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BLOCH-BEILINSON CONJECTURES FOR HECKE CHARACTERS AND EISENSTEIN COHOMOLOGY OF PICARD SURFACES

JITENDRA BAJPAI AND MATTIA CAVICCHI

ABSTRACT. We consider certain families of Hecke characters ϕ over a quadratic imaginary field F . The order of vanishing of the L -function $L(\phi, s)$ at the central point $s = -1$, according to the Beilinson conjectures, should be equal to the dimension of the space of extensions of the Tate motive $\mathbb{Q}(1)$ by the motive associated with ϕ .

In this article, candidates for the corresponding extensions of Hodge structure are constructed, under the assumptions that the sign of the functional equation of $L(\phi, s)$ is -1 and that $L'(\phi, -1) \neq 0$. This is achieved by means of the cohomology of variations of Hodge structures over Picard modular surfaces attached to F and Harder's theory of Eisenstein cohomology. Moreover, we provide a criterion for the non-triviality of these extensions, based on the non-vanishing of a height pairing that we define and study.

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1. INTRODUCTION

Let E be an elliptic curve over \mathbb{Q} and denote by r the rank of the finitely generated abelian group $E(\mathbb{Q})$. Suppose that the sign of the functional equation of the (completed) L -function of E is -1 . One of the striking consequences of the celebrated Gross-Zagier formula ([10]), in this case, is that if $L'(E, 1) \neq 0$, then $r \geq 1$. In fact, Gross and Zagier manage to exhibit an explicit element, *of infinite order*, of $E(\mathbb{Q})$. Since by the hypothesis on the sign, $L(E, s)$ vanishes at the central point $s = 1$, this

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obviously agrees with the Birch and Swinnerton-Dyer conjecture, which predicts in particular that r should be precisely equal to the order of vanishing of $L(E, s)$ at $s = 1$.

In this article, we are interested in certain conjectural analogues of the latter results, when $L(E, s)$ is replaced by the L -function of some algebraic Hecke characters ϕ of a quadratic imaginary field F , of odd weight w . To such a ϕ , one can associate a Chow motive M_ϕ over \mathbb{Q} of weight w , whose L -function coincides with that of ϕ , denoted $L(\phi, s)$ from now on; notice that the central point for $L(\phi, s)$ is then $s = \frac{w+1}{2}$. This construction gives rise in particular to a pure Hodge structure H_ϕ of weight w .

The starting point allowing to single out an object, which plays the role of $E(\mathbb{Q})$ in this new context, is the following observation. According to Deligne ([8]), the group $E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is canonically isomorphic to

$$\mathrm{Ext}_{\mathrm{MM}_1(\mathbb{Q})}^1 \left(\mathbf{1}, H^1(E)(1) \right)$$

where $\mathrm{MM}_1(\mathbb{Q})$ is the abelian category of *1-motifs* over \mathbb{Q} with rational coefficients. The above-mentioned consequence of Gross-Zagier formula can be then expressed by saying that if the sign of the functional equation of $L(E, s)$ is -1 , and if $L'(E, 1) \neq 0$, then

$$(1) \quad \dim_{\mathbb{Q}} \mathrm{Ext}_{\mathrm{MM}_1(\mathbb{Q})}^1 \left(\mathbf{1}, H^1(E)(1) \right) \geq 1$$

The expected Hodge-theoretic analogue of this latter statement for Hecke characters is the following:

Conjecture 1. *Let $\mathrm{MHS}_{\overline{\mathbb{Q}}}$ be the category of mixed \mathbb{Q} -Hodge structures with coefficients¹ in $\overline{\mathbb{Q}}$. Let ϕ be a Hecke character of the quadratic imaginary field F , of odd weight w . If the sign of the functional equation of $L(\phi, s)$ is -1 , and if $L'(\phi, \frac{w+1}{2}) \neq 0$, then*

$$(2) \quad \dim_{\mathbb{Q}} \mathrm{Ext}_{\mathrm{MHS}_{\overline{\mathbb{Q}}}}^1 \left(\mathbf{1}, H_\phi \left(\frac{w+1}{2} \right) \right) \geq 1$$

This statement is predicted by Beilinson's conjectures, which propose a vast generalization of properties of the kind of (1) in terms of *mixed motives* (cfr. Scholl's reformulation of these conjectures in [26, Conjecture B, p. 379]), and by the conjectural injectivity, according to Bloch and Beilinson, of certain *Abel-Jacobi maps* ([4, 5.6]). In fact, the latter basically implies that the expected non-trivial extensions of motives should give rise to non-trivial extensions of Hodge structures, *since we are working over a number field*.

The ultimate aim of our work is to prove the above conjecture for some special Hecke characters ϕ of F of weight -3 : namely, those of *infinity type* $(k, -(k+3))$, with k a positive integer, which have the property that their restriction $\phi_{\mathbb{Q}}$ to \mathbb{Q} is such that

$$(3) \quad \phi_{\mathbb{Q}} = \epsilon_{F|\mathbb{Q}} \cdot | \cdot |_{\mathbb{Q}}^3$$

¹See section 3 for a general definition of mixed Hodge structures with coefficients and our conventions about the category $\mathrm{MHS}_{\overline{\mathbb{Q}}}$.

where $\epsilon_{F|\mathbb{Q}}$ is the quadratic character attached to the extension $F|\mathbb{Q}$ and $|\cdot|_{\mathbb{Q}}$ denotes the norm on the idèles of \mathbb{Q} . Let us explain our plan of attack, thereby shedding light on the reason for these specific assumptions.

Showing (2) for such a ϕ means constructing, under the stated hypotheses, a non-trivial extension E of $\mathbf{1}$ by $H_\phi(-1)$ in $\text{MHS}_{\overline{\mathbb{Q}}}$. The strategy that we adopt is based on ideas and results by Harder ([12, 13]) and makes use of the fact that starting from the Hecke character ϕ , one can construct a class in the boundary cohomology, with coefficients in an appropriate local system, of a *Picard modular surface* S_K , that is, a non-projective Shimura variety, attached to a \mathbb{Q} -algebraic group G such that $G(\mathbb{R}) \simeq \text{SU}(2,1)$. This cohomology class is furthermore lifted, through Langlands' theory of Eisenstein operators, to a class in the cohomology of S_K itself, as an Eisenstein series whose constant term is controlled by the behaviour of $L(\phi, s)$ at $s = -1$. This is where we need the special shape of our character and the *vanishing* hypothesis on $L(\phi, s)$. Thanks to the *sign* hypothesis, Rogawski's study of the automorphic spectrum of unitary groups in three variables ([21, 22, 23]) allows us to find the other piece for our candidate extension inside the *interior cohomology* of Picard surfaces. Then we use the computations of degenerating variations of Hodge structures on S of [1] and the localization exact sequence, involving interior cohomology and boundary cohomology, in order to prove our first result:

Theorem 1. *Let k be an integer ≥ 0 and let ϕ be a Hecke character of the quadratic imaginary field F of infinity type $(k, -(k+3))$. Suppose that the restriction $\phi_{\mathbb{Q}}$ to \mathbb{Q} has the shape (3). If the sign of the functional equation of $L(\phi, s)$ is -1 , and if $L'(\phi, 1) \neq 0$, then there exists an element of*

$$(4) \quad \text{Ext}_{\text{MHS}_{\overline{\mathbb{Q}}}}^1(\mathbf{1}, H_\phi(-1))$$

of geometric origin.

Theorem 1 is the consequence of Theorem 6.2 and Corollary 6.3, and actually, our method gives rise to a family of extensions E_x lying in the space (4), depending on the choice of a cohomology class x in a subspace $I_{\phi, \Theta}^K$ of boundary cohomology of S_K , intuitively defined by a condition of *support* in a subspace Θ of the boundary of the *Baily-Borel compactification* of S_K . These extensions should be seen as the Hodge-theoretic analogues of the extensions of Galois representations constructed in [3], under the hypothesis of *odd* order of vanishing of $L(\phi, s)$ at the central point, for families of Hecke characters ϕ of F which are slightly different from ours (see also [14] for the case of *even* order of vanishing). The source of the Galois representations considered in these works is again the cohomology of Picard modular surfaces, but in order to obtain the desired extensions, the authors use p -adic families of automorphic forms, without looking at the geometry of the boundary of the Baily-Borel compactification. It would be interesting to study the Galois-theoretic counterpart of our constructions, via étale cohomology, and to put it in relation with their methods.

Our second contribution is to take inspiration from the ideas of Scholl on *height pairings*, from [25], to provide the following construction, which we state in a slightly informal way at this point:

Theorem 2. *There exist subspaces $I_{\phi, \Theta}^K, I_{\theta\phi, \Sigma}^K$ of boundary cohomology of S_K , and a bilinear pairing*

$$b(x, y) : I_{\phi, \Theta}^K \times (I_{\theta\phi, \Sigma}^K)^\vee \rightarrow \mathbb{C}$$

with the property that, given $x \in I_{\phi, \Theta}^K$, if there exists $y \in (I_{\theta\phi, \Sigma}^K)^\vee$ such that $b(x, y) \neq 0$, then the extension E_x is non trivial.

For the precise statement of this result and its proof, see Theorem 7.5. We provide then a method to check the non-vanishing of this pairing in Proposition 7.8. Ongoing work of the authors shows that making this method explicit leads to computations in Lie algebra cohomology with coefficients in the principal series of $SU(2, 1)$, and that by employing the detailed description of these principal series representation available in [29], the value $L'(\phi, -1)$ (non-zero by hypothesis) will appear in a crucial way in the final expression for $b(x, y)$. This is what makes it possible to show the non-triviality of E_x for a careful choice of x and to give a proof for Conjecture 1. Details will appear in the forthcoming article [2].

1.1. Content of the article. In Section 2 we gather all the preliminary facts that we need about the structure of the group G , its parabolic subgroups, its representation theory, and about its associated locally symmetric spaces, i.e. the Picard surfaces S_K , together with their compactifications (Baily-Borel, denoted by S_K^* , and Borel-Serre, denoted by \bar{S}_K). Then, Section 3 is devoted to some formal aspects of mixed Hodge structures with coefficients which will be needed in the sequel. In Section 4, we recall the known results about the computation of boundary cohomology of Picard surfaces ([12]) and its associated Hodge structure, by means of S_K^* ([1]), adding the complements we need. We proceed then by recalling in detail the contents of Harder's paper [12] on which we rely crucially: in Section 4.5 we record the description of boundary cohomology, by means of \bar{S}_K , in terms of induced representations from Hecke characters, and in Section 5 we explain the description of Eisenstein cohomology, where the L -functions of the aforementioned Hecke characters play an essential role. In Section 6, we pass to construct the desired extension of Hodge structures; this is inspired by the ideas in [13] and made possible by the information given before, by the localization exact sequence associated to the open immersion $S_K \hookrightarrow S_K^*$, and by the results of Rogawski from [21, 22, 23]. In Section 7, we give the construction of the pairing $b(x, y)$ following Scholl's work [25], and provide a non-vanishing criterion for it, which gives in turn a non-triviality criterion for our extensions.

2. PRELIMINARIES

This section quickly reviews the basic properties of G and familiarizes the reader with the notations to be used throughout the article. We discuss the corresponding locally symmetric space, Weyl group, the associated spectral sequence and Kostant representatives of the minimal parabolic subgroup.

2.1. Structure Theory. Let F be a quadratic imaginary extension of \mathbb{Q} . Let V be a 3-dimensional F -vector space, and let J be an F -valued, non degenerate hermitian

form on V , such that $J \otimes_F \mathbb{C}$ is of signature $(2, 1)$. We consider then the algebraic group $G := \mathrm{SU}(V, J)$ over \mathbb{Q} . In other terms, G is defined, for every \mathbb{Q} -algebra R , by

$$G(R) = \{g \in \mathrm{SL}_{F \otimes_{\mathbb{Q}} R}(V \otimes_{\mathbb{Q}} R) \mid J(g \cdot, g \cdot) = J(\cdot, \cdot)\}$$

Notation 2.1. *Throughout the article, let α denote the only non-trivial element of $\mathrm{Gal}(F|\mathbb{Q})$, and $|x|^2$ denote the norm $x \cdot \alpha(x)$ of an element $x \in F^\times$.*

Remark 2.2. We have an isomorphism

$$(5) \quad V \otimes_{\mathbb{Q}} F \simeq V^+ \times V^-$$

where

$$\begin{aligned} V^+ &:= \{v \in V \otimes_{\mathbb{Q}} F \mid (x \otimes 1)v = (1 \otimes x)v \ \forall x \in F\} \\ V^- &:= \{v \in V \otimes_{\mathbb{Q}} F \mid (x \otimes 1)v = (1 \otimes \alpha(x))v \ \forall x \in F\}. \end{aligned}$$

The restriction to V of the projection on V^+ and on V^- induces an F -linear and F -antilinear isomorphisms of \mathbb{Q} -vector spaces

$$(6) \quad V \simeq V^+, \quad V \simeq V^-.$$

Lemma 2.3. *There exists an isomorphism $G_F \simeq \mathrm{SL}_{3,F}$.*

Proof. Define the desired morphism on F -points by sending $g \in G(F)$ to $g|_{V^+}$, using (5). To see that it is an isomorphism, fix a F -basis of V and use abusively the same symbol J for the matrix which represents the form J in the chosen basis. Then, by definition of G , and using the identification (6), the relation $g|_{V^-} = J^{-1t} g|_{V^+}^{-1} J$ holds for any $g \in G(F)$. Hence $g \mapsto (g, J^{-1t} g^{-1} J)$ defines an inverse by using (5). \square

The following is standard:

Lemma 2.4. *Let (V, J) be as above.*

- (1) *There exist infinitely many isotropic vectors for J in V .*
- (2) *For any non-zero isotropic vector $\bar{v} \in V$, there exist $\beta \in \mathbb{Q}_{>0}^\times$ and an isomorphism $(V, J) \simeq (F^3, J_\beta)$ such that \bar{v} is sent to the first vector of the canonical basis of F^3 and J_β is (the Hermitian form represented by) the matrix*

$$\begin{pmatrix} & & 1 \\ & \beta & \\ 1 & & \end{pmatrix}$$

Definition 2.5. *The basis of V corresponding to the canonical one of F^3 through the isomorphism of the above lemma, with first vector \bar{v} isotropic for J , is called a parabolic basis of V (adapted to \bar{v}).*

We are now going to describe the locally symmetric spaces associated with G . Observe that $G(\mathbb{R}) \simeq \mathrm{SU}(2, 1)$. By fixing a basis of $V_{\mathbb{C}}$, we get an embedding $G(\mathbb{R}) \hookrightarrow \mathrm{GL}_3(\mathbb{C})$, with respect to which we can write an element $g \in G(\mathbb{R})$ in block-matrix form as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A is a 2-by-2 matrix, B and C are respectively a 2-by-1 and a 1-by-2 matrix, and D is a scalar. Choose now a basis such that the corresponding matrix of J is $\text{diag}(1, 1, -1)$. There is a standard maximal compact subgroup K_∞ of $G(\mathbb{R})$, given by the subgroup of elements

$$(7) \quad \left\{ \begin{pmatrix} A & \\ & \det(A)^{-1} \end{pmatrix} \mid A \in U_2(\mathbb{R}) \right\}$$

where $\det A \in U_1(\mathbb{R}) \simeq S^1$. Then, $K_\infty \simeq U_2(\mathbb{R})$, and each other maximal compact subgroup of $G(\mathbb{R})$ is conjugated to K_∞ .

We can now define the symmetric space $S = G(\mathbb{R})/K_\infty$ and associate to any arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$ the locally symmetric space $S_\Gamma := \Gamma \backslash S$. Any finite dimensional representation \mathcal{M} of G naturally defines a sheaf $\widetilde{\mathcal{M}}$ on S_Γ . One therefore has an isomorphism

$$(8) \quad H^\bullet(\Gamma, \mathcal{M}) \cong H^\bullet(S_\Gamma, \widetilde{\mathcal{M}}),$$

Moreover, if \overline{S}_Γ is the Borel-Serre compactification of S_Γ , then there exists an isomorphism between the cohomology spaces $H^\bullet(\overline{S}_\Gamma, \widetilde{\mathcal{M}})$ and $H^\bullet(S_\Gamma, \widetilde{\mathcal{M}})$.

2.2. The associated Shimura variety. By writing the elements of $G(\mathbb{R})$ with respect to a basis of $V_{\mathbb{C}}$ in which the form J is represented by $\text{diag}(1, 1, -1)$, one sees that $G(\mathbb{R})$ acts transitively by generalized fractional transformations on the matrix space

$$\mathcal{D}_{2,1} := \{U \in M_{2,1}(\mathbb{C}) \mid \bar{U}U - 1 < 0\}$$

and that the stabilizer of 0 is precisely K_∞ . Hence, the symmetric space S is diffeomorphic to $\mathcal{D}_{2,1}$. This yields then a complex analytic structure on the locally symmetric spaces S_Γ . The previous action can also be identified to the transitive action of $G(\mathbb{R})$ on the set of J -negative lines in the 3-dimensional F -vector space $V(\mathbb{Q})$, providing a complex analytic isomorphism between $\mathcal{D}_{2,1}$ and a complex 2-ball in \mathbb{C}^2 .

A purely group-theoretic description of S can be obtained by making use of the group $\tilde{G} := GU(V, J)$ of *unitary similitudes* (with *rational* similitude factor). One shows that the morphism $h : \mathbb{S} \rightarrow \tilde{G}_{\mathbb{R}}$ (where $\mathbb{S} := \text{Res}_{\mathbb{C}|\mathbb{R}} \mathbb{G}_{m, \mathbb{R}}$ is the *Deligne torus*) defined on \mathbb{R} -points, with respect to a parabolic basis, by

$$x + iy \mapsto \begin{pmatrix} x & 0 & iy \\ 0 & x + iy & 0 \\ iy & 0 & x \end{pmatrix}$$

defines a *pure Shimura datum*. Then, the $\tilde{G}(\mathbb{R})$ -conjugacy class X of h is in canonical bijection with a disjoint union of copies of S and therefore acquires a canonical topology. A reference for all these facts is² [16, 3.2].

²To make the comparison with Lan's setting, one has to notice that $i \cdot \begin{pmatrix} & 1 \\ i & \end{pmatrix} \cdot J_\beta \cdot \begin{pmatrix} & -i \\ 1 & \end{pmatrix} = \begin{pmatrix} & i\beta \\ -1 & \end{pmatrix}$, where the latter matrix represents a *skew-hermitian* form, and that conjugating the above Shimura datum by $\begin{pmatrix} & -i \\ 1 & \end{pmatrix}$ gives the *complex conjugate* of Lan's Shimura datum (*loc. cit.*, Eq. (3.2.5.4)).

2.2.1. Take a compact open subgroup $K \subset \tilde{G}(\mathbb{A}_f)$ which is *neat* in the sense of [19, Sec. 0.6]. Then, the set of double classes

$$\tilde{G}(\mathbb{Q}) \backslash (X \times \tilde{G}(\mathbb{A}_f) / K)$$

equipped with its natural topology, is canonically homeomorphic to a finite disjoint union of locally symmetric spaces

$$\bigsqcup_i S_{\Gamma_i}$$

where the Γ_i 's are *congruence* subgroups of $\tilde{G}(\mathbb{Q})$. It therefore acquires a canonical structure of complex analytic space, which can be shown to be isomorphic to the analytification of a canonical *smooth* and *quasi-projective* algebraic surface S_K , called a *Picard surface* of level K , defined over F . As a consequence, the locally symmetric spaces $\Gamma_i \backslash S$ are identified to the analytifications of smooth, quasi-projective surfaces defined over an abelian extension of F depending on K .

2.3. **Root System.** Consider the maximal torus H of $SL_{3,F}$ defined by the subgroup whose F -points are

$$H(F) = \left\{ \begin{pmatrix} a & & \\ & a^{-1}b & \\ & & b^{-1} \end{pmatrix} \right\}.$$

We obtain then, through the isomorphism in Lemma 2.3, a maximal torus H of G_F , still denoted by H . The group of characters of H is identified with \mathbb{Z}^2 , by associating to the vector (k_1, k_2) the character defined on points by

$$(9) \quad (k_1, k_2) : \left(\begin{pmatrix} a & & \\ & a^{-1}b & \\ & & b^{-1} \end{pmatrix} \right) \mapsto a^{k_1 - k_2} b^{k_2}$$

By means of these identifications, the root system Φ of the (split) group G_F is of type A_2 . The system Δ of simple roots is the set $\{(1, -1), (1, 2)\}$ and the set Φ^+ of positive roots is the set $\{(1, -1), (1, 2), (2, 1)\}$. The element $\frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = (2, 1)$ will be often denoted by δ in what follows.

Consider the group \tilde{G} defined in Remark 2.2. One sees, along the same lines of the proof of Lemma 2.3, that there is an isomorphism

$$(10) \quad \tilde{G}_F \simeq GL_{3,L} \times G_{m,F}$$

defined on points by sending an element $g \in \tilde{G}(F)$, seen as an element of $GL(V^+ \times V^-)$, to

$$(g|_{V^+}, \nu(g))$$

where $\nu(g)$ is the similitude factor of g . Note that such a g verifies

$$(11) \quad g|_{V^-} = \nu(g) J^{-1t} g|_{V^+}^{-1} J$$

A character of the standard maximal torus of \tilde{G}_F can be identified with a vector (k_1, k_2, c, r) of integers such that

$$k_1 \geq k_2 \geq 0, \quad c = k_1 + k_2 \pmod{2}, \quad r = \frac{c + k_1 + k_2}{2} \pmod{2}$$

by associating (k_1, k_2, c, r) with the character

$$(12) \quad \left(\begin{pmatrix} a & & \\ & a^{-1}b & \\ & & b^{-1}q \end{pmatrix}, f \right) \mapsto a^{k_1-k_2} b^{k_2} q^{\frac{c-(k_1+k_2)}{2}} \cdot f^{-\frac{1}{2}(r+\frac{3c-(k_1+k_2)}{2})}$$

We will often choose to lift a character $\lambda = (k_1, k_2)$ of the maximal torus of G_F to the character

$$(13) \quad \tilde{\lambda} := (k_1, k_2, k_1 + k_2, (k_1 + k_2))$$

of the maximal torus of \tilde{G}_F .

2.4. Kostant Representatives. We will need to use a classical result of Kostant. In order to state it, fix a split reductive group \mathcal{G} over a field of characteristic zero, with root system \mathfrak{r} and Weyl group Υ . Denote by \mathfrak{r}^+ the set of positive roots and fix moreover a parabolic subgroup Q with its unipotent radical W . Let \mathfrak{w} be the Lie algebra of W and \mathfrak{r}_W the set of roots appearing inside \mathfrak{w} (necessarily positive). For every $w \in \Upsilon$, we define:

$$\begin{aligned} \mathfrak{r}^+(w) &:= \{\alpha \in \mathfrak{r}^+ \mid w^{-1}\alpha \notin \mathfrak{r}^+\}, \\ l(w) &:= |\mathfrak{r}^+(w)|, \\ \Upsilon' &:= \{w \in \Upsilon \mid \mathfrak{r}^+(w) \subset \mathfrak{r}_W\}. \end{aligned}$$

Then, Kostant's theorem reads as follows:

Theorem 2.6. [27, Thm. 3.2.3] *Let V_λ be an irreducible \mathcal{G} -representation of highest weight λ , and let δ be the half-sum of the positive roots of \mathcal{G} . Then, as (Q/W) -representations,*

$$H^q(W, V_\lambda) \simeq \bigoplus_{w \in \Upsilon' \mid l(w)=q} U_{w \cdot (\lambda + \delta) - \delta},$$

where U_μ denotes an irreducible (Q/W) -representation of highest weight μ .

Coming back to our situation, the Weyl group \mathcal{W} of G_F is isomorphic to the symmetric group on 3 elements. Denote by λ a fixed character (k_1, k_2) of H . Then, for an element $w \in \mathcal{W}$, we compute as follows the “twisted” Weyl action

$$(14) \quad w \star \lambda := w \cdot (\lambda + \delta) - \delta$$

appearing in Kostant's theorem:

$$\begin{aligned} id \star \lambda &= \lambda \\ (1\ 2) \star \lambda &= (k_2 - 1, k_1 + 1) \\ (2\ 3) \star \lambda &= (k_1 - k_2 - 1, -k_2 - 2) \\ (1\ 2\ 3) \star \lambda &= (-k_2 - 3, k_1 - k_2) \\ (1\ 3\ 2) \star \lambda &= (k_2 - k_1 - 3, -k_1 - 3) \\ (1\ 3) \star \lambda &= (-k_1 - 4, k_2 - k_1 - 2) \end{aligned}$$

The only element of length 0 is the identity, the elements (1 2) and (2 3) have length 1, the elements (1 2 3) and (1 3 2) have length 2, and the element (1 3) has length 3.

2.5. The Irreducible Representations. The irreducible representations of G_F coincide with the irreducible representations of $SL_{3,F}$, and are therefore parametrized by characters (k_1, k_2) which are *dominant*, i.e. such that $k_1 \geq k_2 \geq 0$.

Consider the irreducible representation V_λ of G_F , of highest weight λ . Lift it to a irreducible representation $V_{\tilde{\lambda}}$ of \tilde{G}_F as in (13). Then, the space

$$(15) \quad V_{\tilde{\lambda}, \mathbb{Q}} := \text{Res}_{F|\mathbb{Q}} V_{\tilde{\lambda}}$$

is a representation of $\text{Res}_{F|\mathbb{Q}} \tilde{G}_F$, and hence, through the canonical morphism, adjoint to the identity of \tilde{G}_F ,

$$\tilde{G} \rightarrow \text{Res}_{F|\mathbb{Q}} \tilde{G}_F.$$

it is also a representation of \tilde{G} . It is not absolutely irreducible, and in fact, once we extend scalars back to F , we have (with analogous notation as in (5)) a decomposition

$$(16) \quad V_{\tilde{\lambda}, \mathbb{Q}} \otimes_{\mathbb{Q}} F \simeq V_{\tilde{\lambda}}^+ \times V_{\tilde{\lambda}}^-$$

into irreducible representations of \tilde{G}_F , where the representation $V_{\tilde{\lambda}}^+$ is canonically isomorphic to the original $V_{\tilde{\lambda}}$. As far as $V_{\tilde{\lambda}}^-$ is concerned, its highest weight can be easily identified as follows. If $\tilde{\lambda} = (k_1, k_2, k_1 + k_2, k_1 + k_2)$, then the highest weight through which the maximal torus of \tilde{G}_F acts on $V_{\tilde{\lambda}}$ is

$$\left(\begin{pmatrix} a & & \\ & a^{-1}b & \\ & & b^{-1}q \end{pmatrix}, f \right) \mapsto a^{k_1 - k_2} b^{k_2} f^{-(k_1 + k_2)}.$$

In particular, this is how the maximal torus acts on the isotropic line W^+ inside $V_{\tilde{\lambda}}^+$ stabilized by the standard Borel. Then, consider the similarly defined isotropic line W^- inside $V_{\tilde{\lambda}}^-$. Since the action of \tilde{G}_F on $V_{\tilde{\lambda}}^-$ is described by the same formula as in (11), the maximal torus acts on W^- via the character

$$\left(\begin{pmatrix} a & & \\ & a^{-1}b & \\ & & b^{-1}q \end{pmatrix}, f \right) \mapsto a^{k_2} b^{k_1 - k_2} q^{-k_1} f^{-(k_1 + k_2)}.$$

This is then the highest weight of $V_{\tilde{\lambda}}^-$, which is therefore isomorphic to the representation $V_{\tilde{\lambda}^-}$ of highest weight

$$(17) \quad \tilde{\lambda}^- := (k_1, k_1 - k_2, -k_2, k_1 + k_2).$$

When $\lambda = (k, 0)$, we will denote

$$(18) \quad V_k := V_{\tilde{\lambda}, \mathbb{Q}}.$$

Since $V_k \simeq \text{Sym}^k V$, we will look at the action of the \mathbb{Q} -points of \tilde{G} on V_k as the restriction of the classical action of $SL_3(F)$ on homogeneous polynomials of degree k in 3 variables, with coefficients in F . The decomposition (16) becomes in this case

$$(19) \quad V_k \otimes_{\mathbb{Q}} F \simeq V_k \times V_k^\vee$$

with V_k^\vee being associated with the highest weight $\lambda^- = (k, k)$.

2.6. Standard \mathbb{Q} -Parabolic Subgroups. The standard \mathbb{Q} -parabolic subgroups of G are easily described by employing the notion of parabolic basis.

Lemma 2.7. *The only \mathbb{Q} -conjugacy class of parabolic subgroups of G is the class of the Borel subgroup B whose \mathbb{Q} -points are given, in a parabolic basis, by*

$$B(\mathbb{Q}) = G(\mathbb{Q}) \cap \mathcal{B}(\mathbb{Q})$$

where \mathcal{B} is the subgroup of upper triangular matrices in $\text{Res}_{F|\mathbb{Q}} \text{SL}_{3,F}$.

In such a basis, a maximal \mathbb{Q} -split torus T of G has \mathbb{Q} -points given by

$$T(\mathbb{Q}) = \left\{ \begin{pmatrix} t & & \\ & 1 & \\ & & t^{-1} \end{pmatrix} \mid t \in \mathbb{Q}^\times \right\}$$

Proof. It is clear that B is a \mathbb{Q} -parabolic. Any other one has to stabilize a subspace W inside V , hence its orthogonal W^\perp , and by dimension considerations we see that $W \cap W^\perp$ cannot be trivial and that it has to be a line. Hence any \mathbb{Q} -parabolic P is the stabilizer of an isotropic line, and since there is a transitive action of $G(\mathbb{Q})$ on the set of isotropic lines, P has to be conjugated to B .

As far as the maximal torus is concerned, we see that, for any element

$$g = \begin{pmatrix} z_1 & & \\ & z_1^{-1}z_2 & \\ & & z_2^{-1} \end{pmatrix}$$

of the standard diagonal maximal torus of $\text{SL}_3(F)$, g belongs to $G(F)$ if and only if

$$\begin{pmatrix} & & \alpha(z_1)z_2^{-1} \\ & |z_1|^{-2}|z_2|^2\beta & \\ z_1\alpha(z_2)^{-1} & & \end{pmatrix} = \begin{pmatrix} & & 1 \\ & \beta & \\ 1 & & \end{pmatrix}$$

(by definition of G). But this implies that g is of the form

$$\begin{pmatrix} z_1 & & \\ & z_1^{-1}\alpha(z_1) & \\ & & \alpha(z_1)^{-1} \end{pmatrix}$$

The above description provides us with a maximal torus T^M of G and an isomorphism $T^M \simeq \text{Res}_{F|\mathbb{Q}} \mathbb{G}_{m,F}$. Hence, T^M is a non-split torus, whose maximal \mathbb{Q} -split subtorus is a copy of \mathbb{G}_m , embedded as in the statement. \square

Remark 2.8. The Levi component of the maximal parabolic B is precisely the torus T^M . Denote by A_G the identity component of the real points of a maximal \mathbb{Q} -split subtorus of T^M . By the above proof, there are isomorphisms of Lie groups $T^M(\mathbb{R}) \simeq \mathbb{C}^\times$ and $A_G \simeq \mathbb{R}_{>0}^\times$. It follows that there exists an isomorphism of Lie groups ${}^0T^M := A_G \backslash T^M(\mathbb{R}) \simeq S^1$.

3. MIXED HODGE STRUCTURES WITH COEFFICIENTS

Before giving the description of boundary cohomology of our locally symmetric spaces, we discuss some formal aspects of mixed Hodge structures which will be used

throughout the article, and more importantly we will put our focus on the description of mixed Hodge structures with coefficients.

Denote by $\text{MHS}_{\mathbb{Q}}$ the abelian category of mixed \mathbb{Q} -Hodge structures, whose objects are in particular the triples $\mathbb{V} = (V, \mathcal{W}_{\bullet}, \mathcal{F}^{\bullet})$ where V is a \mathbb{Q} -vector space, \mathcal{W}_{\bullet} is an increasing filtration on V , and \mathcal{F}^{\bullet} is a decreasing filtration on $V \otimes_{\mathbb{Q}} \mathbb{C}$, subject to a series of conditions. This definition can be generalized by replacing \mathbb{Q} with any subfield R of \mathbb{R} and by using R -vector spaces as first entries of the objects \mathbb{V} , thus obtaining a category MHS_R .

In this article, we will need to make use of mixed Hodge structures with coefficients in subfields of \mathbb{C} which don't admit any embedding into \mathbb{R} . Fix a number field \mathbb{K} . Mixed \mathbb{K} -Hodge structures are defined via a two-step process. First, one considers the category $\text{MHS}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{K}$ whose objects are the same as the objects of $\text{MHS}_{\mathbb{Q}}$ and whose spaces of morphisms are defined by

$$\text{Hom}_{\text{MHS}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{K}}(A, B) := \text{Hom}_{\text{MHS}_{\mathbb{Q}}}(A, B) \otimes_{\mathbb{Q}} \mathbb{K}$$

Then, the desired category $\text{MHS}_{\mathbb{Q}, \mathbb{K}}$ of mixed \mathbb{Q} -Hodge structures with coefficients in \mathbb{K} ³ is defined as the *pseudo-abelian completion*⁴ of $\text{MHS}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{K}$. In particular, the objects of $\text{MHS}_{\mathbb{Q}, \mathbb{K}}$ are couples (\mathbb{V}, p) where \mathbb{V} is an object of $\text{MHS}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{K}$ (hence of $\text{MHS}_{\mathbb{Q}}$) and p is an idempotent element of $\text{End}_{\text{MHS}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{K}}(\mathbb{V})$. All of these constructions can be repeated starting with a category of mixed R -Hodge structures, with R a subfield of \mathbb{R} and \mathbb{K} a finite extension of R , thus obtaining categories $\text{MHS}_{R, \mathbb{K}}$. There are fully faithful embeddings

$$(20) \quad \text{MHS}_R \otimes_{\mathbb{Q}} \mathbb{K} \hookrightarrow \text{MHS}_{R, \mathbb{K}}$$

Convention 3.1.

- (1) Let R be a subfield of \mathbb{R} and \mathbb{K} a finite extension of R . By saying that a \mathbb{K} -vector space V is endowed with a mixed R -Hodge structure with coefficients in \mathbb{K} , we will mean that there exist a R -vector space V_R , a mixed R -Hodge structure $\mathbb{V} = (V_R, \mathcal{W}_{\bullet}, \mathcal{F}^{\bullet})$ and an idempotent element p of $\text{End}_{\text{MHS}_R}(\mathbb{V}) \otimes_{\mathbb{Q}} \mathbb{K}$, such that $p(V_R \otimes_{\mathbb{Q}} \mathbb{K}) \simeq V$. The filtrations \mathcal{W}_{\bullet} and \mathcal{F}^{\bullet} induce then filtrations on V and $V \otimes_{\mathbb{K}} \mathbb{C}$, which allow us to speak about the *weights* and the *types* of the mixed \mathbb{K} -Hodge structure on V . Of course, the types of a mixed \mathbb{K} -Hodge structure will not, in general, satisfy Hodge symmetry.
- (2) If given a $\overline{\mathbb{Q}}$ -vector space V , there exist an extension of number fields $\mathbb{K}|R$, and a \mathbb{K} -vector space $V_{\mathbb{K}}$ endowed with a mixed R -Hodge structure with coefficients in \mathbb{K} , such that $V \simeq V_{\mathbb{K}} \otimes_{\mathbb{K}} \overline{\mathbb{Q}}$, then we will say that V is endowed with a mixed Hodge structure with coefficients in $\overline{\mathbb{Q}}$, or that V is an object of $\text{MHS}_{\overline{\mathbb{Q}}}$. We will speak freely of the *category* $\text{MHS}_{\overline{\mathbb{Q}}}$ and of Ext groups in $\text{MHS}_{\overline{\mathbb{Q}}}$, leaving to the reader the technical details of the definitions.

By adopting completely analogous definitions at the level of (bounded) derived categories, the embeddings (20) extend to fully faithful embeddings

$$(21) \quad D^b(\text{MHS}_R \otimes_{\mathbb{Q}} \mathbb{K}) \hookrightarrow D^b(\text{MHS}_{R, \mathbb{K}})$$

³Also called mixed \mathbb{K} -Hodge structures when the “base field” \mathbb{Q} of coefficients is understood.

⁴Also known in the literature as *idempotent completion* or *Karoubi envelope*.

This allows one to deduce formulae for the Ext-groups in $\text{MHS}_{R,\mathbb{K}}$ from the formulae for the Ext-groups in MHS_R . We record in a couple of lemmas the ones which will be important for us. In the following, the symbol $\mathbb{1}$ will denote the unit object in the appropriate categories of mixed Hodge structures, and $\mathbb{1}(n)$, for $n \in \mathbb{Z}$, will denote the n -th Tate twist.

Lemma 3.2. *For any $n > 0$, there is a canonical isomorphism*

$$(22) \quad \text{Ext}_{\text{MHS}_{R,\mathbb{K}}}^1(\mathbb{1}, \mathbb{1}(n)) \simeq (\mathbb{C}/(2\pi i)^n R) \otimes \mathbb{K}$$

Proof. We have

$$\text{Ext}_{\text{MHS}_{R,\mathbb{K}}}^1(\mathbb{1}, \mathbb{1}(n)) = \text{Hom}_{\text{D}^b(\text{MHS}_{R,\mathbb{K}})}(\mathbb{1}, \mathbb{1}(n)[1]) = \text{Hom}_{\text{D}^b(\text{MHS}_R)}(\mathbb{1}, \mathbb{1}(n)[1]) \otimes \mathbb{K}$$

where the last equality comes from the full faithfulness of (21) and from the definition of morphisms in the category $\text{D}^b(\text{MHS}_R) \otimes \mathbb{K}$. Now the result is implied by the fact that

$$\text{Hom}_{\text{D}^b(\text{MHS}_R)}(\mathbb{1}, \mathbb{1}(n)[1]) = \text{Ext}_{\text{MHS}_R}^1(\mathbb{1}, \mathbb{1}(n)) \simeq \mathbb{C}/(2\pi i)^n R$$

where the last isomorphism is obtained as follows ([18, Thm. 3.31]). Let $[E]$ be the class of an extension

$$0 \rightarrow \mathbb{1}(n) \rightarrow E \rightarrow \mathbb{1} \rightarrow 0$$

and let 1 denote a generator of the R -vector space underlying $\mathbb{1}$. Choose a lift $s_W(1)$ of 1 in the R -vector space underlying E , and a lift $s_F(1)$ in the \mathbb{C} -vector space $F^0 E$. Then one sends $[E]$ to the class modulo $\mathbb{1}(n)$ of the element $s_F(1) - s_W(1)$. \square

Lemma 3.3. *Let H be a \mathbb{R} -Hodge structure with coefficients in \mathbb{C} , of weight -1 . Then,*

$$\text{Ext}_{\text{MHS}_{\mathbb{R},\mathbb{C}}}^1(\mathbb{1}, H) = 0$$

Proof. It is a classical fact, again following from [18, Thm. 3.31], that if H is a \mathbb{R} -Hodge structure of weight -1 then

$$\text{Ext}_{\text{MHS}_{\mathbb{R}}}^1(\mathbb{1}, H) = 0$$

To obtain the same result with coefficients in \mathbb{C} , it is enough to observe the general fact, holding for any R, \mathbb{K} as in Convention 3.1, that if (\mathbb{V}, p) , (\mathbb{U}, q) are objects of $\text{MHS}_{R,\mathbb{K}}$, then the group $\text{Ext}_{\text{MHS}_{R,\mathbb{K}}}^1((\mathbb{U}, q), (\mathbb{V}, p))$ is a subgroup of $\text{Ext}_{\text{MHS}_R}^1(\mathbb{U}, \mathbb{V}) \otimes \mathbb{K}$.⁵ \square

We conclude this section by formulating the consequence of the above principles that will be concretely used later.

Proposition 3.4. *Let W_1, W_2 be 1-dimensional \mathbb{C} -vector spaces such that the underlying \mathbb{R} -vector space is endowed with a pure \mathbb{R} -Hodge structure, of weights w_1 , resp. w_2 with $w_1 < w_2$, whose (p, q) -decompositions are induced by the isomorphisms*

$$\begin{aligned} \iota_1 : W_1 \otimes_{\mathbb{R}} \mathbb{C} &\simeq W_1^+ \times W_1^- \\ \iota_2 : W_2 \otimes_{\mathbb{R}} \mathbb{C} &\simeq W_2^+ \times W_2^- \end{aligned}$$

⁵More precisely, it coincides with the subgroup

$$q^* p_*(\text{Ext}_{\text{MHS}_R}^1(\mathbb{U}, \mathbb{V}) \otimes \mathbb{K})$$

where q^* and p_* are respectively the pullback and pushout functors.

(cfr. (5)). Let p_i^+, p_i^- , for $i \in \{1, 2\}$, be the projections on W_i^+, W_i^- defined via ι_i .
Let

$$0 \rightarrow W_1 \rightarrow W \xrightarrow{\pi} W_2 \rightarrow 0$$

be an extension of \mathbb{R} -Hodge structures and let be W^+ the extension in $\text{MHS}_{\mathbb{R}, \mathbb{C}}$ obtained from W via pushout by p_1^+ and pullback by p_2^+ , W^- the extension in $\text{MHS}_{\mathbb{R}, \mathbb{C}}$ obtained from W via pushout by p_1^- and pullback by p_2^- .

Fix a \mathbb{R} -basis $\{v, w\}$ of W_1 and a \mathbb{R} -basis $\{u, \omega\}$ of W_2 . Fix moreover a section σ of π , and a section σ_F of $\pi_{\mathbb{C}}$ respecting the Hodge filtration. Then the class of W^+ is trivial \iff the class of W^- is trivial \iff the coefficients of $\sigma_F(u \otimes 1)$ with respect to $v \otimes 1$ and of $\sigma_F(\omega \otimes 1)$ with respect to $w \otimes 1$ are conjugated to each other.

4. BOUNDARY COHOMOLOGY

In this section, we fix one of the Picard modular surfaces S_K introduced in Remark 2.2, and we let S_{Γ} be a connected component of $S_K(\mathbb{C})$. Our aim is to study the boundary cohomology of S_{Γ} with coefficients in local systems coming from representations of the underlying group G , together with their mixed Hodge structures and $G(\mathbb{A}_f)$ -actions. To do so, we will consider two different compactifications, the Baily-Borel compactification S_{Γ}^* (a complex projective variety) and the Borel-Serre compactification \overline{S}_{Γ} (a real analytic manifold with corners).

4.1. Boundary of the Baily-Borel compactification. As follows from [20, Sections 3.6-3.7], the boundary of the Baily-Borel compactification of S_{Γ} is the disjoint union of (connected components of the complex points of) Shimura varieties associated to a certain subgroup of the Levi component of a representative of each conjugacy class of \mathbb{Q} -parabolics in G . Since there is only one such class (the one of the Borel) and since the corresponding Levi component is a torus, we see that there is only one type of stratum in the boundary, which is 0-dimensional (a disjoint union of *cusps*).

4.2. Boundary of the Borel-Serre compactification. From the structure theory of the Borel-Serre compactification of S_{Γ} , we know that it admits a projection to the Baily-Borel one, in such a way that the fiber over each cusp is a fibration over a locally symmetric space $S_{\Gamma}^{\text{T}^M}$ for the group ${}^0\text{T}^M$ defined in Remark 2.8. Each fiber is diffeomorphic to a nilmanifold, obtained as a quotient U_{Γ} of the real points of the unipotent radical U of B by a discrete subgroup.

In our case, U is inserted into an exact sequence

$$0 \rightarrow W \rightarrow U \rightarrow \tilde{U} \rightarrow 0$$

where W is the commutator subgroup of U , isomorphic to \mathbb{G}_a , and \tilde{U} is isomorphic to $\text{Res}_{L|\mathbb{Q}} \mathbb{G}_{a,L}$ ([12, p. 567]). If Γ is torsion free, this yields a diffeomorphism of U_{Γ} with a fiber bundle over an elliptic curve, whose fibers are homeomorphic to S^1 ([12, pp. 569-570]).

Moreover, we have the following:

Lemma 4.1. *Suppose Γ to be neat. Then, the locally symmetric space $S_{\Gamma}^{\text{T}^M}$ is a point.*

Proof. Let K'_∞ be the conjugate of the maximal compact subgroup K_∞ of $G(\mathbb{R})$ of (7) under the element

$$\gamma := \begin{pmatrix} \frac{1}{\sqrt{2}} & & \frac{1}{\sqrt{2}} \\ & \sqrt{\beta} & \\ \frac{1}{\sqrt{2}} & & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

of $G(\mathbb{R})$ (notice that the hermitian form $\text{diag}(1, 1, -1)$ is congruent to J_β via γ). By definition, the locally symmetric space that we want to describe is diffeomorphic to $\Gamma_T \backslash {}^0\mathbb{T}^M / (K'_\infty \cap {}^0\mathbb{T}^M)$, where Γ_T is the projection to ${}^0\mathbb{T}^M$ of $\Gamma \cap \mathbb{T}^M(\mathbb{R})$. By neatness of Γ , such a projection is again arithmetic, and Remark 2.8 tells us that ${}^0\mathbb{T}^M$ is homeomorphic to S^1 , a compact torus, so that it has no non-trivial arithmetic subgroups. We leave it to the reader to check that on the other hand, $K'_\infty \cap {}^0\mathbb{T}^M$ is the whole of ${}^0\mathbb{T}^M$. \square

4.3. Hodge structures at the boundary. Let us denote by S_Γ^* the (complex analytic) Baily-Borel compactification of S_Γ and by $j : S_\Gamma \hookrightarrow S_\Gamma^*$, resp. $i : \partial S_\Gamma^* \hookrightarrow S_\Gamma^*$ the natural open, resp. closed immersions. Fix a sheaf $\widetilde{\mathcal{M}}$ on S_Γ coming from a representation \mathcal{M} of G_F . Suppose for simplicity that \mathcal{M} is an irreducible representation of highest weight λ . Fix a cusp c in $\partial S_\Gamma^* \hookrightarrow S_\Gamma^*$. For any element w of the Weyl group of G_F , recall the action $w \star \lambda$ defined in (14), and let $\mathcal{N}_{w \star \lambda}$ be the irreducible representation of \mathbb{T}_F^M of highest weight $w \star \lambda$. Since \mathbb{T}_F^M is a split torus, $\mathcal{N}_{w \star \lambda}$ is a 1-dimensional F -vector space. It follows from results of Harder for general locally symmetric spaces (cfr. [17, Cor. (6.6)] for a published proof in the context of Shimura varieties) and from Kostant's theorem 2.6 that for each positive integer n

$$(23) \quad R^n i^* j_* \widetilde{\mathcal{M}}|_{\{c\}} \simeq \bigoplus_{p+q=n} \bigoplus_{\ell(w)=q} \mathcal{N}_{w \star \lambda}$$

Notice that the sheaf on the left hand side is over a point, hence it makes sense to identify it with the vector space on the right hand side. Since the sheaf $\widetilde{\mathcal{M}}$ underlies a variation of Hodge structure, the latter vector space is endowed with a canonical mixed Hodge structure, which can be understood thanks to [7, Thms. 2.6-2.9], where the analogue (for general Shimura varieties) of the above isomorphism of sheaves is upgraded to a canonical isomorphism of *mixed Hodge modules*. This provides the following result:

Proposition 4.2. *Let Γ be a neat arithmetic subgroup of $G(\mathbb{Q})$. Let λ be a dominant weight of G_F of the form (k_1, k_2) . Let $\widetilde{\mathcal{M}}_\lambda$ be the local system on S_Γ associated to the highest weight representation \mathcal{M}_λ of G_F , and denote by r the weight of the canonical pure Hodge structure on \mathcal{M}_λ provided by the Shimura datum and by a choice of lift of λ to \widetilde{G}_F . Then, the following holds:*

- (1) *the spaces $R^n i^* j_* \widetilde{\mathcal{M}}_\lambda$ are trivial for n outside the set $\{0, \dots, 3\}$.*
- (2) *For each cusp c , we have isomorphisms*

$$R^0 i^* j_* \widetilde{\mathcal{M}}_\lambda|_{\{c\}} \cong \mathcal{N}_\lambda \text{ (a pure } F\text{-Hodge structure of weight } r - k_1)$$

$$R^1 i^* j_* \widetilde{\mathcal{M}}_\lambda|_{\{c\}} \simeq \mathcal{N}_{(1\ 2) \star \lambda} \oplus \mathcal{N}_{(2\ 3) \star \lambda}$$

(a semisimple F -Hodge structure of weights $r + 1 - k_2, r + 1 - (k_1 - k_2)$)

$$R^2 i^* j_* \widetilde{\mathcal{M}}_\lambda \Big|_{\{c\}} \simeq \mathcal{N}_{(1\ 2\ 3)\star\lambda} \oplus \mathcal{N}_{(1\ 3\ 2)\star\lambda}$$

(a semisimple F -Hodge structure of weights $(r + 2) + k_2 + 1, (r + 2) + (k_1 - k_2) + 1$)

$$R^3 i^* j_* \widetilde{\mathcal{M}}_\lambda \Big|_{\{c\}} \simeq \mathcal{N}_{(1\ 3)\star\lambda} \quad (\text{a pure } F\text{-Hodge structure of weight } (r + 3) + k_1 + 1)$$

Remark 4.3. The above assertions on the weights of the objects $R^n i^* j_* \widetilde{\mathcal{M}}_\lambda$ can be found in [28, proof of Theorem 3.8] and are the content of the main result of [1]. The topological description of the local systems $R^n i^* j_* \widetilde{\mathcal{M}}_\lambda$ was already determined in [12, Eq. (2.1.2)]. In the notation of *loc. cit.*, the elements $w_\alpha, w_{\bar{\alpha}}, w_{\bar{\alpha}} w_\alpha, w_\alpha w_{\bar{\alpha}}$ and θ correspond respectively to our (2 3), (1 2), (1 2 3), (1 3 2) and (1 3).

Remark 4.4. To make explicit the weight r in Proposition 4.2, one needs a choice of lifting of the character λ from G_F to \tilde{G}_F . Using the parameterization of characters of the maximal torus of \tilde{G}_F set up in Section 2.3, the weight of the Hodge structure on the representation of highest weight (k_1, k_2, c, r) coincides in fact with r , thus making the notation consistent. If one adopts the choice of lifting fixed in (13), the irreducible representation of G_F of highest weight λ is then endowed with a pure Hodge structure of weight $r = k_1 + k_2$.

For our purposes we will need to know something more precise about the above Hodge structures.

Proposition 4.5. *Keep the notation of Proposition 4.2 and lift λ to \tilde{G}_F as in (13).*

(1) *The one-dimensional F -Hodge structures described in Proposition 4.2 verify the following:*

- (a) \mathcal{N}_λ has type $(0, k_2)$;
- (b) $\mathcal{N}_{(1\ 2)\star\lambda}$ has type $(0, k_1 + 1)$;
- (c) $\mathcal{N}_{(2\ 3)\star\lambda}$ has type $(k_2 + 1, k_2)$;
- (d) $\mathcal{N}_{(1\ 2\ 3)\star\lambda}$ has type $(k_2 + 1, k_1 + k_2 + 2)$;
- (e) $\mathcal{N}_{(1\ 3\ 2)\star\lambda}$ has type $(k_1 + 2, k_1 + 1)$;
- (f) $\mathcal{N}_{(1\ 3)\star\lambda}$ has type $(k_1 + 2, k_1 + k_2 + 2)$.

(2) *Let $V_{\tilde{\lambda}}^-$ be obtained from V_λ as in (16). Then the types of F -Hodge structures on the cohomology of the local system associated with $V_{\tilde{\lambda}}^-$ are the conjugated of the types described in part (1).*

Proof. Recall the group \tilde{G} introduced in Remark 2.2. By employing [7, Thms. 2.6-2.9] and exploiting the formalism of [19, Ch. 4], the Hodge structure on $\mathcal{N}_{w\star\lambda}$ is seen to be induced by the cocharacter $\omega : \mathbb{S} \rightarrow \tilde{G}_{\mathbb{R}}$ defined on \mathbb{C} -points by

$$(z_1, z_2) \mapsto \left(\begin{pmatrix} z_1 z_2 & 0 & z_1 z_2 - 1 \\ 0 & z_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, z_1 z_2 \right)$$

using the isomorphism (10). Consider a character $\mu = (\kappa_1, \kappa_2, \gamma, \rho)$ of the maximal torus of \tilde{G}_L . Our previous computation of the Kostant action of the Weyl group \mathcal{W}

on a character of the maximal torus of G_F extends to μ as follows:

$$\begin{aligned}
id \star \mu &= \mu \\
(1\ 2) \star \mu &= (\kappa_2 - 1, \kappa_1 + 1, \gamma, \rho) \\
(2\ 3) \star \mu &= (\kappa_1 - \kappa_2 - 1, -\kappa_2 - 2, \gamma - \kappa_2 - 1, \rho) \\
(1\ 2\ 3) \star \mu &= (-\kappa_2 - 3, \kappa_1 - \kappa_2, \gamma - \kappa_2 - 1, \rho) \\
(1\ 3\ 2) \star \mu &= (\kappa_2 - \kappa_1 - 3, -\kappa_1 - 3, \gamma - \kappa_1 - 2, \rho) \\
(1\ 3) \star \mu &= (-\kappa_1 - 4, \kappa_2 - \kappa_1 - 2, \gamma - \kappa_1 - 2, \rho)
\end{aligned}$$

Fix any $w \in \mathcal{W}$ and denote $w \star \mu := (\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{c}, \mathfrak{r})$. Denote by π the projection modulo the unipotent radical of the standard Borel of $\tilde{G}_{\mathbb{R}}$. Then, the composition of $w \star \mu$ with $\pi \circ \omega$ sends $(z_1, z_2) \in \mathbb{S}(\mathbb{C})$ to

$$(24) \quad z_1^{\frac{\mathfrak{k}_1 + \mathfrak{k}_2}{4} + \frac{1}{4}\mathfrak{c} - \frac{1}{2}\mathfrak{r}} z_2^{\frac{3\mathfrak{k}_1 - \mathfrak{k}_2}{4} - \frac{1}{4}\mathfrak{c} - \frac{1}{2}\mathfrak{r}}.$$

By choosing μ to be the lifting $\tilde{\lambda}$ of λ fixed in (13), we perform the computation and we get the result of part (1). If we perform the same computation by choosing μ to be the highest weight of $V_{\tilde{\lambda}}^-$ as described in (17), we get part (2). \square

4.4. The boundary cohomology spaces. Boundary cohomology of S_{Γ} with coefficients in $\tilde{\mathcal{M}}$ in degree n can be computed in several ways. Recall that, choosing any compactification $j : S_{\Gamma} \hookrightarrow S_{\Gamma}^*$ with boundary $i : \partial S_{\Gamma}^* \hookrightarrow S_{\Gamma}^*$, it is defined as hypercohomology of a complex of sheaves over ∂S_{Γ}^* :

$$\partial H^n(S_{\Gamma}, \tilde{\mathcal{M}}) := \mathbb{H}^n(\partial S_{\Gamma}^*, i^* Rj_* \tilde{\mathcal{M}})$$

There is a natural long exact sequence

$$(25) \quad \cdots \rightarrow H_c^n(S_{\Gamma}, \tilde{\mathcal{M}}) \rightarrow H^n(S_{\Gamma}, \tilde{\mathcal{M}}) \rightarrow \partial H^n(S_{\Gamma}, \tilde{\mathcal{M}}) \rightarrow \cdots$$

Choosing as S_{Γ}^* the Baily-Borel compactification, we see that boundary cohomology in degree n is isomorphic to the sum over the cusps of the spaces $R^n i^* j_* \tilde{\mathcal{M}}|_{\{c\}}$ described in the previous section.

Choosing the Borel-Serre compactification \bar{S}_{Γ} , the sheaf $\tilde{\mathcal{M}}$ canonically extends to a sheaf on \bar{S}_{Γ} , denoted by the same symbol, and there is no need to compute hypercohomology: the spaces $\partial H^n(S_{\Gamma}, \tilde{\mathcal{M}})$ are canonically isomorphic to $H^n(\partial \bar{S}_{\Gamma}, \tilde{\mathcal{M}})$. In our case, as follows from Subsection 4.2 and by applying van Est's and Kostant's theorem, the latter coincide with a direct sum (over the connected components of the boundary) of spaces isomorphic to $\bigoplus_{p+q=n} H^p(S_{\Gamma}^{\text{T}^M}, \bigoplus_{\ell(w)=q} \widetilde{N_{w\star\lambda}})$, where by Lemma 4.1,

the manifold $S_{\Gamma}^{\text{T}^M}$, i.e. a locally symmetric space attached to ${}^0\text{T}^M$, is just a point. The identification of each one of the latter spaces with $R^n i^* j_* \tilde{\mathcal{M}}|_{\{c\}}$ is then clear.

In this work, we will use both descriptions: the use of the Baily-Borel compactification allows to describe the mixed Hodge structures on the boundary cohomology

spaces, whereas the use of the Borel-Serre one allows to describe them by means of automorphic forms.⁶

4.5. Automorphic structure of boundary cohomology. We end this section by describing the automorphic structure associated to boundary cohomology. For this purpose, we switch to the adelic setting, and work with the spaces

$$S_K(\mathbb{C}) := \tilde{G}(\mathbb{Q}) \backslash (X \times \tilde{G}(\mathbb{A}_f) / K)$$

defined in Remark 2.2. As in the non-adelic case, to any algebraic representation $\tilde{\mathcal{M}}_\lambda$ of \tilde{G}_L of highest weight λ there corresponds functorially a local system $\tilde{\mathcal{M}}_{\lambda,K}$ over $S_K(\mathbb{C})$, underlying a variation of Hodge structure. Letting K vary among the compact open subgroups of $\tilde{G}(\mathbb{A}_f)$, one can define a projective system $(S_K(\mathbb{C}), \tilde{\mathcal{M}}_{\lambda,K})_K$ of spaces and sheaves, the cohomology of whose projective limit $(S(\mathbb{C}), \tilde{\mathcal{M}}_\lambda)$ is such that

$$H^\bullet(S(\mathbb{C}), \tilde{\mathcal{M}}_\lambda) = \varprojlim_K H^\bullet(S_K(\mathbb{C}), \tilde{\mathcal{M}}_{\lambda,K})$$

and is endowed with a canonical structure of $\tilde{G}(\mathbb{A}_f)$ -module.

By taking the boundary $\partial S_K(\mathbb{C})$ of the Borel-Serre compactification at any level K , we obtain an analogous projective system and, in the projective limit, a $\tilde{G}(\mathbb{A}_f)$ -module $H^\bullet(\partial S(\mathbb{C}), \tilde{\mathcal{M}}_\lambda)$. To describe the “automorphic structure of the boundary cohomology” means, in a first approximation, to describe the structure of this $\tilde{G}(\mathbb{A}_f)$ -module.

This description will be given in terms of Hecke characters. For us, a Hecke character on a torus T will be a continuous homomorphism

$$T(\mathbb{Q}) \backslash T(\mathbb{A}) \rightarrow \mathbb{C}^\times.$$

We will often refer to a Hecke character of $\mathbb{G}_{m,F}$ as a Hecke character of F . Any *algebraic* Hecke character ϕ has an associated *type*, i.e. a character

$$\chi : T \rightarrow \mathbb{G}_m$$

and the finite part ϕ_f of such a ϕ takes then values in $\overline{\mathbb{Q}}^\times$, see [11, Sec. 2.5] for complete definitions and proofs.

We denote by \tilde{T}^M and \tilde{B} the standard maximal torus and Borel of \tilde{G} containing T^M and B . For any algebraic Hecke character $\phi : \tilde{T}^M(\mathbb{A}) \rightarrow \mathbb{C}^\times$ on \tilde{T}^M , we denote by $\overline{\mathbb{Q}}_\phi$ the one-dimensional $\overline{\mathbb{Q}}$ -vector space on which $\tilde{T}^M(\mathbb{A}_f)$ operates through ϕ_f , and we make it a $\tilde{B}(\mathbb{A}_f)$ -module in the standard way. Then, the theorem we are interested in reads as follows:

Theorem 4.6. [12, Thm. 1] *There exists a canonical isomorphism of $\tilde{G}(\mathbb{A}_f)$ -modules*

$$(26) \quad H^\bullet(\partial S(\mathbb{C}), \tilde{\mathcal{M}}_\lambda) \simeq \bigoplus_{w \in \mathcal{W}} \bigoplus_{\text{type}(\phi) = w \star \lambda} \text{Ind}_{\tilde{B}(\mathbb{A}_f)}^{\tilde{G}(\mathbb{A}_f)} \overline{\mathbb{Q}}_\phi$$

⁶Observe that for higher-rank groups, the Borel-Serre compactification, which doesn't necessitate the passage to hypercohomology, has a strong computational advantage over the Baily-Borel one. In our situation, the structure of the boundary being as simple as possible, determining boundary cohomology with either choice is equally easy.

5. EISENSTEIN COHOMOLOGY

In this section, we revisit the article [12] of Harder and study the Eisenstein cohomology of the Picard modular surfaces S_K associated to our group \tilde{G} . Recall that $\tilde{G}(\mathbb{R}) = \mathrm{GU}(2, 1)$. We summarize the main results and set up the notations and background required to establish our main results in Sections 6 and 7.

Following the notations fixed in Sections 2 and 4, we define $Z := Z(\tilde{G})(\mathbb{R})^0$ and $\tilde{K}_\infty := Z \cdot K_\infty$. Denote by ω the central character of the representation $\tilde{\mathcal{M}}_\lambda$ and define the space $C^\infty(\tilde{G}(\mathbb{Q}) \backslash \tilde{G}(\mathbb{A}))(\omega^{-1})$ as the space of those C^∞ -functions f on $\tilde{G}(\mathbb{Q}) \backslash \tilde{G}(\mathbb{A})$ which satisfy $f(zg) = \omega^{-1}(z)f(g)$ for any $z \in Z(\tilde{G})(\mathbb{A})^0$, $g \in \tilde{G}(\mathbb{A})$. We define the archimedean component $C^\infty(\tilde{G}(\mathbb{Q}) \backslash \tilde{G}(\mathbb{R}))(\omega^{-1})$ and the non-archimedean component $C^\infty(\tilde{G}(\mathbb{Q}) \backslash \tilde{G}(\mathbb{A}_f))(\omega^{-1})$ of this space in an analogous way. Then, if we denote by \mathfrak{g} the complexified Lie algebra of $\tilde{G}(\mathbb{R})$, there is a canonical isomorphism

$$H^\bullet(S(\mathbb{C}), \tilde{\mathcal{M}}_{\lambda, \mathbb{C}}) \simeq H^\bullet(\mathfrak{g}, \tilde{K}_\infty, C^\infty(\tilde{G}(\mathbb{Q}) \backslash \tilde{G}(\mathbb{A}))(\omega^{-1}) \otimes \tilde{\mathcal{M}}_{\lambda, \mathbb{C}})$$

which induces an isomorphism of $\tilde{G}(\mathbb{A}_f)$ -modules

$$(27) \quad H^\bullet(S(\mathbb{C}), \tilde{\mathcal{M}}_{\lambda, \mathbb{C}}) \simeq C^\infty(\tilde{G}(\mathbb{Q}) \backslash \tilde{G}(\mathbb{A}_f))(\omega^{-1}) \otimes H^\bullet(\mathfrak{g}, \tilde{K}_\infty, C^\infty(\tilde{G}(\mathbb{Q}) \backslash \tilde{G}(\mathbb{R}))(\omega^{-1}) \otimes \tilde{\mathcal{M}}_{\lambda, \mathbb{C}}).$$

The aim of this section is to exploit the latter description in order to determine *Eisenstein cohomology*, i.e. the image of the map of $\tilde{G}(\mathbb{A}_f)$ -modules

$$(28) \quad r : H^\bullet(S(\mathbb{C}), \tilde{\mathcal{M}}_{\lambda, \mathbb{C}}) \rightarrow H^\bullet(\partial S(\mathbb{C}), \tilde{\mathcal{M}}_{\lambda, \mathbb{C}}).$$

For any algebraic Hecke character $\phi : \tilde{T}^M(\mathbb{A}) \rightarrow \mathbb{C}^\times$ on \tilde{T}^M , denote

$$(29) \quad I_\phi := \mathrm{Ind}_{\tilde{B}(\mathbb{A}_f)}^{\tilde{G}(\mathbb{A}_f)} \overline{\mathbb{Q}}_\phi$$

and define the *Harish-Chandra module*

$$I_{\phi, \infty} := \left\{ f : \tilde{G}(\mathbb{R}) \rightarrow \mathbb{C} \mid f \text{ is } \tilde{K}_\infty\text{-finite and s.t. } f(bg) = \phi_\infty(b)f(g) \forall b \in \tilde{B}(\mathbb{R}), g \in \tilde{G}(\mathbb{R}) \right\}.$$

We then get the space $I_\phi^* := \mathrm{Ind}_{\tilde{B}(\mathbb{A})}^{\tilde{G}(\mathbb{A})} \phi = I_{\phi, \mathbb{C}} \otimes I_{\phi, \infty}$. The aim is to define, through Langlands' theory of Eisenstein series, a suitable operator

$$\mathrm{Eis}_\phi^* : I_\phi^* \rightarrow \mathcal{A}(\tilde{G}(\mathbb{Q}) \backslash \tilde{G}(\mathbb{A})),$$

where the space on the right-hand side is the space of *automorphic forms*, and get through (27) an induced morphism

$$(30) \quad \mathrm{Eis}_\phi : I_{\phi, \mathbb{C}} \otimes H^\bullet(\mathfrak{g}, \tilde{K}_\infty, I_{\phi, \infty} \otimes \tilde{\mathcal{M}}_{\lambda, \mathbb{C}}) \rightarrow H^\bullet(S(\mathbb{C}), \tilde{\mathcal{M}}_{\lambda, \mathbb{C}}).$$

This is then used to show that, when considering the restriction Eis'_ϕ of the latter morphism to suitable submodules $I'_{\phi, \mathbb{C}} \otimes H^\bullet(\mathfrak{g}, \tilde{K}_\infty, I'_{\phi, \infty} \otimes \tilde{\mathcal{M}}_{\lambda, \mathbb{C}})$, one gets an isomorphism

$$\bigoplus_{\phi} \mathrm{Im}(r \circ \mathrm{Eis}'_\phi) \simeq \mathrm{Im}(r).$$

5.1. The constant term of the Eisenstein operator. We begin by defining the Hecke character

$$\begin{aligned} |\delta| : \tilde{\mathbb{T}}(\mathbb{A}) &\rightarrow \mathbb{R}_{>0}^\times \\ t &\mapsto |\delta(t)|, \end{aligned}$$

where δ is the element defined in Subsection 2.3, and by recalling that for any other Hecke character ϕ , one can define, for any $s \in \mathbb{C}$ with $\Re(s) \gg 0$, an *Eisenstein operator*

$$(31) \quad \text{Eis}_{\phi,s}^* : I_{\phi|\delta|^s}^* \rightarrow \mathcal{A}(\tilde{\mathbb{G}}(\mathbb{Q}) \backslash \tilde{\mathbb{G}}(\mathbb{A}))$$

which, as a function of s , admits a meromorphic continuation to the whole of \mathbb{C} . Denote by $\tilde{\mathbb{U}}$ the unipotent radical of $\tilde{\mathbb{B}}$, and by θ the longest element in the Weyl group of $\tilde{\mathbb{G}}$. It has a canonical representative in $\tilde{\mathbb{G}}(\mathbb{Q})$. Then a standard calculation shows that for every $\Psi \in I_{\phi|\delta|^s}^*$, the constant term of the operator $\text{Eis}_{\phi,s}^*$ is given, for every $g \in \tilde{\mathbb{G}}(\mathbb{A})$, by

$$\int_{\tilde{\mathbb{U}}(\mathbb{Q}) \backslash \tilde{\mathbb{U}}(\mathbb{A})} \text{Eis}_{\phi,s}^*(\Psi)(ug) du = \Psi(g) + \int_{\tilde{\mathbb{U}}(\mathbb{A})} \Psi(\theta \cdot ug) du$$

and it is also known that $\text{Eis}_{\phi,s}^*$ has a pole at $s = 0$ if and only if this is the case for the above-described constant term.

To study whether such a pole occurs, we introduce the following:

Definition 5.1.

(1) For any $s \in \mathbb{C}$ with $\Re(s) \gg 0$, the *intertwining operator*

$$T(\phi, s) : I_{\phi|\delta|^s}^* \rightarrow I_{\phi^{-1}|\delta|^{-2-s}}^*$$

is defined, for every $\Psi \in I_{\phi|\delta|^s}^*$ and for every $g \in \tilde{\mathbb{G}}(\mathbb{A})$, by

$$T(\phi, s)(\Psi(g)) = \int_{\tilde{\mathbb{U}}(\mathbb{A})} \Psi(\theta \cdot ug) du.$$

(2) For any place v of \mathbb{Q} , we define

$$I_{\phi_v|\delta_v|^s}^* := \text{Ind}_{\tilde{\mathbb{B}}(\mathbb{Q}_v)}^{\tilde{\mathbb{G}}(\mathbb{Q}_v)} \phi_v|\delta_v|^s$$

and write $I_{\phi|\delta|^s}^* = \bigotimes_v I_{\phi_v|\delta_v|^s}^*$. Then, for any $s \in \mathbb{C}$ with $\Re(s) \gg 0$, the local components $T(\phi_v, s)$ of the intertwining operator $T(\phi, s)$ are defined, for every $\Psi_v \in I_{\phi_v|\delta_v|^s}^*$ and for every $g_v \in \tilde{\mathbb{G}}(\mathbb{Q}_v)$, by

$$T(\phi_v, s)(\Psi_v(g_v)) = \int_{\tilde{\mathbb{U}}(\mathbb{Q}_v)} \Psi_v(\theta_v \cdot u_v g_v) du_v$$

and we have $T(\phi, s) = \bigotimes_v T(\phi_v, s)$.

The pole at $s = 0$ of the intertwining operator will turn out to be controlled by an object of arithmetic nature:

Definition 5.2. (1) Let $\phi^{(1)}$ be the restriction of the Hecke character ϕ to the maximal torus T^M of G , $\phi_{\mathbb{Q}}^{(1)}$ be the restriction of $\phi^{(1)}$ to the copy of \mathbb{G}_m embedded into T^M , and $\epsilon_{F|\mathbb{Q}}$ be the quadratic character associated to the extension $F|\mathbb{Q}$. Then we define the meromorphic function $c(\phi, s)$ of the complex variable $s \in \mathbb{C}$ as

$$c(\phi, s) := \frac{L(\phi^{(1)}, s-1) \cdot L(\phi_{\mathbb{Q}}^{(1)} \cdot \epsilon_{F|\mathbb{Q}}, 2s-2)}{L(\phi^{(1)}, s) \cdot L(\phi_{\mathbb{Q}}^{(1)} \cdot \epsilon_{F|\mathbb{Q}}, 2s-1)}.$$

We denote by $\tilde{c}(\phi, s)$ the product of $c(\phi, s)$ with the appropriate Γ -factors of the occurring L -functions.

(2) For every place v of \mathbb{Q} , the local factor $c_v(\phi, s)$ of $c(\phi, s)$ is defined as

$$c_v(\phi, s) := \frac{L_p(\phi^{(1)}, s-1) \cdot L_p(\phi_{\mathbb{Q}}^{(1)} \cdot \epsilon_{F|\mathbb{Q}}, 2s-2)}{L_p(\phi^{(1)}, s) \cdot L_p(\phi_{\mathbb{Q}}^{(1)} \cdot \epsilon_{F|\mathbb{Q}}, 2s-1)}$$

if v corresponds to a finite prime p , and as the product of the appropriate Γ -factors of the occurring L -functions, if v is an infinite place. Here $L_p(\cdot, \cdot)$ denotes the corresponding factor in the Euler product of the L -function under consideration.

Lemma 5.3. *The intertwining operator $T(\phi, s)$ has a pole at $s = 0$ if and only if this is the case for $c(\phi, s)$.*

Proof. Let $v = p$ be an odd prime which does not ramify in F and at which ϕ is not ramified. Then consider the *spherical function* $\Psi_v^{(0)} \in I_{\phi_v}^* | \delta|_v^s$ defined for every $g_v = b_v k_v$ with $b_v \in \tilde{B}(\mathbb{Q}_v)$ and $k_v \in \tilde{G}(\mathbb{Z}_v)$, by $\Psi_v^{(0)}(g_v) = \phi_v | \delta|_v^s(b_v)$. The statement then follows from *Lai's formula* from [15, Sec. 3], which says that

$$T(\phi_v, s)(\Psi_v^{(0)}) = c_v(\phi, s) \cdot \Psi_v^{(0)}.$$

□

5.2. Recap on L -functions of Hecke characters. In order to study the behaviour of the term $c(\phi, s)$, it will be useful to recall some basic terminology and facts about L -functions of Hecke characters. For a character $\chi : T^M \rightarrow \mathbb{G}_m$, given on \mathbb{R} -points by

$$\begin{pmatrix} \bar{z} & & \\ & \bar{z}^{-1}z & \\ & & z^{-1} \end{pmatrix} \mapsto z^\nu \bar{z}^\mu,$$

we say that the *weight* of an Hecke character $\phi^{(1)}$ of T^M of type χ is $w(\phi^{(1)}) := \nu + \mu$.⁷ Such a Hecke character will also be called of *Hodge type* (ν, μ) .

Let now $\phi^{(1)}$ be a Hecke character of type $\chi = (\kappa_1, \kappa_2)$ - this implies in particular that we have $w(\phi^{(1)}) = \kappa_1$ and Hodge type $(\kappa_2, \kappa_1 - \kappa_2)$. The region in which the L -function $L(\phi^{(1)}, s)$ is expressed as an absolutely convergent Euler product is the half-plane $\operatorname{Re} s > 1 + \frac{w(\phi^{(1)})}{2}$. In this region, the L -function is therefore holomorphic, and moreover known to be non-zero. Multiplying the Euler product by appropriate Γ -factors, we obtain a *completed L -function* $\Lambda(\phi^{(1)}, s)$ which admits a meromorphic

⁷This is the *opposite* of the quantity denoted by $g(\phi^{(1)})$ in [12, Lemma 2.3.1].

continuation to the whole of \mathbb{C} . The function $\Lambda(\phi^{(1)}, s)$ and the meromorphically continued $L(\phi^{(1)}, s)$ are entire unless $\phi^{(1)}(\cdot) = |\cdot|_{\mathbb{F}}^t$ for some $t \in \mathbb{C}$; in the latter case, $\Lambda(\phi^{(1)}, s)$ has first-order poles precisely in $s = -t$ and $s = 1 - t$, and $L(\phi^{(1)}, s)$ has a unique first-order pole in $s = 1 - t$. Finally, there exists a holomorphic function $\epsilon(s)$ such that $\Lambda(\phi^{(1)}, s)$ satisfies the functional equation

$$(32) \quad \Lambda(\phi^{(1)}, s) = \epsilon(s)\Lambda((\phi^{(1)})^{-1}, 1 - s).$$

By applying the previous considerations to the specific Hecke characters appearing in our cohomology spaces, and remembering Subsection 2.4, we get:

Lemma 5.4. *Let $\lambda = (k_1, k_2)$ be a character of \mathbb{T}^M and w an element of the Weyl group of \mathbb{G}_F of length ≥ 2 . Fix an Hecke character $\phi^{(1)}$ of \mathbb{T}^M of type $w \star \lambda$.*

(1) *The half-plane of convergence for the Euler product of $L(\phi^{(1)}, s)$ is given by:*

- $\operatorname{Re} s > -\frac{1}{2} - \frac{k_2}{2}$ if $w = (1 \ 2 \ 3)$;
- $\operatorname{Re} s > -\frac{1}{2} - \frac{k_1 - k_2}{2}$ if $w = (1 \ 3 \ 2)$;
- $\operatorname{Re} s > -1 - \frac{k_1}{2}$ if $w = (1 \ 3)$.

(2) *The L -function $L(\phi^{(1)}, s)$ is entire, unless $(k_1, k_2) = (0, 0)$, $w = (1 \ 3)$ and $\phi^{(1)}(\cdot) = |\cdot|_{\mathbb{F}}^2$. In this case, it has a unique first-order pole at $s = -1$.*

Now, taking $\phi^{(1)}$ as above, its restriction $\phi_{\mathbb{Q}}^{(1)}$ to $\mathbb{I}_{\mathbb{Q}}$ has weight $2w(\phi^{(1)})$. The same then holds for $\phi_{\mathbb{Q}}^{(1)} \cdot \epsilon_{F|\mathbb{Q}}$. Hence, we obtain the following:

Lemma 5.5. *Let $\lambda = (k_1, k_2)$ be a character of \mathbb{T}^M and w an element of the Weyl group of \mathbb{G}_F of length ≥ 2 . Fix an Hecke character $\phi^{(1)}$ of \mathbb{T}^M of type $w \star \lambda$.*

(1) *The half-plane of convergence for the Euler product of $L(\phi_{\mathbb{Q}}^{(1)} \cdot \epsilon_{F|\mathbb{Q}}, s)$ is given by:*

- $\operatorname{Re} s > -2 - k_2$ if $w = (1 \ 2 \ 3)$;
- $\operatorname{Re} s > -2 - (k_1 - k_2)$ if $w = (1 \ 3 \ 2)$;
- $\operatorname{Re} s > -3 - k_1$ if $w = (1 \ 3)$.

(2) *The L -function $L(\phi_{\mathbb{Q}}^{(1)} \cdot \epsilon_{F|\mathbb{Q}}, s)$ is entire, unless*

$$k_2 = 0, \ w = (1 \ 2 \ 3) \text{ and } \phi_{\mathbb{Q}}^{(1)} \cdot \epsilon_{F|\mathbb{Q}}(\cdot) = |\cdot|_{\mathbb{I}_{\mathbb{Q}}}^3,$$

or

$$k_1 = k_2, \ w = (1 \ 3 \ 2) \text{ and } \phi_{\mathbb{Q}}^{(1)} \cdot \epsilon_{F|\mathbb{Q}}(\cdot) = |\cdot|_{\mathbb{I}_{\mathbb{Q}}}^3.$$

In each of the above two cases, it has a unique first-order pole at $s = -2$.

5.3. Automorphic description of Eisenstein cohomology. Finally, we are ready to describe the main result of this section. We begin by stating the following:

Definition 5.6. For every place v , we define

$$T^{\text{loc}}(\phi_v, s) := c_v(\phi, s)^{-1} \cdot T(\phi_v, s).$$

We then put

$$T^{\text{loc}}(\phi, s) := \bigotimes_v T^{\text{loc}}(\phi_v, s).$$

Remark 5.7. *The operator $T^{\text{loc}}(\phi, s)$ is also defined for $s = 0$, as follows from [12, Lemma 2.3.1].*

Definition 5.8. The operator $\text{Eis}_\phi^* : I_\phi^* \rightarrow \mathcal{A}(\tilde{G}(\mathbb{Q}) \backslash \tilde{G}(\mathbb{A}))$ is defined as

- (1) the evaluation $\text{Eis}_{\phi,0}^*$ at $s = 0$ of the Eisenstein operator $\text{Eis}_{\phi,s}^*$ of (31), if $c(\phi, s)$ does not have a pole at $s = 0$;
- (2) the residue at $s = 0$ of the operator $s \cdot \text{Eis}_{\phi,s}^*$, if $c(\phi, s)$ has a pole at $s = 0$.

Notation 5.9.

- (1) Denote by Φ_{np} the set of those ϕ 's such that $c(\phi, s)$ does not have a pole at $s = 0$, and by Φ_p the set of those ϕ 's such that $c(\phi, s)$ has a pole at $s = 0$.
- (2) For each $\phi \in \Phi_{\text{np}}$, let Eis_ϕ be the operator defined as in (30) from the operator Eis_ϕ^* of Definition 5.8.1.
- (3) For each $\phi \in \Phi_p$, define $I'_\phi := \text{Ker } T^{\text{loc}}(\phi, 0)$ and $J_\phi := \text{Im } T^{\text{loc}}(\phi, 0)$ and let Eis'_ϕ be the restriction to I'_ϕ of the operator Eis_ϕ defined as in (30) from the operator Eis_ϕ^* of Definition 5.8.2.

Theorem 5.10. [12, Thm. 2] *Let r be the restriction morphism of (28) and following Notation 5.9. Then, $r \circ (\bigoplus_{\phi \in \Phi_{\text{np}}} \text{Eis}_\phi \oplus \bigoplus_{\phi \in \Phi_p} \text{Eis}'_\phi)$ induces an isomorphism*

$$(33) \quad \text{Im}(r) \simeq \bigoplus_{\substack{w \in \mathcal{W} \\ \ell(w) \geq 2 \\ \phi \in \Phi_{\text{np}}}} \bigoplus_{\text{type}(\phi) = w \star \lambda} I_{\phi, \mathbb{C}} \oplus \bigoplus_{\substack{\text{type}(\phi) = w \star \lambda \\ \ell(w) \geq 2 \\ \phi \in \Phi_p}} I'_{\phi, \mathbb{C}} \oplus J_{\phi, \mathbb{C}}$$

Proof. The case $\phi \in \Phi_{\text{np}}$ is treated in [12, pp. 581-583] and the case $\phi \in \Phi_p$ in [12, pp. 583-584]. \square

Remark 5.11. In the above theorem, the spaces $I_{\phi, \mathbb{C}}$ and $I'_{\phi, \mathbb{C}}$ contribute to cohomology in degrees 2 and 3, whereas the spaces $J_{\phi, \mathbb{C}}$ contribute to cohomology in degrees 0 and 1.

6. CONSTRUCTION OF HECKE EXTENSIONS

Consider the algebraic groups G and \tilde{G} introduced in Section 2, with maximal tori T^M and \tilde{T}^M . From now on, we will fix an algebraic Hecke character ϕ of \tilde{T}^M , whose restriction $\phi^{(1)}$ to the maximal torus T^M (which is identified to an algebraic Hecke character of F) is of type $(1 \ 2 \ 3) \star \lambda = (-3, k)$ for some integer $k \geq 0$, and has the following two properties:

- (1) $\phi_{\mathbb{Q}}^{(1)} \cdot \epsilon_{F|\mathbb{Q}}(\cdot) = |\cdot|_{\mathbb{Q}}^3$;
- (2) the sign of the functional equation of $L(\phi^{(1)}, s)$ is -1 .

Note that such a ϕ has *infinity type* $(k, -(k+3))$ in the sense of [24, Ch. 0, 1] and that it has weight -3 . Hence, the central point of the functional equation of $L(\phi^{(1)}, s)$ is $s = -1$, and condition (2) above implies that $L(\phi^{(1)}, -1) = 0$, with *odd* order of vanishing. In fact, we will ask a more precise condition on ϕ :

- (2b) $L(\phi^{(1)}, s)$ vanishes at $s = -1$ at the first order.

Let F' be the field of values of ϕ . Denote by $\text{CHM}(F)_{F'}$ the category of *Chow motives* over F with coefficients in F' and let M_ϕ be the object of $\text{CHM}(F)_{F'}$ attached to ϕ in [9, Prop. 3.5].

The absolute Hodge motive corresponding to M_ϕ is then the one attached to ϕ in [24, Ch. 1, Thm. 4.1] and we have $L(\phi, s) = L(M_\phi, s)$. The Hodge realization of M_ϕ is a rank-1 F' -Hodge structure H_ϕ , of Hodge type $(k, -(k+3))$.

The aim of the present section is to construct a non-zero element E inside $\text{Ext}_{\text{MHS}_{\mathbb{Q}}}^1(\mathbb{1}, H_\phi(-1))$ of motivic origin, which will be called a *Hecke extension*, whose existence is predicted by Beilinson's conjectures ([26]) and the conjectural injectivity of Abel-Jacobi maps over number fields ([4]), as discussed in Section 1.

Let us fix a Picard modular surface S_K corresponding to a neat level K and a local system $\widetilde{\mathcal{M}}_\lambda$ on $S_K(\mathbb{C})$ associated to an irreducible representation of G of highest weight λ .

The idea for constructing the desired extension consists in exploiting the natural exact sequence (25) expressing $\widetilde{\mathcal{M}}_\lambda$ -valued cohomology of S_K as an extension of its Eisenstein cohomology by its *interior* cohomology

$$(34) \quad H_c^\bullet(S(\mathbb{C}), \widetilde{\mathcal{M}}_\lambda)$$

defined as the image of the natural morphism $H_c^\bullet(S(\mathbb{C}), \widetilde{\mathcal{M}}_\lambda) \rightarrow H^\bullet(S(\mathbb{C}), \widetilde{\mathcal{M}}_\lambda)$.

A relevant subspace of Eisenstein cohomology useful for our purposes will be obtained thanks to Theorem 5.10. In order to select a suitable subspace of interior cohomology, we need to analyze further its automorphic structure, which has been studied by Rogawski, as we are now going to recall.

6.1. A summary of Rogawski's results. In [13, p. 99], the following has been observed: it is a consequence of [21, Thms. 13.3.6, 13.3.7], as corrected in [23, Thm. 1.1], that under our condition (2) on ϕ , there exists, for an appropriate λ , a $\widetilde{G}(\mathbb{A}_f)$ -submodule $J_{\phi, !}$ of $H_c^2(S(\mathbb{C}), \widetilde{\mathcal{M}}_{\lambda, \overline{\mathbb{Q}}})$, whose local components are isomorphic to $J_{\phi, v}$ for infinitely many places v (see Notation 5.9.3). Let us now explain this in detail.

Following the penultimate equation of [12, p. 579], our Hecke character $\phi^{(1)}$ can be written as

$$(35) \quad \phi^{(1)} = |\delta|^{\frac{3}{2}} \cdot \phi_u^{(1)}$$

where $\phi_u^{(1)}$ is a *unitary* Hecke character. Then, as in the last equation of *loc. cit.*, we have an equality of $G(\mathbb{A}_f)$ -representations

$$(36) \quad \text{Ind}_{\mathbb{B}(\mathbb{A}_f)}^{\mathbb{G}(\mathbb{A}_f)} \phi^{(1)} = \text{IndUn}_{\mathbb{B}(\mathbb{A}_f)}^{\mathbb{G}(\mathbb{A}_f)} \phi_u^{(1)} \cdot |\delta|^{\frac{1}{2}}$$

where IndUn denotes the *unitary* induction functor.

Denote by $U(n)$, for $n \in \mathbb{N}$, the quasi-split form over \mathbb{Q} of the unitary group in n variables (defined with respect to the extension $F|\mathbb{Q}$), and write \widetilde{G}_0 for the subgroup of \widetilde{G} of elements with trivial similitude character. Fix a choice of a Hecke character μ of F whose restriction to \mathbb{Q} is $\epsilon_{F|\mathbb{Q}}(\cdot)$. We choose a couple of characters on the norm-one subgroup $C_F^{(1)}$ of the idèle class group of F as follows: define ρ_1 to be the restriction of $\mu \cdot \phi_u^{(1)}$ to $C_F^{(1)}$, and ρ' to be the restriction of $\phi \cdot (\phi^{(1)})^{-1}$ to the maximal torus of \widetilde{G}_0 . Such a ρ' actually factors through the image of that maximal torus via the determinant, i.e. it is truly a character of $U(1)$. Once we observe that

$$(37) \quad |\delta| = |\cdot|_F$$

where $|\cdot|_F$ is the norm character on the idèles of F , and once we remember (36), we then see that for any finite place v of \mathbb{Q} , the unitarily induced representation called $\text{ind}(\eta_v)$ in [23, p. 395] is nothing but our I_{ϕ_v} , or more precisely, its restriction to a representation of \tilde{G}_0 (remark that in our notation, the character called ϕ in *loc. cit.* becomes $\phi_u^{(1)}$). To the couple (ρ_1, ρ') that we have just chosen, one attaches a character ρ on $U(2) \times U(1)$ as in *loc. cit.*. From what we have just said, it follows that for any v which is inert in F , the representation denoted by $\pi^n(\rho_v)$ in *loc. cit.*, i.e. the Langlands quotient of $\text{ind}(\eta_v)$, coincides with (the restriction to \tilde{G}_0) of $J_{\phi,v}$. For v ramified, we still denote by $\pi^n(\rho_v)$ the Langlands quotient of $\text{ind}(\eta_v)$. For split v , we define $\pi^n(\rho_v)$ as in [23, p. 396].

At the infinite place of \mathbb{Q} , we consider the discrete series representation $\pi^s(\rho)$ of [22, p. 77, first line]. Adopt the notation $\tilde{\mathcal{M}}_{\lambda, \mathbb{Q}}$ of (15), and denote by $\tilde{\mathcal{M}}_{\lambda, \mathbb{Q}, 0}$ the restriction of $\tilde{\mathcal{M}}_{\lambda, \mathbb{Q}}$ to a representation of \tilde{G}_0 . In our notation, the representation $\tilde{\mathcal{M}}_{\lambda, \mathbb{Q}, 0}$ is identified with a triple (k_1, k_2, c) , whereas in [22, p. 79] it is identified with a triple (m, r, n) ; under the change of variables between the two parametrizations, one sees that $k_2 = 0$ if and only if $r - n = 1$, and that $k_1 = k_2$ if and only if $m - r = 1$. Then, by [22, 3.2], the representation $\pi^s(\rho)$ is $\tilde{\mathcal{M}}_{\lambda, \mathbb{Q}, 0}$ -cohomological in degree 2 when either $k_1 = k_2$ or $k_2 = 0$. In the first case, it coincides with the discrete series representation denoted by π^+ in *loc. cit.*; in the second case, it coincides with the one denoted by π^- . In both cases, we put $\pi_\infty := \pi^s(\rho)$.

Proposition 6.1. *Consider the $\tilde{G}_0(\mathbb{A}_f)$ -representation $\pi_f(\phi)$ and the automorphic representation $\pi(\phi)$ of \tilde{G}_0 defined by*

$$(38) \quad \begin{aligned} \pi_f(\phi) &:= \bigotimes_v \pi^n(\rho_v), \\ \pi(\phi) &:= \pi_f(\phi) \otimes \pi_\infty \end{aligned}$$

where the local components $\pi^n(\rho_v)$ and π_∞ are as defined above. Then we have the following

(1) *The automorphic representation $\pi(\phi)$ has multiplicity 1 in the cuspidal spectrum of \tilde{G}_0 .*

(2) *Recall the \tilde{G} -representation V_k defined by (18) and denote by the same symbol the associated local system. There exist a $\bar{\mathbb{Q}}$ -model of $\pi_f(\phi)$, denoted by the same symbol, and 1-dimensional $\bar{\mathbb{Q}}$ -Hodge structures $H_{J_{\phi, !}}$, which verify the following: for any neat open compact K small enough, there exist isomorphic Hecke submodules $H^2(\pi_f(\phi))$ of $H_1^2(S_K(\mathbb{C}), V_k \otimes_F \bar{\mathbb{Q}})$ and of $H_1^2(S_K(\mathbb{C}), V_k^\vee \otimes_F \bar{\mathbb{Q}})$ such that*

$$(39) \quad H^2(\pi_f(\phi)) \simeq \pi_f(\phi)^K \otimes H_{J_{\phi, !}}$$

as a Hecke module and as Hodge structure. The Hodge structure $H_{J_{\phi, !}}$ attached to the submodule of $H_1^2(S_K(\mathbb{C}), V_k \otimes_F \bar{\mathbb{Q}})$ is of type $(k+2, 0)$ and the one attached to the submodule of $H_1^2(S_K(\mathbb{C}), V_k^\vee \otimes_F \bar{\mathbb{Q}})$ is of type $(0, k+2)$.

Proof. (1) The automorphic representation $\pi(\phi)$ belongs to the A -packet $\Pi(\rho)$ of [23, p. 396], and by construction, the number $n(\pi)$ of *loc. cit.* is equal to 1. Now write ϕ_R for the character denoted ϕ in *loc. cit.*. We have already observed above that $\phi_R = \phi_u^{(1)}$, so that by (35) and (37)

$$L\left(\phi_R, \frac{1}{2}\right) = L\left(\phi \cdot |\cdot|_L^{-\frac{3}{2}}, \frac{1}{2}\right) = L(\phi, -1)$$

which shows that our hypothesis (2) on ϕ is precisely equivalent to $\epsilon(\frac{1}{2}, \phi_R) = -1$. By [23, Thm. 1.1], we obtain that the multiplicity of $\pi(\phi)$ in the discrete spectrum of \tilde{G}_0 is 1. By [21, Thm. 13.3.6 (a)], it is actually a representation appearing in the *cuspidal* spectrum.

(2) For any local system $\tilde{\mathcal{M}}_\lambda$, recall that the *cuspidal cohomology* $H_{\text{cusp}}^\bullet(S(\mathbb{C}), \tilde{\mathcal{M}}_{\lambda, \mathbb{C}})$ is the sub- $\tilde{G}(\mathbb{A}_f)$ -module of the interior cohomology $H_!^\bullet(S(\mathbb{C}), \tilde{\mathcal{M}}_{\lambda, \mathbb{C}})$ (34), defined as the direct sum of all the irreducible sub- $\tilde{G}(\mathbb{A}_f)$ -modules of $H_!^\bullet(S(\mathbb{C}), \tilde{\mathcal{M}}_{\lambda, \mathbb{C}})$ isomorphic to the non-archimedean component of some cuspidal automorphic representation of $\tilde{G}(\mathbb{A})$. We denote the set of these isomorphism classes by $\mathcal{C}_{\text{cusp}}$. By multiplicity one for the discrete spectrum of $\tilde{G}(\mathbb{A})$ ([21, Thm. 13.3.1]), cuspidal cohomology decomposes as

$$(40) \quad H_{\text{cusp}}^\bullet(S(\mathbb{C}), \tilde{\mathcal{M}}_{\lambda, \mathbb{C}}) \simeq \bigoplus_{\substack{\pi = \pi_f \otimes \pi_\infty \\ \text{s. t. } \pi_f \in \mathcal{C}_{\text{cusp}}}} \pi_f \otimes H^\bullet(\mathfrak{g}, \tilde{K}_\infty, \pi_\infty \otimes \tilde{\mathcal{M}}_{\lambda, \mathbb{C}})$$

Upon choosing a $\overline{\mathbb{Q}}$ -model for each submodule π_f , denoted by the same symbol, we get a $\overline{\mathbb{Q}}$ -model for cuspidal cohomology, described as

$$(41) \quad H_{\text{cusp}}^\bullet(S(\mathbb{C}), \tilde{\mathcal{M}}_{\lambda, \overline{\mathbb{Q}}}) := \bigoplus_{\substack{\pi = \pi_f \otimes \pi_\infty \\ \text{s. t. } \pi_f \in \mathcal{C}_{\text{cusp}}}} \pi_f \otimes \text{Hom}_{\tilde{G}(\mathbb{A}_f)}(\pi_f, H_!^\bullet(S(\mathbb{C}), \tilde{\mathcal{M}}_{\lambda, \overline{\mathbb{Q}}}))$$

where each space $\text{Hom}_{\tilde{G}(\mathbb{A}_f)}(\pi_f, H_!^\bullet(S(\mathbb{C}), \tilde{\mathcal{M}}_{\lambda, \overline{\mathbb{Q}}}))$ is endowed with a canonical $\overline{\mathbb{Q}}$ -Hodge structure. Now take for $\tilde{\mathcal{M}}_{\lambda, \overline{\mathbb{Q}}}$ either $V_k \otimes_F \overline{\mathbb{Q}}$ or $V_k^\vee \otimes_F \overline{\mathbb{Q}}$ and consider the submodule corresponding to $\pi = \pi(\phi)$. By the comparison isomorphism between singular interior cohomology and étale interior cohomology, the dimension of the Hodge structure

$$H_{J_\phi, !} := \text{Hom}_{\tilde{G}(\mathbb{A}_f)}(\pi(\phi)_f, H_!^2(S(\mathbb{C}), \tilde{\mathcal{M}}_{\lambda, \overline{\mathbb{Q}}}))$$

is the same as the dimension of the space $V^2(\pi_f)$ of Case 5 of [22, pp. 91-92], which according to *loc. cit.* is equal to 1, by our choice of the archimedean component of $\pi(\phi)$ mentioned above. We obtain then the desired Hecke submodule. Finally, to get the assertion on the type of $H_{J_\phi, !}$, we apply the following two facts: (a) if r is the weight of the pure Hodge structure on $\tilde{\mathcal{M}}_\lambda$ (so that $r = k$ in both the cases we are treating, by Remark 4.4 and (17)), then $H_!^2(S(\mathbb{C}), \tilde{\mathcal{M}}_\lambda)$ has weight $r + 2$. Its Hodge substructure $H_{J_\phi, !}$ has thus the same weight; (b) the Hodge types of $H_{J_\phi, !}$ are those (p, q) 's appearing in the (p, q) -decomposition of the space $H^2(\mathfrak{g}, \tilde{K}_\infty, \pi(\phi)_\infty \otimes \tilde{\mathcal{M}}_{\lambda, \mathbb{C}})$, which is in turn determined by the (p, q) -decomposition of the discrete series

representation $\pi(\phi)_\infty$, and the discrete series representations π^+ and π^- discussed above, are respectively holomorphic and antiholomorphic ([5, 1.3]). \square

6.2. Construction of the extension. We have now all the ingredients needed to construct the desired extensions.

Theorem 6.2. *Consider the \tilde{G} -representation V_k of (18).*

(1) *For any level K small enough, the extension of scalars to $\overline{\mathbb{Q}}$ over \mathbb{Q} of the exact sequence obtained from (25)*

$$(42) \quad H_c^2(S_K(\mathbb{C}), V_k) \rightarrow H^2(S_K(\mathbb{C}), V_k) \xrightarrow{r^2} \partial H^2(S_K(\mathbb{C}), V_k)$$

has a subquotient short exact sequence of Hecke modules and Hodge structures

$$(43) \quad 0 \rightarrow H^2(\pi_f(\phi)) \rightarrow \tilde{H}_0 \rightarrow I_\phi^K \rightarrow 0$$

where $H^2(\pi_f(\phi))$ is as defined in (39) and endowed with a pure $\overline{\mathbb{Q}}$ -Hodge structure of type $(k+2, 0)$ and I_ϕ^K is defined as in (29) and endowed with a pure $\overline{\mathbb{Q}}$ -Hodge structure of type $(1, k+2)$.

(2) *Let n be the dimension of I_ϕ^K . Then the short exact sequence (43) induces an extension of $\overline{\mathbb{Q}}$ -Hodge structures*

$$(44) \quad 0 \rightarrow H_\phi(-1) \rightarrow H_0 \rightarrow \mathbb{1}^{\oplus n} \rightarrow 0$$

where H_ϕ is the 1-dimensional Hodge structure of type $(k, -(k+3))$ attached to ϕ .

Proof. (1) Consider our assumption (1) on ϕ and recall that it is of type $w \star \lambda$, with $w = (1 \ 2 \ 3)$ and $\lambda = (k, 0)$. Then, because of the shape of λ , the first case of part (2) of Lemma 5.5 applies and tells us that the L -function $L(\phi_{\mathbb{Q}}^{(1)} \cdot \epsilon_{L|\mathbb{Q}}, 2s-2)$ has a first-order pole at $s=0$. But thanks to the vanishing of $L(\phi^{(1)}, s-1)$ at $s=0$ given by our assumption (2) on ϕ , our Hecke character belongs to the set Φ_{np} appearing in the statement of Theorem 5.10. Keeping into account that w is of length 2, the latter theorem then implies that after extending scalars to $\overline{\mathbb{Q}}$, we have $I_\phi \hookrightarrow \text{Im}(r^2)$, because by the decomposition (16), the cohomology of $\widetilde{\mathcal{M}}_\lambda$ is a direct summand of the cohomology of $V_k \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$. By Proposition 4.2 and part (4) of Proposition 4.5, I_ϕ^K is endowed with an Hodge structure of type $(1, k+2)$.

Now, again by the decomposition (16), the cohomology of the representation $\widetilde{\mathcal{M}}_{\lambda^-}$ of equation (17) is also a direct summand of the cohomology of $V_k \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$, so that, by Proposition 6.1.(2), after extending scalars to $\overline{\mathbb{Q}}$, we get a direct summand $H^2(\pi_f(\phi))$ of $\ker(r^2)$, that is of interior cohomology in degree 2, endowed with a Hodge structure of type $(k+2, 0)$.

The desired short exact sequence is then obtained by taking pullback via the inclusion $I_\phi \hookrightarrow \text{Im}(r^2)$ and pushout via the projection $\ker(r^2) \rightarrow H^2(\pi_f(\phi))$.

(2) The Hodge structure on I_ϕ^K is a direct sum of copies of the 1-dimensional pure $\overline{\mathbb{Q}}$ -Hodge structure of type $(1, k+2)$. By tensoring with the dual of this latter Hodge structure the extension of Hodge structures defined by (43), one obtains the desired extension. \square

Corollary 6.3. *The choice of an element of I_ϕ^K provides, by pullback of the extension (44), a 1-extension of $\mathbb{1}$ by $H_\phi(-1)$ in the category $\text{MHS}_{\overline{\mathbb{Q}}}$. Hence we obtain a morphism*

$$(45) \quad I_\phi^K \rightarrow \text{Ext}_{\text{MHS}_{\overline{\mathbb{Q}}}}^1(\mathbb{1}, H_\phi(-1)).$$

It thus becomes our task to select an element of I_ϕ^K in such a way to produce a non-trivial extension E , whose class will be the desired element of $\text{Ext}_{\text{MHS}_{\overline{\mathbb{Q}}}}^1(\mathbb{1}, H_\phi(-1))$.

7. NON-TRIVIALITY OF HECKE EXTENSIONS

Throughout this section, we maintain the notation of the previous section. In particular, we consider a fixed Hecke character ϕ satisfying the hypotheses stated at the beginning of Section 6. It is of infinity type $(k, -(k+3))$ for a fixed integer $k \geq 0$ and it has an associated Hodge structure H_ϕ . Moreover, we make use of the \tilde{G} -representation V_k of (18).

We will now give a method to detect non-triviality of elements of $\text{Ext}_{\text{MHS}_{\overline{\mathbb{Q}}}}^1(\mathbb{1}, H_\phi(-1))$ constructed via the morphism (45).

7.1. The Scholl pairing. We begin with the construction of a *dual* extension which will play an auxiliary role in the proof of desired non-triviality of Hecke extension.

Proposition 7.1. (1) *For any level K small enough, the extension of scalars to $\overline{\mathbb{Q}}$ of the exact sequence obtained from (25)*

$$(46) \quad \partial H^1(S_K(\mathbb{C}), V_k) \rightarrow H_c^2(S_K(\mathbb{C}), V_k) \rightarrow H^2(S_K(\mathbb{C}), V_k)$$

provides a short exact sequence of Hecke modules and Hodge structures

$$(47) \quad 0 \rightarrow I_{\theta\phi}^K \rightarrow \tilde{H}'_0 \rightarrow H^2(\pi_f(\phi)) \rightarrow 0$$

where $H^2(\pi_f(\phi))$ is defined as in (39) and endowed with a pure $\overline{\mathbb{Q}}$ -Hodge structure of type $(k+2, 0)$ and $I_{\theta\phi}$ is defined as in (29) and endowed with a pure $\overline{\mathbb{Q}}$ -Hodge structure of type $(0, k+1)$.

(2) *Let n be the dimension of $I_{\theta\phi}^K$. Then the short exact sequence (47) induces an extension of $\overline{\mathbb{Q}}$ -Hodge structures*

$$(48) \quad 0 \rightarrow \mathbb{1}(1)^{\oplus n} \rightarrow H'_0 \rightarrow H_\phi(-1) \rightarrow 0$$

where H_ϕ is the 1-dimensional Hodge structure of type $(k, -(k+3))$ attached to ϕ .

Proof. By remembering that the type of $\theta\phi$ is $(1\ 2) \star \lambda$ with $\lambda = (k, 0)$ and by using part (2) of Proposition 4.5, the statements follow by arguing as in the proof of Proposition 6.2 (or by applying Poincaré duality to the statement of that same proposition). \square

We also get a morphism

$$(49) \quad I_{\theta\phi}^K \rightarrow \text{Ext}_{\text{MHS}_{\overline{\mathbb{Q}}}}^1(H_\phi(-1), \mathbb{1})$$

analogous to (45).

Now recall the notation i, j from Subsection 4.4. Fix a decomposition of the boundary $\partial := \partial S_K^*$ of S_K^* as the disjoint union of two sets of connected components

$$(50) \quad \partial = \Theta \sqcup \Sigma.$$

Proposition 7.2. (1) For any decomposition as in (50) and $\Delta \in \{\Theta, \Sigma\}$, there exist subalgebras $\mathcal{H}_\Delta(G, K)$ of the Hecke algebra $\mathcal{H}(G, K)$, and sub- $\mathcal{H}_\Delta(G, K)$ -modules

$$I_{\phi, \Theta}^K \leq I_\phi^K \quad (\text{for } \Delta = \Theta)$$

$$I_{\theta\phi, \Sigma}^K \leq I_{\theta\phi}^K \quad (\text{for } \Delta = \Sigma)$$

such that the elements of $I_{\phi, \Theta}^K$ are precisely the cohomology classes in I_ϕ^K supported on Θ . and the elements of $I_{\theta\phi, \Sigma}^K$ are precisely the cohomology classes in $I_{\theta\phi}^K$ supported on Σ .

(2) The extensions (43) and (47) provide, by pullback, extensions

$$(51) \quad 0 \rightarrow H^2(\pi_f(\phi)) \rightarrow \tilde{H} \rightarrow I_{\phi, \Theta}^K \rightarrow 0$$

$$(52) \quad 0 \rightarrow I_{\theta\phi, \Sigma}^K \rightarrow \tilde{H}' \rightarrow H^2(\pi_f(\phi)) \rightarrow 0$$

For any $\Delta \in \{\Theta, \Sigma\}$, consider the pair of complementary, open resp. closed immersions

$$j^\Delta : S_K(\mathbb{C}) \hookrightarrow S_K(\mathbb{C}) \sqcup \Delta \hookleftarrow \Delta : i^\Delta$$

and the open immersion

$$j_\Delta : S_K(\mathbb{C}) \sqcup \Delta \hookrightarrow S_K^*(\mathbb{C})$$

and define $i_\Delta := j_\Delta \circ i^\Delta$. There are canonical exact triangles

$$j_{\Delta, *} j_!^\Delta \rightarrow j_* \rightarrow i_{\Delta, *} i^{\Delta, *} j_*^\Delta \rightarrow$$

Denote

$$H_{c, \partial \setminus \Delta}^i(S_K(\mathbb{C}), \widetilde{\mathcal{M}}_\lambda) := H^i(S_K^*(\mathbb{C}), j_{\Delta, *} j_!^\Delta \widetilde{\mathcal{M}}_\lambda),$$

$$H_\Delta^i(\partial S_K(\mathbb{C}), \widetilde{\mathcal{M}}_\lambda) := H^i(S_K^*(\mathbb{C}), i_{\Delta, *} i^{\Delta, *} j_*^\Delta \widetilde{\mathcal{M}}_\lambda).$$

We get a long exact sequence

$$(53) \quad \cdots \rightarrow H_{c, \partial \setminus \Delta}^i(S_K(\mathbb{C}), \widetilde{\mathcal{M}}_\lambda) \rightarrow H^i(S_K(\mathbb{C}), \widetilde{\mathcal{M}}_\lambda) \rightarrow H_\Delta^i(\partial S_K(\mathbb{C}), \widetilde{\mathcal{M}}_\lambda) \rightarrow \cdots$$

Moreover, there are canonical morphisms

$$(54) \quad i_* i^* j_* \xrightarrow{r_\Delta} i_{\Delta, *} i^{\Delta, *} j_*^\Delta$$

such that the diagram

$$\begin{array}{ccc} j_* & & \\ \downarrow & \searrow & \\ & & i_* i^* j_* \\ & \swarrow r_\Delta & \\ & & i_{\Delta, *} i^{\Delta, *} j_*^\Delta \end{array}$$

commutes.

Proposition 7.3. *Fix a decomposition as in (50). Then there exists a canonical mixed $\overline{\mathbb{Q}}$ -Hodge structure \tilde{H}^\sharp such that the extensions (51) and (52) fit in a commutative diagram of mixed $\overline{\mathbb{Q}}$ -Hodge structures, with exact rows and columns*

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & I_{\theta\phi,\Sigma}^K & \xlongequal{\quad} & I_{\theta\phi,\Sigma}^K & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{H}' & \longrightarrow & \tilde{H}^\sharp & \longrightarrow & I_{\phi,\Theta}^K \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & H^2(\pi_f(\phi)) & \longrightarrow & \tilde{H} & \longrightarrow & I_{\phi,\Theta}^K \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

Proof. Denote again by V_k the local system induced by the \tilde{G} -representation of (18) and consider the exact sequence

$$(55) \quad H_\Sigma^1(\partial S_K(\mathbb{C}), V_k) \rightarrow H_{c,\Theta}^2(S_K(\mathbb{C}), V_k) \rightarrow H^2(S_K(\mathbb{C}), V_k) \rightarrow H_\Sigma^2(\partial S_K(\mathbb{C}), V_k)$$

obtained from the long exact sequence (53). We have inclusions

$$I_{\phi,\Theta}^K \hookrightarrow H_\Theta^2(\partial S_K(\mathbb{C}), V_k \otimes_F \overline{\mathbb{Q}}), \text{ resp. } I_{\theta\phi,\Sigma}^K \hookrightarrow H_\Sigma^1(\partial S_K(\mathbb{C}), V_k \otimes_F \overline{\mathbb{Q}})$$

which factor as the composition of

$$I_{\phi,\Theta}^K \hookrightarrow H^2(\partial S_K(\mathbb{C}), V_k \otimes_F \overline{\mathbb{Q}}), \text{ resp. } I_{\theta\phi,\Sigma}^K \hookrightarrow H^1(\partial S_K(\mathbb{C}), V_k \otimes_F \overline{\mathbb{Q}})$$

and of the morphisms induced by (54). By taking pushout via the projection

$$H_\Sigma^1(\partial S_K(\mathbb{C}), V_k \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \rightarrow I_{\theta\phi,\Sigma}^K$$

(which exists by semi-simplicity of $H_\Sigma^1(\partial S_K(\mathbb{C}), \widetilde{\mathcal{M}}_{\lambda,\overline{\mathbb{Q}}})$), the extension of scalars to $\overline{\mathbb{Q}}$ of the sequence (55) yields an exact sequence

$$0 \rightarrow I_{\theta\phi,\Sigma}^K \rightarrow \tilde{H}_\Sigma \rightarrow H^2(S_K(\mathbb{C}), V_k \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$$

Denoting by \tilde{H} the pullback of

$$H^2(S_K(\mathbb{C}), V_k \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \rightarrow H^2(\partial S_K(\mathbb{C}), V_k \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$$

along

$$I_{\phi,\Theta}^K \hookrightarrow H^2(\partial S_K(\mathbb{C}), V_k \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$$

we obtain, by pullback via $\tilde{H} \rightarrow H^2(S_K(\mathbb{C}), V_k \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$, an extension

$$(56) \quad 0 \rightarrow I_{\theta\phi,\Sigma}^K \rightarrow \tilde{H}_\Sigma \rightarrow \tilde{H} \rightarrow 0$$

Consider the space \tilde{H} defined in (51). Pullback of (56) along the induced morphism

$$\tilde{H} \rightarrow \tilde{H}$$

provides an extension

$$0 \rightarrow I_{\theta\phi,\Sigma}^K \rightarrow \tilde{H}^\sharp \rightarrow \tilde{H} \rightarrow 0$$

which gives the right four-term column of the diagram in the statement. \square

Corollary 7.4. *Fix a decomposition as in (50) and let r be the dimension of $I_{\phi,\Theta}^K$ and s the dimension of $I_{\theta\phi,\Sigma}^K$. Consider the Hodge structure on $I_{\phi,\Theta}^K$, a direct sum of r copies of a 1-dimensional pure $\overline{\mathbb{Q}}$ -Hodge structure of type $(1, k+2)$. Tensoring with the dual of the latter the diagram of Proposition 7.3, we obtain a diagram of mixed $\overline{\mathbb{Q}}$ -Hodge structures*

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \mathbb{1}(1)^{\oplus s} & \xlongequal{\quad} & \mathbb{1}(1)^{\oplus s} & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H' & \longrightarrow & H^\sharp & \longrightarrow & \mathbb{1}^{\oplus r} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & H_\phi(-1) & \longrightarrow & H & \longrightarrow & \mathbb{1}^{\oplus r} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Notice that the weight filtration of the mixed Hodge structure H^\sharp has three non-trivial steps, with graded quotients $\mathbb{1}(1)^{\oplus n}$ (weight -2), $H_\phi(-1)$ (weight -1) and $\mathbb{1}^{\oplus m}$ (weight 0). Thus, we have precisely put ourselves on the Hodge-theoretic side of the situation considered by Scholl in [25, 3] (with our H^\sharp playing the role of E in *loc. cit.*).

Theorem 7.5. (1) *Fix a decomposition as in (50) and an isomorphism*

$$\mathrm{Ext}_{\mathrm{MHS}_{\mathbb{R},\mathbb{C}}}^1(\mathbb{1}, \mathbb{1}(1)) \simeq \mathbb{C}$$

Then the diagram of Corollary 7.4 defines a pairing

$$(57) \quad b : I_{\phi,\Theta}^K \times (I_{\theta\phi,\Sigma}^K)^\vee \rightarrow \mathbb{C}$$

(2) *Denote by \mathcal{E}_1 , resp. by \mathcal{E}_2 , the image of the restriction to $I_{\phi,\Theta}^K$ of the morphism (45), resp. of the restriction to $I_{\theta\phi,\Sigma}^K$ of the morphism (49). Choose sections s_1, s_2 of the latter morphisms (thought of as having target in $\mathcal{E}_1, \mathcal{E}_2$). The pairing b induces a pairing*

$$\tilde{b} : \mathcal{E}_1 \times \mathcal{E}_2 \rightarrow \mathbb{C}$$

such that for any $E_1 \in \mathcal{E}_1, E_2 \in \mathcal{E}_2$

$$\tilde{b}(E_1, E_2) = b(s_1(E_1), s_2(E_2))$$

Proof. (1) We follow the construction given in [25, 3.3]. In the diagram of Corollary 7.4, extend scalars to \mathbb{C} and consider the result as a diagram of mixed \mathbb{R} -Hodge structures with coefficients in \mathbb{C} . Then, since $H_{\phi, \mathbb{C}}(-1)$ has weight -1 , Lemma 3.3 implies that the space $\text{Ext}_{\text{MHS}_{\mathbb{R}, \mathbb{C}}}^1(\mathbf{1}, H_{\phi, \mathbb{C}}(-1))$ is trivial, so that there is a splitting

$$(58) \quad H_{\mathbb{C}}^{\sharp} = H_{\phi, \mathbb{C}}(-1) \oplus W$$

where W is an extension

$$(59) \quad 0 \rightarrow \mathbf{1}(1)^{\oplus s} \rightarrow W \rightarrow \mathbf{1}^{\oplus r} \rightarrow 0$$

which defines an element of

$$\text{Ext}_{\text{MHS}_{\mathbb{R}, \mathbb{C}}}^1(\mathbf{1}^{\oplus r}, \mathbf{1}(1)^{\oplus s}) = \text{Hom}(I_{\phi, \theta}^K, I_{\theta, \Sigma}^K) \otimes \text{Ext}_{\text{MHS}_{\mathbb{R}, \mathbb{C}}}^1(\mathbf{1}, \mathbf{1}(1))$$

where we have remembered that before extending scalars to \mathbb{C} , the underlying vector spaces of the Hodge structures $\mathbf{1}^{\oplus m}$ and $\mathbf{1}(1)^{\oplus n}$ are respectively $I_{\phi, \theta}^K$ and $I_{\theta, \Sigma}^K$. Lemma 22 implies that there exists an isomorphism

$$\text{Ext}_{\text{MHS}_{\mathbb{R}, \mathbb{C}}}^1(\mathbf{1}, \mathbf{1}(1)) \simeq \mathbb{C}$$

so that the extension class of V can be interpreted as a bilinear pairing

$$b : I_{\phi, \Theta}^K \times (I_{\theta, \Sigma}^K)^{\vee} \rightarrow \mathbb{C}$$

(2) The pairing \tilde{b} is obtained in the same way as b , by taking pullback via s_1 and pushout via the dual of s_2 . The property in the statement comes from [25, Prop. 3.11]. \square

We call \tilde{b} the *Scholl pairing*; the reader may wish to consult the proof of [25, Thm. 7.7] in order to appreciate the underlying motivic inspiration (in terms of height pairings on algebraic cycles). We conclude that, in order to check non-triviality of an extension class $E \in \mathcal{E}_1$, obtained by evaluation of the morphism (45) on an element $x \in I_{\phi, \Theta}^K$, it is enough to check that there exists $y \in (I_{\theta, \Sigma}^K)^{\vee}$ such that

$$b(x, y) \neq 0$$

7.2. A non-vanishing criterion for the Scholl pairing. Let us now give a concrete recipe, which will allow us to test whether the Scholl pairing of two given elements vanishes. We will use the notations set up at the beginning of Section 5 and write

$$(60) \quad I_{\phi_{\infty}} \otimes V_k := I_{\phi_{\infty}} \otimes_{\mathbb{C}} V_{k, \mathbb{R}}$$

We recall that according to [29, Sec. 4.1], the space $I_{\phi_{\infty}}$ decomposes as an (infinite) direct sum of 1-dimensional spaces generated by *Wigner D-functions*

$$(61) \quad W_{m_1, m_2}^{j, n}$$

where j, n, m_1, m_2 are half-integers subject to a series of conditions.

Define ϕ^{-} to be the Hecke character obtained from ϕ by composing it with complex conjugation, so that in particular, we have

$$(62) \quad \phi_{\infty}^{-} : z \mapsto \overline{\phi_{\infty}(z)}$$

and ϕ^{-} is of infinity type $-(k+3), k$.

Lemma 7.6. (1) The \mathbb{C} -vector spaces $H^2(\mathfrak{g}, \tilde{K}_\infty, I_{\phi_\infty} \otimes V_k)$ and $H^1(\mathfrak{g}, \tilde{K}_\infty, I_{\theta\phi_\infty} \otimes V_k)$ are 1-dimensional.

(2) The (p, q) decomposition of the pure \mathbb{R} -Hodge structure on the underlying \mathbb{R} -vector space of $H^2(\mathfrak{g}, \tilde{K}_\infty, I_{\phi_\infty} \otimes V_k)$ is defined by an isomorphism

$$(63) \quad H^2(\mathfrak{g}, \tilde{K}_\infty, I_{\phi_\infty} \otimes V_{k, \mathbb{C}}) \simeq H^2(\mathfrak{g}, \tilde{K}_\infty, I_{\phi_\infty} \otimes V_k) \oplus H^2(\mathfrak{g}, \tilde{K}_\infty, I_{\phi_\infty} \otimes V_k^\vee)$$

Projection on the first summand on the right defines a 1-dimensional \mathbb{R} -Hodge structure with coefficients in \mathbb{C} on $H^2(\mathfrak{g}, \tilde{K}_\infty, I_{\phi_\infty} \otimes V_k)$.

(3) Fix a generator u of the \mathbb{C} -vector space $H^2(\mathfrak{g}, \tilde{K}_\infty, I_{\phi_\infty} \otimes V_k)$. The \mathbb{R} -Hodge structure with coefficients in \mathbb{C} on $I_{\phi, \Theta}^K \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ is induced by the 1-dimensional \mathbb{R} -Hodge structure with coefficients in \mathbb{C} on $H^2(\mathfrak{g}, \tilde{K}_\infty, I_{\phi_\infty} \otimes V_k)$ via the isomorphism

$$I_{\phi, \Theta}^K \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \simeq I_{\phi, \Theta}^K \otimes_{\overline{\mathbb{Q}}} H^2(\mathfrak{g}, \tilde{K}_\infty, I_{\phi_\infty} \otimes V_k)$$

given by the element u .

Proof. (1) Let \mathfrak{t} , resp. \mathfrak{t}^K denote the complexified Lie algebras of a maximal torus \tilde{T}^M of \tilde{G} , resp. of its intersection with a maximal compact subgroup of $\tilde{G}(\mathbb{R})$, and let \mathfrak{u} denote the complexified Lie algebra of the unipotent radical of a Borel of \tilde{G} . Write \mathbb{C}_{ϕ_∞} for the 1-dimensional \mathbb{C} -vector space on which $\tilde{T}^M(\mathbb{C})$ acts via multiplication by ϕ_∞ . By the so-called Delorme isomorphism, see [6, Thm. 3.3] and also [11, p. 68], we have that

$$H^\bullet(\mathfrak{g}, \tilde{K}_\infty, I_{\phi_\infty} \otimes V_k) \simeq \text{Hom}(\Lambda^\bullet(\mathfrak{t}/\mathfrak{t}^K), (H^\bullet(\mathfrak{u}, V_{k, \mathbb{R}}) \otimes \mathbb{C}_{\phi_\infty})(0))$$

where (0) indicates the weight-zero space for the action of \mathfrak{t} . Now a computation as in the end of the proof of Lemma 4.1 shows that $\mathfrak{t}/\mathfrak{t}^K$ is a 1-dimensional \mathbb{C} -vector space. This fact and an application of Kostant's theorem show that in the left-hand side of the above isomorphism, the degree-2 summand corresponds to a summand on the right, which is a 1-dimensional \mathbb{C} -vector space, on which \tilde{T}^M acts via the character $(1 \ 2 \ 3) \star \lambda$.

An analogous computation, or application of Poincaré duality to the previous computation, shows that $H^1(\mathfrak{g}, \tilde{K}_\infty, I_{\theta\phi_\infty} \otimes V_k)$ is also 1-dimensional.

(2) We look at the complex

$$\cdots \rightarrow \text{Hom}_{\mathfrak{l}}(\Lambda^i(\mathfrak{g}/\mathfrak{l}), I_{\phi_\infty} \otimes V_k) \xrightarrow{d^i} \text{Hom}_{\mathfrak{l}}(\Lambda^{i+1}(\mathfrak{g}/\mathfrak{l}), I_{\phi_\infty} \otimes V_k) \rightarrow \cdots$$

which computes relative Lie algebra cohomology of the (\mathfrak{g}, K_∞) -module $I_{\phi_\infty} \otimes V_k$. The decomposition (19) induces $G(\mathbb{C})$ -equivariant isomorphisms

$$\begin{aligned} (I_{\phi_\infty} \otimes V_k)^+ &\simeq I_{\phi_\infty} \otimes V_k \\ (I_{\phi_\infty} \otimes V_k)^- &\simeq \overline{I_{\phi_\infty}} \otimes V_k^\vee \end{aligned}$$

where $\overline{I_{\phi_\infty}}$ denotes the subspace of those complex-valued functions on $G(\mathbb{R})$ obtained as *complex conjugates* of the elements of I_{ϕ_∞} . Now we represent elements of I_{ϕ_∞} via the Wigner D-functions of (61) and observe that by [29, p. 9, (3)], we have

$$\overline{W_{m_1, m_2}^{j, n}} = (-1)^{m_2 - m_1} W_{-m_1, -m_2}^{j, -n}$$

We get a $G(\mathbb{C})$ -equivariant isomorphism

$$\overline{I_{\phi_\infty}} \simeq I_{\phi_\infty^-}$$

sending $W_{m_1, m_2}^{j, n}$ to $(-1)^{m_2 - m_1} W_{-m_1, -m_2}^{j, -n}$.

Putting everything together, we get a $G(\mathbb{C})$ -equivariant isomorphism

$$(64) \quad I_{\phi_\infty} \otimes V_{k, \mathbb{C}} \simeq (I_{\phi_\infty} \otimes V_k) \oplus (I_{\phi_\infty^-} \otimes V_k^\vee)$$

As a consequence, we get

$$(65) \quad \mathrm{Hom}_l(\Lambda^i(\mathfrak{g}/\mathfrak{l}), I_{\phi_\infty} \otimes V_{k, \mathbb{C}}) \simeq \mathrm{Hom}_l(\Lambda^i(\mathfrak{g}/\mathfrak{l}), I_{\phi_\infty} \otimes V_k) \oplus \mathrm{Hom}_l(\Lambda^i(\mathfrak{g}/\mathfrak{l}), I_{\phi_\infty^-} \otimes V_k^\vee)$$

which induces the desired isomorphism. The latter, in its turn, defines the (p, q) -decomposition by Proposition 4.5.

(3) The \mathbb{R} -Hodge structure with coefficients in \mathbb{C} on $I_{\phi, \Theta}^K \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ is defined by remembering that $I_{\phi, \Theta}^K$ is a subspace of the $V_k \otimes_F \overline{\mathbb{Q}}$ -valued singular cohomology of $\partial \overline{S}_K$, by extending the scalars of the latter to \mathbb{C} and by considering the De Rham isomorphism between the space so obtained and the cohomology of $V_{k, \mathbb{R}}$ -valued differential forms over $\partial \overline{S}_K$. By tracing back all our identifications, this De Rham isomorphism and the choice of a generator of the \mathbb{C} -vector space $H^2(\mathfrak{g}, \tilde{K}_\infty, I_{\phi_\infty} \otimes V_k)$ induce precisely the isomorphism given in the statement. \square

Remark 7.7. Fix a decomposition as in (50). With an abuse of notation, call again W the direct factor appearing in the splitting

$$(66) \quad \tilde{H}_{\mathbb{C}}^\sharp = H^2(\pi_f(\phi))_{\mathbb{C}} \oplus W$$

induced by the splitting (58). Then, by Lemma 7.6(3), W is an extension

$$(67) \quad 0 \rightarrow I_{\theta\phi, \Sigma}^K \otimes_{\overline{\mathbb{Q}}} H^2(\mathfrak{g}, \tilde{K}_\infty, I_{\theta\phi_\infty} \otimes V_k) \xrightarrow{i_W} W \xrightarrow{p_W} I_{\phi, \Theta}^K \otimes_{\overline{\mathbb{Q}}} H^2(\mathfrak{g}, \tilde{K}_\infty, I_{\phi_\infty} \otimes V_k) \rightarrow 0$$

which is trivial if and only if the extension (59) is trivial.

Denoting by π the projection from \tilde{H}^\sharp to V , the extension of scalars to \mathbb{C} of the diagram of Proposition 7.3 provides a diagram of mixed \mathbb{R} -Hodge structures with coefficients in \mathbb{C}

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & I_{\theta\phi, \Sigma}^K \otimes_{\overline{\mathbb{Q}}} \mathbb{C} & \xlongequal{\quad} & I_{\theta\phi, \Sigma}^K \otimes_{\overline{\mathbb{Q}}} \mathbb{C} & & \\
 & & \downarrow i' & & \downarrow i^\perp & \searrow i_W & \\
 & & & & & & W \\
 & & & & & \nearrow \pi & \\
 0 & \longrightarrow & \tilde{H}'_{\mathbb{C}} & \xrightarrow{i^\sharp} & \tilde{H}^\sharp_{\mathbb{C}} & \xrightarrow{p^\sharp} & I_{\phi, \Theta}^K \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \longrightarrow 0 \\
 & & \downarrow p' & & \downarrow p^\perp & & \parallel \\
 0 & \longrightarrow & H^2(\pi_f(\phi))_{\mathbb{C}} & \xrightarrow{i} & \tilde{H}_{\mathbb{C}} & \xrightarrow{p} & I_{\phi, \Theta}^K \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

By redoing the same constructions starting with the Hecke character ϕ^- of (62), we obtain an extension

$$(68) \quad 0 \rightarrow I_{\theta\phi^-, \Sigma}^K \otimes_{\overline{\mathbb{Q}}} H^2(\mathfrak{g}, \tilde{K}_\infty, I_{\theta\phi^-} \otimes V_k^\vee) \xrightarrow{i_{\overline{W}}} \overline{W} \xrightarrow{p_{\overline{W}}} I_{\phi^-, \Theta}^K \otimes_{\overline{\mathbb{Q}}} H^2(\mathfrak{g}, \tilde{K}_\infty, I_{\phi^-} \otimes V_k^\vee) \rightarrow 0$$

so that we get an isomorphism

$$(69) \quad W_{\mathbb{C}} \simeq W^+ \times W^-$$

with a \mathbb{C} -linear isomorphism $W \simeq W^+$ and $\overline{W} \simeq W^-$ (that we will implicitly use in the following). Hence, W and \overline{W} are identified with extensions of the form W^+ , W^- , obtained, as in Proposition 3.4, from the extension of scalars (to \mathbb{C} over \mathbb{R}) of (67).

The image of an element x under one of the \mathbb{C} -antilinear isomorphisms

$$W \simeq \overline{W}, I_{\theta\phi^-, \Sigma}^K \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \simeq I_{\theta\phi, \Sigma}^K \otimes_{\overline{\mathbb{Q}}} \mathbb{C}, I_{\phi, \Theta}^K \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \simeq I_{\phi^-, \Theta}^K \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$$

will be denoted by \bar{x} .

Proposition 7.8. *Fix a decomposition as in (50) and generators u and v of the \mathbb{C} -vector spaces $H^2(\mathfrak{g}, \tilde{K}_\infty, I_{\phi_\infty} \otimes V_k)$ and $H^1(\mathfrak{g}, \tilde{K}_\infty, I_{\theta\phi_\infty} \otimes V_k)$. Moreover, following the notations from Remark 7.7, fix a section σ of p_W and a section σ_F of*

$$(p_W, p_{\overline{W}}) : W \times \overline{W} \rightarrow (I_{\phi, \Theta}^K \otimes_{\overline{\mathbb{Q}}} \mathbb{C}) \times (I_{\phi^-, \Theta}^K \otimes_{\overline{\mathbb{Q}}} \mathbb{C})$$

respecting the Hodge filtration.

For $x \in I_{\phi, \Theta}^K$ and $y \in (I_{\theta\phi, \Sigma}^K)^\vee$, take

$$(\epsilon_x, \eta_x), (\epsilon'_x, \eta'_x) \in (I_{\theta\phi, \Sigma}^K \otimes_{\overline{\mathbb{Q}}} \mathbb{C}) \times (I_{\theta\phi^-, \Sigma}^K \otimes_{\overline{\mathbb{Q}}} \mathbb{C})$$

such that

$$(i_W, i_{\overline{W}})(\epsilon_x, \eta_x) = \sigma_F((x \otimes u, \bar{x} \otimes \bar{u})) - \sigma \otimes 1((x \otimes u, \bar{x} \otimes \bar{u}))$$

$$(i_W, i_{\overline{W}})(\epsilon'_x, \eta'_x) = \sigma_F((x \otimes iu, -\bar{x} \otimes i\bar{u})) - \sigma \otimes 1((x \otimes iu, -\bar{x} \otimes i\bar{u}))$$

Then if b is the pairing of (57), we have

$$b(x, y) = 0$$

if and only if the complex numbers

$$(y \otimes v^\vee, \bar{y} \otimes \bar{v}^\vee)(\eta_x, \epsilon_x)$$

$$(iy \otimes v^\vee, -i\bar{y} \otimes \bar{v}^\vee)(\eta'_x, \epsilon'_x)$$

are conjugated to each other.

Proof. Let $x \in I_{\phi, \Theta}^K$ and $y \in (I_{\theta\phi, \Sigma}^K)^\vee$. Unraveling the construction of the pairing b of (57), we see that b sends (x, y) to the class of the 1-extension of $\mathbf{1}$ by $\mathbf{1}(1)$ obtained from the extension W of (59) by pullback via x and pushout via y . This extension is trivial if and only if the analogous one obtained by replacing W with the extension \overline{W} of Remark 7.7 is trivial. Following the method described in Proposition 3.4, we obtain the statement. \square

The above Proposition gives the promised recipe for checking the non-vanishing of the pairing b when evaluated on classes x and y . Making it explicit becomes then a matter of finding concrete representatives for the Lie algebra cohomology classes u and v , for the lifts of $x \otimes u$ in the space $W_{\mathbb{C}}$, and for their duals. This will be the object of the forthcoming paper [2], aiming at a proof of Conjecture 1 in Section 1.

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