PURITY AND TORSORS OVER PRÜFER BASES

Ning Guo, Fei Liu

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Abstract. We establish Zariski–Nagata purity theorem concerning finite étale covers on smooth schemes over Prüfer rings by proving Auslander's flatness criterion in this non-Noetherian context. Inspired by Gabber–Ramero's upper bound of projective dimensions over Prüfer bases, we present an Auslander–Buchsbaum formula. On the basis of the analysis of reflexive sheaves, we prove various purity theorems for torsors under reductive group algebraic spaces. Specifically, by parafactorial results in [EGA IV$_4$] on smooth schemes over normal bases, we prove the purity for cohomology groups of multiplicative type groups at this level of generality. Subsequently, we take advantage of aforementioned purity results to give affirmative answer to the Grothendieck–Serre conjecture for torsors on smooth schemes over semilocal Prüfer rings in certain cases. Along the way, inspired by the recent preprint of Česnavičius [Čes22c], we also prove several versions of Nisnevich conjecture in our context.

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1. Purity and the Grothendieck–Serre on schemes smooth over Prüfer bases

1.1. Purity and regularity. In algebraic geometry, purity refers to a diverse range of phenomena in which certain invariants or categories associated to geometric objects are insensitive to the removal of closed subsets of large codimensions. In the classical Noetherian world, purities, say, for vector bundles (and even torsors), or for finite étale covers, are intimately related to the regularities measured by lengths of regular sequences of geometric objects. For a concrete instance, the Auslander–Buchsbaum formula

$$\operatorname{depth}_R M + \operatorname{proj.dim}_R M = \operatorname{depth}_R R \qquad ([\text{AB57}, \text{Theorem 3.7}])$$

controls the projective dimension of the finite type module M over the Noetherian local ring R via depths, leading to the purity for vector bundles on regular local rings of dimension two ([Sam64, Proposition 2]). Granted this, Colliot-Thélène and Sansuc [CTS79, Théorème 6.13] established the purity for reductive torsors over arbitrary regular local ring R of dimension two by bootstrapping from the vector bundle case:

$$\text{the restriction} \quad H^1_{\text{ét}}(\operatorname{Spec} R, G) \xrightarrow{\sim} H^1_{\text{ét}}(\operatorname{Spec} R\backslash\{\mathfrak{m}_R\}, G) \quad \text{is bijective}$$

for every reductive R-group scheme G. Nevertheless, not only does the term 'regularity' make sense for Noetherian rings, its non-Noetherian generalization can still enlighten us to contemplate purity problems.

1.2. Regularity of Prüfer rings. Originally formulated by Bertin [Ber71], [Ber72, Définition 3.5] for coherent local rings, we say that a ring R is *regular* if every finitely generated ideal of R has finite projective dimension. This coincides with the classical notion of regularity when restricting to Noetherian rings by Serre's homological characterization [Ser56, Théorème 3]. A typical non-Noetherian example can be sought in Prüfer rings, namely, the rings whose all local rings are valuation rings. By definition, an integral domain V is a valuation ring if every $x \in (\operatorname{Frac} V)\backslash V$ satisfies $x^{-1} \in V$. Beyond fields, Noetherian valuation rings are exactly discrete valuation rings. The regularity of Prüfer rings thus follows from the fact that all finitely generated ideals of valuation rings are principal. In addition to the regularity and other nature (Lemma 3.1.1), the ubiquity of Prüfer rings in the study of nonarchimedean geometry, Zariski–Riemann spaces, among others, motivates us to investigate their algebro-geometric properties.

1.3. Basic setup I. The purity part of the present article focuses on a semilocal affine Prüfer scheme S with $\dim S > 0$ (and with $\dim S < \infty$ if necessary), an S-flat finite type algebraic space X with regular S-fibers, and a closed subset $Z \subset X$ such that $j\colon X\backslash Z \hookrightarrow X$ is quasi-compact. For a point $x \in X$ lying in an open subscheme, the local ring of X at x makes sense and we denote $A := \mathscr{O}_{X,x}$. When involving torsors on X, we let G be an X-group algebraic space that étale-locally permits an embedding $G \hookrightarrow \mathrm{GL}_n$ such that GL_n/G is X-affine. This condition is fulfilled if G is X-reductive[1], or finite and locally free.

1.4. Auslander–Buchsbaum over Prüfer bases. Gabber–Ramero's upper bound of projective dimensions of coherent modules over X unveils a glimpse of the Prüferian Auslander–Buchsbaum formula Theorem 2.8.1: if $x \in X$ lies over a closed point $s \in S$, then every finitely presented A-module M satisfies

$$\operatorname{proj.dim}_A M + \operatorname{depth}_A M = \operatorname{depth}_A A = d + 1, \quad \text{where } d = \dim \mathscr{O}_{X_s,x}.$$

Here $\operatorname{proj.dim}_A(0) = -\infty$ and $\operatorname{depth}_A M$ is the smallest i such that the i-th local cohomology of M be nonzero (§2.4). Our proof is significantly different from the classical case [AB57, Theorem 3.7]. Specifically, taking Gabber–Ramero's boundness [GR18, Proposition 11.4.1] as an input, we bypass the interpretation of projective dimensions in terms with Tor functors, which is a crucial ingredient in Auslander–Buchsbaum's argument. In the sequel, we will only use Gabber–Ramero's part of Proposition 3.2.7(i).

1.5. Purity for torsors on smooth relative curves over Prüfer rings. Once the projective dimensions of reflexive sheaves on X are controlled, by imposing codimensional constraints on Z, we may extend vector bundles on $X\backslash Z$ to X, as in Noetherian scenarios. Subsequently, this allows us to obtain the purity Theorem 6.1.4 for G-torsors: if Z satisfies

$$Z_\eta = \varnothing \quad \text{for each generic point } \eta \in S \quad \text{and} \quad \operatorname{codim}(Z_s, X_s) \geqslant 1 \text{ for all } s \in S,$$

and X is an S-curve, then restriction induces the following equivalence of categories of G-torsors

$$\mathbf{Tors}(X_{\text{ét}}, G) \xrightarrow{\sim} \mathbf{Tors}((X\backslash Z)_{\text{ét}}, G).$$

[1]By this we mean a smooth affine X-group algebraic space G whose X-geometric fibers are (connected) reductive algebraic groups. Then, étale-locally on X, G splits so admits a closed immersion $G \hookrightarrow \mathrm{GL}_{n,X}$ for some integer n; by [Alp14, 9.4.1], the reductivity of G implies that the quotient $\mathrm{GL}_{n,X}/G$ is X-affine of finite type.

In particular, passing to isomorphism classes of objects, we have the following bijection of pointed sets

$$H^1_{\text{ét}}(X, G) \simeq H^1_{\text{ét}}(X\backslash Z, G).$$

Meanwhile, a local version Theorem 6.2.1 allows us to loose constraints on the relative dimension of X: if

$$\text{either } x \in X_\eta \text{ with } \dim \mathscr{O}_{X_\eta, x} = 2, \quad \text{or} \quad x \in X_s \text{ with } s \neq \eta \text{ and } \dim \mathscr{O}_{X_s, x} = 1,$$

then every G-torsor over $\operatorname{Spec} \mathscr{O}_{X,x}\backslash\{x\}$ extends uniquely to a G-torsor over $\operatorname{Spec} \mathscr{O}_{X,x}$. This permits us to iteratively extend reductive torsors beyond a closed subset of higher fiberwise codimensions.

1.6. Zariski–Nagata over Prüfer bases. The Zariski–Nagata purity, known as "purity of branch locus", states that every finite extension $A \subset B$ of rings with A regular Noetherian and B normal is unramified if and only if so it is in codimension one on $\operatorname{Spec} B$. This purity was settled by Zariski [Zar58] in a geometric context, and more algebraically by Nagata [Nag59] based on Chow's local Bertini theorem. In contrast to them, Auslander gave an alternative proof [Aus62, Theorem 1.4] by skillful homological methods leading to a criterion for flatness. In [SGA 2_{new}, Exposé X, §3], Grothendieck reformulated their results into a purity concerning finite étale covers and proved this purity on Noetherian local rings that is a complete intersection of dimension $\geqslant 3$ by reducing the assertion to hypersurfaces via several passages involving formal completions. Nevertheless, a practical deficiency of the later argument is that, even over a rank-one valuation ring V with pseudo-uniformizer ϖ, the coherence of the ϖ-adic completion $\widehat{A}$ of A is unknown to us, not to mention the primary decomposition on it. To circumvent this technical obstacle, we revert to Auslander's argument by establishing a Prüferian counterpart Theorem 4.1 of the criterion for flatness [Aus62, Theorem 1.3]. Granted this, we acquire the Prüferian Zariski–Nagata Theorem 6.4.2:

$$\text{the pullback} \quad \text{FÉt}_X \xrightarrow{\sim} \text{FÉt}_{X\backslash Z} \quad \text{is an equivalence}$$

for every closed subset $Z \subset X$ in the basic setup §1.3 that satisfies the following condition

$$\operatorname{codim}(Z_\eta, X_\eta) \geqslant 2 \quad \text{for each generic point } \eta \in S \quad \text{and} \quad \operatorname{codim}(Z_s, X_s) \geqslant 1 \text{ for all } s \in S.$$

In particular, for every geometric point $\overline{x}\colon \operatorname{Spec} \Omega \to X\backslash Z$ with a separably closed field Ω, the map

$$\pi_1^{\text{ét}}(X\backslash Z, \overline{x}) \to \pi_1^{\text{ét}}(X, \overline{x}) \quad \text{is an isomorphism.}$$

1.7. Grothendieck–Serre on semilocal Prüfer rings. The Grothendieck–Serre conjecture predicts that, for a regular local ring R and a reductive R-group scheme G, every generically trivial G-torsor is trivial, that is, the following restriction map of nonabelian cohomology pointed sets has trivial kernel:

$$\ker\left(H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(\operatorname{Frac} R, G)\right) = \{*\}.$$

The conjecture was settled in the equicharacteristic case and in certain unramified mixed characteristic cases, see the histrical summary below. Thanks to the purity for cohomology of groups of multiplicative type, we prove the non-Noetherian counterpart of Colliot-Thélène–Sansuc's result for tori and then obtain a product formula for tori. Based on this, the similar argument in [Guo20] leads to a passage from the semilocal case to the local case. Hence, we settle the Grothendieck–Serre on semilocal Prüfer rings in §9.

1.8. Basic setup II. The second half of this article deals mainly with the following. For a semilocal Prüfer ring R, an irreducible R-smooth scheme X, the semilocalization $A := \mathscr{O}_{X,\mathbf{x}}$ of X at a finite subset $\mathbf{x} \subset X$ contained in a single affine open of X, and a reductive A-group scheme G, we study the trivialization behaviour of G-torsors.

1.9. Grothendieck–Serre on smooth projective schemes. This result was proved by the second author and simultaneously by an unpublished work of Panin and the first author in the Noetherian case. We show that, when X is R-projective in §1.8 and G has a reductive model over X, every generically trivial G-torsor on A is trivial, that is,

$$\ker\left(H^1(A, G) \to H^1(\operatorname{Frac} A, G)\right) = \{*\}.$$

To prove this, we use crucially our purity Theorem 6.1.4 after spreading out to extend the domain of the torsor in question to an open subset as large as possible: according to that purity, a generically trivial torsor on $\mathscr{O}_{X,\mathbf{x}}$ extends to a torsor on an open neighbourhood of $\mathbf{x}$ whose complementary closed has codimension $\geqslant 3$ (resp., $\geqslant 2$) in the generic (resp., non-generic) R-fibers of X, see Corollary 6.3.2. This codimension bound is sharp enough for us to apply the geometric presentation Lemma 7.1.1 and glueing techniques to reduce the problem to studying torsors on relative affine lines that we treat in detail in §10.

1.10. Grothendieck–Serre under constant reductive groups. Assume that G is 'constant', namely, it is a pullback from the Prüfer base ring R. Then every generically trivial G-torsor on A is trivial, that is,

$$\ker\left(H^1(A,G)\to H^1(\operatorname{Frac}A,G)\right)=\{*\}.$$

For this, we first devise a variant (in some aspect, a stronger form) of Lindel's lemma (Proposition 7.2.1), which states that, for a closed subscheme $Y\subset X$ that avoids all the maximal points of the R-fibers of X, the pair (Y,X) Zariski-locally on X can be presented as an elementary étale neighbourhood of a similar pair (Y',X'), where X' is an open of some projective R-space. This allows us to use glueing techniques to reduce to studying generically trivial torsors on opens of projective R-spaces, which is done in §1.9.

1.11. Grothendieck–Serre under quasi-split groups. As for the quasi-split case of the Grothendieck–Serre, we will follow a similar strategy of [Čes22a] (with its earlier version given by Fedorov [Fed22b]), where the key input is our toral version of purity Proposition 8.2.5 and Grothendieck–Serre type Proposition 8.3.2 in this context. Precisely, by the valuative criterion of properness, a generically trivial torsor on X, say, reduces to a generically trivial torsor under a Borel B away from a closed subset Z of X that has codimension $\geqslant 2$ (resp., $\geqslant 1$) in the generic (resp., non-generic) R-fiber. Further, utilizing the aforementioned toral purity and Grothendieck–Serre type results, one shows that the above torsor further reduces to a $\mathrm{rad}^u(B)$-torsor on $X\backslash Z$. In conclusion, when G is quasi-split, we prove Theorem 13.1 that

$$\ker\left(H^1(A\otimes_R K,G)\to H^1(\operatorname{Frac}A,G)\right)=\{*\};$$

if R has Krull dimension 1, then every generically trivial G-torsor is trivial, that is,

$$\ker\left(H^1(A,G)\to H^1(\operatorname{Frac}A,G)\right)=\{*\}.$$

1.12. Nisnevich's purity conjecture. Now, we turn to Nisnevich's purity conjecture, where we require the total isotropicity of group schemes. A reductive group scheme G defined over a scheme S is *totally isotropic* at $s\in S$ if every G_i in the decomposition [SGA $3_{\text{III new}}$, Exposé XXIV, Proposition 5.10 (i)]

$$G^{\mathrm{ad}}_{\mathscr{O}_{S,s}}\cong\textstyle\prod_i\operatorname{Res}_{A_i/\mathscr{O}_{S,s}}(G_i)$$

contains a $\mathbb{G}_{m,R_i}$. If this holds for all $s\in S$, then G is *totally isotropic*. Proposed by Nisnevich [Nis89, Conjecture 1.3] and modified due to the anisotropic counterexamples of Fedorov [Fed22b, Proposition 4.1], the Nisnevich conjecure predicts that, for a regular semilocal ring R, a regular parameter $r\in R$ (that is, $r\in\mathfrak{m}\backslash\mathfrak{m}^2$ for every maximal ideal $\mathfrak{m}\subset R$), and a reductive R-group scheme G such that $G_{R/rR}$ is totally isotropic, every generically trivial G-torsor on $R[\frac{1}{r}]$ is trivial, that is, the following map

$$H^1(R[\tfrac{1}{r}],G)\to H^1(\operatorname{Frac}R,G)\qquad\text{has trivial kernel.}$$

The case when R is a local ring of a regular affine variety over a field and $G=\mathrm{GL}_n$ was settled by Bhatwadekar–Rao in [BR83] and was subsequently extended to arbitrary regular local rings containing fields by Popescu [Pop02, Theorem 1]. Nisnevich in [Nis89] proved the conjecture in dimension two, assuming that R is a local ring with infinite residue field and that G is quasi-split. For the state of the art, the conjecure was settled in equicharacteristic case and in several mixed characteristic case by Česnavičius in [Čes22c, Theorem 1.3] (previously, Fedorov [Fed21] proved the case when R contains an infinite field). Besides, the toral case and some low dimensional cases are known and surveyed in [Čes22b, Section 3.4.2 (1)] including Gabber's result [Gab81, Chapter I, Theorem 1] for the local case $\dim R\leqslant 3$ when G is either GL_n or PGL_n. In this article, we prove several variants of Nisnevich conjecture over Prüfer bases, see Theorem 11.1 (ii) and Theorem 12.1 (ii).

1.13. The Grothendieck–Serre conjecture: a history. Since proposed by Serre [Ser58, page 31] and Grothendieck [Gro58, pages 26–27, Remarques 3], [Gro68a, Remarques 1.11 a)], the Grothendieck–Serre conjecture has already various known cases beyond the trivial $\dim R=0$ case for fields, as listed below.

(i) The case when G is a torus is proved by Colliot-Thélène and Sansuc in [CTS87].

(ii) The case when $\dim R=1$, namely, R is a discrete valuation ring, was addressed by Nisnevich in [Nis82] and [Nis84], then is improved and generalized to the semilocal Dedekind case in [Guo22]. Several special cases were proved in [Har67], [BB70], [$\mathrm{BT}_{\mathrm{III}}$] over discrete valuation rings, and in [PS16], [BVG14], [BFF17], [BFFH20] for the semilocal Dedekind case.

(iii) The case when R is Henselian was settled in [BB70] and [CTS79, Assertion 6.6.1] by reducing the triviality of G-torsors to residue fields then inducting on $\dim R$ to reach Nisnevich's resolved case.

(iv) The equicharacteristic case, namely, when R contains a field k, was established by Fedorov and Panin [FP15] when k is infinite (see also [PSV15], [Pan20b] for crucial techniques) and by Panin [Pan20a] when k is finite, which was later simplified by [Fed22a]. Before these, several equicharacteristic subcases were proved in [Oja80],[CTO92], [Rag94], [PS97], [Zaĭ00], [Oja01], [Oja04], [Pan05], [Zai05], [Che10], [PSV15].

(v) When R is of mixed characteristic, Česnavičius [Čes22a] settled the case when G is quasi-split and R is unramified (that is, for $p := \mathrm{char}(R/\mathfrak{m}_R)$, the ring R/pR is regular). Prior to this, Fedorov [Fed22b] proved the split case under additional assumptions on R. Recently, Česnavičius [Čes22c, Theorem 1.3] settled a generalized Nisnevich conjecture under certain conditions, which specializes to the equal and mixed characteristic cases of the Grothendieck–Serre proved in [FP15], [Pan20a], [Čes22a].

(vi) There are sporadic cases where R or G are speical (with possible mixed characteristic condition), see [Gro68a, Remarque 1.11 a)], [Oja82], [Nis89], [Fed22b], [Fir22], [BFFP22], [Pan21].

1.14. Notations and conventions. All rings in this paper are commutative with units, unless stated otherwise. For a point s of a scheme (resp., for a prime ideal $\mathfrak{p}$ of a ring), we let $\kappa(s)$ (resp., $\kappa(\mathfrak{p})$) denote its residue field. For a global section s of a scheme S, we write $S[\frac{1}{s}]$ for the open locus where s does not vanish. For a ring A, we let $\operatorname{Frac} A$ denote its total ring of fractions. For a morphism of algebraic spaces $S' \to S$, we let $(-)_{S'}$ denote the base change functor from S to S'; if $S = \operatorname{Spec} R$ and $S' = \operatorname{Spec} R'$ are affine schemes, we will also write $(-)_{R'}$ for $(-)_{S'}$.

Let S be an algebraic space, and let G be an S-group algebraic space. For an S-algebraic space T, by a G-torsor over T we shall mean a $G_T := G \times_R T$-torsor (see Definition 5.2). Denote by $\mathbf{Tors}(S_{\mathrm{fppf}}, G)$ (resp., $\mathbf{Tors}(S_{\text{ét}}, G)$) the groupoid of G-torsors on S that are fppf-locally (resp., étale-locally) trivial; specifically, if G is S-smooth (e.g., G is S-reductive, see below), then every fppf-locally trivial G-torsor is étale-locally trivial, so we have

$$\mathbf{Tors}(S_{\mathrm{fppf}}, G) = \mathbf{Tors}(S_{\text{ét}}, G).$$

For an algebraic space S, a reductive S-group algebraic space is a smooth affine S-group algebraic space whose geometric S-fibers are (connected) reductive algebraic groups. For a scheme S this coincides with the definition of reductive S-groups schemes given in [SGA $3_{\text{III new}}$].

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2. Auslander–Buchsbaum formula over valuation rings

The goal of this section is to establish Theorem 2.8.1, the Auslander–Buchsbaum formula over finite rank valuation rings as an analogue of the classical regular case [AB57, Theorem 3.7]. Based on the upper-bound of projective dimensions [GR18, Proposition 11.4.1], we induct by using the notion of depths.

2.1. Coherent rings and schemes. For a ring A, a finitely generated A-module M is *coherent* if its any finitely generated A-submodule is finitely presented. A ring A is *coherent* if it is a coherent A-module. For a scheme X, an $\mathscr{O}_X$-module $\mathscr{F}$ is coherent if, for every affine open $U \subset X$, $\Gamma(U, \mathscr{F})$ is a coherent

$\Gamma(U, \mathscr{O}_U)$-module. A scheme X is *locally coherent* if $\mathscr{O}_X$ is a coherent $\mathscr{O}_X$-module. A locally coherent scheme is *coherent* if it is quasi-compact quasi-separated.

Example 2.2. Noetherian rings and Prüfer rings are coherent rings ([SP, 05CY, 0EWV]). Although Noetherian schemes are coherent, open subschemes of affine integral Prüfer schemes are not coherent in general: there exists a valuation ring V such that $\operatorname{Spec} V \backslash \{\mathfrak{m}_V\}$ has no closed points and is not quasi-compact.

Lemma 2.3. *Let A be a coherent ring.*

(i) *For every multiplicative subset $S \subset A$, the localization $S^{-1}A$ is a coherent ring.*

(ii) *Any A-module M is coherent if and only if it is finitely presented over A. Further, the full subcategory of coherent A-modules is an abelian subcategory of the category of A-modules and is closed under taking extensions.*

Proof. For (i), see [Gla89, Theorem 2.4.2]. For the first assertion of (ii), see [FK18, Chapter 0, Corollary 3.3.5]. □

2.4. Depth. For a local ring A and the closed point $x \in \operatorname{Spec} A$, consider the following functor

$$\Gamma_{\{x\}} : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod} \qquad M \mapsto \ker\left(\Gamma(\operatorname{Spec} A, \widetilde{M}) \to \Gamma(\operatorname{Spec} A\backslash\{x\}, \widetilde{M})\right)$$

sending M to its largest A-submodule supported on $\{x\}$. The functor $\Gamma_{\{x\}}$ is left exact so gives rise to a right derived functor $R\Gamma_{\{x\}} \colon D^+(A\text{-}\mathbf{Mod}) \to D^+(A\text{-}\mathbf{Mod})$. The *depth* of $M \in D^+(A\text{-}\mathbf{Mod})$ is

$$\operatorname{depth}_A(M) := \sup\{n \in \mathbf{Z} \mid R^i\Gamma_{\{x\}}M = 0 \quad \text{for all } i < n\} \in \mathbf{Z}_{\geqslant 0} \cup \{+\infty\},$$

For an A-module N supported on $\{x\}$, we also consider the following closely related quantity

$$\tau_N(M) := \sup\{n \in \mathbf{Z} \mid \operatorname{Ext}^i_A(N, M) = 0 \quad \text{for all } i < n\} \in \mathbf{Z}_{\geqslant 0} \cup \{+\infty\}.$$

Lemma 2.5. *For a local ring A, an A-module M, and an M-regular sequence $(f_1, \cdots, f_r)$ in $\mathfrak{m}_A$,*

$$\operatorname{depth}_A(M) = \operatorname{depth}_A(M/\textstyle\sum_{i=1}^r f_iM) + r \quad \text{and} \quad \tau_N(M) = \tau_N(M/\textstyle\sum_{i=1}^r f_iM) + r.$$

Proof. The two equalities are proved similarly, so we only treat the one concerning depth. By induction on r, we are reduced to the case when $r = 1$ and $f_1 = f$ is a nonzerodivisor of M in $\mathfrak{m}_A$. From the short exact sequence $0 \to M \xrightarrow{f} M \to M/fM \to 0$, we derive the following long exact sequence

$$\cdots \to R^{i-1}\Gamma_{\{x\}}M \xrightarrow{f} R^{i-1}\Gamma_{\{x\}}M \to R^{i-1}\Gamma_{\{x\}}(M/fM) \to R^i\Gamma_{\{x\}}M \xrightarrow{f} R^i\Gamma_{\{x\}}M \to \cdots.$$

If $\operatorname{depth}_A M = +\infty$, then $M = 0$ so it suffices to assume that $\operatorname{depth}_A M = d$ for an integer $d \geqslant 0$. If $d = 0$, then there is a nontrivial A-submodule of M supported on $\{x\}$, contradicting to the assumption that $f \in \mathfrak{m}_A$ is a nonzerodivisor of M. Therefore, we have $d \geqslant 1$ and $R^i\Gamma_{\{x\}}M = 0$ for every $0 \leqslant i \leqslant d-1$. The displayed long exact sequence implies that $R^i\Gamma_{\{x\}}(M/fM) = 0$ for every $0 \leqslant i \leqslant d-2$ (if $d-2 < 0$ then such i does not exist). If $R^{d-1}\Gamma_{\{x\}}(M/fM) = 0$, then the map $R^d\Gamma_{\{x\}}M \hookrightarrow R^d\Gamma_{\{x\}}M$ induced by multiplication by f is injective. However, since the nonzero A-module $R^d\Gamma_{\{x\}}M$ is supported on $\{x\}$ and $f \in \mathfrak{m}_A$, we deduce that $R^d\Gamma_{\{x\}}M = 0$, that is, $\operatorname{depth}_A M > d$, a contradiction. Consequently, we have $R^{d-1}\Gamma_{\{x\}}(M/fM) \neq 0$ and $\operatorname{depth}_A(M/fM) = d - 1 = \operatorname{depth}_A M - 1$. □

Example 2.6. Assume that A is Noetherian, and take $N = A/I$ for an ideal I of A (for instance, $N = A/\mathfrak{m}_A$). Then for any finitely generated A-module M we have

$$\operatorname{depth}_A M = \tau_N(M).$$

Indeed, utilizing Lemma 2.5, one verifies easily that both of them equals the length of any maximal M-regular sequence in $\mathfrak{m}_A$ (so the length is independent of all choices). However, this is false when A is non-Noetherian. For instance, we let $A := V$ be a non-discrete valuation ring of finite rank and let $N := V/\mathfrak{m}_V$ be its residue field. Take $M = V/fV$ for a nonzero $f \in \mathfrak{m}_V$. Then one checks immediately that $\operatorname{depth}_V(V/fV) = 0$, but there are no nonzero elements of V/fV annihilated by $\mathfrak{m}_V$. Thus $\operatorname{Hom}_V(V/\mathfrak{m}_V, V/fV) = 0$, and so $\tau_{V/\mathfrak{m}_V}(V/fV) \geqslant 1 > 0 = \operatorname{depth}_V(V/fV)$.

Lemma 2.7. *For a valuation ring V of finite rank, a V-flat finitely presented scheme X, and a point $x \in X$ with image $s \in \operatorname{Spec} V$ such that $\mathscr{O}_{X_s,x}$ is regular,*

(i) *we have* $\operatorname{depth}_A(A) = d+1$*, where* $d = \dim \mathscr{O}_{X_s,x}$*;*

(ii) *for any* A*-module* N *supported on* $\{x\}$*, we have* $\operatorname{Ext}^i_A(N, A) = 0$ *for all* $i \leqslant d$*.*

Proof.

(i) Since by assumption V has nonzero finite rank, we can pick an element $f \in \mathfrak{m}_V$ such that $\dim V/(f) = 0$. Let $(g_1, \cdots, g_d)$ be a sequence in $\mathfrak{m}_A$ such that their images in the regular local ring $A/\mathfrak{m}_V A$ forms a regular system of parameters, and hence also forms a regular sequence. By the flatness criterion [EGA IV$_3$, Théorème 11.3.8], $(g_1, \cdots, g_d)$ is a regular sequence of A, and the quotient ring $\overline{A} := A/(g_1, \cdots, g_d)$ is V-flat with $\mathfrak{m}_V \overline{A} = \mathfrak{m}_{\overline{A}}$. Therefore, $(g_1, \cdots, g_d, f)$ is a regular sequence of A in $\mathfrak{m}_A$ such that $\dim A/(fA + \sum_{i=1}^d g_i A) = 0$. Applying Lemma 2.5 yields

$$\operatorname{depth}_A(A) = \operatorname{depth}_A(A/(fA + \textstyle\sum_{i=1}^d g_i A)) + d + 1 = d + 1.$$

(ii) Repeating the preceding argument involving Lemma 2.5, we deduce the following inequality

$$\tau_N(A) = \tau_N(A/(fA + \textstyle\sum_{i=1}^d g_i A)) + d + 1 \geqslant d + 1.$$

By the definition of $\tau_N(A)$, this is equivalent to the displayed vanishing. □

Lemma 2.8. *For a local ring* $(A, \mathfrak{m}_A)$*, a nonzero* A*-module* M *supported on* $\{\mathfrak{m}_A\}$*, and a matrix* $H \in M_{m\times n}(A)$*, if the* A*-linear map* $H_M : M^{\oplus n} \to M^{\oplus m}$ *induced by* H *(via left multiplication) is injective, then* H *admits a left inverse, or, equivalently,* H *exhibits* $A^{\oplus n}$ *as a direct summand of* $A^{\oplus m}$*.*

Proof. Recall [SP, 0953] that the assumption on the support of M means that, for any $w \in M$ and any finitely generated ideal $I \subset A$, we have $I^N w = 0$ for large enough N. Let $H = (h_{ij})$, then McCoy's theorem [Gla89, page 211] implies that the ideal generated by the minors of order n of H does not annihilate a nonzero element of M. Indeed, the invertibility of minors already yields a left inverse of H and we are done. Precisely, since M is supported at $\{\mathfrak{m}_A\}$, there exist some i, j for which $h_{ij} \in A^\times$. We may assume that $h_{11} \in A^\times$. By subtracting suitable multiples of the first row of H to other rows (resp. the first column of H to other columns), we may also assume that $h_{1j} = 0$ for $1 < j \leqslant n$ and $h_{i1} = 0$ for $1 < i \leqslant m$ (the assumption and conclusion of the lemma are preserved if we replace H by $H_1 H H_2$, where $H_1 \in M_{m\times m}(A)$ and $H_2 \in M_{n\times n}(A)$). In other words, we have $H = (h_{11}) \oplus H'$, where $H' \in M_{(m-1)\times(n-1)}(A)$. Then the map $H'_M : M^{\oplus(n-1)} \to M^{\oplus(m-1)}$ induced by H' is also injective. So we may assume by induction that H' admits a left inverse $H'' \in M_{(n-1)\times(m-1)}(A)$. Then $(h_{11}^{-1}) \oplus H''$ is a left inverse of H. □

Now, we acquire the Prüferian analogy of the Auslander–Buchsbaum formula [AB57, Theorem 3.7].

Theorem 2.8.1 (Auslander–Buchsbaum formula over valuation rings)**.** *For a valuation ring* V *of finite nonzero rank, a* V*-flat finite type scheme* X*, a point* $x \in X$ *lying over the closed point* $s \in \operatorname{Spec} V$ *such that* $\mathscr{O}_{X_s,x}$ *is regular, and the local ring* $A := \mathscr{O}_{X,x}$*, every finitely presented* A*-module* M *satisfies*

$$\operatorname{proj.dim}_A(M) + \operatorname{depth}_A(M) = \operatorname{depth}_A(A) = d + 1, \quad \text{where } d = \dim \mathscr{O}_{X_s,x}.$$

(By convention, $\operatorname{proj.dim}_A(0) = -\infty$*)*

Proof. Let M be a finitely presented nonzero A-module. We will induct on $\operatorname{proj.dim}_A(M)$ to verify the formula. Note that, by [GR18, Proposition 11.4.1], we have $\operatorname{proj.dim}_A(M) \leqslant d+1$. If $\operatorname{proj.dim}_A(M) = 0$, or, M is A-free, then by Lemma 2.7 we have $\operatorname{depth}_A(M) = d+1$, so $\operatorname{proj.dim}_A(M) + \operatorname{depth}_A(M) = d+1$.

Next, assume that $\operatorname{proj.dim}_A(M) \geqslant 1$, so every partial resolution $0 \to M' \xrightarrow{\iota} A^{\oplus n} \to M \to 0$ is non-split and satisfies $\operatorname{proj.dim}_A(M') = \operatorname{proj.dim}_A(M) - 1$. We exploit the associated long exact sequence

$$\cdots \to R^{i-1}\Gamma_{\{x\}}M' \to R^{i-1}\Gamma_{\{x\}}A^{\oplus n} \to R^{i-1}\Gamma_{\{x\}}M \to R^i\Gamma_{\{x\}}M' \to R^i\Gamma_{\{x\}}A^{\oplus n} \to \cdots.$$

If $\operatorname{proj.dim}_A(M) = 1$, then $M' \simeq A^{\oplus m}$ for some $m \geqslant 1$, and the map $A^{\oplus m} \simeq M' \xrightarrow{\iota} A^{\oplus n}$ is given by an $n \times m$ matrix $H \in M_{n\times m}(A)$. We have known that $\operatorname{proj.dim}_A(M') = d+1$, and so $R^i\Gamma_{\{x\}}M' = 0$ for all $i \leqslant d$. It follows from the above long exact sequence that $R^i\Gamma_{\{x\}}M = 0$ for all $i \leqslant d-1$. If $R^d\Gamma_{\{x\}}M = 0$, then left multiplication by H induces an injection

$$\left(R^{d+1}\Gamma_{\{x\}}A\right)^{\oplus m} = R^{d+1}\Gamma_{\{x\}}(A^{\oplus m}) \simeq R^{d+1}\Gamma_{\{x\}}M' \hookrightarrow R^{d+1}\Gamma_{\{x\}}(A^{\oplus n}) = \left(R^{d+1}\Gamma_{\{x\}}A\right)^{\oplus n}.$$

Since $R^{d+1}\Gamma_{\{x\}}A$ is a nonzero A-module supported on $\{x\}$, we deduce from Lemma 2.8 that H admits a left inverse. This implies that ι splits, and so M is A-free, contradicting our assumption $\operatorname{proj.dim}_A(M) = 1$. Hence, $\operatorname{depth}_A(M) = d$, and we thus obtain the desired formula $\operatorname{proj.dim}_A(M) + \operatorname{depth}_A(M) = d+1$.

If $\operatorname{proj.dim}_A(M) > 1$, then $\operatorname{proj.dim}_A(M') = \operatorname{proj.dim}_A(M) - 1$. Applying the induction hypothesis to M', we have

$$\operatorname{depth}_A(M') = d + 1 - (\operatorname{proj.dim}_A(M) - 1) = d + 2 - \operatorname{proj.dim}_A(M), \quad \text{which is} \quad \leqslant d.$$

It follows from the above long exact sequence and the fact $\operatorname{depth}_A(A^{\oplus n}) = d+1$ that $R^{i-1}\Gamma_{\{x\}}M \simeq R^i\Gamma_{\{x\}}M'$ for all $i \leqslant d$. Combining this with the bound $\operatorname{depth}_A(M') \leqslant d$, we deduce that $\operatorname{depth}_A(M) = \operatorname{depth}_A(M') - 1$. Therefore, by induction hypothesis, we have

$$\operatorname{proj.dim}_A(M) + \operatorname{depth}_A(M) = \big(\operatorname{proj.dim}_A(M') + 1\big) + \big(\operatorname{depth}_A(M') - 1\big) = d + 1.$$

This finishes the induction step. □

3. Geometry of schemes over Prüfer bases

In this section, we recollect useful geometric properties on scheme over Prüfer bases.

3.1. Geometric properties and reduction methods

Lemma 3.1.1. *For a valuation ring V with spectrum S, a finite type irreducible S-scheme X, a point $x \in X$ and its image $s \in S$, the following assertions hold*

(i) *all nonempty S-fibers have the same dimension;*

(ii) *if X is S-flat , then X is finitely presented over S;*

(iii) *if X is S-flat, then for any maximal point $\xi \in X_s$, the local ring $\mathscr{O}_{X,\xi}$ is a valuation ring such that the extension $\mathscr{O}_{S,s} \hookrightarrow \mathscr{O}_{X,\xi}$ of valuation rings induces an isomorphism of value groups;*

(iv) *for $x' \in X$ that is distinct with x whose image is denoted by s', if $x \in \overline{\{x'\}}$, then*

- *either $\operatorname{ht}(s) = \operatorname{ht}(s')$ (i.e., $s = s'$) and then $\dim(\mathscr{O}_{X_{s'},x'}) < \dim(\mathscr{O}_{X_s,x})$;*
- *or $\operatorname{ht}(s') < \operatorname{ht}(s)$ and then $\dim(\mathscr{O}_{X_{s'},x'}) \leqslant \dim(\mathscr{O}_{X_s,x})$.*

Proof. For (i), see [EGA IV$_3$, Lemme 14.3.10]. For (ii), see [Nag66, Theorem 3']. For (iii), see [MB22, Théorème A]. Now, to prove (iv), we may assume that X is affine and of some pure relative dimension, say, n, over V. By assumption, we have $s \in \overline{\{s'\}}$. Assume that we are not in the first case, then $\operatorname{ht}(s') < \operatorname{ht}(s)$. The schematic closure $\overline{\{x'\}}$ is a finite type dominant scheme over $\overline{\{s'\}}$ (the spectrum of a valuation ring), so by (i), all its non-empty fibers have the same dimension. Thus, we deduce from $\overline{\{x'\}} \supset \overline{\{x\}}$ that

$$\dim(\overline{\{x'\}}_{s'}) = \dim(\overline{\{x'\}}_s) \geqslant \dim(\overline{\{x\}}_s).$$

Hence, we have

$$\dim(\mathscr{O}_{X_{s'},x'}) = n - \dim(\overline{\{x'\}}_{s'}) \leqslant n - \dim(\overline{\{x\}}_s) = \dim(\mathscr{O}_{X_s,x}).$$ □

The following Lemma 3.1.2 provides us a passage to the case when there is a section.

Lemma 3.1.2. *For a valuation ring V, an essentially finitely presented (resp., essentially smooth) V-local algebra A, there are an extension of valuation rings V'/V with trivial extension of value groups, and an essentially finitely presented (resp., essentially smooth) V-map $V' \to A$ with finite residue fields extension.*

Proof. Assume $A = \mathscr{O}_{X,x}$ for an affine scheme X finitely presented over V and a point $x \in X$ lying over the closed point $s \in \operatorname{Spec}(V)$. Let $t = \operatorname{tr.deg}(\kappa(x)/\kappa(s))$. As $\kappa(x)$ is a finite extension of $l := \kappa(s)(a_1, \cdots, a_t)$ for a transcendence basis $(a_i)_1^t$ of $\kappa(x)/\kappa(s)$, we have $t = \dim_l \Omega^1_{l/\kappa(s)} \leqslant \dim_{\kappa(x)} \Omega^1_{\kappa(x)/\kappa(s)}$. Choose sections $b_1, \cdots, b_t \in \Gamma(X, \mathscr{O}_X)$ such that $d\overline{b_1}, \cdots, d\overline{b_t} \in \Omega^1_{\kappa(x)/\kappa(s)}$ are linearly independent over $\kappa(x)$, where the bar stands for their images in $\kappa(x)$. Define $p : X \to \mathbb{A}^t_V$ by sending the standard coordinates $T_1, \cdots, T_t$ of $\mathbb{A}^t_V$ to $b_1, \cdots, b_t$, respectively. Since $d\overline{b_1}, \cdots, d\overline{b_t} \in \Omega^1_{\kappa(x)/\kappa(s)}$ are linearly independent, the

image $\eta := p(x)$ is the generic point of $\mathbb{A}^t_{\kappa(s)}$, so $V' := \mathscr{O}_{\mathbb{A}^t_V,\eta}$ is a valuation ring whose value group is $\Gamma_{V'} \simeq \Gamma_V$. Note that $\kappa(x)/\kappa(\eta)$ is finite, the map $V' \to A$ induces a finite residue fields extension.

When $V \to A$ is essentially smooth, the images of $db_1, \cdots, db_t$ under the map $\Omega^1_{X/V} \otimes \kappa(x) \to \Omega^1_{\kappa(x)/\kappa(s)}$ are linearly independent, so are their images in $\Omega^1_{X/V} \otimes \kappa(x)$. Hence, p is essentially smooth at x. □

In the sequel, we will use the following limit argument repeatedly.

Lemma 3.1.3. *Every semilocal Prüfer domain R is a filtered direct union of its subrings R_i such that:*

(i) *for every i, R_i is a semilocal Prüfer domain of finite Krull dimension; and*

(ii) *for i large enough, $R_i \to R$ induces a bijection on the sets of maximal ideals hence is fpqc.*

Proof. Write $\mathrm{Frac}(R) = \cup_i K_i$ as the filtered direct union of the subfields of $\mathrm{Frac}(R)$ that are finitely generated over its prime field $\mathfrak{K}$. Let $R_i := R \cap K_i$. Then $R = \cup_i R_i$. It remains to see that every R_i is a semilocal Prüfer domain whose local rings have finite ranks. Let $\{\mathfrak{p}_j\}_{1\leqslant j\leqslant n}$ be the set of maximal ideals of R. Then $R = \bigcap_{1\leqslant j\leqslant n} R_{\mathfrak{p}_j}$ is the intersection of the valuation rings $R_{\mathfrak{p}_j}$. Thus we have

$$R_i = \textstyle\bigcap_{1\leqslant j\leqslant n} \left(K_i \cap R_{\mathfrak{p}_j}\right).$$

Since $K_i/\mathfrak{K}$ has finite transcendence degree, by Abhyankar's inequality, every $K_i \cap R_{\mathfrak{p}_j}$ is a valuation ring of finite rank. By [BouAC, VI, §7, Proposition 1–2], R_i is a semilocal Prüfer domain, and its local rings at maximal ideals are precisely the minimal elements of the set $\{K_i \cap R_{\mathfrak{p}_j}\}_{1\leqslant j\leqslant n}$ under inclusion. This implies (i). For (ii), it suffices to show that for i large enough there are no strict inclusion relation between $K_i \cap R_{\mathfrak{p}_{j_1}}$ and $K_i \cap R_{\mathfrak{p}_{j_2}}$ for $j_1 \neq j_2$. Indeed, if $\pi_j \in \mathfrak{p}_j \backslash \bigcup_{j'\neq j} \mathfrak{p}_{j'}$ for $1 \leqslant j \leqslant n$, then (ii) holds for any i for which $\{\pi_j\}_{1\leqslant j\leqslant n} \subset K_i$. □

3.2. Reflexive sheaves on schemes over Prüfer bases with regular fibers

3.2.1. Reflexive sheaves. Assume that X is a locally coherent scheme, see 2.1. For an $\mathscr{O}_X$-module $\mathscr{F}$, its *dual* is denote by $\mathscr{F}^\vee := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$. A coherent $\mathscr{O}_X$-module $\mathscr{F}$ is *reflexive* if the canonical map $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is an isomorphism. Since every coherent $\mathscr{O}_X$-module $\mathscr{G}$ is Zariski-locally finitely presented $\mathscr{O}_X^{\oplus m} \to \mathscr{O}_X^{\oplus n} \to \mathscr{G} \to 0$, by taking dual, $\mathscr{G}^\vee$ is finitely copresented as $0 \to \mathscr{G}^\vee \to \mathscr{O}_X^{\oplus n} \to \mathscr{O}_X^{\oplus m}$. In particular, the dual $\mathscr{G}^\vee$ of a coherent $\mathscr{O}_X$-module $\mathscr{G}$ is also coherent (equivalently, finitely presented). Moreover, Lemma 3.2.2 shows that for integral X and every coherent $\mathscr{O}_X$-module $\mathscr{G}$, the double dual $\mathscr{G}^{\vee\vee}$ is $\mathscr{O}_X$-reflexive, hence $\mathscr{G}^{\vee\vee}$ is the *reflexive hull* of $\mathscr{G}$.

Lemma 3.2.2 (reflexive hull)**.** *For a locally coherent integral scheme X and two $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$, if $\mathscr{F}$ is coherent and $\mathscr{G}$ is reflexive, then $\mathscr{H} := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ is reflexive. In particular, the double dual*

$$\mathscr{F}^{\vee\vee} := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X), \mathscr{O}_X) \quad \textit{is a reflexive } \mathscr{O}_X\textit{-module.}$$

Proof. For the coherence of $\mathscr{H}$, it suffices to take a presentation $\mathscr{O}_X^{\oplus m} \to \mathscr{O}_X^{\oplus n} \to \mathscr{F} \to 0$ of $\mathscr{F}$ and its sheaf homomorphism with $\mathscr{G}$ so that $\mathscr{H} = \ker(\mathscr{G}^{\oplus n} \to \mathscr{G}^{\oplus m})$ which is coherent by [SP, 01BY].

Claim 3.2.3. For a domain R, a finitely presented R-module M, and an exact sequence $0 \to M \to M' \to M''$ of finite R-modules, if M' is reflexive and M'' is torsion-free, then M is reflexive.

Proof of the claim. Denote $(-)^\vee = \mathrm{Hom}_R(-, R)$ and consider the following commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \longrightarrow & M' & \longrightarrow & M'' \\ & & \downarrow & & \downarrow & & \downarrow \\ & & M^{\vee\vee} & \longrightarrow & M'^{\vee\vee} & \longrightarrow & M''^{\vee\vee}. \end{array}$$

By [SP, 0AV0], M' is torsion-free, so is M, hence the map $M \hookrightarrow M^{\vee\vee}$ is injective. It remains to show that this map is surjective. For the map $u\colon M'^\vee \to M^\vee$, consider the exact sequence $M'^\vee \to M^\vee \to \mathrm{coker}(u) \to 0$. As M' is reflexive, it is finitely presented, so [SP, 0583] applies, yielding the exact sequence

$$\mathrm{Hom}_R(M' \otimes_R K, K) \to \mathrm{Hom}_R(M \otimes_R K, K) \to \mathrm{coker}(u) \otimes_R K \to 0,$$

where $K := \operatorname{Frac} R$. Since K is R-flat, the injectivity of $M \hookrightarrow M'$ implies that $\operatorname{coker}(u)_K = 0$, hence $\operatorname{coker}(u)$ is R-torsion and $\operatorname{coker}(u)^\vee = 0$. Therefore, $M^{\vee\vee} \hookrightarrow M'^{\vee\vee}$ is injective. Because M'' is torsion-free, the map $M'' \hookrightarrow M''^{\vee\vee}$ is injective. By snake lemma, $M \twoheadrightarrow M^{\vee\vee}$ is surjective so M is reflexive. □

Since $\mathscr{H}$ is coherent, it is finitely presented. The desired reflexivity follows from Claim 3.2.3. □

By reflexive hull, reflexive sheaves extend from quasi-compact open (*cf.* [GR18, Proposition 11.3.8(i)]).

Corollary 3.2.4. *For a coherent reduced scheme X with an quasi-compact open $U \subset X$, the restriction*

$$\mathscr{O}_X\text{-}\mathbf{Rflx} \to \mathscr{O}_U\text{-}\mathbf{Rflx} \quad \textit{is essentially surjective.}$$

Proof. It suffices to assume that X is irreducible, so X is integral. Every reflexive $\mathscr{O}_U$-module $\mathscr{F}$, by [GR18, Lemma 10.3.24 (ii)], extends to a finitely presented quasi-coherent $\mathscr{O}_X$-module $\widetilde{\mathscr{F}}$, which is coherent. Then by Lemma 3.2.2, the reflexive hull $\widetilde{\mathscr{F}}^{\vee\vee}$ is a reflexive extension of $\mathscr{F}$ on X. □

Corollary 3.2.5. *For a locally coherent integral scheme X and two $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$, if $\mathscr{F}$ is coherent and $\mathscr{G}$ is reflexive, then the natural map $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}^{\vee\vee}, \mathscr{G}) \xrightarrow{\sim} \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ is an isomorphism.*

Proof. Locally on X the reflexive $\mathscr{O}_X$-module $\mathscr{G}$ fits into an exact sequence $0 \to \mathscr{G} \to \mathscr{O}_X^{\oplus m} \to \mathscr{O}_X^{\oplus n}$, hence we have the following commutative diagram of $\mathscr{O}_X$-modules with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}^{\vee\vee}, \mathscr{G}) & \longrightarrow & \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}^{\vee\vee}, \mathscr{O}_X^{\oplus m}) & \longrightarrow & \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}^{\vee\vee}, \mathscr{O}_X^{\oplus n}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) & \longrightarrow & \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X^{\oplus m}) & \longrightarrow & \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X^{\oplus n})
\end{array}$$

By Lemma 3.2.2, $\mathscr{F}^\vee$ is reflexive, hence the two rightmost vertical arrows are bijective and so is the leftmost vertical arrow, as desired. □

Lemma 3.2.6. *Let $X \to S$ be a finite type morphism with regular fibers between topologically Noetherian schemes, let $j\colon U \hookrightarrow X$ be a quasi-compact open immersion with complement $Z := X \backslash U$ satisfying*

$$\operatorname{codim}(Z_s, X_s) \geqslant 1 \textit{ for every } s \in S \quad \textit{and} \quad \operatorname{codim}(Z_\eta, X_\eta) \geqslant 2 \textit{ for every generic point } \eta \in S,$$

and let $\mathscr{F}$ be a reflexive $\mathscr{O}_X$-module. Assume that S is a cofiltered inverse limit of integral schemes $(S_\lambda)_{\lambda\in\Lambda}$ with generic point η_λ and surjective transition maps. Then, there is a $\lambda_0 \in \Lambda$, a finite type morphism $X_{\lambda_0} \to S_{\lambda_0}$ with regular fibers such that $X_{\lambda_0} \times_{S_{\lambda_0}} S \simeq X$, a closed subscheme $Z_{\lambda_0} \subset X_{\lambda_0}$ such that $Z_{\lambda_0} \times_{S_{\lambda_0}} S \simeq Z$, the open immersion $j_{\lambda_0}\colon X_{\lambda_0} \backslash Z_{\lambda_0} \hookrightarrow X_{\lambda_0}$ is quasi-compact, and the following

$$\operatorname{codim}((Z_{\lambda_0})_s, (X_{\lambda_0})_s) \geqslant 1 \textit{ for every } s \in S_{\lambda_0} \quad \textit{and} \quad \operatorname{codim}((Z_{\lambda_0})_{\eta_0}, (X_{\lambda_0})_{\eta_0}) \geqslant 2$$

is satisfied. Also, there is a reflexive $\mathscr{O}_{X_{\lambda_0}}$-module $\mathscr{F}_{\lambda_0}$ whose inverse image on X is $\mathscr{F}$.

Proof. The condition that X has regular S-fibers descends to X_{λ_0} by [EGA IV$_2$, Proposition 6.5.3]. The reflexive $\mathscr{O}_X$-module $\mathscr{F}$ descends thanks to [EGA IV$_3$, Théorème 8.5.2] and by applying [EGA IV$_3$, Corollaire 8.5.2.5] to $\mathscr{F} \xrightarrow{\sim} \mathscr{F}^{\vee\vee}$. Because Z is contructible closed, by [EGA IV$_3$, Théorème 8.3.11], it descends to Z_λ such that $p_\lambda^{-1}(Z_\lambda) = Z$. For $f_\lambda\colon X_\lambda \to S_\lambda$, by the transversity of fibers and [EGA IV$_2$, Corollaire 4.2.6], Z_λ does not contain any irreducible components of $f_\lambda^{-1}(s_\lambda)$ for any $s_\lambda \in S_\lambda$. Finally, the image of the generic point $\eta \in S$ is the generic point $\eta_\lambda \in S_\lambda$. By [EGA IV$_2$, Corollaire 6.1.4], we have $\operatorname{codim}((Z_\lambda)_{\eta_\lambda}, (X_\lambda)_{\eta_\lambda}) = \operatorname{codim}(Z_\eta, X_\eta) \geqslant 2$. □

Proposition 3.2.7. *For a valuation ring V with spectrum S and a flat, locally of finite type morphism $f\colon X \to S$ of integral schemes with regular fibers, the following assertions hold.*

(i) *For every $x \in X$ and every coherent $\mathscr{O}_X$-module $\mathscr{F}$ that is reflexive at x, we have*

$$\operatorname{proj.dim}_{\mathscr{O}_{X,x}} \mathscr{F}_x \leqslant \max(0, n-1), \quad \textit{where} \quad n = \dim \mathscr{O}_{f^{-1}(f(x)),x}.$$

(ii) *For a closed subset $Z \subset X$ such that $j\colon X\backslash Z \hookrightarrow X$ is quasi-compact and satisfies the following*

$$\operatorname{codim}(Z_s, X_s) \geqslant 1 \text{ for all } s \in S \quad \text{and} \quad \operatorname{codim}(Z_\eta, X_\eta) \geqslant 2 \text{ for the generic point } \eta \in S,$$

the restriction functors induce the following equivalences of categories.

$$\mathscr{O}_X\text{-}\mathbf{Rflx} \xrightarrow{\sim} \mathscr{O}_{X\backslash Z}\text{-}\mathbf{Rflx} \qquad \mathbf{Pic}\, X \xrightarrow{\sim} \mathbf{Pic}\, X\backslash Z \tag{3.2.7.1}$$

In particular, for every X-affine finite type algebraic space Y, we have a bijection of sets

$$Y(X) \simeq Y(X\backslash Z).$$

(iii) *For a closed subset $Z \subset X$ such that $j\colon X\backslash Z \hookrightarrow X$ is quasi-compact and $X\backslash Z$ contains all the associated points of the generic fiber of X and every X-separated algebraic space Y, the map*

$$Y(X) \hookrightarrow Y(X\backslash Z) \quad \text{is injective.}$$

(iv) *For a closed subset $Z \subset X$ satisfying the assumption in (ii) and a quasi-compact quasi-separated morphism $p\colon W \to X\backslash Z$ such that $p_*\mathscr{O}_W$ is a reflexive $\mathscr{O}_{X\backslash Z}$-module, we have the Cartesian square*

$$\begin{array}{ccc} \operatorname{Aff}_{X\backslash Z} W & \hookrightarrow & \operatorname{Aff}_X W \\ {\scriptstyle p^{\mathrm{aff}}}\downarrow & & \downarrow{\scriptstyle \nu} \\ X\backslash Z & \xhookrightarrow{j} & X, \end{array}$$

where $\operatorname{Aff}_{X\backslash Z} W = \underline{\operatorname{Spec}}_{X\backslash Z}(p_*\mathscr{O}_W)$ and $\operatorname{Aff}_X W = \underline{\operatorname{Spec}}_X(j_*p_*\mathscr{O}_W)$, such that p^{aff} and ν are finite, p^{aff} is the relative normalization [SP, 035H] *of $X\backslash Z$ in W and ν is the relative normalization of X in W. In particular, $\nu_*(\mathscr{O}_{\operatorname{Aff}(W/X)})$ is a reflexive $\mathscr{O}_X$-module.*

(v) *For a closed subset $Z \subset X$ satisfying the assumption in (ii) and a finite flat locally finitely presented morphism $p\colon W \to X\backslash Z$, the morphism $\nu\colon \operatorname{Aff}_X W \to X$ is the relative normalization of X in W such that $(\operatorname{Aff}_X W)_{X\backslash Z} = W$. In particular, $\nu_*(\mathscr{O}_{\operatorname{Aff}_X W})$ is a reflexive $\mathscr{O}_X$-module.*

Proof. The assertion (i) is [GR18, Proposition 11.4.1 (iii)]. For (ii), by Lemmata 3.1.3 and 3.2.6, we may assume that V has finite rank. Since $|X|$ is the finite disjoint union of its S-fibers X_s, which are Noetherian spaces, we know that X is topologically Noetherian. In particular, every open subset of X is quasi-compact. By Corollary 3.2.4, the functors (3.2.7.1) are essentially surjective. For the faithfulness, consider two morphisms $\alpha, \beta\colon \mathscr{F} \to \mathscr{G}$ between reflexive $\mathscr{O}_X$-modules such that $\alpha|_{X\backslash Z} = \beta|_{X\backslash Z}$. To show that $\alpha = \beta$, since it is a local problem, it suffices to check that $\alpha_x = \beta_x\colon \mathscr{F}_x \to \mathscr{G}_x$ for every $x \in Z$. Take a presentation $\mathscr{O}_{X,x}^{\oplus m} \to \mathscr{O}_{X,x}^{\oplus n} \to \mathscr{F}_x \to 0$ and copresentation $0 \to \mathscr{G}_x \to \mathscr{O}_{X,x}^{\oplus m'} \to \mathscr{O}_{X,x}^{\oplus n'}$, then α_x and β_x induce two morphisms between these copresentations. Then we are reduced to the case when $\mathscr{F}_x$ and $\mathscr{G}_x$ are free. We may assume that $\mathscr{F}_x = \mathscr{O}_{X,x}^{\oplus r}$ and $\mathscr{G}_x = \mathscr{O}_{X,x}^{\oplus s}$, so the following isomorphisms lead to $\alpha = \beta$

$$\operatorname{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x) = \operatorname{Hom}_{\mathscr{O}_{X,x}}(\mathscr{O}_{X,x}^{\oplus r}, \mathscr{O}_{X,x}^{\oplus s}) \simeq \operatorname{Hom}_{j^*\mathscr{O}_{X,x}}(j^*\mathscr{O}_{X,x}^{\oplus r}, j^*\mathscr{O}_{X,x}^{\oplus s}).$$

It remains to show that (3.2.7.1) are full. If $\mathscr{F}$ and $\mathscr{G}$ are two reflexive $\mathscr{O}_X$-modules with a morphism $\phi\colon j^*\mathscr{F} \to j^*\mathscr{G}$, then by [GR18, Corollary 11.3.9], taking $j_*(-)$ induces the following morphism

$$\widetilde{\phi}\colon \mathscr{F} \simeq j_*j^*\mathscr{F} \to j_*j^*\mathscr{G} \simeq \mathscr{G}.$$

For the second assertion of (ii), by the sheaf property, the problem is étale local on X, so we can assume that X is affine. Choose an embedding $Y \hookrightarrow \mathbb{A}_X^n$ for some integer n. The assumption implies that $X\backslash Z$ is scheme-theoretically dense in X. Hence, for every morphism $\phi\colon X\backslash Z \to Y$, if ϕ extends uniquely to a morphism $\widetilde{\phi}\colon X \to \mathbb{A}_X^n$, then $\widetilde{\phi}^{-1}(Y)$ is a closed subscheme of X containing $X\backslash Z$ and by [EGA IV$_4$, Lemme 20.3.8.8], coincides with X. In other words, if $\widetilde{\phi}$ exists uniquely, then it factorises as $X \xrightarrow{\psi} Y \hookrightarrow \mathbb{A}_X^n$ such that ψ is the unique extension of ϕ. This reduces us to the case $Y = \mathbb{A}_X^n$. Now, by the reflexivity of $\mathscr{O}_X$ and the full faithfulness of $\mathscr{O}_X\text{-}\mathbf{Rflx} \xrightarrow{\sim} \mathscr{O}_{X\backslash Z}\text{-}\mathbf{Rflx}$, we have the desired bijections

$$\mathbb{A}_X^n(X) = \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{O}_X^{\oplus n}) \simeq \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{O}_{X\backslash Z}, \mathscr{O}_{X\backslash Z}^{\oplus n}) = \mathbb{A}_X^n(X\backslash Z).$$

To prove (iii), we first prove that $X\backslash Z \subset X$ is scheme-theoretically dense in the sense of [SP, 0834]. By [SP, 0836], we need to show that $\mathscr{O}_X \to j_*\mathscr{O}_{X\backslash Z}$ is injective, which through the terminology of [GR18, 10.4.2 and 10.4.19], signifies that $\delta(z, \mathscr{O}_X) > 0$ for all $z \in Z$. It suffices to take étale coverings of X by schemes and use the depth formula [GR18, Corollary 10.4.46] for flat morphisms to deduce that

all $z \in Z$ satisfies $\delta(x, \mathscr{O}_X) > 0$. Since j is quasi-compact, by [SP, 0835], the schematic image of $X\backslash Z$ is X. Therefore, we apply [SP, 084N] to conclude. The (iv) follows from (ii). For (v), note that $p_*\mathscr{O}_W$ is $\mathscr{O}_{X\backslash Z}$-reflexive since by [SP, 02KB], p is finite locally free, hence it is a special case of (iv). □

4. Auslander's flatness criterion on schemes smooth over valuation rings

The goal is to establish Theorem 4.1 as a counterpart of Auslander's flatness criterion [Aus62, Theorem 1.3] on schemes smooth over valuation rings. As expected, our criterion leads to a Zariski–Nagata purity.

Theorem 4.1. *For a valuation ring V with spectrum S and closed point $s \in S$, an S-smooth finite type scheme X, a point $x \in X$ lying over s with local ring $A := \mathscr{O}_{X,x}$, and a reflexive A-module M,*

$$\mathrm{End}_A(M) \text{ is isomorphic to a direct sum of copies of } M \quad \text{if and only if} \quad M \text{ is } A\text{-free.}$$

As Auslander's proof, our strategy relies on an estimate of the length of cohomology groups of M. To begin with, we introduce the length function on torsion modules over valuation rings.

4.2. Lengths of torsion modules. For a nontrivial valuation ring V with fraction field K, value group Γ and a valuation map $\nu\colon K \to \Gamma$, every finitely presented torsion V-module M is of the form

$$M \simeq \bigoplus_i V/a_iV \quad \text{for finitely many } a_i \in V\backslash\{0\}.$$

Define the *length* of M as $\delta(M) = \sum_i \nu(a_i) \in \Gamma_{\geqslant 0}$. The element $\delta(M)$ is well defined, and $\delta(M) = 0$ if and only if $M = 0$. Every acyclic, bounded complex $M^\bullet$ of torsion, finitely presented V-modules satisfies

$$\textstyle\sum_j (-1)^j \delta(M^j) = 0.$$

Lemma 4.3. *For a nontrivial valuation ring V, an essentially smooth V-local algebra $(A, \mathfrak{m}_A)$, and the collection A-$\mathbf{Mod}_{\mathrm{tor,fp}}$ of all finitely presented A-modules M such that $\mathrm{Supp}(M) \subset \{\mathfrak{m}_A\}$, there exist a totally ordered abelian group Γ and a map $l\colon A\text{-}\mathbf{Mod}_{\mathrm{tor,fp}} \to \Gamma_{\geqslant 0}$ satisfying the following properties:*

- *for A-module $M \in A$-$\mathbf{Mod}_{\mathrm{tor,fp}}$, we have $l(M) = 0$ if and only if $M = 0$;*
- *for every acyclic, bounded complex $M^\bullet$ such that $M^j \in A$-$\mathbf{Mod}_{\mathrm{tor,fp}}$ for each j, one has*

$$\textstyle\sum_j (-1)^j l(M^j) = 0.$$

Proof. First we assume that the structural map $V \to A$ admits a section $A \twoheadrightarrow V$. In this case we claim that M is finitely presented over V and is V-torsion, so we can simply let Γ be the valuation group of V and set $l(M) := \delta(M)$, where δ is delivered from 4.2. Indeed, it is clear that M is V-torsion. Any section $\operatorname{Spec} V \to \operatorname{Spec} A$ is a regular immersion [SP, 067R], so there is a finitely generated ideal $J \subset A$ such that $V \simeq A/J$. Hence, since $M \in A$-$\mathbf{Mod}_{\mathrm{tor,fp}}$, we see that $J^nM = 0$ for a large n. On the other hand, the essential smoothness of A over V implies that J/J^2 is a free $V \simeq A/J$-module whose rank equals the rank of the free A-module $\Omega^1_{A/V}$, and there is a natural isomorphism of graded $V \simeq A/J$-algebras

$$\textstyle\bigoplus_{n\geqslant 0} J^n/J^{n+1} \simeq \mathrm{Sym}^\bullet_{A/J}(J/J^2).$$

In particular, A/J^n is a finite free V-module for every $n \geqslant 1$. Therefore, by tensoring a presentation

$$A^N \to A^N \to M \to 0$$

of M with A/J^n for a large enough n, we get a desired finite presentation of the V-module M.

In the general case, we first use Lemma 3.1.2 to reduce to the case when the residue fields extension of $V \to A$ is finite. Then, if B is the integral closure of V in an algebraic closure of $\mathrm{Frac}(V)$, we let V' be a valuation ring of $\mathrm{Frac}(B)$ centered at a maximal ideal of B. It's clear that V' is absolutely integral closed, so it is strictly Henselian and there exists a V-map $\phi : A/\mathfrak{m}_A \to V'/\mathfrak{m}_{V'}$. Let $A' := A \otimes_V V'$. Then ϕ induces a V'-map $\phi' : A' \to V'/\mathfrak{m}_{V'}$; let $\mathfrak{p} \subset A'$ be its kernel. Then $A'_\mathfrak{p}$ is essentially smooth over V' and ϕ' induces a V'-map $A'_\mathfrak{p} \to V'/\mathfrak{m}_{V'}$, which, by the Henselianity of V', lifts to a V'-map $A'_\mathfrak{p} \to V'$. By the previous paragraph, the lemma is true for $A'_\mathfrak{p}$, say, with corresponding map l' valued in Γ, where Γ is the valuation group of V'. Since $A \to A'_\mathfrak{p}$ is faithfully flat, it suffices to define $l(M) := l'(M \otimes_A A'_\mathfrak{p})$. □

Lemma 4.4. *For a valuation ring V, a V-smooth finite type scheme X, a point $x \in X$ that lies over a non-generic point $s \in \mathrm{Spec}(V)$, and a map of finitely presented $\mathscr{O}_{X,x}$-modules $M \to N$ that induces an isomorphism over $\mathrm{Spec}(\mathscr{O}_{X,x})\backslash\{x\}$, we have an isomorphism $\mathrm{Ext}^i_{\mathscr{O}_{X,x}}(N, \mathscr{O}_{X,x}) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathscr{O}_{X,x}}(M, \mathscr{O}_{X,x})$ for every $i < d$ and a monomorphism $\mathrm{Ext}^d_{\mathscr{O}_{X,x}}(N, \mathscr{O}_{X,x}) \hookrightarrow \mathrm{Ext}^d_{\mathscr{O}_{X,x}}(M, \mathscr{O}_{X,x})$, where $d := \dim \mathscr{O}_{X_s,x}$.*

Proof. Let ker, coker, and im be the kernel, cokernel, and image of $M \to N$, respectively. By assumption and the coherence of $\mathscr{O}_{X,x}$, ker and coker are coherent, or, equivalently, finitely presented $\mathscr{O}_{X,x}$-modules ([SP, 05CX]) supported at $\{x\}$. Consider the following short exact sequences

$$0 \to \ker \to M \to \mathrm{im} \to 0,$$

$$0 \to \mathrm{im} \to N \to \mathrm{coker} \to 0.$$

By applying $\mathrm{Hom}_{\mathscr{O}_{X,x}}(-, \mathscr{O}_{X,x})$, we get two long exact sequences concerning Ext's, and the lemma follows from the vanishing $\mathrm{Ext}^i_{\mathscr{O}_{X,x}}(\ker, \mathscr{O}_{X,x}) = 0$ and $\mathrm{Ext}^i_{\mathscr{O}_{X,x}}(\mathrm{coker}, \mathscr{O}_{X,x}) = 0$ for $i \leqslant d$ (Lemma 2.7). □

Lemma 4.5. *For finitely presented $A := \mathscr{O}_{X,x}$-modules M and N, $\mathrm{Ext}^i_A(M, N)$ and $\mathrm{Tor}^A_i(M, N)$ are finitely presented over A for all $i \geqslant 0$ and are zero for $i > d + 1$, where $d = \dim \mathscr{O}_{X_s,x}$.*

Proof. By [GR18, Proposition 11.4.1 (i)], since A is coherent, the coherent A-module ([SP, 05CX]) M has a resolution by finite free A-modules of length $\leqslant d + 1$: $F_\bullet \to M$, $F_i = 0$ for $i > d + 1$. Then

$$\mathrm{Ext}^i_A(M, N) = H^i(\mathrm{Hom}(F_\bullet, N)) \quad \text{and} \quad \mathrm{Tor}^A_i(M, N) = H_i(F_\bullet \otimes N)$$

are all coherent, or equivalently, finitely presented A-modules, and are zero for $i > d + 1$. □

Lemma 4.6. *For a finitely presented $A := \mathscr{O}_{X,x}$-module M, we have a natural isomorphism*

$$\mathrm{End}_A(M)^{\vee\vee} \xrightarrow{\sim} \mathrm{End}_A(M^{\vee\vee}).$$

Proof. First, we define a natural map $\mathrm{End}_A(M)^{\vee\vee} \to \mathrm{End}_A(M^{\vee\vee})$. Note that $M^{\vee\vee}$ is A-reflexive due to Lemma 3.2.2. By Corollary 3.2.5, where $M^{\vee\vee}$ plays the role of $\mathscr{G}$, we get a natural isomorphism

$$\mathrm{Hom}_A(M, M^{\vee\vee}) \xleftarrow{\sim} \mathrm{End}_A(M^{\vee\vee}).$$

It suffices to consider the natural maps $\mathrm{End}_A(M) \to \mathrm{Hom}_A(M, M^{\vee\vee}) \xleftarrow{\sim} \mathrm{End}_A(M^{\vee\vee})$. By Lemma 3.2.2, the two rightmost modules are reflexive. Taking double dual yields the desired map of reflexive A-modules.

It remains to check that the map $\mathrm{End}_A(M)^{\vee\vee} \to \mathrm{End}_A(M^{\vee\vee})$ is an isomorphism. The equivalence of categories of reflexive modules in Proposition 3.2.7(ii) reduces us to checking this at $x \in X$ that is either a one-codimensional point of the generic V-fiber or a maximal point of a non-generic V-fiber, where, by Lemma 3.1.1(iii), A is a valuation ring, so there is an $N \in \mathbf{Z}_{\geqslant 0}$ and finitely many $a_i \in \mathfrak{m}_A\backslash\{0\}$ such that

$$M \simeq A^{\oplus N} \bigoplus (\oplus_i A/a_i A)\,.$$

Consequently, we conclude by the isomorphisms $\mathrm{End}_A(M)^{\vee\vee} \simeq \mathrm{End}_A(A^{\oplus N}) \simeq \mathrm{End}_A(M^{\vee\vee})$. □

Proof of Theorem 4.1. The proof proceeds as the following steps.

Preliminary cases and reductions. First, since X is locally of finite presentation over S and M is finitely presented over A, by a standard limit argument involving Lemmata 3.1.3 and 3.2.6, we are reduced to the case when V is a finite-rank valuation ring. Secondly, if V' is a valuation ring of an algebraic closure of $\mathrm{Frac}(V)$ that dominates V and if $x' \in X' := X \times_V V'$ is a point lying over $x \in X$, then $M_{A'} := M \otimes_A A'$ is a finitely presented reflexive A'-module and $\mathrm{End}_{A'}(M_{A'}) \simeq \mathrm{End}_A(M) \otimes_A A'$ is isomorphic to a direct sum of copies of $M_{A'}$, where $A' := \mathscr{O}_{X',x'}$ (because A' is faithfully flat over A). By faithfully flat descent [SP, 08XD, 00NX], the freeness of M over A is equivalent to the freeness of $M_{A'}$ over A'. Therefore, by replacing V by V', A by A', and M by $M_{A'}$, we are reduced to the case when $\mathrm{Frac}(V)$ is algebraically closed (this assumption will be only used in the very end of the proof).

Set $d_x := \dim(\mathscr{O}_{X_s,x})$ and $r := \mathrm{rank}(V)$. The case $r = 0$ and d_x arbitrary is classical. The case r arbitrary and $d_x = 0$ is trivial, because A is a valuation ring (Lemma 3.1.1(iii)). The case r arbitrary and $d_x = 1$ follows from Proposition 3.2.7(i). Subsequently, we may assume $d_x \geqslant 2$ in the sequel.

Case 1: r is arbitrary and $d_x = 2$. Now, we deal with the crucial case when r arbitrary and $d_x = 2$ by induction on r. The induction hypothesis is that the assertion holds for $d_x = 2$ and $r' \leqslant r-1$. Notice that, for any proper generalization $x' \in X$ of x that lies over, say, $s' \in \mathrm{Spec}(V)$, by Lemma 3.1.1(iv), we have either $s' = s$ and $d_{x'} < 2$, or $\mathrm{ht}(s') < r$ and $d_{x'} \leqslant 2$. Hence, by induction hypothesis and the preliminary cases above, the assertion holds for $\mathscr{O}_{X,x'}$. Since $M_{x'}$ is a finitely presented reflexive $\mathscr{O}_{X,x'}$-module and

$$\mathrm{End}_{\mathscr{O}_{X,x'}}(M_{x'}) = \mathrm{End}_{\mathscr{O}_{X,x}}(M) \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X,x'} \simeq (\bigoplus M) \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X,x'} = \bigoplus M_{x'},$$

the induction hypothesis applies to the $\mathscr{O}_{X,x'}$-module $M_{x'}$, implying that $M_{x'}$ is $\mathscr{O}_{X,x'}$-free. In other words, $\widetilde{M}$ is locally free over $\mathrm{Spec}\, A\backslash\{x\}$. Consider the following evaluation map

$$M^\vee \otimes_A M \to \mathrm{Hom}_A(M, M), \quad f \otimes m \mapsto [m' \mapsto f(m')m],$$

which, by the local freeness of $\widetilde{M}$ over $\mathrm{Spec}\, A\backslash\{x\}$, is an isomorphism over $\mathrm{Spec}\, A\backslash\{x\}$. Since $d_x = 2 > 1$, by Lemma 4.4, we apply $\mathrm{Ext}^1_A(-, A)$ to the above map to obtain the following isomorphism

$$\mathrm{Ext}^1_A(M^\vee \otimes M, A) \simeq \mathrm{Ext}^1_A(\mathrm{End}_A(M), A) \simeq \mathrm{Ext}^1_A(M, A)^{\oplus \mathrm{rk}_M} \tag{4.6.1}$$

of A-modules that are supported on $\{x\}$ by the local freeness of $\widetilde{M}$ over $\mathrm{Spec}\, A\backslash\{x\}$, where $\mathrm{rk}_M = \dim_{\mathrm{Frac}\, A} M \otimes_A \mathrm{Frac}\, A$. By Lemma 4.5, the modules in (4.6.1) are also finitely presented over A.

For the adjunction $\mathrm{Hom}_A(M, \mathrm{Hom}_A(M^\vee, -)) \simeq \mathrm{Hom}_A(M \otimes M^\vee, -)$, we take their derived functors valued at A, so the E_2-page of the associated Grothendieck spectral sequence yields a monomorphism

$$\mathrm{Ext}^1_A(M, M) \hookrightarrow \mathrm{Ext}^1_A(M \otimes M^\vee, A) \overset{(4.6.1)}{\simeq} \mathrm{Ext}^1_A(M, A)^{\oplus \mathrm{rk}_M},$$

where we have used $M^{\vee\vee} \simeq M$; again, by the local freeness of $\widetilde{M}$ over $\mathrm{Spec}\, A\backslash\{x\}$ and Lemma 4.5, they are finitely presented supported on $\{x\}$. In particular, the map l from Lemma 4.3 applies so we have

$$l(\mathrm{Ext}^1_A(M, M)) \leqslant \mathrm{rk}_M \cdot l(\mathrm{Ext}^1_A(M, A)). \tag{4.6.2}$$

Since M is reflexive, by Proposition 3.2.7(i), we have $\mathrm{proj.dim}_A M \leqslant d_x - 1 = 1$. We prove $\mathrm{proj.dim}(M) = 0$ by contradiction. If $\mathrm{proj.dim}(M) = 1$, then M has a free resolution $0 \to F_1 \to F_0 \to M \to 0$ by finite A-modules. As M is not free, the sequence is nonsplit, corresponding to a nontrivial extension class in

$$\mathrm{Ext}^1_A(M, F_1) \simeq \mathrm{Ext}^1_A(M, A)^{\mathrm{rank}(F_1)}.$$

In particular, we have $C := \mathrm{Ext}^1_A(M, A) \neq 0$. Applying $\mathrm{Hom}_A(-, A)$ to $F_\bullet \to M$ yields an exact sequence

$$0 \to M^\vee \to F_0^\vee \to F_1^\vee \to \mathrm{Ext}^1_A(M, A) \to 0.$$

Tensoring it with M, we get an exact sequence $F_0^\vee \otimes_A M \to F_1^\vee \otimes_A M \to \mathrm{Ext}^1_A(M, A) \otimes_A M \to 0$. Since

$$\mathrm{coker}\,(F_0^\vee \otimes M \to F_1^\vee \otimes M) \simeq \mathrm{coker}\,(\mathrm{Hom}_A(F_0, M) \to \mathrm{Hom}_A(F_1, M)) = \mathrm{Ext}^1_A(M, M),$$

we deduce that $\mathrm{Ext}^1_A(M, M) \simeq \mathrm{Ext}^1_A(M, A) \otimes_A M = C \otimes_A M$.

By tensoring $0 \to F_1 \to F_0 \to M \to 0$ with $C = \mathrm{Ext}^1_A(M, A)$ (which is a nonzero finitely presented A-module supported at $\{x\}$, by the locally freeness of $\widetilde{M}$ over $\mathrm{Spec}\, A\backslash\{x\}$), we get an exact sequence

$$0 \to \mathrm{Tor}^A_1(C, M) \to C \otimes_A F_1 \to C \otimes_A F_0 \to C \otimes_A M \to 0$$

of finitely presented A-modules supported on $\{x\}$. Applying the map l from Lemma 4.3, we obtain

$$l(C \otimes_A M) = l(C \otimes_A F_0) - l(C \otimes_A F_1) + l(\mathrm{Tor}^A_1(C, M) = \mathrm{rk}_M \cdot l(C) + l(\mathrm{Tor}^A_1(C, M)), \tag{4.6.3}$$

where $\mathrm{rk}_M = \mathrm{rank}(F_0) - \mathrm{rank}(F_1) > 0$. On the other hand, since $C \otimes_A M \simeq \mathrm{Ext}^1_A(M, M)$, we deduce

$$l(C \otimes_A M) \overset{(4.6.2)}{\leqslant} \mathrm{rk}_M \cdot l(C). \tag{4.6.4}$$

The combination of (4.6.3) and (4.6.4) leads to $l(\mathrm{Tor}^A_1(C, M)) = 0$. So, we have an exact sequence

$$0 \to C \otimes_A F_1 \to C \otimes_A F_0 \to C \otimes_A M \to 0,$$

which combined with Lemma 2.8 implies that the map $F_1 \to F_0$ splits, that is, M is A-free, contradicting our assumption that $\mathrm{proj.dim}_A(M) = 1$. This completes the case when r is arbitrary and $d_x = 2$.

Case 2: r is arbitrary and $d_x > 2$. We deduce by double induction on the pair $(r = \mathrm{ht}(s), d_x)$. By induction hypothesis, the assertion holds for all smooth V-scheme X' and all points $x' \in X'$ such that $\mathrm{ht}(s') \leqslant \mathrm{ht}(s)$ and $d_{x'} \leqslant d_x$, where $s' \in \mathrm{Spec}(V)$ lies below x', and at least one of equalities is strict. In

particular, by Lemma 3.1.1(iv), the induction hypothesis applies to $\mathscr{O}_{X,x'}$ for all proper generalization $x' \in X$ of x. Since $M_{x'}$ is a finitely presented reflexive $\mathscr{O}_{X,x'}$-module and

$$\mathrm{End}_{\mathscr{O}_{X,x'}}(M_{x'}) = \mathrm{End}_{\mathscr{O}_{X,x}}(M)_{x'} \simeq \bigoplus M_{x'},$$

the induction hypothesis gives that $M_{x'}$ is $\mathscr{O}_{X,x'}$-free. In other words, $\widetilde{M}$ is locally free over $\operatorname{Spec} A\backslash\{x\}$.

Claim 4.6.5 ([SP, 057F]). Assume that the residue field extension of $V \to A$ is separable (e.g., this holds if $\kappa(s) := V/\mathfrak{m}_V$ is perfect), then there exists an $a \in A$ such that $\overline{A} := A/(a)$ is essentially V-smooth and

$$\dim(\overline{A}/\mathfrak{m}_V\overline{A}) = d_x - 1.$$

Since our V has algebraically closed fraction field (by the first paragraph), all of its primes have algebraically closed residue fields, so we can choose $a \in A$ as in the above claim. Since a is nonzerodivisor in A and $M = \mathrm{Hom}_A(M^\vee, A)$, we see that a is M-regular. Set $\overline{M} := M/aM$. Applying $\mathrm{Hom}_A(M, -)$ to the short exact sequence $0 \to M \xrightarrow{a} M \to \overline{M} \to 0$, we get an exact sequence

$$0 \to \mathrm{Hom}_A(M, M) \xrightarrow{a} \mathrm{Hom}_A(M, M) \to \mathrm{Hom}_A(M, \overline{M}) \to \mathrm{Ext}^1_A(M, M).$$

Substituting our assumption $\mathrm{Hom}_A(M, M) \cong M^{\oplus \mathrm{rk}_M}$ into it yields an exact sequence of $\overline{A}$-modules

$$0 \to \overline{M}^{\oplus \mathrm{rk}_M} \to \mathrm{Hom}_{\overline{A}}(\overline{M}, \overline{M}) \to T \to 0,$$

where $T \subset \mathrm{Ext}^1_A(M, M)$ is a finitely presented $\overline{A}$-submodule (Lemma 4.5), which, by the locally freeness of $\widetilde{M}$ over $\operatorname{Spec} A\backslash\{x\}$, is supported on $\{x\}$. Since $\dim(\overline{A}/\mathfrak{m}_V\overline{A}) = d_x - 1 \geqslant 2$, taking dual (as $\overline{A}$-modules) of the above short exact sequence and using Lemma 4.4, we see that

$$(\overline{M}^\vee)^{\oplus \mathrm{rk}_M} \simeq \mathrm{Hom}_{\overline{A}}(\overline{M}, \overline{M})^\vee.$$

Taking dual further and invoking Lemma 4.6, we get the following isomorphism

$$(\overline{M}^{\vee\vee})^{\oplus \mathrm{rk}_M} \simeq \mathrm{Hom}_{\overline{A}}(\overline{M}^{\vee\vee}, \overline{M}^{\vee\vee}).$$

Since the double dual $\overline{M}^{\vee\vee}$ is finitely presented over $\overline{A}$ and is reflexive (Lemma 3.2.2), we can apply our induction hypothesis to the $\overline{A}$-module $\overline{M}^{\vee\vee}$ and conclude that it is $\overline{A}$-free. The same lemma also implies that $\overline{M}^\vee$ is $\overline{A}$-reflexive, so $\overline{M}^\vee \simeq \overline{M}^{\vee\vee\vee}$ is $\overline{A}$-free.

Finally, we show that M is A-free. Since $\widetilde{M}$ is locally free over $\operatorname{Spec} A\backslash\{x\}$, the natural map $\overline{M} \to \overline{M}^{\vee\vee}$ is an isomorphism over $\operatorname{Spec} A\backslash\{x\}$, and, since $\dim(\overline{A}/\mathfrak{m}_V\overline{A}) = d_x - 1 > 1$, we may apply Lemma 4.4 to see that $\mathrm{Ext}^1_{\overline{A}}(\overline{M}, \overline{A}) \simeq \mathrm{Ext}^1_{\overline{A}}(\overline{M}^{\vee\vee}, \overline{A}) = 0$. Since a is M-regular, we deduce that

$$\mathrm{Ext}^1_A(M, \overline{A}) \simeq \mathrm{Ext}^1_{\overline{A}}(M \otimes^{\mathbf{L}}_A \overline{A}, \overline{A}) \simeq \mathrm{Ext}^1_{\overline{A}}(\overline{M}, \overline{A}) = 0.$$

Applying $\mathrm{Hom}_A(M, -)$ to the short exact sequence $0 \to A \xrightarrow{a} A \to \overline{A} \to 0$ we get an exact sequence

$$0 \to M^\vee \xrightarrow{a} M^\vee \to \mathrm{Hom}_A(M, \overline{A}) \to \mathrm{Ext}^1_A(M, A) \xrightarrow{a} \mathrm{Ext}^1_A(M, A) \to \mathrm{Ext}^1_A(M, \overline{A}).$$

As all modules are finitely presented over A and $\mathrm{Ext}^1_A(M, \overline{A}) = 0$, Nakayama's lemma gives that $\mathrm{Ext}^1_A(M, A) = 0$. Therefore, $M^\vee/aM^\vee \simeq \mathrm{Hom}_A(M, \overline{A}) = \overline{M}^\vee$ is $\overline{A}$-free (by the previous paragraph). From this we can deduce that M is A-free. Indeed, the $\overline{A}$-free module $M^\vee/aM^\vee$ has projective dimension 1 over A, thus, for any finitely presented A-module N, we can derive from

$$0 \to M^\vee \xrightarrow{a} M^\vee \to M^\vee/aM^\vee \to 0$$

an exact sequence of finitely presented A-modules

$$\mathrm{Ext}^1_A(M^\vee, N) \xrightarrow{a} \mathrm{Ext}^1_A(M^\vee, N) \to \mathrm{Ext}^2_A(M^\vee/aM^\vee, N).$$

As $\mathrm{Ext}^2_A(M^\vee/aM^\vee, N) = 0$, by Nakayama's lemma, we have $\mathrm{Ext}^1_A(M^\vee, N) = 0$. In particular, for any surjection $A^{\oplus n} \twoheadrightarrow M^\vee$ with, say, kernel N, the extension class of the short exact sequence $0 \to N \to A^{\oplus n} \to M^\vee \to 0$ is zero. This implies that $M^\vee$ is A-free, hence so is $M = M^{\vee\vee}$. □

5. Generalities on torsors over algebraic spaces

5.1. Setup. Throughout this section, we let S denote a base scheme, X an algebraic space over S, and G an X-group algebraic space.

Definition 5.2.

(1) A (right) *G-torsor* (for the fppf topology) is an X-algebraic space $\mathcal{P}$ equipped with a G-action $a : \mathcal{P} \times_X G \to \mathcal{P}$ such that the following conditions hold:

  (i) the induced morphism $\mathcal{P} \times_X G \xrightarrow{\sim} \mathcal{P} \times_X \mathcal{P}, (p, g) \mapsto (p, a(p, g))$, is an isomorphism; and

  (ii) there exists a fppf covering $\{X_i \to X\}_{i \in I}$ of algebraic spaces [SP, 03Y8] such that $\mathcal{P}(X_i) \neq \varnothing$ for every $i \in I$.

(2) For G-torsors $\mathcal{P}_1$ and $\mathcal{P}_2$, a *morphism* $\mathcal{P}_1 \to \mathcal{P}_2$ is a G-equivariant morphism $\mathcal{P}_1 \to \mathcal{P}_2$ of X-algebraic spaces.

(3) By a trivialization of a G-torsor $\mathcal{P}$ we mean a G-equivariant isomorphism $t : G \xrightarrow{\sim} \mathcal{P}$, where G acts on itself via right multiplication; this amounts to the choice of a section $t(1_G) \in \mathcal{P}(X)$ (if exists). A G-torsor $\mathcal{P}$ is *trivial* if there exists a trivialization, or, equivalently, if $\mathcal{P}(X) \neq \varnothing$.

Note that every morphism of two G-torsors is an isomorphism. To see this, one may pass to a fppf covering of X to reduce to the case when both torsors are trivial, in this case the assertion is trivial.

Remark 5.3. One can also define a sheaf torsor for an X-group algebraic space G. It is a sheaf

$$\mathcal{P} : (\mathbf{Sch}_{/S})^{\mathrm{opp}}_{\mathrm{fppf}} \to \mathbf{Set}$$

equipped with a map $\mathcal{P} \to X$ of sheaves and a G-action $a : \mathcal{P} \times_X G \to \mathcal{P}$ such that the above two conditions (i) and (ii) in (1) hold. However, it turns out that such a sheaf torsor is necessarily representable by an algebraic space, so working with sheaf torsors adds no more generality. To see this, let $\{X_i \to X\}_{i \in I}$ be a fppf covering as in (ii) that trivializes $\mathcal{P}$. Then every $\mathcal{P} \times_X X_i \simeq G \times_X X_i$ is an algebraic space, and the map

$$\bigsqcup_i \mathcal{P} \times_X X_i \to \mathcal{P}$$

is representable by algebraic spaces and is a fppf covering, because it is the base change of the fppf covering $\bigsqcup_i X_i \to X$ of algebraic spaces via $\mathcal{P} \to X$. Here, all coproducts are taken in the category of sheaves on $(\mathbf{Sch}/S)_{\mathrm{fppf}}$. It follows from (3) of [SP, 04S6] that $\mathcal{P}$ is an algebraic space, as desired.

Let $\mathcal{P}_1, \mathcal{P}_2$ be two G-torsors. Define a functor

$$\underline{\mathrm{Isom}}_X(\mathcal{P}_1, \mathcal{P}_2) : (\mathbf{Sch}_{/X})^{\mathrm{opp}} \to \mathbf{Set}$$

which associates to any scheme T over X the set of G_T-equivariant isomorphisms $\mathcal{P}_{1,T} \to \mathcal{P}_{2,T}$ over T.

Lemma 5.4. *For two G-torsors $\mathcal{P}_1$ and $\mathcal{P}_2$, $\underline{\mathrm{Isom}}_X(\mathcal{P}_1, \mathcal{P}_2)$ is an algebraic space over S. Further, $G \to X$ is quasi-compact (resp., étale, smooth, flat, separated, (locally) of finite type, (locally) of finite presentation, quasi-affine, affine, or finite) if and only if $\underline{\mathrm{Isom}}_X(\mathcal{P}_1, \mathcal{P}_2) \to X$ is so.*

Proof. Since $\underline{\mathrm{Isom}}_X(\mathcal{P}_1, \mathcal{P}_2)$ is fppf locally on X isomorphic to G, it admits a representable fppf covering by algebraic spaces, hence it is an algebraic space by [SP, 04S6].

The list properties of morphisms of algebraic spaces are all stable under base changes and are fppf local on the target, see [SP, 03KG] (resp., [SP, 03XT, 03ZF, 03MM, 03KM, 040Y, 0410, 03WM, 03WG, 03ZQ]). Consequently, since $\underline{\mathrm{Isom}}_X(\mathcal{P}_1, \mathcal{P}_2)$ is fppf locally on X isomorphic to G, the properties of G are inherited by and can be detected from $\underline{\mathrm{Isom}}_X(\mathcal{P}_1, \mathcal{P}_2)$. □

Since every G-torsor $\mathcal{P} \to X$ trivializes over a fppf covering $\{X_i \to X\}$, one may try to obtain $\mathcal{P}$ by glueing the trivial G_{X_i}-torsors $\mathcal{P}_{X_i}$ using the canonical isomorphisms

$$\phi_{ij} : (\mathcal{P}_{X_i})_{X_{ij}} \simeq \mathcal{P}_{X_{ij}} \simeq \mathcal{P}_{X_j})_{X_{ji}}, \quad \text{where} \quad X_{ij} = X_i \times_X X_j.$$

It turns out that, unlike the case of schemes, this is always possible in the framework of algebraic spaces, see Lemma 5.6. Note that, by taking $U := \bigsqcup X_i$, we may assume that $\mathcal{P}_U$ is trivial for a fppf covering $U \to X$ with U an algebraic space.

Definition 5.5 (Descent datum for torsors)**.** Let S, X and G be as in 5.1. Let $U \to X$ be a fppf covering of algebraic spaces over S. For every integer $n \geqslant 0$, denote by $U^{(n)}$ the n-fold fiber product of U over X. The *category of descent datum* for G-torsors relative to $U \to X$, denoted

$$\mathbf{Tors}\left((U^{(2)} \rightrightarrows U)_{\mathrm{fppf}}, G\right),$$

has pairs $(\mathcal{Q}, \phi)$ as objects, where

- $\mathcal{Q} \to U$ is a G_U-torsor; and
- $\phi : \mathrm{pr}_1^*\mathcal{Q} \xrightarrow{\sim} \mathrm{pr}_2^*\mathcal{Q}$ is an isomorphism of $G_{U^{(2)}}$-torsors such that the following diagram commutes (i.e., the cocycle condition holds)

$$\begin{array}{ccccccc}
\mathrm{pr}_{12}^*\mathrm{pr}_1^*\mathcal{Q} & \xrightarrow{\mathrm{pr}_{12}^*(\phi)} & \mathrm{pr}_{12}^*\mathrm{pr}_2^*\mathcal{Q} & \xrightarrow{\simeq} & \mathrm{pr}_{23}^*\mathrm{pr}_1^*\mathcal{Q} & \xrightarrow{\mathrm{pr}_{23}^*(\phi)} & \mathrm{pr}_{23}^*\mathrm{pr}_2^*\mathcal{Q} \\
\downarrow \simeq & & & & & & \downarrow \simeq \\
\mathrm{pr}_{13}^*\mathrm{pr}_1^*\mathcal{Q} & & & \xrightarrow{\qquad\qquad \mathrm{pr}_{13}^*(\phi) \qquad\qquad} & & & \mathrm{pr}_{13}^*\mathrm{pr}_2^*\mathcal{Q}.
\end{array}$$

A *morphism* from a pair $(\mathcal{Q}, \phi)$ to another pair $(\mathcal{Q}', \phi')$ is a morphism $\theta : \mathcal{Q} \to \mathcal{Q}'$ of G_U-torsors compatible with ϕ and ϕ', that is, $\mathrm{pr}_2^*(\theta)\phi = \phi'\mathrm{pr}_1^*(\theta)$.

To every G-torsor $\mathcal{P}$ one can associate a pair $\Psi(\mathcal{P}) := (\mathcal{P}_U, \mathrm{can})$ via base changes, where can is the canonical isomorphism $\mathrm{pr}_1^*(\mathcal{P}_U) \simeq \mathcal{P}_{U^{(2)}} \simeq \mathrm{pr}_2^*(\mathcal{P}_U)$. Thus we obtain a functor

$$\Psi : \mathbf{Tors}(X_{\mathrm{fppf}}, G) \to \mathbf{Tors}((U^{(2)} \rightrightarrows U)_{\mathrm{fppf}}, G).$$

Lemma 5.6 (Descent G-torsors)**.** *Ψ is an equivalence of category.*

In other words, every descent datum $(\mathcal{Q}, \phi)$ for G-torsors are effective in the sense that there exists a G-torsor $\mathcal{P}$ and an isomorphism $\mathcal{Q} \simeq \mathcal{P}_U$ compatible with θ and the canonical descent datum for $\mathcal{P}_U$.

Proof. The full faithfulness of Ψ follows from the sheaf property of the functor $\underline{\mathrm{Isom}}_X(\mathcal{P}_1, \mathcal{P}_2)$ for any G-torsors $\mathcal{P}_1$ and $\mathcal{P}_2$. To show that Ψ is essential surjective, we pick a descent datum $(\mathcal{Q}, \phi)$, and we need to show that there exists a G-torsor $\mathcal{P}$ such that $(\mathcal{P}_U, \mathrm{can}) \simeq (\mathcal{Q}, \phi)$.

When both X and U are schemes, this is proven in [SP, 04U1]. The case of algebraic spaces can be proved similarly, and we repeat the argument for convenience. First we view $\mathcal{Q}$ as a sheaf on the site $(\mathbf{AS}/U)_{\mathrm{fppf}}$ (by the natural equivalence of the topoi associated to $(\mathbf{AS}/U)_{\mathrm{fppf}}$ and $(\mathbf{Sch}/U)_{\mathrm{fppf}}$). Since descent datums for sheaves on any site are always effective [SP, 04TR], we may find a sheaf $\mathcal{P}$ on the site $(\mathbf{AS}/X)_{\mathrm{fppf}}$ and an isomorphism of sheaves $\mathcal{P}_U \simeq \mathcal{Q}$ compatible with the descent datums. Further, since maps of sheaves on any site can be glued [SP, 04TQ], the G_U-action on $\mathcal{Q}$ descent to a G-action on $\mathcal{P}$. All the assumptions (i) and (ii) of Definition 5.2 hold, because they can be checked on the fppf covering $U \to X$. It remains to see that $\mathcal{P}$ is representable by an algebraic space over X. However, this follows from (3) of [SP, 04S6], in view of the fact that the map $\mathcal{Q} \to \mathcal{P}$ is representable by algebraic spaces and is a fppf covering (being a base change of the fppf covering $U \to X$). □

We end this section with the following result, which will be used repeatedly in the sequel.

Lemma 5.7. *Let S be a scheme, X an algebraic space over S, and G an X-group algebraic space. Let $f : Y \to X$ be a morphism of algebraic spaces over S. Assume the following conditions hold:*

(i) *for every fppf covering $T \to X$ with T a scheme, the pullback functor*

$$f_T^* : \mathbf{Tors}(T_{\mathrm{fppf}}, G_T) \to \mathbf{Tors}((Y_T)_{\mathrm{fppf}}, G_{Y_T})$$

is fully faithful, where $Y_T := Y \times_X T$, and $f_T := f \times_X T$; and

(ii) *for every G_Y-torsor $\mathcal{P}$, there is a fppf covering $T \to X$ with T a scheme such that $\mathcal{P}_{Y_T}$ lies in the essential image of f_T^*.*

Then pullback induces an equivalence $f^* : \mathbf{Tors}(X_{\mathrm{fppf}}, G_T) \xrightarrow{\sim} \mathbf{Tors}(Y_{\mathrm{fppf}}, G_Y)$.

Similarly, if $G \to X$ is smooth, then we have an equivalence

$$f^* : \mathbf{Tors}(X_{\text{ét}}, G_T) \xrightarrow{\sim} \mathbf{Tors}(Y_{\text{ét}}, G_Y),$$

provided that one replaces 'fppf' by 'étale' everywhere in the above assumptions.

Proof. We prove the Lemma for fppf torsors. It remains to check that f^* is essentially surjective. Let $\mathcal{P}$ be a G_Y-torsor. By assumption (ii) there is a fppf covering $T \to X$ with T a scheme and a G_T-torsor $\mathcal{Q}$ such that $f_T^*\mathcal{Q} \simeq \mathcal{P}_{Y_T}$. Using this isomorphism we can transform the canonical descent datum on $\mathcal{P}_{Y_T}$ to a descent datum

$$\theta : \mathrm{pr}_1^* f_T^* \mathcal{Q} \xrightarrow{\sim} \mathrm{pr}_2^* f_T^* \mathcal{Q}$$

on $f_T^*\mathcal{Q}$ (relative to the covering $Y_T \to Y$). For every integer $n \geqslant 0$, denote by $T^{(n)}$ the n-fold fiber product of T over X. Using the canonical identifications

$$\mathrm{pr}_1^* f_T^* \mathcal{Q} = f_{T^{(2)}}^* \mathrm{pr}_1^* \mathcal{Q} \quad \text{and} \quad \mathrm{pr}_2^* f_T^* \mathcal{Q} = f_{T^{(2)}}^* \mathrm{pr}_2^* \mathcal{Q},$$

the full faithfulness of $f_{T^{(2)}}$ implies that there is a unique isomorphism

$$\tau : \mathrm{pr}_1^* \mathcal{Q} \xrightarrow{\sim} \mathrm{pr}_2^* \mathcal{Q}$$

of $G_{T^{(2)}}$-torsors such that $f_{T^{(2)}}^*(\tau) = \theta$. Since

$$\mathrm{pr}_{13}^*(\theta) = \mathrm{pr}_{13}^*(f_{T^{(2)}}^*(\tau)) = f_{T^{(3)}}^* \mathrm{pr}_{13}^*(\tau)$$

and

$$\begin{aligned} \mathrm{pr}_{13}^*(\theta) &= \mathrm{pr}_{23}^*(\theta)\mathrm{pr}_{12}^*(\theta) \\ &= \mathrm{pr}_{23}^* \left(f_{T^{(2)}}^*(\tau)\right) \mathrm{pr}_{12}^* \left(f_{T^{(2)}}^*(\tau)\right) \\ &= f_{T^{(3)}}^* \left(\mathrm{pr}_{23}^*(\tau)\right) f_{T^{(3)}}^* \left(\mathrm{pr}_{12}^*(\tau)\right) \\ &= f_{T^{(3)}}^* \left(\mathrm{pr}_{23}^*(\tau)\mathrm{pr}_{12}^*(\tau)\right), \end{aligned}$$

the full faithfulness of $f_{T^{(3)}}^*$ implies that $\mathrm{pr}_{13}^*(\tau) = \mathrm{pr}_{23}^*(\tau)\mathrm{pr}_{12}^*(\tau)$, that is, τ is a descent datum on $\mathcal{Q}$ relative to $T \to X$. By Lemma 5.6, there is a G-torsor $\mathcal{R}$ and an isomorphism $(\mathcal{Q}, \phi) \simeq (\mathcal{R}_T, \mathrm{can})$ of descent datums. Pulling back to Y_T, we get an isomorphism of descent datums

$$(\mathcal{P}_{Y_T}, \mathrm{can}) \simeq f_T^*(\mathcal{Q}, \tau) \simeq (\mathcal{R}_{Y_T}, \mathrm{can}).$$

By Lemma 5.6 again (applied to the covering $Y_T \to Y$), we see that $f^*(\mathcal{R}) = \mathcal{R}_Y \simeq \mathcal{P}$. □

6. Purity for torsors and finite étale covers

We begin with generalities about linear groups that will be fundamental in multiple types of purities for reductive torsors, where the overall argument is bootstrapped from that for vector bundles. Hence, in this process, controlling on the projective dimensions of extended reflexive sheaves leads to relative-dimensional constraints. In particular, we obtain the purity for reductive torsors on relative curves §6.1. We then present local variants of the acquired purity results §6.2, where the constraints on dimensions are more flexible. By virtue of this, we shrink complements of domains of reductive torsors to a higher-codimensional closed subset §6.3, laying the groundwork for later proofs of the Grothendieck–Serre. Finally, by our Auslander's flatness criterion, we present a Prüferian counterpart of the Zariski–Nagata purity in §6.4.

6.0.1. Coaffine locally linear groups. Let X be an algebraic space. An X-group algebraic space G is *linear* if there exists a group monomorphism $G \hookrightarrow \mathrm{GL}(\mathscr{V})$ for a locally free $\mathscr{O}_X$-module $\mathscr{V}$ of finite rank; it is *fppf* (resp., *étale*) *locally linear* if there exists a fppf covering (resp., an étale covering) $X' \to X$ such that $G_{X'}$ is linear. A locally linear X-group algebraic space G is *coaffine*, if it locally has an X-affine coset $\mathrm{GL}(\mathscr{V})/G$. For instance, if a linear group $G \subset \mathrm{GL}(\mathscr{V})$ is reductive or finite locally free, then $\mathrm{GL}(\mathscr{V})/G$ is X-affine. In the sequel, we mainly consider locally linear coaffine X-group algebraic spaces G.

6.1. Purity for reductive torsors on relative curves

Now we study the extension behavior of torsors over relative curves. Motivated by [EGA IV$_4$, Proposition 21.9.4] that every invertible sheaf on a curve over a field extend across finitely many closed points, Proposition 6.1.2 concerns relative curves over valuation rings and generalizes [Guo20, Lemma 7.3].

6.1.1. Torsors on relative curves. For a valuation ring V with spectrum S, a V-flat finite type scheme X with regular one-dimensional V-fibers, and a closed subscheme $D \subset X$ such that

(i) D is finite locall free over V; and

(ii) D factors through an affine open $\operatorname{Spec} R \subset X$,

we consider the completion $\widehat{R} := \varprojlim_n R/I^n$, where $I \subset R$ is the ideal determined by D. Denote $B_D := \operatorname{Spec} \widehat{R}$ as the formal neighborhood of D and $U_D := B_D \backslash D$ for the punctured formal neighborhood.

Proposition 6.1.2. *For a valuation ring V with spectrum S, an S-flat finite type scheme X with regular one-dimensional S-fibers, an S-finite locally free closed subscheme $D \subset X$ inside an affine open $X_0 \subset X$ with complementary open $j\colon X\backslash D \hookrightarrow X$, then the restriction functor between the categories*

$$\mathbf{Vect}_X \to \mathbf{Vect}_{X\backslash D} \quad \textit{is essentially surjective.}$$

In particular, for the formal neighborhood $B_D := \widehat{X}^D$, the punctured neighborhood $U_D := B_D \backslash D$, we have

$$H^1_{\mathrm{Zar}}(U_D, \mathrm{GL}_n) = H^1_{\text{ét}}(U_D, \mathrm{GL}_n) = \{*\}.$$

Proof. Every vector bundle $\mathscr{E}$ on $X\backslash D$ by Corollary 3.2.4 extends to a reflexive sheaf $\widetilde{\mathscr{E}}$ on X. Hence Proposition 3.2.7(i) implies that $\widetilde{\mathscr{E}}$ is a vector bundle. Now let $\mathscr{V}$ be a vector bundle on U_D and denote the Henselization of the pair (X_0, D) by (B_D^{h}, D) with $U_D^{\mathrm{h}} := B_D^{\mathrm{h}} \backslash D$. Then [BČ22, Corollary 2.1.22] descends $\mathscr{V}$ to a vector bundle $\mathscr{V}^{\mathrm{h}}$ on U_D^{h}. Since B_D^{h} is the limit of elementary étale neighorhoods of $D \subset X_0$, by a limiat argument, $\mathscr{V}^{\mathrm{h}}$ descends to a vector bundle $\mathscr{V}'$ on an S-flat finite type scheme X' with regular one-dimesensional S-fibers and the open $X'\backslash D$. Since $\mathbf{Vect}_{X'} \to \mathbf{Vect}_{X'\backslash D}$ is essentially surjective, $\mathscr{V}'$ extends to a vector bundle $\widetilde{\mathscr{V}'}$ on X'. Consequently, there exists a vector bundle $\widetilde{\mathscr{V}}$ on B_D extending $\mathscr{V}$. Since (B_D, D) is a Henselian pair, by [Čes22b, Proposition 6.1.1], we have an isomorphism $\mathbf{Vect}_{B_D} \simeq \mathbf{Vect}_D$. Note that D is semilocal and affine, so $\widetilde{\mathscr{V}}$ is trivial, in particular, $\mathscr{V}$ is trivial. □

Lemma 6.1.3. *For a semilocal affine Prüfer scheme S, an S-flat finite type algebraic space X with regular one-dimensional S-fibers, and its closed subset Z such that $j\colon X\backslash Z \hookrightarrow X$ is quasi-compact and*

$$Z_\eta = \varnothing \quad \textit{for each generic point } \eta \in S \quad \textit{and} \quad \operatorname{codim}(Z_s, X_s) \geqslant 1 \textit{ for all } s \in S,$$

the pushforward $j_*(-)$ and restriction as inverse induce an equivalence between categories of vector bundles

$$\mathbf{Vect}_{X\backslash Z} \xrightarrow{\sim} \mathbf{Vect}_X.$$

Proof. We simply verify the assumptions of Lemma 5.7 for $G = \mathrm{GL}_{n,X}$. For vector bundles $\mathscr{E}_1$ and $\mathscr{E}_2$,

$$Y := \underline{\mathrm{Isom}}_X(\mathscr{E}_1, \mathscr{E}_2)$$

is X-affine of finite type (Lemma 5.4), so $Y(X\backslash Z) = Y(X)$ by Proposition 3.2.7(ii). The same holds when we base change to every étale X-scheme. For (ii), by taking étale atlas, we may assume that X is a scheme. By Proposition 3.2.7(ii), every vector bundle $\mathscr{E}$ on $X\backslash Z$ extends to a reflexive $\mathscr{O}_X$-module $j_*\mathscr{E}$. To show that the reflexive $\mathscr{O}_X$-module $j_*\mathscr{E}$ is a vector bundle, it suffices to exploit Proposition 3.2.7(i). □

Theorem 6.1.4 (*cf.* [CTS79, Théorème 6.13])**.** *For a semilocal affine Prüfer scheme S, an S-flat finite type algebraic space X with regular one-dimensional S-fibers, an X-group algebraic space G that is étale-locally linear and coaffine[2], and a closed subset $Z \subset X$ such that $j\colon X\backslash Z \hookrightarrow X$ is quasi-compact and*

$$Z_\eta = \varnothing \quad \textit{for each generic point } \eta \in S \quad \textit{and} \quad \operatorname{codim}(Z_s, X_s) \geqslant 1 \textit{ for all } s \in S,$$

restriction of torsors induces the following equivalence of categories of G-torsors

$$\mathbf{Tors}(X_{\text{ét}}, G) \xrightarrow{\sim} \mathbf{Tors}((X\backslash Z)_{\text{ét}}, G).$$

In particular, passing to isomorphism classes of objects, we have an isomorphism

$$H^1_{\text{ét}}(X, G) \simeq H^1_{\text{ét}}(X\backslash Z, G).$$

Proof. We simply verify the assumptions of Lemma 5.7.

[2] A special case is when X is an affine scheme and G is X-reductive, as explained in a footnote of the introduction.

(i) Since the assumption on the fiber codimension still holds when we base change to every étale scheme over X, it suffices to verify that the restriction functor

$$\mathbf{Tors}(X_{\text{ét}}, G) \to \mathbf{Tors}((X\backslash Z)_{\text{ét}}, G)$$

is fully faithful. Indeed, for any G-torsors $\mathcal{P}_1$ and $\mathcal{P}_2$, by Lemma 5.4,

$$Y := \underline{\mathrm{Isom}}_X(\mathcal{P}_1, \mathcal{P}_2)$$

is an X-affine algebraic space of finite type, so $Y(X\backslash Z) = Y(X)$ by Proposition 3.2.7(ii).

(ii) Étale locally on X, every G-torsor on $X\backslash Z$ extends to a G-torsor on X. To see this, we may assume that X is affine and $G \subset \mathrm{GL}_{n,X}$, then exploit the commutative diagram with exact rows

$$\begin{array}{ccccc} (\mathrm{GL}_{n,X}/G)(X) & \longrightarrow & H^1_{\text{ét}}(X, G) & \longrightarrow & H^1_{\text{ét}}(X, \mathrm{GL}_{n,X}) \\ \wr\downarrow & & \downarrow & & \downarrow \\ (\mathrm{GL}_{n,X}/G)(X\backslash Z) & \longrightarrow & H^1_{\text{ét}}(X\backslash Z, G) & \longrightarrow & H^1_{\text{ét}}(X\backslash Z, \mathrm{GL}_{n,X}), \end{array}$$

where the bijectivity of the left vertical arrow follows from Proposition 3.2.7(ii) and our assumption $\mathrm{GL}_{n,X}/G$ being affine over X. For every G-torsor $\mathcal{P}$ on $X\backslash Z$, by Lemma 6.1.3, we may replace X by an affine open cover to ensure that the induced $\mathrm{GL}_{n,X\backslash Z}$-torsor $\mathcal{P} \times^{G_{X\backslash Z}} \mathrm{GL}_{n,X\backslash Z}$ is trivial. A diagram chase implies that there exists a G-torsor $\mathcal{Q}$ on X such that $\mathcal{Q}|_{X\backslash Z} \simeq \mathcal{P}$, as claimed. □

6.2. **Local variants of purity results**

The following is a variant of Theorem 6.1.4.

Theorem 6.2.1. *For a finite-rank valuation ring R with spectrum (S, η), an S-flat finite type scheme X with regular fibers, an X-group scheme G that is étale-locally linear and coaffine, and a point x that is*

(i) *either $x \in X_\eta$ with $\dim \mathscr{O}_{X_\eta, x} = 2$, or*

(ii) *$x \in X_s$ with $s \neq \eta$ and $\dim \mathscr{O}_{X_s, x} = 1$,*

every G-torsor over $\mathrm{Spec}\, \mathscr{O}_{X,x}\backslash\{x\}$ extends uniquely to a G-torsor over $\mathrm{Spec}\, \mathscr{O}_{X,x}$.

Proof. The argument of Theorem 6.1.4 reduces us to the case of vector bundles, namely, $G = \mathrm{GL}_n$. Then the assertion (i) follows from the classical purity (see for instance, [Gab81, §1, Lemma 1]). For (ii), by the quasi-compactness of $\mathrm{Spec}(\mathscr{O}_{X,x})\backslash\{x\}$ and Proposition 3.2.7(ii), every vector bundle $\mathscr{E}$ on $\mathrm{Spec}\, \mathscr{O}_{X,x}\backslash\{x\}$, extends to a reflexive sheaf $j_*(\mathscr{E})$ on $\mathrm{Spec}\, \mathscr{O}_{X,x}$, which, by the assumption $\dim \mathscr{O}_{X_s,x} = 1$ and Proposition 3.2.7(i), is projective, hence the assertion follows. □

Lemma 6.2.2. *For an algebraic space S with a finitely presented closed subspace $Z \subset X$ and an affine morphism of algebraic spaces $f\colon X' \to X$, denote $Z' := Z \times_X X'$, $U := X\backslash Z$, and $U' := U \times_X X'$, consider the following Cartesian square*

$$\begin{array}{ccc} U' & \xhookrightarrow{j'} & X' \\ {\scriptstyle f_U}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xhookrightarrow{j} & X, \end{array}$$

where j and j' are open immersions. If f is faithfully flat and induces an isomorphism $Z \simeq Z'$, then

(i) *The restriction $\Psi\colon \mathrm{FÉt}_X \xrightarrow{\sim} \mathrm{FÉt}_U \times_{\mathrm{FÉt}_{U'}} \mathrm{FÉt}_{X'}$ is an equivalence of categories. In particular, if $\mathrm{FÉt}_{X'} \to \mathrm{FÉt}_{U'}$ is essentially surjective (resp., an equivalence), then so is $\mathrm{FÉt}_X \to \mathrm{FÉt}_U$.*

(ii) *If X, X' are schemes, then for a quasi-affine, flat, finitely presented X-group scheme G, the following base change functor is an equivalence of categories of G-torsors*

$$\mathbf{Tors}(X_{\text{fppf}}, G) \xrightarrow{\sim} \mathbf{Tors}(X'_{\text{fppf}}, G) \times_{\mathbf{Tors}(U'_{\text{fppf}}, G)} \mathbf{Tors}(U_{\text{fppf}}, G).$$

Proof.

(i) Consider the fibered category **AFF** over the category of algebraic spaces such that every algebraic space T has the fiber category $\mathbf{AFF}(T)$, the category of T-affine algebraic spaces. By [MB96, Théorème 1.1], then base change induces the following equivalence of categories

$$\Phi_{\mathbf{AFF}}\colon \mathbf{AFF}(X) \xrightarrow{\sim} \mathbf{AFF}(X') \times_{\mathbf{AFF}(U')} \mathbf{AFF}(U).$$

Hence Ψ is fully faithful. For the essential surjectivity, it suffices to patch finite étale covers over U and X' to an X-affine algebraic space, and conclude by using faithfully flat descent for finite étale properties.

(ii) See [Čes22a, Lemma 7.1]. □

Corollary 6.2.3. *For a local scheme X, the closed point x and punctured spectrum $U := X\backslash\{x\}$, if for the Henselization X^h of X at x with punctured spectrum U^h,*

$$\mathrm{F\acute{E}t}_{X^h} \xrightarrow{\sim} \mathrm{F\acute{E}t}_{U^h} \quad \textit{is an equivalence if and only if so is} \quad \mathrm{F\acute{E}t}_X \xrightarrow{\sim} \mathrm{F\acute{E}t}_U.$$

Proposition 6.2.4. *Let $X' \to X$ be a flat morphism of affine schemes that are smooth over a semilocal Prüfer domain R with spectrum (S, η) such that there is a closed subscheme $Z \subset X$ satisfies the following*

(i) $\operatorname{codim}(Z_s, X_s) \geqslant 1$ *for every $s \in S$ and* $\operatorname{codim}(Z_\eta, X_\eta) \geqslant 2$ *for the generic point $\eta \in S$; and*

(ii) *$X' \to X$ induces an isomorphism between Z and its preimage $Z' := Z \times_X X'$.*

Denote $U := X\backslash Z$ and $U' := U \times_X X'$. For an affine, smooth X-group (resp., U-group) F with a filtration

$$F = F_0 \supset F_1 \supset \cdots \supset F_n = 0$$

by affine smooth S-normal subgroups (U-normal subgroups) such that every subquotient F_i/F_{i+1} is a vector group on X (resp., such that F_i/F_{i+1} is a vector group on S and is central in F/F_{i+1}), the map

$$H^1_{\text{ét}}(U, F) \to H^1_{\text{ét}}(U', F) \quad \textit{has trivial kernel (resp., is surjective).}$$

Proof. When F is an X-group, since X and X' are affine, both $H^1(X, F)$ and $H^1(X', F)$ vanish. Then, for every F-torsor $\mathcal{P}$ on U that becomes trivial over U', we utilize Lemma 6.2.2 to patch trivial torsors on X' and U to obtain a trivial F-torsor $\widetilde{\mathcal{P}}$ on X such that $\widetilde{\mathcal{P}}|_U \simeq \mathcal{P}$. Hence, $\mathcal{P}$ is trivial and the displayed map has trivial kernel.

Now assume that F is a U-group and we induct on n. When $n = 1$, then F is associated to a vector bundle $\mathscr{F}$ on U. Let $j\colon U \hookrightarrow X$ denote the open immersion, then for $j_*\mathscr{F}$ we apply [GR18, Lemma 10.4.17 (iii)] to deduce that $R\Gamma_Z(X, j_*\mathscr{F}) \simeq R\Gamma_{Z'}(X', j_*\mathscr{F})$. Consequently, we have $H^i(U, F) \xrightarrow{\sim} H^i(U', F)$. When $n > 1$, we invoke the nonabelian cohomology sequences [Gir71, Chapitre IV, Remarque 4.2.10] for a central extension to acquire the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
H^1(U, F_{n-1}) & \longrightarrow & H^1(U, F) & \longrightarrow & H^1(U, F/F_{n-1}) & \longrightarrow & H^2(U, F_{n-1}) \\
\downarrow\sim & & \downarrow & & \downarrow & & \downarrow\sim \\
H^1(U', F_{n-1}) & \longrightarrow & H^1(U', F) & \longrightarrow & H^1(U', F/F_{n-1}) & \longrightarrow & H^2(U', F_{n-1})
\end{array}$$

by a diagram chase up to twist technique [Gir71, Chapitre III, Proposition 2.6.1(i)], we conclude. □

Theorem 6.2.5. *For a semilocal Prüfer domain R with spectrum S and generic point η, an S-smooth algebraic space X, and a point $x \in X$ that is not any maximal point of S-fibers of X such that $\dim \mathscr{O}_{X,x} \geqslant 2$, then pullback induces an equivalence of categories of finite étale covers*

$$\mathrm{F\acute{E}t}_{\operatorname{Spec}\mathscr{O}_{X,x}} \xrightarrow{\sim} \mathrm{F\acute{E}t}_{\operatorname{Spec}\mathscr{O}_{X,x}\backslash\{x\}}.$$

Further, for a qc open immersion $j\colon U \hookrightarrow X$ such that every $z \in X\backslash U$ satisfies the condition for x,

$$\mathrm{F\acute{E}t}_X \to \mathrm{F\acute{E}t}_U \quad \textit{is essentially surjective.}$$

Proof. If $x \notin X_\eta$ and $\dim \mathscr{O}_{X_s,x} = 1$, then the assertion is due to Theorem 6.1.4. The remained case is proved below. To show that $\mathrm{F\acute{E}t}_X \to \mathrm{F\acute{E}t}_U$ is essentially surjective, let $f\colon \widetilde{U} \to U$ be a finite étale cover and we use Noetherian induction to reduce to showing that the finite étale cover f extends to an open subset of U strictly containing U. Pick a maximal point of $X\backslash U$ so $U^\circ := U \times_X \operatorname{Spec}\mathscr{O}_{X,x} =$

$\operatorname{Spec}\mathscr{O}_{X,x}\backslash\{x\}$. Restricting f over U° to $f^\circ\colon \widetilde{U}^\circ \to U^\circ$, the equivalence $\mathrm{FÉt}_{\operatorname{Spec}\mathscr{O}_{X,x}} \xrightarrow{\sim} \mathrm{FÉt}_{\operatorname{Spec}\mathscr{O}_{X,x}\backslash\{x\}}$ yields a finite étale cover $W \to U$. A spreading out [SP, 0BQ5, 0EY3] provides an open neighborhood $x \in U' \supset U$ with a finite étale cover $W' \to U'$ extending $\widetilde{U} \to U$, as desired. □

Remark 6.2.6. Let S be a semilocal affine geometrically unibranched scheme with total ring of fractions K. For an étale locally constant group scheme E over S of finite type[3], the map $H^1_{\text{ét}}(S,E) \hookrightarrow H^1_{\text{ét}}(K,E)$ has trivial kernel. Let $\mathcal{T}$ be an E-torsor that trivializes over K. This signifies that $\mathcal{T}(K) \neq \varnothing$. Since S is geometrically unibranched, by [SGA 3_{II}, Exposé X, Théorème 5.16] (the Noetherian assumption is removable), E is isotrivial, so there is a finite étale covering $S' \to S$ with total ring of fractions K' such that $E_{S'}$ is a constant group in finite type abelian group. Therefore, we have the commuative diagram

$$\begin{array}{ccccc} \mathcal{T}(S) & \longrightarrow & \mathcal{T}(S') & \rightrightarrows & \mathcal{T}(S'\times S') \\ \downarrow & & \| & & \| \\ \mathcal{T}(K) & \longrightarrow & \mathcal{T}(K') & \rightrightarrows & \mathcal{T}(K'\times K') \end{array}$$

so descent yields the equality $\mathcal{T}(S) = \mathcal{T}(K)$. (If S is the spectrum of a Prüfer domain and E is S-finite, then this is simplier by valuative criterion for properness) In particular, we have $\mathcal{T}(S) \neq \varnothing$ so $\mathcal{T}$ is trivial.

Remark 6.2.7. For a valuation ring V with fraction field K, every reductive K-group scheme G has at most one reductive V-model. To see this, we let $\mathcal{G}$ be a reductive V-model of G and consider the commutative diagram with exact rows

$$\begin{array}{ccccccc} H^0_{\text{ét}}(V,\underline{\mathrm{Out}}(\mathcal{G})) & \longrightarrow & H^1_{\text{ét}}(V,\mathcal{G}^{ad}) & \longrightarrow & H^1_{\text{ét}}(V,\underline{\mathrm{Aut}}(\mathcal{G})) & \longrightarrow & H^1_{\text{ét}}(V,\underline{\mathrm{Out}}(\mathcal{G})) \\ {\scriptstyle f_0}\| & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\ H^0_{\text{ét}}(K,\underline{\mathrm{Out}}(\mathcal{G})) & \longrightarrow & H^1_{\text{ét}}(K,\mathcal{G}^{ad}) & \longrightarrow & H^1_{\text{ét}}(K,\underline{\mathrm{Aut}}(\mathcal{G})) & \longrightarrow & H^1_{\text{ét}}(K,\underline{\mathrm{Out}}(\mathcal{G})). \end{array}$$

The map f_1 is injective by [Guo20]. By diagram chase, f_2 has trivial kernel, so we are done.

6.3. Extending generically trivial torsors

Granted the purity Theorem 6.2.1, we extend reductive torsors outside a closed subset of higher codimension.

Proposition 6.3.1. *For a semilocal affine Prüfer scheme S, an S-flat finite type scheme X with regular S-fibers, a closed subset $Z \subset X$ such that $X\backslash Z \subset X$ is quasi-compact and satisfies the following condition*

$$\operatorname{codim}(Z_\eta, X_\eta) \geqslant 2 \quad \textit{for each generic point } \eta \in S \qquad \textit{and} \qquad \operatorname{codim}(Z_s, X_s) \geqslant 1 \quad \textit{for all } s \in S,$$

and a reductive X-group scheme G, there is a closed subset $Z' \subset Z$ satisfying the following condition

$$\operatorname{codim}(Z'_\eta, X_\eta) \geqslant 3 \quad \textit{for each generic point } \eta \in S \qquad \textit{and} \qquad \operatorname{codim}(Z'_s, X_s) \geqslant 2 \quad \textit{for all } s \in S,$$

such that every G-torsor on $X\backslash Z$ extends to a G-torsor on $X\backslash Z'$.

Proof. Write $R = \operatorname{colim}_{\lambda\in\Lambda} R_\lambda$ as in Lemma 3.1.3. By a standard limit argument ([SP, 0EY1, 0C0C]), for large enough $\lambda \in \Lambda$, the scheme X, the open $X\backslash Z \subset X$, and the reductive X-group scheme G descend to a quasi-compact quasi-separated R_λ-smooth scheme X_λ, a quasi-compact open $(X\backslash Z)_\lambda \subset X_\lambda$, and a reductive X_λ-group scheme G_λ, respectively. Also, up to enlarging λ, the G-torsor over $X\backslash Z$ in question descends to a G_λ-torsor over $(X\backslash Z)_\lambda$. By Lemma 3.2.6 that descends the fiberwise codimension of Z, we are reduced to the case when all local rings of R are valuation rings of finite ranks.

Let $\mathcal{P}_{X\backslash Z}$ be a G-torsor over $X\backslash Z$. Since S has finitely many points and each fiber X_s is Noetherian, there are finitely many points $x \in Z$ satisfying one of the assumptions (i)-(ii) of Theorem 6.2.1; among these points we pick a maximal one under the generalization, say x. The maximality of x implies that $(X\backslash Z) \cap \operatorname{Spec}(\mathscr{O}_{X,x}) = \operatorname{Spec}(\mathscr{O}_{X,x})\backslash\{x\}$, so, by Theorem 6.2.1, the G-torsor $\mathcal{P}_{X\backslash Z}|_{X\backslash Z\cap\operatorname{Spec}(\mathscr{O}_{X,x})}$ extends to a G-torsor $\mathcal{P}_x$ over $\operatorname{Spec}(\mathscr{O}_{X,x})$. As X is topologically Noetherian, we may spread out $\mathcal{P}_x$ to obtain a G-torsor $\mathcal{P}_{U_x}$ over an open neighbourhood U_x of x such that $\mathcal{P}_{X\backslash Z}|_{(X\backslash Z)\cap U_x} \simeq \mathcal{P}_{U_x}|_{(X\backslash Z)\cap U_x}$ as G-torsors over $(X\backslash Z)\cap U_x$. Consequently, we may glue $\mathcal{P}_{X\backslash Z}$ and $\mathcal{P}_{U_x}$ to obtain a G-torsor over $U_1 := (X\backslash Z)\cup U_x$.

[3]This means that after a finite étale covering, the constant group is a finite type abelian group, see [SGA 3_{II}, X, 5.1]

Since $Z_1 := X\backslash U_1$ contains strictly fewer points x satisfying the assumptions (i) or (ii) of Theorem 6.2.1, we extend $\mathcal{P}$ iteratively to find the desired closed subset $Z' \subset X$ such that $\mathcal{P}_{X\backslash Z}$ extends over $X\backslash Z'$. □

Corollary 6.3.2. *For a semilocal Prüfer affine scheme S, an S-flat finite type scheme X with regular S-fibers, finitely many points $\boldsymbol{x} \subset X$ contained in an affine open, a nonzero element $r \in \mathscr{O}_{X,\mathbf{x}}$, and a reductive X-group scheme G, every generically trivial G-torsor over $\mathscr{O}_{X,\boldsymbol{x}}[\frac{1}{r}]$ extends to a G-torsor over an open neighbourhood U of $\operatorname{Spec}(\mathscr{O}_{X,\boldsymbol{x}}[\frac{1}{r}])$ whose complementary closed $Z := X\backslash U$ satisfies the following*

$$\operatorname{codim}(Z_\eta, X_\eta) \geqslant 3 \quad \textit{for each generic point } \eta \in S \qquad \textit{and} \qquad \operatorname{codim}(Z_s, X_s) \geqslant 2 \quad \textit{for all } s \in S.$$

Proof. As in the proof of Proposition 6.3.1, we may assume that S has finite Krull dimension; in particular, X is topologically Noetherian. Let $\mathcal{P}$ be a generically trivial G-torsor over $\mathscr{O}_{X,\mathbf{x}}[\frac{1}{r}]$. By spreading out, $\mathcal{P}$ extends to a G-torsor $\mathcal{P}_U$ over $U := \operatorname{Spec} R[\frac{1}{r}]$ for a subring $R \subset \mathscr{O}_{X,\mathbf{x}}$. It remains to extend U and $\mathcal{P}_U$ to ensure that $Z := X\backslash U$ satisfies the assumptions of Proposition 6.3.1. Let $z \in Z$ be such that either

(i) $z \in X_\eta$ and $\dim \mathscr{O}_{X,z} = 1$, in which case $\operatorname{Spec}(\mathscr{O}_{X,z}) \cap U$ is a maximal point of X, or

(ii) z is a maximal point of X_s with $s \neq \eta$, in which case $\operatorname{Spec}(\mathscr{O}_{X,z})$, and hence also $\operatorname{Spec}(\mathscr{O}_{X,z}) \cap U$, is the spectrum of a valuation ring (Lemma 3.1.1(iii)).

By the Grothendieck–Serre over valuation rings [Guo20], the generically trivial G-torsor $\mathcal{P}_U|_{\operatorname{Spec}(\mathscr{O}_{X,z})\cap U}$ is trivial. Thus, as in the proof of Proposition 6.3.1, we can glue $\mathcal{P}_U$ with the trivial G-torsor over a small enough open neighbourhood of z to extend $\mathcal{P}_U$ across such a point $z \in Z$. Note that Z contains finitely many points z satisfying the above assumption (i) or (ii). Therefore, iteratively extend U and $\mathcal{P}_U$, we may assume that Z does not contain any point z satisfying (i) or (ii), when Proposition 6.3.1 applies. □

6.4. Purity for finite locally free torsors and the Zariski–Nagata

With the purity for reflexive sheaves and Auslander's flatness criterion Theorem 4.1 in hand, we obtain the purity theorem for finite locally free torsors and establish our non-Noetherian Zariski–Nagata.

Theorem 6.4.1 (Purity for finite locally free groups)**.**

(i) *For a semilocal affine Prüfer scheme S, an S-smooth algebraic space X, an X-finite locally free group algebraic space G, and a closed subset $Z \subset X$ such that $j\colon X\backslash Z \hookrightarrow X$ is quasi-compact and*

$$\operatorname{codim}(Z_\eta, X_\eta) \geqslant 2 \quad \textit{for each generic point } \eta \in S \qquad \textit{and} \qquad \operatorname{codim}(Z_s, X_s) \geqslant 1 \textit{ for all } s \in S,$$

the restriction functor induces the following equivalence of categories of G-torsors.

$$\mathbf{Tors}(X_{\mathrm{fppf}}, G) \xrightarrow{\sim} \mathbf{Tors}((X\backslash Z)_{\mathrm{fppf}}, G).$$

In particular, passing to isomorphism classes of objects, we have the following isomorphism

$$H^1_{\mathrm{fppf}}(X, G) \simeq H^1_{\mathrm{fppf}}(X\backslash Z, G).$$

(ii) *For a finite-rank valuation ring R with spectrum S, an S-smooth scheme X, an X-finite locally free group scheme G, and a point $x \in X$ that is not a maximal point of S-fibers of X such that $\dim \mathscr{O}_{X,x} \geqslant 2$, the restriction functor induces the following equivalence of category of G-torsors*

$$\mathbf{Tors}((\operatorname{Spec} \mathscr{O}_{X,x})_{\mathrm{fppf}}, G) \xrightarrow{\sim} \mathbf{Tors}((\operatorname{Spec} \mathscr{O}_{X,x}\backslash\{x\})_{\mathrm{fppf}}, G).$$

In particular, passing to isomorphism classes of objects, we have the following isomorphism

$$H^1_{\mathrm{fppf}}(\operatorname{Spec} \mathscr{O}_{X,x}, G) \simeq H^1_{\mathrm{fppf}}(\operatorname{Spec} \mathscr{O}_{X,x}\backslash\{x\}, G).$$

Proof. (i) We simply verify the assumptions of Lemma 5.7. By considering the space $\underline{\mathrm{Isom}}_X$ of isomorphisms of two torsors (see Lemma 5.4), we deduce from Proposition 3.2.7(ii) that the restriction functor is fully faithful. The same holds when we base change to every étale X-scheme over S.

Next, we show that, étale locally on X, any G-torsor on $X\backslash Z$ extends to a G-torsor over X. For this, we may assume that X is an affine scheme and S is the spectrum of a valuation ring. By a standard limit argument involving Lemma 3.1.3, we reduce to the case when S has finite Krull dimension. Since every R-fiber of X is Noetherian and S has finitely many points, X is topologically Noetherian.

Let $\mathcal{P}$ be a $G_{X\backslash Z}$-torsor. Then $j_*\mathscr{O}_{\mathcal{P}}$ by Proposition 3.2.7(iv) is a reflexive $\mathscr{O}_X$-module. First, we prove the $\mathscr{O}_X$-flatness of $j_*\mathscr{O}_{\mathcal{P}}$. Since X is topologically Noetherian, we use Noetherian induction to reduce to the case when X is local and essentially smooth over R and $Z = \{x\}$ is its closed point. Then, our Auslander's flatness criterion Theorem 4.1 reduces us to showing that the following is an isomorphism

$$\mathscr{H}om_{\mathscr{O}_X}(j_*\mathscr{O}_{\mathcal{P}}, j_*\mathscr{O}_{\mathcal{P}}) \simeq (j_*\mathscr{O}_{\mathcal{P}})^{\oplus r}, \quad \text{where} \quad r = \operatorname{rank}_{\mathscr{O}_X}\mathscr{O}_G.$$

Note that in such local case, we have $\mathscr{O}_G \simeq \mathscr{O}_X^{\oplus r}$, consider the following map of reflexive $\mathscr{O}_X$-modules

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_G, j_*\mathscr{O}_{\mathcal{P}}) \to \mathscr{H}om_{\mathscr{O}_X}(j_*\mathscr{O}_{\mathcal{P}}, j_*\mathscr{O}_{\mathcal{P}}),\ f \mapsto \Big(j_*\mathscr{O}_{\mathcal{P}} \xrightarrow{j_*\rho} \mathscr{O}_G \otimes_{\mathscr{O}_X} j_*\mathscr{O}_{\mathcal{P}} \xrightarrow{(f,\mathrm{id})} j_*\mathscr{O}_{\mathcal{P}}\Big).$$

This is an isomorphism: by Proposition 3.2.7(ii), it suffices to argue over $X\backslash Z$, then its explicit inverse is

$$g \mapsto \Big(\mathscr{O}_{G_{X\backslash Z}} \xrightarrow{\mathrm{id}\otimes 1} \mathscr{O}_{G_{X\backslash Z}} \otimes_{\mathscr{O}_{X\backslash Z}} \mathscr{O}_{\mathcal{P}} \xrightarrow{(\rho,\mathrm{id})^{-1}} \mathscr{O}_{\mathcal{P}} \otimes_{\mathscr{O}_{X\backslash Z}} \mathscr{O}_{\mathcal{P}} \xrightarrow{(g,\mathrm{id})} \mathscr{O}_{\mathcal{P}}\Big).$$

Then, we prove that the G-torsor structure of $\mathcal{P}$ extends uniquely to that of $\underline{\mathrm{Spec}}_X(j_*\mathscr{O}_{\mathcal{P}})$. As G is finite locally free, by projection formula [SP, 01E8], taking j_* of the co-action $\rho : \mathscr{O}_{\mathcal{P}} \to j^*\mathscr{O}_G \otimes_{\mathscr{O}_{X\backslash Z}} \mathscr{O}_{\mathcal{P}}$ yields

$$j_*\rho\colon \quad j_*\mathscr{O}_{\mathcal{P}} \to \mathscr{O}_G \otimes_{\mathscr{O}_X} j_*\mathscr{O}_{\mathcal{P}}.$$

To check that $j_*\rho$ is a co-action, we verify the associativity, the commutativity of the following diagram

$$\begin{array}{ccc} j_*\mathscr{O}_{\mathcal{P}} & \xrightarrow{\quad j_*(\rho) \quad} & \mathscr{O}_G \otimes_{\mathscr{O}_X} j_*\mathscr{O}_{\mathcal{P}} \\ \downarrow{\scriptstyle j_*(\rho)} & & \downarrow{\scriptstyle \mathrm{id}\otimes j_*(\rho)} \\ \mathscr{O}_G \otimes_{\mathscr{O}_X} j_*\mathscr{O}_{\mathcal{P}} & \xrightarrow{\mu_G\otimes\mathrm{id}} & \mathscr{O}_G \otimes_{\mathscr{O}_X} \mathscr{O}_G \otimes_{\mathscr{O}_X} j_*\mathscr{O}_{\mathcal{P}}, \end{array}$$

where $\mu_G : \mathscr{O}_G \to \mathscr{O}_G \otimes_{\mathscr{O}_X} \mathscr{O}_G$ is the co-multiplication of G. Since all sheaves involved are $\mathscr{O}_X$-reflexive, the commutativity over $X\backslash Z$ by Proposition 3.2.7(ii) extends over X. Finally, the following map

$$(j_*\rho, 1\otimes\mathrm{id})\colon \quad j_*\mathscr{O}_{\mathcal{P}} \otimes_{\mathscr{O}_X} j_*\mathscr{O}_{\mathcal{P}} \to \mathscr{O}_G \otimes_{\mathscr{O}_X} j_*\mathscr{O}_{\mathcal{P}},$$

by the $\mathscr{O}_X$-flatness of $j_*\mathscr{O}_{\mathcal{P}}$ and Proposition 3.2.7(ii), is an isomorphism since so is its restriction on $X\backslash Z$.

(ii) This can be proved similarly. For instance, for the essential surjectivity of the restriction functor, the finite rank assumption on V guarantees $j : \operatorname{Spec}\mathscr{O}_{X,x}\backslash\{x\} \hookrightarrow \operatorname{Spec}\mathscr{O}_{X,x}$ to be quasi-compact quasi-separated, and so $j_*\mathscr{O}_{\mathcal{P}}$ is a reflexive $\mathscr{O}_{X,x}$-module (by Proposition 3.2.7(ii)) for any G-torsor $\mathcal{P}$ over $\operatorname{Spec}\mathscr{O}_{X,x}\backslash\{x\}$. Then one uses Auslander's flatness criterion Theorem 4.1 to show that $j_*\mathscr{O}_{\mathcal{P}}$ is $\mathscr{O}_{X,x}$-free and inherits the G-torsor structure on $\mathcal{P}$, giving the desired extension of $\mathcal{P}$ to $\operatorname{Spec}\mathscr{O}_{X,x}$. □

Theorem 6.4.2 (Zariski–Nagata: purity for finite étale covers)**.**

(i) *For a semilocal affine Prüfer scheme S, an S-smooth algebraic space X, and a closed subset $Z \subset X$ such that $X\backslash Z \hookrightarrow X$ is quasi-compact and satisfies the following condition*

$$\operatorname{codim}(Z_\eta, X_\eta) \geqslant 2 \quad \textit{for each generic point } \eta \in S \quad \textit{and} \quad \operatorname{codim}(Z_s, X_s) \geqslant 1 \textit{ for all } s \in S,$$

the pullback functor induces the following equivalence between categories of finite étale covers

$$\mathrm{F\acute{E}t}_X \xrightarrow{\sim} \mathrm{F\acute{E}t}_{X\backslash Z}.$$

In particular, for every geometric point $\overline{x}\colon \operatorname{Spec}\Omega \to X\backslash Z$ with a separably closed field Ω, the map

$$\pi_1^{\text{ét}}(X\backslash Z, \overline{x}) \to \pi_1^{\text{ét}}(X, \overline{x}) \quad \textit{is an isomorphism.}$$

(ii) *For a finite-rank valuation ring R with spectrum S and generic point $\eta \in S$, an S-smooth scheme X, and a point which is either $x \in X_\eta$ with $\dim\mathscr{O}_{X_\eta,x} = 2$, or $x \in X_s$ with $s \neq \eta$ and $\dim\mathscr{O}_{X_s,x} = 1$, the pullback functor induces the following equivalence of categories of finite étale covers*

$$\mathrm{F\acute{E}t}_{\operatorname{Spec}\mathscr{O}_{X,x}} \xrightarrow{\sim} \mathrm{F\acute{E}t}_{\operatorname{Spec}\mathscr{O}_{X,x}\backslash\{x\}}.$$

Proof. (i) Full faithfulness. For two finite étale covers $\pi_i : X_i \to X$, $i = 1, 2$, consider the X-functor

$$Y := \underline{\mathrm{Hom}}_X(X_1, X_2)$$

that parameterizes X-morphisms from X_1 to X_2; it is a subfunctor of $\underline{\mathrm{Hom}}_X(\pi_{2,*}\mathscr{O}_{X_2}, \pi_{1,*}\mathscr{O}_{X_1})$ consisting of sections compatible with algebraic structures of $\pi_{2,*}\mathscr{O}_{X_2}$ and $\pi_{1,*}\mathscr{O}_{X_1}$, which amount to the commutativity of a certain diagram of $\mathscr{O}_X$-modules. So $Y \subset \underline{\mathrm{Hom}}_X(\pi_{2,*}\mathscr{O}_{X_2}, \pi_{1,*}\mathscr{O}_{X_1})$ is a closed subfunctor Zariski-locally. Hence, Y is an algebraic space finite over X. (Using the infinitesimal criterion for formal smoothness, one can check that $Y \to X$ is even finite étale, but we will not need this.) By Proposition 3.2.7(ii), we have $Y(X) \simeq Y(X\backslash Z)$, thereby proving the full faithfulness.

Essential surjectivity. Let $V \to X\backslash Z$ be a finite étale cover. We need to show that it extends to a finite étale cover of X. By the full faithfulness, we may use glueing in the étale topology to reduce to the case that X is an affine scheme. By the S-smoothness of X, X and also V is normal, so, by breaking X and V into connected components, we may assume that both X and V are integral schemes. Let $\widetilde{V} \to X\backslash Z$ be a connected finite étale Galois cover dominating $V \to X\backslash Z$, say with Galois group $G := \mathrm{Gal}(\widetilde{V}/(X\backslash Z)$. Let $H := \mathrm{Gal}(\widetilde{V}/V) \subset G$. By Theorem 6.4.1(i), the G-Galois cover $\widetilde{V} \to X\backslash Z$ extends (uniquely) to a G-Galois cover $\widetilde{W} \to X$. By Grothendieck–Galois correspondence, the subcover $\widetilde{W}/H \to X$ corresponding to the subgroup $H \subset G$ is a finite étale cover that extends $V \to X\backslash Z$.

(ii) This is proved in the same way as (i), using Theorem 6.4.1(ii) in place of Theorem 6.4.1(i). □

7. Geometric lemmata for the Grothendieck–Serre

7.1. Geometric presentation lemma over Prüfer bases

In both of the works of Fedorov and Česnavičius on mixed charateristic Grothendieck–Serre, a certain type geometric results in the style of Gabber-Quillen play a prominent role, see [Fed22b, Proposition 3.18] and [Čes22a, Variant 3.7], respectively. This is also true in our context, and we begin with an analog of [Čes22a, Variant 3.7].

Lemma 7.1.1. *Let R be a semilocal Prüfer ring, X a projective, flat R-scheme with fibers of pure dimension $d > 0$, $\mathscr{O}_X(1)$ a R-ample line bundle on X, $W \subset X^{\mathrm{sm}}$ an open, $\boldsymbol{x} \subset W$ finitely many points, and $Y \subset X$ a closed subscheme that is R-fiberwise of codimension > 0. Upon replacing $\mathscr{O}_X(1)$ by any large power, there exists nonzero*

$$h_0 \in \Gamma(X, \mathscr{O}_X(1)),\ h_1 \in \Gamma(X, \mathscr{O}_X(w_1)),\ \cdots,\ h_{d-1} \in \Gamma(X, \mathscr{O}_X(w_{d-1})) \quad \text{with} \quad w_1, \cdots, w_{d-1} > 0,$$

such that

(i) *the hypersurface $H_0 := V(h_0) \subset X$ is disjoint from $\boldsymbol{x}$;*

(ii) *the hypersurfaces $H_i := V(h_i) \subset X$ satisfy $Y \cap H_0 \cap \cdots \cap H_{d-1} = \varnothing$;*

(iii) *in the following commutative diagram with vertical maps determined by the $h_0, \cdots, h_{d-1}$:*

$$\begin{array}{ccccc} X\backslash H_0 & \hookrightarrow & X\backslash(H_0 \cap \cdots \cap H_{d-1}) & \hookrightarrow & \overline{X} := \mathrm{Bl}_X(h_0, \cdots, h_{d-1}) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \overline{\pi}} & & \downarrow{\scriptstyle \overline{\pi}} \\ \mathbb{A}^{d-1}_R & \hookrightarrow & \mathbb{P}_R(1, w_1, \cdots, w_{d-1}) & = & \mathbb{P}_R(1, w_1, \cdots, w_{d-1}), \end{array}$$

the map π is smooth of relative dimension 1 at $\boldsymbol{x}$;

(iv) *we have $Y \cap H_0 \cap \overline{\pi}^{-1}(\pi(\boldsymbol{x})) = \varnothing$;*

(v) *if $Y\backslash X^{\mathrm{sm}}$ is R-fiberwise of codimension $\geqslant 2$ in X, then π is smooth at $Y \cap \overline{\pi}^{-1}(\pi(\boldsymbol{x}))$;*

(vi) *if $Y\backslash W$ is R-fiberwise of codimension $\geqslant 2$ in X, then $(Y\backslash W) \cap \overline{\pi}^{-1}(\pi(\boldsymbol{x})) = \varnothing$;*

(vii) *if $Y\backslash W$ is R-fiberwise of codimension $\geqslant 2$ in X, then there are affine opens*

$$S \subset \mathbb{A}^{d-1}_R \quad \text{and} \quad \boldsymbol{x} \subset U \subset W \cap \pi^{-1}(S) \subset X\backslash H_0$$

such that $\pi : U \to S$ is smooth of relative dimension 1 and $Y \cap U = Y \cap \pi^{-1}(S)$ is S-finite.

Proof. This can be proved similarly as [Čes22a, Variant 3.7]. □

7.2. **A variant of Lindel's lemma**

According to a lemma of Lindel [Lin81, Proposition 1 *et seq* Lemma], an étale extension of local rings $A \to B$ with trivial extension of residue fields automatically induces isomorphisms

$$A/r^n A \xrightarrow{\sim} B/r^n B, \quad \text{where} \quad n \geqslant 1,$$

for a well-chosen non-unit $r \in A$. In our context in which the prescribed B is essentially smooth over a valuation ring, we will prove the following variant of *loc. cit.* by allowing to fix the $r \in B$ in advance, at the cost of that A is a carefully-chosen local ring of an affine space over that valuation ring. This result will be the key geometric input for dealing with torsors under a reductive group scheme that descends to the Prüfer base ring, and, as the cited work of Lindel on the Bass–Quillen conjecture for vector bundles, it reduces us to studying torsors on opens of affine spaces.

Proposition 7.2.1. *Let Λ be a semilocal Prüfer domain, X an irreducible, Λ-smooth affine scheme of pure relative dimension $d > 0$, $Y \subset X$ a finitely presented closed subscheme that avoids all the maximal points of the Λ-fibers of X, and $\boldsymbol{x} \subset X$ a finite subset. Assume that for every maximal ideal $\mathfrak{m} \subset \Lambda$ with finite residue field, there are at most $\max(\#\,\kappa(\mathfrak{m}), d) - 1$ points of $\boldsymbol{x}$ lying over $\mathfrak{m}$. There are an affine open neighbourhood $W \subset X$ of $\boldsymbol{x}$, an affine open subscheme $U \subset \mathbb{A}^d_\Lambda$, and an étale surjective Λ-morphism $f : W \to U$ such that the restriction $f|_{W\cap Y} : W \cap Y \to U$ is a closed immersion and f induces a Cartesian square:*

$$\begin{array}{ccc} W \cap Y & \hookrightarrow & W \\ \| & & \downarrow f \\ W \cap Y & \hookrightarrow & U. \end{array}$$

Moreover, if Y is a Cartier divisor on X, then $W \cap Y$ is a Cartier divisor on U.

Remark 7.2.2. The assumption on the cardinality of $\mathbf{x}$ holds, for instance, either if $\mathbf{x}$ is a singleton or if $d > \#\,\mathbf{x}$. The latter will be critical to settle the general semilocal case of Theorem 12.1. On the other hand, the following finite field obstruction shows a certain assumption on $\#\mathbf{x}$ is necessary: if $d = 1$ and $\Lambda = k$ is a finite field, then the map f delivered from Proposition 7.2.1 gives a closed immersion $\mathbf{x} \hookrightarrow \mathbb{A}^1_k$, which is impossible as soon as $\#\,\mathbf{x} > \#\,k$.

To prove Proposition 7.2.1 we begin with the following reduction:

Lemma 7.2.3. *The proof of Proposition 7.2.1 reduces to the case when $\boldsymbol{x}$ consists of closed points of the closed Λ-fibers of X.*

Proof. As an initial step, by a standard limit argument involving Lemma 3.1.3, we can reduce to the case when $\mathrm{Spec}(\Lambda)$ has a finite underlying space (which we will assume from now on).

If for each $x \in \mathbf{x}$ the closure $\overline{\{x\}}$ contains a closed point x' of the closed Λ-fibers of X and if the new collection $\{x' : x \in \mathbf{x}\}$ satisfies the same cardinality assumption on $\mathbf{x}$, we can simply replace each x by x' to complete the reduction process. However, it may happen that $\overline{\{x\}}$ does not contain any point of the closed Λ-fibers of X, and even if it does, the new collection $\{x' : x \in \mathbf{x}\}$ may destroy the cardinality assumption on $\mathbf{x}$. To overcome this difficulty, we will use a trick by adding auxiliary primes to $\mathrm{Spec}(\Lambda)$ (and adding the corresponding fibers to X and Y) so that $\overline{\{x\}}$ contains closed points of the closed Λ-fibers of X for all $x \in \mathbf{x}$. More precisely, we will show that there are a semilocal Prüfer domain Λ', an open embedding $\mathrm{Spec}(\Lambda) \subset \mathrm{Spec}(\Lambda')$, an irreducible, affine, Λ'-smooth scheme X' of pure relative dimension d, a closed Λ'-subscheme $Y' \subset X'$ that avoids all the maximal points of the Λ'-fibers of X', and a Λ-isomorphism $X'_\Lambda \simeq X$ that identifies Y'_Λ with Y such that the assumptions of the first sentence of this paragraph hold for our new X' and Y'.

To construct the desired Λ' (and X', Y'), we can first use the specialization technique to reduce to the case when all points of $\mathbf{x}$ are closed in the *corresponding* Λ-fibers of X, that is, if $x \in \mathbf{x}$ lies over $\mathfrak{p} \subset \Lambda$, then x is $\kappa(\mathfrak{p})$-finite. For the rest of proof we will assume, without lose of generality, that there is exactly one point of $\mathbf{x}$, say x, that lies over some *non-maximal* prime of Λ, say $\mathfrak{p}$. Write $\Lambda_\mathfrak{p} = \bigcup A$ as a filtered union of its finitely generated $\mathbf{Z}$-subalgebras A. By a standard limit argument ([SP, 0EY1, 0C0C]), for large enough A,

(a) $X_{\Lambda_\mathfrak{p}}$ descends to an irreducible, affine, A-smooth scheme $\mathcal{X}$ of pure relative dimension d;

(b) the finitely presented closed subscheme $Y_{\Lambda_\mathfrak{p}} \subset X_{\Lambda_\mathfrak{p}}$ descends to a closed A-subscheme $\mathcal{Y} \subset \mathcal{X}$ which, upon enlarging A, avoids all the maximal points of the A-fibers of $\mathcal{X}$: by [EGA IV$_3$, Proposition 9.2.6.1],

$$\text{the subset} \quad \{s \in \operatorname{Spec}(A : \dim \mathcal{Y}_s = d\} \subset \operatorname{Spec}(A \quad \text{is constructible},$$

and its pullback to $\operatorname{Spec}(\Lambda_\mathfrak{p}) = \lim_A \operatorname{Spec}(A$ is empty, hence after enlarging A we can assume that it is already empty;

(c) the $\kappa(\mathfrak{p})$-finite point x descends to a $A/\mathfrak{p}_A$-finite closed subscheme $\widetilde{x} \subset \mathcal{X}_{A/\mathfrak{p}_A}$, where $\mathfrak{p}_A := A \cap \mathfrak{p}$;

For any prime $\Lambda \supset \mathfrak{q} \supset \mathfrak{p}$ with $\operatorname{ht}(\mathfrak{q}) = \operatorname{ht}(\mathfrak{p}) + 1$, choose an element $a_\mathfrak{q} \in \mathfrak{q} \backslash \mathfrak{p}$. We assume that

(d) $a_\mathfrak{q}^{-1} \in A$ for all such $\mathfrak{q}$. (This guarantees the equality $A \cdot \Lambda_\mathfrak{m} = \Lambda_\mathfrak{p}$ for every maximal ideal $\mathfrak{m} \subset \Lambda$ containing $\mathfrak{p}$.)

Since a maximal ideal $\mathfrak{m} \subset \Lambda$ containing $\mathfrak{p}$ gives rise to a non-trivial valuation ring $\Lambda_\mathfrak{m}/\mathfrak{p}\Lambda_\mathfrak{m}$ of $\kappa(\mathfrak{p})$, the field $\kappa(\mathfrak{p})$ is not finite. As $\kappa(\mathfrak{p}) = \bigcup_A A/\mathfrak{p}_A$, by enlarging A we may assume that $A/\mathfrak{p}_A$ is also not a finite field, and therefore we can find a nonzero prime $\mathfrak{p}' \subset A/\mathfrak{p}_A$. (We have used the following fact: for a finite type $\mathbb{Z}$-algebra, a prime ideal is maximal if and only if its residue field is finite.) Choose a valuation ring of $\kappa(\mathfrak{p}_A)$ with center $\mathfrak{p}'$ in $A/\mathfrak{p}_A$, and then extend it to a valuation ring $V_{\mathfrak{p}'}$ of $\kappa(\mathfrak{p})$. Let V be the composite of $\Lambda_\mathfrak{p}$ and $V_{\mathfrak{p}'}$; explicitly, V is the preimage of $V_{\mathfrak{p}'}$ under the reduction map $\Lambda_\mathfrak{p} \twoheadrightarrow \kappa(\mathfrak{p})$. Then V is a valuation ring of $\operatorname{Frac}(\Lambda)$, and, by the above assumption (d), the equality $V \cdot \Lambda_\mathfrak{m} = \Lambda_\mathfrak{p}$ holds for any maximal ideal $\mathfrak{m} \subset \Lambda$ containing $\mathfrak{p}$. Therefore, by [BouAC, VI, §7, Proposition 1-2],

$$\Lambda' := \Lambda \cap V$$

is a semilocal Prüfer domain whose spectrum is obtained by glueing $\operatorname{Spec}(\Lambda)$ with $\operatorname{Spec}(V)$ along their common open $\operatorname{Spec}(\Lambda_\mathfrak{p})$. Consequently, we may glue X with $\mathcal{X}_V$ along $X_{\Lambda_\mathfrak{p}}$ to extend X to an irreducible, affine, Λ'-smooth scheme X' of pure relative dimension d, with a closed Λ'-subscheme $Y' \subset X'$ obtained by glueing Y with $\mathcal{Y}_V$ along $Y_{\Lambda_\mathfrak{p}}$; by construction, Y' avoids all the maximal points of the Λ'-fibers of X'. Since the closed subscheme $\widetilde{x}_V \subset \mathcal{X}_V$ is V-finite, we may specialize x to a point of $\widetilde{x}_V \subset X'$ that lies over the closed point of $\operatorname{Spec}(V)$. Hence, by replacing Λ by Λ', X by X' and Y by Y', we can reduce to the already treated case when all points of $\mathbf{x}$ specialize to closed points of the closed Λ-fibers of X. □

Henceforth, we may assume that $\mathbf{x}$ consists of closed points of the closed Λ-fibers of X. Then, since the relative dimension of X/Λ is $d > 0$, the closed subset $\mathbf{x} \bigcup Y$ does not contain any maximal points of the R-fibers of X, and so, by prime avoidance, there is an $a \in \Gamma(X, \mathscr{O}_X)$ such that a vanishes on $\mathbf{x} \bigcup Y$ but does not vanish at any maximal points of Λ-fibers of X. Since for the proof of Proposition 7.2.1 we are free to enlarge Y to a closed subscheme of X that still avoids all the maximal points of the Λ-fibers of X, by replacing Y by $V(a) \subset X$, we reduce to the case

- $\mathbf{x}$ consists of closed points of the closed Λ-fibers of X, and
- $\mathbf{x} \subset Y = V(a)$ for some $a \in \Gamma(X, \mathscr{O}_X)$.

For the rest of the proof we will assume this throughout.

Lemma 7.2.4. *For a field k, an affine k-variety X, a closed subscheme $Y \subset X$ of pure dimension $e > 0$, a finite subset of closed points $\boldsymbol{x} \subset Y \cap X^{\mathrm{sm}}$, and an arbitrary element $(t(x)) \in \prod_{x \in \boldsymbol{x}} \kappa(x)$, there is a morphism $h : X \to \mathbb{A}^1_k$ that is smooth at $\boldsymbol{x}$ such that $h|_Y$ has fiber dimension $e - 1$ and such that $h(x) = t(x)$ for every $x \in \boldsymbol{x}$.*

Proof. Choose a finite subset of closed points $\mathbf{y} \subset Y$ that is disjoint from $\mathbf{x}$ and that contains precisely 1 point of every irreducible component of Y. For every integer $n > 0$ denote by $\mathbf{x}^{(n)}$ (resp., $\mathbf{y}^{(n)}$) the n^{th} infinitesimal neighbourhood of $\mathbf{x}$ (resp., $\mathbf{y}$) in X. Let $h_\mathbf{x} \in H^0(\mathbf{x}^{(1)}, \mathscr{O}_{\mathbf{x}^{(1)}})$ be such that

$$h_\mathbf{x}(x) = t(x) \quad \text{and} \quad \mathrm{d}h_\mathbf{x}(x) \neq 0 \in \mathfrak{m}_x/\mathfrak{m}_x^2 \quad \text{for every} \quad x \in \mathbf{x}. \tag{7.2.1}$$

By prime avoidance, there exists a $h_\mathbf{y} \in H^0(X, \mathscr{O}_X)$ whose restriction to every irreducible component of Y_{red} is not identically zero. By the faithfully flatness of

$$\mathscr{O}_{Y_{\mathrm{red}},\mathbf{y}} = \prod_{y \in \mathbf{y}} \mathscr{O}_{Y_{\mathrm{red}},y} \to \prod_{y \in \mathbf{y}} \widehat{\mathscr{O}_{Y_{\mathrm{red}},y}} = \lim_n H^0(\mathbf{y}^{(n)} \cap Y_{\mathrm{red}}, \mathscr{O}_{\mathbf{y}^{(n)} \cap Y_{\mathrm{red}}}),$$

we see that for large enough n, the restriction of $h_{\mathbf{y}}$ to every component of $\mathbf{y}^{(n)} \cap Y_{\mathrm{red}}$ is nonzero. Let $h \in H^0(X, \mathscr{O}_X)$ be any element whose restriction to $\mathbf{x}^{(1)}$ is $h_{\mathbf{x}}$ and whose restriction to $\mathbf{y}^{(n)}$ is congruent to $h_{\mathbf{y}}$ for large n. Since X is smooth at $\mathbf{x}$, (7.2.1) implies that the morphism $h : X \to \mathbb{A}^1_k$ (obtained by sending the standard coordinate of $\mathbb{A}^1_k$ to h) is smooth at $\mathbf{x}$ and $h(x) = t(x)$ for every $x \in \mathbf{x}$. Since the restriction of h to every irreducible component of $\mathbf{y}^{(n)} \cap Y_{\mathrm{red}}$ and hence also to Y_{red} is nonzero, the morphism h is non-constant on every irreducible component of Y, so $h|_Y$ has fiber dimension $e-1$. □

Lemma 7.2.5. *There exists a Λ-morphism $g : X \to \mathbb{A}^{d-1}_\Lambda$ such that*

(i) *it smooth of relative dimension 1 at $\boldsymbol{x}$;*

(ii) *the restriction $g|_Y$ is quasi-finite at $\boldsymbol{x}$; and*

(iii) *for $x \in \boldsymbol{x}$ lying over $\mathfrak{m}$, one has $\kappa(\mathfrak{m}) = \kappa(g(x))$.*

In addition, if $d > \#(\boldsymbol{x} \cap X_{\kappa(\mathfrak{m})})$ for every maximal ideal $\mathfrak{m} \subset \Lambda$ with finite residue field, then we may find such a g under which all points of $\boldsymbol{x}$ have pairwise distinct images.

Proof. We first reduce the lemma to the case when $\Lambda = k$ is a field. Assume that for every maximal ideal $\mathfrak{m} \subset \Lambda$ there exists a $\kappa(\mathfrak{m})$-morphism $g_{\mathfrak{m}} : X_{\kappa(\mathfrak{m})} \to \mathbb{A}^{d-1}_{\kappa(\mathfrak{m})}$ that is smooth at $\mathbf{x} \cap X_{\kappa(\mathfrak{m})}$ such that the restriction $g_{\mathfrak{m}}|_{Y_{\kappa(\mathfrak{m})}}$ is quasi-finite at $\mathbf{x} \cap X_{\kappa(\mathfrak{m})}$. We then use Chinese remainder theorem to lift the maps $\{g_{\mathfrak{m}}\}_{\mathfrak{m}}$ simultaneously to obtain a Λ-morphism $g : X \to \mathbb{A}^{d-1}_\Lambda$ which would verify the first assertion of the lemma: only the flatness of g at $\mathbf{x}$ need to be checked, but this follows from the fibral criterion of flatness [EGA IV$_3$, Théorème 11.3.10]. In addition, if all points of $\mathbf{x} \cap X_{\kappa(\mathfrak{m})}$ have pairwise distinct images under $g_{\mathfrak{m}}$, then the resulting morphism g verifies the second assertion of the lemma.

In case $\Lambda = k$ being a field, our assumptions become that X is a k-smooth affine variety of pure dimension $d > 0$ and $Y = V(a)$ is a closed k-subvariety of pure codimension 1 that contains $\mathbf{x}$, and, for the second assertion, our assumption becomes that $d > \#\mathbf{x}$.

For a collection of maps $t_1, \cdots, t_{d-1} : \mathbf{x} \to k$, taking products yields maps $(t_1, \cdots, t_i) : \mathbf{x} \to \mathbb{A}^i_k(k) = k^i$ for $1 \leqslant i \leqslant d-1$. We now apply Lemma 7.2.4 inductively to show:

Claim 7.2.1. For $1 \leqslant i \leqslant d-1$, there exists a k-morphism $g_i : X \to \mathbb{A}^i_k$ such that

- g_i is smooth at $\mathbf{x}$ with $g_i|_{\mathbf{x}} = (t_1, \cdots, t_i)$; and
- every irreducible component of $g_i|_Y^{-1}(g_i(\mathbf{x}))$ intersecting $\mathbf{x}$ has dimension $d-1-i$.

Proof of the claim. Assume the morphism g_{i-1} has been constructed. We apply Lemma 7.2.4, with k being the ring k' of global sections of $g_{i-1}(\mathbf{x})$ here, X being $g_{i-1}^{-1}(g_{i-1}(\mathbf{x}))$ here, Y being the union Y' of all the irreducible components of $g_{i-1}|_Y^{-1}(g_{i-1}(\mathbf{x}))$ meeting $\mathbf{x}$ here, and t being $t_i|_{k'}$, to obtain a k'-morphism $h : g_{i-1}^{-1}(g_{i-1}(\mathbf{x})) \to \mathbb{A}^1_{k'}$ that is smooth at $\mathbf{x}$ such that $h|_{Y'}$ has fiber dimension $d-1-i$ and such that $h|_{\mathbf{x}} = t_i|_{k'}$, where $t_i|_{k'} : \mathbf{x} \xrightarrow{t_i} k \to k'$. It remains to take $g_i := (g_{i-1}, \widetilde{h}) : X \to \mathbb{A}^i_k = \mathbb{A}^{i-1}_k \times_k \mathbb{A}^1_k$ for any lifting $\widetilde{h} \in H^0(X, \mathscr{O}_X)$ of

$$h \in H^0\left(g_{i-1}^{-1}(g_{i-1}(\mathbf{x})), \mathscr{O}_{g_{i-1}^{-1}(g_{i-1}(\mathbf{x}))}\right).$$

□

Starting from a map $(t_1, \cdots, t_{d-1}) : \mathbf{x} \to k^{d-1}$, the map $g := g_{d-1}$ of the Claim 7.2.1 immediately settles the first assertion of the lemma. For the second assertion, it suffices to note that, under the stated assumption, there always exists an injection $\mathbf{x} \hookrightarrow k^{d-1}$: for an infinite field k, k^{d-1} is infinite, and, for a finite field k, $\#\, k^{d-1} \geqslant d-1$. □

Consider the map $(g, a) : X \to \mathbb{A}^d_\Lambda = \mathbb{A}^{d-1}_\Lambda \times_\Lambda \mathbb{A}^1_\Lambda$. By construction, it is quasi-finite at $\mathbf{x}$, and, by the openness of the quasi-finite locus of a finite type morphism, we may shrinking X if needed to assume that it is already quasi-finite; since the generic Λ-fibers of its domain and codomain are irreducible varieties of the same dimension d, it is also dominant. Consequently, by Zariski's main theorem [EGA IV$_4$, Corollaire 18.12.13], (g, a) factors as

$$X \xrightarrow{j} \overline{X} \xrightarrow{h_1} \mathbb{A}^d_\Lambda,$$

where $\overline{X}$ is an integral affine scheme, j is an open immersion, and h_1 is finite, dominant. (Unless Λ is a DVR, $\Gamma(\overline{X}, \mathscr{O}_{\overline{X}})$ is, in general, only a finite type Λ-subalgebra of the integral closure of $\Lambda[t_1, \cdots, t_d]$ in the function field of X.) Denote $\overline{g} := \mathrm{pr}_1 \circ h_1$, where $\mathrm{pr}_1 : \mathbb{A}^d_\Lambda \to \mathbb{A}^{d-1}_\Lambda$ is the projection onto the first $(d-1)$-coordinates, and let $\overline{a} \in \Gamma(\overline{X}, \mathscr{O}_{\overline{X}})$ be the pullback of the last standard coordinate of $\mathbb{A}^d_\Lambda$. Then $h_1 = (\overline{g}, \overline{a})$, and $\overline{g}$ (resp., $\overline{a}$) restricts to g (resp., a) on X. In what follows, we shall identify the points of $j(\mathbf{x})$ with the corresponding points of $\mathbf{x}$ via j.

Write $S \subset \operatorname{Spec} \Lambda$ for the union of the closed points of $\operatorname{Spec} \Lambda$ (with the reduced structure).

Lemma 7.2.6. *There exists an element $b \in \Gamma(\overline{X}, \mathscr{O}_{\overline{X}})$ such that the morphism*

$$h_2 := (\overline{g}, b) : \overline{X} \to \mathbb{A}^d_\Lambda = \mathbb{A}^{d-1}_\Lambda \times_\Lambda \mathbb{A}^1_\Lambda$$

has the following properties:

(i) *set-theoretically we have $h_1^{-1}(h_1(\boldsymbol{x})) \cap h_2^{-1}(h_2(\boldsymbol{x})) = \boldsymbol{x}$;*

(ii) *h_2 is étale around $\boldsymbol{x}$ and induces a bijection $\boldsymbol{x} \xrightarrow{\sim} h_2(\boldsymbol{x})$; and*

(iii) *h_2 induces an isomorphism of residue fields $\kappa(h_2(x)) \xrightarrow{\sim} \kappa(x)$ for every $x \in \boldsymbol{x}$.*

Proof. Since h_1 is finite, surjective, $\overline{g}^{-1}(g(\mathbf{x}))$ is an S-curve that contains $g^{-1}(g(\mathbf{x}))$ as an open subcurve, so it is S-smooth around $\mathbf{x}$. For a point $x \in \mathbf{x}$ lying over a maximal ideal $\mathfrak{m} \subset \Lambda$, its first infinitesimal neighbourhood in $\overline{g}^{-1}(g(\mathbf{x}))$ is isomorphic to $\operatorname{Spec}(\kappa(x)[u_x]/(u_x^2))$, where u_x is an uniformizer of $\overline{g}^{-1}(g(\mathbf{x}))$ at x. Recall the fact that the residue field of a point on a smooth curve over a field is a simple extension of that field, see [Čes22a, Lemma 6.3]. It follows that, for $x \in \mathbf{x}$ lying over $\mathfrak{m}$, there exists a closed $\kappa(\mathfrak{m})$-immersion $x^{(1)} \hookrightarrow \mathbb{A}^1_{\kappa(\mathfrak{m})} = \mathbb{A}^1_{g(x)}$. For a maximal ideal $\mathfrak{m} \subset \Lambda$ with finite residue field, under our assumption that $\#(\mathbf{x} \cap X_{\kappa(\mathfrak{m})}) < \max(\#\,\kappa(\mathfrak{m}), d)$, either $\mathbf{x}$ contains at most $\#\,\kappa(\mathfrak{m}) - 1$ points lying over $\mathfrak{m}$ or the fiber of $g_{\kappa(\mathfrak{m})}$ contains at most 1 point of $\mathbf{x}$ (Lemma 7.2.5). Consequently, we may arrange the above immersions so that they jointly give a closed immersion over $\mathbb{A}^{d-1}_\Lambda$:

$$\bigsqcup_{x \in \mathbf{x}} x^{(1)} \hookrightarrow \mathbb{A}^1_{g(\mathbf{x})} \subset \mathbb{A}^1_{\mathbb{A}^{d-1}_\Lambda} = \mathbb{A}^d_\Lambda, \tag{7.2.1}$$

where we regard $g(\mathbf{x}) \subset \mathbb{A}^{d-1}_\Lambda$ as a closed subscheme. Note that the complement of the image of the morphism (7.2.1) in $\mathbb{A}^d_\Lambda$ has at least 1 rational point $\mathbb{A}^{d-1}_\Lambda$-fiberwisely. Thus, by sending any $y \in (h_1^{-1}(h_1(\mathbf{x}))\backslash\mathbf{x})$ to a suitable rational point of $\mathbb{A}^1_{g(y)}$, we may further extend (7.2.1) to a $\mathbb{A}^{d-1}_\Lambda$-morphism

$$u \colon Z := \left(\textstyle\bigsqcup_{x \in \mathbf{x}} x^{(1)}\right) \bigsqcup \left(\bigsqcup_{y \in h_1^{-1}(h_1(\mathbf{x}))\backslash\mathbf{x}} y\right) \to \mathbb{A}^d_\Lambda$$

such that $u(\mathbf{x}) \cap u(h_1^{-1}(h_1(\mathbf{x}))\backslash\mathbf{x}) = \varnothing$, or, what amounts to the same,

$$h_1^{-1}(h_1(\mathbf{x})) \cap u^{-1}(u(\mathbf{x})) = \mathbf{x}. \tag{7.2.2}$$

As Z is a closed subscheme of the affine scheme $\overline{X}$, we can lift $u^*(t) \in \Gamma(Z, \mathscr{O}_Z)$ to obtain an element $b \in \Gamma(\overline{X}, \mathscr{O}_{\overline{X}})$, where t is the standard coordinate on $\mathbb{A}^1_\Lambda$.

Consider the morphism $h_2 := (\overline{g}, b) : \overline{X} \to \mathbb{A}^d_\Lambda = \mathbb{A}^{d-1}_\Lambda \times_\Lambda \mathbb{A}^1_\Lambda$. Viewing $\overline{X}$ as a $\mathbb{A}^{d-1}_\Lambda$-scheme via $\overline{g}$, the base change of h_2 to $g(\mathbf{x}) \subset \mathbb{A}^{d-1}_\Lambda$ restricts to u on Z, so h_2 is unramified at $\mathbf{x}$. Now (i) follows from (7.2.2), (iii) is a consequence of our choice of the morphism (7.2.1). For (ii), it suffices to argue that h_2 is flat at $\mathbf{x}$; however, since the domain and the codomain of h_2 are Λ-flat of finite presentation, the fibral criterion of flatness [EGA IV$_3$, Théorème 11.3.10] reduces us to checking the flatness of the Λ-fibers of h_2 at $\mathbf{x}$, while the latter follows from the flatness criterion [EGA IV$_2$, Proposition 6.1.5]. □

Let $\Lambda[h_1^*(t_1), \cdots, h_1^*(t_{d-1}), a, b] \subset \Gamma(\overline{X}, \mathscr{O}_{\overline{X}})$ be the Λ-subalgebra generated by a, b and $h_2^*(t_i) (= h_1^*(t_i) = \overline{g}^*(t_i))$ for $1 \leqslant i \leqslant d-1$. We introduce the following notations.

- Let $V := \operatorname{Spec}(\Lambda[h_1^*(t_1), \cdots, h_1^*(t_{d-1}), a, b])$, and let $h_3 : \overline{X} \to V$ be the morphism induced by the inclusion $\Lambda[h_1^*(t_1), \cdots, h_1^*(t_{d-1}), a, b] \subset \Gamma(\overline{X}, \mathscr{O}_{\overline{X}})$.
- Let $v_1 : V \to \mathbb{A}^d_\Lambda$ be the map such that $v_1^*(t_i) = h_1^*(t_i)$ for $1 \leqslant i \leqslant d-1$ and $v_1^*(t_d) = a$.

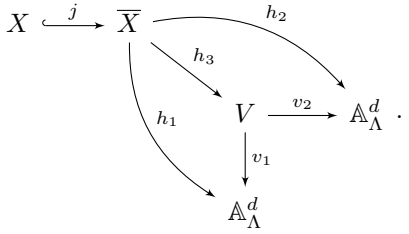
- Let $v_2 : V \to \mathbb{A}^d_\Lambda$ be the map such that $v_2^*(t_i) = h_1^*(t_i)$ for $1 \leqslant i \leqslant d-1$ and $v_2^*(t_d) = b$. Note that there is a natural surjection

$$\Lambda[h_1^*(t_1), \cdots, h_1^*(t_{d-1}), b] \twoheadrightarrow \Lambda[h_1^*(t_1), \cdots, h_1^*(t_{d-1}), a, b]/(a) = \Gamma(V, \mathscr{O}_V)/(a);$$

this implies that v_2 induces a closed immersion

$$\overline{v}_2 : \mathrm{Spec}(\Gamma(V, \mathscr{O}_V)/(a)) \hookrightarrow V \xrightarrow{v_2} \mathbb{A}^d_\Lambda.$$

We have the following commutative diagram of morphisms of affine schemes:



Lemma 7.2.7. *The map h_3 induces a bijection $\boldsymbol{x} \xrightarrow{\sim} h_3(\boldsymbol{x})$ with $h_3^{-1}(h_3(\boldsymbol{x})) = \boldsymbol{x}$. Further, h_3 induces an isomorphism of semilocal rings*

$$\mathscr{O}_{V,h_3(\boldsymbol{x})} \simeq \mathscr{O}_{\overline{X},\boldsymbol{x}} = \mathscr{O}_{X,\boldsymbol{x}}.$$

Proof. By Lemma 7.2.6(ii)-(iii), we see that h_3 induces a bijection $\mathbf{x} \xrightarrow{\sim} h_3(\mathbf{x})$ and an isomorphism of residue fields $\kappa(h_3(x)) \xrightarrow{\sim} \kappa(x)$ for every $x \in \mathbf{x}$. Chasing the above diagram we see that

$$h_3^{-1}(h_3(\mathbf{x})) \subset h_1^{-1}(h_1(\mathbf{x})) \cap h_2^{-1}(h_2(\mathbf{x})) = \mathbf{x},$$

where the last equality is Lemma 7.2.6(i). As h_3 is finite, surjective, we see that $h_3^{-1}(h_3(\mathbf{x})) = \mathbf{x}$. By Lemma 7.2.6(ii), h_3 is unramified at $\mathbf{x}$. It follows that the base change of h_3 to Spec $\mathscr{O}_{V,h_3(\mathbf{x})}$ is

$$\mathrm{Spec}\ \mathscr{O}_{\overline{X},\mathbf{x}} \to \mathrm{Spec}\ \mathscr{O}_{V,h_3(\mathbf{x})},$$

and it is actually an isomorphism: letting J be the Jacobson radical of the semilocal ring $\mathscr{O}_{V,h_3(\mathbf{x})}$, since the natural map

$$\prod_{x\in\mathbf{x}} \kappa(h_3(x)) \simeq \mathscr{O}_{V,h_3(\mathbf{x})}/J \xrightarrow{h_3^*} \mathscr{O}_{\overline{X},\mathbf{x}}/J\mathscr{O}_{\overline{X},\mathbf{x}} \simeq \prod_{x\in\mathbf{x}} \kappa(x)$$

is an isomorphism (in particular, surjective), an application of Nakayama lemma shows

$$h_3^* : \mathscr{O}_{V,h_3(\mathbf{x})} \simeq \mathscr{O}_{\overline{X},\mathbf{x}} = \mathscr{O}_{X,\mathbf{x}}.$$

□

End of the proof of Proposition 7.2.1. Define $f := h_2 \circ j : X \to \mathbb{A}^d_\Lambda$, which we may assume to be étale upon replacing X by an affine open neighbourhood of $\mathbf{x}$. By Lemma 7.2.7, there exists an affine open neighbourhood $W_0' \subset V$ of $h_3(\mathbf{x})$ such that $W_0 := h_3^{-1}(W_0) \subset j(X)$ and $h_3|_{W_0} : W_0 \xrightarrow{\sim} W_0'$. We shall identify W_0 as an open subscheme of X via j. As noted above, v_2 induces a closed immersion

$$\overline{v}_2 : Y' := \mathrm{Spec}(\Gamma(V, \mathscr{O}_V)/(a)) \hookrightarrow \mathbb{A}^d_\Lambda.$$

In particular, the topology of Y' is induced from that of $\mathbb{A}^d_\Lambda$ via $\overline{v}_2$. Note also that, since a vanishes on $\mathbf{x}$, $h_3(\mathbf{x}) \subset Y' \subset V$. Consequently, there exists an affine open neighbourhood $U \subset \mathbb{A}^d_\Lambda$ of $f(\mathbf{x}) = v_2(h_3(\mathbf{x}))$ such that $\overline{v}_2^{-1}(U) \subset W_0'$. Therefore, f induces a closed immersion of affine schemes

$$Y_U := f^{-1}(U) \cap Y = (h_3 \circ j)^{-1}(v_2^{-1}(U) \cap Y') = (h_3 \circ j)^{-1}(\overline{v}_2^{-1}(U)) \simeq \overline{v}_2^{-1}(U) \hookrightarrow U.$$

Since f is separated and étale, any section of $f \times_{\mathbb{A}^d_\Lambda, f} Y_U$, such as the one induced by the inclusion $Y_U \hookrightarrow X$, is an inclusion of a clopen, so

$$X \times_{\mathbb{A}^d_\Lambda, f} Y_U = \widetilde{Y}_1 \sqcup \widetilde{Y}_2 \qquad \text{with} \qquad \widetilde{Y}_1 \xrightarrow{\sim} Y_U.$$

Let $W \subset f^{-1}(U)$ be an affine open whose preimage in $X \times_{\mathbb{A}^d_\Lambda, f} Y_U$ is $\widetilde{Y}_1$. Then $f|_W : W \to U$ is an étale morphism such that $f|_{W\cap Y} : W \cap Y \hookrightarrow U$ is a closed immersion and such that $W \times_{U,f} (W \cap Y) \xrightarrow{\sim} W \cap Y$. As any étale map is open, we may shrink U around $f(\mathbf{x})$ to ensure that $f|_W : W \to U$ is also surjective. This proves the first assertion of Proposition 7.2.1.

The second assertion follows from descent theory, because the ideal sheaf of $W \cap Y$ on U pulls back to that of $W \cap Y$ on W. □

8. Cohomology of groups of multiplicative type

Inspired by the purity results in [ČS21, Theorem 7.2.8], we investigate the parafactoriality over Prüfer bases and then present the purity for cohomology of group schemes of multiplicative type.

8.1. Geometrically parafactorial pairs

8.1.1. Parafactorial pairs. Let $(X, \mathscr{O}_X)$ be a ringed space with a closed subspace $Z \subset X$ and open immersion $j\colon X\backslash Z \hookrightarrow X$, if for every open subspace $U \subset X$ the restriction

$$\mathbf{Pic}\, X \xrightarrow{\sim} \mathbf{Pic}\, X\backslash Z, \quad \mathscr{L} \mapsto \mathscr{L}|_{U\cap(X\backslash Z)} \quad \text{is an equivalence of categories,}$$

then the pair (X, Z) is *parafactorial*. In particular, we have $\mathscr{L} \simeq j_*j^*\mathscr{L}$. A local ring A is *parafactorial* if the pair $(\operatorname{Spec} A, \operatorname{Spec} A/\mathfrak{m}_A)$ is parafactorial. We list several parafactorial pairs (X, Z) and local rings.

(i) when A is a Noetherian factorial local ring, by [EGA IV$_4$, Exemples 21.13.9 (ii)], it is parafactorial;

(ii) by [EGA IV$_4$, Proposition 21.13.8], a local ring A is parafactorial if and only if

$$\operatorname{Pic}(\operatorname{Spec} A\backslash\{x\}) = 0 \quad \text{and} \quad A \simeq \Gamma(\operatorname{Spec} A\backslash\{x\}, \widetilde{A}) \quad \text{for the closed point } x \in \operatorname{Spec} A;$$

(iii) when X is a locally Noetherian and locally complete intersection and Z satisfies $\operatorname{codim}(Z, X) \geqslant 4$, by [SGA 2$_{\text{new}}$, Exposé XI, Théorème 3.13 (ii)], the pair (X, Z) is parafactorial;

(iv) for a normal scheme S, an S-smooth scheme X and a closed subset $Z \subset X$ satisfying

$$\operatorname{codim}(Z_\eta, X_\eta) \geqslant 2 \text{ for each generic point } \eta \in S \quad \text{and} \quad \operatorname{codim}(Z_s, X_s) \geqslant 1 \text{ for every } s \in S,$$

by [EGA IV$_4$, Proposition 21.14.3], the pair (X, Z) is parafactorial.

Now we assume that X is a scheme. A parafactorial pair (X, Z) is *geometrically parafactorial*, if for every X-étale X' with the base change $Z' := Z \times_X X'$, the pair (X', Z') is parafactorial. For a local ring A, if its strict Henselization A^{sh} is parafactorial, then A is *geometrically parafactorial* (*cf.* [ČS21, Theorem 7.2.8]).

Lemma 8.1.2. *For a topologically locally Noetherian scheme X and a closed subscheme $Z \subset X$,*

(i) *the pair (X, Z) is parafactorial if and only if $\mathscr{O}_{X,z}$ is parafactorial for every $z \in Z$;*

(ii) *the pair (X, Z) is geometrically parafactorial if and only if $\mathscr{O}^{\mathrm{sh}}_{X,z}$ is parafactorial for every $z \in Z$.*

Proof. The assertion (ii) follows the same argument of (i), except viewing $\mathscr{O}^{\mathrm{sh}}_{X,z}$ as the inverse limit of étale neighborhoods of $z \in X$. Assume that (X, Z) is parafactorial and for each $z \in Z$, denote $U_z := \operatorname{Spec} \mathscr{O}_{X,z}$ and $U_z^\circ := U_z\backslash\{z\}$. To show that $\mathscr{O}_{X,z}$ is parafactorial, we prove that every invertible $\mathscr{O}_{U_z}$-module $\mathscr{L}_0$ is isomorphic to $\mathscr{O}_{U_z^\circ}$. Then by [EGA IV$_3$, Proposition 8.2.13] and [EGA I, Proposition 2.4.2], U_z° is the inverse limit of $B^\circ := B\backslash(B \cap \overline{\{z\}})$ where B ranges over all open affine neighborhoods of $z \in X$. Since every B° is topologically Noetherian and separated, by a limit argument [SP, 0B8W], there exists an open affine neighborhood B of $z \in X$ and an invertible $\mathscr{O}_{B^\circ}$-module $\mathscr{L}_{B^\circ}$ such that $\mathscr{L}_0 \simeq \mathscr{L}_{B^\circ}|_{U_z^\circ}$. By assumption and [EGA IV$_4$, Corollaire 21.13.6 (i)(ii)], the pair $(B, B \cap \overline{\{z\}})$ is parafactorial. In particular, there exists an invertible $\mathscr{O}_B$-module $\widetilde{\mathscr{L}_B}$ such that $\widetilde{\mathscr{L}_B}|_{B^\circ} \simeq \mathscr{L}_{B^\circ}$. Shrinking B if necessary, we have $\widetilde{\mathscr{L}_B} \simeq \mathscr{O}_B$ hence $\mathscr{L}_0 \simeq \mathscr{O}_{U_z^\circ}$.

For the other side, assume that $\mathscr{O}_{X,z}$ are parafactorial for all $z \in Z$, which, combined with [EGA IV$_4$, Proposition 21.13.5], reduces us to showing that for every invertible $\mathscr{O}_{X\backslash Z}$-module $\mathscr{L}$, the pushforward $j_*\mathscr{L}$ is an invertible $\mathscr{O}_X$-module. For this, we use Noetherian induction. Namely, consider the subset

$$\Omega := \{x \in X \mid j_*\mathscr{L} \text{ is invertible on an open neighborhood of } x\}$$

Then $\Omega \subset X$ is a non-empty open whose complementary closed is $X\backslash\Omega =: Y \subset Z$. By [EGA IV$_2$, Lemme 2.3.1], the quasi-compact quasi-separated morphism j guarantees that the formation of $j_*\mathscr{L}$ commutes with arbitrary flat base changes (in particular, localizations). Pick a maximal point $y \in Y \subset Z$ so $\mathscr{O}_{X,y}$ is parafactorial. The maximality of $y \in Y$ implies that $\Omega \cap U_y = U_y^\circ$, so $\mathscr{L}_0 := (j_*\mathscr{L})|_{U_y^\circ}$ is an invertible $\mathscr{O}_{U_y^\circ}$-module. The parafactoriality of $\mathscr{O}_{X,y}$ yields an extension of $\mathscr{L}_0$ to an invertible $\mathscr{O}_{U_y}$-module $\widetilde{\mathscr{L}_0}$, which, by the limit argument [SP, 0B8W] again, descends to an invertible $\mathscr{O}_W$-module $\widetilde{\mathscr{L}_W}$

for an open neighborhood W of $y \in X$. Shrinking W if necessary, we may assume that the restrictions of $j_*\mathscr{L}$ and $\widetilde{\mathscr{L}_W}$ on $\Omega \cap W$ are equal. With this gluing datum, let $\Omega' := \Omega \cup W$, so there is an invertible Ω'-module $\mathscr{L}'$ such that $\mathscr{L}'|_W = \widetilde{\mathscr{L}_W}$ and $\mathscr{L}'|_\Omega = (j_*\mathscr{L})|_\Omega$. Since $X\backslash Z \subset \Omega'$ and $\mathscr{L}'|_{X\backslash Z} = \mathscr{L}$, hence $\mathscr{O}_X \simeq j_*\mathscr{O}_{X\backslash Z}$ and $(j_*\mathscr{L})|_{\Omega'} \simeq \mathscr{L}'$, which leads to a desired contradiction with the definition of Ω. □

Proposition 8.1.3. *For a normal scheme S and an S-scheme X satisfying one of the following*

(i) *either $X \to S$ is a smooth morphism of topologically locally Noetherian schemes; or*

(ii) *S is semilocal Prüfer of finite dimension and X is S-flat locally of finite type with regular S-fibers*

then every $x \in X$ that does not contain any maximal point of S-fibers of X and $\dim \mathscr{O}_{X,x} \geqslant 2$ satisfies

$$\mathscr{O}_{X,x} \text{ is geometrically parafactorial, namely, } \mathscr{O}^{\mathrm{sh}}_{X,x} \text{ is parafactorial.}$$

Proof. The parafactoriality of $\mathscr{O}^{\mathrm{sh}}_{X,x}$ is that of $(\operatorname{Spec}\mathscr{O}^{\mathrm{sh}}_{X,x}, \{x\})$, which by Lemma 8.1.2(ii), is equivalent to the parafactoriality of $(\operatorname{Spec}\mathscr{O}_{X',x'}, \{x'\})$ for all X-étale X'. Since all X' and x' satisfy the conditions in the statement above ([BS15, Lemma 6.6.10 (3)]), we are reduced to showing that $\mathscr{O}_{X,x}$ is parafactorial. For the Zariski closure $Z := \overline{\{x\}}$, by Lemma 8.1.2 again, we are reduced to finding a small open neighborhood U of $x \in X$ such that $(U, Z \cap U)$ is a parafactorial pair. Now, take an arbitrary open neighborhood U of $x \in X$, by [EGA IV$_3$, Proposition 9.5.3] applied to $Z \subset X$, shrinking U, we may assume that $U \cap Z$ does not contain any irreducible components of S-fibers of X. If a $z \in Z$ lies over a maximal point $\eta \in S$, since x specializes to z, then we have $\dim \mathscr{O}_{X_\eta,z} = \dim \mathscr{O}_{X,z} \geqslant 2$. Consequently, we have $\operatorname{codim}(X_\eta \cap Z, X_\eta) \geqslant 2$ and by §8.1.1(iv) and Proposition 3.2.7(ii), the desired parafactoriality of $(U, Z \cap U)$ follows. □

8.2. Purity for groups of multiplicative type

Now we study purity for groups of multiplicative type in the situation of higher relative dimension. We start with the following generalization of Theorem 6.1.4 when $G = M$ is a X-group algebraic space of multiplicative type.

Lemma 8.2.1. *For an algebraic space X with a closed subspace $Z \subset X$ such that for every geometric point $\overline{z} \to Z$, the strict local ring $\mathscr{O}_{X,\overline{z}}$ is parafactorial, the open immersion $j\colon X\backslash Z \hookrightarrow X$ and a finite type multiplicative type X-group algebraic space $\mathscr{M}$, the following map between fppf sheaves on X*

$$\mathscr{M} \xrightarrow{\sim} j_*j^*\mathscr{M} \quad \text{is an isomorphism.}$$

In particular, we have $H^0_Z(X, \mathscr{M}) = H^1_Z(X, \mathscr{M}) = 0$ and $\Gamma(X, \mathcal{P}) \simeq \Gamma(U, \mathcal{P})$ for every $\mathscr{M}$-torsor $\mathcal{P}$ on X.

Proof. For an $\mathscr{M}$-torsor $\mathcal{P}$, to show that $\Gamma(X, \mathcal{P}) \simeq \Gamma(U, \mathcal{P})$, it suffices to prove that $\mathcal{P} \simeq j_*j^*\mathcal{P}$, which can be checked fppf locally. Hence, it suffices to prove the first assertion in the case when X is a scheme. By [SGA 3$_{\mathrm{II}}$, Exposé X, Corollaire 4.5], $\mathscr{M}$ is quasi-isotrivial, namely, there is an étale surjective morphism $\widetilde{X} \to X$ such that $\mathscr{M} \times_X \widetilde{X}$ splits. We need to show that the morphism $\mathscr{M} \to j_*j^*\mathscr{M}$ is an isomorphism fppf locally at all $z \in Z$. Suppose $f\colon X' \to X$ is a flat morphism inducing $g\colon X'\backslash Z' \to X\backslash Z$, where $Z' := Z \times_X X'$ with the open immersion $j'\colon X'\backslash Z' \hookrightarrow X'$. Taking inverse image of $\mathscr{M} \to j_*j^*\mathscr{M}$, we obtain $f^*\mathscr{M} \to f^*j_*j^*\mathscr{M}$. By [EGA IV$_2$, Lemme 2.3.1], the formation of $j_*(-)$ commutes with flat base change, hence $f^*j_*j^*\mathscr{M} \simeq j'_*g^*j^*\mathscr{M} = j'_*(j')^*f^*\mathscr{M}$ and the inverse image of $\mathscr{M} \to j_*j^*\mathscr{M}$ is $f^*\mathscr{M} \to j'_*(j')^*f^*\mathscr{M}$. We may assume that $X' = \operatorname{Spec}\mathscr{O}^{\mathrm{sh}}_{X,z}$ and $Z' = \{\overline{z}\}$, so the desired isomorphism is reduced to an isomorphism $\mathscr{M} \xrightarrow{\sim} j'_*(j')^*\mathscr{M}$ for a split finite type multiplicative group sheaf $\mathscr{M}$. For an X'-group μ_n, we have the following short exact sequence

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m \to 0,$$

hence $j'_*(j')^*\mu_n = \ker(j'_*(j')^*\mathbb{G}_m \xrightarrow{\times n} j'_*(j')^*\mathbb{G}_m)$, reducing us to the case when $\mathscr{M}_{X'} = \mathbb{G}_m$. Since (X', Z') is parafactorial, we have $\mathscr{O}^\times_{X'} \xrightarrow{\sim} j'_*(j')^*\mathscr{O}^\times_{X'}$, so the assertion follows. □

Proposition 8.2.2. *For a finite-rank valuation ring R with spectrum S and generic point $\eta \in S$, an S-flat finite type scheme X with regular S-fibers, a point $x \in X$, and an $\mathscr{O}_{X,x}$-torus T,*

(1) if either $x \in X_\eta$ with $\dim \mathscr{O}_{X_\eta,x} \geqslant 2$, or $x \in X_s$ with $s \neq \eta$ and $\dim \mathscr{O}_{X_s,x} \geqslant 1$, then we have

$$H^i_{\{x\}}(\mathscr{O}_{X,x}, T) = 0 \quad \text{for} \quad 0 \leqslant i \leqslant 3;$$

(2) otherwise, $\mathscr{O}_{X,x}$ is a valuation ring, then if T is flasque we have

$$H^2_{\{x\}}(\mathscr{O}_{X,x}, T) = 0.$$

Proof. (1) Notice that the finite-rank assumption on R guarantees X being topologically locally Noetherian. By the local-to-global E_2 spectral sequence [SGA 4_{II}, Exposé V, Proposition 6.4],

$$H^p_{\text{ét}}(\mathscr{O}_{X,x}, \mathcal{H}^q_{\{x\}}(T)) \Rightarrow H^{p+q}_{\{x\}}(\mathscr{O}_{X,x}, T),$$

where $\mathcal{H}^q_{\{x\}}(T)$ is the sheafification of the étale presheaf

$$\left(U \xrightarrow{h} \operatorname{Spec}(\mathscr{O}_{X,x})\right) \mapsto H^q_{h^{-1}(x)}(U, T).$$

Therefore, it suffices to prove the vanishing of the sheaves $\mathcal{H}^q_{\{x\}}(T)$ for $0 \leqslant q \leqslant 2$. We calculate their stalks at a geometric point $\overline{x}$ lying over x:

$$\mathcal{H}^q_{\{x\}}(T)_{\overline{x}} = H^q_{\{\overline{x}\}}(\mathscr{O}^{\text{sh}}_{X,x}, T).$$

Now, since $T_{\mathscr{O}^{\text{sh}}_{X,x}} \simeq \mathbb{G}_{m,\mathscr{O}^{\text{sh}}_{X,x}}^{\dim T}$, and, since by Proposition 8.1.3 $\mathscr{O}^{\text{sh}}_{X,x}$ is parafactorial, we have

$$H^q_{\text{ét}}(\operatorname{Spec}(\mathscr{O}^{\text{sh}}_{X,x}), T) \simeq H^q_{\text{ét}}(\operatorname{Spec}(\mathscr{O}^{\text{sh}}_{X,x}) \backslash \{\overline{x}\}, T) \quad \text{for } 0 \leqslant q \leqslant 1;$$

as $\mathscr{O}^{\text{sh}}_{X,x}$ is strictly Henselian, we have

$$H^2_{\text{ét}}(\operatorname{Spec}(\mathscr{O}^{\text{sh}}_{X,x}), T) = 0.$$

Looking at the local cohomology exact sequence for the pair $(\operatorname{Spec}(\mathscr{O}^{\text{sh}}_{X,x}), \bar{x})$ and T, we see that

$$H^q_{\{\bar{x}\}}(\mathscr{O}^{\text{sh}}_{X,x}, T) = 0 \quad \text{for } 0 \leqslant q \leqslant 2.$$

This implies $\mathcal{H}^q_{\{x\}}(T) = 0$ for $0 \leqslant q \leqslant 2$, as desired.

(2) In this case, either $x \in X_\eta$ with $\dim \mathscr{O}_{X_\eta,x} \leqslant 1$, then $\mathscr{O}_{X,x}$ is a discrete valuation ring, or x is a maximal point of some fiber of $X \to S$, then, by Lemma 3.1.1(iii), $\mathscr{O}_{X,x}$ is a valuation ring. The desired vanishing is proven in [Guo20, Lemma 2.3]. □

Lemma 8.2.3 (*cf.* [ČS21, Lemma 7.1.1])**.** *For an algebraic space X, an open subspace $U \subset X$ with complement $i : Z := X \backslash U \hookrightarrow X$, and an abelian sheaf $\mathscr{F}$ on $(\boldsymbol{Sch}_{/X})_{\text{fppf}}$, if for any integer $q \geqslant 0$, $\mathcal{H}^q_Z(\mathscr{F})$ denotes the étale-sheafification of the presheaf $X' \mapsto H^q_{Z'}(X', M)$ where $Z' := Z \times_X X'$, then we have a convergent spectral sequence*

$$E_2^{pq} = H^p_{\text{ét}}(X, \mathcal{H}^q_Z(\mathscr{F})) \Rightarrow H^{p+q}_Z(X, M).$$

Theorem 8.2.4.

(i) *(*cf. [ČS21, Theorem 7.2.8 (a)]*) For an algebraic space X, a quasi-compact open immersion $j : U \hookrightarrow X$ with complement $Z := X \backslash U$, and an X-group algebraic space M of multiplicative type, if for every geometric point $\overline{z} \to Z$, the strict local ring $\mathscr{O}_{X,\overline{z}}$ is parafactorial, then restriction functor*

$$\mathbf{Tors}(X_{\text{fppf}}, M) \xrightarrow{\sim} \mathbf{Tors}(U_{\text{fppf}}, M) \quad \textit{induces an equivalence of categories of } M\textit{-torsors.}$$

In particular, passing to isomorphism classes of objects, we have the following isomorphisms

$$H^i_{\text{fppf}}(X, M) \xrightarrow{\sim} H^i_{\text{fppf}}(U, M) \quad \text{for } i \leqslant 1 \quad \text{and} \quad H^2_{\text{fppf}}(X, M) \hookrightarrow H^2_{\text{fppf}}(U, M).$$

(ii) *For a semilocal Prüfer domain R with spectrum S, a quasi-compact quasi-separated S-smooth scheme X, a quasi-compact open $U \subset X$ with complement $Z := X \backslash U$, and an X-torus T such that $T_{\mathscr{O}_{X,z}}$ is flasque for every $z \in Z$ for which $\mathscr{O}_{X,z}$ is a valuation ring, then we have*

$$H^1_{\text{ét}}(X, T) \twoheadrightarrow H^1_{\text{ét}}(U, T) \quad \text{and} \quad H^2_{\text{ét}}(X, T) \hookrightarrow H^2_{\text{ét}}(U, T).$$

Proof. (i) By the local cohomology exact sequence for the pair (X, Z) and the sheaf M, everything reduces to show the vanishings $H^q_Z(X, M) = 0$ for $0 \leqslant q \leqslant 2$. By the spectral sequence in Lemma 8.2.3, it suffices to show the vanishings of $\mathcal{H}^q_Z(M)$, the étale-sheafification of the presheaf $X' \mapsto H^q_{Z'}(X', M)$ where $Z' := Z \times_X X'$. Further, the quasi-compactness of j allows us to identify the stalk of $\mathcal{H}^q_Z(M)$ at a geometric point $\overline{z} \to Z$ as $H^q_{\{\overline{z}\}}(\mathscr{O}_{X,\overline{z}}, M)$. Hence we may assume that M split as μ_n or $\mathbb{G}_m$, and since $\mu_n = \ker(\mathbb{G}_m \overset{\times n}{\to} \mathbb{G}_m)$, it suffices to show that $H^q_{\{\overline{z}\}}(\mathscr{O}_{X,\overline{z}}, \mathbb{G}_m) = 0$ for $0 \leqslant q \leqslant 2$. Since $\mathscr{O}_{X,\overline{z}}$ is parafactorial, we have

$$H^q(\mathrm{Spec}(\mathscr{O}_{X,\overline{z}}), \mathbb{G}_m) \simeq H^q(\mathrm{Spec}(\mathscr{O}_{X,\overline{z}})\backslash\{\overline{z}\}, \mathbb{G}_m) \quad \text{for } 0 \leqslant q \leqslant 1;$$

as $\mathscr{O}_{X,\overline{z}}$ is strictly Henselian, we have

$$H^2(\mathrm{Spec}(\mathscr{O}_{X,\overline{z}}), \mathbb{G}_m) = 0.$$

Looking at the local cohomology exact sequence for the pair $(\mathrm{Spec}(\mathscr{O}_{X,\overline{z}}), \overline{z})$ and T, we deduce the desired vanishings

$$H^q_{\{\overline{z}\}}(\mathscr{O}_{X,\overline{z}}, \mathbb{G}_m) = 0 \quad \text{for } 0 \leqslant q \leqslant 2.$$

(ii) By the local cohomology exact sequence

$$\cdots \to H^1(X, T) \to H^1(U, T) \to H^1_Z(X, T) \to H^2(X, T) \to H^2(U, T) \to \cdots,$$

the assertion is equivalent to the vanishing $H^2_Z(X, T) = 0$. Since X is quasi-compact quasi-separated and $U \subset X$ is a quasi-compact open, by a limit argument involving Lemma 3.1.3, we reduce to the case R having finite Krull dimension, so X is topologically Noetherian. Recall the coniveau spectral sequence [Gro68b, §10.1]

$$E_2^{pq} = \bigoplus_{z \in Z^{(p)}} H^{p+q}_{\{z\}}(T) \Rightarrow H^{p+q}_Z(X, T);$$

the topological Noetherianness of X allows us to identify

$$H^{p+q}_{\{z\}}(T) := \mathrm{colim}\, H^{p+q}_{\overline{\{z\}} \cap U}(U, T)$$

as $H^{p+q}_{\{z\}}(\mathscr{O}_{X,z}, T)$, where U runs over the open neighbourhoods of z in X. Therefore, it is enough to show $H^2_{\{z\}}(\mathscr{O}_{X,z}, T) = 0$, which has been solved by Proposition 8.2.2. □

Proposition 8.2.5. *For a normal scheme S and an S-algebraic space X satisfying one of the following*

(i) *either $X \to S$ is a smooth morphism of topologically Noetherian algebraic spaces; or*

(ii) *S is semilocal Prüfer of finite dimension and X is S-flat locally of finite type with regular S-fibers,*

a quasi-compact open $U \subset X$ with complementary closed $Z := X\backslash U$ satisfying the following condition

$$\mathrm{codim}(Z_\eta, X_\eta) \geqslant 2 \text{ for every generic point } \eta \in S \quad \text{and} \quad \mathrm{codim}(Z_s, X_s) \geqslant 1 \text{ for all } s \in S,$$

and a finite type X-group algebraic space M of multiplicative type, the following restriction functor

$$\mathbf{Tors}(X_{\mathrm{fppf}}, M) \xrightarrow{\sim} \mathbf{Tors}(U_{\mathrm{fppf}}, M) \quad \text{induces an equivalence of categories of } M\text{-torsors.}$$

In particular, passing to isomorphism classes of objects, we have the following isomorphisms

$$H^0(X, M) \simeq H^0(U, M), \quad H^1_{\mathrm{fppf}}(X, M) \simeq H^1_{\mathrm{fppf}}(U, M), \quad H^2_{\mathrm{fppf}}(X, M) \hookrightarrow H^2_{\mathrm{fppf}}(U, M).$$

Proof. We simply verify the assumptions of Lemma 5.7. First, the restriction functor is fully faithful, because M is X-affine so is $Y := \underline{\mathrm{Isom}}_X(\mathcal{P}_1, \mathcal{P}_2)$ for arbitrary M-torsors $\mathcal{P}_1$ and $\mathcal{P}_1$ on X (Lemma 5.4), which implies that $Y(X) \simeq Y(U)$ (note that Y is an $\underline{\mathrm{Aut}}_G(\mathcal{P}_1) \simeq M$-torsor, so we have $Y(X) \simeq Y(U)$ by Lemma 8.2.1). The same holds when we base change to every scheme étale over X. Next, we show that, fppf locally on X, every M-torsor on U extends on X. For this we may assume that X is affine. Since X is normal, M is isotrivial, so there is an X-torus T and a finite X-group μ of multiplicative type fitting into the short exact sequence

$$1 \to T \to M \to \mu \to 1,$$

From which we leverage the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mu(X) & \longrightarrow & H^1_{\mathrm{fppf}}(X,T) & \longrightarrow & H^1_{\mathrm{fppf}}(X,M) & \longrightarrow & H^1_{\mathrm{fppf}}(X,\mu) \\
\| & & \downarrow & & \downarrow & & \downarrow \\
\mu(U) & \longrightarrow & H^1_{\mathrm{fppf}}(U,T) & \longrightarrow & H^1_{\mathrm{fppf}}(U,M) & \longrightarrow & H^1_{\mathrm{fppf}}(U,\mu),
\end{array}$$

where $\mu(X) \simeq \mu(U)$ follows from the X-affineness of μ. A diagram chase reduces us to showing that

$$H^1_{\mathrm{fppf}}(X,T) \xrightarrow{\sim} H^1_{\mathrm{fppf}}(U,T) \quad \text{and} \quad H^1_{\mathrm{fppf}}(X,\mu) \xrightarrow{\sim} H^1_{\mathrm{fppf}}(U,\mu) \quad \text{are isomorphisms.}$$

Since the extension problem is fppf local, we may assume that M splits, without loss of generalities, say $M \simeq \mathbb{G}_m$ or $M = \mu_n$. By Proposition 3.2.7(ii), the X-affineness of M implies that $M(X') \simeq M(U')$. When $M = \mathbb{G}_m$, we have $\mathbf{Pic}\, X \xrightarrow{\sim} \mathbf{Pic}\, U$ because (X, Z) is a parafactorial pair. It remains to prove that $H^1_{\text{ét}}(X, \mu_n) \simeq H^1_{\text{ét}}(U, \mu_n)$, for which we consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{O}(X)^\times/\mathscr{O}(X)^{\times n} & \longrightarrow & H^1_{\text{ét}}(X,\mu_n) & \longrightarrow & {}_n\operatorname{Pic}(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{O}(U)^\times/\mathscr{O}(U)^{\times n} & \longrightarrow & H^1_{\text{ét}}(U,\mu_n) & \longrightarrow & {}_n\operatorname{Pic}(U) & \longrightarrow & 0
\end{array}$$

A diagram chase leads to the desired isomorphism $H^1_{\text{ét}}(X, \mu_n) \xrightarrow{\sim} H^1_{\text{ét}}(U, \mu_n)$. Finally, all fppf local extension data glue together. Hence we obtain the desired essential surjectivity. □

8.3. **Grothendieck–Serre type results for groups of multiplicative type**

Lemma 8.3.1. *Let $\phi : X \to Y$ be a morphism of schemes. Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. If*

(1) Y is quasi-compact quasi-separated, integral, and normal,

(2) there exist a smooth projective morphism $\overline{\phi} : \overline{X} \to Y$, with geometrically integral fibers, and a quasi-compact open immersion $X \hookrightarrow \overline{X}$ over Y, and

(3) $\mathscr{L}$ is trivial when restricted to the generic fiber of ϕ,

then $\mathscr{L} \simeq \phi^*\mathscr{N}$ for some invertible $\mathscr{O}_Y$-module $\mathscr{N}$.

Proof. When Y is Noetherian, this follows from a much more general result [SP, 0BD6]; for instance, (2) can be replaced by the assumption that $X \to Y$ is faithfully flat of finite presentation, with integral fibers. The general case can be deduced from this via Noetherian approximations. More precisely, we first use [SP, 01ZA] to write $Y = \lim_i Y_i$ for a filtered inverse system $\{Y_i\}$ of finite type integral $\mathbb{Z}$-schemes with affine transition morphisms. Since the normalization of a finite type integral $\mathbb{Z}$-scheme is finite, we may assume that each Y_i is normal. Next, by [SP, 01ZM, 0C0C], for some i_0 there exist a finite type smooth morphism $\overline{\phi}_{i_0} : \overline{X}_{i_0} \to Y_{i_0}$ such that $\overline{X} \simeq \overline{X}_{i_0} \times_{Y_{i_0}} Y$ as Y-schemes, an open subscheme $X_{i_0} \subset \overline{X}_{i_0}$ whose pullback to $\overline{X}$ identifies with X, and, by [SP, 0B8W], there is an invertible $\mathscr{O}_{X_{i_0}}$-module $\mathscr{L}_{i_0}$ whose pullback to X is isomorphic to $\mathscr{L}$. For any $i \geqslant i_0$ denote by $\phi_i : X_i := X_{i_0} \times_{Y_{i_0}} Y_i \to Y_i$ the base change of $\overline{\phi}_{i_0}|_{X_{i_0}}$ to Y_i, and denote by $\mathscr{L}_i$ the pullback of $\mathscr{L}_{i_0}$ to X_i. By [SP, 01ZM, 01ZP], any projective embedding of $\overline{X}$ over Y descends to a projective embedding of $\overline{X}_i$ over Y_i for large enough i; in particular, $\overline{\phi}_i$ is projective for large enough i.

Since Y is normal, the assumption (3) implies that the Stein factorization of $\overline{\phi}$ is itself; in particular, $\mathscr{O}_Y \xrightarrow{\sim} \overline{\phi}_*\mathscr{O}_{\overline{X}}$. This implies that the finite extension $\mathscr{O}_{Y_{i_0}} \hookrightarrow \overline{\phi}_{i_0,*}\mathscr{O}_{\overline{X}_{i_0}}$ is an isomorphism, because its base change to the function field of Y is so and Y_{i_0} is normal. In particular, by Zariski's main theorem, $\overline{\phi}_{i_0}$ has connected geometric fibers; as it is also smooth, all its fibers are even geometrically integral. By limit formalism, for large enough i, $\mathscr{L}_i$ is trivial when restricted to the generic fiber of ϕ_i. Consequently, for large enough i, the morphism $\phi_i : X_i \to Y_i$ and the invertible $\mathscr{O}_{X_i}$-module $\mathscr{L}_i$ satisfy all the assumptions of the Lemma, so $\mathscr{L}_i \simeq \phi_i^*\mathscr{N}_i$ for some invertible $\mathscr{O}_{Y_i}$-module $\mathscr{N}_i$. Then $\mathscr{L} \simeq \phi^*\mathscr{N}$ where $\mathscr{N}$ is the pullback of $\mathscr{N}_i$ to Y. □

Proposition 8.3.2 (*cf.* [CTS87, 4.1–4.3])**.** *For a Prüfer domain R with spectrum (S,η), an irreducible scheme X essentially smooth over S with function field $K(X)$, an X-group scheme M of multiplicative type, and a connected finite étale Galois covering $X' \to X$ splitting M*[4]*, the restriction maps*

$$H^1_{\mathrm{fppf}}(X,M) \to H^1_{\mathrm{fppf}}(K(X),M) \quad \text{and} \quad H^2_{\mathrm{fppf}}(X,M) \to H^2_{\mathrm{fppf}}(K(X),M)$$

are injective in each of the following cases:

(i) *$X = \mathrm{Spec}(A)$ and A is a semilocal ring essentially smooth over R;*

(ii) *For some essentially smooth semilocal R-algebra A, there exists a quasi-compact open immersion $X \hookrightarrow \overline{X}$, where $\overline{X}$ is a smooth projective A-scheme, with geometrically integral fibers, such that $\mathrm{Pic}(X_L) = 0$ for any finite separable fields extension $L/\mathrm{Frac}(A)$, and $M = N_X$ for N an A-group of multiplicative type (for instance, X could be any quasi-compact open subscheme of $\mathbb{P}^N_A$);*

(iii) *any subcovering $X'' \to X$ of $X' \to X$ satisfies $\mathrm{Pic}(X'') = 0$.*

Further, if M is a flasque X-torus, then in all cases (i)-(iii) *the restriction map*

$$H^1_{\text{ét}}(X,M) \xrightarrow{\sim} H^1_{\text{ét}}(K(X),M) \quad \text{is bijective.}$$

Proof. It is clear that (i) is a particular case of (ii). Let us show that (ii) is a particular case of (iii). Let $A \to B$ be a connected finite étale Galois covering that splits N. Take $X' := X \times_A B$. By the normality of A and the smoothness of $X \to \mathrm{Spec}(A)$, X is also normal. Then, since $X \to \mathrm{Spec}(A)$ has geometrically integral generic fiber, the natural map $\pi_1^{\text{ét}}(X) \to \pi_1^{\text{ét}}(\mathrm{Spec}\, A)$ is surjective. This implies that any subcovering $X'' \to X$ of $X' \to X$ is of the form $X'' = X \times_A C$ for some subcovering $A \to C$ of $A \to B$. By assumption, $\mathrm{Pic}(X_{\mathrm{Frac}(C)}) = 0$, so we may apply Lemma 8.3.1 to the morphism $X \times_A C \to \mathrm{Spec}(C)$ to deduce that the pullback map

$$\mathrm{Pic}(\mathrm{Spec}(C)) \to \mathrm{Pic}(X \times_A C) \quad \text{is surjective.}$$

Since C is semilocal, we conclude that $\mathrm{Pic}(\mathrm{Spec}(C)) = 0 = \mathrm{Pic}(X \times_A C)$.

It is thus enough to prove all assertions only for (iii). Assume first that $M = T$ is an X-torus. Take a flasque resolution

$$1 \to F \to P \to T \to 1,$$

where F is a flasque X-torus and P is a quasitrivial X-torus. This yields a commutative diagram

$$\begin{array}{ccccc} H^1_{\text{ét}}(X,P) & \longrightarrow & H^1_{\text{ét}}(X,T) & \longrightarrow & H^2_{\text{ét}}(X,F) \\ & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} \\ & & H^1_{\text{ét}}(K(X),T) & \longrightarrow & H^2_{\text{ét}}(K(X),F) \end{array}$$

with exact rows. Now the quasitrivial torus P is isomorphic to a finite direct product of tori $\mathrm{Res}_{X''/X}\mathbb{G}_{m,X''}$ for finite étale subcoverings $X'' \to X$ of $X' \to X$. Hence, assumption (iii) implies that $H^1_{\text{ét}}(X,P) = 0$, and so the injectivity of ρ_1 reduces to that of ρ_2. To prove that ρ_2 is injective we pick $a \in H^2_{\text{ét}}(X,F)$ for which $a|_{K(X)} = 0$. By spreading out, we may assume that X is a localization of an irreducible, smooth, affine R-scheme $\widetilde{X}$, $F = \widetilde{F}_X$ for a flasque $\widetilde{X}$-torus $\widetilde{F}$, and $a = \widetilde{a}|_X$ for some class $\widetilde{a} \in H^2(\widetilde{X},\widetilde{F})$. Since $\widetilde{a}|_{K(X)} = 0$, for a large enough hypersurface $Z \subset \widetilde{X}$,

$$\widetilde{a}|_{\widetilde{X}\setminus Z} = 0 \in H^2_{\text{ét}}(\widetilde{X}\setminus Z, \widetilde{F}).$$

By Theorem 8.2.4(ii), $\widetilde{a} = 0$, so $a = \widetilde{a}|_X = 0$. This proves the injectivity of ρ_2 and hence also of ρ_1. Now let M be an arbitrary X-group of multiplicative type, then there is an X-subtorus $T \subset M$ such that $\mu := M/T$ is X-finite. Consequently, for any generically trivial M-torsor $\mathcal{P}$, the μ-torsor $\mathcal{P}/T$ is finite over X; as X is normal, this implies $(\mathcal{P}/T)(X) = (\mathcal{P}/T)(K(X))$. Therefore, $\mathcal{P}/T \to X$ has a section that lifts to a generic section of $\mathcal{P} \to X$, that is, $\mathcal{P}$ reduces to a generically trivial T-torsor $\mathcal{P}_T$. By the injectivity of ρ_1, $\mathcal{P}_T$ and hence also $\mathcal{P}$ is trivial. This proves the injectivity of $H^1_{\text{ét}}(X,M) \to H^1_{\text{ét}}(K(X),M)$.

On the other hand, there is a short exact sequence

$$1 \to M \to F \to P \to 1$$

[4]Such a covering always exists, because X is normal and so M is isotrivial.

of X-groups of multiplicative type, where F is flasque and P is quasitrivial, both split after base change by $X' \to X$. This yields the following commutative diagram with exact rows

$$\begin{array}{ccccc} H^1_{\mathrm{fppf}}(X,P) & \longrightarrow & H^2_{\mathrm{fppf}}(X,M) & \longrightarrow & H^2_{\mathrm{fppf}}(X,F) \\ & & \downarrow{\scriptstyle \rho_3} & & \downarrow{\scriptstyle \rho_2} \\ & & H^2_{\mathrm{fppf}}(K(X),M) & \longrightarrow & H^2_{\mathrm{fppf}}(K(X),F) \end{array}$$

Since we have already shown that $H^1_{\mathrm{fppf}}(X,P)=0$ and ρ_2 is injective, the injectivity of ρ_3 follows from a diagram chase.

Finally, if M is a flasque X-torus, the bijectivity of $H^1_{\mathrm{fppf}}(X,M) \to H^1_{\mathrm{fppf}}(K(X),M)$ will follow if one proves its surjectivity, but the latter follows from Theorem 8.2.4(ii) via a limit argument. □

9. Grothendieck–Serre on a semilocal Prüfer domain

The main result of this section is the following mild generalization of [Guo20].

Theorem 9.0.1. *For a semilocal Prüfer domain R with fraction field K, and a reductive R-group scheme G, the following restriction map has trivial kernel:*

$$\ker\left(H^1_{\text{ét}}(R,G) \to H^1_{\text{ét}}(K,G)\right) = \{*\}.$$

9.0.2. Setup. We fix the following notations. For a semilocal Prüfer domain R of finite Krull dimension, all the maximal ideals $(\mathfrak{m}_i)_{i=1}^r$ of R, the local rings $\mathcal{O}_i := R_{\mathfrak{m}_i}$, an element $a \in R$ such that $V(a) = \{\mathfrak{m}_i\}_{i=1}^r$, let $\widehat{R}$ (resp., $\widehat{\mathcal{O}}_i$) denote the a-adic completion of R (resp., of $\mathcal{O}_i$). Then $\widehat{\mathcal{O}}_i$ is an a-adic complete valuation ring of rank 1, and we have $\widehat{R} \simeq \prod_{i=1}^r \widehat{\mathcal{O}}_i$, compatibly with the topologizes. Denote $\widehat{K}_i := \operatorname{Frac}\widehat{\mathcal{O}}_i = \widehat{\mathcal{O}}_i[\frac{1}{a}]$. Topologize $R[\frac{1}{a}]$ by declaring $\{\operatorname{im}(a^n R \to R[\frac{1}{a}])\}_{n\geqslant 1}$ to be a fundamental system of open neighbourhood of 0; the associated completion is

$$R[\tfrac{1}{a}] \to \widehat{R}[\tfrac{1}{a}] \simeq \textstyle\prod_{i=1}^r \widehat{\mathcal{O}}_i[\frac{1}{a}] = \prod_{i=1}^r \widehat{K}_i,$$

where each $\widehat{K}_i$ is a complete valued field, with pseudo-uniformizer (the image of) a. In particular, for an R-scheme X, we have a map

$$\Phi_X \colon X(R[\tfrac{1}{a}]) \to \textstyle\prod_{i=1}^r X(\widehat{K}_i).$$

If X is locally of finite type over R, we endow the right hand side with the product topology where each $X(\widehat{K}_i)$, by, for example, Conrad, has a natural topology induced from that of $\widehat{K}_i$, which we will call the a-adic topology. If moreover X is affine, we can canonically topologize $X(R[\frac{1}{a}])$ by choosing a closed embedding $X \hookrightarrow \mathbb{A}^N_R$ and endowing $X(R[\frac{1}{a}]) \hookrightarrow R[\frac{1}{a}]^N$ with the subspace topology (this is independent of the choices of the embeddings), then Φ_X is a continuous map.

9.1. Lifting maximal tori of reductive group schemes over semilocal rings

Lemma 9.1.1. *For a scheme S, an S-smooth finitely presented group scheme G whose S-fibers are connected and affine, and a finite subset $I \subset S$. If I satisfies the following conditions*

(i) *I is contained in an affine open subset of S;*

(ii) *for each residue field κ_i of S at $i \in I$, the fiber G_{κ_i} is a κ_i-reductive group; and*

(iii) *$\sharp\kappa_i \geqslant \dim(G_{\kappa_i}/Z_i)$ for the center $Z_i \subset G_{\kappa_i}$,*

then there is an open neighborhood U of I such that the following map is surjective

$$\underline{\mathrm{Tor}}(G)(U) \twoheadrightarrow \textstyle\prod_{i\in I} \underline{\mathrm{Tor}}(G)(\kappa_i).$$

Proof. By [SGA 3_{II}, Exposé XVI, Théorème 5.2], there is an open neighborhood U of I such that $G|_U$ is a U-reductive group scheme, so we may replace S by U. By [SGA 3_{II}, Exposé XII, Théorème 4.7 c)], G has a reductive center Z and we have $Z_i = (Z)_{\kappa_i}$ for every $i \in I$. Since $\underline{\mathrm{Tor}}(G) \simeq \underline{\mathrm{Tor}}(G/Z)$, we may replace G by G/Z. By [SGA 3_{II}, Exposé XIV, Théorème 3.18], the maximal tori of G are exactly the subgroups of type (C), which are bijectively assigned by $D \mapsto \mathrm{Lie}(D)$ to the Cartan subalgebras of $\mathfrak{g} := \mathrm{Lie}(G)$ ([SGA 3_{II}, Exposé XIV, Théorème 3.9]). It suffices to lift a Cartan subalgebra $\mathfrak{c}_0 \subset \prod_{i\in I} \mathfrak{g}_{\kappa_i}$ to that of $\mathfrak{g}$. Denote $\mathfrak{c}_i := (\mathfrak{c}_0)_{\kappa_i}$. Since for each $i \in I$, we have $\sharp\kappa_i \geqslant \dim(G/Z) = \dim(G)$, by [Bar67, Theorem 1], $\mathfrak{c}_i$

is of the form $\mathrm{Nil}(a_i) := \bigcup_n \ker(\mathrm{ad}(a_i^n))$ for some $a_i \in \mathfrak{c}_i$. Hence [SGA 3_{II}, Exposé XIII, Corollaire 5.7] implies that each $a_i \in \mathfrak{c}_i$ is a regular element of $\mathfrak{g}_{\kappa_i}$. We take a section a of $\mathfrak{g}$ passing through all a_i and claim that $\mathcal{V} := \{s \in \operatorname{Spec} R \text{ such that } a_s \in \mathfrak{g}_s \text{ is regular}\}$ is an open subset of $\operatorname{Spec} R$. We may assume that R is reduced. Since the nilpotent rank of $\mathfrak{g}$ is locally constant, there is an open neighborhood U of I such that the nilpotent rank of $\mathfrak{g}$ is constant on each connected component U_α of U. On each U_α, the Killing polynomial of $\mathfrak{g}$ at every $s \in U_\alpha$ is uniformly $P_{\alpha,\mathfrak{g}_s}(t) = t^{r_\alpha}(t^{n-r_\alpha} + (c_1)_s t^{n-r_\alpha-1} + \cdots + (c_{n-r_\alpha})_s)$ such that $(c_{n-r_\alpha})_s$ is nonzero. Thus, the regular locus in $\mathfrak{g}$ is the principle open subset $\bigcap_\alpha \{c_{n-r_\alpha} \neq 0\} \subset \mathbf{W}(\mathfrak{g})$ so $\mathcal{V}$ is nonempty and open, hence shrinking U if necessary, we have $\mathcal{V} = U$. In particular, the regular elements $(a_i \in \mathfrak{c}_i)_{i\in I}$ are lifted to a quasi-regular section $a \in \mathfrak{g}$, which by [SGA $3_{\text{III new}}$, Exposé XIV, Corollaire 3.7], is regular. By definition of regular sections, there is a Cartan subalgebra of $\mathfrak{g}$ containing a and is the desired lifting of $\mathfrak{c}_0$. □

Lemma 9.1.2. *For a semilocal Prüfer domain R of finite Krull dimension, we use the notations in the setup §9.0.2. For a reductive R-group scheme G, the scheme $\underline{\mathrm{Tor}}(G)$ of maximal tori of G, and the a-adic topology on $\underline{\mathrm{Tor}}(G)(\widehat{K}_i)$, the image of the following map is dense:*

$$\underline{\mathrm{Tor}}(G)(R[\tfrac{1}{a}]) \to \textstyle\prod_{i=1}^r \underline{\mathrm{Tor}}(G)(\widehat{K}_i).$$

Proof. The proof proceeds in the following steps.

Step 1. The ring $\mathcal{A} := \varinjlim_{k\geqslant 0} \mathrm{Cauchy}^{\geqslant k}(R[\frac{1}{a}])$ is a semilocal ring with residue fields $\operatorname{Frac}\widehat{\mathcal{O}}_i$. Let I be the kernel of the surjection $\mathcal{A} \twoheadrightarrow \prod_{i=1}^r \operatorname{Frac}\widehat{\mathcal{O}}_i$. Since $\mathcal{A}/I$ is a product of fields, it suffices to show that $1 + I \subset \mathcal{A}^\times$. For a sequence $(b_N)_N \in I$, its tail lies in $\mathrm{im}(a^k R \to R[\frac{1}{a}])$ for all $k > 0$, so the tail of $(1 + b_N)_N$ is invertible in $R^\times$. Since $R[\frac{1}{a}]$ is semilocal, the tail of $(1 + b_N)_N$ is termwise invertible in $R[\frac{1}{a}]$ and the inverses form a Cauchy sequence.

Step 2. We combine the Step 1 and Lemma 9.1.1 to obtain the following surjective map

$$\varinjlim_{m\geqslant 0}\Big(\underline{\mathrm{Tor}}(G)\big(\mathrm{Cauchy}^{\geqslant m}(R[\tfrac{1}{a}])\big)\Big) \simeq \underline{\mathrm{Tor}}(G)\Big(\varinjlim_{m\geqslant 0}\big(\mathrm{Cauchy}^{\geqslant m}(R[\tfrac{1}{a}])\big)\Big) \twoheadrightarrow \textstyle\prod_{i=1}^r \underline{\mathrm{Tor}}(G)(\operatorname{Frac}\widehat{\mathcal{O}}_i),$$

which signifies that every Cauchy sequence in the image of $\underline{\mathrm{Tor}}(G)(R[\frac{1}{a}])$ converges in $\prod_{i=1}^r \underline{\mathrm{Tor}}(G)\operatorname{Frac}\widehat{\mathcal{O}}_i$, hence the assertion follows. □

9.2. **Harder's weak approximation**

Lemma 9.2.1. *For a semilocal Prüfer domain R of finite Krull dimension, we use the setup §9.0.2. For a $R[\frac{1}{a}]$-torus T, let $L_i/\widehat{K}_i$ be minimal Galois field extensions splitting $T_{\widehat{K}_i}$ and consider the norm map*

$$N_i \colon T(L_i) \to T(\widehat{K}_i).$$

Then, the image U of $\prod_{i=1}^r N_i$ is a-adically open and is contained in $\overline{\mathrm{im}(T(R[\frac{1}{a}]) \to \prod_{i=1}^r T(\widehat{K}_i))}$.

Proof. The proof proceeds as the following steps.

Step 1. The image U is a-adically open. For each i, there is a short exact sequence of tori

$$1 \to \mathcal{T}_i \to \mathrm{Res}_{L_i/\widehat{K}_i} T_{L_i} \to T_{\widehat{K}_i} \to 1$$

and the norm map $N_i \colon \mathrm{Res}_{L_i/\widehat{K}_i} T_{L_i}(\widehat{K}_i) \to (\mathrm{Res}_{L_i/\widehat{K}_i} T_{L_i}/\mathcal{T}_i)(\widehat{K}_i)$, which by [Česl5, Proposition 4.3 (a) and §2.8 (2)] is a-adically open. As a product of open subsets, U is open in $\prod_{i=1}^r T(\widehat{K}_i)$.

Step 2. We prove that U is contained in the closure of $\mathrm{im}(T(R[\frac{1}{a}]))$. Equivalently, we show that every $u \in U$ and every a-adically open neighborhood $B_u \subset U$ satisfy that $B_u \cap \mathrm{im}(T(R[\frac{1}{a}])) \neq \varnothing$. Let $\widetilde{R}/R[\frac{1}{a}]$ be a minimal Galois cover splitting T. Consider the following commutative diagram

$$\begin{array}{ccc} T(\widetilde{R}) & \longrightarrow & \prod_{i=1}^r T(L_i) \\ {\scriptstyle N_{\widetilde{R}/R[\frac{1}{a}]}}\downarrow & & \downarrow{\scriptstyle \prod_{i=1}^r N_i} \\ T(R[\tfrac{1}{a}]) & \longrightarrow & \prod_{i=1}^r T(\widehat{K}_i). \end{array}$$

Take a preimage $v \in (\prod_{i=1}^r N_i)^{-1}(u) \subset \prod_{i=1}^r T(L_i)$ and let $B_v \subset \prod_{i=1}^r T(L_i)$ be the preimage of B_u. Since $T_{\widetilde{R}}$ splits, the image of $T(\widetilde{R})$ in $\prod_{i=1}^r T(L_i)$ is a-adically dense, hence $T(\widetilde{R}) \times_{\prod_{i=1}^r T(L_i)} B_v \neq \varnothing$, namely, there is $r \in T(\widetilde{R})$ whose image is in B_v. Let $s := N_{\widetilde{R}/R[\frac{1}{a}]}(r) \in T(R[\frac{1}{a}])$, then the image of s under the map $T(R[\frac{1}{a}]) \to \prod_{i=1}^r T(\widehat{K}_i)$ is contained in B_u, so the assertion follows. □

Lemma 9.2.2. *For a semilocal Prüfer domain R of finite Krull dimension, we use the setup §9.0.2. For a reductive R-group scheme G and for each i a fixed maximal torus $T_i \subset G_{\widehat{K}_i}$ with minimal Galois field extension $L_i/\widehat{K}_i$ splitting T_i, consider the following norm map*

$$N_i \colon T(L_i) \to T(\widehat{K}_i).$$

Then the image U of the map $\prod_{i=1}^r N_i$ is an a-adically open subgroup of $\prod_{i=1}^r T(\widehat{K}_i)$ and is contained in the closure of $\mathrm{im}(G(R[\frac{1}{a}]) \to \prod_{i=1}^r G(\widehat{K}_i))$.

Proof. By the same arguement in Lemma 9.2.1, the image U is a-adically open in $\prod_{i=1} T(\widehat{K}_i)$. It remains to show that $U \subset \overline{\mathrm{im}(G(R[\frac{1}{a}]))}$, which proceeds as the following steps.

Step 1. The map $\phi_i \colon G(\widehat{K}_i) \to \underline{\mathrm{Tor}}(G)(\widehat{K}_i)$ defined by $g \mapsto gTg^{-1}$ is a-adically open for each i. Since the image of $T(R[\frac{1}{a}]) \to \prod_{i=1}^r T(\widehat{K}_i)$ is a-adically dense, for every open neighborhood $W \subset \prod_{i=1}^r G(\widehat{K}_i)$ of id, we have $((\prod_{i=1}^r \phi_i)(W)) \cap \mathrm{Im}(\underline{\mathrm{Tor}}(G)(R[\frac{1}{a}]) \to \prod_{i=1}^r \underline{\mathrm{Tor}}(G)(\widehat{K}_i)) \neq \varnothing$. Therefore, there exist a torus $T' \in \underline{\mathrm{Tor}}(G)(R[\frac{1}{a}])$ and a $(g_i)_{i=1}^r \in W$ such that $g_i T_i g_i^{-1} = T'_{\widehat{K}_i}$ for all i.

Step 2. For any $u \in U$, consider the map $\prod_{i=1}^r G(\widehat{K}_i) \to \prod_{i=1}^r G(\widehat{K}_i)$ defined by $g \mapsto g^{-1}ug$. Then, we apply the Step 1 to the preimage W of U under this map: there is a $\gamma = (\gamma_i)_{i=1}^r \in W$ and a torus $T' \in \underline{\mathrm{Tor}}(G)(R[\frac{1}{a}])$ such that $\gamma_i^{-1} T_i \gamma_i = T'_{\widehat{K}_i}$ for each i. Then, $u \in \gamma U \gamma^{-1} = \gamma((\prod_{i=1}^r N_i)(T_i(L_i)))\gamma^{-1}$, which by transport of structure, is $(\prod_{i=1}^r N_i)(T'_{\widehat{K}_i}(L_i))$. By Lemma 9.2.1, the last term is contained in the closure of $\mathrm{im}(T'(R[\frac{1}{a}]) \to \prod_{i=1}^r T(\widehat{K}_i))$, so is contained in $\overline{\mathrm{im}(G(R[\frac{1}{a}]))}$. □

Proposition 9.2.3. *For a semilocal Prüfer domain R of finite Krull dimension, we use the setup §9.0.2. For a reductive group scheme G over R, the closure $\overline{G(R[\frac{1}{a}])}$ of the image of $G(R[\frac{1}{a}]) \to \prod_{i=1}^r (\widehat{K}_i)$,*

$$\overline{G(R[\tfrac{1}{a}])} \quad \textit{contains an open normal subgroup } N \textit{ of } \prod_{i=1}^r G(\widehat{K}_i).$$

Proof. The proof proceeds in the following steps.

(i) For each i, we fix a maximal torus $T_i \subset G_{\widehat{K}_i}$. Then Lemma 9.2.2 provides the open subgroup $U \subset \prod_{i=1}^r T_i(\widehat{K}_i)$. Since each component of the norm map defining U is the image of the $\widehat{K}_i$-points of $\mathrm{Res}_{L_i/\widehat{K}_i}(T_i)_{L_i} \to T_i$, and $\mathrm{Res}_{L_i/\widehat{K}_i}(T_i)_{L_i}$ is a Zariski dense open subset of an affine space over $\widehat{K}_i$, we have $U \cap \prod_{i=1}^r T_i^{\mathrm{reg}}(\widehat{K}_i) \neq \varnothing$.

(ii) Fix an element $\tau \in U \cap \prod_{i=1}^r T_i^{\mathrm{reg}}(\widehat{K}_i)$, by [SGA 3_{II}, Exposé XIII, Corollaire 2.2], for each i,

$$f_i \colon G_{\widehat{K}_i} \times T_i \to G_{\widehat{K}_i}, \quad (g,t) \mapsto gtg^{-1} \quad \text{is smooth at } (\mathrm{id}, \tau).$$

Hence, there is a Zariski open neighborhood B of (id, τ) such that $(\prod_{i=1}^r f_i)|_B \colon B \to \prod_{i=1}^r G_{\widehat{K}_i}$ is smooth. By [GGMB14, Proposition 3.1.4], the map $B(\prod_{i=1}^r \widehat{K}_i) \to \prod_{i=1}^r G(\widehat{K}_i)$ is open. Then the image of $W := B(\prod_{i=1}^r \widehat{K}_i) \cap (\prod_{i=1}^r G(\widehat{K}_i) \times U)$ under $f = \prod_{i=1}^r f_i$ is open. Subsequently, all $\prod_{i=1}^r G(\widehat{K}_i)$ translations of W have open images, so there is an open subset $U_0 \subset U$ such that $E := f(\prod_{i=1}^r G(\widehat{K}_i) \times U_0)$ is open. Now we define N as the subgroup of $\prod_{i=1}^r G(\widehat{K}_i)$ generated by E, then E is an open subgroup. Further, by construction, E is stable under conjugations by $\prod_{i=1}^r G(\widehat{K}_i)$, thus N is normal.

(iii) We prove that N is contained in the closure of $\mathrm{im}(G(R[\frac{1}{a}]) \to \prod_{i=1}^r G(\widehat{K}_i))$. Since E is the union of all conjugates of U_0, which are contained in $\overline{G(R[\frac{1}{a}])}$ by Lemma 9.2.2, so E is in this closure, and so is N. □

Corollary 9.2.4. *For a semilocal Prüfer domain R of finite Krull dimension, we use the setup §9.0.2. For a reductive group scheme G over R, a maximal torus $T_i \subset G_{\widehat{\mathcal{O}}_i}$ for each i, and any a-adically open neighborhood W of $\mathrm{id} \in \prod_{i=1}^r G(\widehat{K}_i)$ such that $W \subset \overline{G(R[\frac{1}{a}])} \cap \prod_{i=1}^r G(\widehat{\mathcal{O}}_i)$, there exist $g = (g_i)_i \in W$ and a maximal torus $T \in \underline{\mathrm{Tor}}(G)(R)$ such that for every i, we have*

$$T_{\widehat{K}_i} = g_i (T_i)_{\widehat{K}_i} g_i^{-1}.$$

Proof. By Proposition 9.2.3, $\overline{G(R[\frac{1}{a}])} \cap \prod_{i=1}^r G(\widehat{\mathcal{O}}_i)$ is an a-adically open neighborhood of $\mathrm{id} \in \prod_{i=1}^r G(\widehat{K}_i)$, so it makes sense to take its subset W such that W is a neighborhood of id. Now consider the a-adically open map $\phi\colon \prod_{i=1}^r G(\widehat{K}_i) \to \prod_{i=1}^r \underline{\mathrm{Tor}}(G)(\widehat{K}_i)$ defined by $g_i \mapsto g_i(T_i)_{\widehat{K}_i} g_i^{-1}$. Then $\phi(W)$ is an a-adically open neighborhood of $(T_i)_i \in \prod_{i=1}^r \underline{\mathrm{Tor}}(G)(\widehat{K}_i)$. Since $\prod_{i=1}^r \underline{\mathrm{Tor}}(G)(\widehat{\mathcal{O}}_i) \subset \prod_{i=1}^r \underline{\mathrm{Tor}}(G)(\widehat{K}_i)$ is also an a-adically open neighborhood of $(T_i)_i$, we have an open intersection $\phi(W) \cap \prod_{i=1}^r \underline{\mathrm{Tor}}(G)(\widehat{\mathcal{O}}_i) \neq \varnothing$. Then the density of the image of $\underline{\mathrm{Tor}}(G)(R[\frac{1}{a}]) \to \prod_{i=1}^r \underline{\mathrm{Tor}}(G)(\widehat{K}_i)$ provided by Lemma 9.1.2 yields an element

$$T \in \underline{\mathrm{Tor}}(G)(R) \xrightarrow{\sim} \underline{\mathrm{Tor}}(G)(R[\tfrac{1}{a}]) \times_{\prod_{i=1}^r \underline{\mathrm{Tor}}(G)(\widehat{K}_i)} \prod_{i=1}^r \underline{\mathrm{Tor}}(G)(\widehat{\mathcal{O}}_i).$$

Therefore, T is a maximal torus of G over R satisfying the conditions. □

Corollary 9.2.5. *With the notations in Proposition 9.2.3, we have*

$$\overline{G(R[\tfrac{1}{a}])} \cdot \prod_{i=1}^r G(\widehat{\mathcal{O}}_i) = \mathrm{im}(G(R[\tfrac{1}{a}]) \to \prod_{i=1}^r G(\widehat{K}_i)) \cdot \prod_{i=1}^r G(\widehat{\mathcal{O}}_i).$$

9.3. Product formula over semilocal Prüfer domains, passage to the local case

Lemma 9.3.1. *For a semilocal Prüfer domain R of finite Krull dimension, we use the notations in the setup §9.0.2. For an R-torus T, we have the following product formula*

$$\prod_{i=1}^r T(\widehat{K}_i) = \mathrm{im}(T(R[\tfrac{1}{a}]) \to \prod_{i=1}^r T(\widehat{K}_i)) \cdot \prod_{i=1}^r T(\widehat{\mathcal{O}}_i).$$

Proof. Let R^{h} denote the Henselization of the pair (R, aR). Then we have the commutative digram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T(R) & \longrightarrow & T(R[\frac{1}{a}]) & \longrightarrow & H^1_{\{a=0\}}(R,T) & \longrightarrow & H^1(R,T) \longrightarrow H^1(R[\frac{1}{a}],T) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \quad\quad \downarrow \\
0 & \longrightarrow & T(R^{\mathrm{h}}) & \longrightarrow & T(R^{\mathrm{h}}[\frac{1}{a}]) & \longrightarrow & H^1_{\{a=0\}}(R^{\mathrm{h}},T) & \longrightarrow & H^1(R^{\mathrm{h}},T) \longrightarrow H^1(R^{\mathrm{h}}[\frac{1}{a}],T),
\end{array}$$

whose exact rows are the local cohomology exact sequences. Since the case of tori for Theorem 9.0.1 is proved, the two horizontal arrows of the rightmost squares are injective, hence the coset $T(R^{\mathrm{h}}[\frac{1}{a}])/T(R^{\mathrm{h}}) \simeq H^1_{\{a=0\}}(R^{\mathrm{h}},T)$. By excision [Mil80, III, 1.28], we have an isomorphism $H^1_{\{a=0\}}(R,T) \cong H^1_{\{a=0\}}(R^{\mathrm{h}},T)$, which leads to a surjection $T(R[\frac{1}{a}]) \twoheadrightarrow H^1_{\{a=0\}}(R^{\mathrm{h}},T)$. Therefore, we obtain the product formula

$$T(R^{\mathrm{h}}[\tfrac{1}{a}]) = \mathrm{im}(T(R[\tfrac{1}{a}]) \to T(R^{\mathrm{h}}[\tfrac{1}{a}])) \cdot T(R^{\mathrm{h}}). \tag{9.3.1}$$

On the other hand, by [BČ22, 2.2.17], the image of $T(R^{\mathrm{h}}[\frac{1}{a}]) \to \prod_{i=1}^r T(\widehat{K}_i)$ is dense in $\prod_{i=1}^r T(\widehat{K}_i)$ with respect to the topology fixed in §9.0.2. Since each $T(\widehat{\mathcal{O}}_i) \subset T(\widehat{K}_i)$ is an open subgroup, we have

$$\mathrm{im}\big(T(R^{\mathrm{h}}[\tfrac{1}{a}]) \to \textstyle\prod_{i=1}^r T(\widehat{K}_i)\big) \cdot \prod_{i=1}^r T(\widehat{\mathcal{O}}_i) = \prod_{i=1}^r T(\widehat{K}_i). \tag{9.3.2}$$

Consequently, the combination of (9.3.1) and (9.3.2) leads to the assertion. □

Proposition 9.3.2. *For a semilocal Prüfer domain R of finite Krull dimension, we use the notations in the setup §9.0.2. For a reductive R-group scheme G, we have*

$$\prod_{i=1}^r G(\widehat{K}_i) = \mathrm{im}\left(G(R[\tfrac{1}{a}]) \to \prod_{i=1}^r G(\widehat{K}_i)\right) \cdot \prod_{i=1}^r G(\widehat{\mathcal{O}}_i).$$

Proof. We will proceed verbatim as in [Guo20, §4]. We choose a minimal parabolic $\widehat{\mathcal{O}}_i$-subgroup P_i for each $G_i := G \times_R \widehat{\mathcal{O}}_i$. Denote $U_i := \mathrm{rad}^u(P_i)$.

(i) for the maximal split torus $T_i \subset P_i$, we have $\prod_{i=1}^r T_i(\widehat{K}_i) \subset \mathrm{im}(G(R[\frac{1}{a}]) \to \prod_{i=1}^r G(\widehat{K}_i)) \cdot \prod_{i=1}^r G(\widehat{\mathcal{O}}_i)$. By [SGA 3$_{\text{III new}}$, Exposé XXVI, Corollaire 6.11], there is a maximal torus $\widetilde{T}_i$ of G_i containing T_i. In particular, $(\widetilde{T}_i)_{\widehat{K}_i}$ is a maximal torus of $G_{\widehat{K}_i}$. Then we apply Corollary 9.2.4 to all $\widetilde{T}_i$: there are a $g = (g_i)_i \in \overline{G(R[\frac{1}{a}])} \cdot \prod_{i=1}^r G(\widehat{\mathcal{O}}_i)$ and a maximal torus $T_0 \subset G$ such that $(T_0)_{\widehat{K}_i} = g_i(\widetilde{T}_i)_{\widehat{K}_i} g_i^{-1}$ for every i, which combined with the product formula Lemma 9.3.1 for T_0 yields

$$\prod_{i=1}^r \widetilde{T}(\widehat{K}_i) = \prod_{i=1}^r g_i^{-1} T_0(\widehat{K}_i) g_i \subset \prod_{i=1}^r \mathrm{im}(G(R[\tfrac{1}{a}])) \cdot G(\widehat{\mathcal{O}}_i) g_i.$$

Since $g \in \overline{\mathrm{im}(G(R[\frac{1}{a}]))} \cap \prod_{i=1}^r G(\widehat{\mathcal{O}}_i)$, the inclusion displayed above implies that $\prod_{i=1}^r \widetilde{T}_i(\widehat{K}_i) \subset \overline{\mathrm{im}(G(R[\frac{1}{a}]))} \cdot \prod_{i=1}^r G(\widehat{\mathcal{O}}_i)$. Therefore, we obtain the following desired inclusion

$$\prod_{i=1}^r T_i(\widehat{K}_i) \subset \prod_{i=1}^r \widetilde{T}_i(\widehat{K}_i) \subset \mathrm{im}(G(R[\tfrac{1}{a}])) \cdot \prod_{i=1}^r G(\widehat{\mathcal{O}}_i).$$

(ii) we have $\prod_{i=1}^r U_i(\widehat{K}_i) \subset \overline{\mathrm{im}(G(R[\frac{1}{a}]) \to \prod_{i=1}^r G(\widehat{K}_i))}$. Consider the T_i-action on G_i defined by

$$T_i \times G_i \to G_i, \quad (t, g) \mapsto tgt^{-1}.$$

Recall the open normal subgroup $N \subset \prod_{i=1}^r G(\widehat{K}_i)$ constructed in Proposition 9.2.3, then each $N \cap U_i(\widehat{K}_i)$ is open in $U_i(\widehat{K}_i)$. The dynamic argument in [Guo20] shows that $U_i(\widehat{K}_i) = N \cap U_i(\widehat{K}_i)$, hence $U_i(\widehat{K}_i) \subset N$ for each i. Therefore, we have $\prod_{i=1}^r U_i(\widehat{K}_i) \subset \overline{\mathrm{im}(G(R[\frac{1}{a}]) \to \prod_{i=1}^r G(\widehat{K}_i))}$.

(iii) we have $\prod_{i=1}^r P_i(\widehat{K}_i) \subset \mathrm{im}(G(R[\frac{1}{a}]) \to \prod_{i=1}^r G(\widehat{K}_i)) \cdot \prod_{i=1}^r G(\widehat{\mathcal{O}}_i)$. The quotient $H_i := L_i/T_i$ is anisotropic, therefore we have $H_i(\widehat{K}_i) = H_i(\widehat{\mathcal{O}}_i)$ for every i. Consider the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & T_i(\widehat{\mathcal{O}}_i) & \longrightarrow & L_i(\widehat{\mathcal{O}}_i) & \longrightarrow & H_i(\widehat{\mathcal{O}}_i) \longrightarrow H^1(\widehat{\mathcal{O}}_i, T_i) = 0 \\
 & & \downarrow & & \downarrow & & \| \qquad\quad \downarrow \\
0 & \longrightarrow & T_i(\widehat{K}_i) & \longrightarrow & L_i(\widehat{K}_i) & \longrightarrow & H_i(\widehat{K}_i) \longrightarrow H^1(\widehat{K}_i, T_i) = 0
\end{array}$$

with exact rows. By diagram chase, we have $L_i(\widehat{K}_i) = T_i(\widehat{K}_i) \cdot L_i(\widehat{\mathcal{O}}_i)$ for every i. Subsequently, the combination of (i) and (ii) yields the inclusion

$$\prod_{i=1}^r P_i(\widehat{K}_i) \subset \mathrm{im}(G(R[\tfrac{1}{a}]) \to \prod_{i=1}^r G(\widehat{K}_i)) \cdot \prod_{i=1}^r G(\widehat{\mathcal{O}}_i).$$

(iv) Recall [SGA 3$_{\text{III new}}$, Exposé XXVI, Théorème 4.3.2 and Corollaire 5.2] that for each P_i, there is a parabolic subgroup Q_i of G_i such that $P_i \cap Q_i = L_i$ fitting into the following surjection

$$\mathrm{rad}^u(P_i)(\widehat{K}_i) \cdot \mathrm{rad}^u(Q_i)(\widehat{K}_i) \twoheadrightarrow G(\widehat{K}_i)/P_i(\widehat{K}_i).$$

This surjection, combined with the result of (ii) gives an inclusion

$$\prod_{i=1}^r G(\widehat{K}_i) \subset \overline{\mathrm{im}(G(R[\tfrac{1}{a}]) \to \prod_{i=1}^r G(\widehat{K}_i))} \cdot \prod_{i=1}^r P_i(\widehat{K}_i).$$

Now we further use the result of (iii) to obtain $\prod_{i=1}^r G(\widehat{K}_i) \subset \overline{\mathrm{im}(G(R[\frac{1}{a}]) \to \prod_{i=1}^r G(\widehat{K}_i))} \cdot \prod_{i=1}^r G(\widehat{\mathcal{O}}_i)$. Hence, we have the following product formula

$$\prod_{i=1}^r G(\widehat{K}_i) = \mathrm{im}\left(G(R[\tfrac{1}{a}]) \to \prod_{i=1}^r G(\widehat{K}_i)\right) \cdot \prod_{i=1}^r G(\widehat{\mathcal{O}}_i). \qquad \square$$

10. Torsors on a smooth affine relative curve

In this section we prove the following result concerning triviality of torsors on a smooth affine relative curve. The idea of the proof ultimately depends on the geometry of affine Grassmannians developed by Fedorov, who proved Theorem 10.1 (i) for $C = \mathbb{A}^1_R$. A similar result can also be found in the recent preprint [Čes22c, Theorem 4.4].

Theorem 10.1 (Section theorem)**.** *Let R be a semilocal domain whose local rings at primes are geometrically unibranch*[5]*, C a smooth, affine, relative R-curve, and G a reductive C-group scheme. Let A be a*

[5]According to [SP, 0BPZ], a local ring A is geometrically unibranch if its reduction $A_{\mathrm{red}} := A/\sqrt{(0)}$ is a domain, and if the integral closure of A_{red} in its fraction field is a local ring whose residue field is purely inseparable over that of A. By [SP, 06DM], A is geometrically unibranch iff its strict Henselization A^{sh} has a unique minimal prime.

R-algebra. Let $\mathcal{P}$ be a G-torsor over $C_A := C \times_R A$ that trivializes over $C_A \backslash Z_A$ for some R-finite closed subscheme $Z \subset C$. For a section $s \in C(R)$, if either

(i) *A is semilocal, or*

(ii) *$s_A^*(G)$ is totally isotropic,*

then the pullback $s_A^*(\mathcal{P})$ is trivial as an $s_A^*(G)$-torsor, where s_A stands for the image of s in $C_A(A)$.

To prove Theorem 10.1, we first use Lemma 10.2 to reduce to the case when G is the base change of a reductive R-group scheme, and then to the case when $C = \mathbb{A}^1_R$, see Lemma 10.3. As for the latter, one can approach it via the geometry of affine Grassmannians.

We start with the following result concerning equating reductive group schemes, which was already known to experts, see also [Čes22c, Lemma 3.5].

Lemma 10.2 (Equating reductive group schemes)**.** *Let B be a semilocal ring whose local rings are geometrically unibranch, and let G_1, G_2 be two reductive B-group schemes whose geometric B-fibers are of the same type. Let $T_1 \subset G_1, T_2 \subset G_2$ be maximal B-tori. Assume that, for some ideal $I \subset B$, there is an isomorphism of B/I-group schemes*

$$\iota : (G_1)_{B/I} \simeq (G_2)_{B/I} \quad \text{such that} \quad \iota((T_1)_{B/I}) = (T_2)_{B/I}.$$

There are a faithfully flat, finite, étale B-algebra B', a section $s : B' \twoheadrightarrow B/I$, and an isomorphism of B'-groups $\iota' : (G_1)_{B'} \simeq (G_2)_{B'}$ such that $\iota((T_1)_{B'}) = (T_2)_{B'}$ and whose s-pullback is ι.

Proof. According to [SGA $3_{\text{III new}}$, Exposé XXIV, Corollaire 2.2], the condition on the geometric B-fibers ensures that the functor

$$X := \underline{\mathrm{Isom}}_B((G_1, T_1), (G_2, T_2))$$

parameterizing the isomorphisms of the pairs (G_1, T_1) and (G_2, T_2) is representable by a B-scheme and is a $H := \underline{\mathrm{Aut}}_B((G_1, T_1))$-torsor. We need to show that, for any $\iota \in X(B/I)$, there are a faithfully flat, finite, étale B-algebra B', an $\iota' \in X(B')$, and a section $s : B' \twoheadrightarrow B/I$ such that $s(\iota') = \iota \in X(B/I)$.

By *loc. cit.*, H is an extension of an étale locally constant B-group scheme by T_1^{ad}, the quotient of T_1 by the scheme-theoretic center of G_1. According to [SGA $3_{\text{III new}}$, Exposé XXIV, Proposition 2.6], T_1^{ad} acts freely on X and the quotient

$$\overline{X} := X/T_1^{\mathrm{ad}}$$

is represented by a faithfully flat B-scheme that is étale locally constant on B. As B is geometrically unibranch, by [SGA $3_{\text{III new}}$, Exposé X, Corollaire 5.14], every connected component of $\overline{X}$ is finite, étale over B. As the image of $\iota : \mathrm{Spec}(B/I) \to X \to \overline{X}$ intersects only finitely many connected components of $\overline{X}$, the union of these components is the spectrum of a finite étale B-algebra A, and there are an $\bar{\iota} \in \overline{X}(A)$ and a section $t : A \twoheadrightarrow B/I$ such that $t(\bar{\iota}) = \iota$. By adding more connected components of $\overline{X}$ into $\mathrm{Spec}(A)$ if needed, we may assume that A is faithfully flat over B. Let

$$Y := X \times_{\overline{X}, \bar{\iota}} \mathrm{Spec}(A);$$

it is a T_A^{ad}-torsor equipped with a point $\iota \in Y(A/J) \subset X(A/J)$, where $J := \ker(A \twoheadrightarrow B/I)$. By [Čes22b, Corollary 6.3.2], there are a faithfully flat, finite, étale A-algebra B', a section

$$s' : B' \twoheadrightarrow A/J \simeq B/I,$$

and an $\iota' \in Y(B') \subset X(B')$ such that $s'(\iota') = \iota$. □

Lemma 10.3. *The proof of the Theorem 10.1 reduces to the case when $C = \mathbb{A}^1_R$ and G is the base change of a reductive R-group scheme.*

Proof. Let B be the semilocal ring of C at the closed points of $\mathrm{im}(s) \cup Z$; its local rings are geometrically unibranch. By abuse of notation, we may view $s : B \twoheadrightarrow R$ as a section of the R-algebra B. As B is semilocal, by [SGA 3_{II}, Exposé XIV, Corollaire 3.20], G_B admits a maximal B-torus T_B. Since the pullbacks of the paris (G_B, T) and $((s^*(G))_B, (s^*(T))_B)$ along s are the same, by Lemma 10.2, there are a faithfully flat, finite, étale B-algebra B', a section $s' : B' \twoheadrightarrow R$ that lifts s, and a B'-isomorphism

$$\iota : (G_{B'}, T_{B'}) \simeq ((s^*(G))_{B'}, (s^*(T)_{B'})$$

whose s-pullback is the identity. We may spread out $\mathrm{Spec}(B') \to \mathrm{Spec}(B)$ to obtain a finite étale covering $C' \to U$ of a small enough affine open neighbourhood U of $\mathrm{im}(s) \cup Z$ in C. By shrinking U if necessary, we may assume that the isomorphism ι is defined over C'. In both cases of Theorem 10.1 we may replace C by C', Z by $C' \times_C Z$, s by s', and $\mathcal{P}$ by $\mathcal{P}|_{C'_A}$ to reduce to the case when G is the base change of the reductive R-group scheme $s^*(G)$.

Next, in order to apply glueing Lemma 6.2.2(ii) to achieve that $C = \mathbb{A}^1_R$, we need to modify C so that Z embeds into $\mathbb{A}^1_R$. For this, we first replace Z by $Z \cup \mathrm{im}(s)$ to assume that s factors through Z. Then we apply Panin's 'finite field tricks' [Čes22a, Proposition 7.4] to obtain a finite morphism $\widetilde{C} \to C$ that is étale at the points in $\widetilde{Z} := \widetilde{C} \times_C Z$ such that s lifts to $\widetilde{s} \in \widetilde{C}(R)$, and there are no finite fields obstruction to embedding $\widetilde{Z}$ into $\mathbb{A}^1_R$ in the following sense: for every maximal ideal $\mathfrak{m} \subset R$,

$$\sharp\left\{z \in \widetilde{Z}_{\kappa(\mathfrak{m})} : [\kappa(z) : \kappa(\mathfrak{m})] = d\right\} < \sharp\left\{z \in \mathbb{A}^1_{\kappa(\mathfrak{m})} : [\kappa(z) : \kappa(\mathfrak{m})] = d\right\} \quad \text{for every} \quad d \geqslant 1.$$

Then, by [Čes22a, Lemma 6.3], there are an affine open $C'' \subset \widetilde{C}$ containing $\mathrm{im}(\widetilde{s})$, a quasi-finite, flat R-map $C'' \to \mathbb{A}^1_R$ that maps Z isomorphically to a closed subscheme $Z' \subset \mathbb{A}^1_R$ with

$$Z \simeq Z' \times_{\mathbb{A}^1_R} C''.$$

(Actually, by shrinking C'' around $\mathrm{im}(\widetilde{s})$, one can show that $C'' \to \mathbb{A}^1_R$ is étale.) For both cases of Theorem 10.1, since $\mathcal{P}|_{C''_A}$ is a G-torsors that trivializes over $C''_A \backslash \widetilde{Z}_A$, we may use Lemma 6.2.2(ii) to glue $\mathcal{P}_{C''_A}$ with the trivial G-torsor over $\mathbb{A}^1_A$ to obtain a G-torsor $\mathcal{P}'$ over $\mathbb{A}^1_A$ that trivializes over $\mathbb{A}^1_A \backslash Z'_A$. Let $s' \in \mathbb{A}^1_R(R)$ be the image of $\widetilde{s}$; then $s'^*(\mathcal{P}') \simeq s^*(\mathcal{P})$. It remains to replace C by $\mathbb{A}^1_R$, Z by Z', s by s', and $\mathcal{P}$ by $\mathcal{P}'$. □

The analysis of torsors on $\mathbb{A}^1_R$ ultimately depends on the geometry of affine Grassmannians. A nice summary of and complement on the relevant techniques can be found in [Čes22b, §5.3]. In particular, we will use the following result; it is a slight variant of [Čes22b, Proposition 5.3.6], which in turn is a mild generalization of [Fed22b, Theorem 6].

Proposition 10.4. *For a semilocal ring R with* connected *spectrum and a reductive R-group scheme G, let*

$$G^{\mathrm{ad}} \simeq \prod_i \mathrm{Res}_{R_i/R}(G_i)$$

be the canonical decomposition of the adjoint quotient G^{ad} [SGA 3$_{\text{III new}}$, Exposé XXIV, Proposition 5.10], where G_i is an adjoint simple R_i-group scheme, and R_i is a finite, étale R-algebra with connected *spectra. Let $Y \subset \mathbb{A}^1_R$ be a R-finite, étale, closed subscheme with the following properties:*

(i) *for every i, there is a clopen $Y_i \subset Y \times_R R_i$ such that $(G_i)_{Y_i}$ contains a copy of $\mathbb{G}_{m,Y_i}$;*

(ii) *for every maximal ideal $\mathfrak{m} \subset R_i$ such that $(G_i)_{\kappa(\mathfrak{m})}$ is isotropic, the line bundle $\mathscr{O}_{\mathbb{P}^1_{\kappa(\mathfrak{m})}}(1)$ is trivial over $\mathbb{P}^1_{\kappa(\mathfrak{m})} \backslash (Y_i)_{\kappa(\mathfrak{m})}$;*

(iii) *the line bundle $\mathscr{O}_{\mathbb{P}^1_R}(1)$ is trivial over $\mathbb{P}^1_R \backslash Y$.*

Let $\mathcal{P}$ be a G-torsor over $\mathbb{P}^1_R$ that trivializes over $\mathbb{P}^1_R \backslash Z$ for some R-finite closed subscheme $Z \subset \mathbb{A}^1_R \backslash Y$. Assume that for every maximal ideal $\mathfrak{m} \subset R$ the G^{ad}-torsor over $\mathbb{P}^1_{\kappa(\mathfrak{m})}$ induced by $\mathcal{P}$ lifts to a generically trivial *$(G^{\mathrm{ad}})^{\mathrm{sc}}$-torsor over $\mathbb{P}^1_{\kappa(\mathfrak{m})}$. Then the restriction $\mathcal{P}|_{\mathbb{P}^1_R \backslash Y}$ is trivial.*

Recall that, by [SGA 3$_{\text{III new}}$, Exposé XXVI, Corollaire 6.12], (i) is equivalent to that the base change of $(G_i)_{Y_i}$ to every connected component of Y_i contains a proper parabolic subgroup scheme. For instance, if G is quasi-split, we can just take $Y_i = Y \times_R R_i$ to ensure (i). In practice, we achieve (i) by guaranteeing base change of $(G_i)_{Y_i}$ to connected components of Y_i contain proper parabolics. For (ii), we can take Y_i so that $Y_i(\kappa(\mathfrak{m})) \neq \varnothing$ for every maximal ideal $\kappa(\mathfrak{m}) \subset R_i$ with $(G_i)_{\kappa(\mathfrak{m})}$ isotropic. For (iii), we just need to choose Y so that it contains finite étale R-schemes of degrees d and $d+1$ for some $d \geqslant 1$, because $\mathscr{O}(d)$ and $\mathscr{O}(n+1)$ are both trivial on $\mathbb{P}^1_R \backslash Y$, and so is $\mathscr{O}(1)$.

Proof. We will deduce Proposition 10.4 from (the proof of) a particular case of [Čes22b, Proposition 5.3.6]. (We remind that the assumption (ii) of *loc. cit.* should read as '$(G_i)_{Y_i}$ contains a copy of $\mathbb{G}_{m,Y_i}$', as its proof shows.)

The R-finite étale Y is the vanishing locus of a monic polynomial t in the standard coordinate of $\mathbb{A}^1_R$; namely, t is the characteristic polynomial of this standard coordinate acting on $\widetilde{R} := \Gamma(Y, \mathscr{O}_Y)$. The formal completion of $\mathbb{P}^1_R$ along Y has coordinate ring $\widetilde{R}[\![t]\!]$. Recall that, by formal glueing, a G-torsor over $\mathbb{P}^1_R$ can be viewed as the glueing of its restriction to $\mathbb{P}^1_R \backslash Y$ and to $\widetilde{R}[\![t]\!]$ along the 'intersection' $\widetilde{R}(\!(t)\!)$; since our torsor $\mathcal{P}$ is trivial over an open neighbourhood $U \subset \mathbb{P}^1_R$ of Y, both of the restriction $\mathcal{P}|_{U\backslash Y}$ and $\mathcal{P}|_{\widetilde{R}[\![t]\!]}$ are trivial, and once a trivialization of the former was chosen, all such glueings are parameterized by elements of $G(\widetilde{R}(\!(t)\!))/G(\widetilde{R}[\![t]\!])$. In particular, since $G(\widetilde{R}(\!(t)\!))$ acts on $G(\widetilde{R}(\!(t)\!))/G(\widetilde{R}[\![t]\!])$ (via left multiplication), an element of $G(\widetilde{R}(\!(t)\!))$ yields a modification of $\mathcal{P}$ along Y: it is the G-torsor over $\mathbb{P}^1_R$ whose restriction to $\mathbb{P}^1_R \backslash Y$ and to $\widetilde{R}[\![t]\!]$ are the same as $\mathcal{P}$, but their corresponding glueings, viewed as elements of $G(\widetilde{R}(\!(t)\!))/G(\widetilde{R}[\![t]\!])$, differ by a left translation by the element of $\widetilde{R}(\!(t)\!)$ we choose.

Denote by $\mathcal{P}^{\mathrm{ad}}$ the G^{ad}-torsor over $\mathbb{P}^1_R$ induced by $\mathcal{P}$. Since the formation of $H^1(\mathbb{P}^1_R, -)$ commutes with taking products, $\mathcal{P}^{\mathrm{ad}}$ corresponds to a collection $(\mathcal{P}_i^{\mathrm{ad}})$, where $\mathcal{P}_i^{\mathrm{ad}}$ is a $\mathrm{Res}_{R_i/R}(G_i)$-torsor over $\mathbb{P}^1_R$ satisfying the analogous assumptions (i)-(iii) of the Proposition 10.4. Since $R \to R_i$ is finite étale and G_i is R_i-smooth, we have $R^1 f_* G_i = 1$ for the map $f : \mathrm{Spec}(R_i) \to \mathrm{Spec}(R)$ induced by $R \to R_i$. By the exact sequence from [Gir71, Chapitre V, Proposition 3.1.3],

$$1 \to H^1(\mathbb{P}^1_R, \mathrm{Res}_{R_i/R}(G_i)) \to H^1(\mathbb{P}^1_{R_i}, G_i) \to H^1(\mathbb{P}^1_R, R^1 f_* G_i).$$

Thus $\mathcal{Q} \mapsto \mathrm{Res}_{R_i/R}(\mathcal{Q})$ defines a bijection of pointed sets $H^1(\mathbb{P}^1_{R_i}, G_i) \xrightarrow{\sim} H^1(\mathbb{P}^1_R, \mathrm{Res}_{R_i/R}(G_i))$. In particular, each $\mathcal{P}_i^{\mathrm{ad}}$ corresponds to a G_i-torsor $\mathcal{Q}_i$ over $\mathbb{P}^1_{R_i}$. As one can see immediately, the assumptions (i)-(iii) of Proposition 10.4 for the $\mathrm{Res}_{R_i/R}(G_i)$-torsor $\mathcal{P}_i^{\mathrm{ad}}$ translate into the assumptions [Čes22b, Proposition 5.3.6] (i)-(iv) for the G_i-torsor $\mathcal{Q}_i$ over $\mathbb{P}^1_{R_i}$. By the proof of *loc. cit.*, for some element

$$\alpha_i \in \mathrm{im}\left(G_i^{\mathrm{sc}}((\widetilde{R} \otimes_R R_i)(\!(t)\!)) \to G_i((\widetilde{R} \otimes_R R_i)(\!(t)\!))\right),$$

the corresponding modification of $\mathcal{Q}_i$ along $Y \times_R R_i$ is trivial. We can view

$$\alpha := (\alpha_i) \in \mathrm{im}\left((G^{\mathrm{ad}})^{\mathrm{sc}}(\widetilde{R}(\!(t)\!)) \to G^{\mathrm{ad}}(\widetilde{R}(\!(t)\!))\right);$$

as $(G^{\mathrm{ad}})^{\mathrm{sc}} \to G^{\mathrm{ad}}$ factors through $(G^{\mathrm{ad}})^{\mathrm{sc}} \to G$, α lifts to $\widetilde{\alpha} \in G(\widetilde{R}(\!(t)\!))$. Denote by $\mathcal{Q}$ the modification of $\mathcal{P}$ along Y using $\widetilde{\alpha}$. By our construction, the G^{ad}-torsor $\mathcal{Q}^{\mathrm{ad}}$ over $\mathbb{P}^1_R$ induced by $\mathcal{Q}$ corresponds to the collection of modifications of the $\mathcal{P}_i^{\mathrm{ad}} = \mathrm{Res}_{R_i/R}(\mathcal{Q}_i)$ along Y using $\alpha_i \in G_i((\widetilde{R} \otimes_R R_i)(\!(t)\!)) = \mathrm{Res}_{R_i/R}(\widetilde{R}(\!(t)\!))$, which is trivial, so that $\mathcal{Q}^{\mathrm{ad}}$ is trivial, to the effect that $\mathcal{Q}$ reduces to a torsor over $\mathbb{P}^1_R$ under the center Z_G of G. Now, as the last paragraph of the proof of [Čes22b, Proposition 5.3.6] shows, any Z_G-torsor over $\mathbb{P}^1_R$ is the sum of a constant torsor (i.e., the pullback of a Z_G-torsor over R) and $\lambda_*(\mathscr{O}(1))$ for a unique cocharacter λ of Z_G. Therefore, by our assumption (iii), $\mathcal{Q}$ is a constant torsor, and, by checking along the infinity section, it is even trivial, so is $\mathcal{P}|_{\mathbb{P}^1_R \backslash Y} = \mathcal{Q}|_{\mathbb{P}^1_R \backslash Y}$, as desired. □

The following result will help us to construct the desired R-finite, étale schemes Y_i and Y from the previous theorem.

Lemma 10.5. *Let R be a semilocal ring with connected spectrum, let R_1 be a finite, étale R-algebra with connected spectrum, let $W \subset \mathbb{A}^1_R$ be a R-finite closed scheme, and let G_1 be a simple R_1-group scheme. There is a R_1-finite, étale scheme Y_1, and a closed immersion $Y_1 \subset \mathbb{A}^1_R \backslash W$ over R such that $(G_1)_{Y_1}$ contains a copy of $\mathbb{G}_{m,Y_1}$, and, for every maximal ideal $\mathfrak{m} \subset R_1$ with $(G_1)_{\kappa(\mathfrak{m})}$ isotropic, the line bundle $\mathcal{O}_{\mathbb{P}^1_{\kappa(\mathfrak{m})}}(1)$ is trivial over $\mathbb{P}^1_{\kappa(\mathfrak{m})} \backslash (Y_1)_{\kappa(\mathfrak{m})}$. (Notice that Y_1 is a clopen of $Y_1 \times_R R_1$, thus naturally embeds into $\mathbb{A}^1_{R_1}$.)*

In addition, there is a R_1-finite, étale scheme Y' and a closed immersion $Y' \hookrightarrow \mathbb{A}^1_R \backslash W$ over R such that the line bundle $\mathcal{O}_{\mathbb{P}^1_R}(1)$ is trivial over $\mathbb{P}^1_R \backslash Y'$.

Proof. Let $\mathrm{Par}' \to \mathrm{Spec}(R_1)$ be the scheme parameterizing *proper* parabolic subgroup schemes of the reductive R_1-group scheme G_1; it is smooth projective over R_1 (*cf.* [SGA $3_{\mathrm{III\ new}}$, Exposé XXVI, Corollaire 3.5]). Fix an embedding $\mathrm{Par}' \hookrightarrow \mathbb{P}^N_{R_1}$ over R_1. Write $\mathrm{Par}' = \bigsqcup_{i=1}^t P_t$ as a disjoint union of its connected components; every P_t has a constant relative dimension d_t over R_1. For every maximal ideal $\mathfrak{m} \subset R_1$ with $(G_1)_{\kappa(\mathfrak{m})}$ isotropic, a proper parabolic subgroup of $(G_1)_{\kappa(\mathfrak{m})}$ gives a point $b_{\mathfrak{m}} \in \mathrm{Par}'(\kappa(\mathfrak{m}))$.

Fix an $i = 1, \cdots, t$. For every maximal ideal $\mathfrak{m} \subset R_1$, by Bertini theorem (including Poonen's version over finite fields), one can find a hypersurface in $\mathbb{P}^N_{\kappa(\mathfrak{m})}$ of large enough degree such that it passes through all points $b_{\mathfrak{m}}$ that lies in P_i and it intersects $(P_i)_{\kappa(\mathfrak{m})}$ transversally. We may assume that the above hypersurfaces have the same degree for all $\mathfrak{m}$. By the Chinese Remainder theorem, one can lift these simultaneously to get a hypersurfaces $H \subset \mathbb{P}^N_{R_1}$. Then $H \cap P_i$ is a smooth projective R_1-scheme of pure relative dimension $d_i - 1$, and $b_{\mathfrak{m}} \in H \cap P_i$ whenever $b_{\mathfrak{m}} \in P_i$. The same argument can be applied to the hypersurface section $H \cap P_i$. Continuing in this way, we finally arrive at a R_1-finite, étale, closed subscheme $Y_i \subset P_i$ such that $b_{\mathfrak{m}} \in Y_i$ whenever $b_{\mathfrak{m}} \in P_i$. Denote $Y_1' := \bigsqcup_{i=1}^t Y_i$. Unfortunately, Y_1' may not embed into $\mathbb{A}^1_R \backslash W$. So let's first modify Y_1' using Panin's 'finite field tricks'.

Let $d > 0$ be a large enough integer such that, for every maximal ideal $\mathfrak{n} \subset R$,

(1) we have $d > \dim_{\kappa(\mathfrak{n})} \Gamma(W_{\kappa(\mathfrak{n})}, \mathscr{O}_{W_{\kappa(\mathfrak{m})}})$;

(2) for every maximal ideal $\mathfrak{n}' \subset \Gamma(Y_1', \mathscr{O}_{Y_1'})$ lying over $\mathfrak{n}$ and every $n \geqslant d$, there are at least $\deg(Y_1'/R)$ (resp., at least one) closed point(s) on $\mathbb{A}^1_{\kappa(\mathfrak{n})}$ (resp., on $\mathbb{A}^1_{\kappa(\mathfrak{n}')}$) of exact degree n.

For every maximal ideal $\mathfrak{n}' \subset \Gamma(Y_1', \mathscr{O}_{Y_1'})$ we choose a monic polynomial $h_{\mathfrak{n}'} \in \kappa(\mathfrak{n}')[u]$ of degree $2d+1$ such that:

(i) if $\kappa(\mathfrak{n}')$ is finite, $h_{\mathfrak{n}'}$ is a product of two irreducible polynomials of degrees d and $d+1$, respectively (which is possible by (2));

(ii) if $\kappa(\mathfrak{n}')$ is infinite, $h_{\mathfrak{n}'}$ is a separable polynomial and has at least one root in $\kappa(\mathfrak{n}')$.

Let $h \in \Gamma(Y_1', \mathscr{O}_{Y_1'})[u]$ be a common monic lifting of $h_{\mathfrak{n}'}$ for all $\mathfrak{n}' \subset \Gamma(Y_1', \mathscr{O}_{Y_1'})$, and define

$$Y_1 = \mathrm{Spec}\left(\frac{\Gamma(Y_1', \mathscr{O}_{Y_1'})[u]}{(h)}\right);$$

it is finite, étale over Y_1', and hence also over R_1. By (1)-(2), there is a closed immersion

$$\bigsqcup_{\mathfrak{n} \subset R} (Y_1)_{\kappa(\mathfrak{n})} \hookrightarrow \mathbb{A}^1_R \backslash W \quad \text{over } R;$$

by Nakayama's lemma, any of its lifting $Y_1 \hookrightarrow \mathbb{A}^1_R \backslash W$ over R (which exists by Chinese Remainder theorem) is also a closed immersion. By construction, the restriction of $(G_1)_{Y_1'}$ to every connected component of Y_1' contains a proper parabolic subgroup scheme. Thus, by [SGA 3$_{\text{III new}}$, Exposé XXVI, Corollaire 6.12], $(G_1)_{Y_1'}$ contains $\mathbb{G}_{m,Y_1'}$, and so $(G_1)_{Y_1}$ contains $\mathbb{G}_{m,Y_1}$. By (i)-(ii), for $\mathfrak{m} \subset R_1$ with $(G_1)_{\kappa(\mathfrak{m})}$ isotropic, the line bundle $\mathscr{O}_{\mathbb{P}^1_{\kappa(\mathfrak{m})}}(1)$ is trivial over $\mathbb{P}^1_{\kappa(\mathfrak{m})} \backslash (Y_1)_{\kappa(\mathfrak{m})}$.

To construct Y', it suffices to produce, for a large enough d, a R-finite, étale, closed subschemes $Y_2 \subset \mathbb{A}^1_R$ of R-degrees d and $d+1$ which are disjoint from W, and then take $Y' := Y_1 \bigsqcup Y_2$. To achieve this, one just need to imitate the above procedure for constructing Y_1 from Y_1'. Details are omitted. □

Proof of Theorem 10.1. By the reduction Lemma 10.3, we may assume throughout that $C = \mathbb{A}^1_R$ and G is a reductive R-group scheme. Up to shifting we may assume that $s = 0_R \in \mathbb{A}^1_R(R)$ is the zero section, and base changing to A reduces us further to the case $A = R$ at the cost that R need not be a domain or geometrically unibranch. Thus, in case (i), our R is semilocal, and, in case (ii), our G is totally isotropic (but R need not be semilocal). By decomposing $\mathrm{Spec}(R)$ into connected components, we can assume that R has connected spectrum.

For both cases (i)-(ii), by glueing $\mathcal{P}$ with the trivial G-torsor over $\mathbb{P}^1_R \backslash Z$ we extend $\mathcal{P}$ to a G-torsor $\mathcal{Q}$ over $\mathbb{P}^1_R$. By [Fed22b, Proposition 2.3] or [Čes22b, Lemma 5.3.5], up to replacing $\mathcal{Q}$ and Z by their pullbacks by $\mathbb{P}^1_R \to \mathbb{P}^1_R, t \mapsto t^d$, where d is divisible by the R-fibral degree of the simply-connected central cover $(G^{\mathrm{ad}})^{\mathrm{sc}} \to G^{\mathrm{ad}}$, we may assume that for every maximal ideal $\mathfrak{m} \subset R$ the G^{ad}-torsor over $\mathbb{P}^1_{\kappa(\mathfrak{m})}$ induced by $\mathcal{Q}$ lifts to a generically trivial $(G^{\mathrm{ad}})^{\mathrm{sc}}$-torsor over $\mathbb{P}^1_{\kappa(\mathfrak{m})}$.

Claim 10.5.1. In both cases (i)-(ii), assume that R is semilocal. For any R-finite closed subscheme $W_0 \subset \mathbb{A}^1_R$, there exists a R-finite, étale, closed subscheme $Y \subset \mathbb{A}^1_R \backslash W_0$ such that $\mathcal{Q}|_{\mathbb{P}^1_R \backslash Y}$ is trivial.

Proof of the claim. We write the canonical decomposition of G^{ad} as in Proposition 10.4. Replacing W_0 by $W_0 \cup Z$, we may assume that $Z \subset W_0$. Applying Lemma 10.5 separately to each simple R_i-group

scheme G_i (with appropriate choices of W's), we get R_i-finite, étale schemes Y_i such that $(G_i)_{Y_i}$ is totally isotropic, a closed immersion $\bigsqcup_i Y_i \hookrightarrow \mathbb{A}^1_R \backslash W_0$ over R such that for every maximal ideal $\mathfrak{m} \subset R_i$ with $(G_i)_{\kappa(\mathfrak{m})}$ isotropic, the line bundle $\mathcal{O}_{\mathbb{P}^1_{\kappa(\mathfrak{m})}}(1)$ is trivial over $\mathbb{P}^1_{\kappa(\mathfrak{m})} \backslash (Y_i)_{\kappa(\mathfrak{m})}$. Applying the second part of Lemma 10.5 to $W := (\sqcup_i Y_i) \bigsqcup W_0$, we get a R-finite, étale, closed subscheme

$$Y' \subset \mathbb{A}^1_R \backslash \Big((\sqcup_i Y_i) \bigsqcup W_0 \Big)$$

such that $\mathscr{O}_{\mathbb{P}^1_R}(1)$ is trivial over $\mathbb{P}^1_R \backslash Y'$. Let $Y := Y' \bigsqcup (\sqcup_i Y_i)$. Then all the assumptions (i)-(iii) of Proposition 10.4 are verified, so we conclude that $\mathcal{Q}|_{\mathbb{P}^1_R \backslash Y}$ is trivial. □

For (i), we take $W_0 = Z \cup 0_R$, then the above Claim 10.5.1 gives a R-finite, étale, closed subscheme $Y \subset \mathbb{A}^1_R \backslash W_0$ such that $\mathcal{Q}|_{\mathbb{P}^1_R \backslash Y}$ is trivial. Since $Y \cap 0_R = \varnothing$, we deduce that the pullback of $\mathcal{Q}$ along $s = 0_R$ is also trivial, as wanted.

For (ii), we will follow [Čes22c, Lemma 4.3] to show that both $\mathcal{P} = \mathcal{Q}|_{\mathbb{A}^1_R}$ and $\mathcal{Q}|_{\mathbb{P}^1_R \backslash 0_R}$ descend to G-torsors over R, and then we are done: both of these descendants agree with the restriction of $\mathcal{Q}$ along $1_R \in \mathbb{A}^1_R(R)$, so they agree with the restriction of $\mathcal{Q}$ along ∞_R, which is trivial, and hence they must be trivial. By Quillen patching [Čes22b, Corollary 5.1.5 (b)], for the descent claim we may replace R by its localizations at maximal ideals to assume that R is local.

Now, since R is local, we may apply the above Claim 10.5.1 to $W_0 = 0_R$ to find a R-finite, étale, closed subscheme $Z' \subset \mathbb{A}^1_R \backslash 0_R$ such that $\mathcal{Q}|_{\mathbb{P}^1_R \backslash Z'}$ is trivial. It remains to apply Proposition 10.4 twice, with $Y = 0_R$ and $Y = \infty_R$ respectively, to show that both $\mathcal{Q}|_{\mathbb{P}^1_R \backslash 0_R}$ and $\mathcal{Q}|_{\mathbb{P}^1_R \backslash \infty_R}$ are trivial. □

11. Torsors under a reductive group scheme over a smooth projective base

The main result of this section is the following:

Theorem 11.1. *For a semilocal Prüfer domain R, an $r \in R \backslash \{0\}$, an irreducible, smooth, projective R-scheme X, a finite subset $\boldsymbol{x} \subset X$ with semilocal ring $A := \mathscr{O}_{X,\boldsymbol{x}}$, and a reductive X-group scheme G,*

(i) *any generically trivial G-torsor over A is trivial, that is,*

$$\ker\left(H^1(A, G) \to H^1(\operatorname{Frac} A, G)\right) = \{*\};$$

(ii) *if $G_{A[\frac{1}{r}]}$ is totally isotropic, then any generically trivial G-torsor over $A[\frac{1}{r}]$ is trivial, that is,*

$$\ker\left(H^1(A[\tfrac{1}{r}], G) \to H^1(\operatorname{Frac} A, G)\right) = \{*\}$$

The case (i) is a version of the Grothendieck–Serre conjecture in the case the relevant reductive group scheme G_A has a reductive model over some smooth projective compactification of $\operatorname{Spec}(A$. The case (ii) provides a version of Nisnevich conjecture for such 'nice' reductive groups satisfying the total isotropicity assumption: if R is a discrete valuation ring with uniformizer r and if $R \to A$ is a local homomorphism of local rings, then $r \in \mathfrak{m}_A \backslash \mathfrak{m}_A^2$, and (ii) says that any generically trivial G-torsor over $A[\frac{1}{r}]$ is trivial (the isotropicity assumption on G_A is essential, see, for instance, [Fed21]).

Remark 11.2. An inspection of the proof below shows that, if $X^{\mathrm{ns}} \subset X$ denotes the loci where a finitely presented morphism $X \to \operatorname{Spec}(R)$ is non-smooth, then Theorem 11.1 still holds provided that X is only a flat projective R-scheme such that X^{ns} is R-fiberwise of codimension $\geqslant 2$ in X, $\mathbf{x} \cap X^{\mathrm{ns}} = \varnothing$, and G is a reductive $X \backslash X^{\mathrm{ns}}$-group scheme.

To prove Theorem 11.1, we first derive from Corollary 6.3.2 and Lemma 7.1.1 the following key result, which reduces the proof of Theorem 11.1 to studying torsors on a smooth affine relative curve.

Lemma 11.3. *For a semilocal Prüfer domain R of finite Krull dimension, an irreducible, smooth, projective R-scheme X of pure relative dimension $d > 0$, a finite subset $\boldsymbol{x} \subset X$, and a reductive X-group scheme G, the following assertions hold.*

(i) *Given a generically trivial G-torsor $\mathcal{P}$ over $A := \mathscr{O}_{X,\boldsymbol{x}}$, there are*

- *a smooth, affine A-curve C, an A-finite closed subscheme $Z \subset C$, and a section $s \in C(A)$;*
- *a reductive C-group scheme $\mathscr{G}$ satisfying $s^*\mathscr{G} \simeq G_A$ and a $\mathscr{G}$-torsor $\mathcal{F}$ such that $\mathcal{F}|_{C \backslash Z}$ is trivial and $s^*\mathcal{F} \simeq \mathcal{P}$.*

(ii) *Given an $r \in R\backslash\{0\}$ and a generically trivial G-torsor $\widetilde{\mathcal{P}}$ over $A[\frac{1}{r}]$, there are*

- *a smooth, affine A-curve C, an A-finite closed subscheme $Z \subset C$, and a section $s \in C(A)$;*
- *a reductive C-group scheme $\mathscr{G}$ such that $s^*\mathscr{G} \simeq G_A$, a $\mathscr{G}$-torsor $\widetilde{\mathcal{F}}$ over $C[\frac{1}{r}] := C \times_A A[\frac{1}{r}]$ such that $\widetilde{\mathcal{F}}|_{C[\frac{1}{r}]\backslash Z[\frac{1}{r}]}$ is trivial and $(s|_{A[\frac{1}{r}]})^*(\widetilde{\mathcal{F}}) \simeq \widetilde{\mathcal{P}}$.*

Proof. By Corollary 6.3.2, $\mathcal{P}$ (resp., $\widetilde{\mathcal{P}}$) extends to a G-torsor $\mathcal{P}_0$ (resp., $\widetilde{\mathcal{P}_0}$) over an open neighbourhood $W \subset X$ of $\mathbf{x}$ (resp., an open neighbourhood $\widetilde{W} \subset X$ of $\mathrm{Spec}(A[\frac{1}{r}])$) such that

$$\mathrm{codim}((X\backslash W)_K, X_K) \geqslant 3 \quad \text{and} \quad \mathrm{codim}((X\backslash W)_s, X_s) \geqslant 2 \text{ for all } s \in \mathrm{Spec}(R);$$

and

$$\mathrm{codim}((X\backslash \widetilde{W})_K, X_K) \geqslant 3 \quad \text{and} \quad \mathrm{codim}((X\backslash \widetilde{W})_s, X_s) \geqslant 2 \text{ for all } s \in \mathrm{Spec}(R).$$

Here, K is the fraction field of R. Let $\mathbf{z} \subset X$ be the set of maximal points of the R-fibers of X; the above codimension bounds implies $\mathbf{z} \subset W$ (resp., $\mathbf{z} \subset \widetilde{W}$). By Lemma 3.1.1(iii), the semilocal ring $\mathscr{O}_{X,\mathbf{z}}$, and hence also $\mathscr{O}_{X,\mathbf{z}}[\frac{1}{r}]$, is a Prüfer domain. By the Grothendieck–Serre on semilocal Prüfer schemes (Theorem 9.0.1), the generically trivial G-torsor $(\mathcal{P}_0)|_{\mathscr{O}_{X,\mathbf{z}}}$ (resp., $(\widetilde{\mathcal{P}_0})|_{\mathscr{O}_{X,\mathbf{z}}[\frac{1}{r}]}$) is actually trivial. Thus there exists a closed subscheme $Y \subset X$ (resp., $\widetilde{Y} \subset X$) that avoids all the maximal points of R-fibers of X such that the restriction $(\mathcal{P}_0)|_{X\backslash Y}$ (resp., $(\widetilde{\mathcal{P}_0})|_{(X\backslash \widetilde{Y})[\frac{1}{r}]}$) is trivial; such a Y (resp., $\widetilde{Y}$) is R-fiberwise of codimension > 0 in X. Now, we treat the two cases (i)–(ii) separately.

(i) By the above, $X\backslash W$ is R-fiberwise of codimension $\geqslant 2$ in X; *a fortiori,* the same codimension bound holds for $Y\backslash W$ in X. Consequently, we can apply Lemma 7.1.1 (vii) to obtain an affine open $S \subset \mathbb{A}_R^{d-1}$, an affine open neighbourhood $U \subset W$ of $\mathbf{x}$, and a smooth morphism $\pi\colon U \to S$ of pure relative dimension 1 such that $U \cap Y$ is S-finite.

Let $\tau : C := U \times_S \mathrm{Spec}\, A \to \mathrm{Spec}\, A$ be the base change of π to $\mathrm{Spec}\, A$. Let Z and $\mathcal{F}$ be the pullbacks of $U \cap Y$ and $(\mathcal{P}_0)|_U$ under $\mathrm{pr}_1 : C \to U$, respectively. Then, via τ, C is a smooth affine A-curve, $Z \subset C$ is a A-finite closed subscheme, and $\mathcal{F}$ is a $\mathscr{G} := \mathrm{pr}_1^*(G_U)$-torsor that trivializes over $C\backslash Z$. Finally, the diagonal in C induces a section $s \in C(A)$ with $s^*\mathcal{F} \simeq \mathcal{P}$ (as $s^*\mathscr{G} = G_A$-torsors).

(ii) Since $\mathrm{Spec}(A[\frac{1}{r}])$ consists of points of $X[\frac{1}{r}] := X \times_R R[\frac{1}{r}]$ that specializes to some point of $\mathbf{x}$, we deduce from the inclusion $\mathrm{Spec}(A[\frac{1}{r}]) \subset \widetilde{W}$ that no points of $(X\backslash\widetilde{W})[\frac{1}{r}] = X[\frac{1}{r}]\backslash\widetilde{W}[\frac{1}{r}]$ specializes to any points of $\mathbf{x}$. Hence, the closure $\overline{(X\backslash\widetilde{W})[\frac{1}{r}]}$ (in X) is disjoint from $\mathbf{x}$, so $\widetilde{W}' := X\backslash\overline{(X\backslash\widetilde{W})[\frac{1}{r}]}$ is an open neighbourhood of $\mathbf{x}$. Notice that X is topological Noetherian, because its R-fibers are projective varieties over fields, and by our assumption $\mathrm{Spec}(R)$ has a finite underlying space. Since by the above $(X\backslash\widetilde{W})[\frac{1}{r}]$ is $R[\frac{1}{r}]$-fiberwise of codimension $\geqslant 2$ in $X[\frac{1}{r}]$, by Lemma 3.1.1(i) applied to the closures of the (finitely many) maximal points of $(X\backslash\widetilde{W})[\frac{1}{r}]$, the closure $\overline{(X\backslash\widetilde{W})[\frac{1}{r}]} = X\backslash\widetilde{W}'$ is R-fiberwise of codimension $\geqslant 2$ in X; *a fortiori*, the same holds for $\widetilde{Y}\backslash\widetilde{W}'$ in X. Consequently, we can apply Lemma 7.1.1 (vii) to obtain an affine open $\widetilde{S} \subset \mathbb{A}_R^{d-1}$, an affine open neighbourhood $\widetilde{U} \subset \widetilde{W}'$ of $\mathbf{x}$, and a smooth morphism $\widetilde{\pi} : \widetilde{U} \to \widetilde{S}$ of pure relative dimension 1 such that $\widetilde{U} \cap \widetilde{Y}$ is $\widetilde{S}$-finite. Notice that $\widetilde{U}[\frac{1}{r}] \subset \widetilde{W}'[\frac{1}{r}] = \widetilde{W}[\frac{1}{r}]$, so we have the restriction $(\widetilde{\mathcal{P}_0})|_{\widetilde{U}[\frac{1}{r}]}$.

Let $\tau : C := \widetilde{U} \times_{\widetilde{S}} \mathrm{Spec}(A \to \mathrm{Spec}(A$ be the base change of $\widetilde{\pi}$ to $\mathrm{Spec}(A$. Let Z be the pullback of $\widetilde{U} \cap \widetilde{Y}$ under $\mathrm{pr}_1 : C \to \widetilde{U}$. Let $\widetilde{\mathcal{F}}$ be the pullback of $(\widetilde{\mathcal{P}_0})|_{\widetilde{U}[\frac{1}{r}]}$ under $\mathrm{pr}_1 : C[\frac{1}{r}] \to \widetilde{U}[\frac{1}{r}]$. Then, via τ, C is a smooth affine A-curve, $Z \subset C$ is a A-finite closed subscheme, and $\widetilde{\mathcal{F}}$ is a $\mathscr{G} := \mathrm{pr}_1^*(G_{\widetilde{U}})$-torsor over $C[\frac{1}{r}]$ that trivializes over $C[\frac{1}{r}]\backslash Z[\frac{1}{r}]$. Finally, the diagonal in C induces a section $s \in C(A)$ with $s^*_{A[\frac{1}{r}]}(\widetilde{\mathcal{F}}) \simeq \widetilde{\mathcal{P}}$, and $s^*\mathscr{G} = G_A$. □

Proof of Theorem 11.1. By a standard limit argument involving Lemma 3.1.3, one easily reduces to the case when R has finite Krull dimension. Now, let $\mathcal{P}$ (resp., $\widetilde{\mathcal{P}}$) be a generically trivial G-torsor over $A := \mathscr{O}_{X,\mathbf{x}}$ (resp., over $A[\frac{1}{r}]$) which we want to trivialize. Let d be the relative dimension of X over R. If $d = 0$, then A and $A[\frac{1}{r}]$ are semilocal Prüfer domains, so, by the Grothendieck–Serre on semilocal Prüfer schemes (Theorem 9.0.1), the torsors $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ are trivial. Hence we may assume that $d > 0$. Then,

by Lemma 11.3, there are a smooth, affine A-curve C, an A-finite closed subscheme $Z \subset C$, a section $s \in C(A)$, a reductive C-group scheme $\mathscr{G}$ with $s^*\mathscr{G} \simeq G_A$,

- a $\mathscr{G}$-torsor $\mathcal{F}$ over C that trivializes over $C\backslash Z$ such that $s^*\mathcal{F} \simeq \mathcal{P}$, and
- a $\mathscr{G}$-torsor $\widetilde{\mathcal{F}}$ over $C[\frac{1}{r}]$ that trivializes over $C[\frac{1}{r}]\backslash Z[\frac{1}{r}]$ such that $(s|_{A[\frac{1}{r}]})^*(\widetilde{\mathcal{F}}) \simeq \widetilde{\mathcal{P}}$.

By Theorem 10.1 (i), the G-torsor $s^*\mathcal{F} \simeq \mathcal{P}$ is trivial. By Theorem 10.1 (ii), in the case $(s|_{A[\frac{1}{r}]})^*(\mathscr{G}) \simeq G_{A[\frac{1}{r}]}$ is totally isotropic, the $G_{A[\frac{1}{r}]}$-torsor $(s|_{A[\frac{1}{r}]})^*(\widetilde{\mathcal{F}}) \simeq \widetilde{\mathcal{P}}$ is trivial. □

12. Torsors under a constant reductive group scheme

In this section we prove the following variant of Theorem 11.1, in which the R-smooth scheme X need not be proper, but the reductive group scheme G is supposed to descend to the Prüfer ring R. Thus, we established the Grothendieck–Serre conjecture and a version of Nisnevich conjecture for 'constant' reductive group schemes. As for the proof, we use a variant of Lindel's Lemma (Proposition 7.2.1) and glueing techniques to reduce to the case already settled by Theorem 11.1.

Theorem 12.1. *For a semilocal Prüfer domain R, a nonzero element $r \in R$, an irreducible affine R-smooth scheme X, a finite subset $\boldsymbol{x} \subset X$, and a reductive R-group scheme G,*

(i) *any generically trivial G-torsor over $A := \mathscr{O}_{X,\boldsymbol{x}}$ is trivial, that is,*

$$\ker\big(H^1(A,G) \to H^1(\operatorname{Frac} A, G)\big) = \{*\};$$

(ii) *if $G_{R[\frac{1}{r}]}$ is totally isotropic, then any generically trivial G-torsor over $A[\frac{1}{r}]$ is trivial, that is,*

$$\ker\big(H^1(A[\tfrac{1}{r}],G) \to H^1(\operatorname{Frac} A, G)\big) = \{*\}.$$

Proof. Let $\mathcal{P}$ (resp., $\widetilde{\mathcal{P}}$) be a generically trivial G-torsor over A (resp., over $A[\frac{1}{r}]$). By shrinking X around $\mathbf{x}$, we may assume that $\mathcal{P}$ is defined over the whole X (resp., $\widetilde{\mathcal{P}}$ is defined over the whole $X[\frac{1}{r}] := X \times_R R[\frac{1}{r}]$). Let d be the relative dimension of X over R. As noted by Česnavičius, since it suffices to argue that $\mathcal{P}$ (resp., $\widetilde{\mathcal{P}}$) is trivial Zariski semilocally on X, we may replace X by $X \times_R \mathbb{A}^N_R$ for large N to assume that $d > \#\mathbf{x}$: by pulling back along the zero section $X \to X \times_R \mathbb{A}^N_R$, the Zariski semilocal triviality of $\mathcal{P}_{X\times_R\mathbb{A}^N_R}$ (resp., $\widetilde{\mathcal{P}}_{X[\frac{1}{r}]\times_R\mathbb{A}^N_R}$) on $X \times_R \mathbb{A}^N_R$ implies that of $\mathcal{P}$ (resp., $\widetilde{\mathcal{P}}$) on X.

By specialization, we may assume that each point of $\mathbf{x}$ is closed in the *corresponding* R-fiber of X (but not necessarily lies in the closed R-fibers of X). Our goal is to show that $\mathcal{P}|_A$ (resp., $\widetilde{\mathcal{P}}|_{A[\frac{1}{r}]}$) is trivial.

If $d = 0$, then A (resp., $A[\frac{1}{r}]$) is a semilocal Prüfer domain, so, by the Grothendieck–Serre conjecture on semilocal Prüfer schemes (Theorem 9.0.1), the torsor $\mathcal{P}|_A$ (resp., $\widetilde{\mathcal{P}}|_{A[\frac{1}{r}]}$) is trivial. Thus we may assume that $d > 0$ for what follows.

Denote by $\pi : X \to S := \operatorname{Spec}(R)$ the structural morphism. Let $\mathbf{y}$ be the set of maximal points of the R-fibers of X.

Claim 12.1.1. No points of $\mathbf{x}$ specializes to any point of $\mathbf{y}$, that is, $\overline{\mathbf{x}} \cap \mathbf{y} = \varnothing$.

Proof of the claim. By Lemma 3.1.1(iii), for any $y \in \mathbf{y}$, $\mathscr{O}_{X,y}$ is a valuation ring having the same value group as $\mathscr{O}_{S,\pi(y)}$; in particular, the map $\pi_y : \operatorname{Spec} \mathscr{O}_{X,y} \to \operatorname{Spec} \mathscr{O}_{S,\pi(y)}$ induced by π is a homeomorphism, and is thus injective. Assume by contradiction that $x \in \mathbf{x}$ specializes to $y \in \mathbf{y}$, so $\operatorname{Spec} \mathscr{O}_{X_{\pi(x)},x}$ is a subset of $\operatorname{Spec} \mathscr{O}_{X,y}$. Since the image of $\operatorname{Spec} \mathscr{O}_{X_{\pi(x)},x}$ under π_y is the singleton $\{\pi(x)\}$, by the injectivity of π_y, we deduce that $\dim \mathscr{O}_{X_{\pi(x)},x} = 0$. This contradicts the fact $\dim \mathscr{O}_{X_{\pi(x)},x} = d > 0$ (because by our assumption x is a closed point in the corresponding π-fiber). □

By Lemma 3.1.1(iii) again, the semilocal ring $\mathscr{O}_{X,\mathbf{y}}$, and hence also $\mathscr{O}_{X,\mathbf{y}}[\frac{1}{r}]$, is a Prüfer domain, so, by the Grothendieck–Serre conjecture on semilocal Prüfer schemes (Theorem 9.0.1), the generically trivial G-torsor $\mathcal{P}|_{\mathscr{O}_{X,\mathbf{y}}}$ (resp., $\widetilde{\mathcal{P}}|_{\mathscr{O}_{X,\mathbf{y}}[\frac{1}{r}]}$) is actually trivial. Therefore, using the above claim and prime avoidance, we can find an element $a \in \Gamma(X, \mathscr{O}_X)$ such that, denoting $Y := V(a) \subset X$, then $\mathbf{x} \subset Y$, $\mathbf{y} \cap Y = \varnothing$, and the restriction $\mathcal{P}|_{X\backslash Y}$ (resp., $\widetilde{\mathcal{P}}|_{(X\backslash Y)[\frac{1}{r}]}$) is trivial. (We just take $a = a_1a_2$, where a_1 is an element

such that $\mathbf{y} \cap V(a_1) = \varnothing$ and $\mathcal{P}|_{X\setminus V(a_1)}$ (resp., $\widetilde{\mathcal{P}}|_{(X\setminus V(a_1))[\frac{1}{r}]}$) is trivial, and a_2 is delivered from prime avoidance utilizing the fact $\overline{\mathbf{x}} \cap \mathbf{y} = \varnothing$ so that $\mathbf{x} \subset V(a_2)$ and $\mathbf{y} \cap V(a_2) = \varnothing$.)

Since $d > \#\,\mathbf{x}$, we may apply Proposition 7.2.1 to obtain an affine open neighbourhood $W \subset X$ of $\mathbf{x}$, an affine open subscheme $U \subset \mathbb{A}^d_R$, and an étale surjective R-map $f : W \to U$ such that the restriction $f|_{W\cap Y}$ is a closed immersion and f induces a Cartesian square

$$\begin{array}{ccc} W \cap Y & \hookrightarrow & W \\ \| & & \downarrow f \\ W \cap Y & \hookrightarrow & U. \end{array}$$

Applying $(-) \times_R R[\frac{1}{r}]$ yields a similar Cartesian square. By glueing Lemma 6.2.2 (ii),

(i) we may (non-canonically) glue $\mathcal{P}|_W$ and the trivial G-torsor over $U\setminus f(W \cap Y)$ to descend $\mathcal{P}|_W$ to a G-torsor $\mathcal{Q}$ over U that trivializes over $U\setminus f(W \cap Y)$. Since U has a smooth, projective compactification $\mathbb{P}^d_R$, we may apply Theorem 11.1 (i) to deduce that $\mathcal{Q}|_{\mathscr{O}_{U,f(\mathbf{x})}}$ is trivial, so $\mathcal{P}|_A = \mathcal{P}|_{\mathscr{O}_{W,\mathbf{x}}}$ is trivial, as desired.

(ii) we may (non-canonically) glue $\widetilde{\mathcal{P}}|_{W[\frac{1}{r}]}$ and the trivial G-torsor over $(U\setminus f(W \cap Y))[\frac{1}{r}]$ to descend $\widetilde{\mathcal{P}}|_{W[\frac{1}{r}]}$ to a G-torsor $\widetilde{\mathcal{Q}}$ over $U[\frac{1}{r}]$ that trivializes over $U[\frac{1}{r}]\setminus f(W \cap Y)[\frac{1}{r}]$. Since U has a smooth, projective compactification $\mathbb{P}^d_R$, we may apply Theorem 11.1 (ii) to conclude that $\widetilde{\mathcal{Q}}|_{\mathscr{O}_{U,f(\mathbf{x})}[\frac{1}{r}]}$ is trivial, so $\widetilde{\mathcal{P}}|_{A[\frac{1}{r}]} = \widetilde{\mathcal{P}}|_{\mathscr{O}_{W,\mathbf{x}}[\frac{1}{r}]}$ is trivial, as desired. □

13. Torsors under a quasi-split reductive group scheme

In this section we study generically trivial torsors under quasi-split reductive group schemes. The main result is the following Theorem 13.1, in which (i) is a version of Nisnevich conjecture that is inspired by the recent preprint of Česnavičius [Čes22c, Theorem 1.3 (2)], who proved it in the case R is a Dedekind domain, and (ii) is the Grothendieck–Serre conjecture over one-dimensional Prüfer bases. As for the proof, we will follow the strategy of [Čes22a] (with its earlier version given by Fedorov [Fed22b]), which goes through because the main tools, such as toral version of purity (Proposition 8.2.5) and the Grothendieck–Serre conjecture (Proposition 8.3.2(i)) in our context, are available now.

Theorem 13.1. *For a semilocal Prüfer domain R with fraction field K, an irreducible, semilocal, and essentially smooth R-algebra A, and a quasi-split reductive A-group scheme G,*

(i) *every generically trivial G-torsor over $A \otimes_R K$ is trivial, that is,*

$$\ker\left(H^1(A \otimes_R K, G) \to H^1(\operatorname{Frac} A, G)\right) = \{*\};$$

(ii) *if R has Krull dimension 1, then every generically trivial G-torsor is trivial, that is,*

$$\ker\left(H^1(A, G) \to H^1(\operatorname{Frac} A, G)\right) = \{*\}.$$

We start with the following consequence of Lemma 7.1.1, which is the key geometric input permitting a series of reductions that eventually lead to Theorem 13.1.

Lemma 13.2 (*cf.* [Čes22a, Proposition 4.1])**.** *For*

(i) *a semilocal Prüfer domain R of Krull dimension 1 with fraction field K;*

(ii) *a smooth, faithfully flat, R-algebra A of pure relative dimension $d \geqslant 1$ over R;*

(iii) *a finite subset $\mathbf{x} \subset X := \operatorname{Spec} A$;*

(iv) *a closed subscheme $Y \subset X$ that satisfies*

$$\operatorname{codim}(Y_K, X_K) \geqslant 2 \quad \text{and} \quad \operatorname{codim}(Y_s, X_s) \geqslant 1 \text{ for all } s \in \operatorname{Spec} R;$$

there are an affine open $U \subset \operatorname{Spec} A$ containing $\mathbf{x}$, an affine open $S \subset \mathbb{A}^{d-1}_R$, and a smooth R-morphism $\pi : U \to S$ of relative dimension 1 such that $Y \cap U$ is S-finite.

Moreover, if in (i) *R is allowed to be of arbitrary finite Krull dimension, then the same conclusion holds provided* (iv) *is replaced by the stronger assumption that Y is R-fiberwise of codimension $\geqslant 2$ in X.*

Proof. Choosing an embedding of X into some affine space over R and taking schematic closure in the corresponding projective space, we get a projective compactification $\overline{X}$ of X. Since $\overline{X}$ is flat and projective over R, by Lemma 3.1.1(i), all its R-fibers have the same dimension d. Denote by $\overline{Y} \subset \overline{X}$ the schematic closure of Y. To apply Lemma 7.1.1 (vii) and conclude, in which X is $\overline{X}$ here, W is X here, and Y is $\overline{Y}$ here, we need to check that the boundary $\overline{Y}\backslash Y$ is R-fiberwise of codimension $\geqslant 2$ in $\overline{X}$.

By [SP, 01R8], set-theoretically we have $\overline{Y} = \bigcup_y \overline{\{y\}}$, where y runs through the generic points of Y.

In the case Y is R-fiberwise of codimension $\geqslant 2$ in X, the same holds for $\overline{Y}$ in $\overline{X}$; *a fortiori,* $\overline{Y}\backslash Y$ is R-fiberwise of codimension $\geqslant 2$ in $\overline{X}$. Indeed, by Lemma 3.1.1(i), $\overline{X}$ has equal R-fiber dimension d and all non-empty R-fibers of $\overline{\{y\}}$ have the same dimension, so, if y lies over $s_y \in \operatorname{Spec} R$, then

$$\operatorname{codim}(\overline{\{y\}}_s, \overline{X}_s) = \operatorname{codim}(\overline{\{y\}}_{s_y}, \overline{X}_{s_y}) \geqslant 2 \quad \text{for any specialization } s_y \rightsquigarrow s \in \operatorname{Spec} R.$$

Next, we assume that R has Krull dimension 1 and Y is of codimension $\geqslant 2$ (resp., $\geqslant 1$) in the generic (resp., closed) R-fiber of X. If $y \in Y_\eta$, then, by Lemma 3.1.1(i) again, we see that

$$\operatorname{codim}(\overline{\{y\}}_s, \overline{X}_s) = \operatorname{codim}(\overline{\{y\}}_\eta, \overline{X}_\eta) \geqslant 2 \quad \text{for all } s \in \operatorname{Spec} R;$$

a fortiori, the contribution of such a y to the R-fiber codimension of $\overline{Y}\backslash Y$ in $\overline{X}$ is $\geqslant 2$.

Otherwise, y lies over a height 1 prime (i.e., a closed point) $s_1 \in \operatorname{Spec} R$, then $\overline{\{y\}}_{s_1} = \overline{\{y\}} \subset \overline{Y_{s_1}}$; by assumption $\operatorname{codim}(\overline{Y_{s_1}}, \overline{X}_{s_1}) = \operatorname{codim}(Y_{s_1}, X_{s_1}) \geqslant 1$, so we have $\operatorname{codim}(\overline{\{y\}}_{s_1}, \overline{X}_{s_1}) \geqslant 1$. But since the generic point y of $\overline{\{y\}}_{s_1}$ is not contained in $\overline{Y}\backslash Y$, we deduce that the contribution of such a y to the s_1-fiber codimension of $\overline{Y}\backslash Y$ in $\overline{X}$ is again $\geqslant 2$. □

Lemma 13.3 (Lifting the torsor to a smooth relative curve; *cf.* [Čes22a, Proposition 4.2])**.** *For a semilocal Prüfer domain R with fraction field K, the semilocalization A of an irreducible, R-smooth algebra A' at a finite subset $\boldsymbol{x} \subset \operatorname{Spec}(A')$, and a quasi-split reductive A-group scheme G with a Borel subgroup B,*

(1) given a generically trivial G-torsor P_K over $A_K := A \otimes_R K$, there are

- (i) *a smooth, affine relative A-curve C with a section $s \in C(A)$;*
- (ii) *an A-finite closed subscheme $Z \subset C$;*
- (iii) *a quasi-split reductive C-group scheme $\mathscr{G}$ with a Borel subgroup $\mathscr{B} \subset \mathscr{G}$ whose s-pullback is $B \subset G$, compatible with the quasi-pinnings;*
- (iv) *a $\mathscr{G}$-torsor $\mathcal{P}_K$ over $C_K := C \times_R K$ whose s_{A_K}-pullback is P_K such that $\mathcal{P}_K$ reduces to a $\operatorname{rad}^u(\mathscr{G})$-torsor over $C_K\backslash Z_K$ (here s_{A_K} stands for the image of s in $C(A_K)$).*

(2) if R has Krull dimension 1, given a generically trivial G-torsor P, then there are

- (i) *a smooth, affine relative A-curve C with a section $s \in C(A)$;*
- (ii) *an A-finite closed subscheme $Z \subset C$;*
- (iii) *a quasi-split reductive C-group scheme $\mathscr{G}$ with a Borel subgroup $\mathscr{B} \subset \mathscr{G}$ whose s-pullback is $B \subset G$, compatible with the quasi-pinnings;*
- (iv) *a $\mathscr{G}$-torsor $\mathcal{P}$ whose s-pullback is P such that $\mathcal{P}$ reduces to a $\operatorname{rad}^u(\mathscr{G})$-torsor over $C\backslash Z$.*

Proof. In case (1) we can first use a limit argument involving Lemma 3.1.3 to reduce to the case when R has finite Krull dimension.

If A' is of relative dimension 0 over R, then $A_K = \operatorname{Frac}(A)$ and A is a semilocal Prüfer domain. Thus, P_K is trivial, and, by the Grothendieck–Serre conjecture on semilocal Prüfer schemes (Theorem 9.0.1), P is also trivial. In this case we simply take $C = \mathbb{A}^1_A$, $s = 0 \in \mathbb{A}^1_A(A)$, $Z = \varnothing$, $(\mathscr{G}, \mathscr{B}) = (G_{\mathbb{A}^1_A}, B_{\mathbb{A}^1_A})$, and $\mathcal{P}_K = (P_K)_{\mathbb{A}^1_{A_K}}$ (resp., $\mathcal{P} = P_{\mathbb{A}^1_A}$). Thus, for what follows, we can assume that the relative dimension of A' over R is $d > 0$.

By spreading out and localizing A', we may assume that our quasi-split G (in particular, the Borel B) and torsor P all live over A', and P_K live over A'_K. By [SGA 3$_{\text{III new}}$, Exposé XXVI, Corollaire 3.6 and

Lemme 3.20], the quotient P_K/B_K (resp., P/B) is representable by a smooth projective scheme over A'_K (resp., over A'). Now we treat the cases (1)-(2) separately.

(1) By the generic triviality of P_K, applying the valuative criterion of properness to $P_K/B_K \to \mathrm{Spec}(A'_K)$ yields a closed subscheme $Y_K \subset \mathrm{Spec}(A'_K)$ of codimension $\geqslant 2$ such that $P_K/B_K \to \mathrm{Spec}(A'_K)$ has a section over $\mathrm{Spec}(A'_K)\backslash Y_K$ that lifts to a generic section of P_K. In other words, $(P_K)_{\mathrm{Spec}(A'_K)\backslash Y_K}$ reduces to a generically trivial $B_{\mathrm{Spec}(A'_K)\backslash Y_K}$-torsor P_K^B. Consider the A'-torus $T := B/\mathrm{rad}^u(B)$ and the induced T-torsor

$$P_K^T := P_K^B/\mathrm{rad}^u(B)_K \quad \text{over} \quad \mathrm{Spec}(A'_K)\backslash Y_K.$$

Since P_K^T is generically trivial, by Corollary 6.3.2, it extends to a T-torsor $\widetilde{P_K^T}$ over $\mathrm{Spec}(A')\backslash F$ for a closed subscheme $F \subset \mathrm{Spec}(A')$ satisfying

$$\mathrm{codim}(F_K, \mathrm{Spec}(A')_K) \geqslant 2 \quad \text{and} \quad \mathrm{codim}(F_s, \mathrm{Spec}(A')_s) \geqslant 1 \text{ for all } s \in \mathrm{Spec}(R);$$

by purity for tori (Theorem 8.2.4), this torsor further extends to the whole $\mathrm{Spec}(A')$. As $\widetilde{P_K^T}$ is generically trivial, by the Grothendieck–Serre conjecture for tori (Proposition 8.3.2(i)), we may localize A' around $\mathbf{x}$ to assume that $\widetilde{P_K^T}$, and hence also P_K^T, is already trivial. In other words, $(P_K)_{\mathrm{Spec}(A'_K)\backslash Y_K}$ reduces to a $\mathrm{rad}^u(B)$-torsor over $\mathrm{Spec}(A'_K)\backslash Y_K$.

Denote by Y the schematic closure of Y_K in $\mathrm{Spec}(A')$; by Lemma 3.1.1(i), it is R-fiberwise of codimension $\geqslant 2$ in $\mathrm{Spec}(A')$. Applying Lemma 13.2 to the R-smooth algebra A' and the closed subscheme $Y \subset \mathrm{Spec}(A')$, we obtain an affine open $U \subset \mathrm{Spec}(A')$ containing $\mathbf{x}$, an affine open $S \subset \mathbb{A}_R^{d-1}$, and a smooth R-morphism $\pi : U \to S$ of relative dimension 1 such that $Y \cap U$ is S-finite.

Recall that A is the semilocal ring of U at $\mathbf{x}$. Denote

$$C := U \times_S \mathrm{Spec}\, A \quad \text{and} \quad Z := (Y \cap U) \times_S \mathrm{Spec}\, A.$$

Then C is a smooth affine relative A-curve, the diagonal in C induces a section $s \in C(A)$, and the closed subscheme $Z \subset C$ is A-finite. So (1)(i) and (1)(ii) hold. Let $\mathscr{B} \subset \mathscr{G}$ be the pullback of $B_U \subset G_U$ under the first projection $\mathrm{pr}_1 : C \to U$, and let $\mathcal{P}_K$ be the pullback of $(P_K)_{U_K}$ under the first projection $\mathrm{pr}_1 : C_K \to U_K$. Then, $\mathcal{P}_K$ is a $\mathscr{G}$-torsor over C_K, and, by construction, the s-pullback (resp., s_{A_K}-pullback) of $\mathscr{B} \subset \mathscr{G}$ (resp., of $\mathcal{P}_K$) is $B \subset G$ (resp., P_K). Finally, since P_K reduces to a $\mathrm{rad}^u(B)$-torsor over $\mathrm{Spec}(A'_K)\backslash Y_K$, $\mathcal{P}_K$ reduces to a $\mathrm{rad}^u(\mathscr{B})$-torsor over $C_K\backslash Z_K$. So (1)(iii) and (1)(iv) also hold.

(2) Recall that, by Lemma 3.1.1(iii), the local rings of all maximal points of R-fibers of $\mathrm{Spec}(A')$ are valuation rings. By the generic triviality of P, applying the valuative criterion of properness to $P/B \to \mathrm{Spec}(A')$ yields a closed subscheme $Y \subset \mathrm{Spec}(A')$, which avoids all the codimension 1 points of the generic fiber $\mathrm{Spec}(A'_K)$ and all the maximal points of R-fibers of $\mathrm{Spec}(A')$, such that $P/B \to \mathrm{Spec}(A')$ has a section over $\mathrm{Spec}(A')\backslash Y$ that lifts to a generic section of P. In other words, Y satisfies

$$\mathrm{codim}(Y_K, \mathrm{Spec}(A')_K) \geqslant 2 \quad \text{and} \quad \mathrm{codim}(Y_s, \mathrm{Spec}(A')_s) \geqslant 1 \text{ for all } s \in \mathrm{Spec}(R).$$

Therefore, $P_{\mathrm{Spec}(A')\backslash Y}$ reduces to a generically trivial $B_{\mathrm{Spec}(A')\backslash Y}$-torsor P^B. Consider the A'-torus $T := B/\mathrm{rad}^u(B)$ and the induced T-torsor

$$P^T := P^B/\mathrm{rad}^u(B) \quad \text{over} \quad \mathrm{Spec}(A')\backslash Y.$$

By purity for tori (Theorem 8.2.4), P^T extends to a T-torsor $\widetilde{P^T}$. As $\widetilde{P^T}$ is generically trivial, by the Grothendieck–Serre conjecture for tori (Proposition 8.3.2(i)), we may localize A' around $\mathbf{x}$ to assume that $\widetilde{P^T}$, and hence also P^T, is already trivial. In other words, $P_{\mathrm{Spec}(A')\backslash Y}$ reduces to a $\mathrm{rad}^u(B)_{\mathrm{Spec}(A')\backslash Y}$-torsor.

Now, applying Lemma 13.2 to the R-smooth algebra A' and the closed subscheme $Y \subset \mathrm{Spec}(A')$, we obtain an affine open $U \subset \mathrm{Spec}(A')$ containing $\mathbf{x}$, an affine open $S \subset \mathbb{A}_R^{d-1}$, and a smooth R-morphism $\pi : U \to S$ of relative dimension 1 such that $Y \cap U$ is S-finite.

Recall that A is the semilocal ring of U at $\mathbf{x}$. Denote

$$C := U \times_S \mathrm{Spec}\, A \quad \text{and} \quad Z := (Y \cap U) \times_S \mathrm{Spec}\, A.$$

Then C is a smooth affine relative A-curve, the diagonal in C induces a section $s \in C(A)$, and the closed subscheme $Z \subset C$ is A-finite. So (2)(i) and (2)(ii) hold. Let $\mathscr{B} \subset \mathscr{G}$ and $\mathcal{P}$ be the pullback of $B_U \subset G_U$ and P_U under the first projection $\mathrm{pr}_1 : C \to U$, respectively. Then, $\mathcal{P}$ is a $\mathscr{G}$-torsor over C, and, by construction, the s-pullback of $\mathscr{B} \subset \mathscr{G}$ and $\mathcal{P}$ are $B \subset G$ and P, respectively. Finally, since P reduces to

a $\mathrm{rad}^u(B)$-torsor over $\mathrm{Spec}(A')\backslash Y$, $\mathcal{P}$ reduces to a $\mathrm{rad}^u(\mathscr{B})$-torsor over $C\backslash Z$. So (2)(iii) and (2)(iv) also hold. □

Lemma 13.4 ([Čes22a, Lemma 5.2])**.** *For a semilocal ring A whose local rings are geometrically unibranch, an ideal $I \subset A$, reductive A-groups G and G' that on geometric A-fibers have the same type, fixed quasi-pinnings of G and G' extending Borel A-subgroup $B \subset G$ and $N' \subset G'$ and an A/I-group isomorphism*

$$\iota : G_{A/I} \xrightarrow{\sim} G'_{A/I} \quad \text{respecting the quasi-pinnings; in particular,} \quad \iota(B_{A/I}) = B'_{A/I},$$

there are

(i) *a faithfully flat, finite, étale A-algebra $\widetilde{A}$ equipped with an A/I-point $a : \widetilde{A} \twoheadrightarrow A/I$; and*

(ii) *an $\widetilde{A}$-group isomorphism $\tilde{\iota}$: $G_{\widetilde{A}} \xrightarrow{\sim} G'_{\widetilde{A}}$ respecting the quasi-pinnings such that $a^*(\tilde{\iota}) = \iota$.*

Notice that the original version [Čes22a, Proposition 5.1] assumed further A to be Noetherian, but the Noetherianess of A was not used anywhere in the proof.

Lemma 13.5 (Changing the relative curve C to equate $\mathscr{G}$ and G_C; *cf.* [Čes22a, Proposition 5.2])**.** *In the setting of Lemma 13.3, for both cases (1) and (2) we may replace C by an étale neighbourhood of* $\mathrm{im}(s)$ *to achieve further that $(\mathscr{G}, \mathscr{B}) = (G_C, B_C)$.*

Proof. Consider the semilocalization $\mathrm{Spec}(D)$ of C at the closed points of $\mathrm{im}(s) \cup Z$; since C is normal, all the local rings of D are geometrically unibranch. The image of the section $s : \mathrm{Spec}\, A \to \mathrm{Spec}(D)$ gives rise to a closed subscheme $\mathrm{Spec}(D/I) \subset \mathrm{Spec}(D)$. By the conclusion of Lemma 13.3, the restriction of $\mathscr{B}_D \subset \mathscr{G}_D$ and $B_D \subset G_D$ to $\mathrm{Spec}(D/I)$ agree with each other in a way compatible with their quasi-pinnings. Thus, by Lemma 13.4, there is a faithfully flat, finite, étale D-algebra $\widetilde{D}$, a point $\tilde{s} : \widetilde{D} \twoheadrightarrow D/I \simeq A$ lifting $s : D \twoheadrightarrow D/I \simeq A$ such that $\mathscr{B}_{\widetilde{D}} \subset \mathscr{G}_{\widetilde{D}}$ is isomorphic to $B_{\widetilde{D}} \subset G_{\widetilde{D}}$ compatibly with the fixed identification of $\tilde{s}$-pullbacks. We then spread out the finite étale morphism $\mathrm{Spec}(\widetilde{D}) \to \mathrm{Spec}(D)$ to a finite étale morphism $\widetilde{C} \to C'$ for an open $C' \subset C$ that contains $\mathrm{im}(s) \cup Z$, while preserving an $\tilde{s} \in \widetilde{C}(A)$, and an isomorphism between $\mathscr{B}_{\widetilde{C}} \subset \mathscr{G}_{\widetilde{C}}$ and $B_{\widetilde{C}} \subset G_{\widetilde{C}}$. Now it remains to replace C, s, Z and $\mathcal{P}_K$ (resp., $\mathcal{P}$) by $\widetilde{C}$, $\tilde{s}$, $Z \times_C \widetilde{C}$ and $(\mathcal{P}_K)_{\widetilde{C}_K}$ (resp., $\mathcal{P}_{\widetilde{C}}$). □

Lemma 13.6 (Changing the relative smooth curve C for descending to $\mathbb{A}^1_A$; [Čes22a, Proposition 6.5])**.** *In the setting of Lemma 13.3, for both cases (1) and (2), in addition to $(\mathscr{G}, \mathscr{B}) = (G_C, B_C)$, we may change C to achieve further that there is a flat A-map $C \to \mathbb{A}^1_A$ that maps Z isomorphically to a closed subscheme $Z' \subset \mathbb{A}^1_A$ with*

$$Z \simeq Z' \times_{\mathbb{A}^1_A} C.$$

Proof. Assume that, in both cases (1) and (2) of Lemma 13.3, we have achieved the conclusion of Lemma 13.5. We have the data of a smooth affine relative A-curve C, a section $s \in C(A)$, and an A-finite closed subscheme $Z \subset C$; replacing Z by $Z \cup \mathrm{im}(s)$, we may assume that s factors through Z. However, in general, the A-finite scheme Z may be too large to embed into $\mathbb{A}^1_A$. (For instance, if $R = k$ is a finite field, then Z can't be embedded into $\mathbb{A}^1_k$ as soon as $\sharp\, Z(k) > \sharp\, k$.) For this, we first apply Panin's 'finite fields tricks' [Čes22a, Proposition 7.4] to obtain a finite morphism $\widetilde{C} \to C$ that is étale at the points in $\tilde{Z} := \widetilde{C} \times_C Z$ such that s lifts to $\tilde{s} \in \widetilde{C}(A)$, and there are no finite fields obstruction to embedding $\tilde{Z}$ into $\mathbb{A}^1_A$ in the following sense: for every maximal ideal $\mathfrak{m} \subset A$,

$$\sharp \left\{ z \in \tilde{Z}_{\kappa(\mathfrak{m})} : [\kappa(z) : \kappa(\mathfrak{m})] = d \right\} < \sharp \left\{ z \in \mathbb{A}^1_{\kappa(\mathfrak{m})} : [\kappa(z) : \kappa(\mathfrak{m})] = d \right\} \quad \text{for every} \quad d \geqslant 1.$$

Then, by [Čes22a, Lemma 6.3], there are an affine open $C' \subset \widetilde{C}$ containing $\mathrm{im}(\tilde{s})$, a quasi-finite, flat A-map $C' \to \mathbb{A}^1_A$ that maps Z isomorphically to a closed subscheme $Z' \subset \mathbb{A}^1_A$ with

$$Z \simeq Z' \times_{\mathbb{A}^1_A} C'.$$

It remains to replace C by C', Z by $\tilde{Z}$, s by $\tilde{s}$, $\mathcal{P}_K$ by $(\mathcal{P}_K)_{C'_K}$ (resp., $\mathcal{P}$ by $\mathcal{P}_{C'}$). □

Lemma 13.7 (Descend to $\mathbb{A}^1_A$ via patching; *cf.* [Čes22a, Proposition 7.4])**.** *In the setting of Lemma 13.3, for both cases (1) and (2), we may achieve further that $(\mathscr{G}, \mathscr{B}) = (G_C, B_C)$, $C = \mathbb{A}^1_A$, and $s = 0 \in \mathbb{A}^1_A(A)$.*

Proof. By the reduction given in Lemma 13.6, we have a flat A-curve C, a section $s \in C(A)$, an A-finite closed subscheme $Z \subset C$, a quasi-finite, affine, flat A-map $C \to \mathbb{A}^1_A$ that maps Z isomorphically to a closed subscheme $Z' \subset \mathbb{A}^1_A$ with $Z = Z' \times_{\mathbb{A}^1_A} C$, and a G-torsor $\mathcal{P}_K$ over C_K whose s_{A_K}-pullback is P_K (resp., a G-torsor $\mathcal{P}$ over C whose s-pullback is P) and whose restriction to $C_K \backslash Z_K$ (resp., $C \backslash Z$) reduces to a $\mathrm{rad}^u(B)$-torsor. Now, since $Z = Z' \times_{\mathbb{A}^1_A} C \simeq Z'$, [Čes22a, Lemma 7.2] implies the pullback maps

$$H^1(\mathbb{A}^1_A \backslash Z', \mathrm{rad}^u(G)) \twoheadrightarrow H^1(C \backslash Z, \mathrm{rad}^u(G))$$

and

$$H^1(\mathbb{A}^1_{A_K} \backslash Z'_K, \mathrm{rad}^u(G)) \twoheadrightarrow H^1(C_K \backslash Z_K, \mathrm{rad}^u(G))$$

are surjective. Combining these, we see that $\mathcal{P}_K|_{C_K \backslash Z_K}$ (resp., $\mathcal{P}|_{C \backslash Z}$) descends to a G-torsor $\mathcal{Q}_K$ (resp., $\mathcal{Q}$) over $\mathbb{A}^1_{A_K} \backslash Z'_K$ (resp., $\mathbb{A}^1_A \backslash Z'$) that reduces to a $\mathrm{rad}^u(B)$-torsor. By the glueing Lemma 6.2.2(ii), we may (non-canonically) glue $\mathcal{P}_K$ with $\mathcal{Q}_K$ (resp., $\mathcal{P}$ with $\mathcal{Q}$) to descend $\mathcal{P}_K$ (resp., $\mathcal{P}$) to a G-torsor $\widetilde{\mathcal{P}_K}$ (resp., $\widetilde{\mathcal{P}}$) over $\mathbb{A}^1_{A_K}$ (resp., over $\mathbb{A}^1_A$) that reduces to a $\mathrm{rad}^u(B)$-torsor over $\mathbb{A}^1_{A_K} \backslash Z'_K$ (resp., over $\mathbb{A}^1_A \backslash Z'$). It remains to replace C by $\mathbb{A}^1_A$, Z by Z', $s \in C(A)$ by its image in $\mathbb{A}^1_A(A)$, and $\mathcal{P}_K$ by $\widetilde{\mathcal{P}_K}$ (resp., $\mathcal{P}$ by $\widetilde{\mathcal{P}}$). Finally, by shifting, we may assume even that $s = 0 \in \mathbb{A}^1_A(A)$. □

Proof of Theorem 13.1. Let P_K (resp., P) be a generically trivial G_{A_K}-torsor (resp., G-torsor). By the reduction Lemma 13.7, we get an A-finite closed subscheme $Z \subset \mathbb{A}^1_A$, and a $G_{\mathbb{A}^1_{A_K}}$-torsor $\mathcal{P}_K$ (resp., $G_{\mathbb{A}^1_A}$-torsor $\mathcal{P}$) whose pullback along the zero section is P_K (resp., P) such that $(\mathcal{P}_K)|_{\mathbb{A}^1_{A_K} \backslash Z_K}$ (resp., $\mathcal{P}|_{\mathbb{A}^1_A \backslash Z}$) reduces to a $\mathrm{rad}^u(B)$-torsor. Since any A-finite closed subscheme of $\mathbb{A}^1_A$ is contained in $\{f = 0\}$ for some monic polynomial f, we may enlarge Z to assume that $\mathbb{A}^1_A \backslash Z$ is affine, to the effect that any $\mathrm{rad}^u(B)$-torsor over $\mathbb{A}^1_{A_K} \backslash Z_K$ (resp., over $\mathbb{A}^1_A \backslash Z$), such as $(\mathcal{P}_K)|_{\mathbb{A}^1_{A_K} \backslash Z_K}$ (resp., $\mathcal{P}|_{\mathbb{A}^1_A \backslash Z}$), is trivial. By section Theorem 10.1, the pullback of $\mathcal{P}_K$ (resp., of $\mathcal{P}$) along the section $s \in \mathbb{A}^1_A(A)$ is trivial, that is, P_K (resp., P) is trivial, as desired. □

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