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# EXACT CONTROLLABILITY FOR A RAYLEIGH BEAM WITH PIEZOELECTRIC ACTUATOR\*

YUBO BAI<sup>†</sup>, CHRISTOPHE PRIEUR<sup>‡</sup>, AND ZHIQIANG WANG<sup>§</sup>

**Abstract.** In this paper, exact controllability for a Rayleigh beam with piezoelectric actuator is considered. Controllability results show that the space of controllable initial data depends on the regularity of the control function and the location of the actuator. Two different spaces of control function,  $L^2(0, T)$  and  $(H^1(0, T))'$ , correspond to two different controllability,  $L^2$ -controllability and  $(H^1)'$ -controllability. The approach to prove controllability results is based on Hilbert Uniqueness Method. Some non-controllability results are also obtained. In particular, non-controllability in short control time is studied by using Riesz basis property of exponential family in  $L^2(0, T)$ . Due to this non-controllability result, minimal time for the exact controllability is obtained.

**Key words.** exact controllability, Rayleigh beam, piezoelectric actuator

**MSC codes.** 93C20, 93B05, 35B65

## 1. Introduction and main results.

**1.1. History and problem statement.** In recent decades, there have been a large number of papers concerning the study of flexible structures. Three main directions of research can be considered, i.e., the modelling problem, the controllability problem and the stabilization problem. Modelling a flexible structure as a beam equation or a plate equation is an essential research field. In [11], the author summarized and analysed four types of model for the transversely vibrating uniform beam, i.e., the Euler-Bernoulli beam, Rayleigh beam, shear beam and Timoshenko beam. In the past few decades, the study of elastic structures with a piezoelectric actuator or sensor has gained a lot of attention. The paper on modelling elastic structures with a piezoelectric actuator or sensor as a PDE can refer to [7, 9].

Concerning controllability for PDEs, [17] proposed an important method, Hilbert Uniqueness Method (HUM), to study controllability for infinite-dimensional systems. Using HUM, boundary controllability for wave equation and plate equation was studied in [17]. There were plenty of works on controllability for beam and plate based on HUM. In [13, 17], boundary controllability for Kirchhoff plate equation was fully investigated. Exact controllability was obtained in sufficiently large control time with a single boundary control (active on a sufficiently large portion of the boundary) in the case of clamped boundary conditions. As for Euler-Bernoulli beam equation, in 1996, [21] firstly considered the exact controllability for Euler-Bernoulli beam hinged at both ends with piezoelectric actuator. Since the space dimension is one, Fourier series was used in [21]. Then [8] studied the exact controllability for the same beam equation with piezoelectric actuator in a different physical configuration: the clamped-free boundary conditions, i.e. a beam clamped at one end and free at the other end. In [19], Ingham inequality (see [4, 12]) was used to obtain the exact controllability for Rayleigh beam equation with a single boundary control among four different boundary

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conditions.

In this paper, we consider the control problem modelling the transverse deflection of a Rayleigh beam which is subject to the action of an attached piezoelectric actuator. If we suppose that the beam is hinged at both ends, the equation of Rayleigh beam can be written as (see, for instance, [7, 9])

(1.1a)

$$w_{tt}(x, t) - \alpha w_{xxtt}(x, t) + w_{xxxx}(x, t) = u(t) \frac{d}{dx} [\delta_\eta(x) - \delta_\xi(x)], \quad 0 < x < \pi, \quad t > 0,$$

$$(1.1b) \quad w(0, t) = w(\pi, t) = w_{xx}(0, t) = w_{xx}(\pi, t) = 0, \quad t > 0,$$

$$(1.1c) \quad w(x, 0) = w^0(x), \quad w_t(x, 0) = w^1(x), \quad 0 < x < \pi.$$

In the equations above  $w$  represents the transverse deflection of the beam,  $\alpha > 0$  is a physical constant,  $\xi$  and  $\eta$  stands for the ends of the actuator ( $0 < \xi < \eta < \pi$ ), and  $\delta_y$  is the Dirac mass at the point  $y$ . The control is given by the function  $u : [0, T] \rightarrow \mathbb{R}$  standing for the time variation of the voltage applied to the actuator.

Our main purpose is to find the initial data that can be steered to rest by means of the control function  $u$ . Here we give some precise definitions.

**DEFINITION 1.1.** *The initial data  $(w^0, w^1)$  is exactly  $L^2$ -controllable in  $(\xi, \eta)$  at time  $T$  if there exists  $u$  in  $L^2(0, T)$  such that the solution  $w$  of (1.1) satisfies the condition*

$$w(x, T) = w_t(x, T) = 0, \quad 0 < x < \pi.$$

**DEFINITION 1.2.** *The initial data  $(w^0, w^1)$  is exactly  $(H^1)'$ -controllable in  $(\xi, \eta)$  at time  $T$  if there exists  $u$  in  $(H^1(0, T))'$  such that the solution  $w$  of (1.1) satisfies the condition*

$$w(x, T) = w_t(x, T) = 0, \quad 0 < x < \pi.$$

In Definition 1.2,  $(H^1(0, T))'$  is the dual space of  $H^1(0, T)$  with respect to the space  $L^2(0, T)$ . This definition is inspired by [17] which studied the controllability of changing the norm for wave equation and plate equation. Note that the system (1.1) is a time-reversible linear system, so the exact controllability is equivalent to null controllability (see [6]).

The paper is organized as follows. In the remaining part of this section, we present the main results of the paper, i.e. controllability, non-controllability and minimal time for the exact controllability. In Section 2 we give some preliminaries on the theory of diophantine approximation and Riesz basis property of exponential family. The well-posedness results for the control problem (1.1) are showed in Section 3. The main results are proved in Section 4, respectively. Appendix A provides the proof of a technical lemma which is used in the proof of non-controllability in short control time.

**1.2. Controllability results.** To state the exact controllability results, let us introduce for any  $\omega$  in  $\mathbb{R}$  the functional space  $Y_\omega$  as follows. Let  $Y_0 = L^2(0, \pi)$ . For  $\omega > 0$ , let  $Y_\omega$  be the closure in  $H^\omega(0, \pi)$  of the  $y$  in  $C^\infty([0, \pi])$  satisfying the conditions

$$(1.2) \quad y^{(2n)}(0) = y^{(2n)}(\pi) = 0 \quad \forall n \geq 0.$$

For  $\omega < 0$ , let  $Y_\omega$  be the dual space of  $Y_{-\omega}$  with respect to the space  $Y_0$ .

Our exact controllability results are the following.

- THEOREM 1.3. 1. *There exists a set  $A$  contained in  $(0,1)$  (uncountable but with Lebesgue measure zero, defined in Section 2) such that for every  $\xi$  and  $\eta$  with  $\frac{\eta+\xi}{2\pi}$  and  $\frac{\eta-\xi}{2\pi}$  belonging to the set  $A$  and for any  $T > 2\pi\sqrt{\alpha}$ , all initial data in  $Y_4 \times Y_3$  are exactly  $L^2$ -controllable in  $(\xi, \eta)$  at time  $T$ .*
2. *Let  $\varepsilon > 0$ , there exists a set  $B_\varepsilon$  contained in  $(0,1)$  (with Lebesgue measure equal to 1, defined in Section 2) such that for every  $\xi$  and  $\eta$  with  $\frac{\eta+\xi}{2\pi}$  and  $\frac{\eta-\xi}{2\pi}$  belonging to the set  $B_\varepsilon$  and for any  $T > 2\pi\sqrt{\alpha}$ , all initial data in  $Y_{4+\varepsilon} \times Y_{3+\varepsilon}$  are exactly  $L^2$ -controllable in  $(\xi, \eta)$  at time  $T$ .*

The method for proving Theorem 1.3 is inspired by the ideas and methods used in [21] for Euler-Bernoulli beam with piezoelectric actuator. Theorem 1.3 gives us two exact  $L^2$ -controllability results. The first result of Theorem 1.3 shows that, for the end of the piezoelectric actuator in an uncountable zero measure set, we have the exact  $L^2$ -controllability in space  $Y_4 \times Y_3$ . The second result of Theorem 1.3 shows that, for almost all choices of the end of the piezoelectric actuator, we have the exact  $L^2$ -controllability in Sobolev spaces more regular than  $Y_4 \times Y_3$ .

In order to obtain the exact controllability in less regular space, we propose the following Theorem.

- THEOREM 1.4. 1. *Let the set  $A$  is same as the set in Theorem 1.3. Then for every  $\xi$  and  $\eta$  with  $\frac{\eta+\xi}{2\pi}$  and  $\frac{\eta-\xi}{2\pi}$  belonging to the set  $A$  and for any  $T > 2\pi\sqrt{\alpha}$ , all initial data in  $Y_3 \times Y_2$  are exactly  $(H^1)'$ -controllable in  $(\xi, \eta)$  at time  $T$ .*
2. *Let  $\varepsilon > 0$  and the set  $B_\varepsilon$  is same as the set in Theorem 1.3. Then for every  $\xi$  and  $\eta$  with  $\frac{\eta+\xi}{2\pi}$  and  $\frac{\eta-\xi}{2\pi}$  belonging to the set  $B_\varepsilon$  and for any  $T > 2\pi\sqrt{\alpha}$ , all initial data in  $Y_{3+\varepsilon} \times Y_{2+\varepsilon}$  are exactly  $(H^1)'$ -controllable in  $(\xi, \eta)$  at time  $T$ .*

As we have already mentioned, the definition of exact  $(H^1)'$ -controllability is inspired by [17]. Similar to Theorem 1.3, the method for proving Theorem 1.4 is inspired by [21]. The differences between Theorem 1.4 and Theorem 1.3 are the space of the control function and the space of the controllable initial data. In Theorem 1.4, the control function belongs to  $(H^1(0, T))'$  rather than  $L^2(0, T)$  and the space of the controllable initial data is larger than the space in Theorem 1.3 with the same choice of  $\xi$  and  $\eta$ . Roughly speaking, the larger (less regular) the space of control function is, the larger (less regular) the space of controllable initial data is. To the best knowledge of the authors, such a result has not been developed yet for beam equation with piezoelectric actuator or interior control.

Theorems 1.3 and 1.4 give some sufficient conditions for exact controllability. All the results show the dependence of the space of exactly controllable initial data on the location of the actuator. The proofs of these two Theorems are quite similar. First we use the HUM to claim the controllability is equivalent to the observability of its adjoint problem (see Propositions 4.1 and 4.4). Then we use the preliminaries introduced in Section 2 to prove the observability inequalities. The main difference between the proofs of Theorem 1.3 and Theorem 1.4 is that the measurements in Theorem 1.3 and Theorem 1.4 are different (see Propositions 4.1 and 4.4).

**1.3. Non-controllability results.** After the controllability results, we show some non-controllability results. In Section 4, from Propositions 4.1 and 4.4 and the solution (3.5) of the adjoint problem, we can see that condition

$$(1.3) \quad \frac{\eta - \xi}{2\pi}, \frac{\eta + \xi}{2\pi} \in \mathbb{R} \setminus \mathbb{Q}$$

is necessary to have any exact controllability result. Inspired by the study of [21] for Euler-Bernoulli beam with piezoelectric actuator, we prove the following non-

controllability results. They show that the condition (1.3) is not sufficient in the sense that there are  $\xi$  and  $\eta$  satisfying (1.3) that do not allow the control of arbitrary regular initial data in any time  $T$ .

**THEOREM 1.5.** 1. *For any  $\beta \geq -1$ , there exist  $\xi$  and  $\eta$  satisfying (1.3) such that for any  $T > 0$ , the space  $Y_{\beta+3} \times Y_{\beta+2}$  contains initial data that are not exactly  $L^2$ -controllable in  $(\xi, \eta)$  at time  $T$ .*  
 2. *For any  $\beta \geq -2$ , there exist  $\xi$  and  $\eta$  satisfying (1.3) such that for any  $T > 0$ , the space  $Y_{\beta+3} \times Y_{\beta+2}$  contains initial data that are not exactly  $(H^1)'$ -controllable in  $(\xi, \eta)$  at time  $T$ .*

**Remark 1.6.** In Theorem 1.5, the initial data space is different between exact  $L^2$ -controllability and exact  $(H^1)'$ -controllability because the well-posedness result is in space  $Y_2 \times Y_1$  as  $u$  belongs to  $L^2(0, T)$  while the well-posedness result is in space  $Y_1 \times Y_0$  as  $u$  belongs to  $(H^1(0, T))'$  (see subsections 3.1 and 3.2).

Theorem 1.3 gives no information on the exact  $L^2$ -controllability of initial data in  $Y_{\beta+3} \times Y_{\beta+2}$  for  $\beta < 1$  and Theorem 1.4 gives no information on the exact  $(H^1)'$ -controllability of initial data in  $Y_{\beta+3} \times Y_{\beta+2}$  for  $\beta < 0$ . A partial answer is given by the following results.

**THEOREM 1.7.** *Suppose that  $\varepsilon > 0$ ,  $T > 0$  and  $\xi, \eta$  in  $(0, \pi)$  are arbitrary.*  
 1. *The set  $Y_{3-\varepsilon} \times Y_{2-\varepsilon}$  contains some initial data that are not exactly  $L^2$ -controllable in  $(\xi, \eta)$  at time  $T$ .*  
 2. *The set  $Y_{2-\varepsilon} \times Y_{1-\varepsilon}$  contains some initial data that are not exactly  $(H^1)'$ -controllable in  $(\xi, \eta)$  at time  $T$ .*

Notice that Theorems 1.3 and 1.4 require  $T > 2\pi\sqrt{\alpha}$ , however, in [21] the exact controllability results for Euler-Bernoulli beam have no requirement for control time. Consequently, a huge difference between Rayleigh beam and Euler-Bernoulli beam is revealed and the reason lies in various distributions of their eigenvalues. More precisely, under same boundary condition (1.1b), the eigenvalues of Rayleigh beam equation are  $\frac{k^4}{1+\alpha k^2}$  for  $k$  in  $\mathbb{N}^*$  (see subsection 3.1) while the eigenvalues of Euler-Bernoulli beam equation are  $k^4$  for  $k$  in  $\mathbb{N}^*$  (see [21]). Roughly speaking, this fact makes that Rayleigh beam equation possesses finite propagation speed and that Euler-Bernoulli beam equation possesses infinite propagation speed. For this reason, the exact controllability results of Rayleigh beam all require  $T > 2\pi\sqrt{\alpha}$  while the exact controllability results of Euler-Bernoulli beam hold for all  $T > 0$  (see [21]). Based on this fact, we give the non-controllability results for  $0 < T < 2\pi\sqrt{\alpha}$ . Inspired by [2], we propose and prove Theorem 1.8. In [2], the simultaneous controllability for two elastic strings was studied and the lack of approximate controllability was obtained when the control time is smaller than the critical time.

**THEOREM 1.8.** *Assume  $0 < T < 2\pi\sqrt{\alpha}$  and  $\xi, \eta$  in  $(0, \pi)$  are arbitrary.*  
 1. *For any  $\beta \geq -1$ , the space  $Y_{\beta+3} \times Y_{\beta+2}$  contains initial data that are not exactly  $L^2$ -controllable in  $(\xi, \eta)$  at time  $T$ .*  
 2. *For any  $\beta \geq -2$ , the space  $Y_{\beta+3} \times Y_{\beta+2}$  contains initial data that are not exactly  $(H^1)'$ -controllable in  $(\xi, \eta)$  at time  $T$ .*

**Remark 1.9.** For the question  $T = 2\pi\sqrt{\alpha}$ , whether the exact controllability still holds remains open.

Notice that in Theorem 1.5, the lack of controllability holds for some special  $\xi$  and  $\eta$  which are related to the space of initial data. However, in Theorems 1.7 and 1.8, non-controllability holds for any  $\xi$  and  $\eta$ . From Theorem 1.8, we can see that

$T \geq 2\pi\sqrt{\alpha}$  is necessary for exact controllability for Rayleigh beam equation and that minimal time for the exact controllability is obtained. As far as we know, this is the first result stating a lack of controllability for Rayleigh beam in short control time.

So far, the exact controllability for Rayleigh beam equation with piezoelectric actuator has been fully considered. As said in Remark 1.9, exact controllability in critical time is an open problem. Controllability for other type of beam equation with piezoelectric actuator, such as shear beam equation, also remains open.

**2. Preliminaries.** In this section, we provide some known results on the theory of diophantine approximation (see [5, 14]) and Riesz basis property of exponential family (see [3]).

For a real number  $\rho$ , we denote by  $\|\rho\|_{\mathbb{Z}}$  the difference, taken positively, between  $\rho$  and the nearest integer, i.e.,

$$\|\rho\|_{\mathbb{Z}} = \min_{n \in \mathbb{Z}} |\rho - n|.$$

Let us denote by  $A$  the set of all irrationals  $\rho$  in  $(0, 1)$  such that if  $[0, a_1, \dots, a_n \dots]$  is the expansion of  $\rho$  as a continued fraction, then  $(a_n)$  is bounded. Its Lebesgue measure is equal to zero (see [5]). The following property of this set is essentially useful in our work (see [14]).

**PROPOSITION 2.1.** *A number  $\rho$  is in  $A$  if and only if there exists a constant  $C > 0$  such that*

$$(2.1) \quad \|q\rho\|_{\mathbb{Z}} \geq \frac{C}{q}$$

for all strictly positive integer  $q$ .

The next proposition, which is proved in [5], shows that an inequality slightly weaker than (2.1) holds for almost all points in  $(0, 1)$ . This proposition is the definition of set  $B_{\varepsilon}$ .

**PROPOSITION 2.2.** *For any  $\varepsilon > 0$  there exists a set  $B_{\varepsilon} \subseteq (0, 1)$  having Lebesgue measure equal to 1 and a constant  $C > 0$ , such that for any  $\rho$  in  $B_{\varepsilon}$ ,*

$$(2.2) \quad \|q\rho\|_{\mathbb{Z}} \geq \frac{C}{q^{1+\varepsilon}}$$

for all strictly positive integer  $q$ .

The following proposition on simultaneous approximation (see [5]) used in [21] is quite important to prove Theorem 1.7.

**PROPOSITION 2.3.** *Let  $\rho_1, \dots, \rho_k$  be  $k$  irrationals in  $(0, 1)$ . Then there exists a strictly increasing sequence of natural numbers  $q_n$  such that*

$$q_n^{\frac{1}{k}} \max_{i=1, \dots, k} (\|q_n \rho_1\|_{\mathbb{Z}}, \dots, \|q_n \rho_i\|_{\mathbb{Z}}, \dots, \|q_n \rho_k\|_{\mathbb{Z}}) \leq \frac{k}{k+1} \quad \forall n \geq 1.$$

The next proposition (see Theorem II.4.18 of [3]) on Riesz basis property of exponential family in  $L^2(0, T)$  is essential for us to prove Theorem 1.8.

**PROPOSITION 2.4.** *Let  $\{\lambda_n\}_{n \in \mathbb{Z}}$  be a sequence of complex numbers such that*

$$\sup_{n \in \mathbb{Z}} |\operatorname{Im} \lambda_n| < \infty, \quad \inf_{n \neq m} |\lambda_m - \lambda_n| > 0.$$

Set

$$N(x, r) := \sharp\{\lambda_n | x \leq \operatorname{Re}\lambda_n < x + r\}, \quad x \in \mathbb{R}, r > 0,$$

where  $\sharp A$  is the number of elements in the set  $A$ . Assume that for some  $T > 0$ ,

$$\frac{N(x, r)}{r} \rightarrow \frac{T}{2\pi}, \quad \text{as } r \rightarrow \infty$$

uniformly relative to  $x \in \mathbb{R}$ . Then for any  $T'$  in  $(0, T)$ , family  $\{e^{i\lambda_n t}\}_{n \in \mathbb{Z}}$  contains a subfamily  $G_{T'}$  that forms a Riesz basis in  $L^2(0, T')$ .

**3. Well-posedness of (1.1).** In subsection 3.1, we show the well-posedness result of system (1.1) with  $L^2(0, T)$  control function which has been proved in [23]. In subsection 3.2, we prove the well-posedness and regularity results of system (1.1) with  $(H^1(0, T))'$  control function.

**3.1. Well-posedness of (1.1) with  $L^2(0, T)$  control function.** We state the well-posedness result and show the proof here, because the process of the proof is also used in other sections.

**THEOREM 3.1.** Suppose that  $(w^0, w^1)$  belongs to  $Y_2 \times Y_1$ . For any  $u$  in  $L^2(0, T)$  and for any  $\xi$  and  $\eta$  in  $(0, \pi)$ , the initial and boundary value problem (1.1) admits a unique solution having the regularity

$$(3.1) \quad w \in C([0, T]; Y_2) \cap C^1([0, T]; Y_1).$$

In order to prove Theorem 3.1, let us first consider the adjoint problem of (1.1)

$$(3.2a) \quad \phi_{tt}(x, t) - \alpha \phi_{xxtt}(x, t) + \phi_{xxx}(x, t) = 0, \quad 0 < x < \pi, t > 0,$$

$$(3.2b) \quad \phi(0, t) = \phi(\pi, t) = \phi_{xx}(0, t) = \phi_{xx}(\pi, t) = 0, \quad t > 0,$$

$$(3.2c) \quad \phi(x, 0) = \phi^0(x), \phi_t(x, 0) = \phi^1(x), \quad 0 < x < \pi.$$

The following lemma proved in [23] shows the well-posedness of the adjoint problem (3.2) and some trace regularities needed in the proof of Theorem 3.1.

**LEMMA 3.2.** For any initial data  $(\phi^0, \phi^1)$  in  $Y_2 \times Y_1$ , there exists a unique weak solution  $\phi$  of (3.2) in the class  $C([0, T]; Y_2) \cap C^1([0, T]; Y_1)$ . Moreover, for all  $b$  in  $(0, \pi)$  we have  $\phi_x(b, \cdot)$  belongs to  $H^1(0, T)$  and there exist  $C, C' > 0$  such that

$$(3.3) \quad \|\phi_x(b, \cdot)\|_{H^1(0, T)}^2 \leq C(\|\phi^0\|_{H^2(0, \pi)}^2 + \|\phi^1\|_{H^1(0, \pi)}^2),$$

$$(3.4) \quad \|\phi_x(b, \cdot)\|_{L^2(0, T)}^2 \leq C'(\|\phi^0\|_{H^1(0, \pi)}^2 + \|\phi^1\|_{L^2(0, \pi)}^2).$$

*Proof.* It is easy to see, by the semigroup method, that the problem (3.2) is well-posed in the space  $Y_2 \times Y_1$  (see [20]).

Next we prove (3.3) and (3.4). Since  $\{x \mapsto \sin(kx)\}_{k \in \mathbb{N}^*}$  are eigenfunctions of (3.2a) and (3.2b), we put

$$\phi^0(x) = \sum_{k \geq 1} a_k \sin(kx), \quad \phi^1(x) = \sum_{k \geq 1} b_k \sin(kx),$$

with  $(k^2 a_k)$  and  $(k b_k)$  in  $l^2(\mathbb{R})$ . Obviously, we have

$$(3.5) \quad \phi(x, t) = \sum_{k \geq 1} \left[ a_k \cos\left(\frac{k^2}{\sqrt{1 + \alpha k^2}} t\right) + \frac{b_k \sqrt{1 + \alpha k^2}}{k^2} \sin\left(\frac{k^2}{\sqrt{1 + \alpha k^2}} t\right) \right] \sin(kx).$$

250 Then for all  $T > 0$ ,  $\phi_x(b, \cdot)$  belongs to  $H^1(0, T)$  and

$$251 \quad \int_0^T |\phi_{xt}(b, t)|^2 dx \leq C \sum_{k \geq 1} k^2 \left( a_k^2 \frac{k^4}{1 + \alpha k^2} + b_k^2 \right) \leq C \sum_{k \geq 1} k^2 (a_k^2 k^2 + b_k^2),$$

252 which yields (3.3). And simultaneously we have

$$253 \quad \int_0^T |\phi_x(b, t)|^2 dx \leq C' \sum_{k \geq 1} k^2 \left( a_k^2 + b_k^2 \frac{1 + \alpha k^2}{k^4} \right) \leq C' \sum_{k \geq 1} (a_k^2 k^2 + b_k^2),$$

254 which clearly yields (3.4).  $\square$

255 *Proof of Theorem 3.1.* Thanks to Lemma 3.2, the following backward adjoint  
256 problem is well-posed in  $Y_2 \times Y_1$  for every  $\tau > 0$  and  $g$  in  $Y_1$ .

$$257 \quad (3.6a) \quad v_{tt}(x, t) - \alpha v_{xxtt}(x, t) + v_{xxxx}(x, t) = 0, \quad 0 < x < \pi, \quad t \in (0, \tau),$$

$$258 \quad (3.6b) \quad v(0, t) = v(\pi, t) = v_{xx}(0, t) = v_{xx}(\pi, t) = 0, \quad t \in (0, \tau),$$

$$259 \quad (3.6c) \quad v(x, \tau) = 0, \quad v_t(x, \tau) = g(x), \quad 0 < x < \pi.$$

261 Moreover, for any  $b$  in  $(0, \pi)$  we have

$$262 \quad (3.7) \quad \|v_x(b, \cdot)\|_{L^2(0, \tau)} \leq C \|g\|_{Y_0}.$$

263 Since (1.1a) is linear, it is enough to consider the case  $w^0 = w^1 = 0$ . Suppose  
264 again  $g$  belongs to  $C_0^\infty(0, \pi)$ , and let  $v$  be the solution of (3.6). Define a linear operator  
265  $\mathcal{L} := I - \alpha \partial_{xx}$ . It is well-known that operator  $\mathcal{L}$  is an isomorphism from  $Y_2$  to  $Y_0$  and  
266 an isomorphism from  $Y_1$  to  $Y_{-1}$  by Lax-Milgram Theorem. If we multiply (1.1a) by  
267  $v$  and integrate by parts we obtain

$$268 \quad (3.8) \quad \int_0^\pi \mathcal{L}w(x, \tau)g(x)dx = \int_0^\tau u(t)(v_x(\eta, t) - v_x(\xi, t))dt.$$

269 Trace regularity (3.7) implies that

$$270 \quad \left| \int_0^\tau u(t)(v_x(\eta, t) - v_x(\xi, t))dt \right| \leq C \|u\|_{L^2(0, T)} \|g\|_{Y_0},$$

271 so by (3.8), we obtain  $\mathcal{L}w(\cdot, \tau)$  belongs to  $Y_0$ , and hence  $w(\cdot, \tau)$  belongs to  $Y_2$ , for all  
272  $\tau$  in  $[0, T]$ . By replacing  $\tau$  by  $\tau + h$  in (3.8) we easily get that

$$273 \quad (3.9) \quad w \in C([0, T]; Y_2).$$

274 Denote  $\mathcal{R} := (I - \alpha \partial_{xx})^{-1}$ . It follows from Lax-Milgram Theorem that operator  $\mathcal{R}$  is  
275 an isomorphism from  $Y_{-2}$  to  $Y_0$  and an isomorphism from  $Y_{-1}$  to  $Y_1$ . Applying  $\mathcal{R}$  to  
276 both sides of (1.1a) yields

$$277 \quad (3.10) \quad w_{tt}(x, t) + \mathcal{R}w_{xxxx}(x, t) = u(t)\mathcal{R} \frac{d}{dx} [\delta_\eta(x) - \delta_\xi(x)].$$

278 Regularity (3.9) implies that

$$279 \quad (3.11) \quad \mathcal{R}w_{xxxx} \in C([0, T]; Y_0).$$



As  $w$  satisfies (3.10) and  $\frac{d\delta_b}{dx}$  belongs to  $Y_{-2}$  for all  $b$  in  $(0, \pi)$ , we obtain from (3.11) that

$$(3.12) \quad w_{tt} \in L^2(0, T; Y_0).$$

From (3.9) and (3.12), by applying the intermediate derivative theorem (see [18]) it follows that

$$(3.13) \quad w_t \in L^2(0, T; Y_1).$$

The conclusion (3.1) is now a consequence of (3.9) and (3.13) and of the general lifting result from [16].  $\square$

**3.2. Well-posedness of (1.1) with  $(H^1(0, T))'$  control function.** As control function  $u$  belongs to  $(H^1(0, T))'$ , we need to define the solution of (1.1) in the weak form. First notice that  $u$  is not a distribution. The definition of a function  $u$  in  $(H^1(0, T))'$  is given by following.

**DEFINITION 3.3.** *A function  $u$  is said to belong to  $(H^1(0, T))'$  if there exist some functions  $u_0$  and  $u_1$  in  $L^2(0, T)$  such that for all  $\phi$  in  $H^1(0, T)$ , we have*

$$(3.14) \quad \langle u, \phi \rangle = \int_0^T (u_0 \phi + u_1 \phi_t) dt.$$

We denote

$$(3.15) \quad u = u_0 - \frac{\partial u_1}{\partial t}.$$

Note that in (3.15),  $\frac{\partial u_1}{\partial t}$  is not taken in the sense of distribution, but taken in the sense between  $H^1(0, T)$  and its dual space.

Inspired by [17], we define the weak solution of (1.1) by transposition and prove the well-posedness. We explain the results in three steps.

1. We prove the well-posedness and trace regularity of a non-homogeneous problem (3.16).
2. Using the solution of non-homogeneous problem (3.16), we define the weak solution of (1.1) by transposition.
3. We prove the well-posedness of (1.1) with control function  $u$  in  $(H^1(0, T))'$ .

*Step 1.* In order to define the weak solution of (1.1) by transposition, we need the following well-posedness and trace regularity of a non-homogeneous problem. Assume  $f$  belongs to  $L^1(0, T; Y_{-1})$  and  $(\theta^0, \theta^1)$  belongs to  $Y_2 \times Y_1$ , let us consider the following backward non-homogeneous problem

$$(3.16a) \quad \theta_{tt}(x, t) - \alpha \theta_{xxtt}(x, t) + \theta_{xxxx}(x, t) = f(x, t), \quad 0 < x < \pi, t \in (0, T),$$

$$(3.16b) \quad \theta(0, t) = \theta(\pi, t) = \theta_{xx}(0, t) = \theta_{xx}(\pi, t) = 0, \quad t \in (0, T),$$

$$(3.16c) \quad \theta(x, T) = \theta^0, \theta_t(x, T) = \theta^1, \quad 0 < x < \pi.$$

The following proposition proves the well-posedness and trace regularity of (3.16) which is needed in the definition of weak solution of (1.1). Our approach to prove this proposition is inspired by ideas and methods used in [1, 23].

**PROPOSITION 3.4.** *For any initial data  $(\theta^0, \theta^1)$  in  $Y_2 \times Y_1$  and  $f$  in  $L^1(0, T; Y_{-1})$ , there exists a unique weak solution  $\theta$  of (3.16) in the class  $C([0, T]; Y_2) \cap C^1([0, T]; Y_1)$ . Moreover, for all  $b$  in  $(0, \pi)$  we have  $\theta_x(b, \cdot)$  belongs to  $H^1(0, T)$  and there exists  $C > 0$  such that*

$$(3.17) \quad \|\theta_x(b, \cdot)\|_{H^1(0, T)} \leq C(\|\theta^0\|_{H^2(0, \pi)} + \|\theta^1\|_{H^1(0, \pi)} + \|f\|_{L^1(0, T; Y_{-1})}).$$

*Proof.* Applying  $\mathcal{R}$  to both sides of (3.16a) to obtain

$$\theta_{tt}(x, t) + \mathcal{R}\theta_{xxxx}(x, t) = \mathcal{R}f(x, t).$$

Notice that  $\mathcal{R}f$  belongs to  $L^1(0, T; Y_1)$ , then the problem (3.16) admits a unique solution  $\theta$  in  $C([0, T]; Y_2) \cap C^1([0, T]; Y_1)$  by the classical semigroup method (see [20]). And there exists a constant  $C_T > 0$  such that

$$(3.18) \quad \|\theta\|_{C([0, T]; Y_2)} \leq C_T(\|\theta^0\|_{H^2(0, \pi)} + \|\theta^1\|_{H^1(0, \pi)} + \|f\|_{L^1(0, T; Y_{-1})}).$$

Then we need to prove trace regularity (3.17). The following lemma proved in [1] and [23] shows that operator  $\mathcal{R}\partial_{xxxx}$  is “similar” to a elliptic operator  $-\frac{1}{\alpha}\partial_{xx}$ .

LEMMA 3.5. *The linear operator  $L = -\frac{1}{\alpha}\partial_{xx} - \mathcal{R}\partial_{xxxx}$  is bounded from  $Y_2$  to  $Y_2$ .*

Using this lemma, we can reduce the proof of (3.17) to a regularity property for a string equation. We consider the initial value problem

$$\begin{aligned} \theta_{1,tt}(x, t) - \frac{1}{\alpha}\theta_{1,xx}(x, t) &= \mathcal{R}f(x, t), \quad 0 < x < \pi, \quad t \in (0, T), \\ \theta_1(0, t) &= \theta_1(\pi, t) = 0, \quad t \in (0, T), \\ \theta_1(x, T) &= \theta^0, \quad \theta_{1,t}(x, T) = \theta^1, \quad 0 < x < \pi. \end{aligned}$$

The relations above imply that  $\theta_2 = \theta - \theta_1$  satisfies

$$\begin{aligned} \theta_{2,tt}(x, t) - \frac{1}{\alpha}\theta_{2,xx}(x, t) &= L\theta, \quad 0 < x < \pi, \quad t \in (0, T), \\ \theta_2(0, t) &= \theta_2(\pi, t) = 0, \quad t \in (0, T), \\ \theta_2(x, T) &= 0, \quad \theta_{2,t}(x, T) = 0, \quad 0 < x < \pi. \end{aligned}$$

Since  $\theta$  belongs to  $C([0, T]; Y_2)$  and  $L$  is bounded from  $Y_2$  to  $Y_2$ ,  $L\theta$  belongs to  $C([0, T]; Y_2)$ . Then by the classical theory for evolution equations of hyperbolic type (see [15]), we obtain that  $\theta_2$  belongs to  $C([0, T]; H^3(0, \pi)) \cap C^1([0, T]; H^2(0, \pi))$  and there exists a constant  $C_T > 0$  such that

$$\|(\theta_2, \theta_{2,t})\|_{C([0, T]; H^3(0, \pi) \times H^2(0, \pi))} \leq C_T\|\theta\|_{C([0, T]; Y_2)}.$$

This inequality, combined with (3.18) and the standard trace theorem, implies that for any  $b$  in  $(0, \pi)$ ,

$$(3.21) \quad \|\theta_{2,x}(b, \cdot)\|_{H^1(0, T)} \leq C_T(\|\theta^0\|_{H^2(0, \pi)} + \|\theta^1\|_{H^1(0, \pi)} + \|f\|_{L^1(0, T; Y_{-1})}).$$

As for  $\theta_1$ , it is already proved by using multiplier methods in [10] that there exists a constant  $C_T > 0$  such that

$$(3.22) \quad \begin{aligned} \|\theta_{1,x}(b, \cdot)\|_{H^1(0, T)} &\leq C_T(\|\theta^0\|_{H^2(0, \pi)} + \|\theta^1\|_{H^1(0, \pi)} + \|\mathcal{R}f\|_{L^1(0, T; Y_1)}) \\ &\leq C_T(\|\theta^0\|_{H^2(0, \pi)} + \|\theta^1\|_{H^1(0, \pi)} + \|f\|_{L^1(0, T; Y_{-1})}). \end{aligned}$$

Then trace regularity (3.17) follows from (3.21) and (3.22).  $\square$

*Step 2.* Now we give the definition of the weak solution of (1.1). Denote by  $\mathcal{X}$  a Hilbert space consisting of  $\theta$ , the solution of (3.16). And give  $\mathcal{X}$  a natural Hilbert

structure such that  $\{f, \theta^0, \theta^1\} \rightarrow \theta$  is an isomorphism from  $L^1(0, T; Y_{-1}) \times Y_2 \times Y_1$  to  $\mathcal{X}$ . From Proposition 3.4 we have

$$(3.23) \quad \begin{aligned} \mathcal{X} &\subseteq C([0, T]; Y_2) \cap C^1([0, T]; Y_1), \\ \theta \in \mathcal{X} &\Rightarrow \theta_x(b, \cdot) \in H^1(0, T) \quad \forall b \in (0, \pi). \end{aligned}$$

Assume that  $\{f, \theta^0, \theta^1\}$  belongs to  $L^1(0, T; Y_{-1}) \times Y_2 \times Y_1$  and that  $\{u, w^0, w^1\}$  belongs to  $L^2(0, T) \times Y_2 \times Y_1$ . Denote by  $\theta$  the solution of (3.16) and by  $w$  the solution of (1.1) given by Theorem 3.1. Multiplying (1.1a) by  $\theta$ , we obtain

$$(3.24) \quad \int_0^T \int_0^\pi (w_{tt} - \alpha w_{xxtt} + w_{xxxx}) \theta dx dt = \int_0^T \int_0^\pi u(t) \frac{d}{dx} [\delta_\eta(x) - \delta_\xi(x)] \theta dx dt.$$

Integrating by parts we get

$$(3.25) \quad \begin{aligned} &\int_0^T \int_0^\pi w f dx dt + \int_0^\pi [\mathcal{L} w_t(T) \theta^0 - \mathcal{L} w(T) \theta^1] dx \\ &= - \int_0^T u(t) (\theta_x(\eta, t) - \theta_x(\xi, t)) dt + \int_0^\pi [\mathcal{L} w^1 \theta(0) - \mathcal{L} w^0 \theta_t(0)] dx. \end{aligned}$$

Now relaxing the assumption of  $u$  belonging to  $L^2(0, T)$  to  $u$  belonging to  $(H^1(0, T))'$ , we obtain from (3.15) that

$$(3.26) \quad \begin{aligned} &\int_0^T \int_0^\pi w f dx dt + \int_0^\pi [\mathcal{L} w_t(T) \theta^0 - \mathcal{L} w(T) \theta^1] dx \\ &= - \int_0^T [u_0(t) (\theta_x(\eta, t) - \theta_x(\xi, t)) + u_1(t) (\theta_{xt}(\eta, t) - \theta_{xt}(\xi, t))] dt \\ &\quad + \int_0^\pi [\mathcal{L} w^1 \theta(0) - \mathcal{L} w^0 \theta_t(0)] dx. \end{aligned}$$

Now we set (3.24) as the definition of weak solution.

**DEFINITION 3.6.** Let  $T > 0$ ,  $u$  in  $(H^1(0, T))'$  and  $(w^0, w^1)$  in  $Y_1 \times Y_0$  be given. A solution of the (1.1) is a function  $w$  in  $C([0, T]; Y_1)$  such that, for every  $\{f, \theta^0, \theta^1\}$  in  $L^1(0, T; Y_{-1}) \times Y_2 \times Y_1$ , (3.24) holds and  $(w(T), w_t(T))$  belongs to  $Y_1 \times Y_0$ .

*Step 3.* Then we are able to prove the well-posedness of (1.1) when  $u$  belongs to  $(H^1(0, T))'$ .

**THEOREM 3.7.** Suppose  $(w^0, w^1)$  belongs to  $Y_1 \times Y_0$ . For any  $u$  in  $(H^1(0, T))'$  and for any  $\xi$  and  $\eta$  in  $(0, \pi)$ , the initial and boundary value problem (1.1) admits a unique weak solution in sense of Definition 3.6. And the map  $\{w^0, w^1, u\} \mapsto \{w, w(T), w_t(T)\}$  is linear and continuous with respect to the corresponding norm.

*Proof.* Since  $u$  belongs to  $(H^1(0, T))'$ , there exist  $u_0, u_1$  in  $L^2(0, T)$  such that  $u = u_0 - \frac{\partial u_1}{\partial t}$ , where  $\frac{\partial}{\partial t}$  is in sense of  $(H^1(0, T))'$  derivative. Moreover,  $\{f, \theta^0, \theta^1\} \rightarrow \theta$  is an isomorphism from  $L^1(0, T; Y_{-1}) \times Y_2 \times Y_1$  to  $\mathcal{X}$ . Therefore, we define a linear form  $\Gamma$  on  $\mathcal{X}$  such that

$$(3.27) \quad \begin{aligned} \Gamma(\theta) &= - \int_0^T [u_0(t) (\theta_x(\eta, t) - \theta_x(\xi, t)) + u_1(t) (\theta_{xt}(\eta, t) - \theta_{xt}(\xi, t))] dt \\ &\quad + \int_0^\pi [\mathcal{L} w^1 \theta(0) - \mathcal{L} w^0 \theta_t(0)] dx. \end{aligned}$$

Since  $(w^0, w^1)$  belongs to  $Y_1 \times Y_0$ , then  $(\mathcal{L}w^0, \mathcal{L}w^1)$  belongs to  $Y_{-1} \times Y_{-2}$ . From Proposition 3.4 we have  $\Gamma$  is a continuous linear form on  $\mathcal{X}$ . Denote by  $(f_1, f_2)$  the linear form between  $Y_\gamma$  and  $Y_{-\gamma}$  for any  $\gamma \geq 0$  and denote by  $\langle g_1, g_2 \rangle$  the linear form between  $L^\infty(0, T; Y_1)$  and  $L^1(0, T; Y_{-1})$ . Therefore, for the linear form  $\Gamma$  in  $\mathcal{X}'$ , there exists unique  $\{w, \zeta_*, \zeta\}$  in  $L^\infty(0, T; Y_1) \times Y_{-2} \times Y_{-1}$  such that

$$(3.26) \quad \langle w, f \rangle + (\zeta_*, \theta^0) + (-\zeta, \theta) = \Gamma(\theta) \quad \forall \theta \in \mathcal{X}.$$

Next we claim that  $w$  above is actually the weak solution of (1.1). It is sufficient to prove that  $w$  satisfies (1.1) in weak sense,  $\mathcal{L}w(T) = \zeta$  and  $\mathcal{L}w_t(T) = \zeta_*$ .

Notice that  $\{x \mapsto \sin(kx)\}_{k \in \mathbb{N}^*}$  are eigenfunctions of  $\mathcal{R}\partial_{xxxx}$ . Let  $m(x) = \sin(kx)$  for some  $k$  in  $\mathbb{N}^*$ . Firstly we set  $\mathcal{R}f(t) = h(t)m$ ,  $\theta^0 = 0$  and  $\theta^1 = 0$ , then  $f(t) = h(t)\mathcal{L}m$ . Denote by  $\lambda = k^4/(1 + \alpha k^2)$  the corresponding eigenvalue. Then we obtain from Proposition 3.4 for  $t$  in  $(0, T)$

$$\theta(t) = q(t)m,$$

where  $q$  satisfies

$$q_{tt} + \lambda q = h, \quad q(T) = q_t(T) = 0,$$

namely

$$q(t) = -\frac{1}{\sqrt{\lambda}} \int_t^T \sin((t - \sigma)\sqrt{\lambda}) h(\sigma) d\sigma.$$

Then

$$\begin{aligned} \Gamma(\theta) &= -(m_x(\eta) - m_x(\xi)) \int_0^T [u_0(t)q(t) + u_1(t)q_t(t)] dt \\ &\quad + q(0)(\mathcal{L}w^1, m) - q_t(0)(\mathcal{L}w^0, m), \end{aligned}$$

and we have

(3.27)

$$\begin{aligned} \langle w, f \rangle &= \int_0^T (w, \mathcal{L}m)(q_{tt} + \lambda q) dt = -(m_x(\eta) - m_x(\xi)) \int_0^T [u_0(t)q(t) + u_1(t)q_t(t)] dt \\ &\quad + q(0)(\mathcal{L}w^1, m) - q_t(0)(\mathcal{L}w^0, m). \end{aligned}$$

Notice that  $(w, \mathcal{L}m) = (\mathcal{L}w, m)$ , then (3.27) implies that

$$\begin{aligned} (\mathcal{L}w, m)_{tt} + \lambda(\mathcal{L}w, m) &= -(m_x(\eta) - m_x(\xi)) \left( u_0 - \frac{\partial u_1}{\partial t} \right), \\ (\mathcal{L}w, m)(0) &= (\mathcal{L}w^0, m), \\ (\mathcal{L}w, m)_t(0) &= (\mathcal{L}w^1, m), \end{aligned} \quad (3.28)$$

where  $\frac{\partial u_1}{\partial t}$  is taken in the sense of  $(H^1(0, T))'$ . Since  $m(x) = \sin(kx)$  and  $k \geq 1$  is an arbitrary natural number, then  $w$  satisfies (1.1) in weak sense.

Now if we set  $f = 0$ ,  $\theta^0 = 0$  and  $\theta^1 = -m$ , then for  $t$  in  $(0, T)$

$$\theta(t) = \frac{1}{\sqrt{\lambda}} \sin((T - t)\sqrt{\lambda}) m.$$

Then (3.26) implies that

$$\begin{aligned} (\zeta, m) = \Gamma(\theta) &= -(m_x(\eta) - m_x(\xi)) \int_0^T \left[ u_0(t) \frac{1}{\sqrt{\lambda}} \sin((T - t)\sqrt{\lambda}) \right. \\ &\quad \left. - u_1(t) \cos((T - t)\sqrt{\lambda}) \right] dt \\ &\quad + (\mathcal{L}w^1, m) \frac{1}{\sqrt{\lambda}} \sin(T\sqrt{\lambda}) + (\mathcal{L}w^0, m) \cos(T\sqrt{\lambda}). \end{aligned} \quad (3.29)$$

Moreover, (3.28) implies that

$$\begin{aligned}
 (\mathcal{L}w, m)(T) &= (\mathcal{L}w^0, m) \cos(T\sqrt{\lambda}) + (\mathcal{L}w^1, m) \frac{\sin(T\sqrt{\lambda})}{\sqrt{\lambda}} \\
 &\quad - (m_x(\eta) - m_x(\xi)) \int_0^T \frac{1}{\sqrt{\lambda}} \sin((T - \sigma)\sqrt{\lambda}) \left( u_0 - \frac{\partial u_1}{\partial \sigma} \right) d\sigma.
 \end{aligned}
 \tag{3.30}$$

Note the definition of  $\frac{\partial u_1}{\partial t}$  and compare to (3.29), we obtain  $(\mathcal{L}w, m)(T) = (\zeta, m)$ , which proves  $\mathcal{L}w(T) = \zeta$ .

Next we set  $f = 0$ ,  $\theta^0 = m$  and  $\theta^1 = 0$ , then for  $t$  in  $(0, T)$

$$\theta(t) = \cos((T - t)\sqrt{\lambda})m.$$

Then (3.26) implies that

$$\begin{aligned}
 (\zeta_*, m) &= \Gamma(\theta) = - (m_x(\eta) - m_x(\xi)) \int_0^T [u_0(t) \cos((T - t)\sqrt{\lambda}) \\
 &\quad + u_1(t) \sqrt{\lambda} \sin((T - t)\sqrt{\lambda})] dt \\
 &\quad + (\mathcal{L}w^1, m) \cos(T\sqrt{\lambda}) - (\mathcal{L}w^0, m) \sqrt{\lambda} \sin(T\sqrt{\lambda}).
 \end{aligned}
 \tag{3.31}$$

Moreover, (3.28) implies that

$$\begin{aligned}
 (\mathcal{L}w, m)_t(T) &= - (\mathcal{L}w^0, m) \sqrt{\lambda} \sin(T\sqrt{\lambda}) + (\mathcal{L}w^1, m) \cos(T\sqrt{\lambda}) \\
 &\quad - (m_x(\eta) - m_x(\xi)) \int_0^T \cos((T - \sigma)\sqrt{\lambda}) \left( u_0 - \frac{\partial u_1}{\partial \sigma} \right) d\sigma.
 \end{aligned}
 \tag{3.32}$$

Note the definition of  $\frac{\partial u_1}{\partial t}$  and compare to (3.31), we obtain  $(\mathcal{L}w, m)_t(T) = (\zeta_*, m)$ , which implies  $\mathcal{L}w_t(T) = \zeta_*$ .

Now we have proved there exists unique  $\{w, w(T), w_t(T)\}$  in  $L^\infty(0, T; Y_1) \times Y_1 \times Y_0$  such that (3.24) holds and the map  $\{w^0, w^1, u\} \rightarrow \{w, w(T), w_t(T)\}$  is linear and continuous with respect to the corresponding norm. In fact we have property  $w$  belongs to  $C([0, T]; Y_1)$ . Since when the known data  $\{w^0, w^1, u\}$  belongs to  $Y_2 \times Y_1 \times L^2(0, T)$ , we have (3.1), then according to the results we obtained above, this property is preserved in the density argument. It concludes the proof of Theorem 3.7.  $\square$

**4. Proofs of the main results.** In this section, we prove the main results. For exact controllability results, Theorems 1.3 and 1.4, we use the HUM, introduced in [17], to illustrate the controllability problem is equivalent to the observability problem of the adjoint system. Then we prove the observability inequality in each case. For non-controllability results, Theorems 1.5, 1.7, and 1.8, we show the observability inequalities are false. Essentially, we obtained some sufficient conditions and some necessary conditions for the observability inequality.

**4.1. Exact  $L^2$ -controllability (Proof of Theorem 1.3).** Choose  $(\phi^0, \phi^1)$  in  $(C^\infty[0, \pi])^2$  satisfying the compatible conditions (1.2) and denote by  $\phi(x, t)$  the solution of (3.2) with initial value  $(\phi^0, \phi^1)$ .

Consider a backward adjoint system

$$(4.1a)$$

$$\psi_{tt}(x, t) - \alpha \psi_{xxtt}(x, t) + \psi_{xxxx}(x, t) = u(t) \frac{d}{dx} [\delta_\eta(x) - \delta_\xi(x)], \quad 0 < x < \pi, \quad t > 0,$$

$$(4.1b) \quad \psi(0, t) = \psi(\pi, t) = \psi_{xx}(0, t) = \psi_{xx}(\pi, t) = 0, \quad t > 0,$$

$$(4.1c) \quad \psi(x, T) = \psi_t(x, T) = 0, \quad 0 < x < \pi,$$

where  $u$  in  $L^2(0, T)$  will be chosen later. Problem (4.1) is well-posed according to Theorem 3.1. Then, multiplying (4.1a) by  $\phi$  and integrating by parts, we get

$$(4.2) \quad \int_0^\pi \phi^0(x) \mathcal{L}\psi_t(x, 0) - \phi^1(x) \mathcal{L}\psi(x, 0) dx = \int_0^T u(t) (\phi_x(\eta, t) - \phi_x(\xi, t)) dt.$$

Let  $u(t) = \phi_x(\eta, t) - \phi_x(\xi, t)$ . Since (3.4),  $u$  belongs to  $L^2(0, T)$ . Define a linear operator  $\Lambda$  satisfying

$$(4.3) \quad \Lambda(\phi^0, \phi^1) = (\mathcal{L}\psi_t(\cdot, 0), -\mathcal{L}\psi(\cdot, 0)).$$

Since  $(\mathcal{L}\psi_t(\cdot, 0), -\mathcal{L}\psi(\cdot, 0))$  belongs to  $Y_{-1} \times Y_0$  by Theorem 3.1, operator  $\Lambda$  is well defined. In particular,

$$(4.4) \quad \langle \Lambda(\phi^0, \phi^1), (\phi^0, \phi^1) \rangle = \int_0^T |\phi_x(\eta, t) - \phi_x(\xi, t)|^2 dt.$$

Therefore, we can define a seminorm

$$(4.5) \quad \|(\phi^0, \phi^1)\|_F := \left( \int_0^T |\phi_x(\eta, t) - \phi_x(\xi, t)|^2 dt \right)^{\frac{1}{2}},$$

for all  $(\phi^0, \phi^1)$  in  $(C^\infty[0, \pi])^2$  satisfying the compatible conditions (1.2).

A classical argument in HUM implies the following proposition.

**PROPOSITION 4.1.** *All initial data in  $Y_{\beta+3} \times Y_{\beta+2}$  are exactly  $L^2$ -controllable in  $(\xi, \eta)$  at time  $T$  if and only if there exists a constant  $c > 0$  such that*

$$(4.6) \quad \int_0^T |\phi_x(\eta, t) - \phi_x(\xi, t)|^2 dt \geq c(\|\phi^0\|_{H^{-\beta}}^2 + \|\phi^1\|_{H^{-\beta-1}}^2)$$

for all  $(\phi^0, \phi^1)$  in  $(C^\infty[0, \pi])^2$  satisfying the compatible conditions (1.2).

Equation (4.6) is called observability inequality. As in the proof of Lemma 3.2, the solution  $\phi$  of the adjoint problem (3.2) has the form of (3.5), which implies that

$$(4.7) \quad \begin{aligned} & \int_0^T |\phi_x(\eta, t) - \phi_x(\xi, t)|^2 dt \\ &= 4 \int_0^T \left| \sum_{k \geq 1} k \sin\left(\frac{k(\eta + \xi)}{2}\right) \sin\left(\frac{k(\eta - \xi)}{2}\right) \left\{ a_k \cos\left(\frac{k^2}{\sqrt{1 + \alpha k^2}} t\right) \right. \right. \\ & \quad \left. \left. + \frac{b_k \sqrt{1 + \alpha k^2}}{k^2} \sin\left(\frac{k^2}{\sqrt{1 + \alpha k^2}} t\right) \right\} \right|^2 dt. \end{aligned}$$

**Remark 4.2.** From (4.7), we can see that (1.3) is necessary for controllability.

To prove observability inequality (4.6) for some  $\beta$ , we use the following Ingham inequality (see [12, 4]) to our problem.

**LEMMA 4.3.** *Let  $(\nu_k)_{k \in \mathbb{Z}}$  be a strictly increasing sequence of real numbers and let  $\gamma_\infty > 0$  be defined by*

$$(4.7) \quad \gamma_\infty = \lim_{|k| \rightarrow \infty} |\nu_{k+1} - \nu_k|.$$

For any real  $T > 2\pi/\gamma_\infty$ , there exist two constants  $C_1, C_2 > 0$  such that, for any sequence  $(x_k)_{k \in \mathbb{Z}}$  in  $l^2(\mathbb{C})$ ,

$$C_1 \sum_{k \in \mathbb{Z}} |x_k|^2 \leq \int_0^T \left| \sum_{k \in \mathbb{Z}} x_k e^{i\nu_k t} \right|^2 dt \leq C_2 \sum_{k \in \mathbb{Z}} |x_k|^2.$$

We apply Lemma 4.3 with

$$\begin{aligned} \nu_k &= -\nu_{-k} = \frac{k^2}{\sqrt{1 + \alpha k^2}}, \\ 2x_k &= 2\overline{x_{-k}} = \left( a_k - i \frac{b_k \sqrt{1 + \alpha k^2}}{k^2} \right) k \sin \left( \frac{k(\eta + \xi)}{2} \right) \sin \left( \frac{k(\eta - \xi)}{2} \right). \end{aligned}$$

As  $\lim_{|k| \rightarrow \infty} |\nu_{k+1} - \nu_k| = 1/\sqrt{\alpha}$ , then for any real  $T > 2\pi\sqrt{\alpha}$ , there exist two constants  $C_1, C_2 > 0$  such that,

$$\begin{aligned} & C_1 \sum_{k \geq 1} k^2 \left( a_k^2 + \frac{b_k^2(1 + \alpha k^2)}{k^4} \right) \left[ \sin \left( \frac{k(\eta + \xi)}{2} \right) \sin \left( \frac{k(\eta - \xi)}{2} \right) \right]^2 \\ (4.8) \quad & \leq \int_0^T |\phi_x(\eta, t) - \phi_x(\xi, t)|^2 dt \\ & \leq C_2 \sum_{k \geq 1} k^2 \left( a_k^2 + \frac{b_k^2(1 + \alpha k^2)}{k^4} \right) \left[ \sin \left( \frac{k(\eta + \xi)}{2} \right) \sin \left( \frac{k(\eta - \xi)}{2} \right) \right]^2. \end{aligned}$$

When  $\frac{\eta + \xi}{2\pi}$  and  $\frac{\eta - \xi}{2\pi}$  belong to  $A$ , from (2.1) we see that there exists a constant  $C > 0$  such that for all  $k \geq 1$  we have

$$(4.9) \quad \left| \sin \left( \frac{k(\eta \pm \xi)}{2} \right) \right| = \left| \sin \left\{ \pi \left[ \frac{k(\eta \pm \xi)}{2\pi} - p \right] \right\} \right| \geq \left| \sin \left( \frac{\pi C}{k} \right) \right| \geq \frac{C}{k}.$$

Inequalities (4.8) and (4.9) imply that

$$\int_0^T |\phi_x(\eta, t) - \phi_x(\xi, t)|^2 dt \geq c \sum_{k \geq 1} \left( \frac{a_k^2}{k^2} + \frac{b_k^2(1 + \alpha k^2)}{k^6} \right),$$

which is exactly (4.6) when  $\beta = 1$ . This fact completes the proof of the first part of Theorem 1.3.

When  $\frac{\eta + \xi}{2\pi}$  and  $\frac{\eta - \xi}{2\pi}$  belong to  $B_\varepsilon$ , from (2.2) we see that there exists a constant  $C > 0$  such that for all  $k \geq 1$  we have

$$(4.10) \quad \left| \sin \left( \frac{k(\eta \pm \xi)}{2} \right) \right| \geq \frac{C}{k^{1+\varepsilon}}.$$

Inequalities (4.8) and (4.10) imply that

$$\int_0^T |\phi_x(\eta, t) - \phi_x(\xi, t)|^2 dt \geq c \sum_{k \geq 1} \left( \frac{a_k^2}{k^{2+\varepsilon}} + \frac{b_k^2(1 + \alpha k^2)}{k^{6+\varepsilon}} \right),$$

which is exactly (4.6) when  $\beta = 1 + \varepsilon$ . This completes the proof of the second part of Theorem 1.3.

**4.2. Exact  $(H^1)'$ -controllability (Proof of Theorem 1.4).** Similarly to subsection 4.1, we use the HUM to study the controllability problem. Let  $(\phi^0, \phi^1)$  in  $(C^\infty[0, \pi])^2$  satisfy the compatible conditions (1.2). Denote by  $\phi$  the solution of (3.2) with initial value  $(\phi^0, \phi^1)$ .

Consider a backward system (4.1) where  $u$  in  $(H^1(0, T))'$  will be chosen later. Problem (4.1) is well-posed according to Theorem 3.7. Then multiplying (4.1a) by  $\phi$  and integrating by parts, we obtain

$$(4.11) \quad \int_0^\pi \phi^0(x) \mathcal{L}\psi_t(x, 0) - \phi^1(x) \mathcal{L}\psi(x, 0) dx = \int_0^T u(t) (\phi_x(\eta, t) - \phi_x(\xi, t)) dt.$$

Recall the definition (3.15) of  $u$ , this equation is equivalent to

$$(4.12) \quad \begin{aligned} & \int_0^\pi \phi^0(x) \mathcal{L}\psi_t(x, 0) - \phi^1(x) \mathcal{L}\psi(x, 0) dx \\ &= \int_0^T [u_0(t) (\phi_x(\eta, t) - \phi_x(\xi, t)) + u_1(t) (\phi_{xt}(\eta, t) - \phi_{xt}(\xi, t))] dt. \end{aligned}$$

Let  $u_0(t) = 0$  and  $u_1(t) = \phi_{xt}(\eta, t) - \phi_{xt}(\xi, t)$ . Since (3.3),  $u_1$  belongs to  $L^2(0, T)$ , and therefore,  $u = -\frac{\partial u_1}{\partial t}$  belongs to  $(H^1(0, T))'$ , where the derivative  $\frac{\partial}{\partial t}$  is taken in sense of  $(H^1(0, T))'$ . Define a linear operator  $\Lambda_*$  satisfying

$$(4.13) \quad \Lambda_*(\phi^0, \phi^1) = (\mathcal{L}\psi_t(\cdot, 0), -\mathcal{L}\psi(\cdot, 0)).$$

Since  $(\mathcal{L}\psi_t(\cdot, 0), -\mathcal{L}\psi(\cdot, 0))$  belongs to  $Y_{-2} \times Y_{-1}$  because of Theorem 3.7, operator  $\Lambda_*$  is well defined. In particular,

$$(4.14) \quad \langle \Lambda_*(\phi^0, \phi^1), (\phi^0, \phi^1) \rangle = \int_0^T |\phi_{xt}(\eta, t) - \phi_{xt}(\xi, t)|^2 dt.$$

Therefore, we can define a seminorm

$$(4.15) \quad \|(\phi^0, \phi^1)\|_{F_*} := \left( \int_0^T |\phi_{xt}(\eta, t) - \phi_{xt}(\xi, t)|^2 dt \right)^{\frac{1}{2}},$$

for all  $(\phi^0, \phi^1)$  in  $(C^\infty[0, \pi])^2$  satisfying the compatible conditions (1.2).

A classical argument in HUM shows the following proposition.

**PROPOSITION 4.4.** *All initial data in  $Y_{\beta+3} \times Y_{\beta+2}$  are exactly  $(H^1)'$ -controllable in  $(\xi, \eta)$  at time  $T$  if and only if there exists a constant  $c > 0$  such that*

$$(4.16) \quad \int_0^T |\phi_{xt}(\eta, t) - \phi_{xt}(\xi, t)|^2 dt \geq c (\|\phi^0\|_{H^{-\beta}}^2 + \|\phi^1\|_{H^{-\beta-1}}^2)$$

for all  $(\phi^0, \phi^1)$  in  $(C^\infty[0, \pi])^2$  satisfying the compatible conditions (1.2).

After a similar argument in subsection 4.1. We have the following result. Assume  $T > 2\pi\sqrt{\alpha}$ . When  $\frac{\eta+\xi}{2\pi}$  and  $\frac{\eta-\xi}{2\pi}$  belong to  $A$ , there exists a constant  $c > 0$  such that

$$(4.17) \quad \int_0^T |\phi_{xt}(\eta, t) - \phi_{xt}(\xi, t)|^2 dt \geq c \sum_{k \geq 1} \left( \frac{a_k^2 k^2}{1 + \alpha k^2} + \frac{b_k^2}{k^2} \right),$$



which is exactly (4.16) when  $\beta = 0$ . This completes the proof of the first part of Theorem 1.4.

When  $\frac{\eta+\xi}{2\pi}$  and  $\frac{\eta-\xi}{2\pi}$  belong to  $B_\varepsilon$ , there exists a constant  $c > 0$  such that

$$(4.18) \quad \int_0^T |\phi_{xt}(\eta, t) - \phi_{xt}(\xi, t)|^2 dt \geq c \sum_{k \geq 1} \left( \frac{a_k^2 k^2}{(1 + \alpha k^2) k^\varepsilon} + \frac{b_k^2}{k^{2+\varepsilon}} \right),$$

which is exactly (4.16) when  $\beta = \varepsilon$ . This completes the proof of the second part of Theorem 1.4.

**4.3. Condition (1.3) is not sufficient (Proof of Theorem 1.5).** As mentioned in Remark 4.2, condition (1.3) is necessary for controllability. Thanks to Propositions 4.1 and 4.4, to show condition (1.3) is not sufficient for  $L^2$ -controllability, it is sufficient to show there exist  $\xi$  and  $\eta$  satisfying (1.3) such that (4.6) is false for any  $\beta \geq -1$  and  $c > 0$ . Similarly, to show condition (1.3) is not sufficient for  $(H^1)'$ -controllability, it is sufficient to show there exist  $\xi$  and  $\eta$  satisfying (1.3) such that (4.16) is false for any  $\beta \geq -2$  and  $c > 0$ .

For any fixed  $\beta \geq -1$  and

$$(4.19) \quad \nu > \max \left( \frac{3}{2}\beta + 1, 2 \right),$$

we choose

$$(4.20) \quad \frac{\eta + \xi}{2\pi} = \sum_{n=1}^{\infty} \frac{a_n}{10^{n!}},$$

where  $a_n$  belongs to  $\{0, 1, \dots, 9\}$  for all  $n \geq 1$ , and  $a_n$  is not identically zero for great  $n$ . According to [22] the right-hand side of (4.20) is a Liouville number, i.e., it is transcendental and there exists a strictly increasing sequence of integers  $q_n$  such that

$$(4.21) \quad \left| \sin \left( q_n \frac{\eta + \xi}{2\pi} \right) \right| \leq \frac{\pi}{q_n^\nu} \quad \forall n \geq 1.$$

Now consider the sequence of initial data

$$(4.22) \quad \phi_n^0(x) = q_n^\mu \sin(q_n x), \quad \phi_n^1(x) = 0 \quad \forall x \in (0, \pi),$$

where  $\mu = \frac{3}{2}\beta$  if  $\beta > 0$  and  $\mu = 1$  if  $-1 \leq \beta \leq 0$ . Obviously  $(\phi_n^0, \phi_n^1)$  belongs to  $(C^\infty[0, \pi])^2$  and satisfies compatible conditions (1.2) and

$$(4.23) \quad \|\phi_n^0\|_{H^{-\beta}}^2 + \|\phi_n^1\|_{H^{-\beta-1}}^2 \rightarrow \infty \quad \forall \beta \geq -1.$$

So by (4.7), (4.19), and (4.21), we have

$$(4.24) \quad \begin{aligned} & \int_0^T |\phi_{n,x}(\eta, t) - \phi_{n,x}(\xi, t)|^2 dt \\ &= 4 \int_0^T \left| q_n \sin \left( q_n \frac{\eta + \xi}{2\pi} \right) \sin \left( q_n \frac{\eta - \xi}{2\pi} \right) q_n^\mu \cos \left( \frac{q_n^2}{\sqrt{1 + \alpha q_n^2}} t \right) \right|^2 dt \\ &\leq 4T q_n^{2(\mu+1)} \left| \sin \left( q_n \frac{\eta + \xi}{2\pi} \right) \right|^2 \\ &\leq 4\pi T q_n^{2(\mu+1-\nu)} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Relations (4.23) and (4.24) show that (4.6) is false for any  $\beta \geq -1$  and  $c > 0$ .

Then we show that there exist  $\xi$  and  $\eta$  in  $(0, \pi)$  satisfying (1.3) such that (4.16) is false for any  $\beta \geq -2$  and  $c > 0$ . The proof is quite similar to the proof above in this subsection. For any fixed  $\beta \geq -2$ , we only need to change  $\nu$  as

$$(4.25) \quad \nu > \max\left(\frac{3}{2}\beta + 2, 3\right),$$

and to set  $\mu = \frac{3}{2}\beta$  if  $\beta > 0$  and  $\mu = 1$  if  $-2 \leq \beta \leq 0$ . By similar calculation, we obtain that (4.16) is false for any  $\beta \geq -2$  and  $c > 0$ .

**4.4. Non-controllability for less regular initial data set (Proof of Theorem 1.7).** Similar to last subsection, we focus on (4.6) and (4.16). By applying Proposition 2.3, for any  $\xi$  and  $\eta$  in  $(0, \pi)$  we obtain the existence of a strictly increasing sequence of positive integers  $\{q_n\}_{n \geq 1}$  such that

$$(4.26) \quad \left| \sin\left(q_n \frac{\eta + \xi}{2}\right) \right| \leq \frac{\pi}{\sqrt{q_n}}, \quad \left| \sin\left(q_n \frac{\eta - \xi}{2}\right) \right| \leq \frac{\pi}{\sqrt{q_n}} \quad \forall n \geq 1.$$

First consider the sequence of initial data

$$(4.27) \quad \phi_n^0(x) = \sin(q_n x), \quad \phi_n^1(x) = 0 \quad \forall x \in (0, \pi).$$

We note that

$$(4.28) \quad \|\phi_n^0\|_{H^\varepsilon}^2 + \|\phi_n^1\|_{H^{\varepsilon+1}}^2 = C q_n^{2\varepsilon} \rightarrow \infty, \quad n \rightarrow \infty,$$

where  $C$  is a positive constant. By (4.7) and (4.26) we have

$$(4.29) \quad \int_0^T |\phi_{n,x}(\eta, t) - \phi_{n,x}(\xi, t)|^2 dt \\ = 4 \int_0^T \left| q_n \sin\left(q_n \frac{\eta + \xi}{2\pi}\right) \sin\left(q_n \frac{\eta - \xi}{2\pi}\right) \cos\left(\frac{q_n^2}{\sqrt{1 + \alpha q_n^2}} t\right) \right|^2 dt \leq K \quad \forall n \geq 1,$$

where  $K$  is a positive constant. So (4.28) and (4.29) show that (4.6) is false for  $\beta = -\varepsilon$  and arbitrary  $c > 0$ .

If we choose the sequence of initial data

$$(4.30) \quad \phi_n^0(x) = q_n^{-1} \sin(q_n x), \quad \phi_n^1(x) = 0 \quad \forall x \in (0, \pi).$$

We note that

$$(4.31) \quad \|\phi_n^0\|_{H^{\varepsilon+1}}^2 + \|\phi_n^1\|_{H^{\varepsilon+2}}^2 = C q_n^{2\varepsilon} \rightarrow \infty, \quad n \rightarrow \infty,$$

where  $C$  is a positive constant. By (4.26) we have

$$(4.32) \quad \int_0^T |\phi_{n,xt}(\eta, t) - \phi_{n,xt}(\xi, t)|^2 dt \\ = 4 \int_0^T \left| q_n \sin\left(q_n \frac{\eta + \xi}{2\pi}\right) \sin\left(q_n \frac{\eta - \xi}{2\pi}\right) \frac{q_n^2}{\sqrt{1 + \alpha q_n^2}} \cos\left(\frac{q_n}{\sqrt{1 + \alpha q_n^2}} t\right) \right|^2 dt \\ \leq K \quad \forall n \geq 1,$$

where  $K$  is a positive constant. So (4.31) and (4.32) show that (4.16) is false for  $\beta = -\varepsilon - 1$  and arbitrary  $c > 0$ .

**4.5. Non-controllability in short control time (Proof of Theorem 1.8).**

We study the lack of controllability when  $0 < T < 2\pi\sqrt{\alpha}$  in this subsection. For any  $\beta \geq -1$ , we need to find  $\{(\phi_m^0, \phi_m^1)\}_{m \in \mathbb{N}^*}$  such that

$$\int_0^T |\phi_{m,x}(\eta, t) - \phi_{m,x}(\xi, t)|^2 dt \rightarrow 0, \quad \text{as } m \rightarrow \infty$$

and

$$\|\phi_m^0\|_{H^{-\beta}}^2 + \|\phi_m^1\|_{H^{-\beta-1}}^2 \geq c > 0$$

for any  $m \geq 1$  and  $\beta \geq -1$ .

As in subsection 4.1, denote

$$(4.33) \quad \nu_k = -\nu_{-k} = \frac{k^2}{\sqrt{1 + \alpha k^2}}.$$

Obviously,  $\{\nu_k\}_{k \in \mathbb{Z}^*}$  is a strictly increasing sequence and  $\lim_{|k| \rightarrow \infty} |\nu_{k+1} - \nu_k| = 1/\sqrt{\alpha} > 0$ . Define  $N(x, r)$  as in Proposition 2.4 corresponding to  $\{\nu_k\}_{k \in \mathbb{Z}^*}$ , we have the following lemma.

LEMMA 4.5. Let  $\{\nu_k\}_{k \in \mathbb{Z}^*}$  and  $N(x, r)$  be defined above. We have

$$(4.34) \quad \frac{N(x, r)}{r} \rightarrow \sqrt{\alpha}, \quad \text{as } r \rightarrow \infty$$

uniformly relative to  $x$  in  $\mathbb{R}$ .

We will prove this lemma in Appendix A. For any  $0 < T < 2\pi\sqrt{\alpha}$ , we can choose a  $T'$  such that  $0 < T < T' < 2\pi\sqrt{\alpha}$ . Let  $f$  in  $L^2(0, 2\pi\sqrt{\alpha})$  be a real valued function such that  $f(t) = 0$  if  $0 \leq t \leq T$  and  $\|f\|_{L^2(0, T')} \neq 0$ . According to Lemma 4.5 and Proposition 2.4, the family  $\{e^{i\nu_k t}\}_{k \in \mathbb{Z}^*}$  contains a subfamily  $G_{T'}$  which forms a Riesz basis in  $L^2(0, T')$ .

Remark 4.6. The algorithm proposed in the proof of Proposition 2.4 (see Theorem II.4.18 of [3]) allows us to construct a subfamily  $G_{T'}$  with symmetric spectrum, i.e.,

$$e^{i\lambda t} \in G_{T'} \Rightarrow e^{-i\lambda t} \in G_{T'}.$$

This fact is used in our proof.

According to the remark above, there exists a strictly increasing sequence of integer  $\{q_n\}_{n \in \mathbb{Z}^*}$  satisfying  $q_n = -q_{-n}$ , such that  $G_{T'} = \{e^{i\lambda_{q_n} t}\}_{n \in \mathbb{Z}^*}$ . Then for  $f$  in  $L^2(0, T')$  defined above, there exists a sequence  $\{l_n\}_{n \in \mathbb{Z}^*}$  in  $l^2(\mathbb{C})$  such that

$$f(t) = \sum_{n \in \mathbb{Z}^*} l_n e^{i\lambda_{q_n} t} \quad \text{in } L^2(0, T')$$

and

$$0 < \sum_{n \in \mathbb{Z}^*} |l_n|^2 < \infty.$$

Since  $f(t)$  is a real valued function,  $l_n = \overline{l_{-n}}$ . Now we can defined a sequence  $\{(\phi_m^0, \phi_m^1)\}_{m \in \mathbb{N}^*}$  of initial data such that

$$(4.35) \quad \begin{aligned} \phi_m^0(x) &= 2 \sum_{n=1}^m \operatorname{Re}(l_n) \left[ q_n \sin\left(q_n \frac{\eta + \xi}{2}\right) \sin\left(q_n \frac{\eta - \xi}{2}\right) \right]^{-1} \sin(q_n x), \\ \phi_m^1(x) &= -2 \sum_{n=1}^m \operatorname{Im}(l_n) \left[ q_n \sin\left(q_n \frac{\eta + \xi}{2}\right) \sin\left(q_n \frac{\eta - \xi}{2}\right) \right]^{-1} \frac{q_n^2}{\sqrt{1 + \alpha q_n^2}} \sin(q_n x). \end{aligned}$$

As mentioned in Remark 4.2, we assume (1.3) holds. This fact implies the sequence  $\{(\phi_m^0, \phi_m^1)\}_{m \in \mathbb{N}^*}$  of initial data is well-defined.

Since  $0 < \sum_{n \in \mathbb{Z}^*} |l_n|^2 < \infty$  and  $l_n = \overline{l_{-n}}$ , then there exists a  $m_0 \geq 1$  such that  $l_{m_0} \neq 0$ . So for any  $\beta \geq -1$  and  $m \geq m_0$ , we have

$$(4.36) \quad \|\phi_m^0\|_{H^{-\beta}}^2 + \|\phi_m^1\|_{H^{-\beta-1}}^2 \geq \|\phi_{m_0}^0\|_{H^{-\beta}}^2 + \|\phi_{m_0}^1\|_{H^{-\beta-1}}^2 = c > 0.$$

Moreover, thanks to (4.7), we have

$$(4.37) \quad \int_0^T |\phi_{m,x}(\eta, t) - \phi_{m,x}(\xi, t)|^2 dt = 4 \int_0^T \left| \sum_{n=-m, n \neq 0}^m l_n e^{\lambda_{q_n} t} \right|^2 dt.$$

Since  $0 = f(t) = \sum_{n \in \mathbb{Z}^*} l_n e^{\lambda_{q_n} t}$  in  $L^2(0, T)$ , we obtain that

$$(4.38) \quad \int_0^T |\phi_{m,x}(\eta, t) - \phi_{m,x}(\xi, t)|^2 dt \rightarrow 0, \quad \text{as } m \rightarrow \infty.$$

Relations (4.36) and (4.38) finish the proof of the lack of exact  $L^2$ -controllability for any  $\beta \geq -1$ ,  $0 < T < 2\pi\sqrt{\alpha}$  and  $\xi, \eta$  in  $(0, \pi)$ .

As for the lack of exact  $(H^1)'$ -controllability for any  $\beta \geq -2$ ,  $0 < T < 2\pi\sqrt{\alpha}$  and  $\xi, \eta$  in  $(0, \pi)$ , we only need to change the initial data in (4.35) to

$$(4.39) \quad \begin{aligned} \phi_m^0(x) &= 2 \sum_{n=1}^m \operatorname{Re}(l_n) \left[ q_n \sin\left(q_n \frac{\eta + \xi}{2}\right) \sin\left(q_n \frac{\eta - \xi}{2}\right) \right]^{-1} \frac{\sqrt{1 + \alpha q_n^2}}{q_n^2} \sin(q_n x), \\ \phi_m^1(x) &= -2 \sum_{n=1}^m \operatorname{Im}(l_n) \left[ q_n \sin\left(q_n \frac{\eta + \xi}{2}\right) \sin\left(q_n \frac{\eta - \xi}{2}\right) \right]^{-1} \sin(q_n x). \end{aligned}$$

Similarly, (4.38) still holds for any  $\beta \geq -2$  and  $m \geq m_0$ . Moreover, we have

$$(4.40) \quad \int_0^T |\phi_{m,x}(\eta, t) - \phi_{m,x}(\xi, t)|^2 dt = 4 \int_0^T \left| \sum_{n=-m, n \neq 0}^m l_n e^{\lambda_{q_n} t} \right|^2 dt \rightarrow 0, \quad \text{as } m \rightarrow \infty.$$

Relations (4.36) and (4.40) finish the proof of the lack of exact  $(H^1)'$ -controllability for any  $\beta \geq -2$ ,  $0 < T < 2\pi\sqrt{\alpha}$  and  $\xi, \eta$  in  $(0, \pi)$ .

**Appendix A. Proof of Lemma 4.5.** Solve  $\frac{k^2}{\sqrt{1+\alpha k^2}} < r$ , we have  $k < [(\alpha r^2 + r\sqrt{\alpha^2 r^2 + 4})/2]^{\frac{1}{2}}$ . Thus we define a real function  $g : (0, +\infty) \rightarrow (0, +\infty)$  as  $g(r) = [(\alpha r^2 + r\sqrt{\alpha^2 r^2 + 4})/2]^{\frac{1}{2}}$  and define  $\{x\} = q(x) - 1$ , where  $q(x) = \min_{q \in \mathbb{Z}} \{q \geq x\}$ . Obviously,  $x - 1 \leq \{x\} < x$ . Then we have  $N(0, r) = \{g(r)\}$ . Notice that for  $x \geq 0$ , we have  $N(x, r) = N(0, x+r) - N(0, x) = \{g(x+r)\} - \{g(x)\}$ . Therefore,

$$(4.41) \quad g(x+r) - g(x) - 1 \leq N(x, r) \leq g(x+r) - g(x) + 1.$$

Now we need to estimate  $g(x+r) - g(x)$  for  $x, r > 0$ . Let  $f(x) = \frac{x^2}{\sqrt{1+\alpha x^2}}$  for  $x > 0$ . Then

$$(4.42) \quad \begin{aligned} f'(x) &= 2x(1 + \alpha x^2)^{-\frac{1}{2}} - \alpha x^3(1 + \alpha x^2)^{-\frac{3}{2}}, \\ f''(x) &= 2(1 + \alpha x^2)^{-\frac{1}{2}} - 5\alpha x^2(1 + \alpha x^2)^{-\frac{3}{2}} + 3\alpha^2 x^4(1 + \alpha x^2)^{-\frac{5}{2}}. \end{aligned}$$

Solve  $f''(x) > 0$ , we obtain  $0 < x < \sqrt{2/\alpha}$ . Therefore, there exists a  $N_0$  in  $\mathbb{N}$  satisfying  $N_0 > \sqrt{2/\alpha} + 1$  such that for all  $n \geq N_0$ ,  $\lambda_{n+1} - \lambda_n$  is decreasing to  $1/\sqrt{\alpha}$ . Then for all  $r > 0$ ,  $N(x, r)$  is increasing in  $x \geq N_0$ . Then for  $x \geq N_0$ ,  $N(N_0, r) \leq N(x, r) \leq \lim_{x \rightarrow +\infty} N(x, r)$ . Simple calculation shows that  $\lim_{x \rightarrow +\infty} [g(x+r) - g(x)] = \sqrt{\alpha}r$ . Therefore, we obtain that for  $x \geq N_0$ ,

$$g(N_0 + r) - g(N_0) - 1 \leq N(x, r) \leq \sqrt{\alpha}r + 1.$$

For  $0 \leq x \leq N_0$ , we have

$$\begin{aligned} N(x, r) &\leq N(x, N_0 - x) + N(N_0, r) = N(0, N_0) - N(0, x) + N(N_0, r) \\ &\leq N(0, N_0) + N(N_0, r) \leq \sqrt{\alpha}r + 1 + g(N_0). \end{aligned}$$

Assume that  $r > N_0$ , then we have

$$\begin{aligned} N(x, r) &\geq N(N_0, r - N_0 + x) = N(0, x + r) - N(0, N_0) \\ &\geq g(x + r) - g(N_0) - 1 \geq \min_{x \in [0, N_0]} g(x + r) - g(N_0) - 1. \end{aligned}$$

Then for all  $x \geq 0$  and  $r > N_0$ , we have

$$\min_{x \in [0, N_0]} g(x + r) - g(N_0) - 1 \leq N(x, r) \leq \sqrt{\alpha}r + 1 + g(N_0),$$

and hence,

$$(A.1) \quad \lim_{r \rightarrow \infty} \frac{N(x, r)}{r} = \sqrt{\alpha}$$

is uniformly relative to  $x \geq 0$ .

For  $x < 0$ , if  $|x| \leq r$ , then

$$N(0, |x|) + N(0, r - |x|) \leq N(x, r) \leq N(0, |x|) + N(0, r - |x|) + 1.$$

Let  $\theta$  belong to  $[0, 1]$ , and we consider  $g(\theta r) + g((1 - \theta)r)$ . Notice that

$$g(\theta r) = \left( \frac{\alpha\theta^2 r^2 + \theta r \sqrt{\alpha^2 \theta^2 r^2 + 4}}{2} \right)^{\frac{1}{2}} = \frac{r}{\sqrt{2}} \left( \alpha\theta^2 + \theta \sqrt{\alpha^2 \theta^2 + 4/r^2} \right)^{\frac{1}{2}}.$$

Therefore, we obtain  $\lim_{r \rightarrow \infty} g(\theta r)/r = \sqrt{\alpha}\theta$ . Consequently, (A.1) holds uniformly relative to  $-r \leq x < 0$ .

If  $|x| > r$ , then

$$N(|x| - r, r) \leq N(x, r) \leq N(|x| - r, r) + 1.$$

Set  $t = |x| - r \geq 0$ , then as same as in the situation  $x \geq 0$ , we have (A.1) holds uniformly relative to  $t \geq 0$ , which means that (A.1) holds uniformly relative to  $|x| > r$ .

Combining all the situations, we obtain that (A.1) holds uniformly relative to  $x$  in  $\mathbb{R}$ . Lemma 4.5 is thus proved.

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