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Alexei Ya. Kanel-Belov, Igor Melnikov and Ivan Mitrofanov

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Algebra / Algèbre

On cogrowth function of algebras and its logarithmical gap

Sur la fonction de co-croissance des algèbres et son écart logarithmique

Alexei Ya. Kanel-Belov^a, Igor Melnikov^b and Ivan Mitrofanov^{*,c}

^a Bar Ilan University, Ramat-Gan, Israel

^b Moscow Institute of Physics and Technology, Dolgoprudny, Russia

^c C.N.R.S., École Normale Supérieure, PSL Research University, France

E-mails: kanelster@gmail.com, melnikov_ig@mail.ru, phortim@yandex.ru

Abstract. Let $A \cong k\langle X \rangle / I$ be an associative algebra. A finite word over alphabet X is *I-reducible* if its image in A is a k -linear combination of length-lexicographically lesser words. An *obstruction* is a subword-minimal I -reducible word. If the number of obstructions is finite then I has a finite Gröbner basis, and the word problem for the algebra is decidable. A *cogrowth* function is the number of obstructions of length $\leq n$. We show that the cogrowth function of a finitely presented algebra is either bounded or at least logarithmical. We also show that a uniformly recurrent word has at least logarithmical cogrowth.

Résumé. Soit $A \cong k\langle X \rangle / I$ une algèbre associative. Un mot fini sur l'alphabet X est *I-réductible* si son image dans A est une combinaison linéaire k de mots de longueur lexicographiquement moindre. Une *obstruction* dans un mot minimal I -réductible. Si le nombre d'obstructions est fini, alors I a une base finie Gröbner, et le mot problème pour l'algèbre est décidable. Une fonction *co-croissance* est le nombre d'obstructions de longueur $\leq n$. Nous montrons que la fonction de co-croissance d'une algèbre finement présentée est soit bornée, soit au moins logarithmique. Nous montrons également qu'un mot uniformément récurrent a au moins une co-croissance logarithmique.

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* Corresponding author.

1. Cogrowth of associative algebras

Let A be a finitely generated associative algebra over a field k . Then $A \cong k\langle X \rangle / I$, where $k\langle X \rangle$ is a free algebra with generating set $X = \{x_1, \dots, x_s\}$ and I is a two-sided *ideal of relations*. Further we assume the generating set is fixed. Let “ $<$ ” be a well-ordering of X , $x_1 < \dots < x_s$. This order can be extended to a linear order on the set $\langle X \rangle$ of monomials of $k\langle X \rangle$, i.e. finite words in alphabet X : $u_1 < u_2$ if $|u_1| < |u_2|$ or $|u_1| = |u_2|$ and $u_1 <_{lex} u_2$. Here $|\cdot|$ denotes the length of a word, i.e. the degree of a monomial, and $<_{lex}$ is the lexicographical order. For $f \in k\langle X \rangle$ we denote by \widehat{f} its leading (with respect to $<$) monomial. An algebra $k\langle X \rangle / I$ is said to be *finitely presented* if I is a finitely generated ideal.

We call a monomial $w \in \langle X \rangle$ *I-reducible* if $w = \widehat{f}$ for some relation $f \in I$. In the opposite case, we call w *I-irreducible*. Denote the set of all monomials of degree at most n by $\langle X \rangle_{\leq n}$. Let $A_n \subseteq A$ be the image of $\langle X \rangle_{\leq n}$ under the canonical map. The *growth* $V_A(n)$ is the dimension of the linear span of A_n . It is easily shown that $V_A(n)$ is equal to the number of *I-irreducible* monomials in $\langle X \rangle_{\leq n}$.

We call a monomial $w \in \langle X \rangle$ an *obstruction* for I if w is *I-reducible*, but any proper subword of w is *I-irreducible*. The *cogrowth* of algebra A is defined as the function $O_A(n)$, the number of obstructions of length $\leq n$.

The celebrated Bergman gap theorem says that the growth function $V_A(n)$ is either constant, linear or no less than $(n+1)(n+2)/2$ [2]. In this section we give a non-trivial bound on the cogrowth function for finitely presented algebras.

Theorem 1. *Let A be a finitely presented algebra. Then the cogrowth function $O_A(n)$ is either constant or no less than logarithmic: $O_A(n) \geq \log_2(n) - C$. The constant C depends only on the maximal length of a relation.*

Recall that a *Gröbner basis* of an ideal I is a subset $G \subseteq I$ such that for any $f \in I$ there exists $g \in G$ such that the leading monomial of f contains the leading monomial of g as a subword. One of Gröbner bases can be obtained by taking for each obstruction u a relation $f_u \in I$ such that $\widehat{f_u} = u$.

If f and g are two elements of $k\langle X \rangle$, $g \in I$ and the word \widehat{g} is a subword of \widehat{f} , then f can be replaced by f' such that $f' - f \in I$ and $\widehat{f'} < \widehat{f}$. This operation is called a *reduction*.

Let f and g be two elements of $k\langle X \rangle$. If $u_1 u_2 = \widehat{f}$ and $u_2 u_3 = \widehat{g}$ for some $u_1, u_2, u_3 \in \langle X \rangle$, then the word $u_1 u_2 u_3$ is called a *composition* of f and g , and the normed element $f u_3 - u_1 g$ is the *result of this composition*.

Lemma 2 (Diamond Lemma [3]). *Let two-sided ideal I be generated by a subset U of a free associative algebra $k\langle X \rangle$. Suppose that*

- (i) *there are no $f, g \in U$ such that \widehat{g} is a proper subword of \widehat{f} , and*
- (ii) *for any two elements $f, g \in U$ the result of any their composition can be reduced to 0 after finitely many reductions with elements from U .*

Then the set U is a Gröbner basis of I .

Example. Consider the associative algebra $A \cong k\langle x, y \rangle / I$, where I is a two-sided ideal generated by $f = x^2 - yx$. It can be shown that the set $\{xy^i x - y^{i+1} x \mid i \geq 0\}$ is a Gröbner basis of I , so $O_A(n) = n - 1$ for $n \geq 2$. A monomial is *I-irreducible* if and only if it contains at most one letter x , hence $V_A(n) = (n+1)(n+2)/2$.

Theorem 1 directly follows from

Lemma 3. *Let $A \cong k\langle X \rangle / I$ be a finitely presented algebra and let N be greater than the maximal length of its defining relation. Suppose there are no obstructions of length from the interval $[N, 2N]$. Then I has a finite Gröbner basis.*

Proof. Let S be the set of all obstructions in $\langle X \rangle_{\leq N}$. Take for each monomial $w \in S$ a relation f_w such that $\widehat{f}_w = w$. Let us show that this set $\{f_w \mid w \in S\}$ forms a Gröbner basis for I . Indeed, I is generated by the set $\{f_w \mid w \in S\}$. The condition (i) of the Diamond Lemma holds automatically because no obstruction can be a proper subword of another obstruction. Let us check the condition (ii).

Let $u, v \in S$ and let h be the result of some composition of f_u and f_v . It is clear that the leading monomial of h has length less than $2N$. We start reducing h with elements from $\{f_w \mid w \in S\}$. After finally many steps we obtain either 0 or an element h' such that \widehat{h}' does not contain subwords from S . But since there are no obstructions from $[N, 2N]$, the second case is impossible. \square

The *word problem* for a finitely presented algebra, i.e. the question whether a given element $f \in k\langle X \rangle$ lies in I , is undecidable in the general case. But if I has a finite Gröbner basis G , then A has a decidable word problem. Note also that the problem whether a given element in a finitely presented associative algebra is a zero divisor (or is it nilpotent) is undecidable, even if we are given a finite Gröbner basis [6]. But if the ideal of relations is generated by monomials and has a finite Gröbner basis, the nilpotency problem is algorithmically decidable [2].

2. Colength of a period

A *monomial algebra* is a finitely generated associative algebra whose defining relations are monomials. Let u be a finite word in alphabet X and let A_u be the algebra $k\langle X \rangle/I$, where I is generated by the set of monomials that are not subwords of the periodic sequence u^∞ . Such algebras A_u play important role in the study of monomial algebras [2].

Let W be a sequence on alphabet X , i.e. a map $X^{\mathbb{N}}$. A finite word v is an *obstruction* for W if v is not a subword of W but any proper subword v' of v is a subword of W . If u is a finite word, the number of obstructions for u^∞ is always finite. We call this number the *colength* of the period u . We say that the period is *defined* by the set of obstructions.

In [5], G. R. Chelnokov proved that a sequence of minimal period n cannot be defined by fewer than $\log_2 n + 1$ obstructions. G. R. Chelnokov also gave for infinitely many n_i an example of a binary sequence with minimal period n_i and colength of the period $\log_\varphi n_i$, where $\varphi = \frac{\sqrt{5}+1}{2}$. P. A. Lavrov found the precise lower estimation for colength of period.

Theorem 4 (cf. [7]). *Let $A = \{a, b\}$ be a binary alphabet. Let u be a word of length n and colength c , then $\varphi_c \geq n$, where φ_c is the c -th Fibonacci number ($\varphi_1 = 1, \varphi_2 = 2, \varphi_3 = 3, \varphi_4 = 5$ etc.).*

The case of an arbitrary alphabet was considered in [8] by P. A. Lavrov and independently in [4] by I. I. Bogdanov and G. R. Chelnokov.

3. Cogrowth function for an uniformly recurrent sequence

A sequence of letters W on a finite alphabet is called *uniformly recurrent* (u.r. for brevity) if for any finite subword u of W there exists a number $C(u, W)$ such that any subword of W having length $C(u, W)$ contains u as a subword. This property can be considered as a generalization of periodicity [9].

For a sequence of letters W denote by A_W the algebra $k\langle X \rangle/I_W$, where I_W is generated by the set of monomials that are not subwords of W . A monomial algebra A is called *almost simple* if each of its proper factor algebras $B = A/I$ is nilpotent. In [2] it was shown that almost simple monomial algebras are algebras of the form A_W , where W is an u.r. sequence.

Again, a finite word u is an *obstruction* for W if it is not a subword of W but any its proper subword is a subword of W . The *cogrowth function* $O_W(n)$ is the number of obstructions with length $\leq n$.

Theorem 5. *Let W be an u.r. non-periodic sequence on a binary alphabet. Then $\lim_{n \rightarrow \infty} O_W(n) / \log_3 n \geq 1$.*

A *factorial language* is a set \mathcal{U} of finite words such that for any $u \in \mathcal{U}$ all subwords of u also belong to \mathcal{U} . Denote by \mathcal{U}_k the words of \mathcal{U} having length k . A finite word u is called an *obstruction* for \mathcal{U} if $u \notin \mathcal{U}$, but any proper subword belongs to \mathcal{U} . Denote the factorial language consisting of all subwords of a given sequence W by $\mathcal{L}(W)$. To prove Theorem 5 we will assume the contrary and construct an infinite factorial language that is a proper subset of $\mathcal{L}(W)$.

Let \mathcal{U} be a factorial language and k be an integer. The *Rauzy graph* $R_k(\mathcal{U})$ of order k is the directed graph with vertex set \mathcal{U}_k and edge set \mathcal{U}_{k+1} . Two vertices u_1 and u_2 of $R_k(\mathcal{U})$ are connected by an edge u_3 if and only if $u_3 \in \mathcal{U}$, u_1 is a prefix of u_3 , and u_2 is a suffix of u_3 .

For a sequence W we denote the graph $R_k(\mathcal{L}(W))$ by $R_k(W)$. Further the word *graph* will always mean a directed graph, the word *path* will always mean a *directed path* in a directed graph. The *length* $|p|$ of a path p is the number of its vertices, i.e. the number of edges plus one. If a path p_2 starts at the end of a path p_1 , we denote their concatenation by $p_1 p_2$. Recall that a directed graph is *strongly connected* if for every pair of vertices $\{v_1, v_2\}$ it contains a directed path from v_1 to v_2 and a directed path from v_2 to v_1 . It is clear that any Rauzy graph of an u.r. non-periodic sequence is a strongly connected digraph and is not a cycle.

Given a directed graph H , its *directed line graph* $L(H)$ is a directed graph such that each vertex of $L(H)$ represents an edge of H , and two vertices of $L(H)$ that represent edges e_1 and e_2 of H are connected by an arrow from e_1 to e_2 if and only if the head of e_1 meets the tail of e_2 . For any $k > 0$ there is one-to-one correspondence between paths of length k in $L(H)$ and paths of length $k + 1$ in H .

Let \mathcal{U} be a factorial language and let $m \geq n$. A word $a_1 \dots a_m \in \mathcal{U}_m$ corresponds to a path of length $m - n + 1$ in $R_n(\mathcal{U})$, this path visits vertices $a_1 \dots a_n, a_2 \dots a_{n+1}, \dots, a_{m-n+1} \dots a_m$. The graph $R_m(\mathcal{U})$ can be considered as a subgraph of $L^{m-n}(R_n(\mathcal{U}))$. Moreover, the graph $R_{n+1}(\mathcal{U})$ is obtained from $L(R_n(\mathcal{U}))$ by removing edges that correspond to obstructions of length $n + 1$.

We call a vertex v of a directed graph H a *fork* if v has out-degree more than one. Furthermore we assume that all forks have out-degrees exactly 2 (this is the case of a binary alphabet). For a directed graph H we define its *entropy regulator*: $er(H)$ is the minimal integer such that any directed path of length $er(H)$ in H contains at least one vertex that is a fork in H .

Proposition 6. *Let H be a strongly connected digraph that is not a cycle. Then $er(H) < \infty$.*

Proof. Assume the contrary. Let n be the total number of vertices in H . Consider a path of length $n + 1$ in H that does not contain forks. Note that this path visits some vertex v at least twice. This means that starting from v it is possible to obtain only vertices of this cycle. Since the graph H is strongly connected, H coincides with this cycle. \square

Lemma 7. *Let H be a strongly connected digraph, $er(H) = K$. Then $er(L(H)) = K$.*

Proof. The forks of the digraph $L(H)$ are edges in H that end at forks. Consider K vertices forming a path in $L(H)$. This path corresponds to a path of length $K + 1$ in H . Since $er(H) \leq K$, there exists an edge of this path that ends at a fork. \square

Lemma 8. *Let H be a strongly connected digraph, $er(H) = K$, let v be a fork in H , the edge e starts at v . Let the digraph H^* be obtained from H by removing the edge e . Let G be a subgraph of H^* that consists of all vertices and edges reachable from v . Then G is a strongly connected digraph. Also G is either a cycle of length at most K , or $er(G) \leq 2K$.*

Proof. First we prove that the digraph G is strongly connected. Let v' be an arbitrary vertex of G , then there is a path in G from v to v' . Consider a path p of minimum length from v' to v in H . Such a path exists, for otherwise H is not strongly connected. The path p does not contain the

edge e , for otherwise it could be shortened. This means that p connects v' to v in the digraph G . From any vertex of G we can reach the vertex v , hence G is strongly connected.

Consider an arbitrary path p of length $2K$ in the digraph G , suppose that p does not have forks. Since $\text{er}(H) = K$, then in p there are two vertices v_1 and v_2 which are forks in H and there are no forks in p between v_1 and v_2 . The out-degrees of all vertices except v coincide in H and G . If $v_1 \neq v$ or $v_2 \neq v$, then we find a vertex of p that is a fork in G . If $v_1 = v_2 = v$, then there is a cycle C in G such that $|C| \leq K$ and C does not contain forks of G . Since G is a strongly connected graph, it coincides with this cycle C . \square

Corollary 9. *Let W be a binary u.r. non-periodic sequence, then for any n*

$$\text{er}(R_{n-1}(W)) \leq 2^{O_W(n)}.$$

Proof. We prove this by induction on n . The base case $n = 0$ is obvious. Let $\text{er}(R_{n-1}(W)) = K$ and suppose W has exactly a obstructions of length $n + 1$. These obstructions correspond to paths of length 2 in the graph $R_{n-1}(W)$, i.e. edges of the graph $H := L(R_{n-1}(W))$. From Lemma 7 we have that $\text{er}(H) = K$. The graph $R_n(W)$ is obtained from the graph H by removing some edges e_1, e_2, \dots, e_a . Since W is a u.r. sequence, the digraphs H and $H - \{e_1, e_2, \dots, e_a\}$ are strongly connected. This means that the edges e_1, \dots, e_a start at different forks of H . We also know that $R_n(W)$ is not a cycle. The graph $R_n(W)$ can be obtained by removing edges e_i from H one by one. Applying Lemma 8 a times, we show that $\text{er}(R_n(W)) \leq 2^a K$, which completes the proof. \square

Lemma 10. *Let H be a strongly connected digraph, $\text{er}(H) = K$, $k \geq 3K$. Let u be an arbitrary edge of the graph $L^k(H)$. Then the digraph $L^k(H) - u$ contains a strongly connected subgraph B such that $\text{er}(B) \leq 3K$.*

Proof. Consider in H the path p_u of length $k + 2$, corresponding to u . Divide first k vertices of p_u into three subpaths of length at least K . Since $\text{er}(H) = K$, each of these subpaths contains a fork (some of these forks can coincide). Next, we consider three cases.

Case 1. Assume that the path p_u visits at least two different forks of H . Then p_u contains a subpath of the form pe , where p is a path connecting two different forks v_1 and v_2 (and not containing other forks) and e is an edge starting at v_2 . It is clear that the length of p_1 does not exceed $K + 1$. Lemma 8 implies that there is a strongly connected subgraph G of H such that G contains the vertex v_2 but does not contain the edge e_2 .

If G is not a cycle, then $\text{er}(G) \leq 2K$. Hence, the graph $B := L^k(G)$ is a subgraph of $L^k(H)$, and from Lemma 7 we have $\text{er}(B) \leq 2K$. It is also clear that the digraph B does not contain the edge u .

If G is a cycle, we denote it by p_1 and denote its first edge by e_1 (we assume that v_2 is the first and last vertex of p_1). The length of p_1 does not exceed K . Among the vertices of p_1 there are no forks of H besides v_2 . Therefore, $v_1 \notin p_1$. Call a path t in H good, if t does not contain the subpath pe . Let us show that for any good path s in H there are two different paths s_1 and s_2 starting at the end of s such that $|s_1| = |s_2| = 3K$ and the paths ss_1, ss_2 are also good.

It is clear that for any good path we can add an edge such that the new path is also good. There is a path $t_1, |t_1| < K$ such that st_1 is a good path and ends at some fork v . If $v \neq v_2$, then two edges e_i, e_j start at v , the paths st_1e_i and st_1e_j are good, and each of them can be prolonged further to a good path of arbitrary length. If $v = v_2$, then the paths st_1p_1e and $st_1p_1e_1$ are good and can be extended.

Consider in $L^k(H)$ a subgraph that consists of all vertices and edges that are good paths in H , let B be a strongly connected component of this subgraph. It is clear that $\text{er}(B) \leq 3K$ and the digraph B does not contain the edge u .

Case 2. Assume that the path p_u visits exactly one fork v_1 (at least 3 times), but there are forks besides v_1 in H . There are two edges e_1 and e_2 that start at v_1 . Starting with these edges and

moving until forks, we obtain two paths p_1 and p_2 . The edge e_1 is the first edge of p_1 , the edge e_2 is the first of p_2 , and $|p_1|, |p_2| \leq K$. We can assume that p_1 is a subpath of p_u . Then p_1 ends at v_1 (and is a cycle) and p_2 ends at some fork $v_2 \neq v_1$ (if $v_1 = v_2$, then v_1 is the only fork reachable from v_1). We complete the proof as in the previous case: $p_1 e_1$ is a subpath of p_u . We call a path *good* if it does not contain $p_1 e_1$. As above, we can show that if s is a good path in H , then there are two different paths s_1 and s_2 such that $|s_1| = |s_2| = 3L$ and the paths ss_1, ss_2 are also good.

As above, B will be a strongly connected component in the subgraph of $L^k(H)$ that consists of vertices and edges corresponding to good paths in H .

Case 3. Assume that there is only one fork v in H . Then there are two cycles p_1 and p_2 of length $\leq K$ that start and end at v . Let e_1 be the first edge of p_1 and let e_2 be the first edge of p_2 . The path p_u contains one of the following subpaths: $p_1 e_1, p_2 e_2, p_1 p_1 e_2$ or $p_2 p_2 e_1$. Denote this path by t . Call a path *good* if it does not contain t . A simple check shows that we can complete the proof as in the previous cases. \square

Proof of Theorem 5. Arrange all the obstructions u_i of the u.r. binary sequence W by their length in non-descending order. If $\lim_{k \rightarrow \infty} \frac{\log_3 |u_k|}{k} \leq 1$, then the statement of the Theorem holds. If $\lim_{k \rightarrow \infty} \frac{\log_3 |u_k|}{k} > 1$ then the sequence $|u_k|/3^k$ tends to infinity. Hence, there exists n_0 such that $|u_{n_0}|/3^{n_0} > 10$ and $|u_n|/3^n > |u_{n_0}|/3^{n_0}$ for all $n > n_0$. In this situation, $|u_{n_0+k}| > |u_{n_0}| + 4 \cdot 2^{n_0} \cdot 3^k$ for any $k > 0$.

Let $v_i = u_i$ if $1 \leq i \leq n_0$ and let v_i be a subword of u_i of length $|u_{n_0}| + 4 \cdot 2^{n_0} \cdot 3^{i-n_0}$ if $i > n_0$. Denote by \mathcal{U} the set of all finite binary words that do not contain subwords from $\{v_i\}$. It is clear that \mathcal{U} is a proper subset of $\mathcal{L}(W)$. We get a contradiction with the uniform recurrence of W if we show that the language \mathcal{U} is infinite. The Rauzy graph $R_{u_{n_0-1}}(\mathcal{U})$ is equal to $R_{u_{n_0-1}}(W)$, and from Corollary 9 we have $\text{er}(R_{u_{n_0-1}}(\mathcal{L})) \leq 2^{n_0}$.

By induction on n we show that for all $n \geq n_0$ the graph $R_{|v_n|-1}(\mathcal{U})$ contains a strongly connected subgraph H_n such that $\text{er}(H_n) \leq 3^{n-n_0} \cdot 2^{n_0}$. We already have the base case $n = n_0$. The graph $R_{|v_{n+1}|-1}(\mathcal{U})$ is obtained from $L^{|v_{n+1}|-|v_n|}(R_{|v_n|-1})$ by removing at most one edge. Note that $|v_{n+1}| - |v_n| > 3 \cdot \text{er}(H_n)$, so we can use Lemma 10 for the digraph H_n and $k = |v_{n+1}| - |v_n|$. This completes the inductive step.

All the graphs $R_{|v_n|-1}(\mathcal{U})$ are nonempty and, therefore, the language \mathcal{U} is infinite. \square

For a sequence W over an alphabet $A = \{a_1, \dots, a_k\}$ of size k , we replace in W each letter a_i by $0^i 1$ and obtain a binary sequence W' . If W is u.r. and non-periodic, then W' is also u.r. and non-periodic. It is clear that all long enough obstructions of W' correspond to some of the obstructions of W , so we obtain

Corollary 11. *Let W be an u.r. non-periodic sequence on a finite alphabet. Then $\lim_{n \rightarrow \infty} O_W(n)/\log_3 n \geq 1$.*

Example. Consider a finite alphabet $\{0, 1\}$ and the sequence of words u_i , defined recursively as $u_0 = 0, u_1 = 01, u_k = u_{k-1}u_{k-2}$ for $k \geq 2$. Since u_i is a prefix of u_{i+1} , the sequence (u_i) has a limit, called a *Fibonacci word* $F = 0100101001001\dots$. In Example 25 of [1] the set $\{11, 000, 10101, 00100100, \dots\}$ of obstructions of F is described. These words have lengths equal to Fibonacci numbers. Since the Fibonacci word is u.r., in Theorem 5 we cannot replace the constant 3 by a number smaller than $\frac{\sqrt{5}+1}{2}$.

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