



**HAL**  
open science

# Long-run cycles in a growth model with natural externalities

Stefano Bosi, Thai Ha-Huy

► **To cite this version:**

Stefano Bosi, Thai Ha-Huy. Long-run cycles in a growth model with natural externalities. 2023. hal-03968528v3

**HAL Id: hal-03968528**

**<https://hal.science/hal-03968528v3>**

Preprint submitted on 19 Jun 2023 (v3), last revised 9 Aug 2023 (v4)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Copyright

# Long-run cycles in a growth model with natural externalities\*

Stefano BOSI, Thai HA-HUY

June 19, 2023

## Abstract

We develop an AK model augmented in two directions. First, by modeling nature as a productive externality, and, second, by considering a process of natural capital accumulation where a reproduction force is moderated by the negative impact of productive pollution. Differently from the existing literature, more focused on the planner's solution, we study here a market economy and the effects of its imperfections.

We introduce two alternative natural accumulation processes: a power law, more usual in economics, and a logistic law, more popular in ecology. In both cases, we observe the occurrence of cycles of period two and of period a power of two, and chaos through a flip bifurcation and period-doubling bifurcations. In the case of the power law, dynamics are even richer because of the occurrence of limit cycles through a Neimark-Sacker bifurcation. Worth to note, in this case, cycles occur under a larger Total Factor Productivity, which amplifies the negative impact of pollution on natural regeneration. In the case of a logistic law, cycles take place when the saturation effect of nature has a larger negative impact on the reproduction process.

**Keywords:** AK model, reproduction law, pollution, two-period and limit cycles.

**JEL codes:** C61, E32, 044.

## 1 Introduction

Nature is an economic good. As such, nature can be viewed either as a production factor or as a consumption good. In fact, nature contributes to production in terms of renewable and non-renewable resources, it provides with ecosystemic services; going beyond the various material uses we make of it, experiencing nature also gives us pleasure; and, finally, the health of nature determines and

---

\*The authors gratefully thank Carmen Camacho and David Desmarchelier for his worth remarks and suggestions, and acknowledge the financial support of the LABEX MME-DII (ANR-11-LBX-0023-01).

drives our own health from a one-health perspective. Nature has been introduced in economic models and, in particular, in dynamic general equilibrium theory. There are different straight ways to give value to nature, the most direct is to recognize its role in the production function or in the utility function and, depending on the decision maker (producer, consumer, social planner), to consider the stock of natural capital either as an externality or a choice. Besides all its different roles, any model aiming at portraying nature needs to describe the feedback from the economy. Simply put, exploiting nature and/or polluting, reduces the stock of nature. However, it is also true that when the natural resource under study is renewable, its stock can grow innately by itself, or due to artificial maintenance. Because of all these possible roles and feedbacks, and because of the urgency of current environmental issues, the number of economic models involving nature is large and intended to grow.

The present paper aims at shedding light on some basic mechanisms linking economic activity and nature, while avoiding unnecessary complexity in as much as possible. We focus on the productive effects of nature and the negative impact of productive pollution on the reproduction process of nature, viewed as a renewable resource. We build a market economy model à la Ramsey-Cass-Koopmans in discrete time (RCK hereafter), augmented with a law of nature accumulation, which incorporates both the natural regeneration and the pollution process coming from production activities. Then, we consider two distinct natural accumulation processes: a power law, more standard in economics, and a logistic law, more usual in biology, demography and ecology, discovered by Verhulst in 1838. Recall that the basic RCK model is two-dimensional, so that nature accumulation adds a third dimension and generates more complex trajectories. In our paper, we study these nonlinear dynamics with a focus on the occurrence of long-run cycles.

The interaction between natural resources and economic activity has been modelled by theorists since the early 70s. Broadly speaking, most papers see nature as a production factor or a good in the utility function, and show how economic activities (consumption or production) generate pollution, which in turn harms natural resources. In a seminal paper, Beltratti et al. (1994) study a RCK model where natural capital is both a factor in the production function and a good in the utility function. In their paper, pollution generated by consumption activities reduces the stock of renewable resources. The authors prove the existence of a unique and stable steady state (saddle point). Ayong Le Kama (2001) finds the same results when pollution comes instead from production activities. In Wirl (2004), pollution affects the utility function and the reproduction of the natural resource is represented by a logistic function. He proves the existence of two steady states and the occurrence of a limit cycle through a Hopf bifurcation around the lower steady state. The existence of endogenous cycles entails intergenerational inequalities: some generations experience a higher level of natural resource while others, a lower one. Interestingly, Beltratti et al. (1994), Ayong Le Kama (2001) and Wirl (2004) focus on the central planner's solution which internalizes the pollution externalities.

Our paper differs from Beltratti et al. (1994), Ayong Le Kama (2001), Wirl

(2004) in many respects. Not only we consider a market economy where nature and pollution are externalities, that is market imperfections, but also we represent the regeneration mechanism of nature using a more general logistic law and an alternative process, a power law. Finally, our discrete-time time approach gives rise to richer dynamics through period-doubling and, even, chaotic bifurcations. We observe the occurrence of cycles of period two under both the reproduction laws, cycles of period a power of two, and chaos through a flip bifurcation and period-doubling bifurcations. Moreover, in the case of a power law, dynamics are even richer because of the possibility of limit cycles through a Neimark-Sacker bifurcation, which is the discrete-time equivalent of the Hopf bifurcation obtained in continuous time by Wirl (2004) and Bosi and Desmarchelier (2018). However, differently from them, we deal here with a unique steady state.

Considering instead a market economy within an overlapping generations model, John and Pecchenino (1994) highlights the potential conflict between economic growth and the environmental maintenance. They introduce a simple evolution rule for environmental quality with constant depreciation rate, negative impact of consumption level and positive impact of maintenance. Many are also the differences between their model, more focused on Pareto-optimality and policy issues, and ours: not only the OLG structure, but, most importantly, the fact that environmental quality affects utility instead of production, and that pollution is consumption-driven. Because of these differences, they obtain multiple steady states and very different dynamics around. Also in an OLG setting, Seegmuller and Verchère (2007) model an economy in which pollution both depends on the stock of capital and it lowers utility. They prove that deterministic cycles and equilibrium indeterminacy can arise when the emission rate of pollution is sufficiently low.

Xepapadeas (2005) presents different continuous-time growth models. In the main one, pollution is a by-product of production and a negative externality in the utility function. He finds a unique and stable steady state, and raises some policy issues such as the optimal emissions taxes.

Antoci et al. (2011) consider an economic growth model with environmental externalities. Their model is the closest to ours, but differently from us, they introduce a production function strictly concave in capital and obtain multiple steady states, while our AK formulation leads to a unique steady state. We study alternative forms of natural reproduction law, namely, a more general logistic function, with richer dynamics. They perform a global analysis in continuous time, complemented by simulations to investigate the stability (supercriticality) of a limit cycle, and prove the existence of global indeterminacy, while our local analysis highlights equilibrium uniqueness. Last but not least, our discrete-time approach lets us prove that cycles can emerge, through flip and period-doubling bifurcations. These results are missed in the continuous-time framework of Antoci et al. (2011).<sup>1</sup>

---

<sup>1</sup>For an excellent survey of the main growth models with environmental variables (pollution and natural capital), the reader is referred to Caravaggio and Sodini (2018). These authors

Worth to note, cycles occur under a larger TFP in the case of a power law because the TFP amplifies the negative impact of pollution on natural regeneration. In the case of a logistic law, cycles take place when the saturation effect of nature has a larger negative impact on the reproduction process.

The rest of the paper is organized as follows. Section 2 introduces the fundamentals of a market economy. In Sections 3 and 4, we study the equilibrium when the natural accumulation process is a power law or a logistic function respectively. Section 5 concludes. All proofs are gathered in the Appendix.

## 2 Fundamentals

In this section, we introduce the market economy and its fundamentals under the assumption of Nature In the Production Function (NIPF). Then, we compare: (1) the competitive equilibrium with the planner's solution and (2) the NIPF approach with the alternative model of Nature In the Utility Function (NIUF). Finally, we focus on the particular case of full capital depreciation to provide an explicit equilibrium solution, necessary at the end of the article to simulate and represent cycles in discrete time.

### 2.1 Market economy

Nature  $N_t$  is a positive externality which increases the productivity  $AN_t^\alpha$  of the capital stock  $K_t$ , with  $0 < \alpha < 1$ . The aggregate production is given by:

$$F(K_t, N_t) = AN_t^\alpha K_t = AN_t^\alpha \sum_j k_{jt} \quad (1)$$

where  $k_{jt}$  denotes the capital in the firm  $j$ . The production function is linear in  $K_t$  and, in this sense, our framework can be viewed as a further version of the AK model.

Producers are small price-takers. The individual firm maximizes its profit:  $\max_{k_{jt}} (AN_t^\alpha k_{jt} - r_t k_{jt})$  where  $r_t$  denotes the rental price of capital. At equilibrium, as in the AK model, a zero-profit condition holds:

$$r_t = AN_t^\alpha \quad (2)$$

Because of this linearity, the solution to individual profit maximization coincides with that of an aggregate firm.

The representative consumer maximizes an intertemporal logarithmic utility

$$\sum_{t=0}^{\infty} \beta^t \ln c_t$$

under the budget constraint:

$$c_t + k_{t+1} - (1 - \delta) k_t \leq r_t k_t \quad (3)$$

---

explore the variety of nonlinear dynamics (stability and indeterminacy issues) either through local or global analysis.

where  $\delta$  represents the capital depreciation rate. Notice that, here,  $k_t$  refers to consumer's capital supply, while in formula (1),  $K_t \equiv \sum_j k_{jt}$  represents the firms' aggregate demand.

Maximizing the Lagrangian function

$$\sum_{t=0}^{\infty} \beta^t \ln c_t + \sum_{t=0}^{\infty} \lambda_t [r_t k_t - c_t - k_{t+1} + (1 - \delta) k_t]$$

where  $(\lambda_t)_{t=0}^{\infty}$  is a sequence of non-negative Lagrangian multiplier, we obtain the first-order conditions  $\lambda_t / \lambda_{t+1} = (1 - \delta + r_{t+1})$ , with  $\lambda_t = \beta^t / c_t$ , and the budget constraint (3), now binding. The second-order conditions are also satisfied because of the strict concavity of the utility function  $\ln c_t$ . The transversality condition  $\lim_{t \rightarrow \infty} \lambda_t k_{t+1} = \lim_{t \rightarrow \infty} \beta^t k_{t+1} / c_t = 0$  holds when the steady state is stable or unstable with a stable cycle around because the sequences  $(c_t)_{t=0}^{\infty}$  and  $(k_t)_{t=0}^{\infty}$  are uniformly bounded.

Thus, utility maximization entails the intertemporal consumption smoothing:

$$\frac{c_{t+1}}{c_t} = \beta (1 - \delta + r_{t+1}) \quad (4)$$

The size of population is normalized to one. The aggregate capital coincides with the aggregate one:  $k_t = K_t$ .

Putting together (2), (3), (4), we obtain the two-dimensional RCK block of the dynamic system.

$$\frac{c_{t+1}}{c_t} = \beta (1 - \delta + AN_{t+1}^{\alpha}) \quad (5)$$

$$c_t + k_{t+1} = (1 - \delta + AN_t^{\alpha}) k_t \quad (6)$$

The law of nature accumulation adds a third dimension and complete this dynamic system.

We plausibly conceive an accumulation process driven by two forces: a reproduction mechanism *stricto sensu*, say  $\Phi$ , which depends on the state of nature, and a pollution effect, say  $\Pi_t$ , which always dampens natural accumulation:

$$N_{t+1} - N_t = \Phi(N_t) - \Pi_t \quad (7)$$

The pollution effect depends on human activities, for instance: (1) on production ( $\Pi_t = bF(K_t, N_t)$ ) or, alternatively, (2) on consumption ( $\Pi_t = bc_t$ ). At the steady state, equations (5) and (6) lead to the modified golden rule and, in both the cases, according to (7), we get  $\Pi = \Phi(N)$ , but the mechanisms at work are quite different.<sup>2</sup>

---

<sup>2</sup>More explicitly, equation (5), that is  $\beta(1 - \delta + AN^{\alpha}) = 1$ , determines the same natural level  $N$ , while (6) pegs the consumption-capital ratio  $c/k = 1/\beta - 1$ . However, in case (1),  $k$  is given by  $\Pi = \Phi(N)$ , that is by  $k = \Phi(N) / (bAN^{\alpha})$  and  $c$  by  $c = k(1 - \beta) / \beta$ ; conversely, in case (2),  $c$  is given by  $\Pi = \Phi(N)$ , that is by  $c = \Phi(N) / b$ , and  $k$  by  $k = c\beta / (1 - \beta)$ . Transitional dynamics and, namely, conditions for cycles are also different.

In the following, we consider laws of natural reproduction where pollution comes from production. More explicitly, we study two alternative rules of nature accumulation: (1) power law; (2) generalized logistic.

(1) Power law:

$$N_{t+1} = aN_t^\varepsilon - bAN_t^\alpha K_t \quad (8)$$

with  $a, b > 0$  and  $0 < \varepsilon < 1$ , where  $a$  is the regeneration rate and  $b$  is the pollution rate.

In a world with no humans,  $K_t = 0$  and  $N_{t+1} = aN_t^\varepsilon$ . The natural dynamics

$$N_t = a^{\frac{1-\varepsilon^t}{1-\varepsilon}} N_0^{\varepsilon^t}$$

converge to the steady state in the long run:

$$\lim_{t \rightarrow \infty} \left( a^{\frac{1-\varepsilon^t}{1-\varepsilon}} N_0^{\varepsilon^t} \right) = a^{\frac{1}{1-\varepsilon}}$$

For instance, if  $a = \varepsilon = 1/2$  and  $N_0 = 1/16$ , dynamics are represented in Figure 1 with  $N_\infty = 1/4$ .

(2) Generalized logistic law:

$$N_{t+1} - N_t = aN_t^\varepsilon (\bar{N} - N_t) - bAN_t^\alpha K_t \quad (9)$$

with  $0 < a < 1$ ,  $0 \leq \varepsilon \leq 1$  and  $b, \bar{N} > 0$ .

In a world with no humans,  $K_t = 0$  and  $N_{t+1} - N_t = aN_t^\varepsilon (\bar{N} - N_t)$ .

When  $\varepsilon = 0$ , we obtain a linear law  $N_{t+1} - N_t = a(\bar{N} - N_t)$  and the natural dynamics

$$N_t = (1 - a)^t N_0 + [1 - (1 - a)^t] \bar{N}$$

converge to the steady state  $\lim_{t \rightarrow \infty} N_t = \bar{N}$  in the long run. For instance, if  $a = 1/2$  and  $N_0 = 1/16$  and  $\bar{N} = 1/4$ , dynamics are represented in Figure 2 with  $N_\infty = 1/4$ .

When  $\varepsilon = 1$ , we get the pure logistic law, often considered in biology to represent population dynamics.

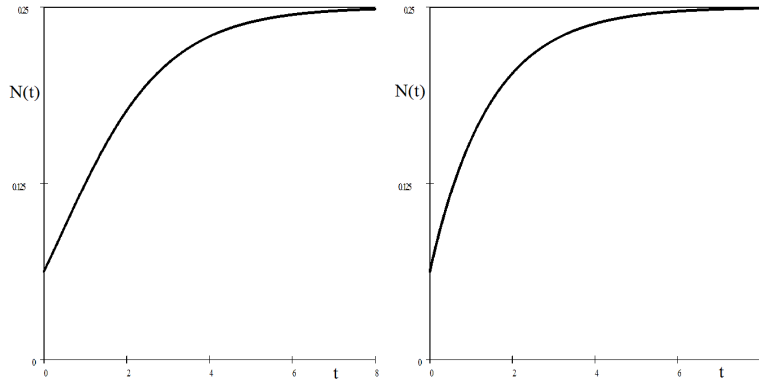


Fig. 1 Power law

Fig. 2 Linear law

It is interesting to compare processes (8) and (9).

We observe that (8) is equivalent to  $N_{t+1} - N_t = aN_t^\varepsilon - N_t - bAN_t^\alpha K_t$ . Thus, the two processes write

$$N_{t+1} - N_t = \Phi_i(N_t) - bAN_t^\alpha K_t \quad (10)$$

with  $i = P, L$ , where

$$\begin{aligned} \Phi_P(N_t) &\equiv aN_t^\varepsilon - N_t \\ \Phi_L(N_t) &\equiv aN_t^\varepsilon (\bar{N} - N_t) \end{aligned}$$

are both concave functions since

$$\begin{aligned} \Phi_P''(N_t) &= a\varepsilon(\varepsilon - 1)N_t^{\varepsilon-2} < 0 \\ \Phi_L''(N_t) &= a\varepsilon(\varepsilon - 1)N_t^{\varepsilon-2}(\bar{N} - N_t) - 2a\varepsilon N_t^{\varepsilon-1} < 0 \end{aligned}$$

with  $\Phi_P(0) = \Phi_L(0) = 0$ ,  $\Phi_P'(0) = \Phi_L'(0) = +\infty$  and

$$\Phi_P\left(a^{\frac{1}{1-\varepsilon}}\right) = \Phi_L(\bar{N}) = 0$$

Thus, they look similar. However, as we will see, the power law is richer in terms of nonlinear dynamics because of the possibility of limit cycles.

## 2.2 Market inefficiency

The planner internalizes the positive externalities of nature and maximizes the welfare, that is the representative agent's intertemporal utility function:

$$\max \sum_{t=0}^{\infty} \beta^t \ln c_t \quad (11)$$

under a sequence of resource constraints instead of budget constraints:

$$c_t + k_{t+1} - (1 - \delta)k_t \leq F(k_t, N_t) = AN_t^\alpha k_t \quad (12)$$

As above, the population of consumers is normalized to one and  $k_t = K_t$ . The planner faces also a sequence of natural resource constraints:

$$N_{t+1} - N_t \leq \Phi_i(N_t) - bAN_t^\alpha k_t \quad (13)$$

where the reproduction law depends on the model:  $i = P, L$ .

This program is far from being trivial, because of the convex term  $-bAN_t^\alpha k_t$  in the law of natural reproduction appearing in the Lagrangian function. In the following proposition, we prove the existence of an optimal solution using a boundedness argument.



**Proposition 1** *An optimal solution to the planner's program (11) under the sequences of constraints (12) and (13) exists.*

*This solution satisfies the first-order conditions*

$$\frac{c_{t+1}}{c_t} = \beta (1 - \delta + AN_{t+1}^\alpha) - b\beta AN_{t+1}^\alpha \frac{\mu_{t+1}}{\lambda_{t+1}} \quad (14)$$

*jointly with the sequences of constraints (12) and (13), now binding.  $\lambda_t$  and  $\mu_t$  are the positive multipliers associated to constraints (12) and (13) respectively.*

We observe that the optimal sequence  $(k_t, c_t, N_t)^*$  satisfies necessarily the first-order conditions: (12) and (13) with equality, and (14), but other inefficient sequences also satisfying them could exist.

Let us focus on the Euler equation and compare equations (5) and (62), that is, respectively, the market solution  $M$  and the planner's one  $P$ :

$$\begin{aligned} \frac{c_{t+1}^M}{c_t^M} &= \beta (1 - \delta + AN_{t+1}^{M\alpha}) \\ \frac{c_{t+1}^P}{c_t^P} &= \beta (1 - \delta + AN_{t+1}^{P\alpha}) - b\beta AN_{t+1}^{P\alpha} \frac{\mu_{t+1}^P}{\lambda_{t+1}^P} \end{aligned}$$

In the long run (steady state),  $c_t = c_{t+1}$  and, therefore,

$$\beta (1 - \delta + AN^{P\alpha}) > 1 = \beta (1 - \delta + AN^{M\alpha})$$

since the ratio  $\mu_t^P/\lambda_t^P$  is constant and positive over time, or, equivalently,  $N^P > N^M$ : the market does not internalize the positive externalities of nature while the planner does: this leads to a lower level of natural resources in a market economy and to an inefficient allocation (underprovision of public good).

### 2.3 An alternative approach: Nature In the Utility Function

In our model, nature externalities enter the production function and increase the productivity of capital. Alternatively, they can enter the utility function and increase or decrease the marginal utility of consumption (complementarity vs substitutability effects).

Consider an AK model with  $F(K_t) = AK_t$  as a production function and  $N_{t+1} - N_t \leq \Phi_i(N_t) - bAK_t$  as a reproduction function, where  $i = P, L$ , depending on the model we adopt. As above, a zero-profit condition holds at equilibrium:  $r_t = A$ . Nature enters the utility function and consumer maximizes the intertemporal utility  $\sum_{t=0}^{\infty} \beta^t u(c_t, N_t)$  under a sequence of budget constraints  $c_t + k_{t+1} - (1 - \delta)k_t \leq r_t k_t$ . Utility is increasing in consumption and natural externalities:  $u_c(c_t, N_t) > 0$  and  $u_N(c_t, N_t) > 0$ .

Solving the model, we obtain the dynamic general equilibrium system:<sup>3</sup>

$$\frac{u_c(c_t, N_t)}{u_c(c_{t+1}, N_{t+1})} = \beta(1 - \delta + A) \quad (15)$$

$$c_t + k_{t+1} = (1 - \delta + A)k_t \quad (16)$$

$$N_{t+1} - N_t = \Phi_i(N_t) - bAk_t \quad (17)$$

Comparing the models NIPF (Nature In the Production Function) and NIUF (Nature In the Utility Function), we observe the following.

(1) The TFP  $AN_t^\alpha$  depends on nature externalities in equations (5), (6) and (10), and it becomes constant in equations (15), (16) and (17). In other terms, the new TFP is a particular case of the previous one with  $\alpha = 0$ , and the externality channel of the former model stops functioning now.

(2) A new mechanism takes place instead in the Euler equation (15): the sequence of natural externalities  $(N_t)_{t=0}^\infty$  directly affects the intertemporal smoothing by distorting the consumption-saving arbitrage. Its impact depends on the complementarity or substitution effects between consumption and nature. Bosi et al. (2018) have considered a similar mechanism where negative externalities of pollution  $P_t$  replace the positive externalities of nature  $N_t$  in the utility function. They have shown that the cross effects  $u_{cN}(c_t, N_t) \geq 0$  (complementarity or substitution) can promote the occurrence of cycles.

Clearly, if the utility function is separable  $u(c_t, N_t) = v(c_t) + w(N_t)$ , the cross effect vanishes  $u_{cN}(c_t, N_t) = 0$  and we recover the basic AK model independent on the natural reproduction process:  $v'(c_t)/v'(c_{t+1}) = \beta(1 - \delta + A)$  and  $c_t + k_{t+1} = (1 - \delta + A)k_t$ . Conversely, the reproduction process remains affected by human activities according to (17), that is by capital accumulation  $(k_t)_{t=0}^\infty$  solution to the basic AK model.

## 2.4 Particular solution under full capital depreciation

In the rest of the article, we study a NIPF market economy.

Evidence shows that natural cycles take a long time. In this respect, a period in our model can be considered as long and a full capital depreciation can be justified. From a mathematical point of view, this full depreciation allows us to compute an explicit solution, that is the trajectories of consumption, capital and nature over time.

Consider the system (5)-(6) with  $\delta = 1$ :

$$\begin{aligned} \frac{c_{t+1}}{c_t} &= \beta AN_{t+1}^\alpha \\ c_t + k_{t+1} &= AN_t^\alpha k_t \\ N_{t+1} &= \Phi_i(N_t) - bAN_t^\alpha k_t + N_t \end{aligned}$$

---

<sup>3</sup>Maximizing the Lagrangian function  $\sum_{t=0}^\infty \beta^t u(c_t, N_t) + \sum_{t=0}^\infty \lambda_t [r_t k_t - c_t - k_{t+1} + (1 - \delta)k_t]$ , we derive the first-order conditions:  $\lambda_t = \beta^t u_c(c_t, N_t)$  and  $\lambda_t/\lambda_{t+1} = 1 - \delta + r_{t+1}$ , jointly with the budget constraint and the regeneration rule, now binding, and the transversality condition:  $\lim_{t \rightarrow \infty} \lambda_t k_{t+1} = 0$ .

where  $i = P, L$ . If

$$c_t = (1 - \beta) AN_t^\alpha k_t \quad (18)$$

then  $k_{t+1} = AN_t^\alpha k_t - c_t = \beta AN_t^\alpha k_t$  and

$$c_{t+1} = \beta AN_{t+1}^\alpha c_t = \beta AN_{t+1}^\alpha k_{t+1} \frac{c_t}{k_{t+1}} = (1 - \beta) AN_{t+1}^\alpha k_{t+1}$$

If  $c_0 = (1 - \beta) AN_0^\alpha k_0$ , the first-order conditions imply  $c_t = (1 - \beta) AN_t^\alpha k_t$  and  $k_{t+1} = \beta AN_t^\alpha k_t$  for any  $t \geq 0$ .

In other terms, the starting point  $(k_0, N_0, c_0) = (k_0, N_0, (1 - \beta) AN_0^\alpha k_0)$  implies a reduced two-dimensional dynamics for the pair  $(k_t, N_t)$ :

$$k_{t+1} = \beta AN_t^\alpha k_t \quad (19)$$

$$N_{t+1} = \Phi_i(N_t) - bAN_t^\alpha k_t + N_t \quad (20)$$

### 3 Power law

The dynamic system becomes:

$$\frac{c_{t+1}}{c_t} = \beta (1 - \delta + AN_{t+1}^\alpha) \quad (21)$$

$$c_t + k_{t+1} = (1 - \delta + AN_t^\alpha) k_t \quad (22)$$

$$N_{t+1} = aN_t^\varepsilon - bAN_t^\alpha k_t \quad (23)$$

More explicitly,

$$\begin{aligned} c_{t+1} &= \beta [1 - \delta + A(aN_t^\varepsilon - bAN_t^\alpha k_t)^\alpha] c_t \\ k_{t+1} &= (1 - \delta + AN_t^\alpha) k_t - c_t \\ N_{t+1} &= aN_t^\varepsilon - bAN_t^\alpha k_t \end{aligned}$$

#### 3.1 Steady state

Let

$$\underline{A} \equiv \frac{1}{\gamma a^{\frac{1}{1-\varepsilon}}} \quad (24)$$

a critical TFP value in the isoelastic case, where

$$\gamma \equiv \frac{\beta}{1 - \beta(1 - \delta)} \geq \beta \quad (25)$$

Notice that  $\gamma = \beta$  when  $\delta = 1$  (full depreciation). The following assumption ensures the positivity of variables at the steady state.

**Assumption 1**  $A > \underline{A}$ .

At the steady state, we recover the modified golden rule.

**Proposition 2** Under Assumption 1, the steady state of dynamic system (21)-(23) is given by:

$$AN^\alpha = \frac{1}{\gamma} \quad (26)$$

$$k = \frac{\gamma}{b} (aN^\varepsilon - N) > 0 \quad (27)$$

$$c = \frac{1-\beta}{\beta} k > 0 \quad (28)$$

The steady state is unique. More explicitly,

$$N = (\gamma A)^{-\frac{1}{\alpha}} \quad (29)$$

$$k = \frac{\gamma}{b} \left[ a(\gamma A)^{-\frac{\varepsilon}{\alpha}} - (\gamma A)^{-\frac{1}{\alpha}} \right] \quad (30)$$

$$c = \frac{1-\beta}{\beta} \frac{\gamma}{b} \left[ a(\gamma A)^{-\frac{\varepsilon}{\alpha}} - (\gamma A)^{-\frac{1}{\alpha}} \right] \quad (31)$$

The uniqueness of the steady state comes from the linear structure of the AK model. Indeed, the modified golden rule:  $\beta(1 - \delta + AN^\alpha) = 1$ , determines a unique natural stock of steady state.

### 3.2 Local dynamics

The following lemma provides the three eigenvalues of local dynamics around the steady state.

**Lemma 3** Dynamics (21)-(23) are approximated by the following linear system around the steady state:

$$\begin{bmatrix} \frac{dk_{t+1}}{k} \\ \frac{dN_{t+1}}{N} \\ \frac{dc_{t+1}}{c} \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta} & \frac{\alpha}{\gamma} & \frac{\beta-1}{\beta} \\ -P & Q & 0 \\ -\frac{\alpha\beta}{\gamma}P & \frac{\alpha\beta}{\gamma}Q & 1 \end{bmatrix} \begin{bmatrix} \frac{dk_t}{k} \\ \frac{dN_t}{N} \\ \frac{dc_t}{c} \end{bmatrix} \quad (32)$$

where

$$P \equiv \frac{b}{\gamma} \frac{k}{N} \quad (33)$$

$$Q \equiv \varepsilon + (\varepsilon - \alpha)P \quad (34)$$

The eigenvalues are given by

$$\lambda_1 = \frac{1}{2} \left[ 1 + Q - \sqrt{(1-Q)^2 - 4\frac{\alpha\beta}{\gamma}P} \right] \quad (35)$$

$$\lambda_2 = \frac{1}{2} \left[ 1 + Q + \sqrt{(1-Q)^2 - 4\frac{\alpha\beta}{\gamma}P} \right] \quad (36)$$

$$\lambda_3 = \frac{1}{\beta} > 1$$

Our model represents a dynamic market economy. Dynamics are driven by a sequence of general equilibrium prices  $(r_t)_{t=0}^{\infty}$ . It is known that the equilibrium is unique in the RCK model. Our model is an extension of the RCK model where equilibrium uniqueness is locally preserved (local determinacy). Indeed, the dynamic system is three-dimensional. Since  $\lambda_3 > 1$ , the stable manifold (union of converging paths to the steady state) is not full-dimensional: its dimension is less than three. We observe that the initial conditions  $k_0$  and  $N_0$  are given (predetermined variables), while  $c_0$  is an individual choice (non-predetermined variable). If the stable manifold is two-dimensional and  $(k_0, N_0)$  lies in a neighborhood of the steady state  $(k, N)$ , there is a unique value of  $c_0$  such that  $(k_0, N_0, c_0)$  belongs to this stable manifold. In this case, the equilibrium path  $(k_t, N_t, c_t)_{t=0}^{\infty}$  is a locally unique (locally determined) saddle-path: it starts from  $(k_0, N_0, c_0)$  and converges to the unique steady state  $(k, N, c)$ . When a supercritical Neimark-Sacker bifurcation occurs, the steady state loses its saddle-path stability (all the eigenvalues lie outside the unit circle) and a stable limit cycle arises around  $(\lambda_3$  remains real, while the other two eigenvalues are nonreal and conjugate). The new stable manifold is the union of converging paths to this limit cycle. As above, there is a unique  $c_0$  such that  $(k_0, N_0, c_0)$  belongs to this stable manifold. The equilibrium path is locally unique (locally determined), it diverges from the steady state and converges to the limit cycle around.

Define three critical values:

$$\varepsilon_1 \equiv \alpha\beta(1 - \delta), \varepsilon_2 \equiv \alpha \frac{4\gamma - 3\beta}{4\gamma + \alpha\beta} \text{ and } \varepsilon_3 \equiv \alpha \frac{1 + \beta(1 - \delta)}{2}$$

It is easy to check that  $\varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \alpha$ .

We introduce also the following critical values of the TFP:

$$A_N \equiv \frac{1}{\gamma} \left( \frac{1 - \varepsilon_1}{a \varepsilon - \varepsilon_1} \right)^{\frac{\alpha}{1-\varepsilon}} \quad (37)$$

$$A_F \equiv \frac{1}{\gamma} \left( \frac{1 - \varepsilon_3}{a \varepsilon_3 - \varepsilon} \right)^{\frac{\alpha}{1-\varepsilon}} \quad (38)$$

We observe that, if  $\varepsilon > \varepsilon_1$ , then  $A_N > \underline{A}$ . Moreover,  $A_F > \underline{A}$  if  $\varepsilon < \varepsilon_3$ , and  $A_F > A_N$  if  $\varepsilon_2 < \varepsilon < \varepsilon_3$ .

The following proposition characterizes generically the local bifurcations of our economy around the unique steady state (29)-(31).

**Proposition 4** *Let  $\varepsilon \neq a$ .*

(1) *If  $0 < \varepsilon < \varepsilon_2$ ,  $A_F > \underline{A}$  is a flip bifurcation value. More precisely,  $\lambda_1$  and  $\lambda_2$  are in the unit circle for  $A \in (\underline{A}, A_F)$  and both outside for  $A > A_F$ . The system generically undergoes a flip bifurcation at  $A = A_F$  and a period-two cycle arises around the steady state.*

(2) *If  $\varepsilon_2 < \varepsilon < \varepsilon_3$ ,  $A_N > \underline{A}$  is a Neimark-Sacker bifurcation value and  $A_F > A_N$  is a flip bifurcation value. More precisely,  $\lambda_1$  and  $\lambda_2$  are in the unit circle for  $A \in (\underline{A}, A_N)$ , both outside for  $A \in (A_N, A_F)$ , one inside and the*

other outside for  $A > A_F$ . The system generically undergoes a Neimark-Sacker bifurcation at  $A = A_N$  and a limit cycle arises around the steady state. The system generically undergoes a flip bifurcation at  $A = A_F$  and a two-period cycle arises around the steady state.

(3) If  $\varepsilon_3 < \varepsilon < 1$ ,  $A_N > \underline{A}$  is a Neimark-Sacker bifurcation value. More precisely,  $\lambda_1$  and  $\lambda_2$  are in the unit circle for  $A \in (\underline{A}, A_N)$  and both outside for  $A > A_N$ . The system generically undergoes a Neimark-Sacker bifurcation at  $A = A_N$  and a limit cycle arises around the steady state.

In case (1) but we do not know whether the flip bifurcation is sub or supercritical, that is we do not know whether an unstable cycle exists below  $A_F$  or a stable cycle above  $A_F$ . In case (2), we do not know whether the Neimark-Sacker bifurcation is sub or supercritical, that is we do not know whether an unstable cycle exists below  $A_N$  or a stable cycle above  $A_N$ . Moreover, we do not know whether the flip bifurcation is super or subcritical, that is we do not know whether an stable cycle exists below  $A_F$  or a unstable cycle above  $A_F$ . In case (3), we do not know whether the Neimark-Sacker bifurcation is sub or supercritical, that is we do not know whether an unstable cycle exists below  $A_N$  or a stable cycle above  $A_N$ . However, numerical simulations allows us to illustrate the stability properties (sub and supercriticality) of two-period and limit cycles.

Let us provide the main lines of the proof of Proposition 4. The reader is referred to the appendix for a complete proof.

Dynamics are three-dimensional. However, we know that one eigenvalue is always outside the unit circle ( $\lambda_3 = 1/\beta > 1$ ) and, thus, the factorization of the characteristic polynomial  $P(\lambda)$  allows us to focus on a two-dimensional sub-polynomial  $\tilde{P}(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$  with  $P(\lambda) = (\lambda - 1/\beta)\tilde{P}(\lambda)$ . We can apply to this sub-polynomial the two-dimensional geometrical method introduced by Samuelson (1942) and developed later by Grandmont et al. (1998).

To apply their method, focus now on the two-dimensional subspace spanned by the eigenvectors associated to  $\lambda_1$  and  $\lambda_2$  regardless of the one-dimensional unstable subspace associated to  $\lambda_3$ . We can study the stability properties considering  $T \equiv \lambda_1 + \lambda_2$  and  $D \equiv \lambda_1\lambda_2$  instead of the eigenvalues  $\lambda_1$  and  $\lambda_2$ .

In the spirit of Samuelson (1942), we can represent the stability properties in the  $(T, D)$ -plane instead of the Argand plan (Fig. 3).

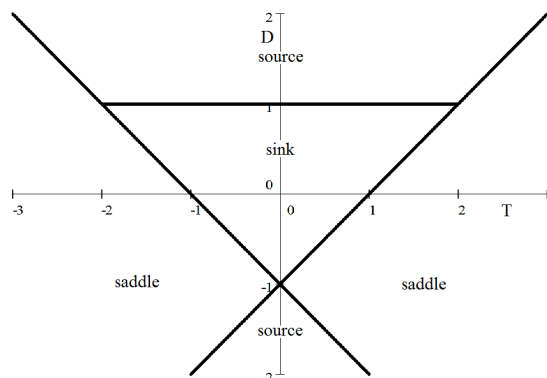


Fig. 3 Stability triangle

Notice that, here, "sink" refers to the two-dimensional subspace we are considering (in the original three-dimensional space, because of  $\lambda_3$ , in fact, this "sink" is a saddle with two eigenvalues inside the unit circle and one outside).

We evaluate the characteristic sub-polynomial  $\tilde{P}(\lambda) \equiv \lambda^2 - T\lambda + D$  at  $-1$  and  $1$ : along the line  $D = T - 1$ , one eigenvalue is equal to  $1$  because  $\tilde{P}(1) = 1 - T + D = 0$ ; along the line  $D = -T - 1$ , one eigenvalue is equal to  $-1$  because  $\tilde{P}(-1) = 1 + T + D = 0$ . Along the segment  $D = 1$  with  $|T| < 2$ , the eigenvalues are nonreal and conjugate with unit modulus. Focus on points outside these lines and the segment. Inside the triangle, the steady state is a sink ( $D < 1$  and  $|T| < 1 + D$ ). It is a saddle point if  $(T, D)$  lies on the left cone of the lines  $D = T - 1$  and  $D = -T - 1$ , or on the right cone of these lines ( $|1 + D| < |T|$ ). It is a source otherwise. A local bifurcation arises when one eigenvalue crosses the unit circle in the Argand plan or, equivalently, the pair  $(T, D)$  goes through one of the lines  $D = T - 1$  or  $D = -T - 1$ , or the segment  $D = 1$  with  $|T| \leq 2$ . When a bifurcation parameter changes,  $(T, D)$  moves in the  $(T, D)$ -plane. Generically, a generic saddle-node occurs when  $(T, D)$  crosses  $D = T - 1$ , a flip when  $(T, D)$  goes through the line  $D = -T - 1$ , whereas a Neimark-Sacker when  $(T, D)$  crosses the segment  $D = 1$  with  $|T| \leq 2$ .

In our case, we apply the method introduced by Grandmont et al. (1998) and based on the linearity property of the locus obtained in the  $(T, D)$ -plane when the TFP  $A$  varies, to have a complete characterization of the local bifurcations arising in our economy.

As explained in the appendix, the origins of the three half-lines in Figure 4 correspond to the starting value  $\underline{A}$ , while their slopes are determined by  $\varepsilon$ . The intersections of the half-lines with the horizontal side of the triangle correspond to the critical value  $A_N$  of Neimark-Sacker bifurcation; the intersections of the half-lines with the line  $D = -T - 1$  correspond to the critical value  $A_F$  of flip

bifurcation.

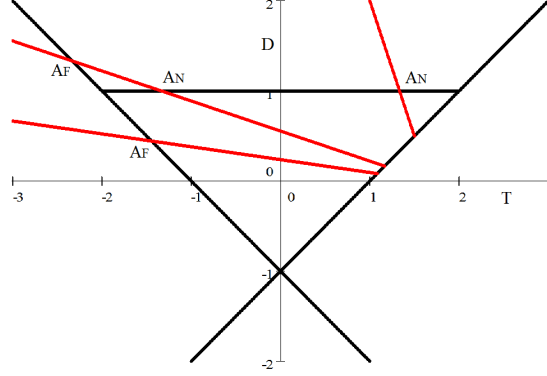


Fig. 4 Bifurcations

Consider for instance the case (1) of Proposition 4, that is  $0 < \varepsilon < \varepsilon_2$  (lower half-line in the Figure 4), with  $\underline{A} < A < A_F$ . In this case, two eigenvalues are inside the unit circle and one outside. Then, the dimension of the stable manifold is two. Let  $(k_0, N_0)$  belong to a neighborhood of the steady state. Since  $k_0$  and  $N_0$  are predetermined, this means that there exists a unique value of the non-predetermined variable  $c_0$ , say  $c_0^*$ , such that  $(k_0, N_0, c_0^*)$  belongs to the stable manifold and, therefore, a unique equilibrium trajectory starting from  $(k_0, N_0, c_0^*)$  and converging to the steady state (equilibrium determinacy).

### 3.3 Simulations

Natural cycles takes place in the long run and a period, in our model, can be viewed as long. Capital fully depreciates during a long period. Thus, we can assume  $\delta = 1$  in the following.

Since the system (19)-(20) jointly with (18) is a solution of the original dynamic system in the particular case of full capital depreciation and the locally converging equilibrium is unique, the equilibrium starting point is precisely  $c_0^* = (1 - \beta) AN_0^\alpha k_0$  where  $k_0$  and  $N_0$  are given.

Because of equilibrium determinacy, we can use the system

$$k_{t+1} = \beta AN_t^\alpha k_t \quad (39)$$

$$N_{t+1} = aN_t^\varepsilon - bAN_t^\alpha k_t \quad (40)$$

$$c_t = (1 - \beta) AN_t^\alpha k_t$$

with  $k_0$  and  $N_0$  given and  $c_0 = c_0^* = (1 - \beta) AN_0^\alpha k_0$  to simulate the trajectory from  $(k_0, N_0, c_0^*)$  to the steady state and, when  $A$  is close to the critical values  $A_F$  or  $A_N$ , to a two-period cycle or to a limit cycle respectively.

**Proposition 5** *Under full capital depreciation ( $\delta = 1$ ), the two-dimensional sub-system (39)-(40) exhibits the same bifurcations point of the original three-dimensional system (21)-(23).*



In particular, the same conclusions about the local uniqueness (local determinacy) of equilibrium paths converging to the steady state or to surrounding two-period and limit cycles hold.

Proposition 5 allows us to simulate the trajectory in a two-dimensional space and, interestingly, to know whether the cycles are stable or unstable. We fix the main parameters as follows:

Parameter	$a$	$b$	$\alpha$	$\beta$	$\delta$	$\varepsilon$
Value	1/2	1/2	2/3	9/10	1	1/12

(41)

Figure 4 has been plotted with this parametrization.

Fixing  $\varepsilon = 1/12 \in (0, \varepsilon_2)$ , we find the flip bifurcation value  $A_F = 6.2146$  and, therefore, the steady state  $(k, N) = (0.58967, 0.075599)$ . We illustrate the case (1) of Proposition 4 in Figure 5 and show the convergence of  $k_t$  to a two-period cycle around the steady state from the starting point  $(k_0, N_0) = (0.1, 0.075)$  when  $A = A_F$ .

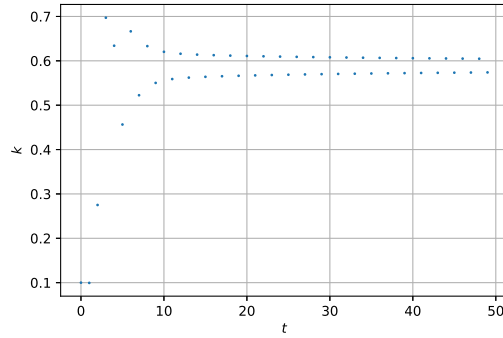


Fig. 5 Supercritical two-period cycle

Then, fixing  $\varepsilon = 1/6 \in (\varepsilon_2, \varepsilon_3)$  jointly with parametrization (41), we obtain the Neimark-Sacker bifurcation value  $A_N = 8.1115$  and, therefore, the stationary state  $(k, N) = (0.45627, 0.050697)$ . Figure 6 illustrates the case (2) of Proposition 4 and shows the limit cycle arising around the steady state through a Neimark-Sacker bifurcation in the  $(k_t, N_t)$ -plane with  $(k_0, N_0) = (0.45, 0.05)$

as initial condition.

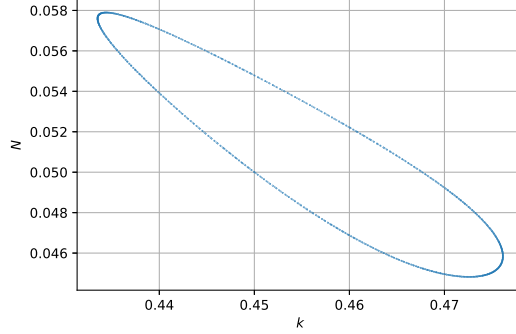


Fig. 6 Limit cycle

The TFP  $A$  plays the main role in the occurrence of cycles, which is twofold. On the one side, it captures the external effect of nature on production; on the other, the impact of pollution on the reproduction process. If  $A$  is sufficiently large, a cycle takes place (through a flip or a Neimark-Sacker bifurcation) because a raise in natural capital implies a higher production level and a larger pollution in turn, which entails a drop in natural capital in the end. Thus, an initial increase in the natural stock is followed by a decrease which, reversely, induces a new increase, and so on.

## 4 Logistic law

Focus now on the second process of nature accumulation (9).

The dynamic system becomes

$$\frac{c_{t+1}}{c_t} = \beta (1 - \delta + AN_{t+1}^\alpha) \quad (42)$$

$$c_t + k_{t+1} = (1 - \delta + AN_t^\alpha) k_t \quad (43)$$

$$N_{t+1} - N_t = aN_t^\varepsilon (\bar{N} - N_t) - bAN_t^\alpha k_t \quad (44)$$

More explicitly,

$$c_{t+1} = \beta (1 - \delta + A [aN_t^\varepsilon (\bar{N} - N_t) - bAN_t^\alpha k_t + N_t]^\alpha) c_t$$

$$k_{t+1} = (1 - \delta + AN_t^\alpha) k_t - c_t$$

$$N_{t+1} = N_t + aN_t^\varepsilon (\bar{N} - N_t) - bAN_t^\alpha k_t$$

### 4.1 Steady state

Let

$$\underline{A} \equiv \frac{1}{\gamma \bar{N}^\alpha}$$

be a critical TFP value under a logistic law, where  $\gamma \geq \beta$  is given by (25). The following assumption ensures the positivity of variables at the steady state.

**Assumption 2**  $A > \underline{A}$ .

At the steady state, we recover the modified golden rule.

**Proposition 6** *The steady state of dynamic system (42)-(44) is given by:*

$$AN^\alpha = \frac{1}{\gamma} \quad (45)$$

$$k = \gamma \frac{a}{b} N^\varepsilon (\bar{N} - N) \quad (46)$$

$$c = \frac{1-\beta}{\beta} k \quad (47)$$

*The steady state is unique. More explicitly,*

$$N = (\gamma A)^{-\frac{1}{\alpha}} \quad (48)$$

$$k = \frac{a}{b} \gamma (\gamma A)^{-\frac{\varepsilon}{\alpha}} \left[ \bar{N} - (\gamma A)^{-\frac{1}{\alpha}} \right] \quad (49)$$

$$c = \frac{1-\beta}{\beta} \frac{a}{b} \gamma (\gamma A)^{-\frac{\varepsilon}{\alpha}} \left[ \bar{N} - (\gamma A)^{-\frac{1}{\alpha}} \right] \quad (50)$$

We observe that  $k, c > 0$  if and only if  $N < \bar{N}$ , that is  $A > \underline{A}$  (sufficiently high TFP).

As above, the linear structure of the AK model entails the uniqueness of the steady state.

## 4.2 Local dynamics

The following lemma provides the three eigenvalues of local dynamics around the steady state.

**Lemma 7** *Dynamics system (42)-(44) are approximated by the following linear system around the steady state:*

$$\begin{bmatrix} \frac{dk_{t+1}}{k} \\ \frac{dN_{t+1}}{N} \\ \frac{dc_{t+1}}{c} \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta} & \frac{\alpha}{\gamma} & \frac{\beta-1}{\beta} \\ -P & \tilde{Q} & 0 \\ -\frac{\alpha\beta}{\gamma} P & \frac{\alpha\beta}{\gamma} \tilde{Q} & 1 \end{bmatrix} \begin{bmatrix} \frac{dk_t}{k} \\ \frac{dN_t}{N} \\ \frac{dc_t}{c} \end{bmatrix} \quad (51)$$

where

$$P \equiv \frac{b}{\gamma} \frac{k}{N} \quad (52)$$

$$\tilde{Q} \equiv 1 + P \left( \varepsilon - \alpha - \frac{N}{\bar{N} - N} \right) \quad (53)$$

The eigenvalues are given by:

$$\lambda_1 = \frac{1}{2} \left[ 1 + \tilde{Q} - \sqrt{(1 - \tilde{Q})^2 - 4 \frac{\alpha\beta}{\gamma} P} \right] \quad (54)$$

$$\lambda_2 = \frac{1}{2} \left[ 1 + \tilde{Q} + \sqrt{(1 - \tilde{Q})^2 - 4 \frac{\alpha\beta}{\gamma} P} \right] \quad (55)$$

$$\lambda_3 = \frac{1}{\beta} > 1$$

Let us introduce two critical values for  $\varepsilon$ :

$$0 < \varepsilon_1 \equiv \alpha\beta(1 - \delta) < \varepsilon_2 \equiv \alpha \frac{1 + \beta(1 - \delta)}{2} < \alpha$$

and three critical values for the TFP:

$$A_i \equiv A(n_i) = A(\varepsilon - \varepsilon_i)$$

for  $i = 1, 2, 3$ , where

$$A(n) \equiv \frac{1}{\gamma} \left[ \frac{1}{\bar{N}} \left( 1 + \frac{1}{n} \right) \right]^\alpha \quad (56)$$

These values are ranked:

$$A_1 < A_2 < A_3$$

where each inequality  $A_i < A_{i+1}$  makes sense only if  $\varepsilon_{i+1} < \varepsilon$  (see the proof of the next proposition).

Finally, we introduce the following critical value for the regeneration rate:

$$a_F \equiv \frac{2(\gamma A)^{\frac{\varepsilon}{\alpha}}}{1 + (\varepsilon - \varepsilon_2) \left[ 1 - \bar{N}(\gamma A)^{\frac{1}{\alpha}} \right]} \quad (57)$$

There are two predetermined variables ( $k_t$  and  $N_t$ ) and one non-predetermined ( $c_t$ ).

The following proposition characterizes generically the local bifurcations of our economy around the unique steady state.

**Proposition 8** *Let  $A > \underline{A}$ . Then, a positive steady state exists.*

(1)  $0 < \varepsilon < \varepsilon_1$ .

*The steady state is stable (two eigenvalues inside the unit circle, one outside) for  $a \in (0, a_F)$  and unstable for  $a > a_F$  (one eigenvalue inside the unit circle, two outside). At  $a = a_F$ , generically, the economic system undergoes a flip bifurcation generating a two-period cycle.*

(2)  $\varepsilon_1 < \varepsilon < \varepsilon_2$ . Then,  $0 < A_1$ .

*(2.1) If  $A < A_1$ , the steady state is stable (two eigenvalues inside the unit circle, one outside) for  $a \in (0, a_F)$  and unstable for  $a > a_F$  (one eigenvalue*

inside the unit circle, two outside). At  $a = a_F$ , generically, the economic system undergoes a flip bifurcation generating a two-period cycle.

(2.2) If  $A_1 < A$ , the steady state is unstable (three eigenvalues outside the unit circle for  $a \in (0, a_F)$  and unstable for  $a > a_F$  (one eigenvalue inside the unit circle, two outside). At  $a = a_F$ , generically, the economic system undergoes a flip bifurcation generating a two-period cycle.

(3)  $\varepsilon_2 < \varepsilon$ . Then,  $0 < A_1 < A_2$ .

(3.1) If  $A < A_1$ , the same results of point (2.1) hold.

(3.2) If  $A_1 < A < A_2$ , the same results of point (2.2) hold.

(3.3) If  $A_2 < A$ , the steady state is unstable (three eigenvalues outside). There is no room for local bifurcations.

As above, we apply the geometrical method introduced by Samuelson (1942) and developed by Grandmont et al. (1998). We plot the half-lines corresponding to the case (3) of Proposition 8.

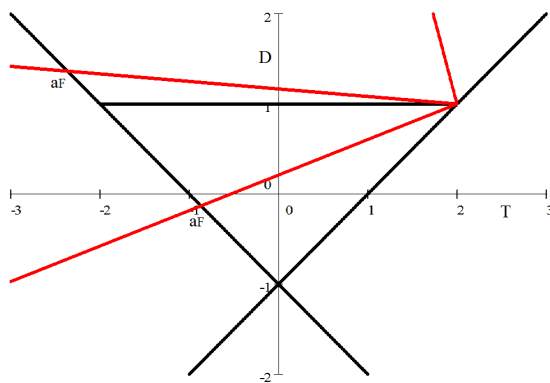


Fig. 7 Flip bifurcations

Figure 7 shows the intersections where the flip bifurcations arise: the half-line in the bottom corresponds to case (3.1); the half-line in the middle, to case (3.2); the half-line in the top, to case (3.3).

As above, we observe that, if  $A < A < A_1$  (case (3.1)) and  $0 < a < a_F$ , two eigenvalues are inside the unit circle and one outside. Then, the dimension of the stable manifold is two. Let  $(k_0, N_0)$  be in a neighborhood of the steady state. Since  $k_0$  and  $N_0$  are predetermined, this means that there exists a unique value of the non-predetermined variable  $c_0$ , say  $c_0^*$ , such that  $(k_0, N_0, c_0^*)$  belongs to the stable manifold and, therefore, a unique equilibrium trajectory starting from  $(k_0, N_0, c_0^*)$  and converging to the steady state (equilibrium determinacy).

At this stage, nothing can be said about the stability (supercriticality) or the instability (subcriticality) of the two-period cycles. More precisely, in cases (1), (2.1) and (3.1) of Proposition 8, if the cycle exists for  $a < a_F$ , it is unstable (subcritical). If it exists for  $a > a_F$ , it is stable (supercritical). In cases (2.2) (3.2), if the cycle exists for  $a > a_F$ , it is stable (supercritical). In order to know more about this stability issue, we perform a numerical simulation. According

to a plausible parametrization, we obtain that an attractive (supercritical) two-period cycle appears around a steady state which becomes unstable.

### 4.3 Simulations

As above, natural cycles takes place in the long run and the length of the period justifies a full capital depreciation ( $\delta = 1$ ).

Since the system (19)-(20) jointly with (18) is a solution of the original dynamic system in the particular case of full capital depreciation and the locally converging equilibrium is unique, the equilibrium starting point is precisely  $c_0^* = (1 - \beta) AN_0^\alpha k_0$  where  $k_0$  and  $N_0$  are given.

Because of equilibrium determinacy, we can use the system

$$k_{t+1} = \beta AN_t^\alpha k_t \quad (58)$$

$$N_{t+1} = aN_t^\varepsilon (\bar{N} - N_t) - bAN_t^\alpha k_t + N_t \quad (59)$$

$$c_t = (1 - \beta) AN_t^\alpha k_t$$

with  $k_0$  and  $N_0$  given and  $c_0 = c_0^* = (1 - \beta) AN_0^\alpha k_0$  to simulate the trajectory from  $(k_0, N_0, c_0^*)$  to the steady state and, when  $a$  is close to the critical values  $a_F$ , to a two-period cycle.

**Proposition 9** *Under full capital depreciation ( $\delta = 1$ ), the two-dimensional sub-system (58)-(59) exhibits the same bifurcations point of the original three-dimensional system (42)-(44).*

In particular, the same conclusions about the local uniqueness (local determinacy) of equilibrium paths converging to the steady state or to surrounding two-period cycles hold.

A numerical example allows us to illustrate the case (3) of Proposition 8. As above, we parametrize the model to simulate the economic dynamics:

Parameter	$A$	$b$	$\alpha$	$\beta$	$\delta$	$\varepsilon$
Value	1.43	1/2	1/3	9/10	1	2/3

We normalize the maximal natural stock:  $\bar{N} = 1$ . Figure 7 has been plotted with this parametrization.

We have  $D = sT + 1 - 2s$  with

$$s = \frac{\varepsilon - n}{\varepsilon - \alpha - n} \text{ and } n \equiv \frac{N}{\bar{N} - N} = \frac{1}{\bar{N} (A\beta)^{\frac{1}{\alpha}} - 1}$$

Then

$$D = 1 + (T - 2) \frac{1458A^3 - 5000}{729A^3 - 4000} \quad (60)$$

Using (56), we get:  $\underline{A} = 1.1111$ ,  $A_1 = 1.508$ ,  $A_2 = 1.6025$ ,  $A_3 = 1.7638$ . To capture the cases (3.1), (3.2) and (3.3) of Proposition 8, fix  $A = B_1 = 1.43$ ,

$A = B_2 = 1.52$ ,  $A = B_3 = 1.68$ . We observe that

$$\begin{aligned} \underline{A} &= 1.1111 < B_1 = 1.43 < A_1 = 1.508 < B_2 = 1.52 \\ &< A_2 = 1.6025 < B_3 = 1.68 < A_3 = 1.7638 \end{aligned}$$

Finally, remember that  $(T(0), D(0)) = (2, 1)$ .

We obtain the flip bifurcation value  $a_F = 7.6308$  and, therefore, the steady state  $(k, N) = (4.4025, 0.46910)$ . We illustrate the case (3.1) of Proposition 8 in Figure 8 and show the convergence of  $k_t$  from the initial condition  $(k_0, N_0) = (4.4, 0.47)$  to an attractive two-period cycle around the steady state when  $a = a_F$ . Therefore, the cycle is stable (supercritical).

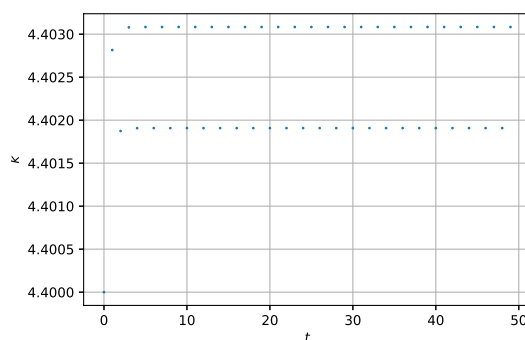


Fig. 8 Supercritical two-period cycle

In order to interpret the occurrence of a two-period cycles, we refer to the reproduction law:  $N_{t+1} = aN_t^\varepsilon (\bar{N} - N_t) - bAN_t^\alpha k_t + N_t$ . We observe that the parameter  $a$  plays now the main role in the occurrence of cycles. They require a sufficiently large value. While in the case of a power law,  $a$  appears in the terms  $aN_t^\varepsilon$  as a monotonic force, now, in the case of a logistic law, it appears in the  $aN_t^\varepsilon (\bar{N} - N_t)$  as a return force because of  $-N_t$ . Indeed, when  $N_t$  is sufficiently large,  $N_t^\varepsilon (\bar{N} - N_t)$  decreases with  $N_t$  and this negative impact is magnified by the size of  $a$  with, possibly, a final negative impact on  $N_{t+1}$ .

## 5 Conclusion

In this paper, we have considered the productive effects of nature and the negative impact of productive pollution on the regeneration process of nature in a discrete-time AK model. We have studied two distinct natural accumulation processes: a power law, more standard in economics, and a logistic law, more popular in biology. In both cases, we have observed the occurrence of cycles of period two, cycles of period a power of two, and chaos through a flip bifurcation and period-doubling bifurcations.

In the case of the power law, we have shown that dynamics are even richer because of the possibility of limit cycles through a Neimark-Sacker bifurcation:

a larger TFP promotes the occurrence of fluctuations by amplifying the negative impact of pollution on natural regeneration. In the case of a logistic law, cycles arise when the saturation effect of nature has a larger negative impact on the reproduction process.

## 6 Appendix

### Proof of Proposition 1

The planner maximizes the following Lagrangian function:

$$\begin{aligned} & \sum_{t=0}^{\infty} \beta^t \ln c_t + \sum_{t=0}^{\infty} \lambda_t [AN_t^\alpha k_t - c_t - k_{t+1} + (1 - \delta) k_t] \\ & + \sum_{t=0}^{\infty} \mu_t [\Phi_i(N_t) - bAN_t^\alpha k_t - N_{t+1} + N_t] \end{aligned}$$

where  $(\lambda_t)_{t=0}^{\infty}$  and  $(\mu_t)_{t=0}^{\infty}$  are sequences of non-negative multipliers.

The first-order conditions are given by

$$\lambda_t = \frac{\beta^t}{c_t} \quad (61)$$

$$\frac{\lambda_t}{\lambda_{t+1}} = 1 - \delta + AN_{t+1}^\alpha - bAN_{t+1}^\alpha \frac{\mu_{t+1}}{\lambda_{t+1}} \quad (62)$$

$$\frac{\mu_t}{\mu_{t+1}} = 1 + \Phi'_i(N_{t+1}) + \alpha \left( \frac{\lambda_{t+1}}{\mu_{t+1}} - b \right) AN_{t+1}^{\alpha-1} k_{t+1} \quad (63)$$

The Lagrangian is not necessarily concave because of the term  $-\mu_t bAN_t^\alpha k_t$ . Let us show that the sequences  $(c_t)$ ,  $(N_t)$  and  $(k_t)$  are uniformly bounded because of the decreasing returns of nature and the pollution effects of capital.

We know that the sequence  $(N_t)$  is uniformly bounded because of our assumptions on  $\Phi_i$ . Thus, we have  $N_t \leq \bar{N}$  for every  $t \geq 0$ . Fix  $\underline{N}$  such that  $A\underline{N}^\alpha < \delta/2$ . Fix a level  $\bar{k} > k_0$  such that

$$\bar{k} > \max_{N \in [\underline{N}, \bar{N}]} \frac{N + \Phi_i(N)}{bAN^\alpha}$$

Hence, since  $N_{t+1} = N_t + \Phi_i(N_t) - bAN_t^\alpha k_t > 0$ , if  $k_t > \bar{k}$ , we have

$$\frac{N_t + \Phi_i(N_t)}{bAN_t^\alpha} > k_t > \max_{N \in [\underline{N}, \bar{N}]} \frac{N + \phi_i(N)}{bAN^\alpha}$$

Since  $N_t \leq \bar{N}$ , this implies  $N_t < \underline{N}$ . Therefore,  $k_{t+1} = (1 - \delta + AN_t^\alpha) k_t - c_t < (1 - \delta + A\underline{N}^\alpha) k_t < (1 - \delta/2) k_t$ .

Now, fix  $\hat{k}$  such that  $(1 - \delta + A\bar{N}^\alpha) \bar{k} < \hat{k}$ . Notice that  $1 - \delta + A\bar{N}^\alpha > 1 - \delta + AN^\alpha = 1/\beta > 1$ . Then,  $\bar{k} < \hat{k}$ . We prove that  $k_t < \hat{k}$  for every



$t$ . Either  $k_t < \bar{k}$  for any  $t$  or there exists a date  $t \geq 1$  such that  $k_{t-1} < \bar{k}$  and  $k_t > \bar{k}$ . We have  $k_t < (1 - \delta + A\bar{N}^\alpha)k_{t-1} < \hat{k}$ . Moreover, we have  $k_{t+1} < (1 - \delta/2)k_t < k_t$ . If  $k_{t+1} > \bar{k}$ , we have  $k_{t+2} < (1 - \delta/2)k_{t+1} < k_{t+1}$ . By induction, we have that  $(k_t)$  decreases until some date  $T$  such that  $k_T < \bar{k}$ .

Hence, the sequence  $(k_t)$  may overcome  $\bar{k}$  at some date, but after this date, it decreases. In any case, it remains below  $\hat{k}$ .

Thus, given the sequences  $(\lambda_t)$  and  $(\mu_t)$  in the space of summable sequences  $\ell^1$ , the Lagrangian always attains a maximum. In this maximum, the first-order conditions (61)-(62) are necessarily satisfied, even if other sequences satisfying them could exist. ■

**Proof of Proposition 2**

Solve system (21)-(23) with  $(k_{t+1}, N_{t+1}, c_{t+1}) = (k_t, N_t, c_t) = (k, N, c)$  for any  $t \geq 0$  to obtain (26), (27) and (28).

Since there is a unique stationary level of nature, the steady state is unique.

Notice that  $k, c > 0$  if and only if  $A > \underline{A}$ , which is true if the TFP  $A$  is sufficiently high. In this case, more explicitly, we obtain (29), (30) and (31). ■

**Proof of Lemma 3**

Noticing that  $\gamma AN^\alpha = 1$ ,  $c/k = 1/\beta - 1$ ,  $aN^\varepsilon = N + bAN^\alpha k$ , the linearization yields:

$$\frac{\alpha\beta}{\gamma} \frac{dN_{t+1}}{N} - \frac{dc_{t+1}}{c} = -\frac{dc_t}{c} \quad (64)$$

$$\beta \frac{dk_{t+1}}{k} = \frac{dk_t}{k} + \frac{\alpha\beta}{\gamma} \frac{dN_t}{N} - (1 - \beta) \frac{dc_t}{c} \quad (65)$$

$$\frac{dN_{t+1}}{N} = -P \frac{dk_t}{k} + Q \frac{dN_t}{N} \quad (66)$$

that is (32), where  $P$  and  $Q$  are given by (33) and (34).

We compute the sums of principal minors:

$$\begin{aligned} S_1 &= \lambda_1 + \lambda_2 + \lambda_3 = \frac{1 + \beta}{\beta} + Q \\ S_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \frac{1}{\beta} + \frac{\alpha\beta}{\gamma}P + \frac{1 + \beta}{\beta}Q \\ S_3 &= \lambda_1\lambda_2\lambda_3 = \frac{\alpha}{\gamma}P + \frac{1}{\beta}Q \end{aligned}$$

where  $S_1$  is the trace and  $S_3$  the determinant of the Jacobian matrix.

The characteristic polynomial is given by:

$$\begin{aligned} P(\lambda) &= \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 \\ &= \lambda^3 - \left(\frac{1 + \beta}{\beta} + Q\right)\lambda^2 + \left(\frac{1}{\beta} + \frac{\alpha\beta}{\gamma}P + \frac{1 + \beta}{\beta}Q\right)\lambda - \frac{\alpha}{\gamma}P - \frac{1}{\beta}Q \end{aligned}$$

We observe that  $P(1/\beta) = 0$ . Then,  $1/\beta$  is an eigenvalue, say,  $\lambda_3$ .

Therefore, we have  $S_1 = \lambda_1 + \lambda_2 + 1/\beta$  and  $S_3 = \lambda_1 \lambda_2 / \beta$ , that is  $\lambda_1 + \lambda_2 = S_1 - 1/\beta = 1 + Q$  and

$$\lambda_1 \lambda_2 = \beta S_3 = \beta \left( \frac{\alpha}{\gamma} P + \frac{1}{\beta} Q \right) = \frac{\alpha \beta}{\gamma} P + Q$$

Focus on the sub-polynomial:

$$\begin{aligned} \tilde{P}(\lambda) &= (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1 \lambda_2 \\ &= \lambda^2 - (1 + Q)\lambda + \frac{\alpha \beta}{\gamma} P + Q = 0 \end{aligned} \quad (67)$$

Thus, the other two eigenvalues are given by (35) and (36). ■

**Proof of Proposition 4**

We observe that, in our model,  $T = 1 + Q$ ,  $D = P\alpha\beta/\gamma + Q$  and  $Q \equiv \varepsilon + (\varepsilon - \alpha)P$ . Then,

$$T = 1 + \varepsilon + (\varepsilon - \alpha)P \quad (68)$$

$$D = \varepsilon + \left( \varepsilon - \alpha + \frac{\alpha\beta}{\gamma} \right) P = \varepsilon + [\varepsilon - \alpha\beta(1 - \delta)]P \quad (69)$$

with

$$P \equiv \frac{b}{\gamma} \frac{k}{N} = a(A\gamma)^{\frac{1-\varepsilon}{\alpha}} - 1 \quad (70)$$

From (68) and (69), we get

$$\begin{aligned} T(P) &= 1 + \varepsilon + (\varepsilon - \alpha)P \\ D(P) &= \left( 1 + \frac{\alpha}{\varepsilon - \alpha} \frac{\beta}{\gamma} \right) T(P) - 1 - (1 + \varepsilon) \frac{\alpha}{\varepsilon - \alpha} \frac{\beta}{\gamma} \end{aligned} \quad (71)$$

In the spirit of Grandmont et al. (1998), we exploit the linearity property of the locus (71) to have a complete characterization of the local bifurcations arising in our economy.

We know that, according to (70) and (24),  $P > 0$  if and only if  $A > \underline{A}$ . Given the other parameters,  $P$  moves from 0 to  $\infty$  as  $A$  moves from  $\underline{A}$  to  $\infty$ .

Consider the half-line  $\{(T(P), D(P))\}_{P>0}$  in the  $(T, D)$ -plane when  $P$  moves from 0 to  $\infty$ .

Its starting point is

$$(T(0), D(0)) = (1 + \varepsilon, \varepsilon)$$

on the line  $D = T - 1$ , with  $0 < \varepsilon < 1$ .

Consider the slope

$$s = 1 + \frac{\alpha}{\varepsilon - \alpha} \frac{\beta}{\gamma} = 1 + \frac{1}{\frac{\varepsilon}{\alpha} - 1} \frac{\beta}{\gamma}$$

of the line

$$D = \left(1 + \frac{\alpha}{\varepsilon - \alpha} \frac{\beta}{\gamma}\right) T - 1 - (1 + \varepsilon) \frac{\alpha}{\varepsilon - \alpha} \frac{\beta}{\gamma} \quad (72)$$

Since  $\alpha, \varepsilon \in (0, 1)$  with  $\alpha \neq \varepsilon$ , we have  $\varepsilon/\alpha \in (0, \infty)$  and  $s \in (-\infty, \beta(1 - \delta)) \cup (1, \infty)$ . Indeed, when  $\varepsilon/\alpha = 0$ , then  $s = 1 - \beta/\gamma = \beta(1 - \delta)$ .

Let  $\varepsilon_1 \equiv \alpha\beta(1 - \delta) < \alpha$ .

There are three cases: (a)  $0 < \varepsilon < \varepsilon_1$ , (b)  $\varepsilon_1 < \varepsilon < \alpha$  (c)  $\alpha < \varepsilon$ .

(a)  $0 < \varepsilon \leq \varepsilon_1$ . In this case,  $T'(P) < 0$  and  $D'(P) < 0$ . Moreover,  $s \in [0, \beta(1 - \delta))$ . Indeed, when  $\varepsilon = 0$ , then  $s = s_0 \equiv \beta(1 - \delta)$  and, when  $\varepsilon = \varepsilon_1$ , then  $s = 0$ . Notice that this case vanishes under full capital depreciation.

The half-line  $\{(T(P), D(P))\}_{P>0}$  crosses the line  $D = -T - 1$  at  $A = A_F$  and the system undergoes a flip bifurcation (but we do not know whether the flip bifurcation is super or subcritical, so we do not know whether a stable cycle exists below  $A_F$  or an unstable cycle above  $A_F$ ).

The flip bifurcation value  $A_F$  corresponds to the intersection of the half-line  $\{(T(P), D(P))\}_{P>0}$  with the line  $D = -T - 1$ . Replacing (68) and (69) in  $D = -T - 1$ , and using (70), we get  $A_F$  as solution to the following equation:

$$P = \frac{2 + 2\varepsilon}{\alpha - 2\varepsilon + \alpha\beta(1 - \delta)} = a(A\gamma)^{\frac{1-\varepsilon}{\alpha}} - 1$$

(b)  $\varepsilon_1 < \varepsilon < \alpha$ . In this case,  $s \in (-\infty, 0)$ ,  $T'(P) < 0$  and  $D'(P) > 0$ .

Let  $s_2 \in (-\infty, 0)$  be the slope such that the half-line  $\{(T(P), D(P))\}_{P>0}$  passes through  $(-2, 1)$ , that is, according to (72),  $\varepsilon = \varepsilon_2$ . Notice that  $\alpha\beta(1 - \delta) < \varepsilon_2 < \alpha$  and

$$s = s_2 \equiv 1 + \frac{\alpha}{\varepsilon_2 - \alpha} \frac{\beta}{\gamma} = -\frac{1 - \alpha\beta(1 - \delta)}{3 + \alpha} \in \left(-\frac{1}{3}, 0\right)$$

(b.1) If  $s \in (-\infty, -1)$ , there exists a Neimark-Sacker bifurcation value such that  $\lambda_1$  and  $\lambda_2$  are in the unit circle for  $A \in (\underline{A}, A_N)$  and outside for  $A > A_N$ . The system generically undergoes a Neimark-Sacker bifurcation at  $A = A_N$  where the half-line  $\{(T(P), D(P))\}_{P>0}$  crosses  $D = 1$  and a limit cycle arises around the steady state (but we do not know whether the Neimark-Sacker bifurcation is sub or supercritical, so we do not know whether an unstable cycle exists under  $A_N$  or a stable cycle beyond  $A_N$ ).

According to (69) and (70), the critical point  $A_N$  is solution to the following equation:

$$P = \frac{1 - \varepsilon}{\varepsilon - \alpha\beta(1 - \delta)} = a(A\gamma)^{\frac{1-\varepsilon}{\alpha}} - 1$$

(b.2) If  $s \in (-1, s_2)$ , there exists a Neimark-Sacker bifurcation value  $A_N$  (expression (37)) and a flip bifurcation value  $A_F > A_N$  (expression (38)) such that  $\lambda_1$  and  $\lambda_2$  are in the unit circle for  $A \in (\underline{A}, A_N)$ , both outside for  $A \in (A_N, A_F)$ , one inside and the other outside if  $A > A_F$ . The system generically undergoes a Neimark-Sacker bifurcation at  $A = A_N$  and a limit cycle arises around the steady state (but we do not know whether the Neimark-Sacker bifurcation is sub

or supercritical, so we do not know whether an unstable cycle exists below  $A_N$  or a stable cycle above  $A_N$ ). The system generically undergoes a flip bifurcation at  $A = A_F$  and a two-period cycle arises around the steady state (but we do not know whether the flip bifurcation is super or subcritical, so we do not know whether a stable cycle exists below  $A_F$  or an unstable cycle above  $A_F$ ).

Since

$$s = 1 + \frac{\alpha}{\varepsilon - \alpha} \frac{\beta}{\gamma} \in (-1, s_2) = \left( -1, -\frac{1 - \alpha\beta(1 - \delta)}{3 + \alpha} \right)$$

we have  $\varepsilon_2 < \varepsilon < \varepsilon_3$ . Then,  $A_F$  is well-defined and  $A_F > A_N > \underline{A}$ .

(b.3) If  $s \in (s_2, 0)$ , there exists a flip bifurcation value  $A_F$  (expression (38)) such that  $\lambda_1$  and  $\lambda_2$  are in the unit circle for  $A \in (\underline{A}, A_F)$  and one inside and one outside for  $A > A_F$ . The system generically undergoes a flip bifurcation at  $A = A_F$  and a period-two cycle arises around the steady state (but we do not know whether the flip bifurcation is sub or supercritical, so we do not know whether an unstable cycle exists below  $A_F$  or a stable cycle above  $A_F$ ).

(c)  $\alpha < \varepsilon$ .

In this case,  $s \in (1, \infty)$ ,  $T'(P) > 0$  and  $D'(P) > 0$ .

There exists a Neimark-Sacker bifurcation value  $A_N$  (expression (37)) such that  $\lambda_1$  and  $\lambda_2$  are in the unit circle for  $A \in (\underline{A}, A_N)$  and both outside for  $A > A_N$ . The system generically undergoes a Neimark-Sacker bifurcation at  $A = A_N$  and a limit cycle arises around the steady state (but we do not know whether the Neimark-Sacker bifurcation is sub or supercritical, so we do not know whether an unstable cycle exists below  $A_N$  or a stable cycle above  $A_N$ ).

We observe that:

(1.1) if  $0 < \varepsilon \leq \varepsilon_1$ , then  $s \in [0, s_0)$  with  $0 < s_0 < 1$  (this case holds under partial depreciation:  $\delta < 1$ );

(1.2) if  $\varepsilon_1 < \varepsilon < \varepsilon_2$ , then  $s \in (s_2, 0)$  with  $-1/3 < s_2 < 0$ ;

(2) if  $\varepsilon_2 < \varepsilon < \varepsilon_3$ , then  $s \in (-1, s_2)$ ;

(3.1) if  $\varepsilon_3 < \varepsilon < \alpha$ , then  $s \in (-\infty, -1)$ ;

(3.2) if  $\alpha < \varepsilon < 1$ , then  $s \in (1, \infty)$ .

Proposition 4 follows. ■

#### **Proof of Proposition 5**

We linearize system (39)-(40) around the steady state:

$$\begin{bmatrix} \frac{dk_{t+1}}{k} \\ \frac{dN_{t+1}}{N} \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ -P & Q \end{bmatrix} \begin{bmatrix} \frac{dk_t}{k} \\ \frac{dN_t}{N} \end{bmatrix}$$

where  $P$  and  $Q$  are given by (33) and (34) with  $\delta = 1$ . The trace and the determinant are given by  $T = 1 + Q$  and  $D = \alpha P + Q$ , and the characteristic polynomial by

$$P(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 - (1 + Q)\lambda + \alpha P + Q$$

that is (67). Therefore, the bifurcation points are the same of the original system under full capital depreciation ( $\delta = 1$ ). ■

**Proof of Proposition 6**

Solve system (42)-(44) with  $(k_{t+1}, N_{t+1}, c_{t+1}) = (k_t, N_t, c_t) = (k, N, c)$  for any  $t \geq 0$  to obtain (45), (46) and (47). Since there is a unique stationary level of nature, the steady state is unique. When the TFP  $A$  is sufficiently high (Assumption 2 or, equivalently,  $N < \bar{N}$ ),  $k, c > 0$  and the steady state is given by (48), (49) and (50). ■

**Proof of Lemma 7**

We linearize the three-dimensional dynamic system around the steady state. Since  $\gamma AN^\alpha = 1$ ,  $c/k = 1/\beta - 1$ ,  $k = N^\varepsilon (\bar{N} - N) \gamma a/b$ , we obtain

$$\begin{aligned} \frac{\alpha\beta}{\gamma} \frac{dN_{t+1}}{N} - \frac{dc_{t+1}}{c} &= -\frac{dc_t}{c} \\ \beta \frac{dk_{t+1}}{k} &= \frac{dk_t}{k} + \frac{\alpha\beta}{\gamma} \frac{dN_t}{N} - (1-\beta) \frac{dc_t}{c} \\ \frac{dN_{t+1}}{N} &= -P \frac{dk_t}{k} + \tilde{Q} \frac{dN_t}{N} \end{aligned}$$

This system is similar to system (64)-(66). The only difference is that, now,  $\tilde{Q}$  replace  $Q$  and the steady state values are given by (48)-(50) instead by (29)-(31). In particular, we observe that  $\lambda_3$  is still equal to  $1/\beta$  and, thus, it lies outside the unit circle.

The proof of Lemma 3 identically applies to obtain the analogous results of Lemma 7. ■

**Proof of Proposition 8**

As in the end of the proof of Lemma 3, we can study the stability properties in the  $(T, D)$ -plane instead of considering  $\lambda_1$  and  $\lambda_2$ . The reader is still referred to Figure 3.

We observe that

$$T = \lambda_1 + \lambda_2 = 1 + \tilde{Q} \tag{73}$$

$$D = \lambda_1 \lambda_2 = \frac{\alpha\beta}{\gamma} P + \tilde{Q} \tag{74}$$

where  $\tilde{Q} \equiv 1 + P(\varepsilon - \alpha - n)$  and  $n \equiv N/(\bar{N} - N) > 0$ . Then,  $T = 2 + P(\varepsilon - n - \alpha)$  and

$$D = 1 + P \left( \frac{\alpha\beta}{\gamma} - \alpha + \varepsilon - n \right) = 1 + P[\varepsilon - n - \alpha\beta(1 - \delta)]$$

We notice that  $N = (\gamma A)^{-1/\alpha}$  does not depend on  $a$ . Moreover,  $P \equiv bk/(\gamma N) = aM$ , where, under Assumption 2,  $M \equiv N^\varepsilon/n > 0$  does not depend on  $a$ .

We have

$$T(a) = 2 + aM(\varepsilon - \alpha - n) \tag{75}$$

$$D(a) = 1 + aM[\varepsilon - \alpha\beta(1 - \delta) - n] \tag{76}$$

Then,  $aM(\varepsilon - n - \alpha) = T - 2$  and

$$D = 1 + (T - 2) \frac{\varepsilon - \alpha\beta(1 - \delta) - n}{\varepsilon - \alpha - n} \quad (77)$$

that is  $D = sT + 1 - 2s$ , where

$$s(n) \equiv \frac{dD}{dT} = \frac{D'(a)}{T'(a)} = \frac{\varepsilon - \alpha\beta(1 - \delta) - n}{\varepsilon - \alpha - n}$$

is the slope of the half-line  $\{(T(a), D(a))\}_{a \geq 0}$ .

The starting point of the half-line ( $a = 0$ ) is  $(T(0), D(0)) = (2, 1)$  that is the right corner of the triangle in Figure 3.

We observe that,

$$n \equiv \frac{N}{\bar{N} - N} = \frac{1}{\bar{N}(\gamma A)^{\frac{1}{\alpha}} - 1} = \frac{1}{\left(\frac{A}{\underline{A}}\right)^{\frac{1}{\alpha}} - 1} > 0 \quad (78)$$

Since  $A > \underline{A}$  (Assumption 2) and  $\underline{A}$  does not depend on  $A$ , we have  $n \in (0, \infty)$  as  $A$  moves from  $\underline{A}$  to  $\infty$ .

We notice that

$$s(0) = \frac{\varepsilon - \alpha\beta(1 - \delta)}{\varepsilon - \alpha}$$

and

$$\begin{aligned} T'(a) &= M(\varepsilon - \alpha - n) \\ D'(a) &= M[\varepsilon - \alpha\beta(1 - \delta) - n] \\ s'(n) &= \alpha \frac{1 - \beta(1 - \delta)}{(\varepsilon - \alpha - n)^2} > 0 \end{aligned}$$

The critical values of  $\varepsilon$  are ranked:  $0 < \varepsilon_1 < \varepsilon_2 < \alpha$ .

(a)  $0 < \varepsilon < \varepsilon_1$ .

In this case,  $\varepsilon - \alpha - n < \varepsilon - \varepsilon_1 - n < 0$ . Then,  $T'(a) < 0$ ,  $D'(a) < 0$  and  $0 < s(0) < s < 1$ .

Sink for  $a < a_F$  and saddle for  $a_F < a$  with a flip bifurcation at  $a = a_F$ .

(b)  $\varepsilon_1 < \varepsilon < \varepsilon_2$ .

In this case,  $\varepsilon - \alpha < 0 < \varepsilon - \varepsilon_1$ . Then,  $T'(a) < 0$  and  $D'(a) < 0$  if and only if  $n > \varepsilon - \alpha\beta(1 - \delta)$ .

(b.1) If  $0 < n < \varepsilon - \varepsilon_1$ , then  $T'(a) < 0$ ,  $D'(a) > 0$  and  $-1 < s(0) < s < 0$ .

Source for  $a < a_F$  and saddle for  $a_F < a$  with a flip bifurcation at  $a = a_F$ .

(b.2) If  $\varepsilon - \varepsilon_1 < n$ , then  $T'(a) < 0$ ,  $D'(a) < 0$  and  $0 < s < 1$ .

Sink for  $a < a_F$  and saddle for  $a_F < a$  with a flip bifurcation at  $a = a_F$ .

(c)  $\varepsilon_2 < \varepsilon < \alpha$ .

In this case,  $\varepsilon - \alpha < 0 < \varepsilon - \varepsilon_1$ . Then,  $T'(a) < 0$  and  $D'(a) < 0$  if and only if  $n > \varepsilon - \varepsilon_1$ .

(c.1) If  $0 < n < \varepsilon - \varepsilon_2$ , then  $T'(a) < 0$ ,  $D'(a) > 0$  and  $s(0) < s < -1$ .

Source for any  $a$ .

(c.2) If  $\varepsilon - \varepsilon_2 < n < \varepsilon - \varepsilon_1$ , then  $T'(a) < 0$ ,  $D'(a) > 0$  and  $-1 < s < 0$ .

Source for  $a < a_F$  and saddle for  $a > a_F$  with a flip bifurcation at  $a = a_F$ .

(c.3) If  $\varepsilon - \varepsilon_1 < n$ , then  $T'(a) < 0$ ,  $D'(a) < 0$  and  $0 < s < 1$ .

Sink for  $a < a_F$  and saddle for  $a > a_F$  with a flip bifurcation at  $a = a_F$ .

(d)  $\alpha < \varepsilon$ .

In this case,  $0 < \varepsilon - \alpha < \varepsilon - \varepsilon_1$ . Then,  $T'(a) < 0$  if and only if  $n > \varepsilon - \alpha$ , and  $D'(a) < 0$  if and only if  $n > \varepsilon - \varepsilon_1$ .

(d.1) If  $0 < n < \varepsilon - \alpha$ , then  $T'(a) > 0$ ,  $D'(a) > 0$  and  $1 < s(0) < s < \infty$ .

Source for any  $a$ .

(d.2) If  $\varepsilon - \alpha < n < \varepsilon - \varepsilon_2 < \varepsilon - \varepsilon_1$ , then  $T'(a) < 0$ ,  $D'(a) > 0$  and  $-\infty < s < -1$ .

Source for any  $a$ .

(d.3) If  $\varepsilon - \varepsilon_2 < n < \varepsilon - \varepsilon_1$ , then  $T'(a) < 0$ ,  $D'(a) > 0$  and  $-1 < s < 0$ .

Source for  $a < a_F$  and saddle for  $a > a_F$  with a flip bifurcation at  $a = a_F$ .

(d.4) If  $\varepsilon - \varepsilon_1 < n$ , then  $T'(a) < 0$ ,  $D'(a) < 0$  and  $0 < s < 1$ .

Sink for  $a < a_F$  and saddle for  $a > a_F$  with a flip bifurcation at  $a = a_F$ .

Summing up and reminding the ranking of critical values:  $0 < \varepsilon_1 < \varepsilon_2 < \alpha$ , we have the following.

(a)  $0 < \varepsilon < \varepsilon_1$ .

Sink for  $a < a_F$  and saddle for  $a_F < a$  with a flip bifurcation at  $a = a_F$ .

(b)  $\varepsilon_1 < \varepsilon < \varepsilon_2$ .

(b.1)  $0 < n < \varepsilon - \varepsilon_1$ .

Source for  $a < a_F$  and saddle for  $a_F < a$  with a flip bifurcation at  $a = a_F$ .

(b.2)  $\varepsilon - \varepsilon_1 < n$ .

Sink for  $a < a_F$  and saddle for  $a_F < a$  with a flip bifurcation at  $a = a_F$ .

(c)  $\varepsilon_2 < \varepsilon < \alpha$ .

(c.1)  $0 < n < \varepsilon - \varepsilon_2$ .

Source for any  $a$ .

(c.2)  $\varepsilon - \varepsilon_2 < n < \varepsilon - \varepsilon_1$ .

Source for  $a < a_F$  and saddle for  $a > a_F$  with a flip bifurcation at  $a = a_F$ .

(c.3) If  $\varepsilon - \varepsilon_1 < n$ .

Sink for  $a < a_F$  and saddle for  $a > a_F$  with a flip bifurcation at  $a = a_F$ .

(d)  $\alpha < \varepsilon$ .

(d.1)  $0 < n < \varepsilon - \alpha$ .

Source for any  $a$ .

(d.2)  $\varepsilon - \alpha < n < \varepsilon - \varepsilon_2$ .

Source for any  $a$ .

(d.3)  $\varepsilon - \varepsilon_2 < n < \varepsilon - \varepsilon_1$ .

Source for  $a < a_F$ , saddle for  $a > a_F$  with a flip bifurcation at  $a = a_F$ .

(d.4)  $\varepsilon - \varepsilon_1 < n$ .

Sink for  $a < a_F$  and saddle for  $a > a_F$  with a flip bifurcation at  $a = a_F$ .

From (78), we get (56) with  $A(0^+) = \infty$  and  $A'(n) < 0$ .

Define  $\varepsilon_0 \equiv 0$  and  $\varepsilon_3 \equiv \alpha$ , and  $n_i \equiv \varepsilon - \varepsilon_i$  for  $i = 0, \dots, 3$ .

Since  $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3$ , we have  $n_3 < n_2 < n_1 < n_0$ .

Define  $A_i \equiv A(n_i)$  for  $i = 0, \dots, 3$  provided that  $n_i > 0$ , that is  $\varepsilon_i < \varepsilon$ .

Notice that

$$A_0 = \frac{1}{\gamma} \left[ \frac{1}{N} \left( 1 + \frac{1}{\varepsilon} \right) \right]^\alpha > 0$$

Since  $A'(n) < 0$ , we obtain the following ranking of critical values of the TFP:  $A_0 < A_1 < A_2 < A_3$ , where each inequality  $A_i < A_{i+1}$  makes sense only if  $\varepsilon_{i+1} < \varepsilon$  (indeed  $n_i, n_{i+1} > 0$  is equivalent to  $\varepsilon_i, \varepsilon_{i+1} < \varepsilon$  and, since  $\varepsilon_i < \varepsilon_{i+1}$ , to  $\varepsilon_{i+1} < \varepsilon$ ).

If  $n_i > 0$  (that is  $\varepsilon > \varepsilon_i$ ), we have  $n < \varepsilon - \varepsilon_i$  if and only if  $n < n_i$ , that is if and only if  $A_i < A$ .

Thus, we obtain the following.

(a)  $0 < \varepsilon < \varepsilon_1$ .

Sink for  $a < a_F$  and saddle for  $a_F < a$  with a flip bifurcation at  $a = a_F$ .

(b)  $\varepsilon_1 < \varepsilon < \varepsilon_2$ .

(b.1)  $A_1 < A < \infty$ .

Source for  $a < a_F$  and saddle for  $a_F < a$  with a flip bifurcation at  $a = a_F$ .

(b.2)  $A < A_1$ .

Sink for  $a < a_F$  and saddle for  $a_F < a$  with a flip bifurcation at  $a = a_F$ .

(c)  $\varepsilon_2 < \varepsilon < \alpha$ .

(c.1)  $A_2 < A < \infty$ .

Source for any  $a$ .

(c.2)  $A_1 < A < A_2$ .

Source for  $a < a_F$  and saddle for  $a > a_F$  with a flip bifurcation at  $a = a_F$ .

(c.3)  $A < A_1$ .

Sink for  $a < a_F$  and saddle for  $a > a_F$  with a flip bifurcation at  $a = a_F$ .

(d)  $\alpha < \varepsilon$ .

(d.1)  $A_3 < A < \infty$ .

Source for any  $a$ .

(d.2)  $A_2 < A < A_3$ .

Source for any  $a$ .

(d.3)  $A_1 < A < A_2$ .

Source for  $a < a_F$  and saddle for  $a > a_F$  with a flip bifurcation at  $a = a_F$ .

(d.4)  $A < A_1$ .

Sink for  $a < a_F$  and saddle for  $a > a_F$  with a flip bifurcation at  $a = a_F$ .

Summing up, we obtain the following.

(a)  $0 < \varepsilon < \varepsilon_1$ .

Sink for  $a < a_F$  and saddle for  $a_F < a$  with a flip bifurcation at  $a = a_F$ .

(b)  $\varepsilon_1 < \varepsilon < \varepsilon_2$ .

In this case,  $0 < A_0 < A_1$ .

If  $A < A_1$ , sink for  $a < a_F$  and saddle for  $a_F < a$  with a flip bifurcation at  $a = a_F$ .

If  $A_1 < A < \infty$ , source for  $a < a_F$  and saddle for  $a_F < a$  with a flip bifurcation at  $a = a_F$ .

(c)  $\varepsilon_2 < \varepsilon < \alpha$ .

In this case,  $0 < A_0 < A_1 < A_2$ .



If  $A < A_1$ , sink for  $a < a_F$  and saddle for  $a > a_F$  with a flip bifurcation at  $a = a_F$ .

If  $A_1 < A < A_2$ , source for  $a < a_F$  and saddle for  $a > a_F$  with a flip bifurcation at  $a = a_F$ .

If  $A_2 < A < \infty$ , source for any  $a$ .

(d)  $\alpha < \varepsilon$ .

In this case,  $0 < A_0 < A_1 < A_2 < A_3$ .

If  $A < A_1$ , sink for  $a < a_F$  and saddle for  $a > a_F$  with a flip bifurcation at  $a = a_F$ .

If  $A_1 < A < A_2$ , source for  $a < a_F$  and saddle for  $a > a_F$  with a flip bifurcation at  $a = a_F$ .

If  $A_2 < A < A_3$ , source for any  $a$ .

If  $A_3 < A < \infty$ , source for any  $a$ .

Therefore, we conclude as follows.

(1)  $0 < \varepsilon < \varepsilon_1$ .

Sink for  $a < a_F$  and saddle for  $a_F < a$  with a flip bifurcation at  $a = a_F$ .

(2)  $\varepsilon_1 < \varepsilon < \varepsilon_2$ . Then,  $0 < A_1$ .

(2.1) If  $A < A_1$ , sink for  $a < a_F$  and saddle for  $a_F < a$  with a flip bifurcation at  $a = a_F$ .

(2.2) If  $A_1 < A$ , source for  $a < a_F$  and saddle for  $a_F < a$  with a flip bifurcation at  $a = a_F$ .

(3)  $\varepsilon_2 < \varepsilon$ . Then,  $0 < A_1 < A_2$ .

(3.1) If  $A < A_1$ , sink for  $a < a_F$  and saddle for  $a > a_F$  with a flip bifurcation at  $a = a_F$ .

(3.2) If  $A_1 < A < A_2$ , source for  $a < a_F$  and saddle for  $a > a_F$  with a flip bifurcation at  $a = a_F$ .

(3.3) If  $A_2 < A$ , source for any  $a$ .

We compute the critical regeneration rate to have a flip bifurcation.

It corresponds to the intersection between the lines (77) and  $D = -T - 1$ . Replacing (75) and (76) in  $D(a) = -T(a) - 1$ , we find

$$a_F \equiv \frac{1}{M} \frac{2}{n - \varepsilon + \varepsilon_2} \quad (79)$$

where  $M$  and  $n$  no longer depend on  $a$ . Since

$$M \equiv \frac{N^\varepsilon}{n}, \quad n \equiv \frac{N}{\bar{N} - N} \quad \text{and} \quad N = (\gamma A)^{-\frac{1}{\alpha}}$$

we get (57).

$a_F$  is well-defined. Indeed, according to (79),  $a_F > 0$  if and only if  $n > \varepsilon - \varepsilon_2$ . This inequality corresponds exactly to the cases (a), (b), (c.2), (c.3), (d.3) and (d.4) above where a flip bifurcation exists.

Proposition 8 follows. ■

**Proof of Proposition 9**

We linearize system (58)-(59) around the steady state:

$$\begin{bmatrix} \frac{dk_{t+1}}{k} \\ \frac{dN_{t+1}}{N} \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ -P & \tilde{Q} \end{bmatrix} \begin{bmatrix} \frac{dk_t}{k} \\ \frac{dN_t}{N} \end{bmatrix}$$

where  $P$  and  $\tilde{Q}$  are given by (52) and (53). The trace and the determinant coincide with (73) and (74). Therefore, the bifurcation points are the same of the original system under full capital depreciation ( $\delta = 1$ ). ■

## 7 References

Antoci A., Galeotti M., Russu P. (2011). Poverty trap and global indeterminacy in a growth model with open-access natural resources. *Journal of Economic Theory* 146, 569-591, DOI: 10.1016/j.jet.2010.12.003.

Ayong Le Kama A. (2001). Sustainable growth, renewable resources and pollution. *Journal of Economic Dynamics and Control* 25, 1911-1918, DOI: 10.1016/S0165-1889(00)00007-5.

Beltratti A., Chichilnisky G., Heal G. (1994). Sustainable growth and the Green Golden Rule. In: Goldin I., Winters L. A. (eds.), *The Economics of Sustainable Development*, Cambridge University Press, Cambridge.

Bosi S., Desmarchelier D. (2018). Natural cycles and pollution. *Mathematical Social Sciences* 96, 10-20, DOI: 10.1016/j.mathsocsci.2018.08.005.

Bosi S., Desmarchelier D., Ragot L. (2018). Pollution effects on preferences: a unified approach. *Journal of Public Economic Theory* 21, 371-399, DOI: 10.1111/jpet.12348.

Caravaggio A., Sodini M. (2018). Nonlinear dynamics in coevolution of economic and environmental systems. *Frontiers in Applied Mathematics and Statistics* 4, 26, DOI: 10.3389/fams.2018.00026.

Grandmont J.-M., Pintus P., de Vilder R. (1998). Capital-labor substitution and competitive nonlinear endogenous business cycles. *Journal of Economic Theory* 80, 14-59, DOI: 10.1006/jeth.1997.2383.

John A., Pecchenino R. (1994). An overlapping generations model of growth and the environment. *Economic Journal* 104, 1393-1410, DOI: 10.2307/2235455.

Ramsey F. (1928). A mathematical theory of saving. *Economic Journal* 38, 543-559, DOI: 10.2307/2224098.

Samuelson P. A. (1942). A method of determining explicitly the coefficients of the characteristic equation. *Annals of Mathematical Statistics* 13, 424-429, DOI: 10.1214/aoms/1177731540.

Seegmuller T., Verchère A. (2007). A note on indeterminacy in overlapping generations economies with environment and endogenous labor supply. *Macroeconomic Dynamics* 11, 423-429, DOI: 10.1017/S1365100506060093.

Verhulst P.-F. (1838). Notice sur la loi que la population suit dans son accroissement. *Correspondance mathématique et physique* 10, 113-121.

Wirl F. (2004). Sustainable growth, renewable resources and pollution: thresholds and cycles. *Journal of Economic Dynamics and Control* 28, 1149-1157, DOI: 10.1007/s001910050035.

Xepapadeas A. (2005). Economic growth and the environment. In: *Handbook of Environmental Economics* 3, 1219-1271, DOI: 10.1016/S1574-0099(05)03023-8.