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Modelling and Numerical Simulation of a Unilateral Contact Problem with Slip-dependent Friction

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Abstract

We construct a mathematical model which describes the contact between an elastic body and an obstacle, the so-called foundation. The contact is frictional and is modelled with normal compliance and unilateral constraint, associated to a slip-dependent version of Coulomb’s law of dry friction. We present a detailed description of the model, then we provide numerical simulations in the study of a two-dimensional example. Our aim is to underline the influence of the parameters involved in the boundary conditions, which could give rise to different status of the material points on the contact surface.

Key words: unilateral contact, normal compliance, slip-dependent friction, Coulomb’s law, weak formulation, numerical simulations

1 Introduction

Phenomena of contact between deformable bodies abound in industry and everyday life. Their complete study involves, generally, four steps: the modelling, the variational analysis, the numerical analysis and the numerical simulations. Various contact boundary conditions have been used to model contact phenomena, both in the engineering and the mathematical in the literature. One of the most popular is the Signorini condition, introduced in (Signorini, 2012), which describes the contact with a perfectly rigid foundation. Expressed in terms of unilateral constraints for the displacement field, this condition leads to highly nonlinear and nonsmooth mathematical problems. For this reason, the normal compliance contact condition was introduced in (Oden et al, 1985), and used in a large number of papers, see (Han et al, 2002), (Klarbring et al, 1988, 1989), (Martins et al, 1985), (Shillor et al, 2004) and the references therein. It represents a regularization of the Signorini contact condition and
it describes the contact with an elastic foundation. A more general contact condition, called the normal compliance condition with unilateral constraint, was introduced in (Jarušek et al, 2008). It contains as particular cases both the Signorini contact condition and the normal compliance condition, and it models the contact with an elastic-rigid foundation.

The variational and numerical analysis of frictional and frictionless contact problems was provided in many papers, see for instance (Han et al, 2002), (Kikuchi et al, 1998), (Shillor et al, 2004), (Sofonea et al, 2012) and the references therein. In particular, results on the unique weak solvability can be found in (Shillor et al, 2004) and (Sofonea et al, 2012); fully discrete schemes for the numerical approximation of the models, including error estimates and numerical simulations can be found in (Han et al, 2002) and (Kikuchi et al, 1998). In all these references the mechanical process was static or quasi-static, the material’s behavior was described either with elastic, viscoelastic or viscoplastic constitutive laws, and the contact was modeled either with the normal compliance condition or the Signorini condition. The analysis of mathematical models formulated by using the contact condition with normal compliance and unilateral constraint can be found in (Barboteu et al, 2012, 2014). The model considered in (Barboteu et al, 2012) was frictionless and was constructed by using a rate-type viscoplastic constitutive law. The model considered in (Barboteu et al, 2014) was frictional; there, the material’s behavior was described with a linear elastic constitutive law and a the friction was modeled with a slip-dependent version of Coulomb’s law; the unique weak solvability of the model was proved by using arguments on pseudomonotone operators; a convergence results was proved and its numerical validation was also provided.

The current paper represents a continuation of (Barboteu et al, 2014). Its aim is to describe a new static contact model with normal compliance, unilateral constraint and slip-dependent friction law for elastic materials, and to provide numerical simulations in the study of this model. The manuscript is structured as follows. In Section 2 we describe the physical setting and present the classical formulation of the problem. Then, in Section 3, we provide numerical simulations in the study of an academic two-dimensional example. Finally, we end this paper with Section 5, in which we present some conclusions, comments and problems for further research.
2 The model

The physical setting is as follows. An elastic body occupies a bounded domain \( \Omega \subset \mathbb{R}^d \) \((d = 1, 2, 3)\) with a Lipschitz continuous boundary \( \Gamma \), divided into three measurable parts \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \). We use the notation \( \mathbf{x} = (x_i) \) for a typical point in \( \Omega \cup \Gamma \) and we denote by \( \mathbf{v} = (v_i) \) the outward unit normal at \( \Gamma \). Here and below the indices \( i, j, k, l \) run between 1 and \( d \) and, unless stated otherwise, the summation convention over repeated indices is used. An index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable \( \mathbf{x} \), e.g. \( u_{i,j} = \partial u_i / \partial x_j \).

Also, \( \varepsilon \) and \( \text{Div} \) will represent the deformation and the divergence operators, respectively, i.e.

\[
\varepsilon(\mathbf{v}) = (\varepsilon_{ij}(\mathbf{v})), \quad \varepsilon_{ij}(\mathbf{v}) = \frac{1}{2} (v_{i,j} + v_{j,i}), \quad \text{Div} \mathbf{\sigma} = (\sigma_{ij,j}).
\]

We denote by \( \mathbb{S}^d \) the space of second order symmetric tensors on \( \mathbb{R}^d \) or, equivalently, the space of symmetric matrices of order \( d \). The inner product and norm on \( \mathbb{R}^d \) and \( \mathbb{S}^d \) are defined by

\[
\mathbf{u} \cdot \mathbf{v} = u_i v_i, \quad \| \mathbf{v} \| = (\mathbf{v} \cdot \mathbf{v})^{\frac{1}{2}} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d, \quad \mathbf{\sigma} \cdot \mathbf{\tau} = \sigma_{ij} \tau_{ij}, \quad \| \mathbf{\tau} \| = (\mathbf{\tau} \cdot \mathbf{\tau})^{\frac{1}{2}} \quad \forall \mathbf{\sigma}, \mathbf{\tau} \in \mathbb{S}^d.
\]

We use the notation \( \mathbf{u} \) and \( \mathbf{\sigma} \) for the displacement field and the stress field, respectively, and, therefore, \( \varepsilon(\mathbf{u}) \) represents the linearized strain tensor. Also, we denote by \( u_\nu \) and \( u_\tau \) the normal and tangential components of \( \mathbf{u} \) on \( \Gamma \) given by \( v_\nu = \mathbf{v} \cdot \mathbf{v}, \quad v_\tau = \mathbf{v} - v_\nu \mathbf{v} \). Finally, \( \sigma_\nu \) and \( \mathbf{\sigma}_\tau \) will represent the normal and the tangential stress on \( \Gamma \), that is \( \sigma_\nu = (\mathbf{\sigma} \mathbf{v}) \cdot \mathbf{v} \) and \( \mathbf{\sigma}_\tau = \mathbf{\sigma} \mathbf{v} - \sigma_\nu \mathbf{v} \).

The elastic body is in equilibrium under the action of body forces of density \( \mathbf{f}_0 \) and surface tractions of density \( \mathbf{f}_2 \) which act on \( \Gamma_2 \). On \( \Gamma_1 \) the displacement field is prescribed. We also assume that the body is in frictional contact on \( \Gamma_3 \) with a deformable obstacle, the so-called foundation. The frictional contact conditions are derived from the following five assumptions:

a) The foundation is made by a rigid body covered by a layer made by of deformable material, say asperities. Therefore, the penetration is restricted, i.e.

\[
u_\nu \leq g, \tag{1}\]

where \( g > 0 \) represents the thickness of the deformable layer.
b) When there is separation, then the reaction of the obstacle vanishes. Therefore, 
\[ u_\nu < 0 \implies \sigma_\nu = 0, \quad \sigma_\tau = 0. \] (2)

c) When there is penetration, as far as the normal displacement does not reach the bound \( g \), the contact is described with a normal compliance condition associated to the static version of Coulomb’s law of dry friction. Therefore,
\[
0 \leq u_\nu < g \implies \begin{cases} 
-\sigma_\nu = p(u_\nu), \\
\|\sigma_\tau\| \leq \mu |\sigma_\nu|, \\
-\sigma_\tau = \mu |\sigma_\nu| \frac{u_\tau}{\|u_\tau\|} \text{ if } u_\tau \neq 0.
\end{cases} \] (3)

Here \( p \) represents a positive normal compliance function such that \( p(0) = 0 \) and \( \mu \) denotes the coefficient of friction.

d) When the normal displacement reaches the bound \( g \), then the normal stress is larger than a given value \( F_b > 0 \) and, moreover, friction follows the static Tresca law with the friction bound \( F_b \). Therefore,
\[
u_\nu = g \implies \begin{cases} 
-\sigma_\nu \geq F_b, \\
\|\sigma_\tau\| \leq F_b, \\
-\sigma_\tau = F_b \frac{u_\tau}{\|u_\tau\|} \text{ if } u_\tau \neq 0.
\end{cases} \] (4)

e) To accommodate the conditions (3) and (4) we assume the compatibility condition
\[ F_b = \mu p(g). \] (5)

Moreover, we assume that the coefficient of friction depends on the slip, that is
\[ \mu = \mu(\|u_\tau\|). \] (6)

Then it is easy to see that assumptions (1)–(6) can be written, equivalently, as follows:
\[
u_\nu \leq g, \quad \sigma_\nu + p(u_\nu) \leq 0, \quad (u_\nu - g)(\sigma_\nu + p(u_\nu)) = 0, \quad \|\sigma_\tau\| \leq \mu(\|u_\tau\|)p(u_\nu), \quad -\sigma_\tau = \mu(\|u_\tau\|)p(u_\nu) \frac{u_\tau}{\|u_\tau\|} \text{ if } u_\tau \neq 0. \] (7) (8)

With these preliminaries, the classical formulation of the contact problem we consider in this paper is as follows.
Problem $\mathcal{P}$. Find a displacement field $u : \Omega \rightarrow \mathbb{R}^d$ and a stress field $\sigma : \Omega \rightarrow \mathbb{S}^d$ such that

\[
\sigma = F\varepsilon(u) \quad \text{in } \Omega, \quad (9)
\]

\[
\text{Div } \sigma + f_0 = 0 \quad \text{in } \Omega, \quad (10)
\]

\[
u = u_1 \quad \text{on } \Gamma_1, \quad (11)
\]

\[
\sigma \nu = f_2 \quad \text{on } \Gamma_2, \quad (12)
\]

\[
u \leq g, \quad \sigma \nu + p(u_\nu) \leq 0,
\]

\[
(u_\nu - g)(\sigma \nu + p(u_\nu)) = 0 \quad \text{on } \Gamma_3, \quad (13)
\]

\[
\|\sigma \tau\| \leq \mu(\|u_\tau\|) p(u_\nu),
\]

\[
-\sigma \tau = \mu(\|u_\tau\|) p(u_\nu) \frac{u_\tau}{\|u_\tau\|} \quad \text{if } u_\tau \neq 0 \quad \text{on } \Gamma_3. \quad (14)
\]

Here and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $x$. Equation (9) represents the elastic constitutive law of the material in which $F$ is a (possibly nonlinear) constitutive function. Equation (10) is the equation of equilibrium; we use it here since we assume that process is static. Conditions (11) and (12) represent the displacement and the traction boundary conditions, respectively, in which $u_1$ is a given displacement field. Finally, conditions (13) and (14) represent the frictional contact condition with normal compliance and unilateral constraint previously described in this section, see (7) and (8).

Under appropriate assumption on the data, it can be shown that if $(u, \sigma)$ represents a regular solution of the contact problem $\mathcal{P}$, then the displacement field satisfies an elliptic quasivariational inequality of the form

\[
u \in K, \quad \int_\Omega F\varepsilon(u) \cdot (\varepsilon(v) - \varepsilon(u)) \, dx + \int_{\Gamma_3} p(u_\nu)(v_\nu - u_\nu) \, dx \quad (15)
\]

\[
+ \int_{\Gamma_3} \mu(\|u_\tau\|) p(u_\nu) \|v_\tau\| \, dx + \int_{\Gamma_3} \mu(\|u_\tau\|) p(u_\nu) \|u_\tau\| \, dx
\]

\[
\geq \int_\Omega f_0 \cdot (v - u) \, dx + \int_{\Gamma_2} f_2 \cdot (v - u) \, da \quad \forall v \in K,
\]

in which $K$ represents the set of admissible displacement fields. Therefore, based on the variational formulation (15), the unique weak solvability of Prob-
lem $P$ can be obtained by using arguments of quasivariational inequalities which can be found, for instance, in (Sofonea et al, 2012). It also can be obtained by using arguments similar to those used in (Barboteu et al, 2014), based on the theory of pseudomonotone operators. The inequality (15) lies the background for the numerical solution of Problem $P$. In the next section we restrict ourselves to present numerical simulations in the study of a two-dimensional example which underline the influence of the different parameters on the solution.

3 Numerical examples

The numerical solution of the contact Problem $P$ is based on arguments similar to those already used in (Barboteu et al, 2013, 2014) and consists on the solution of a sequence of convex programming problems. This approach is realized by using an iterative procedure in which, for each iteration, the coefficient of friction is fixed to a given value which depends on the tangential displacement of the solution found in the previous iteration. Furthermore, the frictional contact conditions are treated by using a numerical approach based on the combination of the penalized method and the augmented Lagrangean method. Then, the nonsmooth convex problems arising during the iterative process are solved by classical numerical methods. More details on the discretization step and the corresponding numerical method can be found in (Alart et al, 1991), (Laursen, 2002) and (Wriggers, 2002).

In order to keep this paper in a reasonable length, we skip the presentation of the numerical algorithm and we describe in what follows a two-dimensional numerical example, which describes the compression of a ball against a foundation. The physical setting is depicted in Figure 1. There, the following notation are used:

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - 10)^2 + (x_2 - 10)^2 \leq 10^2\},$$

$$\Gamma_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 10 - 5\sqrt{2}, (x_1 - 10)^2 + (x_2 - 10)^2 = 10^2\},$$

$$\Gamma_2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \geq 20, (x_1 - 10)^2 + (x_2 - 10)^2 = 10^2\},$$

$$\Gamma_3 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \leq 10 - 5\sqrt{2}, (x_1 - 10)^2 + (x_2 - 10)^2 = 10^2\}.$$

The domain $\Omega$ represents the cross section of a three-dimensional deformable body subjected to the action of tractions in such a way that a plane stress hypothesis is assumed. The horizontal component of the displacement
field vanishes on $\Gamma_1$ and the vertical component of the displacement field is fixed to the value $u_1$. Therefore, $u_1 = (0, u_1)$. Vertical tractions of density $f_2$ act on the part

$$\left\{ (x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 10 + 5\sqrt{2}, (x_1 - 10)^2 + (x_2 - 10)^2 = 10^2 \right\}$$

of the boundary $\Gamma_2$ and the reminder part of $\Gamma_2$ is traction free. No body forces are assumed to act on the body during the process. The body is in friction contact with an obstacle on the part $\Gamma_3$ of its boundary. For the discretization we used 20312 elastic finite elements and 128 contact elements. The total number of degrees of freedom is equal to 20954. We model the material’s behavior with an elastic linear constitutive law in which the elasticity tensor $\mathcal{F}$ satisfies

$$(\mathcal{F}\varepsilon)_{\alpha\beta} = \frac{E\kappa}{(1 + \kappa)(1 - 2\kappa)}(\varepsilon_{11} + \varepsilon_{22})\delta_{\alpha\beta} + \frac{E}{1 + \kappa}\varepsilon_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq 2.$$ 

Here $E$ is the Young modulus, $\kappa$ the Poisson ratio of the material and $\delta_{\alpha\beta}$ denotes the Kronecker symbol.

For the computation below we used the following data:

- $E = 2000 \text{N/m}^2$, $\kappa = 0.3$,
- $u_1 = 1 \text{m}$, $f_0 = (0, 0) \text{N/m}^2$, $f_2 = (-40, 0) \text{N/m}$ on $\Gamma_2$,
- $p(r) = c_\nu r^+$, $c_\nu = 100 \text{N/m}^2$, $g = 0.5 \text{m}$.
where, recall, $r^+ = \max\{0, r\}$. Moreover, we assume that the coefficient of friction satisfies the equality $\mu = \mu(||u_\nu||)$, where $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is the function given by

$$\mu(r) = (a - b) \cdot e^{-\alpha r} + b,$$

with $a, b, \alpha > 0$, $a \geq b$. Note that for $a = b$ this function is constant and for $a \neq b$ it is a decreasing function. Therefore, following the arguments and the terminology in (Barboteu et al, 2013), in the case $a = b$ we refer to the friction law (14) as a monotone friction law and, in the case $a \neq b$, we refer to this law as a non-monotone friction law.

Our results are presented in Figures 2–4 and are described in what follows.

First, in Figure 2, the deformed mesh and the contact interface forces are presented in the case $a = 2$, $b = 0.002$ and $\alpha = 1$ and, therefore, this case corresponds to a non-monotone friction law. We recall that the contact follows a normal compliance condition associated to the Coulomb’s law of dry friction as far as the penetration is less than the bound $g = 0.5m$ and, when this bound is reached, it follows a unilateral condition associated to the Tresca friction law. This behavior of the foundation is an elastic-rigid one and can be interpreted physically as follows: the foundation is assumed to be made of a hard material covered by a layer composed of a soft material with thickness $g$. On the contact interface depicted in Figure 2, we observe the several different status of frictional contact. Thus, in the middle of the contact area $\Gamma_3$, we see that a non-negligible proportion of the contact nodes are in the status of unilateral contact; there, the complete flattening of the asperities of size $g = 0.5m$ was reached and, therefore, $u_\nu = g$; moreover, the magnitude of the
Figure 3: Deformed meshes and contact interface forces (the case $a = b$).

friction force does not reach the friction bound $\mu(\|\tau\|)p(\nu)$ and, therefore, these points are in the stick zone. Next, the contact nodes on the extremities of the boundary $\Gamma_3$ are in the status of normal compliance since, there, the penetration is such that $0 < u_\nu < g$; moreover, the magnitude of the friction force reaches the friction bound $\mu(\|\tau\|)p(\nu)$ and, therefore, these points are in the slip zone. The regions described above are separated by two regions in which the nodes are in stick contact with normal compliance; there, the critical value of penetration $g$ and friction bound are not reached.

In Figure 3, we plotted two deformed meshes and the associated contact interface forces corresponding to a monotone friction law. We note that in the case $a = b = 2$, the behavior of the solution is similar to that obtained in the non-monotone case $a \neq b$. In contrast, in the case $a = b = 0.002$ the behavior of the solution is different.

In Figure 4, the deformed configurations and the associated contact interface forces are plotted for two values of the parameter $g$ which, recall, represents the size of the soft material. Our aim is to show that, with a convenient choice of this parameter, the frictional contact conditions (7)–(8) reduce themselves to some classical contact conditions. First, the case $g = 0m$ corresponds to a frictionless contact problem with Signorini condition; here the value of the Tresca friction bound is equal to zero since $g = 0m$ and, therefore, all the nodes are in the status of slip unilateral contact, with $\sigma_\tau = 0$. Next, the case $g = 10m$ corresponds to a frictional contact problem with normal compliance associated to a Coulomb’s law of dry friction; here the value of $g$ is large enough, is not reached and, therefore, the penetration is positive at each node, and is proportional to the normal component of the contact...
In this paper we presented a model for a frictional process of contact of an elastic body. The contact was governed by a normal compliance condition with unilateral constraint and the friction was described with a slip-dependent version of Coulomb’s law of dry friction. Our aim was to study the behaviour of the solution and to underline the influence of the parameters involved in the boundary conditions. To this end we provided numerical simulations in the study of a two-dimensional model. We used an algorithm based on the Newton method combined with a penalized method for the unilateral constraints. Based on the simulations, we found that the algorithm worked well and the convergence was rapid. Moreover, we conclude that the choice of the different parameters, in both the contact and friction laws, give rise to different status of the material points on the contact surface. Subsequent stages of the research presented in this paper will consist to extend these results to dynamic frictional contact problems with elastic or viscoelastic materials.
5 References


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