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EVERY COMPLEX HÉNON MAP SATISFIES THE CENTRAL LIMIT THEOREM

FABRIZIO BIANCHI AND TIEN-CUONG DINH

ABSTRACT. We consider a measurable dynamical system preserving a probability measure ν . If the system is exponentially mixing of all orders for suitable observables, we prove that these observables satisfy the Central Limit Theorem (CLT) with respect to ν . We show that the measure of maximal entropy of every complex Hénon map is exponentially mixing of all orders for Hölder observables. It follows that the CLT holds for all complex Hénon maps and Hölder observables.

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Notation. The pairing $\langle \cdot, \cdot \rangle$ is used for the integral of a function with respect to a measure or more generally the value of a current at a test form. By (p, p) -currents we mean currents of bi-degree (p, p) . Given $k \geq 1$, we denote by ω_{FS} the Fubini-Study form on $\mathbb{P}^k = \mathbb{P}^k(\mathbb{C})$. The mass of a positive closed (p, p) -current R on \mathbb{P}^k is equal to $\langle R, \omega_{\text{FS}}^{k-p} \rangle$ and is denoted by $\|R\|$. The notations \lesssim and \gtrsim stand for inequalities up to a multiplicative constant. If R and S are two real currents of the same bi-degree, we write $|R| \leq S$ when $S \pm R \geq 0$. Observe that this forces S to be positive.

1. INTRODUCTION

Hénon maps are among the most studied dynamical systems that exhibit interesting chaotic behaviour. They were introduced by Michel Hénon in the real setting as a simplified model of the Poincaré section for the Lorenz model, see, e.g., [7, 39]. Hénon maps are also actively studied in the complex setting, where complex analysis offers additional powerful tools. The reader can find in [3, 4, 5, 27, 33, 34, 52] and the references therein fundamental dynamical properties of these systems. The main goal of this work is to solve a long-standing question proving the Central Limit Theorem (CLT) for the maximal entropy measures of complex Hénon maps. More precisely, we first provide a general criterion to ensure the validity of the CLT in an abstract setting, and apply it to show that every complex Hénon map satisfies such property for all Hölder observables, without any extra assumption on the map.

Let (X, ν) be a probability space and $f: X \rightarrow X$ a measurable map preserving ν (i.e., such that $\nu(f^{-1}(A)) = \nu(A)$ for every measurable set $A \subseteq X$). We denote by f^n the iterate of order n of f . Given an observable $u: X \rightarrow \mathbb{R}$, we denote by $S_n(u)$ the sum

$$S_n(u) := u + u \circ f + \cdots + u \circ f^{n-1}.$$

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A fundamental question in dynamics is to study the orbits of points via the sequence $S_n(u)$, seen as a sequence of random variables on the probability space (X, ν) . By the invariance of ν , all the random variables $u \circ f^n$ have the same distribution. On the other hand, precisely because they arise from a deterministic system, they are not independent. A central problem in dynamics is to show that the functions $u \circ f^n$ behave as close as possible to i.i.d. random variables, for suitable observables u .

At the first order, such property is true and given by Birkhoff's ergodic theorem. For ν ergodic (in particular, for ν mixing) and any $u \in L^1(\nu)$, this result asserts that

$$n^{-1}S_n(u)(x) \rightarrow \langle \nu, u \rangle \quad \text{for } \nu - \text{almost every } x \in X.$$

For i.i.d. random variables, the *Central Limit Theorem* describes the rate of the above convergence. The following is a translation of that property in the setting of dynamical systems.

Definition 1.1. Let (X, f, ν) be a dynamical system as above. Let u be a ν -integrable real function. We say that u satisfies the Central Limit Theorem (CLT) with variance $\sigma^2 \geq 0$ with respect to ν if $n^{-1/2}(S_n(u) - n\langle \nu, u \rangle) \rightarrow \mathcal{N}(0, \sigma^2)$ in law, where $\mathcal{N}(0, \sigma^2)$ denotes the (possibly degenerate, for $\sigma = 0$) Gaussian distribution with mean 0 and variance σ^2 , i.e., for any interval $I \subset \mathbb{R}$ we have

$$\lim_{n \rightarrow \infty} \nu \left\{ \frac{S_n(u) - n\langle \nu, u \rangle}{\sqrt{n}} \in I \right\} = \begin{cases} 1 & \text{when } I \text{ is of the form } I = (-\delta, \delta) \quad \text{if } \sigma^2 = 0, \\ \frac{1}{\sqrt{2\pi\sigma^2}} \int_I e^{-t^2/(2\sigma^2)} dt & \text{if } \sigma^2 > 0. \end{cases}$$

The CLT has been proved for several families of real dynamical systems, see, e.g., [13, 14, 41, 42, 49, 51, 58] for the case of dimension 1 and [2, 8, 15, 16, 17, 18, 37, 43, 59, 60] for higher dimensional cases with assumptions on the hyperbolicity. Among the approaches used in these works, Gordin-Liverani martingale method [21, 35, 40, 44] and Nagaev method based on spectral properties of the Perron-Frobenius operator [1, 36, 46], have also been used in complex dynamics. They allow to prove the CLT for some equilibrium measures of holomorphic endomorphisms of \mathbb{P}^k and other systems with positive Lyapounov exponents [10, 22, 25, 30, 48, 53, 54]. It is still an open interesting and challenging question to know if these approaches can be adapted in the setting of general complex Hénon maps and other non-uniformly hyperbolic systems with both positive and negative Lyapunov exponents.

In this paper, we introduce a new criterion to prove the CLT for general dynamical systems. For the system (X, f, ν) above, we will consider observables in some normed functional space $(E, \|\cdot\|_E)$ satisfying the following property:

(e1) for all $1 \leq p < \infty$ we have $E \subset L^p(\nu)$ and $\|\cdot\|_{L^p(\nu)} \leq c_p \|\cdot\|_E$ for some constant $c_p > 0$.

Our criterion for the CLT is based on the following definition.

Definition 1.2. Let (X, f, ν) be a dynamical system as above. Let $(E, \|\cdot\|_E)$ be a normed space of real functions on X satisfying Property **(e1)**. We say that ν is *exponentially mixing of order* $\kappa \in \mathbb{N}^*$ for observables in E if there exist constants $C_\kappa > 0$ and $0 < \theta_\kappa < 1$ such that, for all g_0, \dots, g_κ in E and integers $0 =: n_0 \leq n_1 \leq \dots \leq n_\kappa$, we have

$$\left| \langle \nu, g_0(g_1 \circ f^{n_1}) \dots (g_\kappa \circ f^{n_\kappa}) \rangle - \prod_{j=0}^{\kappa} \langle \nu, g_j \rangle \right| \leq C_\kappa \cdot \left(\prod_{j=0}^{\kappa} \|g_j\|_E \right) \cdot \theta_\kappa^{\min_{0 \leq j \leq \kappa-1} (n_{j+1} - n_j)}.$$

We say that ν is *exponentially mixing of all orders* for observables in E if it is exponentially mixing of order κ for every $\kappa \in \mathbb{N}$.

Note that Property **(e1)** implies that the integrals in the above definition are meaningful.

We also consider the following further natural property for $(E, \|\cdot\|_E)$:

(e2) there is a constant $c > 0$ such that $gh \in E$ and $\|gh\|_E \leq c\|g\|_E \cdot \|h\|_E$ for all $g, h \in E$.

Our first main result is the following.

Theorem 1.3. *Let (X, f, ν) be a dynamical system as above and $(E, \|\cdot\|_E)$ a normed space of real functions on X satisfying Properties **(e1)** and **(e2)**. Assume also that the operator f^* (i.e., the operator $u \mapsto u \circ f$) preserves E and is bounded with respect to $\|\cdot\|_E$, and that ν is exponentially mixing of all orders for observables in E . Then all observables $u \in E$ satisfy the Central Limit Theorem with respect to ν as in Definition 1.1, with*

$$(1.1) \quad \sigma^2 = \sum_{n \in \mathbb{Z}} \langle \nu, \tilde{u}(\tilde{u} \circ f^{|n|}) \rangle = \lim_{n \rightarrow \infty} \frac{1}{n} \int_X (\tilde{u} + \tilde{u} \circ f + \dots + \tilde{u} \circ f^{n-1})^2 d\nu,$$

where $\tilde{u} := u - \langle \nu, u \rangle$.

Observe that if X is a metric space, the support $\text{supp}(\nu)$ of ν has bounded diameter, and f is Lipschitz continuous on $\text{supp}(\nu)$, then the operator f^* is bounded on the space of γ -Hölder real functions on $\text{supp}(\nu)$ for any $0 < \gamma \leq 1$. This is the case for Hénon maps that we will consider later.

The fact that a sufficiently strong mixing condition on ν implies the CLT is natural. The problem of establishing suitable conditions on a sequence of non-independent random variables to ensure the validity of the CLT dates back to at least a century, since the works of Markov on Markov partitions and Bernstein [9], see also [38, 45, 47, 50] to cite a few further developments. But such criteria do not seem to apply in the setting of general dynamical systems without at least some hyperbolicity condition.

In order to prove Theorem 1.3, we use the method of moments, and prove that all the moments of the limit distribution of $n^{-1/2}(S_n(u) - n\langle \nu, u \rangle)$ coincide with those of the Gaussian distribution. Namely, we show that (assuming $\langle \nu, u \rangle = 0$ for simplicity)

$$\lim_{n \rightarrow \infty} n^{-\kappa/2} \langle \nu, S_n(u)^\kappa \rangle = \begin{cases} 0 & \text{if } \kappa \text{ is odd} \\ (\kappa - 1)!! \sigma^\kappa & \text{if } \kappa \text{ is even.} \end{cases}$$

Up to some combinatorial arguments, this boils down to the problem of computing (sums of) integrals of the form

$$\langle \nu, (u \circ f^{s_1}) \dots (u \circ f^{s_\kappa}) \rangle,$$

where we assume for simplicity that $0 \leq s_1 \leq \dots \leq s_\kappa \leq n-1$. Such terms are controlled thanks to the exponential mixing of all orders. In particular, a key point is to notice that terms for which at least two consecutive gaps among $s_2 - s_1, s_3 - s_2, \dots, s_\kappa - s_{\kappa-1}$ are large in comparison with the other gaps, will be negligible in the limit thanks to the exponential mixing of all order, see Lemma 2.8.

This remark permits to deduce that, in the limit, only the sum of terms with $s_1 \sim s_2 \ll s_3 \sim s_4 \ll \dots \ll s_{\kappa-1} \sim s_\kappa$ can bring a contribution. Such terms are present only for κ even. Moreover, thanks to the mixing of order $\kappa/2 - 1$ applied to the $\kappa/2$ observables $(u \circ f^{s_j})(u \circ f^{s_j+1})$ with odd j , we see that each of these terms is equal, up to a small (controlled) error, to the product

$$\langle \nu, (u \circ f^{s_1})(u \circ f^{s_2}) \rangle \dots \langle \nu, (u \circ f^{s_{\kappa-1}})(u \circ f^{s_\kappa}) \rangle.$$

By the invariance of ν , such product reduces to a product of $\kappa/2$ terms of the form $\langle \nu, u \cdot (u \circ f^{|m|}) \rangle$, for $m \in \mathbb{Z}$. Summing over such terms, and taking into account the combinatorial coefficients and the formula (1.1) for σ^2 , we get the desired result of $(\kappa - 1)!! \sigma^\kappa$.

Let now f be a complex Hénon map on \mathbb{C}^2 . It is a polynomial diffeomorphism of \mathbb{C}^2 . We can associate to f its unique measure of maximal entropy μ [3, 4, 5, 52], see Section 3 for details. The following result ensures that every such system satisfies the conditions of Theorem 1.3. The case of two observables was established by the second author in [23], see also [56, 57]. Similar results were obtained by Liverani [43] in the case of uniformly hyperbolic diffeomorphisms and Dolgopyat [28] for Anosov flows.

Theorem 1.4. *Let f be a complex Hénon map and μ its measure of maximal entropy. Then, for every $\kappa \in \mathbb{N}^*$, μ is exponential mixing of order κ as in Definition 1.2 for \mathcal{C}^γ observables ($0 < \gamma \leq 2$), with $\theta_\kappa = d^{-(\gamma/2)^{\kappa+1}/2}$.*

The following is then a consequence of Theorems 1.3 and 1.4, and is the main result of this paper.

Corollary 1.5. *Let f be a complex Hénon map and μ its measure of maximal entropy. Then all Hölder observables satisfy the Central Limit Theorem with respect to μ .*

Theorem 1.4 and Corollary 1.5 in particular apply to any real Hénon map of maximal entropy [6], i.e., complex Hénon maps with real coefficients and whose measure of maximal entropy is contained in \mathbb{R}^2 . They hold also in the larger settings of regular automorphisms of \mathbb{C}^k in any dimension [52], see Definition 3.1 and Remark 4.3, and invertible horizontal-like maps in any dimension [24, 26], see Remark 4.4. We postpone the proof in the case of compact Kähler manifolds to the forthcoming paper [11], see Remark 4.5.

Our method to prove Theorem 1.4 relies on pluripotential theory and on the theory of positive closed currents. The idea is as follows. Using the classical theory of interpolation [55], we can reduce the problem to the case $\gamma = 2$. For simplicity, assume that $\|g_j\|_{\mathcal{C}^2} \leq 1$ for all j . The measure of maximal entropy μ of a Hénon map f of \mathbb{C}^2 of algebraic degree $d \geq 2$ is the intersection $\mu = T_+ \wedge T_-$ of the two Green currents T_+ and T_- of f [4, 52]. If we identify \mathbb{C}^2 to an affine chart of \mathbb{P}^2 in the standard way, these currents are the unique positive closed $(1, 1)$ -currents of mass 1 on \mathbb{P}^2 , without mass at infinity, satisfying $f^*T_+ = dT_+$ and $f_*T_- = dT_-$.

Consider the automorphism F of \mathbb{C}^4 given by $F := (f, f^{-1})$. Such automorphism also admits Green currents $\mathbb{T}_+ = T_+ \otimes T_-$ and $\mathbb{T}_- = T_- \otimes T_+$. These currents satisfy $(F^n)^*\mathbb{T}_+ = d^2\mathbb{T}_+$ and $(F^n)_*\mathbb{T}_- = d^2\mathbb{T}_-$. Under mild assumptions on their support, other positive closed $(2, 2)$ -currents S of mass 1 of \mathbb{P}^4 satisfy the estimate

$$(1.2) \quad |\langle d^{-2n}(F^n)_*(S) - \mathbb{T}_-, \Phi \rangle| \leq c_{S,\Phi} d^{-n}$$

when Φ is a sufficiently smooth test form. Here, $c_{S,\Phi}$ is a constant depending on S and Φ .

We show that proving the exponential mixing for $\kappa + 1$ observables $g_0, \dots, g_\kappa \in \mathcal{C}^2$ can be reduced to proving the convergence (we assume that n_1 is even for simplicity)

$$(1.3) \quad |\langle d^{-n_1}(F^{n_1/2})_*[\Delta] - \mathbb{T}_-, \Theta \rangle| \lesssim d^{-\min_{0 \leq j \leq \kappa-1} (n_{j+1} - n_j)/2},$$

where $[\Delta]$ denotes the current of integration on the diagonal Δ of $\mathbb{C}^2 \times \mathbb{C}^2$, (z, w) denote the coordinates on $\mathbb{C}^2 \times \mathbb{C}^2$ and

$$\Theta := g_0(w)g_1(z)(g_2 \circ f^{n_2 - n_1}(z)) \dots (g_\kappa \circ f^{n_\kappa - n_1}(z))\mathbb{T}_+.$$

Note that the current $[\Delta]$ is singular and the dependence of the constant $c_{S,\Phi}$ in (1.2) from S makes it difficult to employ regularization techniques to deduce the convergence (1.3) from (1.2).

The key point here is to notice that, when $dd^c\Phi \geq 0$ (on a suitable open set), one can also get the following variation of (1.2):

$$(1.4) \quad \langle d^{-2n}(F^n)_*(S) - \mathbb{T}_-, \Phi \rangle \leq c_\Phi d^{-n}.$$

With respect to (1.2), only the bound from above is present, but the constant c_Φ is now independent of S . This permits to regularize Δ and work as if this current were smooth. Note also that, although Θ is not smooth, we can handle it using a similar regularization.

Working by induction, we show that it is possible to replace both Θ and $-\Theta$ in (1.3) with currents Θ^\pm satisfying $dd^c\Theta^\pm \geq 0$. This permits to deduce the estimate (1.3) from two upper bounds given by (1.4) for Θ^\pm , completing the proof.

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2. THE EXPONENTIAL MIXING OF ALL ORDERS IMPLIES THE CLT

Throughout this section, (X, f, ν) is a dynamical system and E a functional space as in Theorem 1.3. In particular, observables in E satisfy Properties **(e1)**, **(e2)**, and the exponential mixing of all orders as in Definition 1.2.

Define $E' := E + \mathbb{R}$, where \mathbb{R} is identified with the set of constant functions. If $\mathbb{R} \subset E$, we have $E' = E$ and we define $\|\cdot\|_{E'} := \|\cdot\|_E$. Otherwise, for $h \in E'$ we can write in a unique way $h = g + c$ with $g \in E$ and $c \in \mathbb{R}$ and we define

$$\|h\|_{E'} := \|g\|_E + |c|.$$

Lemma 2.1. *We have the following properties:*

- (i) *the vector space $(E', \|\cdot\|_{E'})$ satisfies Properties **(e1)** and **(e2)**;*
- (ii) *the operator f^* preserves E' and is bounded with respect to $\|\cdot\|_{E'}$;*
- (iii) *all observables in E' satisfy the exponential mixing of all orders as in Definition 1.2.*

Proof. The lemma is clear when $E' = E$. Assume now that $E' \neq E$. For $h, h' \in E'$ write $h = g + c$ and $h' = g' + c'$ with $g, g' \in E$ and $c, c' \in \mathbb{R}$. We also assume that $\|h\|_{E'} = \|h'\|_{E'} = 1$. It follows that $\|g\|_E, \|g'\|_E, |c|$, and $|c'|$ are bounded by 1.

(i) It is easy to deduce from the above that $\|h\|_{L^p(\nu)}$ and $\|hh'\|_{E'}$ are bounded. The assertion (i) follows.

(ii) We have $f^*(h) = f^*(g) + c$. Since f^* preserves E and is bounded with respect to $\|\cdot\|_E$, we see that $f^*(g)$ belongs to E and has a bounded norm. It is now clear that $\|f^*(h)\|_{E'}$ is bounded and (ii) follows.

(iii) Fix $\kappa \in \mathbb{N}^*$ and consider $\kappa + 1$ observables $h_j \in E'$ with $\|h_j\|_{E'} = 1$ for $0 \leq j \leq \kappa$. Write each of them (in a unique way) as $h_j = g_j + c_j$, with $g_j \in E$ and $c_j \in \mathbb{R}$. As above, we have that $\|g_j\|_E$ and $|c_j|$ are bounded by 1.

For all $0 =: n_0 \leq n_1 \leq \dots \leq n_\kappa \in \mathbb{N}$, set $m := \min_{0 \leq j \leq \kappa-1} (n_{j+1} - n_j)$. We also define $\theta := \max_{1 \leq j \leq \kappa} \theta_j$ and $C := \max_{1 \leq j \leq \kappa} C_j$, see Definition 1.2. We have to bound the difference between

$$\begin{aligned} \langle \nu, h_0(h_1 \circ f^{n_1}) \dots (h_\kappa \circ f^{n_\kappa}) \rangle &= \langle \nu, (c_0 + g_0)(c_1 + g_1 \circ f^{n_1}) \dots (c_\kappa + g_\kappa \circ f^{n_\kappa}) \rangle \\ &= \sum_{(J_{\mathbb{R}}, J_E)} \left(\prod_{j \in J_{\mathbb{R}}} c_j \right) \langle \nu, \prod_{j \in J_E} g_j \circ f^{n_j} \rangle \end{aligned}$$

and

$$\prod_{j=0}^{\kappa} \langle \nu, h_j \rangle = \sum_{(J_{\mathbb{R}}, J_E)} \left(\prod_{j \in J_{\mathbb{R}}} c_j \right) \prod_{j \in J_E} \langle \nu, g_j \rangle,$$

where the sums are over all partitions $(J_{\mathbb{R}}, J_E)$ of $\{0, \dots, \kappa\}$.

Fix one such partition, assuming for now that J_E is non empty. By the assumption of the exponential mixing for observables in E , we see that

$$\begin{aligned}
& \left| \left(\prod_{j \in J_{\mathbb{R}}} c_j \right) \langle \nu, \prod_{j \in J_E} g_j \circ f^{n_j} \rangle - \left(\prod_{j \in J_{\mathbb{R}}} c_j \right) \prod_{j \in J_E} \langle \nu, g_j \rangle \right| \\
&= \left| \prod_{j \in J_{\mathbb{R}}} c_j \right| \cdot \left| \langle \nu, \prod_{j \in J_E} g_j \circ f^{n_j} \rangle - \prod_{j \in J_E} \langle \nu, g_j \rangle \right| \\
&\leq \left| \prod_{j \in J_{\mathbb{R}}} c_j \right| \cdot C \cdot \left(\prod_{j \in J_E} \|g_j\|_E \right) \cdot \theta^m \\
&\leq C \theta^m.
\end{aligned}$$

The computation above becomes trivial when J_E is empty, as the first line vanishes. As the number of partitions $(J_{\mathbb{R}}, J_E)$ of $\{0, \dots, \kappa\}$ only depends on κ , we obtain the exponential mixing of order κ for the observables h_j . The proof is complete. \square

According to Lemma 2.1, from now on, we can and we do replace E by E' in order to assume for simplicity that constant functions belong to E . We fix an observable $u \in E$ and we show that it satisfies the CLT. Using the invariance of ν , we can add to u any constant and this does not modify the validity of the CLT for u . So, for simplicity, we assume from now on that $\langle \nu, u \rangle = 0$. We have used here that E contains constant functions.

Lemma 2.2. *We have*

$$(2.1) \quad \lim_{n \rightarrow \infty} n^{-1} \langle \nu, S_n(u)^2 \rangle = \sum_{j \in \mathbb{Z}} \langle \nu, u \cdot (u \circ f^{|j|}) \rangle,$$

and both sides are finite and non-negative.

Lemma 2.2 allows us to define

$$\sigma^2 := \lim_{n \rightarrow \infty} n^{-1} \langle \nu, S_n(u)^2 \rangle = \sum_{j \in \mathbb{Z}} \langle \nu, u \cdot (u \circ f^{|j|}) \rangle.$$

We will prove Theorem 1.3 for this value of σ^2 .

Proof. It is clear that the LHS of (2.1) is non-negative. Therefore, we only need to show that the RHS of this identity converges absolutely and that the limit in the LHS exists and is equal to the RHS.

By the exponential mixing of order 1, and with the notations of Definition 1.2, for all $j \in \mathbb{Z}$ we have

$$|\langle \nu, u \cdot (u \circ f^{|j|}) \rangle| \leq C_1 \cdot \|u\|_E^2 \cdot \theta_1^{|j|}.$$

We have used here that $\langle \nu, u \rangle = 0$. It follows that the RHS of (2.1) converges absolutely.

Consider the measure \mathfrak{m} on \mathbb{Z} which is the sum of all Dirac masses on \mathbb{Z} , and the function $\mathcal{U}: \mathbb{Z} \rightarrow \mathbb{R}$ given by $\mathcal{U}(j) := \langle \nu, u \cdot (u \circ f^{|j|}) \rangle$. With these notations, the RHS of (2.1) is equal to the integral $\langle \mathfrak{m}, \mathcal{U} \rangle$ and the function \mathcal{U} is integrable with respect to \mathfrak{m} . We now express the LHS of (2.1) in terms of some integrals with respect to \mathfrak{m} .

For $n \in \mathbb{N}$, define a function $M_n: \mathbb{Z} \rightarrow \mathbb{R}$ as

$$M_n(j) := \begin{cases} n - |j| & \text{for } |j| \leq n - 1 \\ 0 & \text{for } |j| \geq n. \end{cases}$$

Observe that $M_n(j)$ is the number of pairs $(m, l) \in \mathbb{N}^2$ such that $0 \leq m, l \leq n - 1$ and $m - l = j$. Using the invariance of ν we get, for every $n \in \mathbb{N}$,

$$\begin{aligned}
n^{-1} \langle \nu, S_n(u)^2 \rangle &= \frac{1}{n} \sum_{0 \leq m, l \leq n-1} \langle \nu, (u \circ f^m)(u \circ f^l) \rangle = \frac{1}{n} \sum_{0 \leq m, l \leq n-1} \langle \nu, u \cdot (u \circ f^{|m-l|}) \rangle \\
&= \frac{1}{n} \sum_{j \in \mathbb{Z}} M_n(j) \langle \nu, u \cdot (u \circ f^{|j|}) \rangle = \langle \mathfrak{m}, n^{-1} M_n \cdot \mathcal{U} \rangle.
\end{aligned}$$

The sequence of functions $n^{-1}M_n \cdot \mathcal{U}$ tends to \mathcal{U} pointwise on \mathbb{Z} when n goes to infinity. Moreover, we have $|n^{-1}M_n \cdot \mathcal{U}| \leq |\mathcal{U}|$ and the function \mathcal{U} is integrable with respect to \mathfrak{m} . By Lebesgue's dominated convergence theorem, we obtain

$$\langle \mathfrak{m}, n^{-1}M_n \cdot \mathcal{U} \rangle \rightarrow \langle \mathfrak{m}, \mathcal{U} \rangle \quad \text{for } n \rightarrow \infty.$$

The lemma follows. \square

We will use the following lemma for $v := u$.

Lemma 2.3. *Let $v \in L^1(\nu)$ satisfy $\langle \nu, v \rangle = 0$. Assume that there exists $\sigma \geq 0$ such that, for all $\kappa \in \mathbb{N}^*$,*

$$(2.2) \quad \lim_{n \rightarrow \infty} n^{-\kappa/2} \langle \nu, S_n(v)^\kappa \rangle = \begin{cases} 0 & \text{if } \kappa \text{ is odd} \\ (\kappa - 1)!! \sigma^\kappa & \text{if } \kappa \text{ is even.} \end{cases}$$

Then v satisfies the CLT with variance σ^2 with respect to ν , see Definition 1.1.

Proof. Recall that $\mathcal{N}(0, \sigma^2)$ denotes the (possibly degenerate, when $\sigma = 0$) Gaussian distribution of mean 0 and variance σ^2 . A standard computation gives that the moments of $\mathcal{N}(0, \sigma^2)$ are given by

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} x^\kappa e^{-x^2/(2\sigma^2)} dx = \begin{cases} 0 & \text{if } \kappa \text{ is odd} \\ (\kappa - 1)!! \sigma^\kappa & \text{if } \kappa \text{ is even} \end{cases}$$

when $\sigma \neq 0$. When $\sigma = 0$, the Gaussian in the integral should be replaced by the Dirac mass at 0, hence the integral vanishes for all $\kappa \in \mathbb{N}^*$.

By using 2κ instead of κ , we see that the sequence $n^{-\kappa} \langle \nu, S_n(v)^{2\kappa} \rangle$ is bounded and therefore

$$\nu\{|n^{-1/2}S_n(v)| \geq M\} \leq \frac{c_\kappa}{M^{2\kappa}} \text{ for every } M \geq 1,$$

where c_κ is a constant. This and the assumption on v imply that any limit (in law) for $n \rightarrow \infty$ of the random variables $n^{-1/2}S_n(v)$ and $\mathcal{N}(0, \sigma^2)$ have the same moment of order κ . As the normal distribution is uniquely characterized by its moments [12, Theorem 30.1 and Example 30.1], this implies that $n^{-1/2}S_n(v) \rightarrow \mathcal{N}(0, \sigma^2)$ in law, as required. \square

The goal of this section is then to prove that the condition (2.2) in Lemma 2.3 is satisfied when the function v is replaced by our observable u . Note that the case $\kappa = 1$ is trivial because $\langle \nu, u \rangle = 0$ and the case $\kappa = 2$ is given by Lemma 2.2.

We fix $\kappa \geq 3$ in what follows and denote $\theta := \max_{1 \leq j \leq \kappa} \theta_j$ and $C := \max_{1 \leq j \leq \kappa} C_j$, see Definition 1.2. Choose a constant $L > 1$ which is larger than the norm of the operator $f^* : E \rightarrow E$. This norm is finite by hypothesis. Fix a constant $B > 1$ such that $L^{\kappa^2} \theta^B < 1$ and choose another constant $0 < \eta < 1$ such that $L^{\kappa^2} \theta^B < \eta^B$, or equivalently $L^{\kappa^2/B} \theta < \eta$. Finally, define $A := |\log \eta|^{-1}(\kappa + 1)$ and $D := AB^\kappa$.

Set

$$\llbracket 0, n-1 \rrbracket := \{0, 1, \dots, n-1\}$$

and consider an element $\underline{s} = (s_1, \dots, s_\kappa) \in \llbracket 0, n-1 \rrbracket^\kappa$. We also set

$$\mathcal{I}_{\underline{s}} := \langle \nu, u_{s_1} \dots u_{s_\kappa} \rangle \quad \text{and} \quad u_s := u \circ f^s \text{ for } s \in \mathbb{N}.$$

Observe that

$$\langle \nu, S_n(u)^\kappa \rangle = \sum_{\underline{s} \in \llbracket 0, n-1 \rrbracket^\kappa} \mathcal{I}_{\underline{s}}.$$

For $Z \subset \llbracket 0, n-1 \rrbracket^\kappa$ define

$$\mathcal{S}(Z) := \sum_{\underline{s} \in Z} \mathcal{I}_{\underline{s}} \quad \text{and} \quad \mathcal{S}^*(Z) := \sum_{\underline{s} \in Z} |\mathcal{I}_{\underline{s}}|.$$

Define also $\mathcal{S}_n := \mathcal{S}(\llbracket 0, n-1 \rrbracket^\kappa)$, so that the LHS of (2.2) is equal to $n^{-\kappa/2} \mathcal{S}_n$ (recall that in that equation we replace v by u). To study this quantity, the idea is to cover $\llbracket 0, n-1 \rrbracket^\kappa$ by

subsets where our sums \mathcal{S} and \mathcal{S}^* can be evaluated using combinatorial arguments and/or the mixing of order $\leq \kappa$.

Denote by $E_{1,n} \subset \llbracket 0, n-1 \rrbracket^\kappa$ the set of all \underline{s} such that up to a permutation of the components of \underline{s} we have $s_1 \leq s_2 \leq \dots \leq s_\kappa$ and $s_{j+1} - s_j \leq D \log n$ for more than $\kappa/2$ indexes $1 \leq j \leq \kappa-1$, where the constant D was defined above. We will use that, for each such index j , when the value of s_j is fixed there are at most $D \log n + 1$ choices for s_{j+1} .

Lemma 2.4. *We have*

$$\lim_{n \rightarrow \infty} n^{-\kappa/2} \mathcal{S}^*(E_{1,n}) = 0.$$

Proof. Set $\kappa_0 := \kappa/2 + 1$ if κ is even and $\kappa_0 := (\kappa + 1)/2$ if κ is odd. By the Hölder inequality, for each $\underline{s} \in \llbracket 0, n-1 \rrbracket^\kappa$ we have

$$|\mathcal{I}_{\underline{s}}| = |\langle \nu, u_{s_1} \dots u_{s_\kappa} \rangle| \leq \|u \circ f^{s_1}\|_{L^\kappa(\nu)} \dots \|u \circ f^{s_\kappa}\|_{L^\kappa(\nu)} = \|u\|_{L^\kappa(\nu)}^\kappa \lesssim \|u\|_E^\kappa,$$

where we used Property (e1) and the fact that $\|u \circ f\|_{L^\kappa(\nu)} = \|u\|_{L^\kappa(\nu)}$ (which follows from the invariance of ν).

Observe that the sum $\mathcal{S}^*(E_{1,n})$ contains at most $\kappa! \binom{\kappa}{\kappa_0} (D \log n + 1)^{\kappa_0} n^{\kappa - \kappa_0}$ terms, where $\kappa!$ is the number of the permutations of the components of (s_1, \dots, s_κ) , $\binom{\kappa}{\kappa_0}$ is the number of choices of κ_0 indexes j as in the definition of $E_{1,n}$ and $n^{\kappa - \kappa_0}$ is the number of choices of s_l with $l \neq j+1$ for any j as in the definition of $E_{1,n}$. We deduce that

$$n^{-\kappa/2} \mathcal{S}^*(E_{1,n}) \lesssim n^{-\kappa/2} (D \log n)^{\kappa_0} n^{\kappa - \kappa_0} \|u\|_E^\kappa = n^{-\kappa_0 + \kappa/2} (D \log n)^{\kappa_0} \|u\|_E^\kappa.$$

The assertion follows since, by definition, we have $\kappa_0 > \kappa/2$. \square

We will denote in what follows by $0 \leq \delta_1, \dots, \delta_{\kappa-1} \leq n-1$ some integers with the property that $\Delta_{\kappa-1} \leq n-1$, where we set $\Delta_j := \sum_{l=1}^j \delta_l$ for all $1 \leq j \leq \kappa-1$. We also set $\underline{\delta} := (\delta_1, \dots, \delta_{\kappa-1})$ and recall that $u_s := u \circ f^s$ for $s \in \mathbb{N}$.

We will need to study integrals of the form

$$(2.3) \quad \mathcal{J}_{\underline{\delta}} := \mathcal{J}_{\delta_1, \dots, \delta_{\kappa-1}} := \langle \nu, u u_{\Delta_1} \dots u_{\Delta_{\kappa-1}} \rangle.$$

Definition 2.5. Let $\mathcal{D}_+ \neq \emptyset$ and \mathcal{D}_- be two disjoint subsets of $\{\delta_j\}_{1 \leq j \leq \kappa-1}$ with $\mathcal{D}_+ \cup \mathcal{D}_- = \{\delta_j\}_{1 \leq j \leq \kappa-1}$. We say that $(\mathcal{D}_-, \mathcal{D}_+)$ is a *good partition* of $\{\delta_j\}_{1 \leq j \leq \kappa-1}$ if

$$\delta_l \geq 1 \quad \text{and} \quad \delta_l \geq B \delta_j \quad \text{for all } \delta_l \in \mathcal{D}_+ \text{ and } \delta_j \in \mathcal{D}_-,$$

where the constant B has been introduced earlier. We admit here the case where \mathcal{D}_- is empty, and in this case we just mean that $\delta_l \geq 1$ for all $\delta_l \in \mathcal{D}_+$. We set $\max(\mathcal{D}_-) := \max\{\delta_j, \delta_j \in \mathcal{D}_-\}$ when $\mathcal{D}_- \neq \emptyset$ and $\max(\mathcal{D}_-) := 0$ otherwise. We also set $\min(\mathcal{D}_+) := \min\{\delta_j, \delta_j \in \mathcal{D}_+\}$.

Given a good partition $(\mathcal{D}_-, \mathcal{D}_+)$ of $\{\delta_j\}_{1 \leq j \leq \kappa-1}$ as above, write $\mathcal{D}_+ = \{\delta_{j_1}, \dots, \delta_{j_m}\}$, with $j_1 < \dots < j_m$. For simplicity, define $j_0 := 0$, $j_{m+1} := \kappa$, and $\delta_\kappa := 0$. Write

$$(2.4) \quad u u_{\Delta_1} \dots u_{\Delta_{\kappa-1}} = U_0 (U_1 \circ f^{\Delta_{j_1}}) \dots (U_m \circ f^{\Delta_{j_m}}),$$

where

$$U_r := \begin{cases} u & \text{if } j_r + 1 = j_{r+1} \\ u u_{\delta_{j_r+1}} u_{\delta_{j_r+1} + \delta_{j_r+2}} \dots u_{\delta_{j_r+1} + \dots + \delta_{j_{r+1}-1}} & \text{otherwise.} \end{cases}$$

Note that in the last product all indexes $\delta_{j_r+1}, \dots, \delta_{j_{r+1}-1}$ belong to \mathcal{D}_- . In a particular case that we will consider later, we will have $j_{r+1} = j_r + 2$ and then $U_r = u u_{\delta_{j_r+1}}$, which is a product of two functions.

Given the decomposition as in (2.4), we will compare the integral (2.3) with the following quantity

$$\mathcal{J}_{\mathcal{D}_-, \mathcal{D}_+} := \langle \nu, U_0 \rangle \langle \nu, U_1 \circ f^{\Delta_{j_1}} \rangle \dots \langle \nu, U_m \circ f^{\Delta_{j_m}} \rangle = \langle \nu, U_0 \rangle \langle \nu, U_1 \rangle \dots \langle \nu, U_m \rangle,$$

where the second equality is a consequence of the invariance of ν .

We have the following lemma.

Lemma 2.6. *Assume that $\{\delta_j\}_{1 \leq j \leq \kappa-1}$ admits a good partition $(\mathcal{D}_-, \mathcal{D}_+)$. Then we have, for some constant $c > 0$ independent of $\delta_1, \dots, \delta_{\kappa-1}$,*

$$|\mathcal{J}_{\delta_1, \dots, \delta_{\kappa-1}} - \mathcal{J}_{\mathcal{D}_-, \mathcal{D}_+}| \leq c \eta^{\min(\mathcal{D}_+)}.$$

Proof. Recall that u, κ are fixed and that $m \leq \kappa - 1$. The inequality follows from the exponential mixing of order $\leq \kappa - 1$ of ν , applied to the $m + 1$ observables U_0, \dots, U_m . More precisely, with the notations of Definition 1.2, we have

$$|\mathcal{J}_{\delta_1, \dots, \delta_{\kappa-1}} - \mathcal{J}_{\mathcal{D}_-, \mathcal{D}_+}| \leq C \cdot \left(\prod_{r=0}^m \|U_r\|_E \right) \cdot \theta^{\min(\mathcal{D}_+)}.$$

By the definitions of the U_r 's and L and Property **(e2)**, we also have

$$\|U_r\|_E \lesssim L^{(1+\dots+(j_{r+1}-j_r)) \max(\mathcal{D}_-)} \cdot \|u\|_E^{j_{r+1}-j_r} \leq L^{\kappa(j_{r+1}-j_r) \max(\mathcal{D}_-)} \cdot \|u\|_E^{j_{r+1}-j_r} \quad \text{for all } r.$$

Hence, we have

$$|\mathcal{J}_{\delta_1, \dots, \delta_{\kappa-1}} - \mathcal{J}_{\mathcal{D}_-, \mathcal{D}_+}| \lesssim \|u\|_E^\kappa L^{\kappa^2 \max(\mathcal{D}_-)} \theta^{\min(\mathcal{D}_+)} \lesssim L^{\kappa^2 \max(\mathcal{D}_-)} \theta^{\min(\mathcal{D}_+)}.$$

Since $(\mathcal{D}_-, \mathcal{D}_+)$ is a good partition, we have $\max(\mathcal{D}_-) \leq B^{-1} \min(\mathcal{D}_+)$. The lemma is then a consequence of the choices of η and B , which imply that and $L^{\kappa^2/B} \theta \leq \eta$. \square

The following is a consequence of Lemma 2.6. Recall that we are assuming that $\langle \nu, u \rangle = 0$.

Corollary 2.7. *Assume that $\{\delta_j\}_{1 \leq j \leq \kappa-1}$ admits a good partition $(\mathcal{D}_-, \mathcal{D}_+)$. Assume also that at least one of the following conditions holds:*

- (h1) $\delta_1 \in \mathcal{D}_+$, or
- (h2) $\delta_{\kappa-1} \in \mathcal{D}_+$, or
- (h3) there exists j^* such that both δ_{j^*} and δ_{j^*+1} belong to \mathcal{D}_+ , or
- (h4) $|\mathcal{D}_+| > \kappa/2 - 1$.

Then, for some constant $c > 0$ independent of $\delta_1, \dots, \delta_{\kappa-1}$, we have

$$|\mathcal{J}_{\delta_1, \dots, \delta_{\kappa-1}}| \leq c \eta^{\min(\mathcal{D}_+)}.$$

Proof. In each of the cases in the statement, it follows from the definition of the U_r 's that there exists at least one r^* with $U_{r^*} = u$. Indeed, we can take $r^* = 0$ in the first case (where $j_1 = 1$), $r^* = m$ in the second case (where $j_m = \kappa - 1$), and $r^* = j^*$ in the third case. Observe that the fourth condition implies that one among the first three holds, hence we can choose r^* accordingly.

We deduce from the definition of $\mathcal{J}_{\mathcal{D}_-, \mathcal{D}_+}$ and the invariance of ν that

$$\mathcal{J}_{\mathcal{D}_-, \mathcal{D}_+} = \langle \nu, U_{r^*} \rangle \cdot \prod_{r \neq r^*} \langle \nu, U_r \rangle = \langle \nu, u \rangle \cdot \prod_{r \neq r^*} \langle \nu, U_r \rangle.$$

Since by assumption we have $\langle \nu, u \rangle = 0$, we deduce that $\mathcal{J}_{\mathcal{D}_-, \mathcal{D}_+} = 0$. The assertion follows from Lemma 2.6. \square

Denote by $E_{2,n} \subset \llbracket 0, n-1 \rrbracket^\kappa$ the set of all \underline{s} such that, up to a permutation of the components of \underline{s} , we have $s_1 \leq s_2 \leq \dots \leq s_\kappa$ and $\underline{\delta} := (\delta_1, \dots, \delta_{\kappa-1})$ with $\delta_j := s_{j+1} - s_j$ is such that $\{\delta_j\}_{1 \leq j \leq \kappa-1}$ admits a good partition $(\mathcal{D}_-, \mathcal{D}_+)$ with $\min(\mathcal{D}_+) \geq A \log n$ and such that at least one among conditions **(h1)**-**(h4)** in Corollary 2.7 holds.

Lemma 2.8. *We have*

$$\lim_{n \rightarrow \infty} n^{-\kappa/2} \mathcal{S}^*(E_{2,n}) = 0.$$

Proof. Note that $|E_{2,n}| \leq n^\kappa$ because $E_{2,n} \subset \llbracket 0, n-1 \rrbracket^\kappa$. So, it is enough to show that

$$|\langle \nu, u_{s_1} \dots u_{s_\kappa} \rangle| \lesssim n^{-\kappa} \quad \text{for all } \underline{s} \in E_{2,n}.$$

The last integral does not change if we permute the components of \underline{s} . Therefore, we can assume that $s_1 \leq s_2 \leq \dots \leq s_\kappa$. With the above notation, this integral is equal to $\mathcal{J}_{\delta_1, \dots, \delta_{\kappa-1}}$ because ν is invariant.

By the definition of $E_{2,n}$, Corollary 2.7, and the choice of the constant A , we have

$$|\mathcal{J}_{\delta_1, \dots, \delta_{\kappa-1}}| \lesssim \eta^{\min(\mathcal{D}_+)} \leq \eta^{A \log n} = n^{-(\kappa+1)}.$$

This ends the proof of the lemma. \square

We continue the proof of (2.2) for $\kappa \geq 3$ fixed and for $v := u$. We treat the two cases of κ odd and even in the next two propositions.

Proposition 2.9. *Assume that κ is odd. Then*

$$\lim_{n \rightarrow \infty} n^{-\kappa/2} \mathcal{S}_n = 0.$$

Proof. We claim that, since κ is odd, $\llbracket 0, n-1 \rrbracket^\kappa$ is covered by $E_{1,n}$ and $E_{2,n}$. The proposition then follows from Lemmas 2.4 and 2.8.

Take $\underline{s} \in \llbracket 0, n-1 \rrbracket^\kappa$. Since the definitions of $E_{1,n}$ and $E_{2,n}$ do not depend on permutations of the components of \underline{s} , we can assume that $s_1 \leq \dots \leq s_\kappa$. Define $\underline{\delta} := (\delta_1, \dots, \delta_{\kappa-1})$ with $\delta_j := s_{j+1} - s_j$. If $\underline{\delta}$ is such that at least $(\kappa+1)/2$ components δ_j are less than or equal to $D \log n$, we have $\underline{s} \in E_{1,n}$. Hence, we can assume that $\underline{\delta}$ admits at most $(\kappa-1)/2$ components δ_j which are less than or equal to $D \log n$ and at least $(\kappa-1)/2$ components which are larger than $D \log n$. Recall that $D = AB^\kappa > A$. We need to show that $\underline{s} \in E_{2,n}$. It suffices to construct a good partition $(\mathcal{D}_-, \mathcal{D}_+)$ of $\{\delta_j\}_{1 \leq j \leq \kappa-1}$ satisfying the properties in the definition of $E_{2,n}$.

Observe that the $\kappa+1$ points $(AB^j \log n)_{0 \leq j \leq \kappa}$ divide \mathbb{R}^+ in $\kappa+1$ bounded intervals and the half line $\{x > AB^\kappa \log n\}$. Since we only have $\kappa-1$ components δ_j , one of the κ intervals $[AB^l \log n, AB^{l+1} \log n)$, $0 \leq l \leq \kappa-1$, does not contain any δ_j . We fix such an integer l and denote by \mathcal{D}_- (resp. \mathcal{D}_+) the family of all δ_j which are smaller than $AB^l \log n$ (resp. larger than or equal to $AB^{l+1} \log n$). It is clear that $(\mathcal{D}_-, \mathcal{D}_+)$ is a good partition of $\{\delta_1, \dots, \delta_{\kappa-1}\}$. By construction, we also have $|\mathcal{D}_+| > \kappa/2 - 1$, which is the condition **(h4)** of Corollary 2.7, and $\min(\mathcal{D}_+) \geq A \log n$. It follows that $\underline{s} \in E_{2,n}$. \square

From now on, we assume that κ is even.

Proposition 2.10. *Let κ be even. Then*

$$\lim_{n \rightarrow \infty} n^{-\kappa/2} \mathcal{S}_n = (\kappa-1)!! \sigma^\kappa.$$

Denote by $P_n \subset \llbracket 0, n-1 \rrbracket^\kappa$ the family of all \underline{s} such that we can divide the set $\{s_1, \dots, s_\kappa\}$ of the components of \underline{s} into $\kappa/2$ disjoint pairs $p_m = \{s_{i_m}, s_{j_m}\}$, $1 \leq m \leq \kappa/2$, with $i_m < j_m$ and such that for some integer $0 \leq l \leq \kappa-1$ we have

$$(2.5) \quad \max p_m - \min p_m \leq AB^l \log n \quad \text{and} \quad \min p_{m+1} - \max p_m \geq AB^{l+1} \log n \quad \text{for every } m.$$

Note that, as $B > 1$, for every $\underline{s} \in P_n$ and $1 \leq i \leq \kappa$, there is a unique $1 \leq j \leq \kappa$ with $j \neq i$ and such that $|s_i - s_j| \leq AB^l \log n$. So, the choice of the indexes i_m and j_m is uniquely determined by the above conditions.

Lemma 2.11. *We have*

$$\lim_{n \rightarrow \infty} n^{-\kappa/2} \mathcal{S}_n - n^{-\kappa/2} \mathcal{S}(P_n) = 0.$$

Proof. We claim that $\llbracket 0, n-1 \rrbracket^\kappa \setminus P_n$ is covered by $E_{1,n}$ and $E_{2,n}$. The result then follows from Lemmas 2.4 and 2.8.

We proceed as in Proposition 2.9. Take $\underline{s} \in \llbracket 0, n-1 \rrbracket^\kappa$. We can assume that $s_1 \leq \dots \leq s_\kappa$. Define $\underline{\delta} := (\delta_1, \dots, \delta_{\kappa-1})$ with $\delta_j := s_{j+1} - s_j$. If there are at least $\kappa/2 + 1$ components δ_j which are smaller than or equal to $D \log n$, we have $\underline{s} \in E_{1,n}$. So, we can assume that $\delta_j \leq D \log n$ for at most $\kappa/2$ indexes j and $\delta_j > D \log n$ for at least $\kappa/2 - 1$ indexes j . We can choose $0 \leq l \leq \kappa-1$ such that the interval $[AB^l \log n, AB^{l+1} \log n)$ does not contain any δ_j . Then, as above, we

consider the good partition $(\mathcal{D}_-, \mathcal{D}_+)$ of $\{\delta_j\}_{1 \leq j \leq \kappa-1}$, where \mathcal{D}_- (resp. \mathcal{D}_+) is the family of all the δ_j 's which are smaller than $AB^l \log n$ (resp. larger than or equal to $AB^{l+1} \log n$). We have $|\mathcal{D}_+| \geq \kappa/2 - 1$ and $\min(\mathcal{D}_+) \geq A \log n$. If $|\mathcal{D}_+| \geq \kappa/2$, then Condition **(h4)** in Corollary 2.7 holds and we have $\underline{s} \in E_{2,n}$. Hence, we can assume that $|\mathcal{D}_+| = \kappa/2 - 1$, which implies that $|\mathcal{D}_-| = \kappa/2$.

If one of the conditions **(h1)**, **(h2)**, **(h3)** in Corollary 2.7 is satisfied, then $\underline{s} \in E_{2,n}$. We can suppose that this is not the case. Using that $|\mathcal{D}_+| = \kappa/2 - 1$ and $|\mathcal{D}_-| = \kappa/2$, we deduce that

$$\mathcal{D}_- = \{\delta_1, \delta_3, \dots, \delta_{\kappa-1}\} \quad \text{and} \quad \mathcal{D}_+ = \{\delta_2, \delta_4, \dots, \delta_{\kappa-2}\}.$$

Consider $p_m := \{s_{2m-1}, s_{2m}\}$ for $1 \leq m \leq \kappa/2$. We have, by the construction of \mathcal{D}_\pm ,

$$\max p_m - \min p_m = s_{2m} - s_{2m-1} = \delta_{2m-1} < AB^l \log n$$

and

$$\min p_{m+1} - \max p_m = s_{2m+1} - s_{2m} = \delta_{2m} \geq AB^{l+1} \log n.$$

It follows that \underline{s} belongs to P_n . This completes the proof. \square

Denote by $P'_n \subset P_n$ the set of all $\underline{s} \in P_n$ such that $p_m = \{s_{2m-1}, s_{2m}\}$, where the p_m are as in the definition of P_n . Note that we do not require that s_{2m-1} is larger than s_{2m} nor conversely. So, in general, we have $s_1 < s_3 < \dots < s_{\kappa-1}$, but the sequence $(s_1, s_2, \dots, s_\kappa)$ is not increasing. We can define a map $\pi : P_n \rightarrow P'_n$ sending each \underline{s} to the unique element in P'_n obtained from \underline{s} by applying the permutation

$$(s_1, \dots, s_\kappa) \mapsto (s_{i_1}, s_{j_1}, s_{i_2}, s_{j_2}, \dots, s_{i_{\kappa/2}}, s_{j_{\kappa/2}}).$$

It is not difficult to see that each fibre of π is of cardinality $(\kappa-1)!!(\kappa/2)!$. Note that $(\kappa-1)!!$ is the number of possibilities to divide $\{1, \dots, \kappa\}$ into $\kappa/2$ disjoint pairs and $(\kappa/2)!$ is the number of possibilities to order these $\kappa/2$ pairs. Since the value of $\mathcal{I}_{\underline{s}}$ does not change if we permute the components of \underline{s} , we deduce that

$$(2.6) \quad \mathcal{S}(P_n) = (\kappa-1)!!(\kappa/2)!\mathcal{S}(P'_n).$$

Therefore, by Lemma 2.11, in order to establish Proposition 2.10, we need to check that $n^{-\kappa/2}(\kappa/2)!\mathcal{S}(P'_n) \rightarrow \sigma^\kappa$ as $n \rightarrow \infty$.

Recall that κ is assumed to be even. For $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_{\kappa/2}) \in \mathbb{Z}^{\kappa/2}$, let $N_n(\underline{\epsilon})$ be the number of elements $\underline{s} \in P'_n$ such that $s_{2m} - s_{2m-1} = \epsilon_m$ for all m .

Lemma 2.12. *We have*

$$0 \leq n^{-\kappa/2}(\kappa/2)!N_n(\underline{\epsilon}) \leq 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} n^{-\kappa/2}(\kappa/2)!N_n(\underline{\epsilon}) = 1.$$

Proof. We prove the first part of this lemma where only the second inequality is non trivial. Denote by $P'_n(\underline{\epsilon})$ the set of the elements of P'_n such that $s_{2m} - s_{2m-1} = \epsilon_m$ for all m . Then $N_n(\underline{\epsilon})$ is the cardinality of $P'_n(\underline{\epsilon})$. Observe that an element $\underline{s} \in P'_n(\underline{\epsilon})$ is determined by its $\kappa/2$ components of odd index, i.e., by $(s_1, s_3, \dots, s_{\kappa-1}) \in \llbracket 0, n-1 \rrbracket^{\kappa/2}$. Here, we have $s_1 < s_3 < \dots < s_{\kappa-1}$.

Let Q_n denote the set of all elements in $\llbracket 0, n-1 \rrbracket^{\kappa/2}$ whose components are pairwise distinct. Let $Q'_n \subset Q_n$ be the set of such elements whose components are in increasing order. Consider the map $\tau : Q_n \rightarrow Q'_n$ which assigns to each element of Q_n the unique element in Q'_n obtained from the first element by permuting its components. Since each fibre of this map contains exactly $(\kappa/2)!$ elements, we deduce that

$$(\kappa/2)!|Q'_n| = |Q_n| \leq \#\llbracket 0, n-1 \rrbracket^{\kappa/2} = n^{\kappa/2}.$$

Finally, since $N_n(\underline{\epsilon}) = |P'_n(\underline{\epsilon})| \leq |Q'_n|$, we obtain the desired inequality $n^{-\kappa/2}(\kappa/2)!N_n(\underline{\epsilon}) \leq 1$.

It remains to prove the second part of the lemma. We fix $\underline{\epsilon}$ and only consider n big enough. In particular, we have $|\epsilon_m| \ll \log n$ for every m . Denote by $R_n \subset Q_n$ the set of all elements $(s_1, s_3, \dots, s_{\kappa-1})$ such that $|s_{2m-1} - s_{2j-1}| \geq 3AB^\kappa \log n$ for all $m \neq j$. It is not difficult to see that if $(s_1, s_3, \dots, s_{\kappa-1})$ is an element of R_n and if we define

$$(s'_1, s'_3, \dots, s'_{\kappa-1}) := \tau(s_1, s_3, \dots, s_{\kappa-1}) \quad \text{and} \quad s'_{2m} := s'_{2m-1} + \epsilon_m,$$

then

$$\max(s'_{2m-1}, s'_{2m}) - \min(s'_{2m-1}, s'_{2m}) = |\epsilon_m| \leq AB^{\kappa-1} \log n$$

and

$$\begin{aligned} \min(s'_{2m+1}, s'_{2m+2}) - \max(s'_{2m-1}, s'_{2m}) &\geq |s'_{2m+1} - s'_{2m-1}| - |s'_{2m+1} - s'_{2m+2}| - |s'_{2m-1} - s'_{2m}| \\ &\geq 3AB^\kappa \log n - 2AB^{\kappa-1} \log n \geq AB^\kappa \log n. \end{aligned}$$

It follows that $\underline{s}' = (s'_1, \dots, s'_{\kappa-1})$ belongs to $P'_n(\underline{\epsilon})$. By observing that R_n is a union of fibres of τ , we conclude that $|R_n| \leq (\kappa/2)! |P'_n(\underline{\epsilon})|$.

Now, we bound $|R_n|$ from below. We first have n choices for $s_1 \in \llbracket 0, n-1 \rrbracket$. Given a choice for s_1 , we have at least $n - 7AB^\kappa \log n$ choices for s_3 such that $|s_1 - s_3| \geq 3AB^\kappa \log n$. When s_1 and s_3 are fixed, there are at least $n - 14AB^\kappa \log n$ choices for s_5 with $|s_1 - s_5| \geq 3AB^\kappa \log n$ and $|s_3 - s_5| \geq 3AB^\kappa \log n$. Arguing by induction, we get that

$$|R_n| \geq n(n - 7AB^\kappa \log n)(n - 14AB^\kappa \log n) \dots (n - 7(\kappa/2 - 1)AB^\kappa \log n).$$

Thus,

$$\liminf_{n \rightarrow \infty} n^{-\kappa/2} (\kappa/2)! N_n(\underline{\epsilon}) = \liminf_{n \rightarrow \infty} n^{-\kappa/2} (\kappa/2)! |P'_n(\underline{\epsilon})| \geq \liminf_{n \rightarrow \infty} n^{-\kappa/2} |R_n| \geq 1.$$

This and the first part of the lemma imply the result. \square

Recall that κ is even and fixed. For $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_{\kappa/2}) \in \mathbb{Z}^{\kappa/2}$ define

$$\mathcal{K}_{\underline{\epsilon}} := \prod_{m=1}^{\kappa/2} \langle \nu, u \cdot u_{|\epsilon_m|} \rangle.$$

We need the following estimate.

Lemma 2.13. *Given $\underline{s} \in P'_n$, define $\epsilon_m := s_{2m} - s_{2m-1}$. Then we have, for some constant $c > 0$ independent of \underline{s} ,*

$$|\mathcal{I}_{\underline{s}} - \mathcal{K}_{\underline{\epsilon}}| \leq cn^{-\kappa}.$$

Proof. Observe that the value of $\mathcal{I}_{\underline{s}}$ does not change if we permute the components of \underline{s} and the value of $\mathcal{K}_{\underline{\epsilon}}$ does not change if we change the sign of some components of $\underline{\epsilon}$. So, we can assume for simplicity that $s_1 \leq s_2 \leq \dots \leq s_\kappa$ and therefore that $\epsilon_m \geq 0$ for all m . Define also $\underline{\delta} := (\delta_1, \dots, \delta_{\kappa-1})$ with $\delta_j := s_{j+1} - s_j$ for $1 \leq j \leq \kappa - 1$. Observe that $\epsilon_m = \delta_{2m-1}$. Define $\mathcal{D}_- = \{\delta_1, \delta_3, \dots, \delta_{\kappa-1}\}$ and $\mathcal{D}_+ = \{\delta_2, \delta_4, \dots, \delta_{\kappa-2}\}$.

Since $\underline{s} \in P'_n$, the pair $(\mathcal{D}_-, \mathcal{D}_+)$ is a good partition of $\{\delta_1, \dots, \delta_{\kappa-1}\}$ and we also have $\min(\mathcal{D}_+) \geq A \log n$. By Lemma 2.6 and the choice of A , we have

$$|\mathcal{J}_{\delta_1, \dots, \delta_{\kappa-1}} - \mathcal{J}_{\mathcal{D}_-, \mathcal{D}_+}| \lesssim \eta^{\min(\mathcal{D}_+)} \leq \eta^{A \log n} = n^{-(\kappa+1)}.$$

On the other hand, it is easy to see that $\mathcal{I}_{\underline{s}} = \mathcal{J}_{\delta_1, \dots, \delta_{\kappa-1}}$ and $\mathcal{J}_{\mathcal{D}_-, \mathcal{D}_+} = \mathcal{K}_{\underline{\epsilon}}$. The lemma follows. \square

Lemma 2.14. *We have*

$$\lim_{n \rightarrow \infty} \left(n^{-\kappa/2} (\kappa/2)! \mathcal{S}(P'_n) - \sum_{\underline{\epsilon} \in \mathbb{Z}^{\kappa/2}} n^{-\kappa/2} (\kappa/2)! N_n(\underline{\epsilon}) \mathcal{K}_{\underline{\epsilon}} \right) = 0.$$

Proof. For this proof, we use the notation in the statement of Lemma 2.13. Recall that, given $\xi \in \mathbb{Z}^{\kappa/2}$, the number of the elements $\underline{s} \in P'_n$ such that $\underline{\epsilon} = \xi$ is equal to $N_n(\underline{\xi})$. So, we have

$$\begin{aligned}
\mathcal{S}(P'_n) - \sum_{\underline{\xi} \in \mathbb{Z}^{\kappa/2}} N_n(\underline{\xi}) \mathcal{K}_{\underline{\xi}} &= \sum_{\underline{s} \in P'_n} \mathcal{I}_{\underline{s}} - \sum_{\underline{\xi} \in \mathbb{Z}^{\kappa/2}} N_n(\underline{\xi}) \mathcal{K}_{\underline{\xi}} \\
&= \sum_{\underline{\xi} \in \mathbb{Z}^{\kappa/2}} \sum_{\underline{s} \in P'_n: \underline{\epsilon}=\underline{\xi}} \mathcal{I}_{\underline{s}} - \sum_{\underline{\xi} \in \mathbb{Z}^{\kappa/2}} N_n(\underline{\xi}) \mathcal{K}_{\underline{\xi}} \\
&= \sum_{\underline{\xi} \in \mathbb{Z}^{\kappa/2}} \sum_{\underline{s} \in P'_n: \underline{\epsilon}=\underline{\xi}} (\mathcal{I}_{\underline{s}} - \mathcal{K}_{\underline{\xi}}) \\
&= \sum_{\underline{s} \in P'_n} (\mathcal{I}_{\underline{s}} - \mathcal{K}_{\underline{\epsilon}}).
\end{aligned}$$

By Lemma 2.13, we have $|\mathcal{I}_{\underline{s}} - \mathcal{K}_{\underline{\epsilon}}| \lesssim n^{-\kappa}$ for all $\underline{s} \in P'_n$. It follows that

$$\left| \mathcal{S}(P'_n) - \sum_{\underline{\xi} \in \mathbb{Z}^{\kappa/2}} N_n(\underline{\xi}) \mathcal{K}_{\underline{\xi}} \right| \lesssim |P'_n| n^{-\kappa}.$$

As $|P'_n| \leq n^\kappa$ (since $P'_n \subset P_n \subset \llbracket 0, n-1 \rrbracket^\kappa$), we easily get the result from the last estimate. \square

Consider now the space $\mathcal{X} := \mathbb{Z}^{\kappa/2}$ endowed with the positive measure \mathfrak{m} which is the sum of all Dirac masses on \mathcal{X} . Then $\mathcal{K}_{\underline{\epsilon}}$ defines a function on \mathcal{X} that we denote by \mathcal{K} .

Lemma 2.15. *The function \mathcal{K} is integrable with respect to \mathfrak{m} and we have*

$$\langle \mathfrak{m}, \mathcal{K} \rangle = \sigma^\kappa.$$

Proof. By the exponential mixing of order 1, we have that

$$|\langle \nu, u \cdot u_{|\epsilon|} \rangle| \lesssim \theta^{|\epsilon|} \quad \text{for every } \epsilon \in \mathbb{Z}.$$

It follows from this inequality and the definition of \mathcal{K} that

$$\langle \mathfrak{m}, |\mathcal{K}| \rangle = \sum_{\underline{\epsilon} \in \mathbb{Z}^{\kappa/2}} |\mathcal{K}_{\underline{\epsilon}}| = \sum_{\underline{\epsilon} \in \mathbb{Z}^{\kappa/2}} \prod_{m=1}^{\kappa/2} |\langle \nu, u \cdot u_{|\epsilon_m|} \rangle| \lesssim \sum_{\underline{\epsilon} \in \mathbb{Z}^{\kappa/2}} \prod_{m=1}^{\kappa/2} \theta^{|\epsilon_m|} = \left(\sum_{\epsilon \in \mathbb{Z}} \theta^{|\epsilon|} \right)^{\kappa/2}.$$

The last series is convergent since $0 < \theta < 1$. The integrability of \mathcal{K} with respect to \mathfrak{m} follows.

As \mathcal{K} is integrable, we can apply Fubini's theorem and deduce that

$$\langle \mathfrak{m}, \mathcal{K} \rangle = \left(\sum_{\epsilon \in \mathbb{Z}} \langle \nu, u \cdot u_{|\epsilon|} \rangle \right)^{\kappa/2} = \sigma^\kappa,$$

where we have used Lemma 2.2. \square

End of the proofs of Proposition 2.10 and Theorem 1.3. Define the function \mathcal{K}_n on \mathcal{X} by setting

$$\mathcal{K}_n(\underline{\epsilon}) := n^{-\kappa/2} (\kappa/2)! N_n(\underline{\epsilon}) \mathcal{K}_{\underline{\epsilon}}.$$

According to Lemma 2.12, we have $|\mathcal{K}_n| \leq \mathcal{K}$ and \mathcal{K}_n converges pointwise to \mathcal{K} . By Lebesgue's dominated convergence theorem and the fact that \mathcal{K} is integrable, we deduce that

$$\lim_{n \rightarrow \infty} \langle \mathfrak{m}, \mathcal{K}_n \rangle = \langle \mathfrak{m}, \mathcal{K} \rangle = \sigma^\kappa.$$

Proposition 2.10 then follows from Lemmas 2.11, 2.14, and (2.6). As Theorem 1.3 is a consequence of Propositions 2.9, 2.10, and Lemma 2.3, the proof of that theorem is complete. \square

3. REGULAR AUTOMORPHISMS OF \mathbb{C}^k AND CONVERGENCE TOWARDS GREEN CURRENTS

Let F be a polynomial automorphism of \mathbb{C}^k . We still denote by F its extension as a birational map of \mathbb{P}^k . Denote by $\mathbb{H}_\infty := \mathbb{P}^k \setminus \mathbb{C}^k$ the hyperplane at infinity and by $\mathbb{I}^+, \mathbb{I}^-$ the indeterminacy sets of F and F^{-1} respectively. They are analytic sets strictly contained in \mathbb{H}_∞ . If $\mathbb{I}^+ = \emptyset$ or $\mathbb{I}^- = \emptyset$, then both of them are empty and F is given by a linear map and its dynamics is easy to describe. Hence, we assume that $\mathbb{I}^\pm \neq \emptyset$. The following definition is due to Sibony [52].

Definition 3.1. We say that F is a *regular automorphism of \mathbb{C}^k* if $\mathbb{I}^\pm \neq \emptyset$ and $\mathbb{I}^+ \cap \mathbb{I}^- = \emptyset$.

Given F a regular automorphism of \mathbb{C}^k , it is clear that F^{-1} is also regular. We denote by $d_+(F)$ and $d_-(F)$ the algebraic degrees of F and F^{-1} respectively. Observe that $d_\pm(F) \geq 2$, $d_+(F) = d_-(F^{-1})$ and $d_-(F) = d_+(F^{-1})$. Later, we will drop the letter F and just write d_\pm instead of $d_\pm(F)$ for simplicity. We will recall here some basic properties of F and refer the reader to [4, 27, 33, 34, 52] for details.

Proposition 3.2. *Let F be a regular automorphism of \mathbb{C}^k as above.*

- (i) *There exists an integer $1 \leq p \leq k-1$ such that $\dim \mathbb{I}^+ = k-p-1$, $\dim \mathbb{I}^- = p-1$, and $d_+(F)^p = d_-(F)^{k-p}$.*
- (ii) *The analytic sets \mathbb{I}^\pm are irreducible and we have*

$$F(\mathbb{H}_\infty \setminus \mathbb{I}^+) = F(\mathbb{I}^-) = \mathbb{I}^- \quad \text{and} \quad F^{-1}(\mathbb{H}_\infty \setminus \mathbb{I}^-) = F^{-1}(\mathbb{I}^+) = \mathbb{I}^+.$$

- (iii) *For every $n \geq 1$, both F^n and F^{-n} are regular automorphisms of \mathbb{C}^k , of algebraic degrees $d_+(F)^n$ and $d_-(F)^n$, and indeterminacy sets \mathbb{I}^+ and \mathbb{I}^- , respectively.*

Example 3.3. (Generalized) Hénon maps on \mathbb{C}^2 correspond to the case $k=2$ in Definition 3.1. In this case, we have $p = k-p = 1$ and $d_+ = d_- = d$, the algebraic degree of the map, see [4, 32, 52].

The set \mathbb{I}^+ (resp. \mathbb{I}^-) is attracting for F^{-1} (resp. F). Let \widetilde{W}^\pm be the basin of attraction of \mathbb{I}^\pm . Set $W^\pm := \widetilde{W}^\pm \cap \mathbb{C}^k$. Then the sets $\mathbb{K}^+ := \mathbb{C}^k \setminus W^-$ and $\mathbb{K}^- := \mathbb{C}^k \setminus W^+$ are the sets of points (in \mathbb{C}^k) with bounded orbit for F and F^{-1} , respectively. We have $\overline{\mathbb{K}^+} = \mathbb{K}^+ \cup \mathbb{I}^+$ and $\overline{\mathbb{K}^-} = \mathbb{K}^- \cup \mathbb{I}^-$ where the closures are taken in \mathbb{P}^k . We also define $\mathbb{K} := \mathbb{K}^+ \cap \mathbb{K}^-$ which is a compact subset of \mathbb{C}^k .

In the terminology of [27], the set $\overline{\mathbb{K}^+}$ (resp. $\overline{\mathbb{K}^-}$) is *p-rigid* (resp. *(k-p)-rigid*): it supports a unique positive closed (p, p) -current (resp. $(k-p, k-p)$ -current) of mass 1, that we denote by \mathbb{T}_+ (resp. \mathbb{T}_-). The currents \mathbb{T}_\pm have no mass on \mathbb{H}_∞ and satisfy the invariance relations

$$F^*(\mathbb{T}_+) = d_+^p \mathbb{T}_+ \quad \text{and} \quad F_*(\mathbb{T}_-) = d_-^{k-p} \mathbb{T}_-$$

as currents on \mathbb{C}^k or \mathbb{P}^k . We call them *the main Green currents of F* . They can be obtained as intersections of positive closed $(1, 1)$ -currents with local Hölder continuous potentials in \mathbb{C}^k . Therefore, the measure $\mathbb{T}_+ \wedge \mathbb{T}_-$ is well-defined and supported by the compact set \mathbb{K} . This is the unique invariant probability measure of maximal entropy [19, 52], see also [3, 4, 5, 29] for the case of dimension $k=2$.

Using the above description of the dynamics of F , we can fix neighbourhoods U_1, U_2 of $\overline{\mathbb{K}^+}$ and V_1, V_2 of $\overline{\mathbb{K}^-}$ such that $F^{-1}(U_i) \Subset U_i$, $U_1 \Subset U_2 \Subset \mathbb{P}^k \setminus \mathbb{I}^-$, $F(V_i) \Subset V_i$, $V_1 \Subset V_2 \Subset \mathbb{P}^k \setminus \mathbb{I}^+$, and $U_2 \cap V_2 \Subset \mathbb{C}^k$. Let Ω be a real $(p+1, p+1)$ -current with compact support in U_1 . Assume that there exists a positive closed $(p+1, p+1)$ -current Ω' with compact support in U_1 such that $|\Omega| \leq \Omega'$. Define the norm $\|\Omega\|_{*, U_1}$ of Ω as

$$\|\Omega\|_{*, U_1} := \inf \{ \|\Omega'\| : |\Omega| \leq \Omega' \}.$$

Observe that when Ω is a d -exact current we can write $\Omega = \Omega' - (\Omega' - \Omega)$, which is the difference of two positive closed current in the same cohomology class in $H^{p+1, p+1}(\mathbb{P}^k, \mathbb{R})$. Therefore, the norm $\|\cdot\|_{*, U_1}$ is equivalent to the norm given by $\inf \|\Omega^\pm\|$, where Ω^\pm are positive closed currents

with compact support in U_1 such that $\Omega = \Omega^+ - \Omega^-$. Note that Ω^+ and Ω^- have the same mass as they belong to the same cohomology class.

The following property was obtained by the second author, see [23, Proposition 2.1].

Proposition 3.4. *Let R be a positive closed $(k-p, k-p)$ -current of mass 1 with compact support in V_1 and smooth on \mathbb{C}^k . Let Φ be a real-valued (p, p) -form of class \mathcal{C}^2 with compact support in $U_1 \cap \mathbb{C}^k$. Assume that $dd^c\Phi \geq 0$ on V_2 . Then there exists a constant $c > 0$ independent of R and Φ such that*

$$\langle d_-(^{k-p}n)(F^n)_*(R) - \mathbb{T}_-, \Phi \rangle \leq c d_-^n \|dd^c\Phi\|_{*,U_1} \quad \text{for all } n \geq 0.$$

Note that in what follows, since \mathbb{T}_- is an intersection of positive closed $(1, 1)$ -currents with local continuous potentials [52], the intersections $R \wedge \mathbb{T}_-$ and $\mathbb{T}_+ \wedge \mathbb{T}_-$ are well-defined and the former depends continuously on R . In particular, the pairing in the next statement is meaningful and depends continuously on R .

Corollary 3.5. *Let R be a positive closed $(k-p, k-p)$ -current of mass 1 supported in V_1 . Let ϕ be a \mathcal{C}^2 function with compact support on \mathbb{C}^k such that $dd^c\phi \geq 0$ in a neighbourhood of $\mathbb{K}_+ \cap V_2$. Then there exists a constant $c > 0$ independent of R and ϕ such that*

$$(3.1) \quad \langle d_-(^{k-p}n)(F^n)_*(R) - \mathbb{T}_-, \phi\mathbb{T}_+ \rangle \leq c d_-^n \|dd^c\phi \wedge \mathbb{T}_+\|_{*,U_1} \quad \text{for all } n \geq 0.$$

Proof. As \mathbb{P}^k is homogeneous, we will use the group $\text{PGL}(k+1, \mathbb{C})$ of automorphisms of \mathbb{P}^k and suitable convolutions in order to regularize the currents R and $\phi\mathbb{T}_+$ and deduce the result from Proposition 3.4. Choose local coordinates centered at the identity $\text{id} \in \text{PGL}(k+1, \mathbb{C})$ so that a small neighbourhood of id in $\text{PGL}(k+1, \mathbb{C})$ is identified to the unit ball \mathbb{B} of \mathbb{C}^{k^2+2k} . Here, a point of coordinates ϵ represents an automorphism of \mathbb{P}^k that we denote by τ_ϵ . Thus, $\tau_0 = \text{id}$.

Consider a smooth non-negative function ρ with compact support on \mathbb{B} and of integral 1 with respect to the Lebesgue measure and, for $0 < r \leq 1$, define $\rho_r(\epsilon) := r^{-2k^2-4k}\rho(r^{-1}\epsilon)$, which is supported by $\{|\epsilon| \leq r\}$. This function allows us to define an approximation of the Dirac mass at $0 \in \mathbb{B}$ when $r \rightarrow 0$. We define $\Psi := \phi\mathbb{T}_+$ and consider the following regularized currents

$$R_r := \int \rho_r(\epsilon)(\tau_\epsilon)^*(R) \quad \text{and} \quad \Psi_r := \int \rho_r(\epsilon)(\tau_\epsilon)^*(\Psi) = \int \rho_r(\epsilon)(\phi \circ \tau_\epsilon)(\tau_\epsilon)^*(\mathbb{T}_+),$$

where the integrals are with respect to the Lebesgue measure on $\epsilon \in \mathbb{B}$.

When r is small enough and goes to 0, the current R_r is smooth, positive, closed, with compact support in V_1 , and converges to R . Since the RHS of (3.1) depends continuously on R , we can replace R by R_r and assume that R is smooth. When ϵ goes to 0, $\phi \circ \tau_\epsilon$ converges uniformly to ϕ and $(\tau_\epsilon)^*(\mathbb{T}_+)$ converges to \mathbb{T}_+ . Using that R is smooth and \mathbb{T}_- is a product of $(1, 1)$ -currents with continuous potentials, we deduce that the LHS of (3.1) is equal to

$$\lim_{r \rightarrow 0} \langle d_-(^{k-p}n)(F^n)_*(R) - \mathbb{T}_-, \Psi_r \rangle.$$

Since \mathbb{T}_+ is supported by $\overline{\mathbb{K}_+}$ and we have $dd^c\phi \geq 0$ on a neighbourhood of $\mathbb{K}_+ \cap V_2$, we deduce that $dd^c\Psi \geq 0$ on V_2 . By reducing slightly V_2 , we still have $dd^c\Psi_r \geq 0$ on V_2 for r small enough. We will use the last limit and Proposition 3.4 for Ψ_r instead of Φ and U_2 instead of U_1 . Observe that for ϵ small enough, since $U_1 \Subset U_2$, we have $\|(\tau_\epsilon)^*(dd^c\Psi)\|_{*,U_2} \leq \|dd^c\Psi\|_{*,U_1}$. We deduce that the LHS of (3.1) is smaller than or equal to

$$\lim_{r \rightarrow 0} c d_-^n \|dd^c\Psi_r\|_{*,U_2} \leq c d_-^n \|dd^c\Psi\|_{*,U_1} = c d_-^n \|dd^c\phi \wedge \mathbb{T}_+\|_{*,U_1}.$$

This completes the proof of the corollary. \square

In order to use the above corollary, we will need the following lemmas.

Lemma 3.6. *Let $\kappa \geq 1$ be an integer and g_0, \dots, g_κ compactly supported functions on \mathbb{C}^k with $\|g_j\|_{\mathcal{C}^2} \leq 1$. Then there is a constant $c_\kappa > 0$ independent of the g_j 's such that for all $\ell_0, \dots, \ell_\kappa \geq 0$ we have*

$$\|dd^c((g_0 \circ F^{\ell_0}) \dots (g_\kappa \circ F^{\ell_\kappa})) \wedge \mathbb{T}_+\|_{*,U_1} \leq c_\kappa.$$

Proof. Set $\tilde{g}_j := g_j \circ f^{\ell_j}$ for simplicity. We have

$$dd^c(\tilde{g}_0 \dots \tilde{g}_\kappa) = \sum_{j=0}^{\kappa} dd^c \tilde{g}_j \prod_{l \neq j} \tilde{g}_l + \sum_{0 \leq j, l \leq \kappa} i \partial \tilde{g}_j \wedge \bar{\partial} \tilde{g}_l \prod_{m \neq j, l} \tilde{g}_m.$$

Since $\|g_j\|_{\mathcal{C}^2} \leq 1$ we have $|g_j| \leq 1$. Denote by ω_{FS} the Fubini-Study form on \mathbb{P}^k . Then

$$\left| \sum_{j=0}^{\kappa} dd^c \tilde{g}_j \prod_{l \neq j} \tilde{g}_l \right| \lesssim \sum_{j=0}^{\kappa} (F^{\ell_j})^* \omega_{\text{FS}}$$

and an application of Cauchy-Schwarz inequality gives

$$\begin{aligned} \left| \sum_{0 \leq j, l \leq \kappa} i \partial \tilde{g}_j \wedge \bar{\partial} \tilde{g}_l \prod_{m \neq j, l} \tilde{g}_m \right| &\lesssim \sum_{j=0}^{\kappa} i \partial \tilde{g}_j \wedge \bar{\partial} \tilde{g}_j \\ &= \sum_{j=0}^{\kappa} (F^{\ell_j})^* (i \partial g_j \wedge \bar{\partial} g_j) \\ &\lesssim \sum_{j=0}^{\kappa} (F^{\ell_j})^* (\omega_{\text{FS}}). \end{aligned}$$

As we have $dd^c(\tilde{g}_0 \dots \tilde{g}_\kappa) = 0$ near \mathbb{H}_∞ , its intersection with \mathbb{T}_+ can be computed on \mathbb{C}^k . We deduce from the above inequalities and $d_-^{k-p} = d_+^p$ that

$$\begin{aligned} (3.2) \quad |dd^c((g_0 \circ F^{\ell_0}) \dots (g_\kappa \circ F^{\ell_\kappa})) \wedge \mathbb{T}_+| &\lesssim \sum_{j=0}^{\kappa} (F^{\ell_j})^* (\omega_{\text{FS}}) \wedge \mathbb{T}_+ \\ &= \sum_{j=0}^{\kappa} (F^{\ell_j})^* (\omega_{\text{FS}}) \wedge d_+^{-p \ell_j} (F^{\ell_j})^* \mathbb{T}_+ \\ &= \sum_{j=0}^{\kappa} d_-^{-(k-p) \ell_j} (F^{\ell_j})^* (\omega_{\text{FS}} \wedge \mathbb{T}_+). \end{aligned}$$

We will use that the $(p+1, p+1)$ -current $\omega_{\text{FS}} \wedge \mathbb{T}_+$ is positive, closed, of mass 1, and its support is contained in $\mathbb{K}^+ \subset U_1$. We have

$$\|(F^{\ell_j})^* (\omega_{\text{FS}} \wedge \mathbb{T}_+)\| = \langle (F^{\ell_j})^* (\omega_{\text{FS}} \wedge \mathbb{T}_+), \omega_{\text{FS}}^{k-p-1} \rangle = \langle \omega_{\text{FS}} \wedge \mathbb{T}_+, (F^{-\ell_j})^* (\omega_{\text{FS}}^{k-p-1}) \rangle,$$

where the last form is positive closed and smooth outside \mathbb{I}^- . The last pairing only depends on the cohomology classes of ω_{FS} , \mathbb{T}_+ , and $(F^{-\ell_j})^* (\omega_{\text{FS}}^{k-p-1})$. Hence, it is equal to the mass of $(F^{-\ell_j})^* (\omega_{\text{FS}}^{k-p-1})$, which is equal to $d_-^{(k-p-1) \ell_j}$, see [52]. It follows that each term in the last sum in (3.2) is bounded by 1, which implies that the sum is bounded by $\kappa + 1$. The lemma follows. \square

Lemma 3.7. *Let $D \Subset D'$ be two bounded domains in \mathbb{C}^k . Let g be a function with compact support in D and such that $\|g\|_{\mathcal{C}^2} \leq 1$. Then there are a constant $A > 0$ independent of g and functions g^\pm with compact supports in D' and $\|g^\pm\|_{\mathcal{C}^2} \leq 1$ such that*

$$g = A(g^+ - g^-), \quad i \partial g^+ \wedge \bar{\partial} g^+ \leq dd^c g^+ \text{ on } D \quad \text{and} \quad i \partial g^- \wedge \bar{\partial} g^- \leq dd^c g^- \text{ on } D.$$

Proof. Let ρ be a smooth non-negative function, compactly supported on D' and equal to 1 in a neighbourhood of \bar{D} . Observe that $\rho g = g$. We denote by z the coordinates of \mathbb{C}^k . Since $\|g\|_{\mathcal{C}^2} \leq 1$ and g has compact support in D , there exists a constant $A_1 > 0$ independent of g such that $|dd^c g| \leq A_1 dd^c(\|z\|^2)$. Set $g^+ := A^{-1} \rho(g + 2A_1 \|z\|^2)$ and $g^- := 2A^{-1} A_1 \rho \|z\|^2$ for some constant $A > 0$. It is not difficult to check that we have $dd^c g^\pm \geq A^{-1} A_1 dd^c(\|z\|^2)$ on D , $\|i \partial g^\pm \wedge \bar{\partial} g^\pm\|_\infty = O(A^{-2})$ on D , $g = A(g^+ - g^-)$, and $\|g^\pm\|_{\mathcal{C}^2} = O(A^{-1})$. Taking A large enough gives the lemma. \square

4. EXPONENTIAL MIXING OF ALL ORDERS FOR HÉNON MAPS AND FURTHER REMARKS

Throughout this section (except for Remarks 4.2, 4.3, and 4.4), f denotes a Hénon map on \mathbb{C}^2 of algebraic degree $d = d_+ = d_- \geq 2$. Define $F := (f, f^{-1})$. It is not difficult to check that F is a regular automorphism of $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$. We will use the notations and the results of Section 3 with $k = 4$ and $p = 2$. We denote in this section by T_{\pm} the Green $(1, 1)$ -currents of f , and reserve the notation \mathbb{T}_{\pm} for the main Green currents of F . Observe that $\mathbb{T}_+ = T_+ \otimes T_-$ and $\mathbb{T}_- = T_- \otimes T_+$, see [31, Section 4.1.8] for the tensor (or cartesian) product of currents. We denote by K^{\pm} the sets of points of bounded orbit for $f^{\pm 1}$. The wedge product $\mu := T_+ \wedge T_-$ is well defined, and is the measure of maximal entropy of f [3, 4, 5, 52]. Its support is contained in the compact set $K = K^+ \cap K^-$. We have $\mathbb{K}^+ = K^+ \times K^-$ and $\mathbb{K}^- = K^- \times K^+$. Note also that the diagonal Δ of $\mathbb{C}^2 \times \mathbb{C}^2$ satisfies $\overline{\Delta} \cap \mathbb{K}^+ = \emptyset$ and $\overline{\Delta} \cap \mathbb{K}^- = \emptyset$ in \mathbb{P}^4 , see also [23].

We now prove Theorem 1.4. By a standard interpolation [55] (see for instance [23, pp. 262-263] and [28, Corollary 1] for similar occurrences) it is enough to prove the statement for $\gamma = 2$, i.e., in the case where all the functions g_j are of class \mathcal{C}^2 . The statement is clear for $\kappa = 0$, i.e., for one test function. By induction, we can assume that the statement holds for up to κ test functions and prove it for $\kappa + 1 \geq 1$ test functions, i.e., show that

$$\left| \langle \mu, g_0(g_1 \circ f^{n_1}) \dots (g_{\kappa} \circ f^{n_{\kappa}}) \rangle - \prod_{j=0}^{\kappa} \langle \mu, g_j \rangle \right| \lesssim \left(\prod_{j=0}^{\kappa} \|g_j\|_{\mathcal{C}^2} \right) \cdot d^{-\min_{0 \leq j \leq \kappa-1} (n_{j+1} - n_j)/2}.$$

Recall that $n_0 = 0$. The induction assumption implies that we are allowed to modify each g_j by adding a constant. Moreover, using the invariance of ν , the desired estimate does not change if we replace n_j by $n_j - 1$ for $1 \leq j \leq \kappa$ and g_0 by $g_0 \circ f^{-1}$. Therefore, we can for convenience assume that n_1 is even.

We fix a large bounded domain $B \subset \mathbb{C}^2$ satisfying

$$K \subset B, \quad K^- \cap B \subset f(B), \quad \text{and} \quad K^+ \cap B \subset f^{-1}(B).$$

By induction, the inclusions above imply that

$$(4.1) \quad K \subset B, \quad K^- \cap B \subset f^n(B), \quad \text{and} \quad K^+ \cap B \subset f^{-n}(B) \quad \text{for all } n \geq 1.$$

Because of Lemma 3.7 and the fact that we are only interested in the values of the g_j 's on the support of μ , we can assume that all the g_j 's are compactly supported in \mathbb{C}^2 and satisfy

$$(4.2) \quad \|g_j\|_{\mathcal{C}^2} \leq 1 \text{ on } \mathbb{C}^2 \quad \text{and} \quad i\partial g_j \wedge \bar{\partial} g_j \leq dd^c g_j \text{ on } B.$$

For simplicity, write $h := g_1(g_2 \circ f^{n_2 - n_1}) \dots (g_{\kappa} \circ f^{n_{\kappa} - n_1})$. We need to prove that

$$|\langle \mu, g_0(h \circ f^{n_1}) \rangle - \langle \mu, g_0 \rangle \cdot \langle \mu, h \rangle| \lesssim d^{-\min_{0 \leq j \leq \kappa-1} (n_{j+1} - n_j)/2}$$

since this estimate, together with the induction assumption applied to $\langle \mu, h \rangle$, would imply the desired statement. In order to obtain the result, we will prove separately the two estimates

$$(4.3) \quad \langle \mu, g_0(h \circ f^{n_1}) \rangle - \langle \mu, g_0 \rangle \cdot \langle \mu, h \rangle \lesssim d^{-\min_{0 \leq j \leq \kappa-1} (n_{j+1} - n_j)/2}$$

and

$$(4.4) \quad -\langle \mu, g_0(h \circ f^{n_1}) \rangle + \langle \mu, g_0 \rangle \cdot \langle \mu, h \rangle \lesssim d^{-\min_{0 \leq j \leq \kappa-1} (n_{j+1} - n_j)/2}.$$

Set $M := 10\kappa$ and fix a smooth function χ with compact support in \mathbb{C}^2 and equal to 1 in a neighbourhood of \overline{B} . Consider the following four functions, which will later allow us to produce some p.s.h. test functions:

$$g_0^+ := \chi \cdot (g_0 + M) \quad \text{and} \quad h^+ := \chi \cdot (g_1 + M)(g_2 \circ f^{n_2 - n_1} + M) \dots (g_{\kappa} \circ f^{n_{\kappa} - n_1} + M)$$

and

$$g_0^- := \chi \cdot (M - g_0) \quad \text{and} \quad h^- := \chi \cdot ((g_1 + M)(g_2 \circ f^{n_2 - n_1} + M) \dots (g_{\kappa} \circ f^{n_{\kappa} - n_1} + M) - 2(M + 1)^{\kappa}).$$

Recall that $n_0 = 0$. To prove (4.3) and (4.4), it is enough to show that

$$(4.5) \quad \langle \mu, g_0^+(h^+ \circ f^{n_1}) \rangle - \langle \mu, g_0^+ \rangle \cdot \langle \mu, h^+ \rangle \lesssim d^{-n_1/2}$$

and

$$(4.6) \quad \langle \mu, g_0^-(h^- \circ f^{n_1}) \rangle - \langle \mu, g_0^- \rangle \cdot \langle \mu, h^- \rangle \lesssim d^{-n_1/2}.$$

Indeed, we observe that χ does not play any role in (4.5) and (4.6). Hence, the difference between the LHS of (4.5) and the one of (4.3) (resp. of (4.6) and of (4.4)) is a finite combination of expressions involving no more than κ functions among g_0, \dots, g_κ , that we can estimate using the induction hypothesis on the mixing of order up to $\kappa - 1$. It remains to prove the two inequalities (4.5) and (4.6).

Denote by (z, w) the coordinates on $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$ and define

$$\phi^\pm(z, w) := g_0^\pm(w) h^\pm(z).$$

We have the following lemma for a fixed domain U_1 as in Section 3.

Lemma 4.1. *The functions ϕ^\pm satisfy*

- (i) $dd^c \phi^\pm \wedge \mathbb{T}_+ \geq 0$ on $B \times B$;
- (ii) $\|dd^c \phi^\pm \wedge \mathbb{T}_+\|_{*, U_1} \leq c_\kappa$,

where c_κ is a positive constant depending on κ , but not on the g_j 's and the n_j 's.

Proof. (i) For simplicity, we set $\ell_0 = \ell_1 := 0$ and $\ell_j := n_j - n_1$. Define also $\tilde{g}_j := g_j \circ f^{\ell_j}$. In what follows \tilde{g}_0 depends on w and \tilde{g}_j depends on z when $j \geq 1$. Observe that by the invariance property of $K^+ \cap B$ in (4.1) and the constraints in (4.2), the following inequalities hold in a neighbourhood of $K^+ \cap B$:

$$(4.7) \quad i\partial\tilde{g}_j \wedge \bar{\partial}\tilde{g}_j = (f^{\ell_j})^*(i\partial g_j \wedge \bar{\partial} g_j) \leq (f^{\ell_j})^*(dd^c g_j) = dd^c \tilde{g}_j.$$

In particular, we have $dd^c \tilde{g}_j \geq 0$ in a neighbourhood of $K^+ \cap B$. Note that for $\tilde{g}_0 = g_0$ the properties hold on B , which contains $K^- \cap B$.

Now, since \mathbb{T}_+ is closed, positive, and supported by $\overline{\mathbb{K}^+} = \overline{K^+ \times K^-}$, in order to prove the first assertion it is enough to show that $dd^c \phi^\pm \geq 0$ on a neighbourhood of $(K^+ \cap B) \times (K^- \cap B)$ in \mathbb{C}^4 where $\chi = 1$. In what follows, we only work on such a neighbourhood. We have

$$dd^c \phi^+ = \sum_{j=0}^{\kappa} dd^c \tilde{g}_j \prod_{l \neq j} (\tilde{g}_l + M) + \sum_{0 \leq j, l \leq \kappa} i\partial\tilde{g}_j \wedge \bar{\partial}\tilde{g}_l \prod_{m \neq j, l} (\tilde{g}_m + M),$$

where we recall that \tilde{g}_0 is $\tilde{g}_0(w)$ and the other \tilde{g}_j 's are $\tilde{g}_j(z)$ for $1 \leq j \leq \kappa$. For the first term in the RHS of the last expression, we have

$$\sum_{j=0}^{\kappa} dd^c \tilde{g}_j \prod_{l \neq j} (\tilde{g}_l + M) \geq (M - 1)^\kappa \sum_{j=0}^{\kappa} dd^c \tilde{g}_j.$$

For the second term, an application of Cauchy-Schwarz inequality and (4.7) give

$$\begin{aligned} \left| \sum_{0 \leq j, l \leq \kappa} i\partial\tilde{g}_j \wedge \bar{\partial}\tilde{g}_l \prod_{m \neq j, l} (\tilde{g}_m + M) \right| &\leq (\kappa + 1)(M + 1)^{\kappa-1} \sum_{j=0}^{\kappa} i\partial\tilde{g}_j \wedge \bar{\partial}\tilde{g}_j \\ &\leq (\kappa + 1)(M + 1)^{\kappa-1} \sum_{j=0}^{\kappa} dd^c \tilde{g}_j. \end{aligned}$$

It follows that

$$\begin{aligned} dd^c \phi^+ &\geq (M - 1)^\kappa \sum_{j=0}^{\kappa} dd^c \tilde{g}_j - (\kappa + 1)(M + 1)^{\kappa-1} \sum_{j=0}^{\kappa} dd^c \tilde{g}_j \\ &= (M - 1)^\kappa \left[1 - \frac{(\kappa + 1)}{M + 1} \left(1 + \frac{2}{M - 1} \right)^\kappa \right] \sum_{j=0}^{\kappa} dd^c \tilde{g}_j, \end{aligned}$$

which gives $dd^c\phi^+ \geq 0$ since the choice $M = 10\kappa$ implies that $(1 + \frac{2}{M-1})^\kappa < (1 + \frac{1}{\kappa})^\kappa < 3$. Similarly, in the same way, we also have

$$dd^c\phi^- \geq (M-1)^\kappa \sum_{j=0}^{\kappa} dd^c\tilde{g}_j - (M+1)^{\kappa-1} \sum_{0 \leq j, l \leq \kappa} |i\partial\tilde{g}_j \wedge \bar{\partial}\tilde{g}_l|,$$

which gives $dd^c\phi^- \geq 0$. This concludes the proof of the first assertion of the lemma.

(ii) The second assertion of the lemma is a consequence of Lemma 3.6. \square

End of the proof of Theorem 1.4. Recall that it remains to prove (4.5) and (4.6). Since μ is invariant and n_1 is even, we have

$$\langle \mu, g_0^\pm \cdot (h^\pm \circ f^{n_1}) \rangle = \langle \mu, (g_0^\pm \circ f^{-n_1/2})(h^\pm \circ f^{n_1/2}) \rangle.$$

We now transform the last integral on \mathbb{C}^2 to an integral on \mathbb{C}^4 in order to use the dynamical system $F = (f, f^{-1})$ on \mathbb{C}^4 introduced above. We are using the coordinates (z, w) on $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$. We will also use the diagonal of $\mathbb{C}^2 \times \mathbb{C}^2$, which is given by $\Delta = \{(z, w) : z = w\}$.

Recall that we have $\mu = T_+ \wedge T_-$ and that the currents T_\pm have local continuous potentials in \mathbb{C}^2 . It follows that the intersections of \mathbb{T}_\pm with positive closed currents on \mathbb{C}^4 are meaningful. Moreover, the invariance of T_\pm implies that $(F^{n_1/2})_*(\mathbb{T}_+) = d^{-n_1}\mathbb{T}_+$ on \mathbb{C}^4 . Thanks to the above identities, we have

$$\begin{aligned} \langle \mu, g_0^\pm \cdot (h^\pm \circ f^{n_1}) \rangle &= \langle T_+ \wedge T_-, (g_0^\pm \circ f^{-n_1/2})(h^\pm \circ f^{n_1/2}) \rangle \\ &= \langle (T_+ \otimes T_-) \wedge [\Delta], (g_0^\pm \circ f^{-n_1/2})(w)(h^\pm \circ f^{n_1/2}(z)) \rangle \\ (4.8) \quad &= \langle \mathbb{T}_+ \wedge [\Delta], (F^{n_1/2})_*(\phi^\pm) \rangle \\ &= \langle d^{-n_1}\mathbb{T}_+ \wedge (F^{n_1/2})_*[\Delta], \phi^\pm \rangle \\ &= \langle d^{-n_1}(F^{n_1/2})_*[\Delta], \phi^\pm \mathbb{T}_+ \rangle. \end{aligned}$$

We apply Corollary 3.5 with the functions $\phi^\pm = g_0^\pm(w) \cdot h^\pm(z)$ instead of ϕ and the current $[\Delta]$ instead of R . For this purpose, since $\bar{\Delta} \cap \mathbb{T}^+ = \emptyset$, we can choose a suitable open set V_1 containing Δ . We also fix an open set V_2 as in Section 3. Since B is large enough, Lemma 4.1 implies that $dd^c\phi^\pm \geq 0$ on a neighbourhood of $\mathbb{K}^+ \cap V_2$. Thus, we obtain from Corollary 3.5 that

$$\langle d^{-n_1}(F^{n_1/2})_*[\Delta] - \mathbb{T}_-, \phi^\pm \mathbb{T}_+ \rangle \lesssim d^{-n_1/2}$$

or equivalently

$$(4.9) \quad \langle d^{-n_1}(F^{n_1/2})_*[\Delta], \phi^\pm \mathbb{T}_+ \rangle - \langle \mathbb{T}_-, \phi^\pm \mathbb{T}_+ \rangle \lesssim d^{-n_1/2}.$$

Together, (4.8), (4.9), and the fact that

$$\langle \mathbb{T}_-, \phi^\pm \mathbb{T}_+ \rangle = \langle T_+ \wedge T_-, \phi^\pm \rangle = \langle \mu \otimes \mu, g_0^\pm(w) \cdot h^\pm(z) \rangle = \langle \mu, g_0^\pm \rangle \cdot \langle \mu, h^\pm \rangle$$

give the desired estimates (4.5) and (4.6). The proof of Theorem 1.4 is complete. \square

Proof of Corollary 1.5. Fix $0 < \gamma < 1$ and consider the normed space $(E, \|\cdot\|_E)$, where E consists of all γ -Hölder continuous functions on $\text{supp } \mu$, and $\|\cdot\|_E = \|\cdot\|_{\mathcal{C}^\gamma(\text{supp } \mu)}$. This norm satisfies Properties **(e1)** and **(e2)**. In order to apply Theorem 1.3, we need to verify that μ is exponentially mixing of all orders for all observables in E .

Take $g \in E$ and $H := \|g\|_E$. It is easy to check that the function $\tilde{g} : \mathbb{P}^2 \rightarrow \mathbb{R}$ defined as

$$\tilde{g}(x) := \inf_{y \in \text{supp } \mu} \{g(y) + H \text{dist}(x, y)^\gamma\}$$

is γ -Hölder continuous on \mathbb{P}^2 and satisfies $\tilde{g} = g$ on $\text{supp } \mu$ and $\|\tilde{g}\|_{\mathcal{C}^\gamma(\mathbb{P}^2)} \lesssim \|g\|_E$, where the implicit constant does not depend on g .

Fix now $\kappa \in \mathbb{N}^*$ and $g_0, \dots, g_\kappa \in E$. By the previous paragraph, we can construct functions $\tilde{g}_0, \dots, \tilde{g}_\kappa : \mathbb{P}^2 \rightarrow \mathbb{R}$ with $\tilde{g}_j = g_j$ on $\text{supp } \mu$ and $\|\tilde{g}_j\|_{\mathcal{C}^\gamma(\mathbb{P}^2)} \lesssim \|g_j\|_E$. The exponential mixing of order κ of μ is then a consequence of Theorem 1.4. The assertion follows from Theorem 1.3. \square

Remark 4.2. Friedland and Milnor [32] proved that any polynomial automorphism of \mathbb{C}^2 is either conjugate to an elementary automorphism which preserves a fibration by parallel complex lines or to a Hénon map as above. So, our results apply to all automorphisms of \mathbb{C}^2 which are not elementary.

Remark 4.3. When f is a regular automorphism of \mathbb{C}^k with k even and $p = k/2$, the map (f, f^{-1}) is regular on \mathbb{C}^{2k} . The same proof as above gives us the exponential mixing of all orders and the CLT for f . The results still hold for every regular automorphism but the proof requires some extra technical arguments that we choose to do not present here for simplicity, see for instance [20, 56].

Remark 4.4. When f is a horizontal-like map [24, 26] such that the main dynamical degree is larger than the other dynamical degrees, the same strategy gives the exponential mixing of all orders and the CLT, see [24] for the necessary estimates. In particular, these results hold for all Hénon-like maps in dimension 2, see also [29].

Remark 4.5. In the companion paper [11], we explain how to adapt our strategy to get the exponential mixing of all orders and the Central Limit Theorem for automorphisms of compact Kähler manifolds with simple action on cohomology. As the proof in that case requires the theory of super-potentials, which is not needed for Hénon maps, we choose to do not present it here.

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