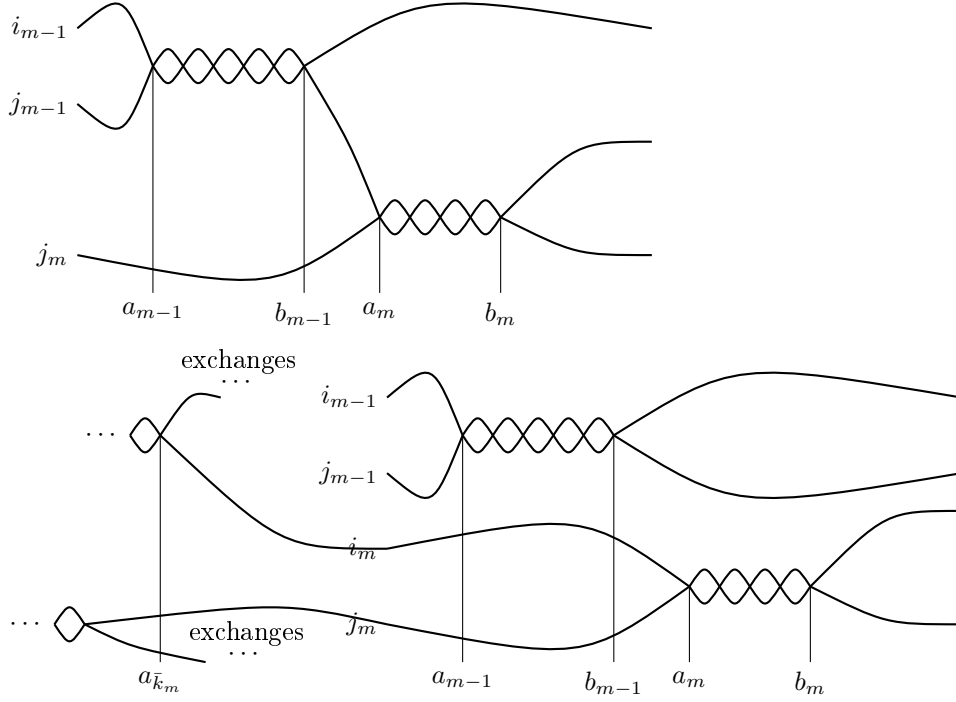


FIGURE 1. Two types of diagrams. Note the different types of exchanges. In the top diagram, $\bar{k}_m = m - 1$ and the m th jump is considered short (the notion of short and long jumps is defined in Section 3.5). In the bottom, the m th jump is considered long (with respect to a given L) if $m - \bar{k}_m > L + 2$. In that case, both paths i_m, j_m will be involved in an intersection not before a_{m-L-2} .

We begin by summing on y_m , which gives a contribution of

$$\sum_{y_m} U_N(b_m - a_m, y_m - x_m) = U_N(b_m - a_m),$$

where $U_N(n)$ is defined in (40). Then summing on x_m gives a factor

$$\begin{aligned} & \sum_{x_m} p(a_m - b_{\bar{k}_m^1}, x_m - y_{\bar{k}_m^1}) p(a_m - b_{\bar{k}_m^2}, x_m - y_{\bar{k}_m^2}) \\ &= p(2a_m - b_{\bar{k}_m^1} - b_{\bar{k}_m^2}, y_{\bar{k}_m^1} - y_{\bar{k}_m^2}) \leq p_{2a_m - b_{\bar{k}_m^1} - b_{\bar{k}_m^2}}^*. \end{aligned}$$

By iterating this process we obtain that the sum on \mathbf{x}, \mathbf{y} is bounded (uniformly on the starting point X) by

$$p_{2a_1}^* U_N(b_m - a_m) \prod_{r=1}^{m-1} U_N(b_r - a_r) p_{2a_{r+1} - b_{\bar{k}_{r+1}^1} - b_{\bar{k}_{r+1}^2}}^*.$$

If we introduce the change of variables $u_i = a_i - b_{i-1}$ and $v_i = b_i - a_i$ with $b_0 = 0$, then equation (50) follows from combining that $2a_{r+1} - b_{\bar{k}_{r+1}^1} - b_{\bar{k}_{r+1}^2} \geq$

$v_r + 2u_{r+1} + 2\tilde{u}_{r+1}$ with the monotonicity of p_n^* in n , which follows from

$$(51) \quad p_{n+1}^* = \sup_y \sum_x p_n(x) p_1(y-x) \leq p_n^*.$$

□

3.3. Estimates on U_N . It is clear from Proposition 3.3 that the function U_N plays a crucial role in our moment estimates, which we will obtain by an induction in the next subsection. In the current subsection, we digress and obtain a-priori estimates on U_N (and $\mathbb{E}[W_N(\beta_N)^2]$). Appendix B contains some improvements that are not needed in the current work but may prove useful in follow up work.

Proposition 3.4. *There exists $N_0 = N_0(\hat{\beta})$ such that for all $N \geq N_0$ and all $n \leq N$,*

$$(52) \quad \mathbb{E} [W_n(\beta_N)^2] \leq \frac{1}{1 - \sigma_N^2 R_n}.$$

Furthermore, there exists $\varepsilon_n = \varepsilon(n, \hat{\beta}) \rightarrow 0$ as $n \rightarrow \infty$, such that for all $N \geq n$,

$$(53) \quad \mathbb{E} [W_n(\beta_N)^2] = (1 + \varepsilon_n) \frac{1}{1 - \hat{\beta}^2 \frac{\log n}{\log N}}.$$

Proof. We first choose $N_0 = N_0(\hat{\beta})$ large enough such that for all $N \geq (n \vee N_0)$, we have $\sigma_N^2 R_n < 1$. That this is possible follows from (18) which yields that

$$(54) \quad \forall n \in \mathbb{N}, \quad R_n = \sum_{s=1}^n p_{2s}(0) \leq \frac{1}{\pi} \sum_{s=1}^n \frac{1}{s} \leq \frac{1}{\pi} \log(n+1).$$

For the rest of the proof, we continue in this setup. Similarly to (41), we have (letting $n_0 = x_0 = 0$) that

$$\begin{aligned} \mathbb{E} [W_n(\beta_N)^2] &= \mathbb{E}_0 \left[e^{\beta_N^2 \sum_{k=1}^n \mathbf{1}_{S_k^1 = S_k^2}} \right] \\ &= \sum_{k=0}^{\infty} \sigma_N^{2k} \sum_{0 < n_1 < \dots < n_k \leq n} \sum_{x_1, \dots, x_k \in \mathbb{Z}^2} \prod_{i=1}^k p_{n_i - n_{i-1}}(x_i - x_{i-1})^2. \end{aligned}$$

Hence, we obtain by letting $n_i - n_{i-1}$ run free in $\llbracket 1, n \rrbracket$ that

$$\mathbb{E} [W_n(\beta_N)^2] \leq \sum_{k=0}^{\infty} \sigma_N^{2k} \left(\sum_{m=1}^n \sum_{x \in \mathbb{Z}^2} p_m(x)^2 \right)^k = \sum_{k=0}^{\infty} \sigma_N^{2k} R_n^k = \frac{1}{1 - \sigma_N^2 R_n},$$

which gives (52). On the other hand, if we let $n_i - n_{i-1}$ run free in $\llbracket 1, n/k \rrbracket$, we have

$$\begin{aligned} \mathbb{E} [W_n(\beta_N)^2] &\geq 1 + \sum_{k=1}^{\infty} \sigma_N^{2k} \left(\sum_{m=1}^{n/k} \sum_{x \in \mathbb{Z}^2} p_m(x)^2 \right)^k \\ &\geq 1 + \sum_{k=1}^{\log n} \sigma_N^{2k} R_{n/\log n}^k = \frac{1 - (\sigma_N^2 R_{n/\log n})^{\log n+1}}{1 - \sigma_N^2 R_{n/\log n}}, \end{aligned}$$

By (35) and the fact that $R_n \sim \frac{1}{\pi} \log n$ as $n \rightarrow \infty$ by (2), we find that for all $N \geq n$,

$$\mathbb{E} [W_n(\beta_N)^2] \geq (1 + \delta_n) \frac{1}{1 - \hat{\beta}^2 \frac{\log n}{\log N}},$$

with $\delta_n = \delta_n(\hat{\beta}) \rightarrow 0$ as $n \rightarrow \infty$. Combining this with (52) entails (53). \square

Proposition 3.5. *For all $M \geq 1$, we have:*

$$(55) \quad \sum_{n=0}^M U_N(n) = \mathbb{E} [W_M^2].$$

Moreover, there is $C(\hat{\beta}) > 0$ such that, as $N \rightarrow \infty$ and for all $n \leq N$,

$$(56) \quad U_N(n) \leq C \frac{1}{\left(1 - \hat{\beta}^2 \frac{\log n}{\log N}\right)^2} \frac{1}{n \log N}.$$

Remark 3.6. *When $n \rightarrow \infty$, one can take the constant C that appears in (56) arbitrarily close to one. See Appendix B.*

Proof. By (42), we have, for $n \geq 1$,

$$(57) \quad \begin{aligned} U_N(n) &= \sigma_N^2 \sum_{k=1}^{\infty} \sigma_N^{2(k-1)} \sum_{0 < n_1 < \dots < n_{k-1} < n_k := n} \sum_{x_1, \dots, x_k \in \mathbb{Z}^2} \prod_{i=1}^k p_{n_i - n_{i-1}}(x_i - x_{i-1})^2 \\ &= \sigma_N^2 \sum_{k=1}^{\infty} \sigma_N^{2(k-1)} \sum_{0 < n_1 < \dots < n_{k-1} < n_k := n} \prod_{i=1}^k p_{2n_i - 2n_{i-1}}(0). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{n=0}^M U_N(n) &= 1 + \sum_{k=1}^{\infty} \sigma_N^{2k} \sum_{0 < n_1 < \dots < n_{k-1} < n_k \leq M} \sum_{x_1, \dots, x_k \in \mathbb{Z}^2} \prod_{i=1}^k p_{n_i - n_{i-1}}(x_i - x_{i-1})^2 \\ &= \mathbb{E}[W_M^2], \end{aligned}$$

which yields (55).

We now prove (56) by expressing U_N as a function of a renewal process, see [6] or [21, Chapter 1] for the general framework in the context of the pinning model. From (57), we have the following representation for $U_N(n)$ when $n \geq 1$:

$$U_N(n) = \sum_{k=1}^{\infty} (\sigma_N^2 R_N)^k \mathbb{P} \left(\tau_k^{(N)} = n \right),$$

where the $\tau_k^{(N)}$ are renewal times defined by

$$\tau_k^{(N)} = \sum_{i \leq k} T_i^{(N)},$$

with $(T_i^{(N)})_i$ being i.i.d. random variables with distribution

$$\mathbb{P} \left(T_i^{(N)} = n \right) = \frac{1}{R_N} p_{2n}(0) \mathbf{1}_{1 \leq n \leq N}, \quad \text{and} \quad R_N = \sum_{n=1}^N p_{2n}(0).$$

By [6, Proposition 1.5], there exists $C > 0$ such that for all $n \leq N$,

$$(58) \quad \mathbb{P}\left(\tau_k^{(N)} = n\right) \leq Ck\mathbb{P}\left(T_1^{(N)} = n\right)\mathbb{P}\left(T_1^{(N)} \leq n\right)^{k-1}.$$

Hence, using that $\sum_{k=1}^{\infty} ka^{k-1} = \frac{1}{(1-a)^2}$ for $a < 1$,

$$\begin{aligned} U_N(n) &\leq C \sum_{k=1}^{\infty} (\sigma_N^2 R_N)^k k \mathbb{P}\left(T_1^{(N)} = n\right) \mathbb{P}\left(T_1^{(N)} \leq n\right)^{k-1} \\ &= C \frac{p_{2n}(0)}{R_N} \frac{\sigma_N^2 R_N}{\left(1 - \sigma_N^2 R_N \frac{R_n}{R_N}\right)^2}, \end{aligned}$$

which gives (56) by (2), (18) and (35). \square

3.4. Summing on the v_i 's. In the following we denote

$$(59) \quad F(u) = \frac{1}{u} \frac{1}{1 - \hat{\beta}^2 \frac{\log(u)}{\log N}}.$$

By differentiation with respect to u one checks that F is non-increasing.

Proposition 3.7. *There exists $N_0(\hat{\beta}) > 0$ and $\varepsilon_N = \varepsilon(N, \hat{\beta}) \searrow 0$ as $N \rightarrow \infty$, such that for all $N \geq N_0(\hat{\beta})$,*

$$(60) \quad \sup_{X \in (\mathbb{Z}^2)^q} \Psi_{N,q}(X) \leq \sum_{m=0}^{\infty} \sigma_N^{2m} \sum_{\mathbf{I} \in \mathcal{D}(m,q)} \left(\frac{1}{\pi}\right)^{m-1} \tilde{A}_{m,N,\mathbf{I}},$$

where, recalling (49),

$$(61) \quad \tilde{A}_{m,N,\mathbf{I}} = \frac{1}{1 - \hat{\beta}^2} \sum_{u_i \in [1,T], 1 \leq i \leq m} (1 + \varepsilon_N)^m p_{2u_1}^* \prod_{r=2}^m F(u_r + \tilde{u}_r) \mathbf{1}_{\sum_{i=1}^r u_i \leq T}.$$

Proof. By (53), (48) and (55), summing over v_m in $A_{m,N,\mathbf{I}}$ gives a factor bounded by $\frac{1}{1 - \hat{\beta}^2} (1 + o(1))$. We will now estimate the sum over the variable v_{m-1} . Let $w = u_m + \tilde{u}_m$. (Note that by definition $3/2 \leq w \leq T \leq N$, and that w might be a non-integer multiple of $1/2$.) Writing $v = v_{m-1}$, the sum over v_{m-1} in (48) gives a factor

$$(62) \quad \sum_{v=0}^T U_N(v) p_{v+2u_m+2\tilde{u}_m}^* =: S_{\leq w} + S_{>w},$$

where $S_{\leq w}$ is the sum on the LHS of (62) restricted to $v \leq \lfloor w \rfloor$. Using (56) and (18), there exists a constant $C = C(\hat{\beta}) > 0$ such that

$$(63) \quad S_{>w} \leq \frac{C}{\log N} \sum_{v=\lfloor w \rfloor+1}^T \frac{1}{v^2} \leq \frac{1}{\log N} \frac{C}{w}.$$

Using (51) and (54),

$$(64) \quad S_{\leq w} \leq p_{2w}^* \sum_{v=0}^{\lfloor w \rfloor} U_N(v) = p_{2w}^* \mathbb{E}[W_{\lfloor w \rfloor}^2] \leq p_{2w}^* \frac{1}{1 - \sigma_N^2 R_{\lfloor w \rfloor}}.$$

where the upper bound holds by (52) for all $N \geq N_0(\hat{\beta})$ since $w \leq N$. Let $\delta_N = \delta(N, \hat{\beta}) \rightarrow 0$ such that (35) holds, and let $N'_0 = N'_0(\hat{\beta}) > N_0(\hat{\beta})$ be such that $\sup_{N \geq N'_0} \sup_{n \leq N} \hat{\beta}^2 \frac{1+\log n}{\log N} (1 + \delta_N) < 1$. By (18) and (54), we obtain that

$$p_{2w}^* \frac{1}{1 - \sigma_N^2 R_w} \leq \frac{1}{\pi} \frac{1}{w} \frac{1}{1 - \hat{\beta}^2 \frac{1+\log w}{\log N} (1 + \delta_N)}.$$

Moreover, as there is $C(\hat{\beta}) \in (0, \infty)$ such that

$$\sup_{N \geq N'_0} \sup_{n \leq N} \frac{1}{1 - \hat{\beta}^2 \frac{1+\log n}{\log N} (1 + \delta_N)} \leq C(\hat{\beta}),$$

we see that there exists $\varepsilon'_N = \varepsilon'(N, \hat{\beta}) \searrow_{N \rightarrow \infty} 0$ such that for all $n \leq N$,

$$\left| \frac{1}{1 - \hat{\beta}^2 \frac{1+\log n}{\log N} (1 + \delta_N)} - \frac{1}{1 - \hat{\beta}^2 \frac{\log n}{\log N}} \right| \leq \frac{\varepsilon'_N}{1 - \hat{\beta}^2 \frac{\log n}{\log N}}.$$

Coming back to (64), we obtain that for all $N \geq N'_0(\hat{\beta})$,

$$(65) \quad S_{\leq w} \leq \frac{1}{\pi w} \frac{1 + \varepsilon'_N}{1 - \hat{\beta}^2 \frac{\log w}{\log N}}.$$

We finally obtain from (65) and (63) that there exists $\varepsilon'_N = \varepsilon'(N, \beta) \searrow_{N \rightarrow \infty} 0$ such that the sum in (62) is smaller than

$$(1 + \varepsilon'_N) \frac{1}{\pi w} \frac{1}{1 - \hat{\beta}^2 \frac{\log w}{\log N}} = (1 + \varepsilon'_N) \frac{1}{\pi} F(u_m + \tilde{u}_m).$$

Repeating the same observation for v_{m-2}, \dots, v_1 leads to Proposition 3.7. \square

3.5. The induction pattern. Our next goal is to sum over $(u_r)_{r \leq m}$ that appear in (61). We will sum by induction starting from $r = m$ and going down to $r = 1$. To do so, we first need to define the notion of *good* and *bad* indices r . While performing the induction, encountering a *bad* index will add some nuisance term to the estimate. We will then show that, for typical diagrams, the *bad* indices are rare enough so that the nuisance can be neglected.

Let $L = L_N \in \mathbb{N} \setminus \{1, 2\}$ to be determined later. Given a diagram $\mathbf{I} \in \mathcal{D}(m, q)$, we say that $r \in \llbracket 1, m \rrbracket$ is a *long jump* if $r - \bar{k}_r > L + 2$, which means that the last times that the two particles i_r, j_r have been involved in an intersection are not too recent. We say that r is a *small jump* if it is not a long jump. (See Figure 1 for a pictorial description of short (top) and long (bottom) jumps.) Since small jumps reduce drastically the combinatorial choice on the new couple that intersects, the diagrams that will contribute to the moments will contain mostly long jumps. Let $K = K(\mathbf{I})$ denote the number of small jumps and $s_1 < \dots < s_K$ denote the indices of small jumps. For all $i \leq K$ such that $s_i - s_{i-1} > L + 1$, we mark the following indices $\{s_i - kL - 1, k \in \mathbb{N}, s_i - kL - 1 > s_{i-1}\}$ as *stopping* indices. We then call any long jump r a *fresh* index if r is stopping or if $r + 1$ is a small jump. Note that any stopping index is a fresh index. If m is a long jump we also mark it as a fresh index. The idea is that all indices smaller than a fresh index avoid nuisance terms, until we stumble on a stopping index or a small jump; we remark that since our induction will be downward from m , these nuisance-avoiding indices occur in the induction *following* a fresh index. Hence we say that an index r is *good* if it is a long jump that is not fresh. An index k is *bad* if it is not good.

For a given diagram, one can easily determine the nature of all indices via the following procedure: (i) mark all small jumps; (ii) mark every stopping index; (iii) mark all fresh indices; (iv) all the remaining indices that have not been marked are good indices.

For all $\mathbf{I} \in \mathcal{D}(m, q)$, we define for all $r < m$

$$\varphi(r) = \varphi(r, \mathbf{I}) = \begin{cases} \inf\{r' \geq r, r' \text{ is fresh}\} - L & \text{if } r \text{ is not a stopping index and } r+1 \text{ is a long jump,} \\ r & \text{otherwise.} \end{cases}$$

We also set $\varphi(m) = m$. Note that because of stopping indices, the function φ satisfies $\varphi(r) \leq r$. Here are a few immediate observations:

Lemma 3.8. (i) *If r is good, then $r+1$ is a long jump.*

(ii) *If $r \in \llbracket 2, m-1 \rrbracket$ is good, then $\varphi(r-1) = \varphi(r)$.*

(iii) *If $r \in \llbracket 2, m \rrbracket$ is fresh, then $\varphi(r-1) = r - L$.*

Proof. Proof of (i). Suppose that r is good. It must be that $r < m$ since by definition m is either fresh or a small jump. Now, $r+1$ must be a long jump otherwise r would be fresh.

Proof of (ii). Let $r \in \llbracket 2, m-1 \rrbracket$ be a good index. We distinguish two cases. First suppose that $r-1$ is not a stopping index. Then $r-1$ cannot be fresh because r is not a small jump. Therefore $\varphi(r-1) = \inf\{r' > r-1, r' \text{ is fresh}\} - L$. Furthermore, by (i), we have that $\varphi(r) = \inf\{r' \geq r, r' \text{ is fresh}\} - L$ and thus $\varphi(r-1) = \varphi(r)$. Now assume that $r-1$ is stopping. Then $\varphi(r-1) = r-1$. Moreover, by definition $r, \dots, r+L-1$ are long jumps and either $r+L-1$ is a stopping index or $r+L$ is a small jump. Therefore $r+L-1$ is a fresh index and $r, \dots, r+L-2$ are good, so that $\varphi(r) = (r+L-1) - L = r-1 = \varphi(r-1)$.

Proof of (iii). Let $r \in \llbracket 2, m \rrbracket$ be a fresh index. We first note that $r-1$ cannot be a stopping index. Indeed, if r is a stopping index, then $r-1$ cannot be stopping by definition; if r is not a stopping index, then as r is fresh, $r+1$ must be a small jump and thus $r-1$ cannot be stopping. Now, as $r-1$ is not stopping and r is fresh, we obtain that $\varphi(r-1) = r - L$. (Note that $r-1$ cannot be fresh because $r-1$ is not stopping and r is a long jump.) \square

For all $v \in [1, T]$, we further let

$$f(v) = \frac{\log N}{\hat{\beta}^2} \log \left(\frac{1 - \hat{\beta}^2 \frac{\log v}{\log N}}{1 - \hat{\beta}^2 \frac{\log T}{\log N}} \right).$$

Note that f is non-increasing. Recall (49) and the definition of F in (59).

Lemma 3.9. *For all $m \geq 2$, $\mathbf{I} \in \mathcal{D}(m, q)$, $k \in \llbracket 1, m-1 \rrbracket$ and $\sum_{i=1}^{m-k} u_i \leq T$ with $u_i \in \llbracket 1, T \rrbracket$,*

$$(66) \quad \begin{aligned} & \sum_{u_i \in \llbracket 1, T \rrbracket, m-k < i \leq m} \prod_{r=m-k+1}^m F(u_r + \tilde{u}_r) \mathbf{1}_{\sum_{i=1}^{m-k+1} u_i \leq T} \\ & \leq \sum_{i=0}^k \frac{c_i^k}{(k-i)!} \frac{1}{(1-\hat{\beta}^2)^i} f \left(\sum_{i=\varphi(m-k)}^{m-k} u_i \right)^{k-i}. \end{aligned}$$

with $c_0^1 = 1$, $c_1^1 = 2$, $c_i^{k+1} \leq c_i^k + 2\gamma_k^m \sum_{j=0}^{i-1} c_j^k$ for $i \leq k+1$ with $\gamma_k^m = \mathbf{1}_{m-k}$ is bad and $c_i^k = 0$ for $i > k$.

Remark 3.10. The c_i^k 's depend on m and $\mathbf{I} \in \mathcal{D}(m, q)$.

Before turning to the proof, we need another result that plays a key role in the proof of Lemma 3.9 and which clarifies the role of good indices.

Lemma 3.11. For all $k \in \llbracket 0, m-2 \rrbracket$, $j \leq k$ and $\sum_{i=1}^{m-k-1} u_i \leq T$ with $u_i \in \llbracket 1, T \rrbracket$,

$$\begin{aligned}
(67) \quad S_k f^j(u_1, \dots, u_{m-k-1}) &:= \\
&\sum_{u_{m-k}=1}^T F(u_{m-k} + \tilde{u}_{m-k}) f\left(\sum_{i=\varphi(m-k)}^{m-k} u_i\right)^j \mathbf{1}_{\sum_{i=1}^{m-k} u_i \leq T} \\
&\leq \frac{1}{j+1} f\left(\sum_{i=\varphi(m-k-1)}^{m-k-1} u_i\right)^{j+1} \\
&\quad + \gamma_k^m \sum_{l=1}^{j+1} \frac{j!}{(j+1-l)!} \frac{2}{(1-\hat{\beta}^2)^l} f\left(\sum_{i=\varphi(m-k-1)}^{m-k-1} u_i\right)^{j+1-l}.
\end{aligned}$$

Remark 3.12. When $m-k$ is good, the right hand side of (67) is reduced to a single term. When $m-k$ is bad, a nuisance term appears.

Proof. We divide the proof into three cases.

Case 1: $m-k$ is good. Necessarily $m-k+1$ is a long jump by Lemma 3.8-(i), so if we let $r_{\text{fresh}} = \inf\{r' > m-k, r' \text{ is fresh}\}$, then $\varphi(m-k) = r_{\text{fresh}} - L$. Because of the presence of stopping points, we have that $r_{\text{fresh}} - (m-k) \leq L-1$. Since also $r_{\text{fresh}} > m-k$, we obtain

$$(m-k) - L \leq \varphi(m-k) \leq (m-k) - 1.$$

Define

$$v := \sum_{i=\varphi(m-k)}^{m-k-1} u_i \in \llbracket 1, T \rrbracket.$$

As $m-k$ is a long jump, we first observe that

$$(68) \quad \tilde{u}_{m-k} \geq u_{m-k-1} + \dots + u_{m-k-L-1} \geq v.$$

Since F and f are non-increasing, see (59), this implies that

$$\begin{aligned}
S_k f^j &\leq \sum_{u_{m-k}=1}^T F(u_{m-k} + v) f(u_{m-k} + v)^j \mathbf{1}_{u_{m-k} + v \leq T} \\
&\leq \int_v^T \frac{1}{u} \frac{1}{1 - \hat{\beta}^2 \frac{\log(u)}{\log N}} f(u)^j du = \left[-\frac{1}{j+1} f(x)^{j+1} \right]_v^T = \frac{1}{j+1} f(v)^{j+1},
\end{aligned}$$

where in the comparison to the integral, we have used that $F(x)f(x)^j$ decreases in $x \in \llbracket 1, \dots, T \rrbracket$. Given that $\varphi(m-k-1) = \varphi(m-k)$ by Lemma 3.8-(ii), we have the identity $v = \sum_{i=\varphi(m-k-1)}^{m-k-1} u_i$. Hence (67) holds.

Case 2: $m - k$ is fresh. By Lemma 3.8-(iii), we have $\varphi(m - k - 1) = m - k - L$. This time, we define

$$v := \sum_{i=\varphi(m-k-1)}^{m-k-1} u_i \in \llbracket 1, T \rrbracket,$$

and decompose

$$(69) \quad S_k f^j = S_k^{\leq v} f^j + S_k^{> v} f^j,$$

where $S_k^{\leq v} f^j$ is the restriction of the sum in $S_k f^j$ to $u_{m-k} \in \llbracket 1, v \rrbracket$. Given that $m - k$ is a long jump, the bounds (68) hold again. Hence, using that F and f are non-increasing, we find that

$$(70) \quad \begin{aligned} S_k^{\leq v} f^j &\leq \sum_{u_{m-k}=1}^v \frac{1}{u_{m-k} + v} \frac{1}{1 - \hat{\beta}^2 \frac{\log(u_{m-k} + v)}{\log N}} f(u_{m-k})^j \mathbf{1}_{u_{m-k} + v \leq T} \\ &\leq \frac{1}{v} \frac{1}{1 - \hat{\beta}^2} \left(f(1)^j + \int_1^v f(x)^j dx \right), \end{aligned}$$

by comparison to an integral. By integrating by part and using that $f'(x) = -\frac{1}{x} \frac{1}{1 - \hat{\beta}^2 \frac{\log(x)}{\log N}}$, we see that for all $j \geq 1$,

$$\begin{aligned} f(1)^j + \int_1^v f(x)^j dx &= v f(v)^j - j \int_1^v x f'(x) f(x)^{j-1} dx \\ &\leq v f(v)^j + \frac{1}{1 - \hat{\beta}^2} \int_1^v f(x)^{j-1} dx. \end{aligned}$$

If we iterate the integration by part, we obtain that

$$f(1)^j + \int_1^v f(x)^j dx \leq v \sum_{i=0}^j \frac{j!}{(j-i)!} \left(\frac{1}{1 - \hat{\beta}^2} \right)^i f(v)^{j-i},$$

and so

$$(71) \quad S_k^{\leq v} f^j \leq \sum_{i=0}^j \frac{j!}{(j-i)!} \left(\frac{1}{1 - \hat{\beta}^2} \right)^{i+1} f(v)^{j-i}.$$

On the other hand, we have

$$\begin{aligned} S_k^{> v} f^j &\leq \sum_{u_{m-k}=v+1}^T \frac{1}{u_{m-k}} \frac{1}{1 - \hat{\beta}^2 \frac{\log(u_{m-k})}{\log N}} f(u_{m-k})^j \\ &\leq \int_v^T \frac{1}{x} \frac{1}{1 - \hat{\beta}^2 \frac{\log(x)}{\log N}} f(x)^j dx = \left[-\frac{1}{j+1} f(x)^{j+1} \right]_v^T \leq \frac{1}{j+1} f(v)^{j+1}. \end{aligned}$$

Combining the two previous estimates yields (67).

Case 3: $m - k$ is a small jump. We have that $f(\sum_{i=\varphi(m-k)}^{m-k} u_i) \leq f(u_{m-k})$. Moreover $\tilde{u}_{m-k} \geq \frac{u_{m-k-1}}{2}$ always holds. Hence, if we use the same decomposition

as in (69) with $v = u_{m-k-1}$, we find that

$$\begin{aligned} S_k^{\leq v} f^j &\leq \sum_{u_{m-k}=1}^v \frac{1}{u_{m-k} + v/2} \frac{1}{1 - \hat{\beta}^2 \frac{\log(u_{m-k} + v/2)}{\log N}} f(u_{m-k})^j \mathbf{1}_{u_{m-k} + v/2 \leq T} \\ &\leq \frac{2}{v} \frac{1}{1 - \hat{\beta}^2} \left(f(1)^j + \int_1^v f(x)^j dx \right) \\ &\leq 2 \sum_{i=0}^j \frac{j!}{(j-i)!} \left(\frac{1}{1 - \hat{\beta}^2} \right)^{i+1} f(v)^{j-i}, \end{aligned}$$

where we have used the integration by part from Case 2. Furthermore, we have $S_k^{>v} \leq \frac{1}{j+1} f(v)^{j+1}$ as in Case 2. Finally, since $m-k$ is bad we have $\varphi(m-k-1) = m-k-1$ and therefore (67) follows. \square

Proof of Lemma 3.9. We prove the lemma by induction on k . The case $k = 1$ follows from Lemma 3.11 with $j = k = 0$.

Assume now that (66) holds for some $k \in \llbracket 1, m-2 \rrbracket$. Then by (67) we obtain that the LHS of (66) for the index $k+1$ is smaller than the sum of all the entries of the following matrix, where we have set $\mu = 1 - \hat{\beta}^2$ and $f = f(v)$ with $v = \sum_{i=\varphi(m-k-1)}^{m-k-1} u_i$:

$$\begin{pmatrix} \frac{c_0^k}{(k+1)!} f^{k+1} & \frac{2\gamma_k^m c_0^k}{k! \mu} f^k & \frac{2\gamma_k^m c_0^k}{(k-1)! \mu^2} f^{k-1} & \dots & \frac{2\gamma_k^m c_0^k}{\mu^k} f & \frac{2\gamma_k^m c_0^k}{\mu^{k+1}} \\ 0 & \frac{c_1^k}{k! \mu} f^k & \frac{2\gamma_k^m c_1^k}{(k-1)! \mu^2} f^{k-1} & \dots & \frac{2\gamma_k^m c_1^k}{\mu^k} f & \frac{2\gamma_k^m c_1^k}{\mu^{k+1}} \\ 0 & 0 & \frac{c_2^k}{(k-1)! \mu^2} f^{k-1} & \dots & \frac{2\gamma_k^m c_2^k}{\mu^k} f & \frac{2\gamma_k^m c_2^k}{\mu^{k+1}} \\ \vdots & \ddots & & & & \\ 0 & 0 & 0 & \dots & \frac{c_k^k}{\mu^k} f & \frac{2\gamma_k^m c_k^k}{\mu^{k+1}} \end{pmatrix},$$

and summing over the columns gives (66) for $k+1$. \square

Recall (61). Lemma 3.9 yields the following.

Proposition 3.13. *There exists $C = C(\hat{\beta}) > 0$ and $\varepsilon_N = \varepsilon(N, \hat{\beta}) \rightarrow 0$ as $N \rightarrow \infty$, such that*

$$(72) \quad \tilde{A}_{m,N,\mathbf{I}} \leq C(1 + |\varepsilon_N|)^m \sum_{i=0}^{m-1} \frac{c_i^{m-1}}{(m-i)!} \times \left(\frac{\log N}{\hat{\beta}^2} \right)^{m-i} \frac{1}{(1 - \hat{\beta}^2)^i} \lambda_{T,N}^{2(m-i)},$$

where $\lambda_{T,N}$ is defined in (15).

Proof of Proposition 3.13. By Proposition 3.7 and Lemma 3.9 applied to $k = m-1$, we have:

$$\begin{aligned} \tilde{A}_{m,N} &\leq \frac{1}{1 - \hat{\beta}^2} (1 + \varepsilon_N)^m \sum_{u_1=1}^T \frac{C}{u_1} \sum_{i=0}^{m-1} \frac{c_i^{m-1}}{(m-1-i)!} f(u_1)^{m-1-i} \frac{1}{(1 - \hat{\beta}^2)^i} \\ &\leq \frac{C}{1 - \hat{\beta}^2} (1 + \varepsilon'_N)^m \sum_{i=0}^{m-1} \frac{c_i^{m-1}}{(m-i)!} f(1)^{m-i} \frac{1}{(1 - \hat{\beta}^2)^i}, \end{aligned}$$

where the second inequality comes from a comparison to an integral. This yields (72). \square

Lemma 3.14. *For all $\mathbf{I} \in \mathcal{D}(m, q)$, for all $k \leq m$:*

$$(73) \quad \forall i \leq k, \quad c_i^k \leq 3^i \prod_{r=1}^{k-1} (1 + \gamma_r^m).$$

Proof. We prove it by induction on k . The estimate holds for $k = 1$ since $c_0^1 = 1$ and $c_1^1 = 2$. Suppose that (73) holds for some $k \leq m - 1$. Then, for all $i \leq k + 1$,

$$c_i^{k+1} \leq c_i^k + 2\gamma_k^m \sum_{j=0}^{i-1} c_j^k \leq \prod_{r=1}^{k-1} (1 + \gamma_r^m) \left(3^i + 2\gamma_k^m \sum_{j=0}^{i-1} 3^j \right) \leq 3^i \prod_{r=1}^k (1 + \gamma_r^m).$$

□

3.6. Proof of Theorem 2.1. By Proposition 3.1, it is enough to show that

$$(74) \quad \sup_{X \in (\mathbb{Z}^2)^q} \Psi_{N,q}(X) \leq c e^{\lambda_{T,N}^2 \binom{q}{2} + o(q^2)},$$

for some $c = c(\hat{\beta})$ when $q \rightarrow \infty$ and $c = c(\hat{\beta}, q_0)$ when $q(N) = q_0$ is a constant. Using Proposition 3.7, we have

$$(75) \quad \sup_{X \in (\mathbb{Z}^2)^q} \Psi_{N,q}(X) \leq \sum_{m=0}^{\infty} \sigma_N^{2m} \left(\frac{1}{\pi} \right)^{m-1} \sum_{\mathbf{I} \in \mathcal{D}(m,q)} \tilde{A}_{m,N,\mathbf{I}},$$

where (72) gives an upper bound on the $\tilde{A}_{m,N,\mathbf{I}}$. Observe that by (73), we have

$$c_i^{m-1} \leq 3^i 2^{\sum_{i=1}^{m-1} \mathbf{1}_{i \text{ is bad}}} \leq 3^i 2^{2n(\mathbf{I}) + m/L + 1},$$

where $n(\mathbf{I})$ is the number of small jumps in \mathbf{I} . Indeed, an index i is bad if it is a small jump or a fresh index. The number of small jumps is $n(\mathbf{I})$. A fresh index is either a stopping index or an index adjacent to a small jump or m , so the number of fresh indices is at most $1 + n(\mathbf{I})$ plus the number of stopping indices. Since stopping indices are spaced at least L steps apart, there are at most m/L stopping indices. Hence there are at most $2n(\mathbf{I}) + m/L + 1$ bad indices. For a fixed $n \leq m$, let us compute the number of diagrams in $\mathcal{D}(m, q)$ such that $n(I) = n$. One has first to choose the location of the bad jumps, which gives $\binom{m}{n}$ possibilities. Now if m is a small jump ($m - \bar{k}_m \leq L + 2$), it means that at least one of the two particles $\{i_m, j_m\}$ is the same as one of the particles $\{i_{m-L+2}, j_{m-L+2}, \dots, i_{m-1}, j_{m-1}\}$, therefore there are at most $2Lq$ choices for the couple (i_m, j_m) . On the other hand, if $\{i_m, j_m\}$ is a long jump, there are at most $\binom{q}{2}$ possibilities. By repeating the argument, we finally find that the number of diagrams in $\mathcal{D}(m, q)$ such that $n(I) = n$ is less than $\binom{m}{n} (2Lq)^n \binom{q}{2}^{m-n}$.

Hence, by (75), Proposition 3.13, Lemma 3.14 and (35), there exists $\varepsilon_N \searrow 0$ such that

$$\begin{aligned} \sup_{X \in (\mathbb{Z}^2)^q} \Psi_{N,q}(X) &\leq C(\hat{\beta}) \times \\ &\sum_{m=0}^{\infty} (1 + \varepsilon_N)^m \sum_{n=0}^m \binom{m}{n} (2Lq)^n \binom{q}{2}^{m-n} \sum_{i=0}^m \frac{3^i 2^{2n+m/L+1}}{(m-i)!} \left(\frac{\hat{\beta}^2}{\log N} \right)^i \frac{\lambda_{T,N}^{2(m-i)}}{(1 - \hat{\beta}^2)^i}. \end{aligned}$$

The sum over n gives a factor of $(8Lq + \binom{q}{2})^m$. Exchanging the sum in i and m entails

$$\begin{aligned} \sup_{X \in (\mathbb{Z}^2)^q} \Psi_{N,q}(X) &\leq C \sum_{i=0}^{\infty} (1 + \varepsilon_N)^i 3^i 2^{i/L} \left(\frac{\hat{\beta}^2}{\log N} \right)^i \left(8Lq + \binom{q}{2} \right)^i \frac{1}{(1 - \hat{\beta}^2)^i} \times \\ &\quad \sum_{m=i}^{\infty} (1 + \varepsilon_N)^{m-i} \times \left(8Lq + \binom{q}{2} \right)^{m-i} \frac{2^{(m-i)/L}}{(m-i)!} \lambda_{T,N}^{2(m-i)}. \end{aligned}$$

So if we assume that

$$(76) \quad r = 3(1 + \varepsilon_N) \frac{2^{1/L}}{(1 - \hat{\beta}^2)} \left(\frac{\hat{\beta}^2}{\log N} \right) \left(8Lq + \binom{q}{2} \right) < 1,$$

we obtain the bound:

$$(77) \quad \sup_{X \in (\mathbb{Z}^2)^q} \Psi_{N,q}(X) \leq \frac{C}{1-r} e^{(1+\varepsilon_N)(8Lq + \binom{q}{2})2^{1/L} \lambda_{T,N}^2}.$$

If $q(N) = q_0 \in \mathbb{N}$ is constant, we can let $L = 3$. Then, the condition (76) is trivially satisfied since $r \rightarrow 0$ as $N \rightarrow \infty$ for any fixed $\hat{\beta} < 1$. Hence (74) holds in this case. (In fact, when q is constant, it is not necessary to introduce the distinction between good and bad indexes, as one can treat every index as a bad index in the induction (Lemma 3.9 and Lemma 3.11) and still arrive to (74) with the same following arguments.)

If $q = q(N) \rightarrow \infty$, then we can choose any $L \rightarrow \infty$ such that $L = o(q)$, so that together with (7), the estimate (74) holds. \square

4. DISCUSSION AND CONCLUDING REMARKS

We collect in this section several comments concerning the results of this paper.

- (1) Our results allow one already to obtain some estimates on the maximum of $Y_N(x) := \log W_N(\beta_N, x)$ over subsets $D \subset [0, 1]^2$. Specifically, let $\gamma > 0$ be given and define $Y_N^* = \sup_{x \in D} Y_N(x)$, where $|D| = N^{2\gamma}$. By Chebyshev's inequality we have that

$$\begin{aligned} \mathbb{P} \left(Y_N^* \geq \delta \sqrt{\log N} \right) &\leq 2N^{2\gamma} \mathbb{P} \left(Y_N(0) \geq \delta \sqrt{\log N} \right) \\ &\leq 2N^{2\gamma} \mathbb{E}[W_N^q] e^{-q\delta\sqrt{\log N}} \leq N^{2\gamma + \frac{q^2\lambda^2}{2\log N} - \frac{q\delta}{\sqrt{\log N}} + o(1)}, \end{aligned}$$

where we used (8) in the last inequality. The optimal q (disregarding the constraint in (7)) is $q/\sqrt{\log N} = \delta/\lambda^2$, and for that value the right side of the last display decays to 0 if $\delta^2 > 4\gamma\lambda^2$. The condition on q in (7) then gives the constraint that $\gamma < \frac{1}{6}\lambda^2 \frac{1-\hat{\beta}^2}{\hat{\beta}^2}$, which for $\hat{\beta}$ small reduces to $\gamma < 1/6$. Thus, our estimates only allow one to consider, for $\hat{\beta}$ small, the maximum over small subsets, if one shoots for the conjectured optimal estimate. (We note that one would hope for $\gamma = 1/2$, which would allow to consider the maximum over $x \in [0, 1]^2$.)

- (2) In view of the last sentence in Remark 1.2, it would be of interest to obtain a lower bound on $\mathbb{E}[W_N^q]$ that matches the upper bound, that is, $\mathbb{E}[W_N^q] \geq e^{\binom{2}{q}\lambda^2(1-\varepsilon_N)}$. This is the topic of work in progress that will be reported elsewhere.

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APPENDIX A. PROOF OF (18)

First note that $p_{2n}^* \leq p_{2n}(0)$ since, by the Cauchy-Schwarz inequality,

$$p_{2n}(x) = \sum_y p_n(x-y)p_n(y) \leq \sum_y p_n(y)^2 = p_{2n}(0).$$

Let $p_{2n}^{(d)}$ be the return probability of d -dimensional SRW to 0. A direct computation gives that $p_{2n}^{(2)} = (p_{2n}^{(1)})^2$ (see e.g. [19, Page 184]). We will show that $a_n = \sqrt{2n}p_{2n}^{(1)}$ is increasing. We have,

$$a_n = \sqrt{2n}2^{-2n} \binom{2n}{n}.$$

Hence,

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{1}{4} \sqrt{\frac{n+1}{n}} \frac{(2n+2)(2n+1)}{(n+1)^2} \\ &= \sqrt{1/(n(n+1))} (n + (n+1))/2. \end{aligned}$$

Since $(a+b)/2 \geq \sqrt{ab}$, we conclude (using $a = n$ and $b = n+1$) that $a_{n+1}/a_n \geq 1$.

Let $p_{2n+1}^{(1)}$ be the probability of the 1-dimensional SRW to come back to 1 in $2n+1$ steps. By the random walk representation [19, Remark in Pg. 185], we have that $p_{2n+1}^* \leq (p_{2n+1}^{(1)})^2$. A similar line of argument to the above shows that $b_n = \sqrt{2n+1}p_{2n+1}^{(1)}$ is increasing in n . Indeed,

$$\begin{aligned} \frac{b_{n+1}}{b_n} &= \frac{1}{4} \frac{\sqrt{2n+3} (2n+3)(2n+2)}{\sqrt{2n+1} (n+2)(n+1)} \\ &= \frac{2n+3}{2\sqrt{(n+1)(n+2)}} \frac{\sqrt{(2n+3)(n+1)}}{\sqrt{(2n+1)(n+2)}}, \end{aligned}$$

where the first fraction is bigger than 1 by the formula $(a+b)/2 \geq \sqrt{ab}$, as well as the second fraction by expanding the products.

Now, we know from the local limit theorem that a_n and b_n converge to $2/\sqrt{2\pi}$, thus they are always smaller than this limit. This leads to (18). \square

APPENDIX B. IMPROVED ESTIMATES ON U_N

When n is taken large enough, the estimate (56) can be improved as follows.

Proposition B.1. *There exists $\varepsilon_n = \varepsilon(n, \hat{\beta}) \rightarrow 0$ such that as $n \rightarrow \infty$ with $n \leq N$,*

$$(78) \quad U_N(n) = (1 + \varepsilon_n) \frac{\hat{\beta}^2}{\left(1 - \hat{\beta}^2 \frac{\log n}{\log N}\right)^2} \frac{1}{n \log N}.$$

Proof. Since $(S_n^1 - S_n^2) \stackrel{(d)}{=} (S_{2n})$, we can write

$$(79) \quad U_N(n+1) = \sigma_N^2 \mathbf{E}_0 \left[e^{\beta_N^2 \sum_{i=1}^n \mathbf{1}_{S_{2i}=0}} \mathbf{1}_{S_{2n}^1=0} \right].$$

Consider $\ell = \ell_n = n^{1-\varepsilon_n}$ with $\varepsilon_n = \frac{1}{\log \log n}$, so that $\ell_n = o(n)$ and $\varepsilon_n \rightarrow 0$.

First step: As $n \rightarrow \infty$ with $n \leq N$,

$$(80) \quad \mathbf{E}_0 \left[e^{\beta_N^2 \sum_{i=1}^n \mathbf{1}_{S_{2i}=0}} \mathbf{1}_{S_{2n}=0} \right] \sim \mathbf{E}_0 \left[e^{\beta_N^2 (\sum_{i=1}^{\ell} \mathbf{1}_{S_{2i}=0} + \sum_{i=n-\ell}^n \mathbf{1}_{S_{2i}=0})} \mathbf{1}_{S_{2n}=0} \right].$$

We compute the norm of the difference which, using that $|e^{-x} - 1| \leq |x|$ for $x \geq 0$, is less than

$$\begin{aligned} & \mathbf{E}_0 \left[e^{\beta_N^2 \sum_{i=1}^n \mathbf{1}_{S_{2i}=0}} \mathbf{1}_{S_{2n}=0} \times \beta_N^2 \sum_{j=\ell}^{n-\ell} \mathbf{1}_{S_{2j}=0} \right] \\ &= \beta_N^2 \sum_{j=\ell}^{n-\ell} \mathbf{E}_0 \left[e^{\beta_N^2 \sum_{i=1}^j \mathbf{1}_{S_{2i}=0}} \mathbf{1}_{S_{2j}=0} \right] \mathbf{E}_0 \left[e^{\beta_N^2 \sum_{i=1}^{n-j} \mathbf{1}_{S_{2i}=0}} \mathbf{1}_{S_{2(n-j)}=0} \right]. \end{aligned}$$

where we have used Markov's property in the second line. By (79) and (56), the last sum is smaller than

$$C \beta_N^2 \sum_{j=\ell}^{n-\ell} \frac{1}{j} \frac{1}{n-j} \leq 2C \beta_N^2 \sum_{j=\ell}^{n/2} \frac{1}{j} \frac{1}{n/2} \leq \frac{1}{n} C' \beta_N^2 \log \left(\frac{n}{\ell_n} \right) \leq \frac{1}{n} C'' \varepsilon_n = o(n^{-1}).$$

Since the left hand side of (80) is bigger than cn^{-1} for some constant $c > 0$, this shows (80).

Second step: As $n \rightarrow \infty$ with $n \leq N$,

$$(81) \quad \begin{aligned} & \mathbf{E}_0 \left[e^{\beta_N^2 (\sum_{i=1}^{\ell} \mathbf{1}_{S_{2i}=0} + \sum_{i=n-\ell}^n \mathbf{1}_{S_{2i}=0})} \mathbf{1}_{S_{2n}=0} \right] \\ & \sim \mathbf{E}_0 \left[e^{\beta_N^2 \sum_{i=1}^{\ell} \mathbf{1}_{S_{2i}=0}} \right] \mathbf{E}_0 \left[e^{\beta_N^2 \sum_{i=n-\ell}^n \mathbf{1}_{S_{2i}=0}} \mathbf{1}_{S_{2n}=0} \right]. \end{aligned}$$

By Markov's property, we can write the LHS of (81) as

$$\begin{aligned} & \sum_{x \in \mathbb{Z}^2} \mathbf{E}_0 \left[e^{\beta_N^2 \sum_{i=1}^{\ell} \mathbf{1}_{S_{2i}=0}} \mathbf{1}_{S_{2\ell}=x} \mathbf{E}_x \left[e^{\beta_N^2 \sum_{i=n-2\ell}^{n-\ell} \mathbf{1}_{S_{2i}=0}} \mathbf{1}_{S_{2n-\ell}=0} \right] \right] \\ &= \sum_{x \in \mathbb{Z}^2} \mathbf{E}_0 \left[e^{\beta_N^2 \sum_{i=1}^{\ell} \mathbf{1}_{S_{2i}=0}} \mathbf{1}_{S_{2\ell}=x} \mathbf{E}_0 \left[e^{\beta_N^2 \sum_{i=1}^{\ell} \mathbf{1}_{S_{2i}=0}} \mathbf{1}_{S_{2n-\ell}=x} \right] \right]. \end{aligned}$$

Therefore the difference in (81) writes $\sum_{x \in \mathbb{Z}^2} \Delta_x$ with

$$\Delta_x := \mathbf{E}_0 \left[e^{\beta_N^2 \sum_{i=1}^{\ell} \mathbf{1}_{S_{2i}=0}} \mathbf{1}_{S_{2\ell}=x} \mathbf{E}_0 \left[e^{\beta_N^2 \sum_{i=1}^{\ell} \mathbf{1}_{S_{2i}=0}} (\mathbf{1}_{S_{2n-\ell}=0} - \mathbf{1}_{S_{2n-\ell}=x}) \right] \right].$$

Since $\mathbf{E}_0 \left[e^{\beta_N^2 \sum_{i=1}^{\ell} \mathbf{1}_{S_{2i}=0}} \right] \leq C(\hat{\beta})$ by (52), we have

$$\sum_{|x| > \sqrt{\ell n^{\varepsilon/4}}} |\Delta_x| \leq C \sum_{|x| > \sqrt{\ell n^{\varepsilon/4}}} \mathbf{E}_0 \left[e^{\beta_N^2 \sum_{i=1}^{\ell} \mathbf{1}_{S_{2i}=0}} \mathbf{1}_{S_{2\ell}=x} \right].$$

By Hölder's inequality with $p^{-1} + q^{-1} = 1$, and p small enough so that $\sqrt{p} \hat{\beta} < 1$,

$$\mathbf{E}_0 \left[e^{\beta_N^2 \sum_{i=1}^{\ell} \mathbf{1}_{S_{2i}=0}} \mathbf{1}_{S_{2\ell}=x} \right] \leq \mathbf{E}_0 \left[e^{p \beta_N^2 \sum_{i=1}^{\ell} \mathbf{1}_{S_{2i}=0}} \right]^{\frac{1}{p}} p_{2\ell}(x)^{\frac{1}{q}} \leq C(\hat{\beta}) \ell_n^{-1} e^{-\frac{1}{2q} \frac{|x|^2}{\ell_n}},$$

for n large enough. Therefore,

$$\sum_{|x| > \sqrt{\ell} n^{\varepsilon/4}} |\Delta_x| \leq C \sum_{|x| > \sqrt{\ell} n^{\varepsilon/4}} \ell_n^{-1} e^{-\frac{1}{2q} \frac{|x|^2}{\ell_n}} \leq C e^{-\frac{1}{2q} n^{\varepsilon/2}} = o(n^{-1}).$$

We now estimate the sum on Δ_x for $|x| \leq \sqrt{\ell} n^{\varepsilon/4}$. We start by bounding the expectation inside the definition of Δ_x :

$$(82) \quad \begin{aligned} & \mathbb{E}_0 \left[e^{\beta_N^2 \sum_{i=1}^{\ell} \mathbf{1}_{S_{2i}=0}} (\mathbf{1}_{S_{2n-\ell}=0} - \mathbf{1}_{S_{2n-\ell}=x}) \right] \\ &= \sum_{y \in \mathbb{Z}^2} \mathbb{E}_0 \left[e^{\beta_N^2 \sum_{i=1}^{\ell} \mathbf{1}_{S_{2i}=0}} \mathbf{1}_{S_{\ell}=y} \right] (p_{2n-2\ell}(y) - p_{2n-2\ell}(y-x)). \end{aligned}$$

By the same argument as above, we can prove that the above sum restricted to $|y| \geq \sqrt{\ell} n^{\varepsilon/4}$ is negligible with respect to n^{-1} , uniformly for $|x| \leq \sqrt{\ell} n^{\varepsilon/4}$. On the other hand, by the local limit theorem we have

$$\sup_{|x| \leq \sqrt{\ell} n^{\varepsilon/4}, |y| \leq \sqrt{\ell} n^{\varepsilon/4}} |p_{2n-2\ell}(y) - p_{2n-2\ell}(y-x)| = o(n^{-1}).$$

since $\ell_n n^{\varepsilon/2} = n^{1-\varepsilon_n/2} = o(n)$. Thus, the quantity in (82) is bounded uniformly for $|x| \leq \sqrt{\ell} n^{\varepsilon/4}$ by

$$\mathbb{E}_0 \left[e^{\beta_N^2 \sum_{i=1}^{\ell} \mathbf{1}_{S_{2i}=0}} \right] \times o(n^{-1}) = o(n^{-1}).$$

This completes the proof of (81).

Third step: As $n \rightarrow \infty$ with $n \leq N$,

$$(83) \quad \mathbb{E}_0 \left[e^{\beta_N^2 \sum_{i=n-\ell}^n \mathbf{1}_{S_{2i}=0}} \mathbf{1}_{S_{2n}=0} \right] \sim \mathbb{E}_0 \left[e^{\beta_N^2 \sum_{i=1}^{\ell} \mathbf{1}_{S_{2i}=0}} \right] p_{2n}(0).$$

Equivalence (83) can be proven by following the same line of arguments as used to prove (81), hence we omit its proof.

Now, combining the three steps leads to the equivalence

$$\mathbb{E}_0 \left[e^{\beta_N^2 \sum_{i=1}^n \mathbf{1}_{S_{2i}=0}} \mathbf{1}_{S_{2n}=0} \right] \sim \mathbb{E}_0 \left[e^{\beta_N^2 \sum_{i=1}^{\ell} \mathbf{1}_{S_{2i}=0}} \right]^2 p_{2n}(0).$$

By (53), as $\log \ell \sim \log n$, we have

$$\mathbb{E}_0 \left[e^{\beta_N^2 \sum_{i=1}^{\ell} \mathbf{1}_{S_{2i}=0}} \right] \sim \frac{1}{1 - \hat{\beta}^2 \frac{\log n}{\log N}},$$

and so (78) follows from (79) and the last two displays. \square

APPENDIX C. KHAS'MINSKII'S LEMMA FOR DISCRETE MARKOV CHAINS

The following theorem is another discrete analogue of Khas'minskii's lemma, compare with Lemma 2.2.

Theorem C.1. *Let $(Y_n)_n$ be any markov chain on a discrete state-space E and let $f : E \rightarrow \mathbb{R}_+$. Then for all $k \in \mathbb{N}$, if*

$$(84) \quad \eta_0 := \sup_{x \in E} \mathbb{E}_x \left[\sum_{n=1}^k (e^{f(Y_n)} - 1) \right] < 1,$$

one has

$$(85) \quad \sup_{x \in E} \mathbb{E}_x \left[e^{\sum_{n=1}^k f(Y_n)} \right] \leq \frac{1}{1 - \eta_0}.$$

Proof. Denote by $D_n = e^{f(Y_n)} - 1$. We have,

$$\begin{aligned}
\mathbb{E}_x \left[e^{\sum_{n=1}^N f(Y_n)} \right] &= \mathbb{E}_x \left[\prod_{n=1}^N (1 + D_n) \right] = \sum_{p=0}^{\infty} \sum_{1 \leq n_1 < \dots < n_p \leq k} \mathbb{E}_x \left[\prod_{i=1}^p D_{n_i} \right] \\
&= \sum_{p=0}^{\infty} \sum_{1 \leq n_1 < \dots < n_{p-1} \leq k} \mathbb{E}_x \left[\prod_{i=1}^{p-1} D_{n_i} \mathbb{E}_{Y_{n_{p-1}}} \left[\sum_{n=1}^{k-n_{p-1}} D_n \right] \right] \\
&\stackrel{(84)}{\leq} \sum_{p=0}^{\infty} \eta_0 \sum_{1 \leq n_1 < \dots < n_{p-1} \leq k} \mathbb{E}_x \left[\prod_{i=1}^{p-1} D_{n_i} \right] \leq \dots \leq \sum_{p=0}^{\infty} \eta_0^p = \frac{1}{1 - \eta_0}.
\end{aligned}$$

□

Corollary C.2. *Let $(Y_n)_n$ be any markov chain on a discrete state-space E and let $f : E \rightarrow [0, 1]$. Then for all $k \in \mathbb{N}$, if*

$$(86) \quad \eta_1 := \sup_{x \in E} \mathbb{E}_x \left[\sum_{n=1}^k f(Y_n) \right] < 1,$$

one has

$$(87) \quad \sup_{x \in E} \mathbb{E}_x \left[e^{\sum_{n=1}^k f(Y_n)} \right] \leq \frac{1}{1 - \eta_1}.$$

Proof. Simply observe that $e^{f(x)} - 1 \leq e^c f(x)$ and apply Theorem C.1. □

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