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## $\mathbb{A}_{\text{inf}}$ IS INFINITE DIMENSIONAL

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*Abstract* Given a perfect valuation ring  $R$  of characteristic  $p$  that is complete with respect to a rank-1 nondiscrete valuation, we show that the ring  $\mathbb{A}_{\text{inf}}$  of Witt vectors of  $R$  has infinite Krull dimension.

*Keywords:*  $p$ -adic Hodge theory; Fargues-Fontaine curves; Krull dimension; Newton polygons

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Secondary 14F30

### 1. Introduction

Fix a prime  $p$ . Let  $R$  be a perfect valuation ring of characteristic  $p$  and denote the valuation by  $v$ . Assume  $v$  is of rank 1 and nondiscrete and that  $R$  is complete with respect to  $v$ . Let  $\mathbb{A} := \mathbb{A}_{\text{inf}} := W(R)$  be the ring of Witt vectors of  $R$ . This ring plays a central role in  $p$ -adic Hodge theory as it is the basic ring from which all of Fontaine’s  $p$ -adic period rings are built. It is also central to the construction of the (adic) Fargues–Fontaine curve [4]. Recently, Bhatt, Morrow and Scholze constructed  $\mathbb{A}_{\text{inf}}$ -cohomology, a cohomology theory that specializes to étale, de Rham and crystalline cohomology [3]. In these works, there is a useful analogy between  $\mathbb{A}$  and a two-dimensional regular local ring. In this paper, we prove the following theorem.

**Theorem 1.1.** *The ring  $\mathbb{A}$  has infinite Krull dimension.*

Bhatt [2, Warning 2.24] and Kedlaya [5, Remark 1.6] note that the Krull dimension of  $\mathbb{A}$  is at least 3. To see this, fix a pseudouniformizer  $\varpi \in R$  and let  $\kappa$  denote the residue field of  $R$ . Let  $W(\mathfrak{m})$  be the kernel of the natural map  $W(R) \rightarrow W(\kappa)$  and  $[-]: R \rightarrow W(R)$  the Teichmüller map. Then Bhatt and Kedlaya point out that  $\mathbb{A}$  contains the following explicit chain of prime ideals:

$$(0) \subset \mathfrak{p} := \bigcup_{k=0}^{\infty} [\varpi^{1/p^k}] \mathbb{A} \subset W(\mathfrak{m}) \subset (p, W(\mathfrak{m})).$$

As suggested in [5, Remark 1.6], we use Newton polygons to find an infinite chain of prime ideals between  $\mathfrak{p}$  and  $W(\mathfrak{m})$ .

The equal characteristic analogue of Theorem 1.1 is the statement that the power series ring  $R[[X]]$  has infinite Krull dimension. This was first proved by Arnold [1, Theorem 1], and the structure of our argument is very similar to his. We axiomatize Arnold’s argument in Section 3.

**Notation.** We use the convention that the symbols  $<, >, \subset, \supset$  denote strict inequalities and inclusions with the exception that we allow the statement “ $\infty < \infty$ ” to be true. Otherwise, if equality is allowed, it will be explicitly reflected in the notation using the symbols  $\leq, \geq, \subseteq, \supseteq$ . An inequality between two  $(\mathbb{R} \cup \{\pm\infty\})$ -valued functions means that the inequality holds pointwise.

### 2. Review of Newton polygons

As above, let  $R$  be a perfect valuation ring of characteristic  $p$  that is complete with respect to a nondiscrete valuation  $v$  of rank 1. Let  $\mathfrak{m}$  be the maximal ideal of  $R$ , and fix an element  $\varpi \in \mathfrak{m}$  of valuation  $v(\varpi) = 1$ .

Let  $\mathbb{A} := W(R)$  be the ring of Witt vectors of  $R$ . Write  $[-]: R \rightarrow \mathbb{A}$  for the Teichmüller map, which is multiplicative. Recall that every element of  $\mathbb{A}$  can be written uniquely in the form  $\sum_{n \geq 0} [x_n]p^n$  with  $x_n \in R$ .

As in [4, Section 1.5.2], given  $f \in \mathbb{A}$  with  $f = \sum_{n \geq 0} [x_n]p^n$ , we define the *Newton polygon*  $\mathcal{N}(f)$  of  $f$  as the largest decreasing convex polygon in  $\mathbb{R}^2$  lying below the set of points  $\{(n, v(x_n)) : n \geq 0\}$ . We shall often view  $\mathcal{N}(f)$  as the graph of a function  $\mathcal{N}(f): \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ . In particular, if  $n_f$  is the smallest integer such that  $x_{n_f} \neq 0$ , then  $\mathcal{N}(f)(t) = +\infty$  for  $t < n_f$  and  $\mathcal{N}(f)(n_f) = v(x_{n_f})$ . Furthermore,  $\lim_{t \rightarrow \infty} \mathcal{N}(f)(t) = \inf_n v(x_n)$ .

Following the conventions in [4, Section 1.5.2], for any integer  $i \geq 0$  define

$$s_i(f) := \mathcal{N}(f)(i) - \mathcal{N}(f)(i + 1).$$

We call  $s_i(f)$  the *slope* of  $\mathcal{N}(f)$  on the interval  $[i, i + 1]$  even though one would typically call that slope  $-s_i(f)$ . With this convention, the slopes form a nonnegative decreasing sequence; that is,  $s_i(f) \geq s_{i+1}(f) \geq 0$  for all  $i$ . We say that  $n$  is a *node* of  $\mathcal{N}(f)$  if  $\mathcal{N}(f)(n) = v(x_n)$ .

We recall the theory of Legendre transforms from [4, Section 1.5.1]. Given a function  $\varphi: \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$  that is not identically equal to  $+\infty$ , define

$$\begin{aligned} \mathcal{L}(\varphi): \mathbb{R} &\rightarrow \mathbb{R} \cup \{-\infty\} \\ \lambda &\mapsto \inf_{t \in \mathbb{R}} \{\varphi(t) + \lambda t\}. \end{aligned}$$

If  $\varphi$  is a convex function, then one can recover  $\varphi$  from  $\mathcal{L}(\varphi)$  via the formula

$$\varphi(t) = \sup_{\lambda \in \mathbb{R}} \{\mathcal{L}(\varphi)(\lambda) - t\lambda\}.$$

From these definitions, it is easy to see that  $\mathcal{N}(f) \leq \mathcal{N}(g)$  if and only if  $\mathcal{L}(\mathcal{N}(f)) \leq \mathcal{L}(\mathcal{N}(g))$ .

As explained in [4, Section 1.5], for any  $f, g \in \mathbb{A}$ , we have

$$\mathcal{L}(\mathcal{N}(fg)) = \mathcal{L}(\mathcal{N}(f)) + \mathcal{L}(\mathcal{N}(g)). \tag{1}$$

Motivated by this, one defines a convolution product on the set of  $(\mathbb{R} \cup \{+\infty\})$ -valued convex functions on  $\mathbb{R}$  that are not identically  $+\infty$  by

$$(\varphi * \psi)(t) := \sup_{\lambda \in \mathbb{R}} \{\mathcal{L}(\varphi)(\lambda) + \mathcal{L}(\psi)(\lambda) - t\lambda\}.$$

Thus we have  $\mathcal{N}(fg) = \mathcal{N}(f) * \mathcal{N}(g)$ . In particular, if  $\mathcal{N}(f) > 0$ , then  $\mathcal{N}(f^m) < \mathcal{N}(f^{m+1})$  for all  $m \geq 1$ , and for any  $t \in \mathbb{R}$  we have  $\lim_{m \rightarrow \infty} \mathcal{N}(f^m)(t) = +\infty$ .

There is another way of describing  $\mathcal{N}(fg)$  in terms of  $\mathcal{N}(f)$  and  $\mathcal{N}(g)$  without explicitly using Legendre transforms. Write  $f = \sum_{n \geq 0} [x_n]p^n$  and  $g = \sum_{n \geq 0} [y_n]p^n$ , and let  $n_f$  (respectively,  $n_g$ ) be the smallest integer such that  $x_n \neq 0$  (respectively,  $y_n \neq 0$ ). Then  $\mathcal{N}(fg)(t) = +\infty$  for all  $t < n_f + n_g$ , and  $\mathcal{N}(fg)(n_f + n_g) = v(x_{n_f}) + v(y_{n_g})$ . The slopes of  $\mathcal{N}(fg)$  are given by interlacing the slopes of  $\mathcal{N}(f)$  and  $\mathcal{N}(g)$ . That is, the slope sequence of  $\mathcal{N}(fg)$  is given by combining the sequences  $\{s_i(f) : i \geq 0\}$  and  $\{s_i(g) : i \geq 0\}$  into a single decreasing sequence that incorporates all positive elements of both sequences. The relationship between this description and equation (1) is explained in [4, Section 1.5].

**Lemma 2.1.** *Let  $f$  be an element of  $\mathbb{A}$  such that  $\mathcal{N}(f) > 0$ . If  $g$  is an element of  $\mathbb{A}$  and  $t_0 \geq 0$  is such that for all  $t \geq t_0$  we have  $\mathcal{N}(g)(t) \leq \mathcal{N}(f)(t)$ , then for all  $m$  sufficiently large we have  $\mathcal{N}(g) \leq \mathcal{N}(f^m)$ .*

**Proof.** As noted above, since  $\mathcal{N}(f) > 0$ , the sequence  $\{\mathcal{N}(f^m)\}_m$  converges to  $+\infty$ . This convergence is uniform on the compact interval  $[0, t_0]$ . Thus for  $m$  sufficiently large, it follows that  $\mathcal{N}(g)(t) \leq \mathcal{N}(f^m)(t)$  for all  $t \in [0, t_0]$ . On the other hand, for all  $t \geq t_0$  we have

$$\mathcal{N}(g)(t) \leq \mathcal{N}(f)(t) < \mathcal{N}(f^m)(t).$$

Thus  $\mathcal{N}(g) \leq \mathcal{N}(f^m)$  for all  $m$  sufficiently large. □

**Proposition 2.2.** *The ideal  $\mathfrak{p} := \bigcup_{k=0}^{\infty} [\varpi^{1/p^k}] \mathbb{A}$  is a prime ideal of  $\mathbb{A}$ .*

**Proof.** Note that an element  $f$  of  $\mathbb{A}$  lies in  $\mathfrak{p}$  if and only if  $\lim_{t \rightarrow \infty} \mathcal{N}(f)(t) > 0$ . If  $g, g' \in \mathbb{A} \setminus \mathfrak{p}$ , then  $\lim_{t \rightarrow \infty} \mathcal{N}(gg')(t) = \lim_{t \rightarrow \infty} (\mathcal{N}(g) * \mathcal{N}(g'))(t) = 0$  and so  $gg' \notin \mathfrak{p}$ . □

### 3. The strategy

We define infinitely many sequences in  $R$  as follows. For all  $i \geq 0$ , define  $a_{1,i} := \varpi^{1/p^i} \in R$ . For  $n > 1$  and  $i \geq 0$ , define  $a_{n,i}$  recursively by

$$a_{n,i} := a_{n-1,i^2} \in R.$$

Thus  $a_{n,i} = \varpi^{1/p^{n_i}}$ , where  $n_i := i^{2^{n-1}}$ , and  $v(a_{n,i}) = p^{-n_i}$ . For each  $n \geq 1$ , define

$$h_n := \sum_{i=0}^{\infty} [a_{n,i}]p^i \in \mathbb{A}.$$

Note that  $\mathcal{N}(h_n) > 0$ , for any  $n$  we have  $\lim_{t \rightarrow \infty} \mathcal{N}(h_n)(t) = 0$ , and  $\mathcal{N}(h_n)$  has a node at every integer.

Finally, we define the following subsets of  $\mathbb{A}$ . For  $n \geq 1$ , let

$$\mathcal{S}_n := \{g \in \mathbb{A} : 0 < \mathcal{N}(g) \leq \mathcal{N}(h_n^m) \text{ for some } m \geq 1\}.$$

In particular,  $h_n \in \mathcal{S}_n$ .

**Proposition 3.1.** *The sets  $\mathcal{S}_n$  satisfy the following three properties:*

- (1) for all  $n \geq 1$  we have  $\mathcal{S}_{n+1} \subset \mathcal{S}_n$ ;
- (2) each  $\mathcal{S}_n$  is multiplicatively closed;
- (3) for any  $g \in \mathcal{S}_{n+1}$  and  $f \in \mathbb{A}$ , we have that  $g + fh_n \in \mathcal{S}_{n+1}$ .

We prove this proposition in Section 4.

**Theorem 3.2.** *The ring  $\mathbb{A}$  has infinite Krull dimension.*

**Proof.** We follow Arnold’s proof of [1, Theorem 1]. We prove that for any  $n \geq 1$ , there exists a chain of prime ideals of  $\mathbb{A}$ , say  $\mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_n$ , such that  $\mathfrak{p}_n \cap \mathcal{S}_n = \emptyset$ .

For  $n = 1$ , let  $\mathfrak{p}_1 = \mathfrak{p}$ . To see that  $\mathfrak{p} \cap \mathcal{S}_1 = \emptyset$ , note that if  $f \in \mathfrak{p}$ , then  $f \in [\varpi^{1/p^k}]\mathbb{A}$  for some  $k \geq 0$ , and so  $\mathcal{N}(f) \geq 1/p^k$ . On the other hand, if  $f \in \mathcal{S}_1$ , then for some  $m \geq 1$  we have that  $\lim_{t \rightarrow \infty} \mathcal{N}(f)(t) \leq \lim_{t \rightarrow \infty} \mathcal{N}(h_1^m)(t) = 0$ .

Fix  $n \geq 1$  and suppose for induction that there is a chain  $\mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_n$  of prime ideals of  $\mathbb{A}$  such that  $\mathfrak{p}_n \cap \mathcal{S}_n = \emptyset$ . Consider the ideal  $\mathfrak{a}_n := \mathfrak{p}_n + h_n\mathbb{A}$ . Note that  $\mathfrak{a}_n \neq \mathfrak{p}_n$  since  $h_n \in \mathcal{S}_n$  and  $\mathfrak{p}_n \cap \mathcal{S}_n = \emptyset$ . We claim that  $\mathfrak{a}_n \cap \mathcal{S}_{n+1} = \emptyset$ . Indeed, given  $g \in \mathcal{S}_{n+1}$ , we have that  $g + h_n f \in \mathcal{S}_{n+1}$  for all  $f \in \mathbb{A}$  by property (3) of the sets  $\mathcal{S}_n$ . By property (1), it follows that  $g + h_n f \in \mathcal{S}_n$  for all  $f \in \mathbb{A}$ . If  $g \in \mathfrak{a}_n$ , then there is some  $f \in \mathbb{A}$  such that  $g + h_n f \in \mathfrak{p}_n$ . But  $\mathfrak{p}_n \cap \mathcal{S}_n = \emptyset$ , so it follows that  $g \notin \mathfrak{a}_n$ .

Since  $\mathcal{S}_{n+1}$  is multiplicatively closed by property (2), there is a prime ideal  $\mathfrak{p}_{n+1}$  of  $\mathbb{A}$  such that  $\mathfrak{p}_n \subset \mathfrak{a}_n \subseteq \mathfrak{p}_{n+1}$  and  $\mathfrak{p}_{n+1} \cap \mathcal{S}_{n+1} = \emptyset$ . By induction on  $n$ , it follows that  $\mathbb{A}$  has infinite Krull dimension. □

**Remark 3.3.** (a) *Arnold has used an argument as above to show that the ring  $R[[X]]$  has infinite Krull dimension [1, Theorem 1]. In fact given any ring  $A$ , if one can exhibit elements  $h_n$  of  $A$  and sets  $\mathcal{S}_n$  satisfying the properties in Proposition 3.1 together with a prime ideal  $\mathfrak{p}$  such that  $\mathfrak{p} \cap \mathcal{S}_1 = \emptyset$ , then the above argument shows that  $A$  has infinite Krull dimension.*

(b) *There is a rigorous way to view the power series ring  $R[[X]]$  as an equal characteristic version of  $\mathbb{A}$  (see [4, Section 1.3]). Our definitions make sense in this more general setting, and our arguments give another proof that  $R[[X]]$  has infinite Krull dimension.*

### 4. The proof of Proposition 3.1

In this section we prove Proposition 3.1. Recall that  $v$  is the valuation on  $R$  and  $s_i(h_n) := v(a_{n,i-1}/a_{n,i})$  is the  $i$ th slope of  $\mathcal{N}(h_n)$ .

**Proposition 4.1.** Fix  $n, m \geq 1$ . For  $t > 2m^2$  we have that

$$\mathcal{N}(h_{n+1}^m)(t) < \mathcal{N}(h_n)(t).$$

**Proof.** Let  $\ell = km + r \in \mathbb{Z}$  with  $k > 2m$  and  $0 \leq r < m$ . We have

$$\mathcal{N}(h_n)(\ell) = v(a_{n,\ell}) = v(a_{n,km+r})$$

and

$$\mathcal{N}(h_{n+1}^m)(\ell) = mv(a_{n+1,k}) - s_{k+1}(h_{n+1})r \leq mv(a_{n+1,k}) = mv(a_{n,k^2}).$$

To see that  $mv(a_{k^2}) < v(a_{n,km+r})$ , recall that  $v(a_{n,i}) = p^{-i^{2n-1}}$ . Thus we must show that

$$m < p^{k^{2n} - (km+r)^{2n-1}}.$$

Since  $r < m$ , it suffices to show that  $m < p^{k^{2n} - ((k+1)m)^{2n-1}}$ . One checks this quickly using that  $k > 2m$  and therefore  $k^2 - (km + m) > m$ . □

**Corollary 4.2.** For all  $n \geq 1$  we have  $\mathcal{S}_{n+1} \subset \mathcal{S}_n$ .

**Proof.** If  $g \in \mathcal{S}_{n+1}$  then for some  $m \geq 1$  we have  $0 < \mathcal{N}(g) \leq \mathcal{N}(h_{n+1}^m)$ . By Proposition 4.1 and Lemma 2.1, it follows that for  $m'$  sufficiently large (depending on  $m$  and  $n$ ), we have  $\mathcal{N}(h_{n+1}^m) < \mathcal{N}(h_n^{m'})$ , so  $g \in \mathcal{S}_n$ . To see that the inclusion is strict, note that Proposition 4.1 also implies that  $h_n \notin \mathcal{S}_{n+1}$ , but  $h_n \in \mathcal{S}_n$ . □

**Proposition 4.3.** Let  $h$  be an element of  $\mathbb{A}$  such that  $\mathcal{N}(h) > 0$ . Then for any  $f \in \mathbb{A}$ ,  $\mathcal{N}(fh) \geq \mathcal{N}(h)$ .

**Proof.** The Newton polygon  $\mathcal{N}(fh)$  starts at  $n_f + n_h$ . Note that the slopes of  $\mathcal{N}(fh)$  are all positive and form a monotone sequence converging to zero. Therefore all slopes  $s_i(h)$  of  $h$  eventually occur as slopes of  $\mathcal{N}(fh)$ . It follows that for any  $l \geq n_f + n_h$ ,  $\mathcal{N}(fh)(l) \geq \sum_{i \geq l}^\infty s_i(h) = \mathcal{N}(h)(l)$ . □

**Proposition 4.4.** For each  $n \geq 1$ , the set  $\mathcal{S}_n$  is multiplicatively closed.

**Proof.** Let  $f, g \in \mathcal{S}_n$ . Then by Proposition 4.3, we have that  $\mathcal{N}(fg) \geq \mathcal{N}(g) > 0$ .

For  $m$  sufficiently large, we have  $0 < \mathcal{N}(f), \mathcal{N}(g) \leq \mathcal{N}(h_n^m)$ . Thus for any  $\lambda, t \in \mathbb{R}$  we have

$$\mathcal{N}(f)(t) + \lambda t \leq \mathcal{N}(h_n^m)(t) + \lambda t.$$

Taking the infimum over  $t \in \mathbb{R}$ , it follows that  $\mathcal{L}(\mathcal{N}(f))(\lambda) \leq \mathcal{L}(\mathcal{N}(h_n^m))(\lambda)$  for all  $\lambda \in \mathbb{R}$ . Similarly,  $\mathcal{L}(\mathcal{N}(g)) \leq \mathcal{L}(\mathcal{N}(h_n^m))$ . Therefore

$$\mathcal{L}(\mathcal{N}(fg)) = \mathcal{L}(\mathcal{N}(f)) + \mathcal{L}(\mathcal{N}(g)) \leq 2\mathcal{L}(\mathcal{N}(h_n^m)) = \mathcal{L}(\mathcal{N}(h_n^{2m})).$$

Hence, we have that  $\mathcal{L}(\mathcal{N}(fg))(\lambda) - t\lambda \leq \mathcal{L}(\mathcal{N}(h_n^{2m}))(\lambda) - t\lambda$  for all  $t, \lambda \in \mathbb{R}$ . It follows that

$$\mathcal{N}(fg)(t) = \sup_{\lambda} \{\mathcal{L}(\mathcal{N}(fg))(\lambda) - t\lambda\} \leq \sup_{\lambda} \{\mathcal{L}(\mathcal{N}(h_n^{2m}))(\lambda) - t\lambda\} = \mathcal{N}(h_n^{2m})(t)$$

for all  $t \in \mathbb{R}$ . Therefore  $fg \in \mathcal{S}_n$ . □

Let  $f, g \in \mathbb{A}$ , and write  $f = \sum_{n=0}^{\infty} [x_n]p^n$  and  $g = \sum_{n=0}^{\infty} [y_n]p^n$ . In order to prove property (3) from Proposition 3.1 we need to understand the Newton polygon of  $f + g$  in terms of those of  $f$  and  $g$ . For that, we show a property of Witt vector addition in Lemma 4.5 below. First, recall the translation between Teichmüller expansions and Witt coordinates:

$$\sum_{n=0}^{\infty} [x_n]p^n = (x_0, x_1^p, x_2^{p^2}, \dots, x_n^{p^n}, \dots).$$

Recall also that addition of Witt vectors is governed by the polynomials

$$S_n(X_0, \dots, X_n; Y_0, \dots, Y_n),$$

which are defined recursively by

$$S_0(X_0; Y_0) := X_0 + Y_0$$

and

$$\sum_{k=0}^n p^k S_k(X_0, \dots, X_k; Y_0, \dots, Y_k) p^{n-k} = \sum_{k=0}^n p^k (X_k^{p^{n-k}} + Y_k^{p^{n-k}}).$$

Thus

$$\begin{aligned} f + g &= (S_0(x_0; y_0), \dots, S_n(x_0, \dots, x_n^{p^n}; y_0, \dots, y_n^{p^n}), \dots) \\ &= \sum_{n=0}^{\infty} [S_n(x_0, \dots, x_n^{p^n}; y_0, \dots, y_n^{p^n}) p^{-n}] p^n. \end{aligned}$$

**Lemma 4.5.** *For all  $n \geq 0$  we have that*

$$S_n(x_0, \dots, x_n^{p^n}; y_0, \dots, y_n^{p^n}) = x_n^{p^n} + y_n^{p^n} + \Sigma_n,$$

where  $\Sigma_n$  is a sum of terms of the form  $\prod_{k=0}^{n-1} x_k^{p^k i_k} y_k^{p^k j_k}$  such that  $\sum_{k=0}^{n-1} p^k (i_k + j_k) = p^n$ .

**Proof.** Note that if the lemma holds for some  $n$ , then  $S_n^p$  is a sum of terms of the form  $\prod_{k=0}^n x_k^{p^k i_k} y_k^{p^k j_k}$  such that  $\sum_{k=0}^n p^k (i_k + j_k) = p^{n+1}$ . The lemma then follows from the definition of  $S_n$  and induction on  $n$ . □

**Proposition 4.6.** *Let  $f = \sum_{n=0}^{\infty} [x_n]p^n, g = \sum_{n=0}^{\infty} [y_n]p^n \in \mathbb{A}$ . Assume that  $\mathcal{N}(g)$  is strictly decreasing. Suppose there exists a  $t_0 \geq 0$  such that for all  $t \geq t_0$  we have  $\mathcal{N}(g)(t) < \mathcal{N}(f)(t)$ . Then there exists  $t_1 \geq t_0$  such that for all  $t \geq t_1$ , we have that  $\mathcal{N}(g + f)(t) \leq \mathcal{N}(g)(t)$ .*

**Proof.** We first show the desired inequality when  $t \geq t_0$  is a node of  $\mathcal{N}(g)$ ; these exist since  $g$  is strictly decreasing. Let  $n \geq t_0$  be a node of  $\mathcal{N}(g)$ . Since  $\mathcal{N}(g)$  is strictly decreasing, we have that

$$v(y_n) = \mathcal{N}(g)(n) < v(y_m)$$

for all  $m < n$ . Since  $n \geq t_0$  and  $\mathcal{N}(f)$  is decreasing, for all  $m \leq n$  we have that

$$v(y_n) = \mathcal{N}(g)(n) < \mathcal{N}(f)(n) \leq v(x_m).$$

Thus  $v(y_n^{p^n}) < v(x_n^{p^n})$  and for any  $i_0, j_0, \dots, i_{n-1}, j_{n-1}$  such that  $\sum_{k=0}^{n-1} p^k(i_k + j_k) = p^n$ , it follows that

$$v\left(\prod_{k=0}^{n-1} y_k^{p^k i_k} x_k^{p^k j_k}\right) > p^n v(y_n) = v(y_n^{p^n}).$$

By Lemma 4.5, it follows that

$$v(\mathcal{S}_n(y_0, \dots, y_n^{p^n}; x_0, \dots, x_n^{p^n})^{p^{-n}}) = v(y_n).$$

Therefore

$$\mathcal{N}(g + f)(n) \leq v(\mathcal{S}_n(y_0, \dots, y_n^{p^n}; x_0, \dots, x_n^{p^n})^{p^{-n}}) = v(y_n) = \mathcal{N}(g)(n),$$

and the inequality holds at all nodes of  $\mathcal{N}(g)$  beyond  $t_0$ .

Let  $t_1 \geq t_0$  be the first node of  $\mathcal{N}(g)$ . Given  $t \geq t_1$ , let  $n_1$  and  $n_2$  be two consecutive nodes such that  $n_1 \leq t \leq n_2$ . On this segment,  $\mathcal{N}(g)$  is the straight line connecting  $(n_1, v(y_{n_1}))$  and  $(n_2, v(y_{n_2}))$ . Since  $\mathcal{N}(g + f)$  is a convex function lying below  $\mathcal{N}(g)$  at the two end points  $n_1$  and  $n_2$ , it follows that  $\mathcal{N}(g + f)(t) \leq \mathcal{N}(g)(t)$ , as desired.  $\square$

**Corollary 4.7.** *If  $g \in \mathcal{S}_{n+1}$  and  $f \in \mathbb{A}$ , then  $g + fh_n \in \mathcal{S}_{n+1}$ .*

**Proof.** Since  $g \in \mathcal{S}_{n+1}$ , it follows that  $\mathcal{N}(g)$  is strictly decreasing and there exists  $m \geq 0$  such that  $\mathcal{N}(g) \leq \mathcal{N}(h_{n+1}^m)$ . By Propositions 4.1 and 4.3, for all  $t > 2m^2$ , we have

$$\mathcal{N}(g)(t) \leq \mathcal{N}(h_{n+1}^m)(t) < \mathcal{N}(h_n)(t) \leq \mathcal{N}(fh_n)(t).$$

By Proposition 4.6, it follows that for all  $t$  sufficiently large,

$$\mathcal{N}(g + fh_n)(t) \leq \mathcal{N}(g)(t) \leq \mathcal{N}(h_{n+1}^m)(t).$$

By Lemma 2.1, it follows that  $g + fh_n \in \mathcal{S}_{n+1}$ .  $\square$

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