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# Nature as a productive externality\*

Stefano BOSI, Thai HA-HUY

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## Abstract

In this paper, we consider an AK model augmented with a productive externality of nature and a process of natural capital accumulation where a reproduction force is moderated by the negative impact of productive pollution. Differently from the existing literature, more focused on the planner's solution, we consider a market economy and the effects of its imperfections.

We introduce two distinct natural accumulation processes: a power law, more usual in economics, and a logistic law, more popular in ecology.

In both cases, we observe the occurrence of cycles of period two, cycle of period a power of two and chaos through a flip bifurcation and period-doubling bifurcations. In the case of the power law, dynamics are even richer because of the possibility of limit cycles through a Neimark-Sacker bifurcation.

In the case of a power law, cycles occur under a larger Total Factor Productivity because the TFP amplifies the negative impact of pollution on natural regeneration. In the case of a logistic law, cycles take place when the saturation effect of nature as a larger negative impact on the reproduction process.

**Keywords:** AK model, reproduction law, pollution, two-period and limit cycles.

**JEL codes:** C61, E32, 044.

## 1 Introduction

Change.

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Et vu que si  $b=0$ , l'économie converge, dont on peut conclure qu'avec  $b$  suffisamment petit, le système est supercritique?

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Avant de ré-soumettre, pourrait-t-on ajouter quelques commentaires sur le fait que les valeurs de bifurcation ne dépendent pas du paramètre de pollution,  $\beta$ ? De mon avis, bien que  $\beta$  n'a pas de l'influence sur les seuils de bifurcation, mais ce paramètre décide la "largeur" des cycles? Et si  $\beta$  converge à zéro, les cycles convergent vers l'état stationnaire?

\* \* \*

Nature is a good. It can be viewed either as a production factor or as a consumption good. In fact, nature contributes to production in terms of renewable and non-renewable resources, and ecosystemic services; the nature experience gives us pleasure; finally, the health of nature is also our health from a one health perspective.

Nature can be introduced in economic models, namely in dynamic general equilibrium theory. The simplest way to give value to nature is to bring it in the production function or the utility function and, depending on the decision maker (producer, consumer, social planner), the stock of natural capital can be considered an externality or a choice.

Exploiting nature or polluting, reduces the stock of nature. However, when nature is renewable, its stock grows either because of a natural reproduction or artificial maintenance.

Because of all these possible actions and feedbacks, and the urgency of environmental issues, the number of economic models involving nature is large and intended to grow.

The interaction between natural resources and economic activity has been modelled by theorists. The seminal paper with a process of renewable resource in a Ramsey model is Beltratti et al. (1994). Natural capital is both a factor in the production function and a good in the utility function. In their paper, pollution coming from consumption activity reduces the stock of natural resources. These authors prove the existence of a unique and stable steady state (saddle point). Assuming instead that pollution comes from production activity, Ayong Le Kama (2001) obtains also the uniqueness and the saddle-path stability of the steady state.

In Wirl (2004), pollution affects the utility function and the reproduction of the natural resource is represented by a logistic function. He proves the existence of two steady states and the occurrence of a limit cycle through a Hopf bifurcation around the lower steady state. The existence of endogenous cycles entails intergenerational inequalities: some generations experience a higher level of natural resource while others, a lower one.

Beltratti et al. (1994), Ayong Le Kama (2001) and Wirl (2004) focus only on the central planner's solution which internalizes the pollution externalities.

In our paper, in order to shed light on some basic mechanism and to avoid unnecessary complexity, we focus on the productive effects of nature and the negative impact of productive pollution on the reproduction process of nature, viewed as a renewable resource. The reference model is a market economy à la Ramsey-Cass-Koopmans in discrete time, augmented with a law of nature

accumulation incorporating both the natural regeneration and the pollution process coming from production activities. Since the basic RCK model is two-dimensional, nature accumulation adds a third dimension to nonlinear dynamics and generates more complex trajectories.

We consider two distinct natural accumulation processes: a power law, more standard in economics, and a logistic law, more usual in biology, demography and ecology, discovered by Verhulst (1838).

Our paper differs from Beltratti et al. (1994), Ayong Le Kama (2001), Wirl (2004) in many respects. We consider a market economy where nature and pollution are externalities, that is market imperfections. Differently from Beltratti et al. (1994), Ayong Le Kama (2001), Wirl (2004) and Bosi and Desmarchelier (2018), we consider a more general logistic law and an alternative process, a power law, to represent the regeneration mechanism of nature. Finally, our discrete-time time approach gives rise to richer dynamics through period-doubling and, even, chaotic bifurcations.

More precisely, in both cases, we observe the occurrence of cycles of period two, cycles of period a power of two and chaos through a flip bifurcation and period-doubling bifurcations. Moreover, in the case of a power law, dynamics turns out to be even richer because of the possibility of limit cycles through a Neimark-Sacker bifurcation, which is the discrete-time equivalent of the Hopf bifurcation obtained in continuous time by Wirl (2004) and Bosi and Desmarchelier (2018). Moreover, differently from them, in our case, the steady state remains unique.

In our model, in the case of a power law, cycles occur under a larger TFP because the TFP amplifies the negative impact of pollution on natural regeneration. In the case of a logistic law, cycles take place when the saturation effect of nature as a larger negative impact on the reproduction process.

The rest of the paper is organized as follows. In Section 2, we introduce the fundamentals of a market economy. In Section 3, we study the equilibrium when the natural accumulation process is a power law. In section 4, we study the equilibrium when the reproduction function becomes logistic. Section 5 concludes. All the proofs are gathered in the appendix.

## 2 Fundamentals

Nature  $N_t$  is a positive externality which increases the productivity  $AN_t^\alpha$  of the capital stock  $K_t$ , with  $0 < \alpha < 1$ :

$$F(K_t, N_t) = AN_t^\alpha K_t = \sum_j AN_t^\alpha k_{jt}$$

where  $k_{jt}$  denotes the capital in the firm  $j$ .

Producers are small and price-takers. Profit maximization implies, at equilibrium:

$$r_t = AN_t^\alpha \tag{1}$$

where  $r_t$  denotes the interest rate. Because of the linearity of this AK model, the solution to individual profit maximization is equivalent to that of an aggregate firm.

The representative consumer maximizes a logarithmic utility

$$\sum_{t=0}^{\infty} \beta^t \ln c_t$$

under the budget constraint:

$$c_t + k_{t+1} \leq r_t k_t \quad (2)$$

where, without loss of generality, the capital is supposed to fully depreciate at every period.

Utility maximization entails the intertemporal consumption smoothing:

$$\frac{c_{t+1}}{c_t} = \beta r_{t+1} \quad (3)$$

The size of population is normalized to one. The aggregate capital coincides with the aggregate one:  $k_t = K_t$ .

Putting together (1), (2), (3), we obtain the two-dimensional Ramsey block of the dynamic system.

$$\begin{aligned} \frac{c_{t+1}}{c_t} &= \beta AN_{t+1}^\alpha \\ c_t + k_{t+1} &= AN_t^\alpha k_t \end{aligned}$$

The law of nature accumulation adds a third dimension and complete this dynamic system. We consider two alternative laws of nature accumulation: (1) power; (2) generalized logistic.

(1) Power law:

$$N_{t+1} = aN_t^\varepsilon - bAN_t^\alpha K_t \quad (4)$$

with  $a, b > 0$  and  $0 < \varepsilon < 1$ , where  $a$  is the regeneration rate and  $b$  is the pollution rate.

In a world with no humans,  $K_t = 0$  and  $N_{t+1} = aN_t^\varepsilon$ . The natural dynamics

$$N_t = a^{\frac{1-\varepsilon^t}{1-\varepsilon}} N_0^{\varepsilon^t}$$

converge to the steady state in the long run:

$$\lim_{t \rightarrow \infty} \left( a^{\frac{1-\varepsilon^t}{1-\varepsilon}} N_0^{\varepsilon^t} \right) = a^{\frac{1}{1-\varepsilon}}$$

For instance, if  $a = \varepsilon = 1/2$  and  $N_0 = 1/16$ , dynamics are represented in Figure 1 with  $N_\infty = 1/4$ .

(2) Generalized logistic law:

$$N_{t+1} - N_t = aN_t^\varepsilon (\bar{N} - N_t) - bAN_t^\alpha K_t \quad (5)$$

with  $0 < a < 1$ ,  $0 \leq \varepsilon \leq 1$  and  $b, \bar{N} > 0$ .

In a world with no humans,  $K_t = 0$  and  $N_{t+1} - N_t = aN_t^\varepsilon (\bar{N} - N_t)$ .

When  $\varepsilon = 0$ , we obtain a linear law  $N_{t+1} - N_t = a(\bar{N} - N_t)$  and the natural dynamics

$$N_t = (1 - a)^t N_0 + \left[1 - (1 - a)^t\right] \bar{N}$$

converge to the steady state  $\lim_{t \rightarrow \infty} N_t = \bar{N}$  in the long run. For instance, if  $a = 1/2$  and  $N_0 = 1/8$  and  $\bar{N} = 1/4$ , dynamics are represented in Figure 2 with  $N_\infty = 1/4$ .

When  $\varepsilon = 1$ , we get the pure logistic law, often considered in biology to represent population dynamics.

It is interesting to compare processes (4) and (5).

We observe that (4) is equivalent to  $N_{t+1} - N_t = aN_t^\varepsilon - N_t - bAN_t^\alpha K_t$ . Thus, the two processes write

$$\begin{aligned} N_{t+1} - N_t &= \Phi_P(N_t) - bAN_t^\alpha K_t \\ N_{t+1} - N_t &= \Phi_L(N_t) - bAN_t^\alpha K_t \end{aligned}$$

where

$$\begin{aligned} \Phi_P(N_t) &\equiv aN_t^\varepsilon - N_t \\ \Phi_L(N_t) &\equiv aN_t^\varepsilon (\bar{N} - N_t) \end{aligned}$$

are both concave functions since

$$\begin{aligned} \Phi_P''(N_t) &= a\varepsilon(\varepsilon - 1)N_t^{\varepsilon-2} < 0 \\ \Phi_L''(N_t) &= a\varepsilon(\varepsilon - 1)N_t^{\varepsilon-2}(\bar{N} - N_t) - 2a\varepsilon N_t^{\varepsilon-1} < 0 \end{aligned}$$

with  $\Phi_P(0) = \Phi_L(0) = 0$ ,  $\Phi_P'(0) = \Phi_L'(0) = +\infty$  and

$$\Phi_P\left(a^{\frac{1}{1-\varepsilon}}\right) = \Phi_L(\bar{N}) = 0$$

Thus, they look similar. However, as we will see, the power law is richer in terms of nonlinear dynamics because of the possibility of limit cycles.

## 2.1 Particular solution

Consider the system

$$\begin{aligned} \frac{c_{t+1}}{c_t} &= \beta AN_{t+1}^\alpha \\ c_t + k_{t+1} &= AN_t^\alpha k_t \\ N_{t+1} &= \Phi_i(N_t) - bAN_t^\alpha k_t + N_t \end{aligned}$$

where  $i = P, L$ . If

$$c_t = (1 - \beta) AN_t^\alpha k_t \tag{6}$$

then  $k_{t+1} = AN_t^\alpha k_t - c_t = \beta AN_t^\alpha k_t$  and

$$\begin{aligned} c_{t+1} &= \beta AN_{t+1}^\alpha c_t = \beta AN_{t+1}^\alpha k_{t+1} \frac{c_t}{k_{t+1}} = \beta AN_{t+1}^\alpha k_{t+1} \frac{(1-\beta) AN_t^\alpha k_t}{\beta AN_t^\alpha k_t} \\ &= (1-\beta) AN_{t+1}^\alpha k_{t+1} \end{aligned}$$

Thus, if  $c_0 = (1-\beta) AN_0^\alpha k_0$ , the first-order conditions imply  $c_t = (1-\beta) AN_t^\alpha k_t$  and  $k_{t+1} = \beta AN_t^\alpha k_t$  for any  $t \geq 0$ .

In other terms, the starting point  $(k_0, N_0, c_0) = (k_0, N_0, (1-\beta) AN_0^\alpha k_0)$  implies a reduced two-dimensional dynamics for the pair  $(k_t, N_t)$ :

$$k_{t+1} = \beta AN_t^\alpha k_t \quad (7)$$

$$N_{t+1} = \Phi_i(N_t) - bAN_t^\alpha k_t + N_t \quad (8)$$

### 3 Power law

The dynamic system becomes

$$\frac{c_{t+1}}{c_t} = \beta AN_{t+1}^\alpha \quad (9)$$

$$c_t + k_{t+1} = AN_t^\alpha k_t \quad (10)$$

$$N_{t+1} = aN_t^\varepsilon - bAN_t^\alpha k_t \quad (11)$$

More explicitly,

$$c_{t+1} = \beta A (aN_t^\varepsilon - bAN_t^\alpha k_t)^\alpha c_t$$

$$k_{t+1} = AN_t^\alpha k_t - c_t$$

$$N_{t+1} = aN_t^\varepsilon - bAN_t^\alpha k_t$$

#### 3.1 Steady state

Let

$$A_0 \equiv \frac{1}{\beta a^{\frac{\alpha}{1-\varepsilon}}} \quad (12)$$

a critical TFP value in the isoelastic case.

The following assumption ensures the positivity of variables at the steady state.

**Assumption 1**  $A > A_0$ .

At the steady state, we recover the modified golden rule.

**Proposition 1** *The steady state of dynamic system (9)-(11) is given by:*

$$AN^\alpha = \frac{1}{\beta} \quad (13)$$

$$k = \frac{\beta}{b} (aN^\varepsilon - N) > 0 \quad (14)$$

$$c = \frac{1-\beta}{\beta} k > 0 \quad (15)$$

The steady state is unique.  
More explicitly,

$$N = \left( \frac{1}{A\beta} \right)^{\frac{1}{\alpha}} \quad (16)$$

$$k = \frac{\beta}{b} \left[ a \left( \frac{1}{A\beta} \right)^{\frac{\varepsilon}{\alpha}} - \left( \frac{1}{A\beta} \right)^{\frac{1}{\alpha}} \right] \quad (17)$$

$$c = \frac{1-\beta}{b} \left[ a \left( \frac{1}{A\beta} \right)^{\frac{\varepsilon}{\alpha}} - \left( \frac{1}{A\beta} \right)^{\frac{1}{\alpha}} \right] \quad (18)$$

### 3.2 Local dynamics

The following lemma provides the three eigenvalues of local dynamics around the steady state.

**Lemma 2** *Dynamic system (9)-(11) is locally represented by the following Jacobian matrix:*

$$\begin{bmatrix} \frac{dk_{t+1}}{dk_t} \\ \frac{dN_{t+1}}{dN_t} \\ \frac{dc_{t+1}}{dc_t} \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta} & \frac{\alpha}{\beta} & \frac{\beta-1}{\beta} \\ -P & Q & 0 \\ -\alpha P & \alpha Q & 1 \end{bmatrix} \begin{bmatrix} \frac{dk_t}{k_t} \\ \frac{dN_t}{N_t} \\ \frac{dc_t}{c_t} \end{bmatrix} \quad (19)$$

where

$$P \equiv \frac{b}{\beta} \frac{k}{N} \quad (20)$$

$$Q \equiv \varepsilon + (\varepsilon - \alpha) P \quad (21)$$

The eigenvalues are given by

$$\lambda_1 = \frac{1}{2} \left( 1 + Q - \sqrt{(1-Q)^2 - 4\alpha P} \right) \quad (22)$$

$$\lambda_2 = \frac{1}{2} \left( 1 + Q + \sqrt{(1-Q)^2 - 4\alpha P} \right) \quad (23)$$

$$\lambda_3 = \frac{1}{\beta} > 1$$

Our model represents a dynamic market economy. Dynamics are driven by a sequence of general equilibrium prices  $(r_t)_{t=0}^{\infty}$ . It is known that the equilibrium is unique in the Ramsey model. Our model is an extension of the Ramsey model where equilibrium uniqueness is locally preserved (local determinacy). Indeed, the dynamic system is three-dimensional. Since  $\lambda_3 > 1$ , the stable manifold (union of converging paths to the steady state) is not full-dimensional: its dimension is less than three. We observe that the initial conditions  $k_0$  and  $N_0$  are given (predetermined variables), while  $c_0$  is an individual choice (non-predetermined variable). If the stable manifold is two-dimensional and  $(k_0, N_0)$



lies in a neighborhood of the steady state  $(k, N)$ , there is a unique value of  $c_0$  such that  $(k_0, N_0, c_0)$  belongs to this stable manifold. In this case, the equilibrium path  $(k_t, N_t, c_t)_{t=0}^{\infty}$  is a locally unique (locally determined) saddle-path: it starts from  $(k_0, N_0, c_0)$  and converges to the unique steady state  $(k, N, c)$ . When a supercritical Neimark-Sacker bifurcation occurs, the steady state loses its saddle-path stability (all the eigenvalues lie outside the unit circle) and a stable limit cycle arises around  $(\lambda_3$  remains real, while the other two eigenvalues are nonreal and conjugate). The new stable manifold is the union of converging paths to this limit cycle. As above, there is a unique  $c_0$  such that  $(k_0, N_0, c_0)$  belongs to this stable manifold. The equilibrium path is locally unique (locally determined), it diverges from the steady state and converges to the limit cycle around.

Let

$$\varepsilon_1 \equiv \frac{\alpha}{\alpha + 4} < \varepsilon_2 \equiv \frac{\alpha}{2}$$

We introduce the following critical values of the TFP:

$$A_N \equiv \frac{1}{\beta (a\varepsilon)^{\frac{\alpha}{1-\varepsilon}}} > A_0 \quad (24)$$

$$A_F \equiv \frac{1}{\beta \left( a \frac{\alpha-2\varepsilon}{\alpha+2} \right)^{\frac{\alpha}{1-\varepsilon}}} \quad (25)$$

with  $A_F > A_0$  if  $\varepsilon < \varepsilon_2$  and  $A_F > A_N$  if  $\varepsilon_1 < \varepsilon < \varepsilon_2$ .

The following proposition characterizes generically the local bifurcations of our economy around the unique steady state (16)-(18).

**Proposition 3** *Let  $\varepsilon \neq a$ .*

(a) *If  $0 < \varepsilon < \varepsilon_1$ ,  $A_F > A_0$  is a flip bifurcation value. More precisely,  $\lambda_1$  and  $\lambda_2$  are in the unit circle for  $A \in (A_0, A_F)$  and both outside for  $A > A_F$ . The system generically undergoes a flip bifurcation at  $A = A_F$  and a period-two cycle arises around the steady state.*

(b) *If  $\varepsilon_1 < \varepsilon < \varepsilon_2$ ,  $A_N > A_0$  is a Neimark-Sacker bifurcation value and  $A_F > A_N$  is a flip bifurcation value. More precisely,  $\lambda_1$  and  $\lambda_2$  are in the unit circle for  $A \in (A_0, A_N)$ , both outside for  $A \in (A_N, A_F)$ , one inside and the other outside for  $A > A_F$ . The system generically undergoes a Neimark-Sacker bifurcation at  $A = A_N$  and a limit cycle arises around the steady state. The system generically undergoes a flip bifurcation at  $A = A_F$  and a two-period cycle arises around the steady state.*

(c) *If  $\varepsilon_2 < \varepsilon < 1$ ,  $A_N > A_0$  is a Neimark-Sacker bifurcation value. More precisely,  $\lambda_1$  and  $\lambda_2$  are in the unit circle for  $A \in (A_0, A_N)$  and both outside for  $A > A_N$ . The system generically undergoes a Neimark-Sacker bifurcation at  $A = A_N$  and a limit cycle arises around the steady state.*

In case (a) but we do not know whether the flip bifurcation is sub or supercritical, that is we do not know whether an unstable cycle exists below  $A_F$  or

a stable cycle above  $A_F$ . In case (b), we do not know whether the Neimark-Sacker bifurcation is sub or supercritical, that is we do not know whether an unstable cycle exists below  $A_N$  or a stable cycle above  $A_N$ . Moreover, we do not know whether the flip bifurcation is super or subcritical, that is we do not know whether a stable cycle exists below  $A_F$  or an unstable cycle above  $A_F$ . In case (c), we do not know whether the Neimark-Sacker bifurcation is sub or supercritical, that is we do not know whether an unstable cycle exists below  $A_N$  or a stable cycle above  $A_N$ . However, numerical simulations allow us to illustrate the stability properties (sub and supercriticality) of two-period and limit cycles.

Let us provide the main lines of the proof of Proposition 3. The reader is referred to the appendix for a complete proof.

The proof rests on a two-dimensional geometrical method introduced by Samuelson (1942) and developed later by Grandmont et al. (1998). To apply their method, focus now on the two-dimensional subspace spanned by the eigenvectors associated to  $\lambda_1$  and  $\lambda_2$  regardless of the one-dimensional unstable subspace associated to  $\lambda_3$ . We can study the stability properties considering  $T \equiv \lambda_1 + \lambda_2$  and  $D \equiv \lambda_1 \lambda_2$  instead of the eigenvalues  $\lambda_1$  and  $\lambda_2$ .

In the spirit of Samuelson (1942), we can represent the stability properties in the  $(T, D)$ -plane instead of the Argand plan (Fig. 3).

Notice that, here, "sink" refers to the two-dimensional subspace we are considering (in the original three-dimensional space, because of  $\lambda_3$ , in fact, this "sink" is a saddle with two eigenvalues inside the unit circle and one outside).

We evaluate the characteristic sub-polynomial  $\tilde{P}(\lambda) \equiv \lambda^2 - T\lambda + D$  at  $-1$  and  $1$ : along the line  $D = T - 1$ , one eigenvalue is equal to  $1$  because  $\tilde{P}(1) = 1 - T + D = 0$ ; along the line  $D = -T - 1$ , one eigenvalue is equal to  $-1$  because  $\tilde{P}(-1) = 1 + T + D = 0$ . Along the segment  $D = 1$  with  $|T| < 2$ , the eigenvalues are nonreal and conjugate with unit modulus. Focus on points outside these lines and the segment. Inside the triangle, the steady state is a sink ( $D < 1$  and  $|T| < 1 + D$ ). It is a saddle point if  $(T, D)$  lies on the left cone of the lines  $D = T - 1$  and  $D = -T - 1$ , or on the right cone of these lines ( $|1 + D| < |T|$ ). It is a source otherwise. A local bifurcation arises when one eigenvalue crosses the unit circle in the Argand plan or, equivalently, the pair  $(T, D)$  goes through one of the lines  $D = T - 1$  or  $D = -T - 1$ , or the segment  $D = 1$  with  $|T| \leq 2$ . When a bifurcation parameter changes,  $(T, D)$  moves in the  $(T, D)$ -plane. Generically, a generic saddle-node occurs when  $(T, D)$  crosses  $D = T - 1$ , a flip when  $(T, D)$  goes through the line  $D = -T - 1$ , whereas a Neimark-Sacker when  $(T, D)$  crosses the segment  $D = 1$  with  $|T| \leq 2$ .

In our case, we apply the method introduced by Grandmont et al. (1998) and based on the linearity property of the locus obtained in the  $(T, D)$ -plane when the TFP  $A$  varies, to have a complete characterization of the local bifurcations arising in our economy.

As explained in the appendix, the origins of the three half-lines in Figure 4 correspond to the starting value  $A_0$ , while their slopes are determined by  $\varepsilon$ . The intersections of the half-lines with the horizontal side of the triangle correspond to the critical value  $A_N$  of Neimark-Sacker bifurcation; the intersections of the

half-lines with the line  $D = -T - 1$  correspond to the critical value  $A_F$  of flip bifurcation.

Consider for instance the case (a) of Proposition 3, that is  $0 < \varepsilon < \varepsilon_1$  (lower half-line in the Figure 4), with  $A_0 < A < A_F$ . In this case, two eigenvalues are inside the unit circle and one outside. Then, the dimension of the stable manifold is two. Let  $(k_0, N_0)$  belong to a neighborhood of the steady state. Since  $k_0$  and  $N_0$  are predetermined, this means that there exists a unique value of the non-predetermined variable  $c_0$ , say  $c_0^*$ , such that  $(k_0, N_0, c_0^*)$  belongs to the stable manifold and, therefore, a unique equilibrium trajectory starting from  $(k_0, N_0, c_0^*)$  and converging to the steady state (equilibrium determinacy).

Since the system (7)-(8) jointly with (6) is a solution of the original dynamic system and the locally converging equilibrium is unique, the equilibrium starting point is precisely  $c_0^* = (1 - \beta) AN_0^\alpha k_0$  where  $k_0$  and  $N_0$  are given.

Because of equilibrium determinacy, we can use the system

$$k_{t+1} = \beta AN_t^\alpha k_t \quad (26)$$

$$N_{t+1} = aN_t^\varepsilon - bAN_t^\alpha k_t \quad (27)$$

$$c_t = (1 - \beta) AN_t^\alpha k_t$$

with  $k_0$  and  $N_0$  given and  $c_0 = c_0^* = (1 - \beta) AN_0^\alpha k_0$  to simulate the trajectory from  $(k_0, N_0, c_0^*)$  to the steady state and, when  $A$  is close to the critical values  $A_F$  or  $A_N$ , to a two-period cycle or to a limit cycle respectively.

**Proposition 4** *The two-dimensional sub-system (26)-(27) has the same bifurcations point of the original three-dimensional system (9)-(11).*

In particular, the same conclusions about the local uniqueness (local determinacy) of equilibrium paths converging to the steady state or to surrounding two-period and limit cycles hold.

Proposition 4 allows us to simulate the trajectory in a two-dimensional space and, interestingly, to know whether the cycles are stable or unstable. We calibrate the main parameters as follows:

Parameter	$a$	$b$	$\alpha$	$\beta$	$\varepsilon$	(28)
Value	1/2	1/2	2/3	9/10	1/12	

Figure 4 has been plotted with this calibration.

Fixing  $\varepsilon = 1/12 \in (0, \varepsilon_1)$ , we find the flip bifurcation value  $A_F = 6.2146$  and, therefore, the steady state  $(k, N) = (0.58967, 0.075599)$ . Using Python 3.9, we illustrate the case (a) of Proposition 3 in Figure 5 and show the convergence of  $k_t$  to a two-period cycle around the steady state from the starting point  $(k_0, N_0) = (0.1, 0.075)$  when  $A = A_F$ .

Then, fixing  $\varepsilon = 1/6 \in (\varepsilon_1, \varepsilon_2)$  jointly with calibration (28), we obtain the Neimark-Sacker bifurcation value  $A_N = 8.1115$  and, therefore, the stationary state  $(k, N) = (0.45627, 0.050697)$ . Figure 6 illustrates the case (b) of Proposition 3 and shows the limit cycle arising around the steady state through a

Neimark-Sacker bifurcation in the  $(k_t, N_t)$ -plane with  $(k_0, N_0) = (0.45, 0.05)$  as initial condition.

The TFP  $A$  plays the main role in the occurrence of cycles, which is twofold. On the one side, it captures the external effect of nature on production; on the other, the impact of pollution on the reproduction process. If  $A$  is sufficiently large, a cycle takes place (through a flip or a Neimark-Sacker bifurcation) because a raise in natural capital implies a higher production level and a larger pollution in turn, which entails a drop in natural capital in the end. Thus, an initial increase in the natural stock is followed by a decrease which, reversely, induces a new increase, and so on.

Finally, we observe that, even if the parameter  $b$  plays the main role in pollution dynamics according to (4), it does not enter the bifurcation values for  $A$  (see expressions (24) and (25)). However, it enters the expression for  $P$  and  $Q$ , and, then, it affects the size of the cycles through the eigenvalues (22) and (23).

## 4 Logistic law

Focus now on the second process of nature accumulation (5).

The dynamic system becomes

$$\frac{c_{t+1}}{c_t} = \beta AN_{t+1}^\alpha \quad (29)$$

$$c_t + k_{t+1} = AN_t^\alpha k_t \quad (30)$$

$$N_{t+1} - N_t = aN_t^\varepsilon (\bar{N} - N_t) - bAN_t^\alpha k_t \quad (31)$$

More explicitly,

$$c_{t+1} = \beta A [aN_t^\varepsilon (\bar{N} - N_t) - bAN_t^\alpha k_t + N_t]^\alpha c_t$$

$$k_{t+1} = AN_t^\alpha k_t - c_t$$

$$N_{t+1} = aN_t^\varepsilon (\bar{N} - N_t) - bAN_t^\alpha k_t + N_t$$

### 4.1 Steady state

Let

$$A_0 \equiv \frac{1}{\beta \bar{N}^\alpha}$$

be a critical TFP value under a logistic law. The following assumption ensures the positivity of variables at the steady state.

**Assumption 2**  $A > A_0$ .

At the steady state, we recover the modified golden rule.

**Proposition 5** *The steady state of dynamic system (29)-(31) is given by:*

$$AN^\alpha = \frac{1}{\beta} \quad (32)$$

$$k = \frac{a\beta}{b} N^\varepsilon (\bar{N} - N) > 0 \quad (33)$$

$$c = \frac{1-\beta}{\beta} k > 0 \quad (34)$$

*The steady state is unique.*

*In the isoelastic case,  $\alpha(A) = \alpha > 0$ , a constant, and the steady state is explicitly given by:*

$$N = \left( \frac{1}{A\beta} \right)^{\frac{1}{\alpha}} \quad (35)$$

$$k = \beta \frac{a}{b} \left( \frac{1}{A\beta} \right)^{\frac{\varepsilon}{\alpha}} \left[ \bar{N} - \left( \frac{1}{A\beta} \right)^{\frac{1}{\alpha}} \right] \quad (36)$$

$$c = (1-\beta) \frac{a}{b} \left( \frac{1}{A\beta} \right)^{\frac{\varepsilon}{\alpha}} \left[ \bar{N} - \left( \frac{1}{A\beta} \right)^{\frac{1}{\alpha}} \right] \quad (37)$$

## 4.2 Local dynamics

The following lemma provides the three eigenvalues of local dynamics around the steady state.

**Lemma 6** *Dynamic system (29)-(31) is locally represented by the following Jacobian matrix:*

$$\begin{bmatrix} \frac{dk_{t+1}}{dk_t} \\ \frac{dN_{t+1}}{dN_t} \\ \frac{dc_{t+1}}{dc_t} \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta} & \frac{\alpha}{\beta} & \frac{\beta-1}{\beta} \\ -P & \tilde{Q} & 0 \\ -\alpha P & \alpha \tilde{Q} & 1 \end{bmatrix} \begin{bmatrix} \frac{dk_t}{k_t} \\ \frac{dN_t}{N_t} \\ \frac{dc_t}{c_t} \end{bmatrix} \quad (38)$$

where

$$P \equiv \frac{b}{\beta} \frac{k}{N} \quad (39)$$

$$\tilde{Q} \equiv 1 + P \left( \varepsilon - \alpha - \frac{N}{\bar{N} - N} \right) \quad (40)$$

*The eigenvalues are given by:*

$$\lambda_1 = \frac{1}{2} \left[ 1 + \tilde{Q} - \sqrt{(1 - \tilde{Q})^2 - 4\alpha P} \right] \quad (41)$$

$$\lambda_2 = \frac{1}{2} \left[ 1 + \tilde{Q} + \sqrt{(1 - \tilde{Q})^2 - 4\alpha P} \right] \quad (42)$$

$$\lambda_3 = \frac{1}{\beta} > 1$$

Let

$$A(n) \equiv \frac{1}{\beta} \left( \frac{1+n}{n\bar{N}} \right)^\alpha \quad (43)$$

and

$$A_1 \equiv A(\varepsilon) < A_2 \equiv A(\max\{0, \varepsilon - \alpha/2\}) \leq A_3 \equiv A(\max\{0, \varepsilon - \alpha\})$$

Notice that  $A_2$  and  $A_3$  can be equal to  $\infty$  if, respectively,  $\varepsilon < \alpha/2$  and  $\varepsilon < \alpha$ . If  $A_2 = \infty$  then  $A_3 = \infty$ .

We introduce the following critical value for the regeneration rate:

$$a_F \equiv \frac{4(A\beta)^{\frac{\varepsilon}{\alpha}}}{2(1+\varepsilon) - \alpha + (\alpha - 2\varepsilon)\bar{N}(A\beta)^{\frac{1}{\alpha}}} \quad (44)$$

There are two predetermined variables ( $k_t$  and  $N_t$ ) and one non-predetermined ( $c_t$ ).

The following proposition characterizes generically the local bifurcations of our economy around the unique steady state.

**Proposition 7** *Let  $A > A_0$ . Then, a positive steady state exists.*

(a) *If  $A_0 < A < A_1$ , then the steady state is stable (two eigenvalues inside the unit circle, one outside) for  $a \in (0, a_F)$  and unstable for  $a > a_F$  (one eigenvalue inside the unit circle, two outside). At  $a = a_F$ , generically, the economic system undergoes a flip bifurcation generating a two-period cycle. If the cycle exists for  $a < a_F$ , it is unstable (subcritical). If it exists for  $a > a_F$ , it is stable (supercritical).*

(b) *If  $A_1 < A < A_2$ , then the steady state is unstable (three eigenvalues outside the unit circle for  $a \in (0, a_F)$  and unstable for  $a > a_F$  (one eigenvalue inside the unit circle, two outside). At  $a = a_F$ , generically, the economic system undergoes a flip bifurcation generating a two-period cycle. If the cycle exists for  $a > a_F$ , it is stable (supercritical).*

(c) *If  $A_2 < A < A_3$  or  $A_3 < A < \infty$ , then the steady state is unstable (three eigenvalues outside). There is no room for local bifurcations.*

As above, we apply the geometrical method introduced by Samuelson (1942) and developed by Grandmont et al. (1998).

We plot the half-lines corresponding to the cases of Proposition 7.

Figure 7 shows the intersections where the flip bifurcations arise: the half-line in the bottom corresponds to case (a); the half-line in the middle, to case (b); the half-line in the top, to case (c).

As above, we observe that, if  $A_0 < A < A_1$  (case (a)) and  $0 < a < a_F$ , two eigenvalues are inside the unit circle and one outside. Then, the dimension of the stable manifold is two. Let  $(k_0, N_0)$  be in a neighborhood of the steady state. Since  $k_0$  and  $N_0$  are predetermined, this means that there exists a unique value of the non-predetermined variable  $c_0$ , say  $c_0^*$ , such that  $(k_0, N_0, c_0^*)$  belongs to the stable manifold and, therefore, a unique equilibrium trajectory starting from  $(k_0, N_0, c_0^*)$  and converging to the steady state (equilibrium determinacy).

Since the system (7)-(8) jointly with (6) is a solution of the original dynamic system and the locally converging equilibrium is unique, the equilibrium starting point is precisely  $c_0^* = (1 - \beta) AN_0^\alpha k_0$  where  $k_0$  and  $N_0$  are given.

Because of equilibrium determinacy, we can use the system

$$k_{t+1} = \beta AN_t^\alpha k_t \quad (45)$$

$$N_{t+1} = aN_t^\varepsilon (\bar{N} - N_t) - bAN_t^\alpha k_t + N_t \quad (46)$$

$$c_t = (1 - \beta) AN_t^\alpha k_t$$

with  $k_0$  and  $N_0$  given and  $c_0 = c_0^* = (1 - \beta) AN_0^\alpha k_0$  to simulate the trajectory from  $(k_0, N_0, c_0^*)$  to the steady state and, when  $a$  is close to the critical values  $a_F$ , to a two-period cycle.

**Proposition 8** *The two-dimensional sub-system (45)-(46) has the same bifurcations point of the original three-dimensional system (29)-(31).*

In particular, the same conclusions about the local uniqueness (local determinacy) of equilibrium paths converging to the steady state or to surrounding two-period cycles hold.

A numerical example allows us to illustrate Proposition 7. As above, we calibrate the model to simulate the economic dynamics:

Parameter	$A$	$b$	$\alpha$	$\beta$	$\varepsilon$
Value	1.43	1/2	1/3	9/10	2/3

We normalize the maximal natural stock:  $\bar{N} = 1$ . Figure 7 has been plotted with this calibration.

We have  $D = sT + 1 - 2s$  with

$$s = \frac{\varepsilon - n}{\varepsilon - \alpha - n} \text{ and } n \equiv \frac{N}{\bar{N} - N} = \frac{1}{\bar{N} (A\beta)^{\frac{1}{\alpha}} - 1}$$

Then

$$D = 1 + (T - 2) \frac{1458A^3 - 5000}{729A^3 - 4000} \quad (47)$$

Using (43), we get:  $A_0 = 1.1111$ ,  $A_1 = 1.508$ ,  $A_2 = 1.6025$ ,  $A_3 = 1.7638$ . To capture the three cases of proposition 7, fix  $A = B_1 = 1.43$ ,  $A = B_2 = 1.52$ ,  $A = B_3 = 1.68$ . We observe that

$$\begin{aligned} A_0 &= 1.1111 < B_1 = 1.43 < A_1 = 1.508 < B_2 = 1.52 \\ &< A_2 = 1.6025 < B_3 = 1.68 < A_3 = 1.7638 \end{aligned}$$

Finally, remember that  $(T(0), D(0)) = (2, 1)$ .

We obtain the flip bifurcation value  $a_F = 7.6308$  and, therefore, the steady state  $(k, N) = (4.4025, 0.46910)$ . Using Python 3.9, we illustrate the case (a) of Proposition 7 in Figure 8 and show the convergence of  $k_t$  from the initial

condition  $(k_0, N_0) = (4.4, 0.47)$  to an attractive two-period cycle around the steady state when  $a = a_F$ . Therefore, the cycle is stable (supercritical).

In order to interpret the occurrence of a two-period cycles, we refer to the reproduction law:  $N_{t+1} = aN_t^\varepsilon (\bar{N} - N_t) - bAN_t^\alpha k_t + N_t$ . We observe that the parameter  $a$  plays now the main role in the occurrence of cycles. They require a sufficiently large value. While in the case of a power law,  $a$  appears in the terms  $aN_t^\varepsilon$  as a monotonic force, now, in the case of a logistic law, it appears in the  $aN_t^\varepsilon (\bar{N} - N_t)$  as a return force because of  $-N_t$ . Indeed, when  $N_t$  is sufficiently large,  $N_t^\varepsilon (\bar{N} - N_t)$  decreases with  $N_t$  and this negative impact is magnified by the size of  $a$  with, possibly, a final negative impact on  $N_{t+1}$ .

**As in the case of a power law, even if the parameter  $b$  plays the main role in pollution dynamics according to (5), it does not enter the flip bifurcation value (44) for  $a$ . However, it enters the expressions for  $P$  and  $\hat{Q}$ , and, then, it affects the size of the cycles.**

## 5 Conclusion

In this paper, we have considered the productive effects of nature and the negative impact of productive pollution on the regeneration process of nature in a discrete-time AK model.

We have studied two distinct natural accumulation processes: a power law, more standard in economics, and a logistic law, more popular in biology.

In both cases, we have observed the occurrence of cycles of period two, cycle of period a power of two and chaos through a flip bifurcation and period-doubling bifurcations. In the case of the power law, we have shown that dynamics are even richer because of the possibility of limit cycles through a Neimark-Sacker bifurcation.

In our model, in the case of a power law, cycles occur under a larger TFP because the TFP amplifies the negative impact of pollution on natural regeneration. In the case of a logistic law, cycles take place when the saturation effect of nature as a larger negative impact on the reproduction process.

## 6 Appendix

### Proof of Proposition 1

Solve system (9)-(11) with  $(k_{t+1}, N_{t+1}, c_{t+1}) = (k_t, N_t, c_t) = (k, N, c)$  for any  $t \geq 0$  to obtain (13), (14) and (15).

Since there is a unique stationary level of nature, the steady state is unique.

Notice that  $k, c > 0$  if and only if

$$N < a^{\frac{1}{1-\varepsilon}}$$

that is, if and only if  $A > A_0$ , which is true if the TFP  $A$  is sufficiently high. In this case, more explicitly, we obtain (16), (17) and (18). ■

### Proof of Lemma 2



Noticing that  $\beta AN^\alpha = 1$ ,  $c/k = AN^\alpha - 1$ ,  $aN^\varepsilon = N + bAN^\alpha k$ , the linearization yields:

$$-\alpha \frac{dN_{t+1}}{N} + \frac{dc_{t+1}}{c} = \frac{dc_t}{c} \quad (48)$$

$$\beta \frac{dk_{t+1}}{k} = \frac{dk_t}{k} + \alpha \frac{dN_t}{N} - (1 - \beta) \frac{dc_t}{c} \quad (49)$$

$$\frac{dN_{t+1}}{N} = -P \frac{dk_t}{k} + Q \frac{dN_t}{N} \quad (50)$$

that is (19), where  $P$  and  $Q$  are given by (20) and (21).

We compute the sums of principal minors:

$$S_1 = \lambda_1 + \lambda_2 + \lambda_3 = \frac{1 + \beta}{\beta} + Q$$

$$S_2 = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 = \frac{1}{\beta} + \alpha P + \frac{1 + \beta}{\beta} Q$$

$$S_3 = \lambda_1 \lambda_2 \lambda_3 = \frac{\alpha P + Q}{\beta}$$

where  $S_1$  is the trace and  $S_3$  the determinant of the Jacobian matrix.

The characteristic polynomial is given by:

$$\begin{aligned} P(\lambda) &= \lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 \\ &= \lambda^3 - \left( \frac{1 + \beta}{\beta} + Q \right) \lambda^2 + \left( \frac{1}{\beta} + \alpha P + \frac{1 + \beta}{\beta} Q \right) \lambda - \frac{\alpha P + Q}{\beta} \end{aligned}$$

We observe that  $P(1/\beta) = 0$ . Then,  $1/\beta$  is an eigenvalue, say,  $\lambda_3$ .

Therefore, we have  $S_1 = \lambda_1 + \lambda_2 + 1/\beta$  and  $S_3 = \lambda_1 \lambda_2 / \beta$ , that is  $\lambda_1 + \lambda_2 = S_1 - 1/\beta = 1 + Q$  and  $\lambda_1 \lambda_2 = \beta S_3 = \alpha P + Q$ .

Focus on the sub-polynomial:

$$\begin{aligned} \tilde{P}(\lambda) &= (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1 \lambda_2 \\ &= \lambda^2 - (1 + Q)\lambda + \alpha P + Q = 0 \end{aligned} \quad (51)$$

Thus, the other two eigenvalues are given by (22) and (23). ■

### Proof of Proposition 3

We observe that, in our model,  $T = 1 + Q$ ,  $D = \alpha P + Q$  and  $Q \equiv \varepsilon + (\varepsilon - \alpha)P$ . Then,

$$T = 1 + \varepsilon + (\varepsilon - \alpha)P \quad (52)$$

$$D = \varepsilon + \varepsilon P \quad (53)$$

with

$$P \equiv \frac{b}{\beta} \frac{k}{N} = a(A\beta)^{\frac{1-\varepsilon}{\alpha}} - 1 \quad (54)$$

From (52) and (53), we get

$$\begin{aligned} T(P) &= 1 + \varepsilon + (\varepsilon - \alpha)P \\ D(P) &= \frac{\varepsilon}{\varepsilon - \alpha}T(P) - (1 + \alpha)\frac{\varepsilon}{\varepsilon - \alpha} \end{aligned} \quad (55)$$

In the spirit of Grandmont et al. (1998), we exploit the linearity property of the locus (55) to have a complete characterization of the local bifurcations arising in our economy.

We know that, according to (12),  $P > 0$  if and only if  $A > A_0$ . Given the other parameters,  $P$  moves from 0 to  $\infty$  as  $A$  moves from  $A_0$  to  $\infty$ .

Consider the half-line  $\{(T(P), D(P))\}_{P>0}$  in the  $(T, D)$ -plane when  $P$  moves from 0 to  $\infty$ .

Its starting point is

$$(T(0), D(0)) = (1 + \varepsilon, \varepsilon)$$

on the line  $D = T - 1$ , with  $0 < \varepsilon < 1$ .

Consider the slope

$$s \equiv \frac{\varepsilon}{\varepsilon - \alpha} = \frac{1}{1 - \frac{\alpha}{\varepsilon}}$$

of the line

$$D = \frac{\varepsilon}{\varepsilon - \alpha}T - (1 + \alpha)\frac{\varepsilon}{\varepsilon - \alpha} \quad (56)$$

Since  $\alpha, \varepsilon \in (0, 1)$  with  $\alpha \neq \varepsilon$ , we have  $\alpha/\varepsilon \in (0, \infty)$  and  $s \in (-\infty, 0) \cup (1, \infty)$ .

There are two cases: (1)  $\varepsilon < \alpha$ , (2)  $\varepsilon > \alpha$ .

(1)  $\varepsilon < \alpha$ . In this case,  $s \in (-\infty, 0)$ ,  $T'(P) < 0$  and  $D'(P) > 0$ .

Let  $s_1 \in (-\infty, 0)$  be the critical slope such that the half-line  $\{(T(P), D(P))\}_{P>0}$  passes through  $(-2, 1)$ , that is, according to (56):

$$\varepsilon = \varepsilon_1 \equiv \frac{\alpha}{\alpha + 4}$$

or, equivalently,

$$s = s_1 \equiv \frac{\varepsilon_1}{\varepsilon_1 - \alpha} = -\frac{1}{\alpha + 3}$$

(1.1) If  $s \in (-\infty, -1)$ , there exists a Neimark-Sacker bifurcation value such that  $\lambda_1$  and  $\lambda_2$  are in the unit circle for  $A \in (A_0, A_N)$  and outside for  $A > A_N$ . The system generically undergoes a Neimark-Sacker bifurcation at  $A = A_N$  where the half-line  $\{(T(P), D(P))\}_{P>0}$  crosses  $D = 1$  and a limit cycle arises around the steady state (but we do not know whether the Neimark-Sacker bifurcation is sub or supercritical, so we do not know whether an unstable cycle exists under  $A_N$  or a stable cycle beyond  $A_N$ ).

According to (53) and (54), the critical point  $A_N$  is solution to the following equation:

$$1 = D = \varepsilon + \varepsilon P = \varepsilon a (A\beta)^{\frac{1-\varepsilon}{\alpha}}$$

(1.2) If  $s \in (-1, s_1)$ , there exists a Neimark-Sacker bifurcation value  $A_N$  (expression (24)) and a flip bifurcation value  $A_F > A_N$  such that  $\lambda_1$  and  $\lambda_2$  are in the unit circle for  $A \in (A_0, A_N)$ , both outside for  $A \in (A_N, A_F)$ , one inside and the other outside if  $A > A_F$ . The system generically undergoes a Neimark-Sacker bifurcation at  $A = A_N$  and a limit cycle arises around the steady state (but we do not know whether the Neimark-Sacker bifurcation is sub or supercritical, so we do not know whether an unstable cycle exists below  $A_N$  or a stable cycle above  $A_N$ ). The system generically undergoes a flip bifurcation at  $A = A_F$  and a two-period cycle arises around the steady state (but we do not know whether the flip bifurcation is super or subcritical, so we do not know whether a stable cycle exists below  $A_F$  or an unstable cycle above  $A_F$ ).

The flip bifurcation value  $A_F$  corresponds to the intersection of the half-line  $\{(T(P), D(P))\}_{P>0}$  with the line  $D = -T - 1$ . Replacing (52) and (53) in  $D = -T - 1$ , and using (54), we get  $A_F$  as solution to the following equation:

$$P = \frac{2 + 2\varepsilon}{\alpha - 2\varepsilon} = a(A\beta)^{\frac{1-\varepsilon}{\alpha}} - 1$$

Since

$$s \equiv \frac{\varepsilon}{\varepsilon - \alpha} \in (-1, s_1) = \left(-1, -\frac{1}{\alpha + 3}\right)$$

we have

$$\varepsilon_1 \equiv \frac{\alpha}{\alpha + 4} < \varepsilon < \frac{\alpha}{2} \equiv \varepsilon_2$$

Then,  $A_F$  is well-defined and  $A_F > A_N > A_0$ .

(1.3) If  $s \in (s_1, 0)$ , there exists a flip bifurcation value  $A_F$  (expression (25)) such that  $\lambda_1$  and  $\lambda_2$  are in the unit circle for  $A \in (A_0, A_F)$  and one inside and one outside for  $A > A_F$ . The system generically undergoes a flip bifurcation at  $A = A_F$  and a period-two cycle arises around the steady state (but we do not know whether the flip bifurcation is sub or supercritical, so we do not know whether an unstable cycle exists below  $A_F$  or a stable cycle above  $A_F$ ).

(2)  $\varepsilon > \alpha$ .

In this case,  $s \in (1, \infty)$ ,  $T'(P) > 0$  and  $D'(P) > 0$ .

There exists a Neimark-Sacker bifurcation value  $A_N$  (expression (24)) such that  $\lambda_1$  and  $\lambda_2$  are in the unit circle for  $A \in (A_0, A_N)$  and both outside for  $A > A_N$ . The system generically undergoes a Neimark-Sacker bifurcation at  $A = A_N$  and a limit cycle arises around the steady state (but we do not know whether the Neimark-Sacker bifurcation is sub or supercritical, so we do not know whether an unstable cycle exists below  $A_N$  or a stable cycle above  $A_N$ ).

We observe that:

- (a) if  $0 < \varepsilon < \varepsilon_1$ , then  $s \in (s_1, 0)$ ;
- (b) if  $\varepsilon_1 < \varepsilon < \varepsilon_2$ , then  $s \in (-1, s_1)$ ;
- (c) if  $\varepsilon_2 < \varepsilon < 1$ , then  $s \in (-\infty, -1)$ .

Proposition 3 follows. ■

**Proof of Proposition 4**

We linearize system (26)-(27) around the steady state:

$$\begin{bmatrix} \frac{dk_{t+1}}{k} \\ \frac{dN_{t+1}}{N} \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ -P & Q \end{bmatrix} \begin{bmatrix} \frac{dk_t}{k} \\ \frac{dN_t}{N} \end{bmatrix}$$

where  $P$  and  $Q$  are given by (20) and (21). The trace and the determinant are given by  $T = 1 + Q$  and  $D = \alpha P + Q$ , and the characteristic polynomial by

$$P(\lambda) = \lambda^2 - T\lambda + D = \lambda^2 - (1 + Q)\lambda + \alpha P + Q$$

that is (51). Therefore, the bifurcation points are the same of the original system. ■

**Proof of Proposition 5**

Solve system (29)-(31) with  $(k_{t+1}, N_{t+1}, c_{t+1}) = (k_t, N_t, c_t) = (k, N, c)$  for any  $t \geq 0$  to obtain (32), (33) and (34).

Since there is a unique stationary level of nature, the steady state is unique.

Notice that  $k, c > 0$  if and only if  $N < \bar{N}$  and, in the isoelastic case:  $A(N) = AN^\alpha$ , if and only if  $A > A_0$ , which is true if the TFP  $A$  is sufficiently high. In this case, more explicitly, we obtain (35), (36) and (37). ■

**Proof of Lemma 6**

We linearize the three-dimensional dynamic system around the steady state. Using since  $\beta A(N) = 1$ ,  $c/k = A(N) - 1$ ,  $aN^{\varepsilon-1}(\bar{N} - N) = bA(N)k/N$ , we obtain

$$\begin{aligned} -\alpha \frac{dN_{t+1}}{N} + \frac{dc_{t+1}}{c} &= \frac{dc_t}{c} \\ \beta \frac{dk_{t+1}}{k} &= \frac{dk_t}{k} + \alpha \frac{dN_t}{N} - (1 - \beta) \frac{dc_t}{c} \\ \frac{dN_{t+1}}{N} &= -P \frac{dk_t}{k} + \tilde{Q} \frac{dN_t}{N} \end{aligned}$$

This system is similar to system (48)-(50). The only difference is that, now,  $\tilde{Q}$  replace  $Q$  and the steady state values are given by (35)-(37) instead by (16)-(18). In particular, we observe that  $\lambda_3$  is still equal to  $1/\beta$  and, thus, it lies outside the unit circle.

The proof of Lemma 2 identically applies to obtain the analogous results of Lemma 6. ■

**Proof of Proposition 7**

As above, we can study the stability properties in the  $(T, D)$ -plane instead of considering  $\lambda_1$  and  $\lambda_2$ . The reader is still referred to Figure 3.

We observe that

$$T = 1 + \tilde{Q} \tag{57}$$

$$D = \alpha P + \tilde{Q} \tag{58}$$

where  $\tilde{Q} \equiv 1 + P(\varepsilon - \alpha - n)$  and

$$n \equiv \frac{N}{\bar{N} - N} > 0$$

Then,  $T = 2 + P(\varepsilon - \alpha - n)$  and  $D = 1 + P(\varepsilon - n)$ . We notice that

$$N = A^{-1} \left( \frac{1}{\beta} \right)$$

does not depend on  $a$ . Moreover,

$$P \equiv \frac{b}{\beta} \frac{k}{N} = aM$$

where, under Assumption 2,  $M \equiv N^\varepsilon/n > 0$  does not depend on  $a$ .

We have  $T(a) = 2 + aM(\varepsilon - n - \alpha)$  and  $D(a) = 1 + aM(\varepsilon - n)$ . Then,  $aM(\varepsilon - n - \alpha) = T - 2$  and

$$D = 1 + (T - 2) \frac{\varepsilon - n}{\varepsilon - n - \alpha}$$

that is  $D = sT + 1 - 2s$  where

$$s \equiv \frac{dD}{dT} = \frac{D'(a)}{T'(a)} = \frac{\varepsilon - n}{\varepsilon - \alpha - n}$$

is the slope of the half-line  $\{(T(a), D(a))\}_{a \geq 0}$ .

The starting point of the half-line ( $a = 0$ ) is  $(T(0), D(0)) = (2, 1)$  that is the right corner of the triangle in Figure 3.

We observe that, in the isoelastic case,

$$n \equiv \frac{N}{\bar{N} - N} = \frac{1}{\bar{N}(A\beta)^{\frac{1}{\alpha}} - 1} = \frac{1}{\left(\frac{A}{A_0}\right)^{\frac{1}{\alpha}} - 1}$$

Since  $A > A_0$  (Assumption 2) and  $A_0$  does not depend on  $A$ , we have  $n \in (0, \infty)$  as  $A$  moves from  $A_0$  to  $\infty$ .

We notice that  $T'(a) = M(\varepsilon - \alpha - n)$ ,  $D'(a) = M(\varepsilon - n)$  and

$$n = \varepsilon + \alpha \frac{s}{1 - s}$$

Consider the three cases: (1)  $\varepsilon < \alpha/2$ , (2)  $\alpha/2 < \varepsilon < \alpha$  and (3)  $\varepsilon > \alpha$ .

(1)  $\varepsilon < \alpha/2$ .

(1.1)  $0 < n < \varepsilon$ .

Then,  $T'(a) < 0$ ,  $D'(a) > 0$  and  $s < 0$ .  $0 < n < \varepsilon$  is equivalent to

$$-1 < -\frac{\varepsilon}{\alpha - \varepsilon} < s < 0$$

Then, flip at  $a = a_F$ .

(1.2)  $\varepsilon < n$ .

Then,  $T'(a) < 0$ ,  $D'(a) < 0$  and  $s > 0$ .  $\varepsilon < n$  is equivalent to  $0 < s < 1$ .

Then, flip at  $a = a_F$ .

(2)  $\alpha/2 < \varepsilon < \alpha$ .

(2.1)  $0 < n < \varepsilon - \alpha/2$ .

Then,  $T'(a) < 0$ ,  $D'(a) > 0$  and  $s < 0$ .  $0 < n < \varepsilon - \alpha/2$  is equivalent to

$$-\frac{\varepsilon}{\alpha - \varepsilon} < s < -1$$

Then, no bifurcations.

(2.2)  $\varepsilon - \alpha/2 < n < \varepsilon$ .

Then,  $T'(a) < 0$ ,  $D'(a) > 0$  and  $s < 0$ .  $\varepsilon - \alpha/2 < n < \varepsilon$  is equivalent to  $-1 < s < 0$ .

Then, flip at  $a = a_F$ .

(2.3)  $\varepsilon < n$ .

Then,  $T'(a) < 0$ ,  $D'(a) < 0$  and  $s > 0$ .  $\varepsilon < n$  is equivalent to  $0 < s < 1$ .

Then, flip at  $a = a_F$ .

(3)  $\alpha < \varepsilon$ .

(3.1)  $0 < n < \varepsilon - \alpha$ .

We have  $T'(a) > 0$ ,  $D'(a) > 0$  and  $s > 0$ .  $0 < n < \varepsilon - \alpha$  is equivalent to

$$1 < \frac{\varepsilon}{\varepsilon - \alpha} < s$$

No bifurcations at all.

(3.2)  $\varepsilon - \alpha < n < \varepsilon - \alpha/2$ .

We have  $T'(a) < 0$ ,  $D'(a) > 0$  and  $s < 0$ .  $\varepsilon - \alpha < n < \varepsilon - \alpha/2$  is equivalent to  $s < -1$ .

Then, no bifurcations at all.

(3.3)  $\varepsilon - \alpha/2 < n < \varepsilon$ .

We have  $T'(a) < 0$ ,  $D'(a) > 0$  and  $s < 0$ .  $\varepsilon - \alpha/2 < n < \varepsilon$  is equivalent to  $-1 < s < 0$ .

Then, flip at  $a = a_F$ .

(3.4)  $\varepsilon < n$ .

We have  $T'(a) < 0$ ,  $D'(a) < 0$  and  $s > 0$ .  $\varepsilon < n$  is equivalent to  $0 < s < 1$ .

Flip at  $a = a_F$ .

Summing up, we have the following.

(1)  $\varepsilon < \alpha/2$ .

(1.1)  $0 < n < \varepsilon \Leftrightarrow -1 < -\varepsilon/(\alpha - \varepsilon) < s < 0$  with  $T'(a) < 0$  and  $D'(a) > 0$ .

(1.2)  $\varepsilon < n \Leftrightarrow 0 < s < 1$  with  $T'(a) < 0$  and  $D'(a) < 0$ .

(2)  $\alpha/2 < \varepsilon < \alpha$ .

(2.1)  $0 < n < \varepsilon - \alpha/2 \Leftrightarrow -\varepsilon/(\alpha - \varepsilon) < s < -1$  with  $T'(a) < 0$  and  $D'(a) > 0$ .

(2.2)  $\varepsilon - \alpha/2 < n < \varepsilon \Leftrightarrow -1 < s < 0$  with  $T'(a) < 0$  and  $D'(a) > 0$ .

(2.3)  $\varepsilon < n \Leftrightarrow 0 < s < 1$  with  $T'(a) < 0$  and  $D'(a) < 0$ .

(3)  $\alpha < \varepsilon$ .

(3.1)  $0 < n < \varepsilon - \alpha \Leftrightarrow 1 < \varepsilon/(\varepsilon - \alpha) < s$  with  $T'(a) > 0$  and  $D'(a) > 0$ .

(3.2)  $\varepsilon - \alpha < n < \varepsilon - \alpha/2 \Leftrightarrow s < -1$  with  $T'(a) < 0$  and  $D'(a) > 0$ .

(3.3)  $\varepsilon - \alpha/2 < n < \varepsilon \Leftrightarrow -1 < s < 0$  with  $T'(a) < 0$  and  $D'(a) > 0$ .

(3.4)  $\varepsilon < n \Leftrightarrow 0 < s < 1$  with  $T'(a) < 0$  and  $D'(a) < 0$ .

Thus,

- (a)  $0 < n < \max\{0, \varepsilon - \alpha\} \Rightarrow 1 < s$  with  $T'(a) > 0$  and  $D'(a) > 0$ .
- (b)  $\max\{0, \varepsilon - \alpha\} < n < \max\{0, \varepsilon - \alpha/2\} \Rightarrow s < -1$  with  $T'(a) < 0$  and  $D'(a) > 0$ .
- (c)  $\max\{0, \varepsilon - \alpha/2\} < n < \varepsilon \Rightarrow -1 < s < 0$  with  $T'(a) < 0$  and  $D'(a) > 0$ .
- (d)  $\varepsilon < n \Rightarrow 0 < s < 1$  with  $T'(a) < 0$  and  $D'(a) < 0$ .

We observe that  $n$  is strictly increasing with  $N$  and, when  $N$  goes from 0 to  $\bar{N}$ , then  $n$  goes from 0 to  $\infty$ . But  $N$  is strictly decreasing with  $A$  and, when  $A$  goes from  $A_0$  to  $\infty$ , then  $N$  goes from  $\bar{N}$  to 0.

Then, using

$$n \equiv \frac{N}{\bar{N} - N} > 0 \text{ and } N = \left( \frac{1}{A\beta} \right)^{\frac{1}{\alpha}}$$

we get

$$A(n) \equiv \frac{1}{\beta} \left( \frac{1+n}{n\bar{N}} \right)^{\alpha}$$

with  $A'(n) < 0$ .

Let  $n_3 \equiv \max\{0, \varepsilon - \alpha\} \leq n_2 \equiv \max\{0, \varepsilon - \alpha/2\} < \varepsilon$  and  $A_1 \equiv A(\varepsilon) < A_2 \equiv A(n_2) \leq A_3 \equiv A(n_3)$ .

Notice that  $A_2$  and  $A_3$  can be infinite.

We obtain

- (a)  $0 < n < n_3 \Rightarrow 1 < s$  with  $T'(a) > 0$  and  $D'(a) > 0$ .
- (b)  $n_3 < n < n_2 \Rightarrow s < -1$  with  $T'(a) < 0$  and  $D'(a) > 0$ .
- (c)  $n_2 < n < \varepsilon \Rightarrow -1 < s < 0$  with  $T'(a) < 0$  and  $D'(a) > 0$ .
- (d)  $\varepsilon < n \Rightarrow 0 < s < 1$  with  $T'(a) < 0$  and  $D'(a) < 0$ .

Thus,

- (d)  $A < A_1 \Rightarrow 0 < s < 1$  with  $T'(a) < 0$  and  $D'(a) < 0 \Rightarrow$  flip at  $a = a_F$ .
- (c)  $A_1 < A < A_2 \Rightarrow -1 < s < 0$  with  $T'(a) < 0$  and  $D'(a) > 0 \Rightarrow$  flip at  $a = a_F$ .

(b)  $A_2 < A < A_3 \Rightarrow s < -1$  with  $T'(a) < 0$  and  $D'(a) > 0 \Rightarrow$  no local bifurcation.

(a)  $A_3 < A < \infty \Rightarrow 1 < s$  with  $T'(a) > 0$  and  $D'(a) > 0 \Rightarrow$  no local bifurcation.

We compute the critical regeneration rate to have a flip bifurcation.

It corresponds to the intersection between the lines  $D = sT + 1 - 2s$  and  $D = -T - 1$ , that is  $D(a) = -T(a) - 1$  where  $T(a) = 2 + aM(\varepsilon - n - \alpha)$  and  $D(a) = 1 + aM(\varepsilon - n)$ .

We get

$$a_F = \frac{4}{M[\alpha - 2(\varepsilon - n)]} \quad (59)$$

where  $M$  and  $n$  no longer depend on  $a$ .

Since

$$M \equiv \frac{N^\varepsilon}{n}, \quad n \equiv \frac{N}{\bar{N} - N} \text{ and } N = \left( \frac{1}{A\beta} \right)^{\frac{1}{\alpha}}$$

we get (44).

$a_F$  is well-defined. Indeed, according to (59),  $a_F > 0$  if and only if  $n > \varepsilon - \alpha/2$ . Since  $n_2 \equiv \max\{0, \varepsilon - \alpha/2\}$ , this inequality corresponds exactly to the case (c) and (d) above where a flip bifurcation exists.

Proposition 7 follows. ■

**Proof of Proposition 8**

We linearize system (45)-(46) around the steady state:

$$\begin{bmatrix} \frac{dk_{t+1}}{k} \\ \frac{dN_{t+1}}{N} \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ -P & \tilde{Q} \end{bmatrix} \begin{bmatrix} \frac{dk_t}{k} \\ \frac{dN_t}{N} \end{bmatrix}$$

where  $P$  and  $\tilde{Q}$  are given by (39) and (40). The trace and the determinant coincide with (57) and (58). Therefore, the bifurcation points are the same of the original system. ■

## 7 References

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