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TOTAL ORDERS ON COMPACT METRIC SPACES AND COVERING DIMENSION

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ABSTRACT. We prove that for a compact metric space the property of having finite covering dimension is equivalent to the existence of a total order with finite snake number.

Keywords: total order · covering dimension · small inductive dimension.

1. INTRODUCTION

The following concept was introduced in [2]. Let M be a metric space and let T be a total order on M (that is, for any distinct points $x, y \in M$ either $x <_T y$ or $y <_T x$). Then for the ordered space (M, T) one calculates *the order breakpoint* $\text{Br}(M, T) \in \mathbb{N} \cup \{+\infty\}$. The value $\min_T \text{Br}(M, T)$ is a quasi-isometry invariant for uniformly discrete metric spaces.

It was shown in [3] that a metric space with finite Assouad-Nagata dimension admits an order T for which $\text{Br}(M, T) < \infty$. It was also shown that some properties contradicting to finiteness of Assouad-Nagata dimension also forbid orders with finite Br , and it was raised a question whether the existence of orders with finite Br is equivalent to finiteness of Assouad-Nagata dimension.

For metric spaces there are various definitions of dimension based on coverings by sets with bounded intersection multiplicity. In order to establish similar results for *Lebesgue covering dimension*, we introduce the notion of *snake number* as an analogue of Br .

Definition 1. *Let M be a metric space, T be a total order on M , and let (U_1, U_2) be an ordered pair of subsets of M . We say that (U_1, U_2) contains a snake of length s if there exists a sequence of points*

$$a_0 <_T a_1 <_T \cdots <_T a_s$$

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such that U_1 contains the points $a_0, a_2, \dots, a_{2\lfloor s/2 \rfloor}$ and U_2 contains the points $a_1, a_3, \dots, a_{2\lfloor s/2 \rfloor - 1}$.

Definition 2. Let (M, T) be an ordered metric space and let x, y be distinct points of M . The snake number $\text{Snake}_T(x, y)$ is the greatest number s such that for any $\varepsilon > 0$ the pair of neighborhoods $(B_\varepsilon(x), B_\varepsilon(y))$ contains a snake of length s . If the greatest such number does not exist, we say that $\text{Snake}(x, y) = \infty$.

Definition 3. Let M be a metric space and T be a total order on M . The Snake number of the ordered space (M, T) is defined as

$$\text{Snake}(M, T) := \sup_{x, y \in M, x \neq y} \text{Snake}_T(x, y).$$

The main result of this paper is

Theorem 1.1. Let M be a compact metric space. Then the following conditions are equivalent:

- (1) the covering dimension $\dim M < \infty$
- (2) there exists an order T on M such that $\text{Snake}(M, T) < \infty$.

Let us specify the upper and lower bounds.

Theorem 1.2. Let (M, T) be an ordered compact metric space such that $\text{Snake}(M, T) \leq s$. Then $\dim M \leq s$.

Theorem 1.3. Let M be a compact metric space with $\dim M \leq s$. Then there exists an order T for which $\text{Snake}(M, T) \leq 2s + 1$.

One can find similarities with the following classical result.

Theorem 1.4 ([4]). Let M be a compact metric space with covering dimension $\dim M \geq n$. Suppose $f : A \rightarrow M$ is a continuous mapping of a closed subspace A of the Cantor set C onto the space M . Then there exist $n + 1$ points $x_0, \dots, x_n \in A$ such that

$$f(x_1) = \dots = f(x_n)$$

(In fact, Hurewicz proved this result for small inductive dimension ind that is equal to \dim for compact metric spaces). From Theorem 1.2 one can conclude by passing to the limit the following fact

Proposition 1.5. Let M be a compact metric space with $\dim M \geq n$. Suppose $f : A \rightarrow M$ is a continuous mapping of a closed subspace A of \mathbb{R} onto the space M . Then there exist $n + 1$ points $x_0 < \dots < x_n$ such that

$$f(x_0) = f(x_2) = \dots = f(x_{2\lfloor n/2 \rfloor})$$

and

$$f(x_1) = f(x_3) = \cdots = f(x_{2^{\lfloor n/2 \rfloor - 1}}).$$

2. PRELIMINARIES

2.1. Snake numbers of a segment and of a circle.

Example 2.1. Obviously, for the segment $I = [0, 1]$ and the natural order T ($x <_T y$ if $x < y$) we have $\text{Snake}(I, T) = 1$.

Example 2.2. If we glue the ends of the segment, we obtain the circle S^1 . We take the order T from Example 2.1 and specify that $0 = 1 <_T x$ for any other point x . It is easy to see that if $0 < x, y < 1$, then $\text{Snake}(x, y) = 1$, and $\text{Snake}(0, x) = 2$, so $\text{Snake}(S^1, T) = 2$.

Proposition 2.3. $\text{Snake}(S^1, T') \geq 2$ for any order T' .

Proof. For any point $x \in S^1$ there is an antipodal point $A(x)$. We colour each point x blue if $x <_T A(x)$ or red if $x >_T A(x)$. Since S^1 contains points of both colours, there exists x_0 that belongs to the boundary of the blue set. For a natural n , choose in $B_{1/n}(x_0)$ a blue point b_n and a red point r_n .

If $A(b_n) <_T r_n$, then $(B_{1/n}(x_0), B_{1/n}(A(x_0)))$ contains the snake $b_n <_T A(b_n) <_T r_n$ of length 2. Otherwise, $(B_{1/n}(A(x_0)), B_{1/n}(x_0))$ contains the snake $A(r_n) <_T r_n <_T A(b_n)$ of length 2.

One of these two events occurs for infinitely many different n , hence for arbitrarily large n . \square

We will use the following properties of ind and dim ([1]):

- (1) (Monotonicity) if M_1 is a subspace of M then $\text{ind } M_1 \leq \text{ind } M$;
- (2) (The sum theorem) if a separable metric space M can be represented as the union of a sequence F_1, F_2, \dots of closed subspaces such that $\text{ind } F_i \leq n$ for $i = 1, 2, \dots$, then $\text{ind } M \leq n$;
- (3) (The coincidence theorem) for every separable metric space M we have $\text{ind } M = \text{dim } M$.

2.2. Some basics of the topological dimension theory.

Definition 4. Let X be a set and \mathcal{A} a family of subsets of X . By the order of the family \mathcal{A} we mean the largest integer n such that the family \mathcal{A} contains $n + 1$ sets with a non-empty intersection; in no such integer exists, we say that the family \mathcal{A} has order ∞ . A cover \mathcal{B} is a refinement of another cover \mathcal{A} of the same space if for every $B \in \mathcal{B}$ there exists an $A \in \mathcal{A}$ such that $B \subset A$.

Definition 5. To every metric space M one assigns the covering dimension of M , denoted by $\dim M$, which is ∞ or an integer larger than or equal to -1 ; the definition of the function \dim is defined by the following conditions:

- (1) $\dim M \leq n$, where $n = -1, 0, 1, \dots$, if every finite open cover of the space M has a finite open refinement of order $\leq n$;
- (2) $\dim M = n$ if $\dim M \leq n$ and $\dim M > n$, i.e. the inequality $\dim M \leq n - 1$ does not hold;
- (3) $\dim M = \infty$ if $\dim M > n$ for $n = -1, 0, 1, \dots$

Definition 6. To every metric space M one assigns the small inductive dimension of M , denoted by $\text{ind } M$, which is ∞ or an integer larger than or equal to -1 ; the definition of the function ind is defined by the following conditions:

- (1) $\text{ind } M = -1$ if and only if $M = \emptyset$;
- (2) $\text{ind } M \leq n$, where $n = 0, 1, \dots$, if for every point $x \in M$ and each $r > 0$ there exists an open set $U \subset M$ such that $x \in U \subset B_\varepsilon(x)$ and the dimension of its boundary $\text{ind } \partial U \leq n - 1$;
- (3) $\text{ind } M = n$ if $\text{ind } M > n$;
- (4) $\text{ind } M = \infty$ if $\text{ind } M > 0, 1, 2, \dots$

3. PROOF OF THEOREM 1.2

Lemma 3.1. Let (M, T) be an ordered metric space and let

$$\text{Snake}(M, T) \leq n.$$

Let M be a disjoint union $M = A \sqcup B$, and suppose that $x_1 <_T x_2$ for all $x_1 \in A, x_2 \in B$. Then there exists an order T' on the boundary ∂A such that $\text{Snake}(\partial A, T') \leq n - 1$.

Proof. For distinct points $x_0, y_0 \in \partial A$ we set $x <_{T'} y$ if for any $\varepsilon > 0$ there exist $x \in B_\varepsilon(x_0) \cap A$ and some $\delta > 0$ such that $x <_T B_\delta(y_0) \cap A$, i.e.

$$x <_T y \text{ for any } y \in B_\delta(y_0) \cap A.$$

We first show that for any x_0, y_0 at least one of the relations $x_0 <_{T'} y_0$ and $y_0 <_{T'} x_0$ holds. Denote $X_k = B_{1/k}(x_0) \cap A, Y_k = B_{1/k}(y_0) \cap A$. For all sufficiently large k , the pairs (X_k, Y_k) and (Y_k, X_k) do not contain snakes of length $n + 1$. Consider the largest m such that there is a sequence of points $t_1 <_T t_2 <_T \dots <_T t_m$ such that

(i) all t_i with odd i belong to X_k and all t_i with even i belong to Y_k

or

(ii) all t_i with odd i belong to Y_k and all t_i with even i belong to X_k .
holds.

If $t_1 \in X_k$, then $t_1 <_T y$ for any $y \in Y_k$, otherwise the sequence $y <_T t_1 <_T \cdots <_T t_m$ is longer. Similarly, if $t_1 \in Y_k$, then $t_1 <_T x$ for any $x \in Y_k$. If the situation (i) holds for infinitely many k 's, then $x_0 <_{T'} y_0$, otherwise $y_0 <_{T'} x_0$.

Assume that for $m > 1$ there exists a cycle

$$x_0 <_{T'} x_1 <_{T'} \cdots <_{T'} x_m = x_1$$

Take $\varepsilon_0 > 0$. The set $B_{\varepsilon_0}(x_0) \cap A$ contains a point $y_0 <_T B_{\varepsilon_2}(x_1) \cap A$ for some $\varepsilon_2 < \varepsilon_1$. Similarly, in $B_{\varepsilon_1}(x_1) \cap A$ there is a point $y_1 <_T B_{\varepsilon_2}(x_2) \cap A$ for some $\varepsilon_2 < \varepsilon_1$, and so on, we construct a sequence

$$y_0 <_T y_1 <_T y_2 <_T \cdots$$

The points y_0, y_m, y_{2m}, \dots belong to $B_{\varepsilon_0}(x_0) \cap A$, the points $y_1, y_{m+1}, y_{2m+1}, \dots$ belong to $B_{\varepsilon_1}(x_1) \cap A$. Since ε_0 was arbitrary, $\text{Snake}(x_0, x_1) = \infty$. This contradiction shows that the relations $x <_{T'} y$ and $y <_{T'} x$ cannot hold simultaneously and that the relation $<_{T'}$ is transitive. So T' is indeed an order relation.

Finally, assume that $\text{Snake}_{T'}(x, y) \geq n$ for some points $x, y \in \partial A$. Take an arbitrary $\varepsilon > 0$. Then there is a sequence of points

$$t_0 <_{T'} t_1 <_{T'} \cdots <_{T'} t_n,$$

the points with odd indices belong to $B_{\varepsilon}(x) \cap \partial A$, and the points with even indices belong to $B_{\varepsilon}(y) \cap \partial A$.

Choose $\delta > 0$ and as before, we can find a sequence of points $r_0 <_T r_1 <_T \cdots <_T r_n$ such that $y_i \in B_{\delta}(t_i) \cap A$ for $i = 0, 1, \dots, n$. Both $B_{\varepsilon}(x)$ and $B_{\varepsilon}(y)$ contain points from B , hence $(B_{\varepsilon+\delta}(x), B_{\varepsilon+\delta}(y))$ contains a snake of length $n + 1$. This leads us to a contradiction. \square

Definition 7. Let (M, T) be an ordered metric space. We call a subset $X \subset M$ T -convex if for any triple of points $x <_T y <_T z$ such that $x, z \in X$ the point y also $\in X$. For a subset $A \subset M$ we define its T -convex hull as the smallest T -convex subset of M that contains A .

Lemma 3.2. Let (M, T) be an ordered compact metric space such that $\text{Snake}(M, T) < \infty$. Let $x_0 \in M$ and $r > 0$. Then there is a finite family of T -convex subsets X_1, \dots, X_m such that their union covers $M - B_r(x_0)$ and does not intersect some neighbourhood $B_{\varepsilon}(x_0)$ of x_0 .

Proof. Let $\text{Snake}(M, T) = s - 1 < \infty$. For any $y \neq x_0$ we can choose $\varepsilon_y > 0$ such that $(B_{\varepsilon_y}(y), B_{\varepsilon_y}(x_0))$ does not contain a snake of length s . The union of sets $B_{\varepsilon_y}(y)$ covers $M - \{x_0\}$. The set $N = \{x \in X :$

$d(x_0, x) \geq r\}$ is compact, so there exists $\varepsilon > 0$ and a finite set $Y \subset M$ such that

- (1) $N \subset \bigcup_{y \in Y} B_\varepsilon(y)$,
- (2) for any $y \in Y$ the pair $(B_\varepsilon(y), B_\varepsilon(x_0))$ does not contain a snake of length s .

Denote $U := B_\varepsilon(x_0)$. We introduce an equivalence relation on the points of the set N :

$a \sim b$ if the T -convex hull of the set $\{a, b\}$ does not intersect U .

It is easy to see that \sim is indeed an equivalence relation. Suppose that for some $y \in Y$ the ball $B_\varepsilon(y)$ intersects at least $s + 1$ of \sim -equivalence classes. Choose a point in each class:

$$c_0 <_T c_2 <_T \cdots <_T c_s.$$

Since c_0 and c_1 belong to different \sim -equivalence classes, U contains a point b_0 such that $c_0 <_T b_0 <_T c_1$. Similarly, we found points b_1, b_2, \dots, b_{s-1} and get that $(B_\varepsilon(y), B_\varepsilon(x_0))$ contains a snake of length $2s$, we obtain a contradiction.

Since each $B_\varepsilon(y)$ intersects at most s \sim -equivalence classes, the total number of classes m is finite and $m \leq s|Y|$. Hence the T -convex hulls of these \sim -equivalence classes satisfy the requirements of the Lemma. \square

Now we are ready to prove Theorem 1.2.

Proof. We proceed by induction on n . If $\text{Snake}(M, T) \leq 0$, then $|M| \leq 1$ and $\dim M \leq 0$. Suppose the statement is true for $n - 1$. We use the definition of small inductive dimension, we want to show that for every point $x \in M$ and each $r > 0$ there exists an open set $U \subset M$ such that $x \in U \subset B_r(x)$ and $\text{ind } \partial U \leq n - 1$.

Using Lemma 2 we find T -convex sets X_1, \dots, X_m such that their union covers $M - B_r(x)$ and does not intersect some neighborhood $B_\varepsilon(x)$. Let X be the closure of the union of all X_i , $i = 1, \dots, m$. We succeed if we show that $\text{ind } \partial X \leq n - 1$.

Since $\partial X \subset \bigcup \partial X_i$, it's enough to show that $\text{ind } B \leq n - 1$ for any T -convex set B and to use the monotonicity property of ind .

But indeed, M can be partitioned into three sets A , B and C such that all points of B are T -less than all points of A and all points of C are T -greater than all points of A . Since $\partial(A \cup B)$ and $\partial(A \cup C)$ are compact spaces, by Lemma 1 and the induction hypothesis we have $\text{ind}(\partial(A \cup B)) \leq n - 1$ and $\text{ind}(\partial(A \cup C)) \leq n - 1$. So $\text{ind } \partial B \leq n - 1$. \square

4. PROOF OF THEOREM 1.3

Let M be a compact metric space with $\dim M \leq n$.

First we construct a sequence \mathcal{U}_i , $i = 0, 1, \dots$ of open covers of the space M satisfying the following properties

- (1) \mathcal{U}_i is a refinement of \mathcal{U}_{i-1} for $i = 1, 2, \dots$;
- (2) every $U \in \mathcal{U}_i$ has diameter $\leq 2^{-i}$;
- (3) for any i and any point $x \in M$, some neighbourhood $B_\varepsilon(x)$ of x intersects no more than $n + 1$ sets from \mathcal{U}_i .

We construct this sequence of covers by induction. Suppose we have already constructed \mathcal{U}_{i-1} , then it is easy to find a cover of \mathcal{V} that satisfies properties 1 and 2 and has order $\leq n$. To obtain a cover that also satisfies property 3, note that there is $r > 0$ such that for any point $x \in M$ the ball $B_r(x)$ is entirely contained in some set from \mathcal{V} . The cover \mathcal{U}_i is obtained from \mathcal{U}_{i-1} by replacing each U by $U_{-r/2}$, obtained by subtracting from U the closure of the $(r/2)$ -neighborhood of its complement.

For each set $U \in \mathcal{U}_{i+1}$ choose a set $f(U) \in \mathcal{U}_i$ such that $U \subset f(U)$. A *chain* is a sequence of sets (U_0, U_1, \dots) such that $U_{i-1} = f(U_i)$ for $i = 1, 2, \dots$.

Let us show that each point $x \in M$ is the intersection of all elements of some chain. Indeed, finite chains whose intersection contains x form a tree with vertices of finite degrees, this tree admits arbitrarily long paths, hence from König's lemma it follows that this tree contains a ray. Fix for every point x such a chain $c(x) = (U_0(x), U_1(x), \dots)$. Obviously, $c(x) \neq c(y)$ for distinct x and y .

For each i we choose an arbitrary total order T_i on \mathcal{U}_i . We define a total order T on M by the following rule.

$x <_T y$ if k is such that

$$U_i(x) = U_i(y) \text{ for } i = 0, 1, \dots, k-1 \text{ and } U_k(x) <_{T_k} U_k(y),$$

i.e. we compare corresponding chains lexicographically.

Now let $a, b \in M$. Choose k such that $2^{-k+1} < d(a, b)$. Let r be such that $B_r(a)$ and $B_r(b)$ intersect at most $n + 1$ sets in \mathcal{U}_k . Denote these sets A_1, \dots, A_{m_1} and B_1, \dots, B_{m_2} , $m_1, m_2 \leq n + 1$. Note that no A_i intersects with B_j .

Let's prove that $(B_r(a), B_r(b))$ does not contain a snake of length $2n + 2$. Assume the contrary. The snake consists of $n + 2$ points a_1, \dots, a_{n+2} from $B_r(a)$ and of $n + 1$ points b_1, \dots, b_{n+1} from $B_r(b)$.

For $i = 1, \dots, n + 2$ the set $U_k(a_i)$ is one of A_1, \dots, A_{m_1} . Thus there are $i < j$ such that $U_k(a_i) = U_k(a_j)$. Then $U_{k'}(a_i) = U_{k'}(a_j)$ for all

$k' = 0, 1, \dots, k$. Since $a_i <_T b_{i+1} <_T a_j$, then $U_k(b_{i+1}) = U_k(a_i)$. This contradiction shows that $\text{Snake}_T(x, y) \leq 2n + 1$ and $\text{Snake}(M, T) \leq 2n + 1$.

5. CONCLUDING REMARKS

5.1. The compactification theorem says [1] that for every separable metric space X there exists a compact metric space X' which contains a dense subspace homeomorphic to X and satisfies $\dim X' \leq \dim X$. Hence Theorem 1.3 can be extended for all separable metric spaces of covering dimension $\leq n$.

5.2.

Definition 8. *A topological space M is called totally disconnected if for every pair x, y of distinct points of M there exists an open-and-closed set $U \subset M$ such that $x \in U$ and $y \in M - U$.*

It is easy to see that every compact totally disconnected metric space has $\dim = 0$.

Proposition 5.1. *Let M be a totally disconnected separable metric space. Then it admits a total order T for which $\text{Snake}(M, T) \leq 1$.*

Proof. Consider a sequence of open-and-closed sets U_0, U_1, \dots such that for any two distinct points $x, y \in M$ at least one of the sets U_i contains exactly one of x and y . For a point $x \in M$ consider a binary sequence $c(x)$, whose i -th term is 1 if $x \in U_i$ and 0 otherwise. We say that $x <_T y$ if $c(x)$ is lexicographically smaller than $c(y)$.

Suppose $c(x)$ and $c(y)$ differ at i -th term. There exists $r > 0$ such that all points from $B_r(x)$ have the same prefix of $c(\cdot)$ of length i ; the same for $B_r(y)$. In this situation $(B_r(x), B_r(y))$ does not contain snakes longer than 1. \square

Mazurkiewicz showed [5] the existence of completely metrizable totally disconnected separable space of arbitrary dimension $n \geq 1$ (and consequently of infinite dimension). Therefore the compactness property cannot be omitted in Theorems 1.1 and 1.2.

5.3. Examples 2.1 and 2.2 show that two spaces of the same covering dimension can have different possible minimal snake numbers. The following question arises.

Question 1. *For given n , what is the minimal $s(n)$ such that for any compact metric space M s.t. $\dim M \leq n$ there exists an order T for which $\text{Snake}(M, T) \leq s(n)$?*

Since in [3] it was shown that for any order on the product of n tripods there is a pair of points with snake number at least $2n$, the answer is $2n$ or $2n + 1$. It was also shown in [2] that for $n = 1$ the answer is 3.

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