HAL
open science

On cogrowth function of algebras and its logarithmical gap

Alexei Ya. Kanel-Belov, Igor Melnikov, Ivan Mitrofanov

To cite this version:

Alexei Ya. Kanel-Belov, Igor Melnikov, Ivan Mitrofanov. On cogrowth function of algebras and its logarithmical gap. Comptes Rendus. Mathématique, 2021, 359 (3), pp.297-303. 10.5802/crmath.170 . hal-03964302

HAL Id: hal-03964302

https://hal.science/hal-03964302v1

Submitted on 31 Jan 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

Alexei Ya. Kanel-Belov, Igor Melnikov and Ivan Mitrofanov

On cogrowth function of algebras and its logarithmical gap

Volume 359, issue 3 (2021), p. 297-303

Published online: 20 April 2021

https://doi.org/10.5802/crmath.170

This article is licensed under the
Creative Commons Attribution 4.0 International License.
http://creativecommons.org/licenses/by/4.0/

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
2021, Vol. 359, 3, p. 297-303
https://doi.org/10.5802/crmath.170

Algebra / *Algèbre*

On cogrowth function of algebras and its logarithmical gap

Sur la fonction de co-croissance des algèbres et son écart logarithmique

Alexei Ya. Kanel-Belov[a], **Igor Melnikov**[b] **and Ivan Mitrofanov**[*, c]

[a] Bar Ilan University, Ramat-Gan, Israel

[b] Moscow Institute of Physics and Technology, Dolgoprudny, Russia

[c] C.N.R.S., École Normale Superieur, PSL Research University, France

E-mails: kanelster@gmail.com, melnikov_ig@mail.ru, phortim@yandex.ru

Abstract. Let $A \cong k\langle X\rangle / I$ be an associative algebra. A finite word over alphabet X is *I-reducible* if its image in A is a k-linear combination of length-lexicographically lesser words. An *obstruction* is a subword-minimal I-reducible word. If the number of obstructions is finite then I has a finite Gröbner basis, and the word problem for the algebra is decidable. A *cogrowth* function is the number of obstructions of length $\leq n$. We show that the cogrowth function of a finitely presented algebra is either bounded or at least logarithmical. We also show that an uniformly recurrent word has at least logarithmical cogrowth.

Résumé. Soit $A \cong k\langle X\rangle / I$ une algèbre associative. Un mot fini sur l'alphabet X est *I-réductible* si son image dans A est une combinaison linéaire k de mots de longueur lexicographiquement moindre. Une *obstruction* dans un mot minimal I -réductible. Si le nombre d'obstructions est fini, alors I a une base finie Gröbner, et le mot problème pour l'algèbre est décidable. Une fonction *co-croissance* est le nombre d'obstructions de longueur $\leq n$. Nous montrons que la fonction de co-croissance d'une algèbre finement présentée est soit bornée, soit au moins logarithmique. Nous montrons également qu'un mot uniformément récurrent a au moins une co-croissance logarithmique.

Funding. The paper was supported by Russian Science Foundation (grant no. 17-11-01377).

Manuscript received 6 July 2020, revised and accepted 14 December 2020.

* Corresponding author.

1. Cogrowth of associative algebras

Let A be a finitely generated associative algebra over a field k. Then $A \cong k\langle X\rangle / I$, where $k\langle X\rangle$ is a free algebra with generating set $X = \{x_1, \dots, x_s\}$ and I is a two-sided *ideal of relations*. Further we assume the generating set is fixed. Let "$\prec$" be a well-ordering of X, $x_1 \prec \dots \prec x_s$. This order can be extended to a linear order on the set $\langle X\rangle$ of monomials of $k\langle X\rangle$, i.e. finite words in alphabet X: $u_1 \prec u_2$ if $|u_1| < |u_2|$ or $|u_1| = |u_2|$ and $u_1 <_{lex} u_2$. Here $|\cdot|$ denotes the length of a word, i.e. the degree of a monomial, and $<_{lex}$ is the lexicographical order. For $f \in k\langle X\rangle$ we denote by $\widehat{f}$ its leading (with respect to $\prec$) monomial. An algebra $k\langle X\rangle / I$ is said to be *finitely presented* if I is a finitely generated ideal.

We call a monomial $w \in \langle X\rangle$ *I-reducible* if $w = \widehat{f}$ for some relation $f \in I$. In the opposite case, we call w *I-irreducible*. Denote the set of all monomials of degree at most n by $\langle X\rangle_{\leq n}$. Let $A_n \subseteq A$ be the image of $\langle X\rangle_{\leq n}$ under the canonical map. The *growth* $V_A(n)$ is the dimension of the linear span of A_n. It is easily shown that $V_A(n)$ is equal to the number of I-irreducible monomials in $\langle X\rangle_{\leq n}$.

We call a monomial $w \in \langle X\rangle$ an *obstruction* for I if w is I-reducible, but any proper subword of w is I-irreducible. The *cogrowth* of algebra A is defined as the function $O_A(n)$, the number of obstructions of length $\leqslant n$.

The celebrated Bergman gap theorem says that the growth function $V_A(n)$ is either constant, linear of no less than $(n+1)(n+2)/2$ [2]. In this section we give a non-trivial bound on the cogrowth function for finitely presented algebras.

Theorem 1. *Let A be a finitely presented algebra. Then the cogrowth function $O_A(n)$ is either constant or no less than logarithmic: $O_A(n) \geq \log_2(n) - C$. The constant C depends only on the maximal length of a relation.*

Recall that a *Gröbner basis* of an ideal I is a subset $G \subseteq I$ such that for any $f \in I$ there exists $g \in G$ such that the leading monomial of f contains the leading monomial of g as a subword. One of Gröbner bases can be obtained by taking for each obstruction u a relation $f_u \in I$ such that $\widehat{f_u} = u$.

If f and g are two elements of $k\langle X\rangle$, $g \in I$ and the word $\widehat{g}$ is a subword of $\widehat{f}$, then f can be replaced by f' such that $f' - f \in I$ and $\widehat{f'} \prec \widehat{f}$. This operation is called a *reduction*.

Let f and g be two elements of $k\langle X\rangle$. If $u_1 u_2 = \widehat{f}$ and $u_2 u_3 = \widehat{g}$ for some $u_1, u_2, u_3 \in \langle X\rangle$, then the word $u_1 u_2 u_3$ is called a *composition* of f and g, and the normed element $f u_3 - u_1 g$ is the *result of this composition*.

Lemma 2 (Diamond Lemma [3]). *Let two-sided ideal I be generated by a subset U of a free associative algebra $k\langle X\rangle$. Suppose that*

(i) *there are no $f, g \in U$ such that $\widehat{g}$ is a proper subword of $\widehat{f}$, and*
(ii) *for any two elements $f, g \in U$ the result of any their composition can be reduced to* 0 *after finitely many reductions with elements from U.*

Then the set U is a Gröbner basis of I.

Example. Consider the associative algebra $A \cong k\langle x, y\rangle / I$, where I is a two-sided ideal generated by $f = x^2 - yx$. It can be shown that the set $\{xy^i x - y^{i+1}x \mid i \geq 0\}$ is a Gröbner basis of I, so $O_A(n) = n - 1$ for $n \geq 2$. A monomial is I-irreducible if and only if it contains at most one letter x, hence $V_A(n) = (n+1)(n+2)/2$.

Theorem 1 directly follows from

Lemma 3. *Let $A \cong k\langle X\rangle / I$ be a finitely presented algebra and let N be greater than the maximal length of its defining relation. Suppose there are no obstructions of length from the interval $[N, 2N]$. Then I has a finite Gröbner basis.*

Proof. Let S be the set of all obstructions in $\langle X\rangle_{\leq N}$. Take for each monomial $w \in S$ a relation f_w such that $\widehat{f_w} = w$. Let us show that this set $\{f_w \mid s \in S\}$ forms a Gröbner basis for I. Indeed, I is generated by the set $\{f_w \mid w \in S\}$. The condition (i) of the Diamond Lemma holds automatically because no obstruction can be a proper subword of another obstruction. Let us check the condition (ii).

Let $u, v \in S$ and let h be the result of some composition of f_u and f_v. It is clear that the leading monomial of h has length less then $2N$. We start reducing h with elements from $\{f_w \mid w \in S\}$. After finally many steps we obtain either 0 or an element h' such that $\widehat{h'}$ does not contain subwords from S. But since there are no obstructions from $[N, 2N]$, the second case is impossible. □

The *word problem* for a finitely presented algebra, i.e. the question whether a given element $f \in k\langle X\rangle$ lies in I, is undecidable in the general case. But if I has a finite Gröebner basis G, then A has a decidable word problem. Note also that the problem whether a given element in a finitely presented associative algebra is a zero divisor (or is it nilpotent) is undecidable, even if we are given a finite Gröebner basis [6]. But if the ideal of relations is generated by monomials and has a finite Gröebner basis, the nilpotency problem is algorithmically decidable [2].

2. Colength of a period

A *monomial algebra* is a finitely generated associative algebra whose defining relations are monomials. Let u be a finite word in alphabet X and let A_u be the algebra $k\langle X\rangle/I$, where I is generated by the set of monomials that are not subwords of the periodic sequence u^∞. Such algebras A_u play important role in the study of monomial algebras [2].

Let W be a sequence on alphabet X, i.e. a map $X^{\mathbb{N}}$. A finite word v is an *obstruction* for W if v is not a subword of W but any proper subword v' of v is a subword of W. If u is a finite word, the number of obstructions for u^∞ is always finite. We call this number the *colength* of the period u. We say that the period is *defined by* the set of obstructions.

In [5], G. R. Chelnokov proved that a sequence of minimal period n cannot be defined by fewer than $\log_2 n + 1$ obstructions. G. R. Chelnokov also gave for infinitely many n_i an example of a binary sequence with minimal period n_i and colength of the period $\log_\varphi n_i$, where $\varphi = \frac{\sqrt{5}+1}{2}$. P. A. Lavrov found the precise lower estimation for colength of period.

Theorem 4 (cf. [7]). *Let $A = \{a, b\}$ be a binary alphabet. Let u be a word of length n and colength c, then $\varphi_c \geq n$, where φ_c is the c-th Fibonacci number ($\varphi_1 = 1$, $\varphi_2 = 2$, $\varphi_3 = 3$, $\varphi_4 = 5$ etc.).*

The case of an arbitrary alphabet was considered in [8] by P. A. Lavrov and independently in [4] by I. I. Bogdanov and G. R. Chelnokov.

3. Cogrowth function for an uniformly recurrent sequence

A sequence of letters W on a finite alphabet is called *uniformly recurrent* (u.r. for brevity) if for any finite subword u of W there exists a number $C(u, W)$ such that any subword of W having length $C(u, W)$ contains u as a subword. This property can be considered as a generalization of periodicity [9].

For a sequence of letters W denote by A_W the algebra $k\langle X\rangle/I_W$, where I_W is generated by the set of monomials that are not subwords of W. A monomial algebra A is called *almost simple* if each of its proper factor algebras $B = A/I$ is nilpotent. In [2] it was shown that almost simple monomial algebras are algebras of the form A_W, where W is an u.r. sequence.

Again, a finite word u is an *obstruction* for W if it is not a subword of W but any its proper subword is a subword of W. The *cogrowth function* $O_W(n)$ is the number of obstructions with length $\leqslant n$.

Theorem 5. *Let W be an u.r. non-periodic sequence on a binary alphabet. Then* $\overline{\lim}_{n\to\infty} O_W(n)/\log_3 n \geq 1$.

A *factorial language* is a set $\mathscr{U}$ of finite words such that for any $u \in \mathscr{U}$ all subwords of u also belong to $\mathscr{U}$. Denote by $\mathscr{U}_k$ the words of $\mathscr{U}$ having length k. A finite word u is called an *obstruction* for $\mathscr{U}$ if $u \notin \mathscr{U}$, but any proper subword belongs to $\mathscr{U}$. Denote the factorial language consisting of all subwords of a given sequence W by $\mathscr{L}(W)$. To prove Theorem 5 we will assume the contrary and construct an infinite factorial language that is a proper subset of $\mathscr{L}(W)$.

Let $\mathscr{U}$ be a factorial language and k be an integer. The *Rauzy graph* $R_k(\mathscr{U})$ of order k is the directed graph with vertex set $\mathscr{U}_k$ and edge set $\mathscr{U}_{k+1}$. Two vertices u_1 and u_2 of $R_k(\mathscr{U})$ are connected by an edge u_3 if and only if $u_3 \in \mathscr{U}$, u_1 is a prefix of u_3, and u_2 is a suffix of u_3.

For a sequence W we denote the graph $R_k(\mathscr{L}(W))$ by $R_k(W)$. Further the word *graph* will always mean a directed graph, the word *path* will always mean a *directed path* in a directed graph. The *length* $|p|$ of a path p is the number of its vertices, i.e. the number of edges plus one. If a path p_2 starts at the end of a path p_1, we denote their concatenation by p_1p_2. Recall that a directed graph is *strongly connected* if for every pair of vertices $\{v_1, v_2\}$ it contains a directed path from v_1 to v_2 and a directed path from v_2 to v_1. It is clear that any Rauzy graph of an u.r. non-periodic sequence is a strongly connected digraph and is not a cycle.

Given a directed graph H, its *directed line graph* $L(H)$ is a directed graph such that each vertex of $L(H)$ represents an edge of H, and two vertices of $L(H)$ that represent edges e_1 and e_2 of H are connected by an arrow from e_1 to e_2 if and only if the head of e_1 meets the tail of e_2. For any $k > 0$ there is one-to-one correspondence between paths of length k in $L(H)$ and paths of length $k+1$ in H.

Let $\mathscr{U}$ be a factorial language and let $m \geqslant n$. A word $a_1 \dots a_m \in \mathscr{U}_m$ corresponds to a path of length $m-n+1$ in $R_n(\mathscr{U})$, this path visits vertices $a_1 \dots a_n$, $a_2 \dots a_{n+1}$, $\dots$, $a_{m-n+1} \dots a_m$. The graph $R_m(\mathscr{U})$ can be considered as a subgraph of $L^{m-n}(R_n(\mathscr{U}))$. Moreover, the graph $R_{n+1}(\mathscr{U})$ is obtained from $L(R_n(\mathscr{U}))$ by removing edges that correspond to obstructions of length $n+1$.

We call a vertex v of a directed graph H *a fork* if v has out-degree more than one. Furthermore we assume that all forks have out-degrees exactly 2 (this is the case of a binary alphabet). For a directed graph H we define its *entropy regulator*: $er(H)$ is the minimal integer such that any directed path of length $\mathrm{er}(H)$ in H contains at least one vertex that is a fork in H.

Proposition 6. *Let H be a strongly connected digraph that is not a cycle. Then* $\mathrm{er}(H) < \infty$.

Proof. Assume the contrary. Let n be the total number of vertices in H. Consider a path of length $n+1$ in H that does not contain forks. Note that this path visits some vertex v at least twice. This means that starting from v it is possible to obtain only vertices of this cycle. Since the graph H is strongly connected, H coincides with this cycle. □

Lemma 7. *Let H be a strongly connected digraph,* $\mathrm{er}(H) = K$. *Then* $\mathrm{er}(L(H)) = K$.

Proof. The forks of the digraph $L(H)$ are edges in H that end at forks. Consider K vertices forming a path in $L(H)$. This path corresponds to a path of length $K+1$ in H. Since $\mathrm{er}(H) \leq K$, there exists an edge of this path that ends at a fork. □

Lemma 8. *Let H be a strongly connected digraph,* $\mathrm{er}(H) = K$, *let v be a fork in H, the edge e starts at v. Let the digraph H^* be obtained from H by removing the edge e. Let G be a subgraph of H^* that consists of all vertices and edges reachable from v. Then G is a strongly connected digraph. Also G is either a cycle of length at most K, or* $\mathrm{er}(G) \leq 2K$.

Proof. First we prove that the digraph G is strongly connected. Let v' be an arbitrary vertex of G, then there is a path in G from v to v'. Consider a path p of minimum length from v' to v in H. Such a path exists, for otherwise H is not strongly connected. The path p does not contain the

edge e, for otherwise it could be shortened. This means that p connects v' to v in the digraph G. From any vertex of G we can reach the vertex v, hence G is strongly connected.

Consider an arbitrary path p of length $2K$ in the digraph G, suppose that p does not have forks. Since $\mathrm{er}(H) = K$, then in p there are two vertices v_1 and v_2 which are forks in H and there are no forks in p between v_1 and v_2. The out-degrees of all vertices except v coincide in H and G. If $v_1 \neq v$ or $v_2 \neq v$, then we find a vertex of p that is a fork in G. If $v_1 = v_2 = v$, then there is a cycle C in G such that $|C| \leq K$ and C does not contain forks of G. Since G is a strongly connected graph, it coincides with this cycle C. □

Corollary 9. *Let W be a binary u.r. non-periodic sequence, then for any n*

$$\mathrm{er}(R_{n-1}(W)) \leq 2^{O_W(n)}.$$

Proof. We prove this by induction on n. The base case $n = 0$ is obvious. Let $\mathrm{er}(R_{n-1}(W)) = K$ and suppose W has exactly a obstructions of length $n+1$. These obstructions correspond to paths of length 2 in the graph $R_{n-1}(W)$, i.e. edges of the graph $H := L(R_{n-1}(W))$. From Lemma 7 we have that $\mathrm{er}(H) = K$. The graph $R_n(W)$ is obtained from the graph H by removing some edges $e_1, e_2, \ldots, e_a$. Since W is a u.r. sequence, the digraphs H and $H - \{e_1, e_2, \ldots, e_a\}$ are strongly connected. This means that the edges $e_1, \ldots, e_a$ start at different forks of H. We also know that $R_n(W)$ is not a cycle. The graph $R_n(W)$ can be obtained by removing edges e_i from H one by one. Applying Lemma 8 a times, we show that $\mathrm{er}(R_n(W)) \leq 2^a K$, which completes the proof. □

Lemma 10. *Let H be a strongly connected digraph,* $\mathrm{er}(H) = K$, $k \geq 3K$. *Let u be an arbitrary edge of the graph $L^k(H)$. Then the digraph $L^k(H) - u$ contains a strongly connected subgraph B such that* $\mathrm{er}(B) \leq 3K$.

Proof. Consider in H the path p_u of length $k+2$, corresponding to u. Divide first k vertices of p_u into three subpaths of length at least K. Since $\mathrm{er}(H) = K$, each of these subpaths contains a fork (some of these forks can coincide). Next, we consider three cases.

Case 1. Assume that the path p_u visits at least two different forks of H. Then p_u contains a subpath of the form pe, where p is a path connecting two different forks v_1 and v_2 (and not containing other forks) and e is an edge starting at v_2. It is clear that the length of p_1 does not exceed $K+1$. Lemma 8 implies that there is a strongly connected subgraph G of H such that G contains the vertex v_2 but does not contain the edge e_2.

If G is not a cycle, then $\mathrm{er}(G) \leq 2K$. Hence, the graph $B := L^k(G)$ is a subgraph of $L^k(H)$, and from Lemma 7 we have $\mathrm{er}(B) \leq 2K$. It is also clear that the digraph B does not contain the edge u.

If G is a cycle, we denote it by p_1 and denote its first edge by e_1 (we assume that v_2 is the first and last vertex of p_1). The length of p_1 does not exceed K. Among the vertices of p_1 there are no forks of H besides v_2. Therefore, $v_1 \notin p_1$. Call a path t in H *good*, if t does not contain the subpath pe. Let us show that for any good path s in H there are two different paths s_1 and s_2 starting at the end of s such that $|s_1| = |s_2| = 3K$ and the paths ss_1, ss_2 are also good.

It is clear that for any good path we can add an edge such that the new path is also good. There is a path t_1, $|t_1| < K$ such that st_1 is a good path and ends at some fork v. If $v \neq v_2$, then two edges e_i, e_j start at v, the paths st_1e_i and st_2e_j are good, and each of them can be prolonged further to a good path of arbitrary length. If $v = v_2$, then the paths st_1p_1e and $st_1p_1e_1$ are good and can be extended.

Consider in $L^k(H)$ a subgraph that consists of all vertices and edges that are good paths in H, let B be a strongly connected component of this subgraph. It is clear that $\mathrm{er}(B) \leq 3K$ and the digraph B does not contain the edge u.

Case 2. Assume that the path p_u visits exactly one fork v_1 (at least 3 times), but there are forks besides v_1 in H. There are two edges e_1 and e_2 that start at v_1. Starting with these edges and

moving until forks, we obtain two paths p_1 and p_2. The edge e_1 is the first edge of p_1, the edge e_2 is the first of p_2, and $|p_1|, |p_2| \le K$. We can assume that p_1 is a subpath of p_u. Then p_1 ends at v_1 (and is a cycle) and p_2 ends at some fork $v_2 \ne v_1$ (if $v_1 = v_2$, then v_1 is the only fork reachable from v_1). We complete the proof as in the previous case: $p_1 e_1$ is a subpath of p_u. We call a path *good* if it does not contain $p_1 e_1$. As above, we can show that if s is a good path in H, then there are two different paths s_1 and s_2 such that $|s_1| = |s_2| = 3L$ and the paths ss_1, ss_2 are also good.

As above, B will be a strongly connected component in the subgraph of $L^k(H)$ that consists of vertices and edges corresponding to good paths in H.

Case 3. Assume that there is only one fork v in H. Then there are two cycles p_1 and p_2 of length $\le K$ that start and end at v. Let e_1 be the first edge of p_1 and let e_2 be the first edge of p_2. The path p_u contains one of the following subpaths: $p_1 e_1$, $p_2 e_2$, $p_1 p_1 e_2$ or $p_2 p_2 e_1$. Denote this path by t. Call a path *good* if it does not contain t. A simple check shows that we can complete the proof as in the previous cases. □

Proof of Theorem 5. Arrange all the obstructions u_i of the u.r. binary sequence W by their length in non-descending order. If $\underline{\lim}_{k\to\infty} \frac{\log_3 |u_k|}{k} \le 1$, then the statement of the Theorem holds. If $\underline{\lim}_{k\to\infty} \frac{\log_3 |u_k|}{k} > 1$ then the sequence $|u_k|/3^k$ tends to infinity. Hence, there exists n_0 such that $|u_{n_0}|/3^{n_0} > 10$ and $|u_n|/3^n > |u_{n_0}|/3^{n_0}$ for all $n > n_0$. In this situation, $|u_{n_0+k}| > |u_{n_0}| + 4 \cdot 2^{n_0} \cdot 3^k$ for any $k > 0$.

Let $v_i = u_i$ if $1 \le i \le n_0$ and let v_i be a subword of u_i of length $|u_{n_0}| + 4 \cdot 2^{n_0} \cdot 3^{i-n_0}$ if $i > n_0$. Denote by $\mathscr{U}$ the set of all finite binary words that do not contain subwords from $\{v_i\}$. It is clear that $\mathscr{U}$ is a proper subset of $\mathscr{L}(W)$. We get a contradiction with the uniform recurrence of W if we show that the language $\mathscr{U}$ is infinite. The Rauzy graph $R_{u_{n_0}-1}(\mathscr{U})$ is equal to $R_{u_{n_0}-1}(W)$, and from Corollary 9 we have $\mathrm{er}(R_{u_{n_0}-1}(\mathscr{L})) \le 2^{n_0}$.

By induction on n we show that for all $n \ge n_0$ the graph $R_{|v_n|-1}(\mathscr{U})$ contains a strongly connected subgraph H_n such that $\mathrm{er}(H_n) \le 3^{n-n_0} \cdot 2^{n_0}$. We already have the base case $n = n_0$. The graph $R_{|v_{n+1}|-1}(\mathscr{U})$ is obtained from $L^{|v_{n+1}|-|v_n|}(R_{|v_n|-1})$ by removing at most one edge. Note that $|v_{n+1}| - |v_n| > 3 \cdot \mathrm{er}(H_n)$, so we can use Lemma 10 for the digraph H_n and $k = |v_{n+1}| - |v_n|$. This completes the inductive step.

All the graphs $R_{|v_n|-1}(\mathscr{U})$ are nonempty and, therefore, the language $\mathscr{U}$ is infinite. □

For a sequence W over an alphabet $A = \{a_1, \dots, a_k\}$ of size k, we replace in W each letter a_i by $0^i 1$ and obtain a binary sequence W'. If W is u.r. and non-periodic, then W' is also u.r. and non-periodic. It is clear that all long enough obstructions of W' correspond to some of the obstructions of W, so we obtain

Corollary 11. *Let W be an u.r. non-periodic sequence on a finite alphabet. Then* $\lim_{n\to\infty} O_W(n)/\log_3 n \ge 1$.

Example. Consider a finite alphabet $\{0,1\}$ and the sequence of words u_i, defined recursively as $u_0 = 0$, $u_1 = 01$, $u_k = u_{k-1}u_{k-2}$ for $k \ge 2$. Since u_i is a prefix of u_{i+1}, the sequence (u_i) has a limit, called a *Fibonacci word* $F = 0100101001001\dots$. In Example 25 of [1] the set $\{11, 000, 10101, 00100100, \dots\}$ of obstructions of F is described. These words have lengths equal to Fibonacci numbers. Since the Fibonacci word is u.r., in Theorem 5 we cannot replace the constant 3 by a number smaller than $\frac{\sqrt{5}+1}{2}$.

References

[1] M.-P. Béal, F. Mignosi, A. Restivo, M. Sciortino, "Forbidden Words in Symbolic Dynamics", *Adv. Appl. Math.* **25** (2020), no. 2, p. 163-193.

[2] A. Y. Belov, V. V. Borisenko, V. N. Latyshev, "Monomial algebras", *J. Math. Sci., New York* **87** (1997), no. 3, p. 3463-3575.

[3] G. M. Bergman, “The diamond lemma for ring theory”, *Adv. Math.* **29** (1978), no. 2, p. 178-218.

[4] I. I. Bogdanov, G. R. Chelnokov, “The maximal length of the period of a periodic word defined by restrictions”, https://arxiv.org/abs/1305.0460, 2013.

[5] G. R. Chelnokov, “On the number of restrictions defining a periodic sequence”, *Model and Analysis of Inform. Systems* **14** (2007), no. 2, p. 12-16, in Russian.

[6] I. Ivanov-Pogodaev, S. Malev, “Finite Gröbner basis algebras with unsolvable nilpotency problem and zero divisors problem”, *J. Algebra* **508** (2018), p. 575-588.

[7] P. A. Lavrov, “Number of restrictions required for periodic word in the finite alphabet”, https://arxiv.org/abs/1209.0220, 2012.

[8] ———, “Minimal number of restrictions defining a periodic word”, https://arxiv.org/abs/1412.5201, 2014.

[9] A. A. Muchnik, Y. L. Pritykin, A. L. Semenov, “Sequences close to periodic”, *Russ. Math. Surv.* **64** (2009), no. 5, p. 805-871.