Taylor Expansion as a Monad in Models of DiLL

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Abstract

Differential Linear Logic (DiLL) adds to Linear Logic (LL) a symmetrization of three out of the four exponential rules, and by doing so allows the expression of a natural notion of differentiation. In this paper, we introduce a *codigging* inference rule for DiLL and study the categorical semantics of DiLL with codigging using differential categories. The addition of codigging makes the rules of DiLL completely symmetrical. We will explain how codigging is interpreted thanks to the exponential function e^x, and in certain cases by the convolutional exponential. In a setting with codigging, every proof is equal to its Taylor series, which implies that every model of DiLL with codigging is *quantitative*. We provide examples of codigging in relational models, as well as models related to game logic and quantum programming. We also construct a graded model of DiLL with codigging in which the indices witness exponential growth. Since codigging makes the exponential of-course connective ! in LL into a *monad*, such that monad axioms enforce Taylor expansion, codigging opens the door to applications in programming languages, as well as further categorical generalizations.

1 Introduction

The quantitative point of view for programming languages consists in measuring through syntax, types or models their usage in time, space or resources. This has in particular led to refined results in λ-calculus [1][2][3] and innovations in probabilistic programming [4][5]. In denotational semantics, this consists of interpreting programs by power series, whose coefficients represent the quantitative information one would like to retrieve. In an analytic context, power series are in particular those functions which equal their Taylor series at 0:

$$\forall x, f(x) = \sum_{n\in\mathbb{N}} \frac{1}{n!} D_0^{(n)}(f)(x).$$

The introduction of differentiation as a core primitive of the λ-calculus was made possible by Linear Logic (LL). We will show that Taylor expansion can be expressed in terms of a *monad* structure on the main connective ! in LL. The monad unit represents *differentiation* at 0, while the monad multiplication will correspond to the *convolutional exponential*.

The introduction of LL by Girard [6] and its development is intertwined with the rise of quantitative semantics [7]. It brought forward the distinction between *linear* proofs and *non-linear* proofs.

The logical interpretation of linear, meaning using an argument exactly once, coincides through dwith the mathematical interpretation from algebra, meaning as a function that preserves sums. The non-linear proofs and functions are retrieved from the introduction of a so-called exponential unary connective denoted !. In LL, there are four exponential laws ruling the use of $!A$ called **weakening** (w), **contraction** (c), **dereliction** (d), and **promotion** (prom):

$$\frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta}\,\mathsf{w} \qquad \frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta}\,\mathsf{c} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta}\,\mathsf{d} \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\,\mathsf{P}$$

Proofs of $A \vdash B$ are interpreted as linear implications $A \multimap B$, while non-linear implication is defined as $A \Rightarrow B := !A \multimap B$. As such, the dereliction rule forgets the linearity of a proof, allowing linear proofs to be considered a special case of non-linear proofs. Categorically speaking, the dereliction is the counit of a comonad !, while promotion is interpreted thanks to the comultiplication of !. Thus P can be replaced by two rules expressing the functoriality of ! (functorial promotion $!_{\mathsf{d}}$) and the comultiplication of !, called **digging** (p):

$$\frac{\Gamma \vdash A}{!\Gamma \vdash !A}\,!_{\mathsf{f}} \qquad \frac{\Gamma, !!A \vdash \Delta}{\Gamma, !A \vdash \Delta}\,\mathsf{p}$$

Differential Linear Logic (DiLL) was introduced by Ehrhard and Regnier [8] as an extension of LL with a syntactical notion of differentiation. It led to a number of work concerning the syntax and semantics of differentiable and probabilistic programming languages [4][9] as well as new proof methods on λ-terms [1]. In classical DiLL there are three extra exponential rules called **coweakening** ($\overline{\mathsf{w}}$), **cocontraction** ($\overline{\mathsf{c}}$), and **codereliction** ($\overline{\mathsf{d}}$):

$$\frac{\vdash}{\vdash !A}\,\overline{\mathsf{w}} \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\,\overline{\mathsf{c}} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\,\overline{\mathsf{d}}$$

The ability to differentiate a proof is encoded in the codereliction rule. Dual to the dereliction, the codereliction takes a non-linear proof and produces a linear proof via *linearization*, that is, by differentiating at zero. The other new rules are necessary for the cut-elimination of DiLL, and have a miraculous symmetric presentation to the usual exponential rules. For an in-depth introduction to DiLL, we refer the reader to [10].

Remarkably, the interactions (i.e. the cut-elimination in sequent calculus or the coherence diagram in categorical models) between the standard exponential rules and the added rules for DiLL are symmetrical. For example, the interaction between the dereliction and cocontraction is a mirror dual to the interaction between the codereliction and the contraction. Furthermore, these interactions are nicely illustrated by the basic rules of differential calculus, as explained in Ehrhard and Regnier's original paper [11]. While DiLL is solidified as an elegant typing system for higher-order functional analysis, the mystery of the symmetrical nature of its rules and their interactions is unexplained. In particular, observe that c, w, and d have their dual rule introduced in DiLL with $\overline{\mathsf{c}}$, $\overline{\mathsf{w}}$, and $\overline{\mathsf{d}}$. Missing is a dual rule for digging. Thus a natural question to ask is if there is such thing as a **codigging** rule, and if it makes sense semantically? By dualizing the digging rule, we easily write a codigging rule ($\overline{\mathsf{p}}$) as follows:

$$\frac{\Gamma \vdash !!A}{\Gamma \vdash !A}\,\overline{\mathsf{p}}$$

Contributions In this work, we define the notion of categorical model of DiLL with codigging, study their properties and express codigging in term of Taylor expansions. We show that the codigging should be interpreted as an *exponential* of distributions, which gives ! a monad structure that enforces a quantitative setting. We exhibit several models of DiLL with codigging, such as the weighted relational model or quantum related examples. We construct a new graded model in a smooth setting based on the notion of the convolutional exponential and exponential growth.

In the rest of the introduction, we give more details about the interpretation of the monad structure on !, and how the quantitative setting follows. In categorical models, formulas are interpreted as objects A, B of a category $\mathcal{L}$ and proofs $A \multimap B$ as morphisms $f : A \to B$ between these objects[1]. As part of the Curry-Howard correspondence, these morphisms should be invariant under the cut-elimination procedure.

Due to its invariance under differentiation and its behavior with respect to sums, we argue that codigging should be interpreted as a sort of generalized version of the exponential function[2] e^x. To justify this last claim, we must consider what e^x in the context of DiLL would even be. The answer to this question comes from an independent categorical exploration by Lemay [12]. We will explain how the axioms of codigging precisely state that $\overline{\mathsf{p}}$ fits in this categorical axiomatization of exponential map.

Now in categorical models of DiLL, each inference rule is interpreted by a natural transformation. Since the digging is a natural transformation of type $\mathsf{p}_A : !A \to !!A$, it follows that the codigging should be a natural transformation of type $\overline{\mathsf{p}}_A : !!A \to !A$. Since we claimed that codigging is a generalization of e^x, we may take inspiration from the power series formula for e^x to provide a formula for $\overline{\mathsf{p}}$ (ignoring any problems infinite sums for now):

$$e^x = \sum_n \frac{x^n}{n!} \qquad \overline{\mathsf{p}} : x \in !!A \mapsto \sum_n \frac{\overline{\mathsf{c}}^n(\mathsf{d}(x)^{\otimes^n})}{n!} \in !A.$$

When $!A$ is interpreted by spaces of distribution [13], $\overline{\mathsf{c}}$ corresponds the the convolution law and so $\overline{\mathsf{p}}$ maps x to the *convolutional exponential* of $\mathsf{d}(x)$.

On the other hand, the codereliction is a natural transformation of type $\overline{\mathsf{d}} : A \to !A$, and precomposing a map $f : !A \to B$ by $\overline{\mathsf{d}}$ results in its differential at 0: $\overline{\mathsf{d}}_A; f = D_0(f) : A \to B$. Since $(!, \overline{\mathsf{p}}, \overline{\mathsf{d}})$ is a comonad, dually we will have that $(!, \overline{\mathsf{p}}, \overline{\mathsf{d}})$ a *monad*. In particular, the monad axioms between $\overline{\mathsf{p}}$ and $\overline{\mathsf{d}}$ relate codigging to its differentiation and Taylor expansion. The law $\overline{\mathsf{d}}_!; \overline{\mathsf{p}} = 1$ is an analogue of the invariance of e^x under differentiation, while the law $!\overline{\mathsf{d}}; \overline{\mathsf{p}} = 1$ accounts for the fact that all non-linear maps are equal to their Taylor series. Therefore, models of DiLL with codigging are strongly related with *quantitative* models. What a quantitative model can lack to have a proper monad structure on ! is the convergence of $\overline{\mathsf{p}}$ on every element x of $!!A$. This is strongly related to the convergence of infinite sums in the model, and the growth allowed to non-linear maps.

[1] We assume the reader is familiar with the basic concepts of category theory such as categories, functors, natural transformations, monoidal categories and (co)monads. In a category we write maps as $f : A \to B$, identity maps as $1_A : A \to A$, and we write composition diagrammatically, that is, the composition of maps $f : A \to B$ and $g : B \to C$ is denoted $f; g : A \to C$.

[2] Beware that we face a difficult overlap in terminology. In LL, the connectives ! and ? are traditionally named "exponential connectives" for the fact they transform additive connectives into the multiplicative ones. Here, we refer to the mathematical exponential function $\mathsf{exp} : x \mapsto e^x$. As much as possible, we will refer to the latter as the "exponential function", as opposed with "exponential rules" or "exponential connectives" in LL.

Related Work In their PhD thesis [14], Gimenez studies codigging as a proof-net construction which is used in the definition of a *super-promotion*. As far as we can see, it does not mentions its denotational interpretation, which is the heart of our work. The notion of Taylor expansion has been otherwise widely studied in denotational semantics [10, 15, 16, 17, 18], exhibiting models which sometimes interpret codigging, and otherwise interpret codigging only on a restricted subset of functions. Quantitative semantics is not restricted to the LL settings, and relates to intersection types [19] and quantitative properties of programs [20]. Differential calculi [11] considers differentiation as a program transformation, leading to resource calculi [21] in which programs computes on quantitative data. To the best of the authors' knowledge, these have never been explained in terms of monads nor exponential functions. In the setting of quantitative algebras [22], Mio and Vignudelli studied the lifting of the probability monad to quantitative equational theories [23]. To the best of our understanding, this is distinct from the codigging story of this paper.

Outline We begin in Section 2 by giving background on the categorical models of DiLL, and also review !-differential exponential maps in Section 3.1. Section 3 is the categorical heart of our paper. In Section 3.2, we define and study monadic differential categories, which are categorical models of DiLL with codigging. We show that codigging is a generalized version of the exponential function e^x. The coherence rules of codigging are defined symmetrically to the one of digging, and we explain why this makes sense semantically in terms of e^x. In Section 3.3, we introduce the novel concept of Taylor differential categories, which are differential categories where Taylor expansion is well-defined, and show that the "illegitimate formula" for codigging holds in such a setting. Section 4 provides examples of models of DiLL with codiging, including the well known relational models, as well as models related to game logic and quantum theory. In Section 5, we give our first counter examples of differential categories that do *not* have a codigging. That said, we will then show in section 5.2 that the convolutional exponential, which is a preexisting notion in functional analysis, interprets codigging in an alternative way and allows the discovery of new smooth, graded and polarized models of DiLL. We explain the symmetry of DiLL in terms of the Laplace transformation. We conclude with Section 6, where we discuss future work on codigging in category theory, the λ-calculus, etc.

2 Differential Categories: the Categorical Semantics of DiLL

In this section we review the categorical semantics of DiLL, which was first developed by Blute, and Cockett under the name differential categories in [24], and later revisited by these three authors along with Lemay in [25], also by Fiore in [26], and Ehrhard in [10]. In this paper, we will mostly be following Ehrhard's notation and terminology in [10], as it takes a more DiLL like perspective (rather than a purely categorical one). For a more in-depth introduction to the categorical semantics of LL, we refer the reader to the introductory source [27].

The underlying category is a **symmetric monoidal category**, which interprets the multiplicative fragment of LL. For an arbitrary symmetric monoidal category, we denote the underlying category as $\mathcal{L}$, the monoidal product as $\otimes$, the monoidal unit as I, and the natural symmetry isomorphism by $\sigma_{A,B} : A \otimes B \to B \otimes A$. For simplicity and following the convention done overall in differential category literature, in this paper we will work in the setting of a symmetric *strict* monoidal category, meaning that the associativity and unit properties of the monoidal product are equalities, so we write $A_1 \otimes A_2 \otimes \ldots \otimes A_n$ and $A \otimes I = A = I \otimes A$. For Classical DiLL, one in fact

needs a *star-autonomous category*, which interprets the involutive linear negation. However, since the closed structure does not play a central role in the story of codigging, we will not assume it in our categorical definitions.

For DiLL, in order to express the product rule for differentiation and that the derivative of a constant function is zero, we will also require the ability of taking sums of maps and having zero maps. So an **additive symmetric monoidal category** [25, Def 3] is a symmetric monoidal category $\mathcal{L}$ which is enriched over the category of commutative monoids, that is, each homset $\mathcal{L}(A, B)$ is a commutative monoid with addition operation $+$ and zero $0 : A \to B$, and such that composition and the monoidal product $\otimes$ are compatible with the additive structure. We will also assume that we have finite products, which interprets the additive fragment of LL. If an additive symmetric monoidal category has finite products, then by the additive structure it follows that said products are in fact *biproducts*, and these distribute with the monoidal product. Recall that a biproduct can be defined as a product that is also a coproduct such that the projection maps and injection maps are compatible. Since the product structure plays a slightly more central role, we use product notation for biproducts. So if an additive symmetric monoidal category $\mathcal{L}$ has finite (bi)products, we denote the binary product as $\times$, with projections $\pi_i : A_0 \times A_1 \to A_i$, and zero object $\top$, and we have that $A \otimes (B \times C) \cong (A \otimes B) \times (A \otimes C)$ and $A \otimes \top \cong \top$.

For the exponential fragment, there are many equivalent ways to provide a categorical interpretation of the ! exponential modality such as a monoidal coalgebra modality (also called a linear exponential modality), an additive bialgebra modality, or a storage modality. We have chosen the latter which is defined in terms of the biproduct structure and the *Seely isomorphisms*.

Definition 2.1. *For an additive symmetric monoidal category $\mathcal{L}$ with finite (bi)products, a **storage modality** [25, Def 10] is a tuple $(!, \mathsf{p}, \mathsf{d}, \mathsf{c}, \mathsf{w})$ consisting of an endofunctor $! : \mathcal{L} \to \mathcal{L}$ and four natural transformations: $\mathsf{p}_A : !A \to !!A$ called the **digging**, $\mathsf{d}_A : !A \to A$ called the **dereliction**, $\mathsf{c}_A : !A \to !A \otimes !A$ called the **contraction**, and $\mathsf{w}_A : !A \to I$ called the **weakening**, and such that:*
1. $(!, \mathsf{p}, \mathsf{d})$ is a comonad:

$$\mathsf{p}_A; !\mathsf{p}_A = \mathsf{p}_A; \mathsf{p}_{!A} \qquad \mathsf{p}_A; !\mathsf{d}_A = 1_{!A} = \mathsf{p}_A; \mathsf{d}_{!A} \tag{1}$$

2. $(!A, \mathsf{c}_A, \mathsf{w}_A)$ is a cocommutative comonoid:

$$\begin{gathered} \mathsf{c}_A; (\mathsf{c}_A \otimes 1_{!A}) = \mathsf{c}_A; (1_{!A} \otimes \mathsf{c}_A) \quad \mathsf{c}_A; \sigma_{!A,!A} = \mathsf{c}_A \\ \mathsf{c}_A; (1_{!A} \otimes \mathsf{w}_A) = 1_{!A} = \mathsf{c}_A; (\mathsf{w}_A \otimes 1_{!A}) \end{gathered} \tag{2}$$

3. The digging p is a comonoid morphism:

$$\mathsf{p}_A; \mathsf{c}_{!A} = \mathsf{c}_A; (\mathsf{p}_A \otimes \mathsf{p}_A) \qquad \mathsf{p}_A; \mathsf{w}_{!A} = \mathsf{w}_A \tag{3}$$

*4. The natural transformation $\chi_{A,B} : !(A \times B) \to !A \otimes !B$, defined as $\chi_{A,B} := \mathsf{c}_{A\times B}; (!(\pi_0) \times !(\pi_1))$, and the weakening $\mathsf{w}_\top : !\top \to I$ are isomorphisms, called the **Seely isomorphisms**, so $!(A \times B) \cong !A \otimes !B$ and $!\top \cong I$.*

From now on we will simply write ! for a storage modality. There are two important canonical natural transformations that can be constructed using the biproduct structure and the inverse of the Seely isomorphisms [25, Sec.7]. These are $\overline{\mathsf{c}}_A : !A \otimes !A \to !A$, and $\overline{\mathsf{w}}_A : !A \to I$, respectively called the **cocontraction** and the **coweakening**. There are many interesting identities that follow

from these extra maps. Of particular importance to the story of this paper is that:
5. $(!A, \overline{\mathsf{c}}_A, \overline{\mathsf{w}}_A)$ is a commutative monoid:

$$\begin{gathered}(\overline{\mathsf{c}}_A \otimes 1_{!A}); \overline{\mathsf{c}}_A = (1_{!A} \otimes \overline{\mathsf{c}}_A); \overline{\mathsf{c}}_A \quad \sigma_{!A,!A}; \overline{\mathsf{c}}_A = \overline{\mathsf{c}}_A \\ (1_{!A} \otimes \overline{\mathsf{w}}_A); \overline{\mathsf{c}}_A = 1_{!A} = (\overline{\mathsf{w}}_A \otimes 1_{!A}); \overline{\mathsf{c}}_A\end{gathered} \tag{4}$$

and in fact, $!A$ is a bimonoid:

$$\begin{gathered}\overline{\mathsf{c}}_A; \mathsf{c}_A = (\mathsf{c}_A \otimes \mathsf{c}_A); (1_{!A} \otimes \sigma_{!A,!A}); (\overline{\mathsf{c}}_A \otimes \overline{\mathsf{c}}_A) \\ \overline{\mathsf{c}}_A; \mathsf{w}_A = \mathsf{w}_A \otimes \mathsf{w}_A \qquad \overline{\mathsf{w}}_A; \mathsf{c}_A = \overline{\mathsf{w}}_A \otimes \overline{\mathsf{w}}_A \\ \overline{\mathsf{w}}_A; \mathsf{w}_A = 1_I\end{gathered} \tag{5}$$

6. The dereliction d is compatible with the monoid structure:

$$\overline{\mathsf{c}}_A; \mathsf{d}_A = \mathsf{w}_A \otimes \mathsf{d}_A + \mathsf{d}_A \otimes \mathsf{w}_A \qquad \overline{\mathsf{w}}_A; \mathsf{d}_A = 0 \tag{6}$$

It is well known that using the Seely isomorphisms, digging, and dereliction, we can construct a natural transformation $\mu_{A,B} : !A \otimes !B \to !(A \otimes B)$ and a map $\mu_I : I \to !I$ which make ! into a lax monoidal functor, but we also that:
7. The digging p is compatible with the monoid structure:

$$\overline{\mathsf{c}}_A; \mathsf{p}_A = (\mathsf{p}_A \otimes \mathsf{p}_A); \mu_{!A,!A}; !\overline{\mathsf{c}}_A \qquad \overline{\mathsf{w}}_A; \mathsf{p}_A = \mu_I; !\overline{\mathsf{w}}_A \tag{7}$$

We may now properly state the definition of a *codereliction*.

Definition 2.2. *A **differential storage category** is an additive symmetric monoidal category with finite (bi)products and a storage modality ! that comes equipped with a **codereliction** [25, Def 9] which is a natural transformation $\overline{\mathsf{d}}_A : A \to !A$ such that the following equalities hold:*

$$\begin{gathered}\overline{\mathsf{d}}_A; \mathsf{p}_A = (\overline{\mathsf{w}}_A \otimes \overline{\mathsf{d}}_A); (\mathsf{p}_A \otimes \overline{\mathsf{d}}_{!(A)}); \overline{\mathsf{c}}_{!(A)} \quad \overline{\mathsf{d}}_A; \mathsf{d}_A = 1_A \\ \overline{\mathsf{d}}_A; \mathsf{c}_A = \overline{\mathsf{w}}_A \otimes \overline{\mathsf{d}}_A + \overline{\mathsf{d}}_A \otimes \overline{\mathsf{w}}_A \qquad \overline{\mathsf{d}}_A; \mathsf{w}_A = 0\end{gathered} \tag{8}$$

To be precise, a categorical model of (Classical) DiLL is a differential storage category that is also monoidal closed (star-autonomous). As discussed in the introduction, the key dynamic in LL is that we have an interpretation of non-linear maps and linear maps. In DiLL, we also have the ability of differentiating the non-linear maps infinitely many times, and therefore non-linear maps are better understood as smooth maps. From a categorical point of view, the non-linear maps are in fact maps of the coKleisli category. For a differential storage category $\mathcal{L}$ with storage modality !, recall that the **coKleisli category** of ! is the category $\mathcal{L}_!$ whose objects are the same as $\mathcal{L}$ but where a map from A to B in $\mathcal{L}_!$ is a map of type $!A \to B$ in $\mathcal{L}$. So from the point of view of DiLL, a non-linear map from A to B is a coKleisli map $!A \to B$, while a linear map A to B is simply of map of type $A \to B$.

All the natural transformation which interpret DiLL proofs have a natural interpretation in term of basic calculus. In particular, for a coKleisli map $f : !A \to B$:

- Precomposing a map $\ell : A \to B$ by the dereliction d forgets that ℓ a linear map, $\mathsf{d}; \ell : !A \to B$.
- The digging p intervenes in the composition of two non-linear maps as usual in coKleisli categories.

- Precomposing by the contraction c turns a function into its composition with the diagonal, $(\mathsf{c}; f)(x) = f(x, x)$.
- Precomposing by the weakening w turns a point $b : I \to B$ into a constant function, $\mathsf{w}; b : !A \to B$.
- Precomposing by the cocontraction $\overline{\mathsf{c}}$ means summing in the domain of the function, $(\overline{\mathsf{c}}_A; f) := (x, y) \mapsto f(x + y)$.
- Precomposing by the coweakening $\overline{\mathsf{w}}$ corresponds to evaluating at 0, $\overline{\mathsf{w}}_A; f = f(0)$.
- Precomposing by the codereliction $\overline{\mathsf{d}}$ means taking the derivative of a function at 0, so $(\overline{\mathsf{d}}_A; f) = D_0(f) : A \to B$ is the linear map mapping v to the differential of f at 0 according to the vector v.

These intuition were discovered in discrete models, but also hold in models based on the classical differential calculus.

3 Codigging

In this section, we introduce the notion of codigging from a categorical point of view. We will demonstrate how codigging fits naturally in the categorical semantics and explain that codigging can be interpreted as a generalization of the classical exponential function e^x, and how it's related to the Taylor series formula for smooth functions.

3.1 Exponentials in Differential Categories

As mentioned above, we will explain below why codigging should be interpreted as a generalized expentional function. To help justify this claim, let us first quickly review the generalization of the exponential function e^x in context of differential storage categories, called a !-differential exponential map, which was introduced by Lemay in [12]. Classically, e^x admits numerous equivalent characterization either as the inverse of the natural logarithm function, or as a limit or converging power series, or even as the unique solution to a differential equation. What is surprising about !-differential exponential maps is that they can be defined for any commutative monoid in a differential storage category without the need of some notion of convergences, or infinite sums, or even unique solutions for differential equations. Instead, their axioms are based on three well-known identities of e^x which are (1) that e^x is its own derivative, (2) $e^{x+y} = e^x e^y$, and (3) $e^0 = 1$.

Definition 3.1. *In a differential storage category, for a commutative monoid $(A, \overline{\mathsf{C}} : A \otimes A \to A, \overline{\mathsf{W}} : I \to A)$, a* ***!-differential exponential map*** *[12, Def 14] is an endomorphism $\mathsf{e} : A \to A$ such that the following equalities hold:*

$$\overline{\mathsf{d}}_A; \mathsf{e} = 1_A \qquad \overline{\mathsf{c}}_A; \mathsf{e} = (\mathsf{e} \otimes \mathsf{e}); \overline{\mathsf{C}} \qquad \overline{\mathsf{w}}_A; \mathsf{e} = \overline{\mathsf{W}} \tag{9}$$

A ***!-differential exponential algebra*** *is a commutative monoid equipped with a !-differential exponential map.*

Categorically speaking, for a !-differential exponential map e, the first axiom says that e is a retract of the codereliction $\overline{\mathsf{d}}$, while the other two say that e is a monoid morphism. From the point of view of DiLL, a !-differential exponential map e is a non-linear map from A to A. For the first axiom, recall that precomposing by the codereliction is interpreted as differentiating and then evaluating at zero. So the first axiom interprets the fact that the derivative of e^x at 0 is x. For the other two axioms, recall that precomposing by the cocontraction corresponds to evaluating at the sum of two arguments, while precomposing by the coweakening corresponds to evaluating at zero. On the other hand, the multiplication $\overline{\mathsf{C}}$ is interpreted as a bilinear multiplication on A, and the unit $\overline{\mathsf{W}}$ is a constant function which gives the mutliplicative unit point of A. Therefore, the other two axioms of e are indeed analogues of $e^{x+y} = e^x e^y$ and $e^0 = 1$.

3.2 Codigging

In this section we introduce the notion of a differential storage category with codigging, which we call a *monadic differential category*. Before giving the definition of codigging, let us first take a step back and remember our original motivation. In the added exponential rules of DiLL there was a cocontraction, coweakening, and codereliction, but there is an astonishing lack of a codigging. The beautiful part of DiLL is that not only $\mathsf{c}, \mathsf{w}, \overline{\mathsf{c}}, \overline{\mathsf{w}}, \overline{\mathsf{d}}$, and d are symmetrical in their types, but they are also symmetrical in their interaction rules. Indeed, equations (2) and (4) are dual of one another, while the two last axioms of equation (8) are dual of equation (6). As such, this naturally leads us to the fact that codigging $\overline{\mathsf{p}}$ should be the dual type of the digging, so $\overline{\mathsf{p}} : !!A \to !A$, and the rules involving $\overline{\mathsf{p}}$ should be symmetrical to the ones of p. So the axioms of codigging can be split into three parts. Since p and d make ! into a comonad (1), symmetrically, we will require that $\overline{\mathsf{p}}$ and $\overline{\mathsf{d}}$ will make ! into a monad, which is where the name monadic differential category comes from. Similarly, since p is a comonoid morphism with respect to c and w (2), we will also have that $\overline{\mathsf{p}}$ is a monoid morphism with respect to $\overline{\mathsf{c}}$ and $\overline{\mathsf{w}}$. Lastly, we will also require that $\overline{\mathsf{p}}$ and d together satisfy the dual of the chain rule (8), which is the compatibility axiom between p and $\overline{\mathsf{d}}$.

Definition 3.2. *A **monadic differential category** is a differential storage category whose storage modality ! comes equipped with a **codigging** which is a natural transformation $\overline{\mathsf{p}}_A : !!A \to !A$, such that the following equalities hold:*
1. $(!, \overline{\mathsf{p}}, \overline{\mathsf{d}})$ *is a monad:*

$$\overline{\mathsf{p}}_{!A}; \overline{\mathsf{p}}_A = !\overline{\mathsf{p}}_A; \overline{\mathsf{p}}_A \qquad \overline{\mathsf{d}}_{!A}; \overline{\mathsf{p}}_A = 1_{!A} = !\overline{\mathsf{d}}_A; \overline{\mathsf{p}}_A \tag{10}$$

2. The codigging $\overline{\mathsf{p}}$ is a monoid morphism:

$$\overline{\mathsf{c}}_{!A}; \overline{\mathsf{p}}_A = (\overline{\mathsf{p}}_A \otimes \overline{\mathsf{p}}_A); \overline{\mathsf{c}}_A \qquad \overline{\mathsf{w}}_{!A}; \overline{\mathsf{p}}_A = \overline{\mathsf{w}}_A \tag{11}$$

3. The codigging $\overline{\mathsf{p}}$ and the dereliction d are compatible in the following sense:

$$\overline{\mathsf{p}}_A; \mathsf{d}_A = \mathsf{c}_{!A}; (\overline{\mathsf{p}}_A \otimes \mathsf{d}_{!A}); (\mathsf{w}_A \otimes \mathsf{d}_A) \tag{12}$$

Let us give some more intuition for these codigging axioms. The type of codigging says that $\overline{\mathsf{p}}$ is a non-linear map from $!A$ to $!A$. The next thing we can say is that codigging is indeed a generalized version of e^x for $!A$. Indeed, (10) and (11) are precisely the requirements which makes $\overline{\mathsf{p}}_A$ a !-differential exponential map.

Lemma 3.3. *In a monadic differential category, the codigging* $\overline{\mathsf{p}}_A : !A \to !!A$ *is a* $!$*-differential exponential map for the commutative monoid* $(!A, \overline{\mathsf{c}}_A, \overline{\mathsf{w}}_A)$.

Proof. One of the monad axioms in (10) is that $\overline{\mathsf{p}}_A$ is a retract of $\overline{\mathsf{d}}_{!A}$, while (11) says that $\overline{\mathsf{p}}_A$ is a monoid morphism between the commutative monoids $(!!A, \overline{\mathsf{c}}_{!A}, \overline{\mathsf{w}}_{!A})$ and $(!A, \overline{\mathsf{c}}_A, \overline{\mathsf{w}}_A)$. These are precisely the requirements which makes $\overline{\mathsf{p}}_A$ a $!$-differential exponential map. □

So the equations of (11), which express the interactions between $\overline{\mathsf{p}}$ and both $\overline{\mathsf{c}}$ and $\overline{\mathsf{w}}$, are indeed analogues of $e^{x+y} = e^x e^x$ and $e^0 = 1$, while the first part of the second equation of (10), expressing the interaction between $\overline{\mathsf{p}}$ and $\overline{\mathsf{d}}$, says that the derivative evaluated at 0 of $\overline{\mathsf{p}}$ is the identity. To help understand the other codigging axioms, since $\overline{\mathsf{p}}$ is a generalization of e^x, it will be usefuly to use a very naive "illicit formula" for $\overline{\mathsf{p}}$ based on the exponential function's power series:

$$e^x = \sum_n \frac{x^n}{n!}$$

In classical DiLL, elements of $!A$ can be interpreted as *distributions*, that is, linear scalar maps acting on non-linear maps, so $[\![!A]\!] := \mathcal{L}(\mathcal{L}_!([\![A]\!], I), I)$, where I is often interpreted as the field of real or complex numbers. From this point of view, cocontraction is interpreted by the *convolution of distributions*:

$$\overline{\mathsf{c}}_A : \phi \otimes \psi \mapsto \phi * \psi := (f \mapsto \phi(x \mapsto \psi(y \mapsto f(x+y))))$$

Now recall that for each element x of A, the dirac distribution at x is distribution which evaluates a non-linear map at x, so $\delta_x : f \mapsto f(x)$. In many cases, it is sufficient to define what a non-linear map does on dirac distributions. As such, the dereliction maps a dirac distribution to the element it tests functions with, $\mathsf{d}_A : \delta_x \mapsto x$, while the contraction duplicates the dirac distribution's test element, $\mathsf{c}_A : \delta_x \mapsto \delta_x \otimes \delta_x$. These intuitions are explained in more details in [13]. Assuming that we have proper convergences and can scalar multiply by rationals, we may generalize x with the dereliction and x^n with applying contraction and cocontraction to the dereliction, to obtain the following formula for codigging:

$$\overline{\mathsf{p}}_A : \delta_\phi \mapsto \exp^*(\phi) = \sum_n \frac{\phi^{*^n}}{n!} \tag{13}$$

where $\phi^{*^n} = \phi * \ldots * \phi$. This is called the convolutional exponential. We will make this formula precise in Section 3.3, and relate it with new models in Section 5.

Now the other monad axioms involving the codigging and the codereliction is that $!\overline{\mathsf{d}}_A; \overline{\mathsf{p}}_A = 1_{!A}$. On dirac distributions, the codereliction gives the differential operator at zero, $\overline{\mathsf{d}}_A : \delta_x \mapsto D_0(\_)(x)$. Then on the left hand side, we have:

$$!\overline{\mathsf{d}}_A; \overline{\mathsf{p}} : \delta_x \mapsto \exp\left(D_0(\_)(x)\right) = \sum_n \frac{D_0(\_)(x)^{*^n}}{n!}$$

However, $D_0(\_)(v)^{*^n}$ is exactly the distribution mapping a function to its n-th differential at 0, $f \mapsto D_0^{(n)}(f)(x)$. Therefore, if $!\overline{\mathsf{d}}_A; \overline{\mathsf{p}}_A = 1_{!A}$ holds, this means that for every $x \in A$ and $f : !A \to B$:

$$\sum_n \frac{D_0^{(n)}(\_)(x)}{n!} = \delta_x \quad \text{thus} \quad \sum_n \frac{D_0^{(n)}(f)(x)}{n!} = f(x)$$

In other words, in a model with codigging, every non-linear map is equal to its Taylor approximation at 0. This implies that any model of DiLL with codigging needs to be a *quantitative model*, with non-linear maps being power series, such that the exponential function series also converges.

The third monad axiom that $\overline{\mathsf{p}}_{!A}; \overline{\mathsf{p}}_A = !\overline{\mathsf{p}}_A; \overline{\mathsf{p}}_A$ essentially explains how to interpret the exponential of the exponential, e^{e^x}. In particular, we have $\exp(\exp(\phi)) = \overline{\mathsf{p}}_A(\exp(\delta_\phi))$. Lastly, equation (12) states the interaction between $\overline{\mathsf{p}}$ and d, which we call the "cochain rule", since the compatibility between $\overline{\mathsf{d}}$ and p is the chain rule. On the one hand,

$$\overline{\mathsf{p}}_A; \mathsf{d}_A : \delta_{\delta_x} \mapsto \sum_n \frac{nx}{n!}$$

By factoring out x, we know that the righthand side should be $e^1 x$, which is indeed what the other side of (12) is. On one hand we have that: $\mathsf{d}_{!A}; \mathsf{d}_A : \delta_{\delta_x} \mapsto x$. While on the other hand, since weakening maps dirac distributions to 1, $\mathsf{w}_A : \delta_x \mapsto 1$, we have also have that $\overline{\mathsf{p}}_A; \mathsf{w}_A : \delta_{\delta_x} \mapsto e^1$. Therefore, (12) precisely tells us that:

$$\sum_n \frac{nx}{n!} = e^1 x.$$

At this point it may be worth discussing how one could argue that codigging $\overline{\mathsf{p}}$ is just a special case of the dereliction d. Indeed, note that $\overline{\mathsf{p}}_A : !!A \to !A$ and $\mathsf{d}_{!A} : !!A \to !A$ have the same types. However, by comparing (6) and (12), we see that the interactions with $\overline{\mathsf{c}}$ and $\overline{\mathsf{w}}$ differ significantly. Intuitively what this means is that the dereliction d and the codigging $\overline{\mathsf{p}}$ are both ways to embed linear maps into non-linear ones, where the dereliction does this by merely forgetting about linearity, while codigging creates non-linearity via exponentiation. This can also be compared to the action of p and $\overline{\mathsf{d}}$, where p has the same type of a restricted $\overline{\mathsf{d}}$. While the digging creates linearity by going to higher-order, dereliction is more radical and creates linearity through differentiation.

A natural question to ask is if there is also any interaction laws between codigging and contraction, or weakening, or even digging. For contraction and weakening, by dualizing the constructions of [25, Sec.7], we use the codigging to construct a natural transformation $\overline{\mu}_{A,B} : !(A \otimes B) \to !A \otimes !B$ and a map $\overline{\mu}_I : !I \to I$ respectively as follows:

$$\begin{aligned} \overline{\mu}_{A,B} &:= !(\overline{\mathsf{d}}_A \otimes \overline{\mathsf{d}}_B); !\chi^{-1}_{!A,!B}; \overline{\mathsf{p}}_{A\times B}; \chi_{A,B} \\ \overline{\mu}_I &:= !(\overline{\mathsf{w}}_I); \overline{\mathsf{p}}_I; \mathsf{w}_I \end{aligned} \tag{14}$$

It is important to point out that while $\overline{\mu}$ and $\overline{\mu}_I$ make ! into a lax comonoidal functor, they are not inverses to μ and μ_I. Indeed, on dirac distributions, we have that $\mu_{A,B} : \delta_x \otimes \delta_y \mapsto \delta_{x\otimes y}$ and $\mu_I : 1 \mapsto \delta_1$. While $\overline{\mu}$ gives a version of partial Taylor approximation in two variables:

$$\overline{\mu}_{A,B} : \delta_{x\otimes y} \mapsto \sum_n \frac{D_0^{(n)}(\_)(x) \otimes D_0^{(n)}(\_)(y)}{n!}$$

We stress that the above formula is not the full Taylor approximation of a smooth function in two variables, and therefore its composition with μ is not equal to the identity in a codigging setting. On the other hand for $\overline{\mu}_I$, recall that the monoidal unit is often interpreted as the field of real or complex numbers $I = \mathbb{K}$. Then $\overline{\mu}_{\mathbb{K}} : !\mathbb{K} \to \mathbb{K}$, interpreted as a non-linear map $\mathbb{K} \to \mathbb{K}$, does indeed recapture the classical exponential function $e^x : \mathbb{K} \to \mathbb{K}$. So on dirac distributions, $\overline{\mu}_I : \delta_x \mapsto e^x$. We can make this precise as:

Lemma 3.4. *In a monadic differential category, the map $\overline{\mu}_I : !I \to I$, as defined in (14), is a !-differential exponential map for I (with respect to the canonical monoid structure on the monoidal unit).*

Proof. By construction, $\overline{\mu}_I$ is the composite of monoid morphisms, and therefore is itself a monoid morphism. Furthermore, since both $\overline{\mathsf{w}}_I \circ \mathsf{w}_I = 1_I$ and $\overline{\mathsf{d}}_{!I} \circ \overline{\mathsf{p}}_I = 1_{!I}$, and by the naturality of $\overline{\mathsf{d}}$, we have that $\eta_I \circ \overline{\mu}_I = 1_I$. Therefore, we conclude that $\overline{\mu}_I$ is a !-differential exponential map on I as desired. □

Turning our attention back to the relation between codigging and the comonoid structure, we can use $\overline{\mu}$ and $\overline{\mu}_I$ to obtain the dual of equation (7) for codigging.

Lemma 3.5. *In a monadic differential category, the codigging $\overline{\mathsf{p}}$ is compatible with the comonoid structure in the sense that the following equalities hold:*

$$\overline{\mathsf{p}}_A; \mathsf{c}_A = !\mathsf{c}_A; \overline{\mu}_{!A,!A}; (\overline{\mathsf{p}}_A \otimes \overline{\mathsf{p}}_A) \quad \overline{\mathsf{p}}_A; \mathsf{w}_A = !\mathsf{w}_A; \overline{\mu}_I \tag{15}$$

Proof. By symmetry of all the axioms, the calculations to prove (15) are precisely dual to the calculations to prove (7), which can be found in [25, App.B]. □

Unfortunately there does not seem to be any obvious compatibility between digging and codigging, specifically what $\overline{\mathsf{p}}_A; \mathsf{p}_A$ may be equal to. Even when investigating in well-behaved models, there does not seem to be any immediate answer. So, for now, we do not require any extra coherence between $\overline{\mathsf{p}}$ and p, and discuss possibilities in the conclusion.

Let us briefly focus our attention back to codigging and its relation to exponential functions. Whenever one has a monad, an important question to ask is what can we say about the algebras of said monad. It turns out that in a monadic differential category, every algebra for the monad ! comes equipped with a natural !-differential exponential map. Recall that an algebra for the monad !, also called a **!-algebra**, is a pair (A, a) consisting of an object A and a map $a : !A \to A$, called the **!-algebra structure map**, such that $\overline{\mathsf{p}}_A; a = !a; a$ and $\overline{\mathsf{d}}_A; a = 1_A$. Then not only does every !-algebra have a canonical commutative monoid structure, the !-algebra structure map is a !-differential exponential map.

Lemma 3.6. *In a monadic differential category, let (A, a) be a !-algebra. Define the maps $\overline{\mathsf{c}}^a_A : A \otimes A \to A$ and $\overline{\mathsf{w}}^a_A : I \to A$ respectively as follows:*

$$\overline{\mathsf{c}}^a_A := (\overline{\mathsf{d}}_A \otimes \overline{\mathsf{d}}_A); \overline{\mathsf{c}}_A; a \qquad \overline{\mathsf{w}}^a_A := \overline{\mathsf{w}}_A; a \tag{16}$$

Then $(A, \overline{\mathsf{c}}^a_A, \overline{\mathsf{w}}_A)$ is a commutative monoid and $a : !A \to A$ is a !-differential exponential map. In other words, every !-algebra is a !-differential exponential algebra.

Proof. A well-known result about storage modalities is that every !-coalgebra (the dual of !-algebra for the comonad !) comes equipped with a canonical cocommutative comonoid structure and the !-coalgebra structure map is a comonoid morphism [27, Prop.28]. The above proposed construction for !-algebras is precisely the dual of the one for !-coalgebras. Therefore, by dualizing the proof, we indeed have that $(A, \overline{\mathsf{c}}^a_A, \overline{\mathsf{w}}_A)$ is a commutative monoid and a is a monoid morphism. Furthermore, by definition of !-algebra, we have that a is a retract of the codereliction $\overline{\mathsf{d}}_A$. So we conclude that a is a !-differential exponential map, as desired. □

3.3 Codigging via Taylor Expansion

In the previous sections, we discussed how codigging was closely linked to Taylor expansion and gave an "illicit formula" for codigging (13), which is based on the Taylor series of the exponential function e^x. The objective of this section is to make the "illicit formula" for codigging legitimate and argue that it makes sense in well-behaved differential categories where Taylor expansion is well-defined. We will justify this even further in Section 4 by providing examples where the "illicit formula" for codigging holds.

Taylor expansion is an important concept in DiLL, as first developped by Ehrhard in Regnier in [28] and later studied by many others, such as Pagini and Tasson in [17] or Boudes et al in [18]. From the categorical point of view, the concept of Taylor expansion in a differential category was first discussed by Ehrhard in [10, Sec 3.1]. However, as discussed above, in order for codigging to properly give a monad, not only do we need Taylor expansions in a differential category but also that every non-linear map is equal to its Taylor series. As such, in this section we introduce the novel concept of a *Taylor* differential category, which is essentially a differential category where if two coKleisli maps have the same Taylor expansion, then they must be equal. This implies that in a Taylor differential category, every coKleisli map is equal to its Taylor series, which can be made even more precise in a setting with some notion of well-defined convergence for infinite sums. The main result of this section is that a Taylor differential category has codigging if and only if there is a non-linear map whose Taylor expansion is precisely given by the "illicit formula" for codigging. In Section 4, we will provide numerous examples of Taylor differential categories with codigging.

As Taylor differential categories are inspired by Ehrhard's work, we will continue using mostly the same notation as in [10]. Let us first define some useful natural transformations. For every $n \in \mathbb{N}$, for an object A or a map f, we denote $A^{\otimes^n}$ and $f^{\otimes^n}$ as a short hand for the monoidal product of n copies of A or f, with the convention that $A^{\otimes^0} = I$ and $A^{\otimes^1} = A$, and that $f^{\otimes^0} = 1_I$ and $f^{\otimes^1} = f$. Now for every $n \in \mathbb{N}$, define $\mathsf{c}^n_A : !A \to !A^{\otimes^n}$ to be the map which comultiplies $!A$ into n-copies of $!A$, and $\overline{\mathsf{c}}^n_A : !A^{\otimes^n} \to !A$ which multiplies n-copies of $!A$ together. By convention, we set that $\mathsf{c}^1_A = 1_{!A}$, $\overline{\mathsf{c}}^1_A = 1_{!A}$, $\mathsf{c}^2_A = \mathsf{c}_A$, and $\overline{\mathsf{c}}^2_A = \overline{\mathsf{c}}_A$. Now define $\mathsf{d}^n_A : !A \to A^{\otimes^n}$ and $\overline{\mathsf{d}}^n_A : A^{\otimes^n} \to !A$ respectively as the composites $\mathsf{d}^n_A := \mathsf{c}^n_A; \mathsf{d}^{\otimes^n}_A$ and $\overline{\mathsf{d}}^n_A := \overline{\mathsf{d}}^{\otimes^n}_A; \overline{\mathsf{c}}^n_A$.

In order to properly define the main natural transformation for Taylor expansion, it is necessary to be able to multiply by $\frac{1}{n!}$, which is an important ingredient in the Taylor expansion formula. As such, we now need to assume we are working in a setting where we can scalar multiplying maps by the non-negative rationals $\mathbb{Q}_{\geq 0}$. Thus for the remainder of this section, we will be working in a $\mathbb{Q}_{\geq 0}$-differential storage category, which means a differential storage category such that each homset is also a $\mathbb{Q}_{\geq 0}$-module. In particular, this implies we may scalar multiply any map $f : A \to B$ by any $\frac{p}{q} \in \mathbb{Q}_{\geq 0}$ to obtain a map $\frac{p}{q} \cdot f$, and scalar multiplication is compatible with composition and the monoidal product. This is not a very heavy requirement, and is often a desirable setting of interest, especially when working with differential categories that have some notion of antiderivatives [10, 29] or integration [30].

Then define $\mathsf{M}^n_A : !A \to !A$ as:

$$\mathsf{M}^n_A := \frac{1}{n!} \cdot \left(\mathsf{d}^n_A; \overline{\mathsf{d}}^n_A \right) \tag{17}$$

Observe that $\mathsf{M}^0_A = \mathsf{w}_A; \overline{\mathsf{w}}_A$ and $\mathsf{M}^1_A = \mathsf{d}_A; \overline{\mathsf{d}}_A$. Intuitively, pre-composing a coKleisli map $f : !A \to B$ gives the n-th term in Taylor series of f at 0, $(\mathsf{M}^n_A; f)(x) = \frac{1}{n!} \cdot D^{(n)}_0(f)(x)$. Here, we call the composite $\mathsf{M}^n_A; f$ the **n-th Taylor monomial** of f. In [10, Sec 3.1], Ehrhard defined the natural

transformation $\mathsf{T}^n_A : !A \to !A$ as the sum $\mathsf{T}^n_A := \sum_{k=0}^n \mathsf{M}^n_A$ and described $\mathsf{T}^n_A; f$ as the **n-th Taylor polynomial** of f. We may now define the notion of a Taylor differential category:

Definition 3.7. *A **Taylor differential category** is a $\mathbb{Q}_{\geq 0}$-differential storage category such that for any pair of parallel coKleisli maps $f : !A \to B$ and $g : !A \to B$, if for all $n \in \mathbb{N}$, $\mathsf{M}^n_A; f = \mathsf{M}^n_A; g$, then $f = g$.*

In other words, if two non-linear maps have the same Taylor monomials (or Taylor polynomials), then they must be equal. This implies that every non-linear map is completely determined by its Taylor expansion. In fact, we will explain how in a Taylor differential category, every Taylor series converges in a well-defined way and how every non-linear map is equal to its Taylor series.

While Taylor differential categories are interesting on their own and merit further exploration, we are particularly interested in when a Taylor differential category has a codigging. So assume that a codigging $\overline{\mathsf{p}}_A : !!A \to !A$ exists. Using all three of the axioms for a !-differential exponential map, it is straightforward to compute that $\overline{\mathsf{d}}^n_{!A}; \overline{\mathsf{p}}_A = \overline{\mathsf{c}}^n_A$. Thus, the Taylor monomials of the codigging are $\mathsf{M}^n_{!A}; \overline{\mathsf{p}}_A = \frac{1}{n!} \cdot (\mathsf{d}^n_{!A}; \overline{\mathsf{c}}^n_A)$. Observe that $\mathsf{M}^0_{!A}; \overline{\mathsf{p}}_A = \mathsf{w}_{!A}; \overline{\mathsf{w}}_A$ and $\mathsf{M}^0_{!A}; \overline{\mathsf{p}}_A = \mathsf{d}_{!A}$. Now observe that the Taylor monomials of the codigging can be defined in any $\mathbb{Q}_{\geq 0}$-differential storage category. Therefore a Taylor differential category has a codigging if there exists maps whose Taylor monomials are those of a codigging.

Proposition 3.8. *A Taylor differential category is a monadic differential category if and only if for every A, there exists a (necessarily unique) map $\overline{\mathsf{p}}_A : !!A \to !A$ such that for every $n \in \mathbb{N}$, the following equality holds:*

$$\mathsf{M}^n_{!A}; \overline{\mathsf{p}}_A = \frac{1}{n!} \cdot (\mathsf{d}^n_{!A}; \overline{\mathsf{c}}^n_A) \tag{18}$$

Proof. Let us start with proving that $\overline{\mathsf{p}}_A$ is a monoid morphism. Starting with preservation of the unit, note that the case $n = 0$ of (18) says that $\mathsf{w}_{!A}; \overline{\mathsf{w}}_{!A}; \overline{\mathsf{p}}_A = \mathsf{w}_{!A}; \overline{\mathsf{w}}_{!A}$. Pre-composing each side by $\overline{\mathsf{w}}_{!A}$, by the bimonoid identity (5), we have that $\overline{\mathsf{w}}_{!A}; \overline{\mathsf{p}}_A = \overline{\mathsf{w}}_{!A}$ as desired. Next to prove that $\overline{\mathsf{p}}_A$ also preserves the multiplication, we will first show that $\chi_{!A,!A}; \overline{\mathsf{c}}_{!A}; \overline{\mathsf{p}}_A$ is equal to $\chi_{!A,!A}; (\overline{\mathsf{p}}_A \otimes \overline{\mathsf{p}}_A); \overline{\mathsf{c}}_A$ using the Taylor property. Then carefully using the bimonoid identities (5) and binoemial coefficient identities, we can compute that:

$$\begin{aligned}
&\mathsf{M}^n_{!A \times !A}; \chi_{!A,!A}; \overline{\mathsf{c}}_{!A}; \overline{\mathsf{p}}_A = \\
&= \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \cdot \left(\mathsf{d}^n_{!A}; (!\pi_0)^{\otimes^k} \otimes (!\pi_1)^{\otimes^{n-k}}; \overline{\mathsf{c}}^n_A \right) \\
&= \mathsf{M}^n_{!A \times !A}; \chi_{!A,!A}; (\overline{\mathsf{p}}_A \otimes \overline{\mathsf{p}}_A); \overline{\mathsf{c}}_A
\end{aligned}$$

So by the Taylor property we have that $\chi_{!A,!A}; \overline{\mathsf{c}}_{!A}; \overline{\mathsf{p}}_A = \chi_{!A,!A}; (\overline{\mathsf{p}}_A \otimes \overline{\mathsf{p}}_A); \overline{\mathsf{c}}_A$. Pre-composing both sides by $\chi^{-1}_{!A,!A}$ we obtain that $\overline{\mathsf{c}}_{!A}; \overline{\mathsf{p}}_A = (\overline{\mathsf{p}}_A \otimes \overline{\mathsf{p}}_A); \overline{\mathsf{c}}_A$. So $\overline{\mathsf{p}}_A$ is indeed a monoid morphism as desired.

Now let us explain why ! is a monad. Note that the case $n = 1$ of (18) says that $\mathsf{d}_{!A}; \overline{\mathsf{d}}_{!A}; \overline{\mathsf{p}}_A = \mathsf{d}_{!A}$. Pre-composing each side by $\overline{\mathsf{d}}_{!A}$, by the codereliction identity (8), we have that $\overline{\mathsf{d}}_{!A}; \overline{\mathsf{p}}_A = 1_{!A}$. Next, by naturality of M^n and d^n, it easy to compute that for all n we have that $\mathsf{M}^n_A; !\mathsf{d}_A; \overline{\mathsf{p}}_A = \mathsf{M}^n_A$. Therefore by the Taylor property, it follows that $!\mathsf{d}_A; \overline{\mathsf{p}}_A = 1_{!A}$. On the other hand, using the naturality of $\overline{\mathsf{d}}^n$, that $\overline{\mathsf{p}}$ is a monoid morphism, and $\overline{\mathsf{d}}_{!A}; \overline{\mathsf{p}}_A = 1_{!A}$, for every n we can show that:

$$\mathsf{M}^n_{!!A}; \overline{\mathsf{p}}_{!A}; \overline{\mathsf{p}}_A = \frac{1}{n!} \cdot \left(\mathsf{d}^n_{!!A}; \overline{\mathsf{p}}^{\otimes^n}_A; \overline{\mathsf{c}}^n_A \right) = \mathsf{M}^n_{!!A}; !\overline{\mathsf{p}}_A; \overline{\mathsf{p}}_A$$

So by the Taylor property, we have that $\overline{\mathsf{p}}_{!A};\overline{\mathsf{p}}_A = !\overline{\mathsf{p}}_A;\overline{\mathsf{p}}_A$. So we have that ! is indeed a monad.

Lastly, using (6), for every n, we can compute that:

$$\mathsf{M}^0_{!A};\overline{\mathsf{p}}_A;\mathsf{d}_A = 0 = \mathsf{M}^0_{!A};\mathsf{c}_{!A};(\overline{\mathsf{p}}_A \otimes \mathsf{d}_{!A});(\mathsf{w}_A \otimes \mathsf{d}_A)$$

$$\begin{aligned}\mathsf{M}^{n+1}_{!A};\overline{\mathsf{p}}_A;\mathsf{d}_A &= \frac{1}{n!}\cdot\left(\mathsf{d}^{n+1}_{!!A};\mathsf{d}_A \otimes \mathsf{w}_A^{\otimes^n}\right)\\ &= \mathsf{M}^{n+1}_{!A};\mathsf{c}_{!A};(\overline{\mathsf{p}}_A \otimes \mathsf{d}_{!A});(\mathsf{w}_A \otimes \mathsf{d}_A)\end{aligned}$$

Note that in the $n+1$ case, the factor $\frac{1}{n!}$ is indeed correct since we obtain n copies of $\mathsf{d}^{n+1}_{!!A};\mathsf{d}_A \otimes \mathsf{w}_A^{\otimes^n}$, which when multiplied by $\frac{1}{(n+1)!}$ gives $\frac{1}{n!}$. So by the Taylor property, we obtain that $\overline{\mathsf{p}}_A;\mathsf{d}_A = \mathsf{c}_{!A};(\overline{\mathsf{p}}_A \otimes \mathsf{d}_{!A});(\mathsf{w}_A \otimes \mathsf{d}_A)$. So we conclude that $\overline{\mathsf{p}}$ is a codigging as desired. □

Now let us explain why in a Taylor differential category $\mathcal{L}$, Taylor series converge. To do so, we must define a metric on the homset $\mathcal{L}(!A, B)$ in which the sequence of Taylor polynomials converges. So define $\mathfrak{D} : \mathcal{L}(!A, B) \times \mathcal{L}(!A, B) \to \mathbb{R}$ as $\mathfrak{D}(f, g) = 2^{-n}$, where n is the smallest natural number such that $\mathsf{M}^n_A;f \neq \mathsf{M}^n_A;g$, and $\mathfrak{D}(f, g) = 0$ if for all n, $\mathsf{M}^n_A;f = \mathsf{M}^n_A;g$. Then $\mathfrak{D}$ is not only a metric but an *ultrametric*, making $\mathcal{L}(!A, B)$ an ultrametric space. At first glance this metric may seem a bit ad hoc, but $\mathfrak{D}$ is in fact a generalization of the metric for power series, which is used to make power series properly converge.

Lemma 3.9. *In a Taylor differential category, for every coKleisli map $f : !A \to B$, the following series converges to f with respect to ultrametric $\mathfrak{D}$:*

$$f = \sum_{n=0}^{\infty} \mathsf{M}^n_A;f \tag{19}$$

Proof. First note that $\mathsf{M}^n;\mathsf{M}^n = \mathsf{M}^n$ while $\mathsf{M}^n;\mathsf{M}^m = 0$ if $n \neq m$. Therefore, it follows that $\mathfrak{D}\left(\sum_{n=0}^{m} \mathsf{M}^n_A;f, f\right) \leq 2^{m+1}$ and so $\lim_{m\to\infty} \mathfrak{D}\left(\sum_{n=0}^{m} \mathsf{M}^n_A;f, f\right) = 0$. So we conclude that the desired series converges to f.

□

As a consequence, the "illicit formula" for codigging is perfectly legitimate in a Taylor differential category.

Corollary 3.10. *In a Taylor differential category that is also a monadic differential category, the following series converges to the codigging $\overline{\mathsf{p}}$ with respect to ultrametric $\mathfrak{D}$:*

$$\overline{\mathsf{p}}_A = \sum_{n=0}^{\infty} \frac{1}{n!}\cdot\left(\mathsf{c}^n_{!A};\mathsf{d}^{\otimes^n}_{!A};\overline{\mathsf{c}}^n_A\right) \tag{20}$$

We leave it to reader to check for themselves that (20) is indeed a proper generalization of (13).

A very natural, and important, question to ask is what if there was already some other established notion of infinite sum or convergence. Would the resulting Taylor series be the same as the one given by the ultrametric $\mathfrak{D}$. Under mild assumption, the answer is yes. Since many of the examples in Section 4 have an algebraic notion of infinite sums, let us focus on this setting.

Briefly, recall that a **countably complete $\mathbb{Q}_{\geq 0}$-module** is a $\mathbb{Q}_{\geq 0}$-module which also has arbitrary countable sums, and such that these countable sums satisfy certain distributivity and partitions axioms (see [31, Chap 23] for more details). Then by a **$\mathbb{Q}^{\Sigma}_{\geq 0}$-differential storage category**, we mean a differential storage category that is enriched over the category of countably complete $\mathbb{Q}_{\geq 0}$-modules, that is, such that each homset is also a countably complete $\mathbb{Q}_{\geq 0}$-module such that both composition and the monoidal product are compatible with the countable sums in the obvious way. In particular, this means we can scalar multiply maps by $\mathbb{Q}_{\geq 0}$ and we have countable infinite sums of maps $\sum_{n=0}^{\infty} f_n : A \to B$. With one other assumption, we obtain both the Taylor property and codigging. In the following lemma, all infinite sums are the one given by the countable additive enrichment.

Lemma 3.11. *Let $\mathcal{L}$ be a $\mathbb{Q}^{\Sigma}_{\geq 0}$-differential storage category such that the following equality holds:*

$$\sum_{n=0}^{\infty} \mathsf{M}^n_A = 1_{!A} \tag{21}$$

Then $\mathcal{L}$ is a Taylor differential category and for every coKleisli map $f : !A \to B$, $f = \sum_{n=0}^{\infty} \mathsf{M}^n_A; f$. Furthermore, $\mathcal{L}$ is also a monadic differential category where the codigging $\overline{\mathsf{p}}_A : !!A \to !A$ is defined as $\overline{\mathsf{p}}_A = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot (\mathsf{d}^n_{!A}; \overline{\mathsf{c}}^n_A)$.

Proof. Since composition preserves countable sums, (21) implies that $f = \sum_{n=0}^{\infty} \mathsf{M}^n_A; f$. It then clearly follows that we have a Taylor differential category. Lastly, it easy to check that $\mathsf{M}^n_{!A}; \overline{\mathsf{p}}_A = \frac{1}{n!} \cdot (\mathsf{d}^n_{!A}; \overline{\mathsf{c}}^n_A)$. Therefore by Proposition 3.8, we have that $\overline{\mathsf{p}}$ is a codigging as desired. □

The above lemma also tells us that for Taylor series, the infinite sum given by the ultrametric $\mathfrak{D}$ is the same as the infinite sum given by the countable additive enrichment.

4 Examples of Codigging

In this section, we provide many examples of models with codigging, some of which are already well known quantitative models of LL and/or DiLL.

4.1 Relations

One of the most important categorical models of LL and DiLL is the relational model. We will now explain how the relational model is also a monadic differential category. Since this model holds such an important role in LL, we take the pain of providing quite a bit of detail for this example.

So let REL be the category of sets and relations, that is, the category whose objects are sets X and whose morphism $R : X \to Y$ are relations, i.e., subsets $R \subseteq X \times Y$. It is already well known that REL is a differential storage category [24, 2.5.1]. The tensor product is given by the Cartesian product of sets, $X \otimes Y = X \times Y$ (which is not the categorical product) and the unit is a chosen singleton $I = \{*\}$. The (bi)product is given by the disjoint union of sets $X \sqcup Y$ and the terminal object is the empty set $\emptyset$. While the additive structure is given by the union of sets, that is, the sum of relations is there union $R + S = R \cup S$ and the zero maps are the empty subsets $0 := \emptyset$.

The storage modality ! is given by finite multisets (also sometimes called finite bags), so $!X = \mathcal{M}_f(X)$, the set of all finite multisets of X. The dereliction $\mathsf{d}_X \subseteq !X \times X$ and codereliction $\overline{\mathsf{d}}_X \subseteq X \times !X$ relates elements of X to the multisets containing that one element:

$$\mathsf{d}_X := \{([x], x) \mid \forall x \in X\} \qquad \overline{\mathsf{d}}_X := \{(x, [x]) \mid \forall x \in X\}$$

The contraction $\mathsf{c}_X \subseteq !X \times (!X \times !X)$ and cocontraction $\overline{\mathsf{c}}_X \subseteq (!X \times !X) \times !X$ relate pairs of finite multisets to their disjoint union, while the weakening $\mathsf{w}_X \subseteq !X \times \{*\}$ and coweakening $\overline{\mathsf{w}}_X \subseteq \{*\} \times !X$ relate $*$ to the empty multiset:

$$\mathsf{c}_X := \{(m, (m_1, m_2)) \mid \forall m, m_1, m_2 \in !X, m = m_1 \sqcup m_2\}$$
$$\overline{\mathsf{c}}_X := \{((m_1, m_2), m_1 \sqcup m_2) \mid \forall m_1, m_2 \in !X\}$$
$$\overline{\mathsf{w}}_X := \{(*, \emptyset)\} \qquad \mathsf{w}_X := \{(\emptyset, *)\}$$

While the digging $\mathsf{p}_X \subseteq !X \times !!X$ relates a finite multiset to all possible finite multiset of finite multisets (of any size) whose disjoint union is the original multiset:

$$\mathsf{p}_X := \left\{ (m, [m_1, \ldots, m_n]) \,|\forall m, m_i \in !X, \text{ s.t. } \bigsqcup_i m_i = m \right\}$$

Now there are many ways to argue why REL is also a monadic differential category. Of particular interest for this paper is using the results of Section 3.3. Now REL is a $\mathbb{Q}^{\Sigma}_{\geq 0}$-differential storage category [30, Ex 7.2], where scalar multiplying by a non-zero rational does nothing, $\frac{p}{q} \cdot R = R$, since in particular $R + R = R$, and where countable sums are given by the countable unions, $\sum_n R_n = \bigcup R_n$. Furthermore, for each n, $\mathsf{M}^n_X \subseteq !X \times !X$ relates finite bags of size n to themselves:

$$\mathsf{M}^n_X = \{(m, m) | \ \forall w \in !X, |m| = n\}$$

where $|m|$ denotes the cardinality of a multiset. Then it is easy to see that (21) holds. Therefore, REL is both a Taylor differential category and a monadic differential category. As such, the codigging $\overline{\mathsf{p}}_X \subseteq !!X \times !X$ relates a finite multiset of finite multisets to its disjoint union:

$$\overline{\mathsf{p}}_X := \{([m_1, \ldots, m_n], m_1 \sqcup \ldots \sqcup m_n) \mid \forall n \in \mathbb{N}, m_i \in !X\}$$

In REL, there is also a notion of transpose, where for a relation $R \subseteq X \times Y$, its transpose $R^\dagger \subseteq Y \times X$ is defined as $R^\dagger = \{(y, x) | \ \forall (x, y) \in R\}$. Observe that the codigging is the transpose of the digging, the codereliction is the transpose of the dereliction, etc. So we have that $\overline{\mathsf{p}}_X = \mathsf{p}^\dagger_X$, $\overline{\mathsf{d}}_X = \mathsf{d}^\dagger_X$, $\overline{\mathsf{c}}_X = \mathsf{c}^\dagger_X$, and $\overline{\mathsf{w}}_X = \mathsf{w}^\dagger_X$. This fits into a more general story which we will discuss in Section 4.4.

4.2 Weighted Relations

In this section we discuss how the weighted relational model, which is a generalization of the relational model, also gives a monadic differential category. While this example is very similar to the one gives above, we still cover this example in some detail to demonstrate how certain coefficients appear in the definition of codigging, which were swept under the rug in the relational model. For an overview on the weighted relational model, we invite the reader to see Ong's paper [32].

Briefly, a complete commutative $\mathbb{Q}_{\leq 0}$-semiring is a commutative semiring R, which is also a $\mathbb{Q}_{\leq 0}$-modules and admits arbitrary set indexed sums, and that are compatible with the $\mathbb{Q}_{\leq 0}$-semiring structure, see [31, Chap 15]. Then define R^{Π} to be the category whose objects are sets X and where a map from X to Y is an arbitrary set function $f : X \times Y \to R$. Intuitively, maps of R^{Π} are interpreted as generalized matrices with coefficients in R. It is already established that R^{Π} is a differential storage category [29, Sec 6], where most of the structure is essentially similar to that of REL. Indeed, the monoidal structure and (bi)product structure of R^{Π} are the same as in REL, while the additive structure of R^{Π} is given by the sum of R. The storage modality ! is also given by finite multisets, so $!X = \mathcal{M}_f(X)$.

To define the remaining structure, first define for any set X and elements $x, y \in X$, the Kroenecker delta $\delta_{x,y}$ as checking if x and y are equal, that is, $\delta_{x,y} = 1$ if $x = y$ and $\delta_{x,y} = 0$ (where 1 and 0 are viewed as elements in R). So the dereliction $\mathsf{d}_X : !X \times X \to R$ and codereliction $\overline{\mathsf{d}}_X : X \times !X \to R$ check for singletons:

$$\mathsf{d}_X(m, x) := \delta_{m,[x]} \qquad\qquad \overline{\mathsf{d}}_X := (x, m) := \delta_{m,[x]}$$

The contraction $\mathsf{c}_X : !X \times (!X \times !X) \to R$ and cocontraction $\overline{\mathsf{c}}_X : (!X \times !X) \times !X \to R$ check if a finite multiset is equal to the disjoint uion of a pair of finite multisets, while on the other hand the weakening $\mathsf{w}_X : !X \times \{*\} \to R$ and coweakening $\overline{\mathsf{w}}_X : \{*\} \times !X \to R$ check for the empty multiset:

$$\begin{aligned} \mathsf{c}_X\,(m, (m_1, m_2)) &= \delta_{m,m_1 \sqcup m_2} \\ \overline{\mathsf{c}}_X\,((m_1, m_2), m) &= \binom{|m_1| + |m_2|}{|m_1|} \delta_{m,m_1 \sqcup m_2} \\ \mathsf{w}_X\,(m, *) := \delta_{m,\emptyset} \qquad & \overline{\mathsf{w}}_X := (*, m) := \delta_{m,\emptyset} \end{aligned}$$

where the binomial coefficient in $\overline{\mathsf{c}}_X$ is necessary for the bimonoid equations (5) to hold. The digging $\mathsf{p}_X : !X \times !!X \to R$ checks if the disjoint union of a finite multiset of finite multisets in the second argument is equal to the first argument:

$$\mathsf{p}_X\,(m, [m_1, \ldots, m_n]) := \delta_{m,\bigsqcup_i m_i}$$

Now R^{Π} is a $\mathbb{Q}^{\Sigma}_{\geq 0}$-differential storage category [29, Thm 6.1], since R is a $\mathbb{Q}_{\geq 0}$-module and has infinite sums. For each n, $\mathsf{M}^n_X : !X \times !X \to R$ checks if a finite multiset is of size n: $\mathsf{M}^n_X(m_1, m_2) = \delta_{m_1,m_2}\delta_{|m_1|,n}$. Then clearly (21) holds, and so by Lemma 3.11, R^{Π} is both a Taylor differential category and a monadic differential category. Using the formula for codigging, one computes that the codigging $\overline{\mathsf{p}}_X : !!X \times !X \to R$ checks if the disjoint union of a finite multiset of finite multisets is equal to the second argument:

$$\overline{\mathsf{p}}_X([m_1, \ldots, m_n], m) = \frac{1}{n!} \cdot \binom{|m|}{|m_1|, \ldots, |m_n|} \cdot \delta_{\bigsqcup_i m_i,m},$$

where the coefficients are necessary for the monad identities (10). Observe that REL is a specific case of the weighted relational model for the Boolean algebra $B = \{0, 1\}$, that is, REL is isomorphic to B^{Π}. Since in B we have that $1 + 1 = 1$, the coefficients in the codigging and cocontraction definitions disappear in REL.

4.3 General Construction

Both of the previous examples are in fact examples of a more general construction. It turns out that the storage modality in both examples is constructed in the same way, since in particular it is a **free exponential modality**, meaning that $!A$ is the cofree cocommutative comonoid over A. So suppose that we are in a setting with infinite products Π and all symmetrized monoidal powers $S^n(A)$, that is, the joint equalizer of all permutations of $A^{\otimes^n}$. Then the free exponential modality can be constructed as $!A = \prod_n S^n(A)$. If one further assumes that Π is an infinite biproduct and we can scalar multiply by $\mathbb{Q}_{\geq 0}$, then $\mathsf{M}^n_A : !A \to !A$ precisely picks out $S^n(A)$ via projection and then injection, $!A \to S^n(A) \to !A$, and so (21) is simply the biproduct identity. Therefore, as a consequence of Lemma 3.11, we may construct a codigging and state the following:

Lemma 4.1. *A symmetric monoidal category which is enriched over $\mathbb{Q}_{\geq 0}$-modules, has countable biproducts which are preserved by the monoidal product, and that has symmetrized monoidal powers, is a Taylor differential category and a monadic differential category.*

Models with infinite biproducts were in particular studied by Laird et al [33] with the objective of building models of DiLL related to game logic. In particular in [33, Sec 5], Laird et al also give a general recipe for how to build differential storage categories that also satisfy precisely the extra assumptions needed for the above. Briefly, for any symmetric monoidal category $\mathcal{L}$, one can first freely make it enriched over $\mathbb{Q}_{\geq 0}$-modules, then taking the countable biproduct completition, and lastly taking the Karoubi envelope to split idempotents in order to obtain symmetrized monoidal powers. After all this, the resulting category is not only a differential storage category, but by the above lemma, also a monadic differential category. By tweaking the construction slightly with regards to enrichment, it is possible to recover both the relational model and the weighted relational models when applying this construction to the terminal category [33, Ex 5.6]. Therefore, thanks to Laird et al's construction, from any symmetric monoidal category, we can construct a monadic differential category, providing us with a bountiful source for examples of codigging.

4.4 Quantum Related Examples

We now very briefly discuss how, surprisingly, there also models of DiLL with codigging that are related to quantum theory. The first is Pagani et al's categorical model of a quantum lambda calculus [34], called $\overline{\mathbf{CPMs}}^{\oplus}$. In particular, $\overline{\mathbf{CPMs}}^{\oplus}$ is a compact closed category, enriched over $\mathbb{R}_{\geq 0}$, has infinite biproducts, and storage modality constructed using symmetrized monoidal powers. Therefore, by Lemma 4.1, it follows that $\overline{\mathbf{CPMs}}^{\oplus}$ is a monadic differential category. The other example is given by examples of Vicary's categorical quantum harmonic oscillator as proposed in [35]. The key to this example is the notion of *dagger* monoidal category, which is a category such that for every map $f : A \to B$, there is a map $f^\dagger : B \to A$ such that $\dagger$ is contravariant, involutive, and preserves the monoidal structure. Then briefly, a categorical quantum harmonic oscillator is a dagger symmetric monoidal category, with $\dagger$-biproducts, and a free exponential modality $!$ such that $!(f^\dagger) = (!f)^\dagger$. Then every categorical quantum harmonic oscillator is also a monadic differential category by setting $\overline{\mathsf{p}} = \mathsf{p}^\dagger$, $\overline{\mathsf{d}} = \mathsf{d}^\dagger$, $\overline{\mathsf{c}} = \mathsf{c}^\dagger$, and $\overline{\mathsf{w}} = \mathsf{w}^\dagger$. The main reason for this fact follows from the contravariant property of $\dagger$ and that the necessary codigging axioms are dual to those of a storage modality. REL is a categorical quantum harmonic oscillator, and in [35, Sec 6], Vicary conjectures that another model based on complex inner product spaces is as well.

4.5 Vector Spaces over $\mathbb{Z}_2$

We now provide a toy example which is important to highlight nonetheless since this is an example without infinite sums and yet still admits a codigging. So let $\mathbb{Z}$ be the ring of integers and let $\mathbb{Z}_2$ be $\mathbb{Z}$ modulo 2, that is, the two element field $\mathbb{Z}_2 = \{0, 1\}$. Let $\mathsf{FVEC}_{\mathbb{Z}_2}$ be the category of finite dimensional vector spaces and $\mathbb{Z}_2$-linear maps between them. The monoidal structure, (bi)product structure, and additive enrichment are all given in the standard algebraic way for vector spaces. The storage modality ! is given by the exterior algebra, so $!V = \mathsf{Ext}(V)$, which recall is a $\mathbb{Z}_2$-algebra with multiplicaiton given by the wedge product $\wedge$. In particular, recall that the wedge product is anti-symmetric, so $x \wedge x = 0$, and elements of $!V$ are given as finite sums of words of the form $w = x_1 \wedge \ldots \wedge x_n$, of any size n. Of course, the exterior algebra can be defined for vector spaces of any dimension over any field. Normally the exterior algebra is anti-commutative, meaning that $x \wedge y = -y \wedge x$. However for $\mathbb{Z}_2$, since $1 = -1$, the exterior algebra is a commutative algebra, and ! is a well-defined (co)monad on $\mathsf{FVEC}_{\mathbb{Z}_2}$.

The dereliction $\mathsf{d}_V : !V \to V$ projects out words of length one, while the digging $\mathsf{p}_V : !V \to !!V$ maps a word to the sum of all possible word of words whose wedge product is the original word:

$$\mathsf{d}_V(w) = \delta_{w,x} \qquad\qquad \mathsf{p}_V(w) := \sum_{\bigwedge w_i = w} [w_1] \wedge \ldots \wedge [w_n]$$

The sum for the digging is finite and well-defined by anti-symmetry of $\wedge$. The (co)contraction and (co)weakening are given by the (co)multiplication and the (co)unit of the canonical $\mathbb{Z}_2$-bialgebra structure of the exterior algebra. The codereliction $\overline{\mathsf{d}}_V : V \to !V$ maps an element of V to the one letter word, while the codigging $\overline{\mathsf{p}}_X : !!V \to !V$ maps a word of word to its wedge product:

$$\overline{\mathsf{d}}_V(x) = x \qquad\qquad \overline{\mathsf{p}}_X([w_1] \wedge \ldots \wedge [w_n]) = \bigwedge w_i$$

One can check that all the necessary identities do indeed hold, and we conclude that $\mathsf{FVEC}_{\mathbb{Z}_2}$ is a monadic differential category, but does not have infinite sums.

5 The codigging in functional analysis

Most examples of Section 4 are settings where any power series converge. In this section, we study the notion of codigging in models of DiLL closer to standard textbook analysis. We show that codigging implies a bound on the growth of functions, explaining why neither Köthe [36] nor quantitative convenient spaces [37] interpret it. By indexing the exponential with the exponential growth of functions, we will show that work by Ouerdiane and al. [38] [39] results in a new higher-order, polarized, graded model of DiLL with codigging.

5.1 The convolutional exponential

In several models of DiLL [13] [37] [40], formulas are interpreted as various sorts of topological vector spaces (tvs) over $\mathbb{R}$ or $\mathbb{C}$, and non-linear proofs are interpreted as higher-order smooth functions. In [13], Blute, Ehrhard and Tasson studied a smooth model of (Intuitionnistic) DiLL, which was later refined to a quantitative simplification by Kerjean and Tasson in [37]. This later version, which we will denote by QMco, only applies to complex Hausdorff and locally convex tvs (lcs). Recall that by distribution we mean the scalar linear morphisms acting on non-linear morphisms, in a sense

to be adapted to each model. Then in both models, ! is interpreted as the completion of the set of distributions generated by all dirac distributions δ_x. In QMco, if one denotes $S(E,\mathbb{C})$ the lcs of all power series between a lcs E and $\mathbb{C}$, the interpretation of the exponential $!E$ is included in the linear dual $S(E,\mathbb{C})'$ of $S(E,\mathbb{C})$, that is, the space of all linear scalar bounded morphisms acting on power series.

$$!E = \overline{\langle \delta_x \rangle_{x \in E}} \subseteq S(E,\mathbb{C})'$$

As always, $!E$ is endowed with a comonad structure and a bialgebra structure, which is nicely defined on the dirac distributions. For example, the dereliction is $d_E : \delta_x \in !E \mapsto x$, the cocontraction is $\overline{\mathsf{c}} : \delta_x \otimes \delta y \in !E \hat{\otimes} !E \mapsto \delta_{x+y}$, the codereliction is $\overline{\mathsf{d}} : v \mapsto \lim_{t \to 0} \frac{\delta_{tv} - \delta_0}{t}$ and the promotion is $\mathsf{p} : \delta_x \to \delta_{\delta_x}$. The codigging however is not here the dual of the diggin.

The category QMco has Taylor expansions [37, Cor 5.18] such that the Taylor series of a coKleisli map f converge to f [37, Cor 5.37]. Furthermore, in QMco, $\mathsf{M}^n_E : !E \to !E$ corresponds exactly to $\frac{1}{n!}\Theta_n$ in [37]. Therefore, it follows that QMco is in fact a Taylor differential category. Now suppose we have some sort of codigging. Then the convolutional exponential formula (13) gives (where $\cdot$ denotes the scalar multiplication):

$$\overline{\mathsf{p}}_E(\delta_{\delta_x}) : f \mapsto \sum_n \frac{1}{n!} f(n \cdot x) \tag{22}$$

Let's observe what happens for the complex exponential power series, $\mathsf{exp} : z \mapsto e^z$.

$$\overline{\mathsf{p}}_E(\delta_{\delta_x})(\mathsf{exp}) = \sum_n \frac{1}{n!} e^{n \cdot x} = \sum_n \frac{1}{n!} (e^x)^n = e^{e^x}$$

This shows that for functions f that *behave like an exponential function*, that $\overline{\mathsf{p}}_E(\delta_{\delta_x})(f)$ would be well defined. Unfortunately, it turns out this codigging cannot converge on every power series, as it cannot converge on tower of exponentials. Indeed, for $f : z \mapsto e^{e^z}$, $\overline{\mathsf{p}}_E(\delta_{\delta_x})(f)$ does not converge. So while $\overline{\mathsf{p}}$ is well defined on exp, it is not on the composition of exp with itself. In fact, in general, the power series which interpret non-linear proofs have uncontrolled growth. This explains why (too) general quantitative models such as Köthe [36] or quantitative convenient spaces do not admit a codigging. This begs the question of whether codigging can properly exists in a smooth setting where infinite sums do not always converge. To solve this issue, we consider a case where ! is *graded*.

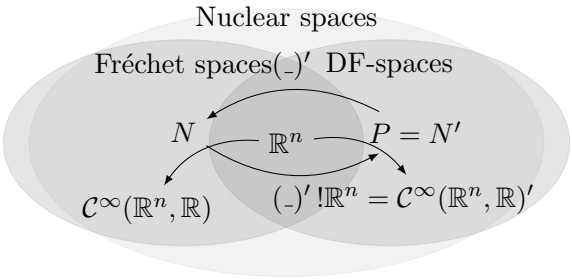
5.2 Making Nuclear spaces go higher-order, quantitatively

Convolutional calculus has been developed for higher order functions in infinite dimensional analysis. It features a nice duality theory [39] and allows for the generalization of power series on distributions by using convolutional powers [38]. We now sketch how these allow for a higher-order extension of a previously known first order model of DiLL based on Fréchet Nuclear spaces [41]. It does not constitute a monadic differential category per say, but rather a graded and polarized version of it, whose proper categorical statement will be explored in future work. Polarization [42] separates LL formulas in two classes, which are interpreted in two categories made equivalent by the interpretation of the negation [43].

The indexation is similar to what can be found in models of graded LL [44]. The indexed exponential rules for ! are interpreted as exponential actions of partially ordered semirings over monoidal closed categories [45]. This includes an indexed comonad (Prop 5.8) interpreting the

usual rules of LL, and a new indexed monad (Prop 5.6) interpreted by the coexponential rules of DiLL, as well as a strong monoidality of the exponential functor which allows for the interpretation of $\mathsf{c}, \mathsf{w}, \overline{\mathsf{c}}$, and $\overline{\mathsf{w}}$ (Prop 5.3). Here, indices are *Young functions*, and bound the exponential growth of functions. We do not yet explore in this paper the full categorical consequences of mixing indexed exponential connectives and coexponential rules, but this is work in progress.

This higher-order development solves the limitations encountered by Kerjean and Lemay in [41]. A first paper by Kerjean [46] interprets formulas of DiLL in specific lcs, that is, Nuclear Fréchet spaces (Nf) for negative formulas of LL, and Nuclear DF-spaces (Ndf) for positive formulas. By extension, Nf and Ndf also denotes the category of Nf (resp. Ndf) lcs and continuous linear maps. In what follows N and M denotes Nf complex lcs while P and Q denote Ndf complex lcs. N' denotes the strong dual of N, that is the tvs $\mathcal{L}(N', \mathbb{C})$ endowed with the topology of uniform convergence on bounded subsets of N'.



This setting provides a denotational model for DiLL up to promotion, meaning that ! is only interpreted when acting on $\mathbb{K}^n$, and not for any higher-order lcs. A partial solution is provided in [41] by using the completeness of the category of Nf to construct a higher-order interpretation for !, which alas does not interpret promotion for technical reasons. The notions of convolutional exponential and functions whose growth allows exponentiation solves these limitations. This was mainly done in a work by Ouerdiane and al. [39] where they defined a space $\mathscr{F}_\theta(P)$ of holomorphic functions with exponential growth. In this definition, $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ may vary but always stands for a Young function [47][48], i.e. it is convex, increasing, null at 0 and $\lim_{x\to\infty} \frac{\theta(x)}{x} = \infty$.

Proposition 5.1. *[49] The topology on any* Nf *space N can be defined through a denumerable family of hilbertian norms $|\_|_p$, $p \in \mathbb{N}$, and if one denote N_p the Hilbert space resulting of the completion of N with respect to $|\_|_p$, we have that:*

$$\bigcap_p N_p = N \qquad \bigcup_p (N_p)' = N'.$$

Definition 5.2. *[39] For a Young function θ and for a Banach space B, let $Exp(B, \theta, m)$ denote the Banach space of holomorphic functions from B to $\mathbb{C}$ such that:*

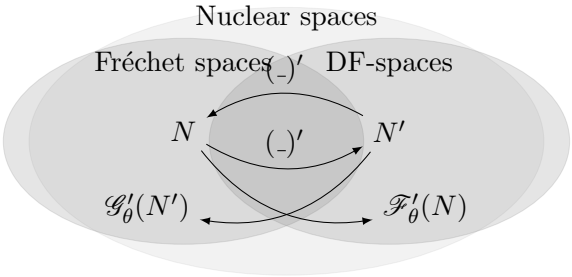
$$|f(z)| \leq K e^{\theta(m||z||)}. \tag{23}$$

The space $Exp(\theta, m, p)$ is Banach when endowed with the norm $f \mapsto sup\{|f(z)|e^{-\theta(m||z||)} | z \in B\}$.

One can define two types of functions with exponential growth on an NF lcs N or its dual:[3]

$$\mathscr{F}_\theta(N)=\bigcap_{m,p}\mathrm{Exp}(N_p,\theta,m)\quad \mathscr{G}_\theta(N')=\bigcup_{m,p}\mathrm{Exp}(N_{-p},\theta,m).$$

Through an isomorphism with spaces of formal power series, one can show that $\mathscr{F}_\theta(N)$ is a NF space [39, Prop 2]. As such, its dual $\mathscr{F}'_\theta(N)$, i.e. the space of distributions acting on $\mathscr{F}_\theta(N)$, is a NDF space. As linear morphisms are bounded, $\mathscr{G}_\theta : \text{NDF} \to \text{NDF}$ and $\mathscr{F}_\theta : \text{NF} \to \text{NF}$ are indeed functors.



Proposition 5.3. *The functor $\mathscr{F}_\theta$ satisfies monoidal laws, depending on θ: $\mathscr{F}'_{\theta_P}(N)\hat{\otimes}\mathscr{F}'_{\theta_Q}(M) \simeq \mathscr{F}'_{\theta_N+\theta_M}(N\times M)$.*

$\mathscr{F}$ and $\mathscr{G}$ enjoy an important duality theorem, which is strongly related to the fact that N and P are reflexive. What the following theorem implies is that distributions on one type of functions $\mathscr{F}$ or $\mathscr{G}$ are functions of the other type $\mathscr{G}$ or $\mathscr{F}$ respectively.

Theorem 5.4. *[39, Thm 1] For the conjugate Young function $\theta^* := sup_{t\geq 0}(tx-\theta(t))$, we have that the Laplace transformations results in an isomorphisms:*

$$\mathscr{L} : \left\{ \begin{array}{ccc} \mathscr{F}'_\theta(N) & \simeq & \mathscr{G}_{\theta^*}(N') \\ \phi & \mapsto & \big(\ell \in N' \mapsto \phi(x\in N' \mapsto e^{\ell(x)}\in\mathbb{C})\big) \end{array}\right.$$

Proof. We show the result for $\mathscr{F}_\theta$. The strong monoidality of the distribution functor is classically interpreted as a variant of Schwart'z Kernel Theorem. We refer to [50] for a detailed proof which we briefly adapt here. The injection $\mathscr{F}_\theta(P)\hat{\otimes}\mathscr{F}_\theta(Q)\to\mathscr{F}_\theta(P\times Q)$ corresponds to $f\otimes g\mapsto f\cdot g$ where $f\cdot g$ denotes the scalar multiplication between two functions with scalar values. The topology on $P\times Q$ is generated by the maximums of semi-norms from P and Q. Consider p,q and $m'>0$. Then for any m' there is K_f and K_g such that:

$$\begin{aligned} |f(z)\cdot g(z')| &\leq K_f K_g e^{\theta_P(m'|z|_{-p})+\theta_Q(m'|z'|_{-q})} \\ &\leq K_f K_g e^{(\theta_P+\theta_Q)(m'(max(|z|_{-p},|z'|_{-q}))} \\ &\leq K_f K_g e^{(\theta_P+\theta_Q)(m'(|(z,z')|_{p,q})} \end{aligned}$$

The surjectivity is done by approximating a function $h\in\mathscr{F}_\theta(P\times Q)$ by polynomials on compact support. It is an adaptation of the proof by Meise (which applies for dual of Fréchet Montel spaces, while our spaces are Fréchet Nuclear so in particular Montel). □

[3]We define $\mathscr{F}'_\theta$ on NF spaces to stay in the chirality [43] (a polarized version of a $*$-autonomous category) used in [46]. Indeed, only spaces $\mathcal{L}(\mathscr{F}'_\theta(N),M)$ stay in NF, and not $\mathcal{L}(\mathscr{F}'_\theta(P),M)$ [46, 3.23].

The conjugate of θ, also called the convex conjugate, is related to inverses: if $\theta = \int \mu(t)dt$, then $\theta^* = \int \mu^{-1}dt$. The Laplace transformation turn the convolution products of distribution into the (pointwise) product of functions. Therefore, it also transforms convolutional power series of distributions into usual power series of functions, where the monoidal law is the scalar pointwise multiplication.

In a following work [38], Ouerdiane and different authors characterize the spaces $\mathscr{F}'_\theta$ on which convolutional generalizations to power series can act. This leads in particular to a convolutional exponential, which interprets codigging, the operation being again recorded in the index θ. The following propositions are shown thanks to the use the isomorphism $\mathscr{L}$ introduced in Theorem 5.4.

Proposition 5.5. *[38, Cor 1] For any $\phi \in \mathscr{F}_\theta(N)'$, its convolutional exponential (as defined in (13)) is an element of $\mathscr{F}_{\left(e^{\theta^*}\right)^*}(N)'$.*

Proposition 5.6. *For any Young functions θ_1, θ_2 we have a natural transformation in the category* NDF*:*

$$\overline{\mathsf{p}}_P : \mathscr{F}'_{\theta_1}\left(\mathscr{F}'_{\theta_2}(N)\right) \to \mathscr{F}'_{\left(\theta_1^* e^{\theta_2^*}\right)^*}(N)$$

Proof. This could by shown by a direct adaptation of [38, Thm 1]. We offer another explanation, thanks to the following intermediate proposition, which corresponds to the composition in the co-Kleisli of a comonad $\mathscr{G}_{\theta_1}$.

Proposition 5.7. *Given linear continuous maps $f : \mathscr{G}'_{\theta_1}(P_1) \to N_2$ and $g : \mathscr{G}'_{\theta_2}(P_2) \to P_3$, then we have a linear continuous map $g \circ f : \mathscr{G}'_{\left(\theta_2 e^{\theta_1}\right)}(P_1) \to P_3$*

This creates an adjunction resulting in $\mathscr{G}'_\theta$ as an indexed comonad, with comultiplication: $\overline{\mathsf{p}}^\perp : \mathscr{G}'_{\theta_2 e^{\theta_1}} \to \mathscr{G}'_{\theta_2}\mathscr{G}'_{\theta_1}$. To obtain Prop 5.6, one takes the dual of $\overline{\mathsf{p}}^\perp$ and runs it through the Laplace isomorphism (Thm5.4). As linear functions are bounded, thanks to the reflexivity of NF and NDF spaces, we also obtain natural transformations $\mathsf{d}_N : \mathscr{F}'_{Id}(N) \to P$ and $\overline{\mathsf{d}}_N : N \to \mathscr{F}'_{Id}(N)$. □

Note that the functions of $\mathscr{F}_\theta(N)$ are only defined to be entire (everywhere holomorphic) and not power series (equal to their Taylor Series at 0). However, they are in direct correspondences with a space of formal power series on N' [39, Prop 1], confirming the intuition that the codigging indeed recaptures some quantitative models. Now the indexed monad structure on $\mathscr{F}'_\theta(\_)$ gives us the type to look for the usual indexed comonad structure on it. The indices in the interpretation of the digging rule is an indication of how functions with exponential growth compose.

Proposition 5.8. *Given linear continuous maps $f : \mathscr{F}'_{\theta_1}(N_1) \to N_2$ and $g : \mathscr{F}'_{\theta_2}(N_2) \to N_3$, we obtain a linear continuous map $g \circ f : \mathscr{F}'_{\left(\theta_2 e^{\theta_1}\right)}(N_1) \to N_3$. This result in a comonad comultiplication:*

$$\mathsf{p} : \mathscr{F}'_{\left(\theta_2 e^{\theta_1}\right)}(N) \to \left(\mathscr{F}'_{\theta_2}(\mathscr{F}'_{\theta_1}(N))\right).$$

Note that the above is hinting at a possible *semiring structure on the set of Young functions*, with $\theta_1 \cdot \theta_2 = (\theta_1^* e^{\theta_2^*})^*$ as a non-commutative multiplication law, while the additive law being the sum of Young functions. The $*$ is optional, and depends on whether you are considering digging or codigging. In particular, when indexing DiLL with graded operators, it might be worth to consider an analogue to $*$ operating on the set of indices. This brings us to our final statement which can be easily checked:

Proposition 5.9. *In any vector spaces over* $\mathbb{R}$ *or* $\mathbb{C}$*, the Laplace transformation* $\mathscr{L}$ *transforms the interpretation of the structural rules* $\mathsf{w}, \mathsf{d}, \mathsf{c}, \mathsf{p}$ *of* LL *into the costructural rules of* DiLL $\overline{\mathsf{w}}, \overline{\mathsf{d}}, \overline{\mathsf{c}}, \overline{\mathsf{p}}$*, when the latter are defined.*

We conjecture that when a Laplace transform is available and invertible on every coKleisli map, a model of LL turns into a model of DiLL.

6 Conclusion

In this paper we constructed and studied the notion of monadic differential categories, which give the ! connective of LL a monad structure on top of its well-known comonad structure. This gives the interpretation of ! a perfectly symmetrical structure. We showed that codigging was naturally interpreted by exponential functions e^x, and we also explained how the monad axioms imply that every non-linear map was equal to its Taylor series. We also related the interpretation of the codigging with the notion of convolutional exponential, allowing us to construct a new graded and polarized model of DiLL with codigging.

Future work This paper only provides the beginning to the story of codigging, and we believe there is still much more to explore on the subject.

A first step would be to find even more examples of monadic differential categories. In particular, it would be quite desirable to understand whether Finiteness spaces [51] or Kothe spaces [36] can somehow be restricted to functions with exponential growth to provide new vectorial models of DiLL with codigging. More generally, we would like to study how codigging is related to $*$-autonomous structure on differential categories.

To develop the theory of differential proof-nets with codigging, one would need to look at Gimenez' work [14], but the presentation could possibly differ. We hope that the categorical structure presented in this paper has been made precise enough to make the cut-elimination procedure in DiLL with codigging unambiguous. In particular, one would need to add a codigging rule $\overline{\mathsf{p}}$ as written in the introduction. In proof nets, this may consist of adding sort of coexponential box. Keeping in mind our description of codigging, one may require some sort of mixed distributive law [52] to express the compatibility between the monad and comonad structures on !. In our case, the mixed distributive law would be a natural transformation of type $\lambda_A : !!A \to !!A$. If one assumes this extra structure, it may be possible to use λ_A to somehow express a compatibility between digging p and codigging $\overline{\mathsf{p}}$. We conjecture that an "illicit formula" for the mixed distributive law would be $\lambda = \sum_n \frac{1}{n!} \cdot \left(\mathsf{d}^n; \mu; !\overline{\mathsf{d}}^n\right)$.

We would also be curious to understand if the monad structure ! adds anything for λ-terms and if it could offer an interesting reformulation of resource calculi. In fact, even in a language whose model does not admit a codigging $\overline{\mathsf{p}}$ as a morphism of the category, one can have morphisms $\texttt{return} : v \mapsto D_0(\_)(v)$ and $\texttt{bind} : \delta_x \to f \to \exp^*(f(x))$.

Lastly, in relation with Section 5, designing a proper syntax and categorical semantics for graded DiLL is currently a work in progress, and related to the indexation of DiLL with differential operators [46]. As Young duality applies to functions θ which are defined on infinite dimensional tvs, we conjecture that this model can be generalized to reflexive tvs without involving semi-norms. Convolutional calculus is also linked with the study of differential equations, and might offer some interesting questions and answers.

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References

[1] D. Barbarossa and G. Manzonetto, "Taylor subsumes scott, berry, kahn and plotkin," *PACMPL*, vol. 4, no. POPL, 2020.

[2] B. Accattoli, S. Graham-Lengrand, and D. Kesner, "Tight typings and split bounds, fully developed," *J. Funct. Program.*, vol. 30, 2020.

[3] J. Laird, G. Manzonetto, G. McCusker, and M. Pagani, "Weighted relational models of typed lambda-calculi," in *28th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2013*. IEEE Computer Society, 2013.

[4] T. Ehrhard, M. Pagani, and C. Tasson, "Measurable cones and stable, measurable functions: a model for probabilistic higher-order programming," *PACMPL*, vol. 2, no. POPL, 2018.

[5] V. Danos and T. Ehrhard, "Probabilistic coherence spaces as a model of higher-order probabilistic computation," *Inf. Comput.*, vol. 209, no. 6, 2011.

[6] J.-Y. Girard, "Linear logic," *Theoret. Comput. Sci.*, vol. 50, no. 1, 1987.

[7] ——, "Normal functors, power series and λ-calculus," *Ann. Pure Appl. Log.*, vol. 37, no. 2, 1988.

[8] T. Ehrhard and L. Regnier, "Differential interaction nets," *Theoretical Computer Science*, vol. 364, no. 2, 2006.

[9] A. Brunel, D. Mazza, and M. Pagani, "Backpropagation in the Simply Typed Lambda-calculus with Linear Negation," *POPL*, 2020.

[10] T. Ehrhard, "An introduction to differential linear logic: proof-nets, models and antiderivatives," *Mathematical Structures in Computer Science*, 2017.

[11] T. Ehrhard and L. Regnier, "The differential lambda-calculus," *Theoretical Computer Science*, vol. 309, no. 1, 2003.

[12] J.-S. P. Lemay, "Exponential functions in cartesian differential categories," *Applied Categorical Structures*, vol. 29, 2021.

[13] R. Blute, T. Ehrhard, and C. Tasson, "A convenient differential category," *Cah. Topol. Géom. Différ. Catég.*, 2012.

[14] S. Gimenez, "Programmer, calculer et raisonner avec les réseaux de la Logique Linéaire," Theses, Université Paris-Diderot - Paris VII, Dec. 2009.

[15] L. Vaux, "Taylor Expansion, lambda-Reduction and Normalization," in *26th EACSL CSL 2017*, ser. LIPIcs, vol. 82, 2017.

[16] T. Tsukada, K. Asada, and C. L. Ong, "Species, profunctors and taylor expansion weighted by SMCC: a unified framework for modelling nondeterministic, probabilistic and quantum programs," in *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2018, Oxford, UK, July 09-12, 2018*, A. Dawar and E. Grädel, Eds. ACM, 2018.

[17] M. Pagani and C. Tasson, "The inverse taylor expansion problem in linear logic," in *2009 24th Annual IEEE Symposium on Logic In Computer Science*. IEEE, 2009.

[18] P. Boudes, F. He, and M. Pagani, "A characterization of the taylor expansion of lambda-terms," in *Computer Science Logic 2013 (CSL 2013)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2013.

[19] M. Coppo and M. Dezani-Ciancaglini, "An extension of the basic functionality theory for the λ-calculus." *Notre Dame Journal of Formal Logic*, vol. 21, no. 4, pp. 685 – 693, 1980.

[20] V. Arrial, G. Guerrieri, and D. Kesner, "Quantitative inhabitation for different lambda calculi in a unifying framework," *Proceedings of the ACM on Programming Languages*, vol. 7, no. POPL, 2023.

[21] P. Tranquilli, "Intuitionistic differential nets and lambda-calculus," *Theor. Comput. Sci.*, vol. 412, no. 20, pp. 1979–1997, 2011.

[22] R. Mardare, P. Panangaden, and G. D. Plotkin, "On the axiomatizability of quantitative algebras," in *32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, Reykjavik, Iceland, June 20-23, 2017*. IEEE Computer Society, 2017.

[23] M. Mio and V. Vignudelli, "Monads and quantitative equational theories for nondeterminism and probability," in *31st International Conference on Concurrency Theory, CONCUR 2020,*, I. Konnov and L. Kovács, Eds. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020.

[24] R. F. Blute, J. R. B. Cockett, and R. A. G. Seely, "Differential categories," *Mathematical structures in computer science*, vol. 16, no. 06, 2006.

[25] R. F. Blute, J. R. B. Cockett, J.-S. P. Lemay, and R. A. G. Seely, "Differential categories revisited," *Applied Categorical Structures*, vol. 28, 2020.

[26] M. P. Fiore, "Differential structure in models of multiplicative biadditive intuitionistic linear logic," in *International Conference on Typed Lambda Calculi and Applications*. Springer, 2007.

[27] P.-A. Melliès, "Categorical semantics of linear logic," *Société Mathématique de France*, 2008.

[28] T. Ehrhard and L. Regnier, "Uniformity and the taylor expansion of ordinary lambda-terms," *Theoretical Computer Science*, vol. 403, no. 2-3, 2008.

[29] J.-S. P. Lemay, "Convenient antiderivatives for differential linear categories," *Mathematical Structures in Computer Science*, vol. 30, no. 5, 2020.

[30] R. Cockett and J.-S. P. Lemay, "Integral categories and calculus categories," *Mathematical Structures in Computer Science*, vol. 29, no. 2, 2019.

[31] J. S. Golan, *Semirings and their Applications*. Springer Science & Business Media, 2013.

[32] C. L. Ong, "Quantitative semantics of the lambda calculus: Some generalisations of the relational model," in *32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, Reykjavik, Iceland, June 20-23, 2017*. IEEE Computer Society, 2017.

[33] J. Laird, G. Manzonetto, and G. McCusker, "Constructing differential categories and deconstructing categories of games," in *International Colloquium on Automata, Languages, and Programming*. Springer, 2011.

[34] M. Pagani, P. Selinger, and B. Valiron, "Applying Quantitative Semantics to Higher-Order Quantum Computing," in *The 41th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL14, San Diego, USA*, P. Sewell, Ed. ACM, 2014.

[35] J. Vicary, "A categorical framework for the quantum harmonic oscillator," *International Journal of Theoretical Physics*, vol. 47, no. 12, 2008.

[36] T. Ehrhard, "On Köthe Sequence Spaces and Linear Logic," *Mathematical Structures in Computer Science*, vol. 12, no. 5, 2002.

[37] M. Kerjean and C. Tasson, "Mackey-complete spaces and power series - a topological model of differential linear logic," *Math. Struct. Comput. Sci.*, vol. 28, no. 4, 2018.

[38] M. B. Chrouda, M. El Oued, and H. Ouerdiane, "Convolution calculus and applications to stochastic differential equations," *Soochow Journal of Mathematics*, vol. 28, no. 4, 2002.

[39] R. Gannoun, R. Hachaichi, H. Ouerdiane, and A. Rezgui, "Un théorème de dualité entre espaces de fonctions holomorphes à croissance exponentielle," *Journal of Functional Analysis*, vol. 171, no. 1, 2000.

[40] Y. Dabrowski and M. Kerjean, "Models of Linear Logic based on the Schwartz epsilon product," *Theory and Applications of Categories*, 2020.

[41] M. Kerjean and J.-S. P. Lemay, "Higher-order distributions for differential linear logic," in *Foundations of Software Science and Computation Structures - 22nd International Conference, FOSSACS 2019, ETAPS 2019,*, ser. Lecture Notes in Computer Science. Springer, 2019.

[42] O. Laurent, "Etude de la polarisation en logique," Thèse de Doctorat, Université Aix-Marseille II, Mar. 2002.

[43] P.-A. Melliès, "Dialogue categories and chiralities," *Publ. Res. Inst. Math. Sci.*, vol. 52, no. 4, 2016.

[44] M. Gaboardi, S. Katsumata, D. Orchard, F. Breuvart, and T. Uustalu, "Combining Effects and Coeffects via Grading," in *Proceedings of the 21st ACM SIGPLAN International Conference on Functional Programming*, ser. ICFP 2016. ACM, 2016.

[45] A. Brunel, M. Gaboardi, D. Mazza, and S. Zdancewic, "A core quantitative coeffect calculus," in *Proceedings of the 23rd European Symposium on Programming Languages and Systems - Volume 8410*, no. 20. Berlin, Heidelberg: Springer-Verlag, 2014.

[46] M. Kerjean, "A Logical Account for Linear Partial Differential Equations," in *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2018*.

[47] I. M. Gelfand and G. E. Shilov, *Generalized functions. Vol. 3: Theory of differential equations*. Academic Press, New York-London, 1967.

[48] M. A. Krasnoselskiĭ and J. B. Rutickiĭ, *Convex functions and Orlicz spaces*. P. Noordhoff Ltd., Groningen, 1961, translated from the first Russian edition by Leo F. Boron.

[49] H. Jarchow, *Locally convex spaces*. B. G. Teubner, 1981.

[50] F. Trèves, *Topological vector spaces, distributions and kernels*. Academic Press, New York-London, 1967.

[51] T. Ehrhard, "Finiteness spaces," *Mathematical Structures in Computer Science*, vol. 15, no. 4, 2005.

[52] B. Mesablishvili and R. Wisbauer, "Bimonads and hopf monads on categories," *Journal of K-Theory*, vol. 7, no. 2, 2011.