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Abstract

We consider an Allen-Cahn equation with nonlinear diffusion, motivated by the study of the scaling limit of certain interacting particle systems. We investigate its singular limit and show the generation and propagation of an interface in the limit. The evolution of this limit interface is governed by mean curvature flow with a novel, homogenized speed in terms of a surface tension-mobility parameter emerging from the nonlinearity in our equation.

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1 Introduction

The Allen-Cahn equation with linear diffusion

$$u_t = \Delta u - \frac{1}{\varepsilon^2} F'(u)$$

was introduced to understand the phase separation phenomena which appears in the construction of polycrystalline materials [4]. Here, u stands for the order parameter which describes the state of the material, F is a double-well potential with two distinct local minima $\alpha_\pm$ at two different phases, and the parameter $\varepsilon > 0$ corresponds to the interface width in the phase separation process. When ε is small, it is expected that u converges to either of the two states $u = \alpha_+$ and $u = \alpha_-$. Thus, the limit $\varepsilon \downarrow 0$ creates a steep interface dividing two phases; this is a phase separation phenomenon and the limiting interface is known to evolve according to mean curvature flow; see [1, 7].

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In this paper, we prove generation and propagation of interface properties for an Allen-Cahn equation with nondegenerate nonlinear diffusion. More precisely, we study the problem

$$(P^\varepsilon) \quad \begin{cases} u_t = \Delta\varphi(u) + \dfrac{1}{\varepsilon^2} f(u) & \text{in } D \times \mathbb{R}^+ \\ \dfrac{\partial \varphi(u)}{\partial \nu} = 0 & \text{in } \partial D \times \mathbb{R}^+ \\ u(x,0) = u_0(x) & \text{for } x \in D \end{cases}$$

where the unknown function u denotes a phase function, D is a smooth bounded domain in $\mathbb{R}^N, N \geq 2$, ν is the outward unit normal vector to the boundary ∂D and $\varepsilon > 0$ is a small parameter. The nonlinear functions φ and f satisfy the following properties.

We assume that f has exactly three zeros $f(\alpha_-) = f(\alpha_+) = f(0) = 0$ where $\alpha_- < 0 < \alpha_+$, and

$$f \in C^2(\mathbb{R}),\ f'(\alpha_-) < 0,\ f'(\alpha_+) < 0,\ f'(0) > 0 \tag{1}$$

so that

$$f(s) > 0 \text{ for } s < \alpha_-,\ f(s) < 0 \text{ for } s > \alpha_+. \tag{2}$$

The conditions (1)-(2) express the fact that f is a bistable function, so that the $\alpha_\pm$ are the stable steady states of the corresponding ordinary differential equation, whereas 0 is an unstable steady state. We suppose that

$$\varphi \in C^4(\mathbb{R}),\quad \varphi' \geq C_\varphi \tag{3}$$

for some positive constant C_φ, so that the partial differential equation (P^ε) is uniformly parabolic. We impose a relation between f and φ, namely

$$\int_{\alpha_-}^{\alpha_+} \varphi'(s) f(s) ds = 0. \tag{4}$$

This condition implies the existence of the standing wave solution, which we will use extensively in the proof of Theorem 1.2. An experimental example is the case that $\varphi(s) = e^s, f(s) = e^{-s}s(1-s^2)$ [19]. Note that in the case of the standard Allen-Cahn equation with linear diffusion, the condition (4) becomes $\int_{\alpha_-}^{\alpha_+} f(s) ds = 0$.

As for the initial condition $u_0(x)$ we assume that $u_0 \in C^2(\overline{D})$. Throughout the paper, we define C_0 and C_1 as follows:

$$C_0 := ||u_0||_{C^0(\overline{D})} + ||\nabla u_0||_{C^0(\overline{D})} + ||\Delta u_0||_{C^0(\overline{D})} \tag{5}$$

$$C_1 := \max_{|s| \leq I} |\varphi(s)| + \max_{|s| \leq I} \varphi'(s) + \max_{|s| \leq I} |\varphi''(s)|,\quad I = C_0 + \max(|\alpha_-|, \alpha_+). \tag{6}$$

Furthermore, we define Γ_0 by

$$\Gamma_0 := \{x \in D : u_0(x) = 0\}.$$

In addition, we suppose Γ_0 is a $C^{4+\nu}, 0 < \nu < 1$ hypersurface without boundary, which is needed such that

$$\Gamma_0 \Subset D, \nabla u_0(x) \cdot n(x) \neq 0 \text{ if } x \in \Gamma_0 \tag{7}$$

$$u_0 > 0 \text{ in } D_0^+, \qquad u_0 < 0 \text{ in } D_0^-, \tag{8}$$

where D_0^- denotes the region enclosed by Γ_0, D_0^+ is the region enclosed between ∂D and Γ_0, and n is the outward normal vector to D_0^-. It is standard that the above formulation, referred to as Problem (P^ε),

possesses a unique classical solution u^ε. The regularity of the hypersurface Γ_0 is needed to insure the existence, up to a certain time, of a smooth interface Γ_t which moves according to mean curvature (13).

The present paper is originally motivated by the study of the scaling limit of a Glauber+Zero-range particle system. In this microscopic system of interacting random walks, the Zero-range part governs the rates of jumps, while the Glauber part prescribes creation and annihilation rates of the particles. In a companion paper [9], we show that the system exhibits a phase separation and, under a certain space-time scaling limit, an interface arises, in the limit macroscopic density field of particles, evolving in time according to the motion by mean curvature. The system is indeed well approximated from macroscopic viewpoint by the Allen-Cahn equation with nonlinear diffusion (P^ε), or more precisely by its discretized equation. Although, in this paper, we study (P^ε) under the Neumann boundary conditions, the formulation under periodic boundary conditions, used in the particle system setting in [9], can be treated similarly; see Remark 1 below.

In some other physical situations, it is expected that the mathematical modeling also involves nonlinear diffusion. In the experimental article [19], Wagner suggested that for metal alloys the diffusion depends on the concentration. In [2, 8], the authors considered degenerate diffusion such as porous medium diffusion instead of linear diffusion. In [11], Fife and Lacey generalized the Allen-Cahn equation, which leads them to a parameter dependent diffusion Allen-Cahn equation. Recently, [12] considered an Allen-Cahn equation with density dependent diffusion in 1 space dimension and showed a slow motion property. However, no rigorous proof on the motion of the interface in the nonlinear diffusion context has been given for larger space dimensions $N \geq 2$.

In this context, the purpose of this article is to study the singular limit of u^ε as $\varepsilon \downarrow 0$. We first present a result on the generation of the interface. We use the following notation:

$$t^\varepsilon = \varepsilon^2 |\ln \varepsilon| / f'(0), \quad \eta_0 = \min(|\alpha_-|, \alpha_+). \tag{9}$$

Theorem 1.1. *Let u^ε be the solution of the problem (P^ε), η be an arbitrary constant satisfying $0 < \eta < \eta_0$. Then, there exist positive constants ε_0 and M_0 such that, for all $\varepsilon \in (0, \varepsilon_0)$, the following holds:*

(i) for all $x \in D$

$$\alpha_- - \eta \leq u^\varepsilon(x, t^\varepsilon) \leq \alpha_+ + \eta; \tag{10}$$

(ii) if $u_0(x) \geq +M_0\varepsilon$, then

$$u^\varepsilon(x, t^\varepsilon) \geq \alpha_+ - \eta; \tag{11}$$

(iii) if $u_0(x) \leq -M_0\varepsilon$, then

$$u^\varepsilon(x, t^\varepsilon) \leq \alpha_- + \eta. \tag{12}$$

After the interface has been generated, the diffusion term has the same order as the reaction term. As a result the interface starts to propagate. Later, we will prove that the interface moves according to the following motion equation:

$$(IP) \begin{cases} V_n = -(N-1)\lambda_0 \kappa & \text{on } \Gamma_t \\ \Gamma_t|_{t=0} = \Gamma_0, \end{cases} \tag{13}$$

where Γ_t is the interface at time $t > 0$, V_n is the normal velocity on the interface, κ denotes its mean curvature, and λ_0 is a positive constant which will be defined later (see (26) and (30)). A typical example

is a case of spherical symmetry, where we assume that Γ_0 is the circle, namely $\Gamma_0 = \{x \in R^N, |x| = 1\}$. Then the circle shrinks with the motion equation

$$\begin{cases} \dfrac{dR}{dt} = -\dfrac{(N-1)\lambda_0}{R}, \\ R(0) = 1, \end{cases}$$

where $R(t)$ denotes the radius of the circle. It is well known that Problem (IP) possesses locally in time a unique smooth solution. Fix $T > 0$ such that the solution of (IP), in (13), exists in $[0, T]$ and denote the solution by $\Gamma = \cup_{0 \le t < T}(\Gamma_t \times \{t\})$. From Proposition 2.1 of [7] such a $T > 0$ exists, and one can deduce that $\Gamma \in C^{4+\nu, \frac{4+\nu}{2}}$ in $[0, T]$, given that $\Gamma_0 \in C^{4+\nu}$.

The second main theorem states a result on the generation and the propagation of the interface.

Theorem 1.2. *Under the conditions given in Theorem 1.1, for any given $0 < \eta < \eta_0$ there exist $\varepsilon_0 > 0$ and $C > 0$ such that*

$$u^\varepsilon \in \begin{cases} [\alpha_- - \eta, \alpha_+ + \eta] & \text{for } x \in D \\ [\alpha_+ - \eta, \alpha_+ + \eta] & \text{if } x \in D_t^+ \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t) \\ [\alpha_- - \eta, \alpha_- + \eta] & \text{if } x \in D_t^- \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t) \end{cases} \tag{14}$$

for all $\varepsilon \in (0, \varepsilon_0)$ and $t \in [t^\varepsilon, T]$, where D_t^- denotes the region enclosed by Γ_t, D_t^+ is enclosed between ∂D and Γ_t, and $\mathcal{N}_r(\Gamma_t) := \{x \in D, dist(x, \Gamma_t) < r\}$.

This theorem implies that, after generation, the interface propagates with a width of order $\mathcal{O}(\varepsilon)$. Note that Theorems 1.1 and 1.2 extend similar results known for linear diffusion Allen-Cahn equations [1].

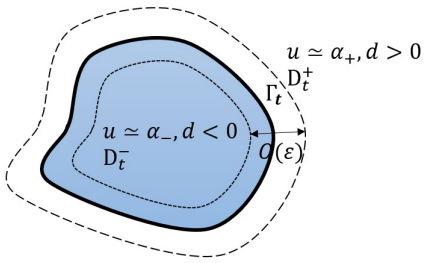


Figure 1: The interface of time t

We now state an approximation result inspired by a similar result proved in [3].

Theorem 1.3. *(i) Let the assumptions of Theorem 1.2 hold and $\rho > 1$. Then, the solution u^ε of (P^ε) satisfies*

$$\lim_{\varepsilon \to 0} \sup_{\rho t^\varepsilon \le t \le T,\ x \in D} \left| u^\varepsilon(x,t) - U_0\left(\frac{d^\varepsilon(x,t)}{\varepsilon}\right) \right| = 0, \tag{15}$$

where U_0 is the standing wave solution defined in (20) and d^ε denotes the signed distance function associated with $\Gamma_t^\varepsilon := \{x \in D : u^\varepsilon(x,t) = 0\}$, defined by

$$d^\varepsilon(x,t) = \begin{cases} dist(x, \Gamma_t^\varepsilon) & \text{if } x \in D_t^{\varepsilon,+} \\ -dist(x, \Gamma_t^\varepsilon) & \text{if } x \in D_t^{\varepsilon,-} \end{cases}$$

where $D_t^{\varepsilon,-}$ denotes the region enclosed by Γ_t^ε and $D_t^{\varepsilon,+}$ denotes the region between ∂D and Γ_t^ε.

(ii) For small enough $\varepsilon > 0$ and for any $t \in [\rho t_\varepsilon, T]$, Γ_t^ε can be expressed as a graph over Γ_t.

Remark 1. *Theorems 1.1, 1.2 and 1.3 hold not only for the Neumann boundary condition of Problem* (P^ε) *but also for periodic boundary conditions with* $D = \mathbb{T}^N$*, with similar proofs as given in Sections 3, 4 and 5.*

The paper is organized as follows. In Section 2, the interface motion (IP) is formally derived from the problem (P^ε) as $\varepsilon \downarrow 0$. In particular, the constant λ_0 is obtained. Section 3 studies the generation of interface and gives the proof of Theorem 1.1. In a short time, the reaction term f governs the system and the solution of (P^ε) behaves close to that of an ordinary differential equation. Section 4 discusses the propagation of interface and Theorem 1.2 is proved. The sub- and super-solutions are constructed by means of two functions U_0 and U_1 formally introduced in asymptotic expansions in Section 2. Section 5 gives the proof of Theorem 1.3. Finally, in the Appendix, we define the mobility μ_{AC} and the surface tension σ_{AC} of the interface, especially in our nonlinear setting, and show the relation $\lambda_0 = \mu_{AC}\sigma_{AC}$.

2 Formal derivation of the interface motion equation

In this section, we formally derive the interface motion equation corresponding to the Problem (P^ε) by applying the method of matched asymptotic expansions. To this purpose, we first define the interface Γ_t and then derive its motion equation.

Suppose that u^ε converges to a step function u where

$$u(x,t) = \begin{cases} \alpha_+ & \text{in } D_t^+ \\ \alpha_- & \text{in } D_t^-. \end{cases}$$

Let

$$\Gamma_t = \overline{D_t^+} \cap \overline{D_t^-}, \overline{D_t^+} \cup \overline{D_t^-} = D, \ t \in [0,T].$$

Let also $\overline{d}(x,t)$ be the signed distance function to Γ_t defined by

$$\overline{d}(x,t) := \begin{cases} -dist(x,\Gamma_t) & \text{for } x \in \overline{D_t^-} \\ dist(x,\Gamma_t) & \text{for } x \in D_t^+. \end{cases} \tag{16}$$

Assume that u^ε has the expansions

$$u^\varepsilon(x,t) = \alpha_\pm + \varepsilon u_1^\pm(x,t) + \varepsilon^2 u_2^\pm(x,t) + \cdots$$

away from the interface Γ and that

$$u^\varepsilon(x,t) = U_0(x,t,\xi) + \varepsilon U_1(x,t,\xi) + \varepsilon^2 U_2(x,t,\xi) + \cdots \tag{17}$$

near Γ, where $\xi = \dfrac{\overline{d}}{\varepsilon}$. Here, the variable ξ is given to describe the rapid transition between the regions $\{u^\varepsilon \simeq \alpha^+\}$ and $\{u^\varepsilon \simeq \alpha^-\}$. In addition, we normalize U_0 and U_k so that

$$\begin{aligned} U_0(x,t,0) &= 0 \\ U_k(x,t,0) &= 0. \end{aligned} \tag{18}$$

To match the inner and outer expansions, we require that

$$U_0(x,t,\pm\infty) = \alpha_\pm, \quad U_k(x,t,\pm\infty) = u_k^\pm(x,t) \tag{19}$$

for all $k \geq 2$.

After substituting the expansion (17) into (P^ε), we collect the ε^{-2} terms, to obtain

$$\varphi(U_0)_{zz} + f(U_0) = 0.$$

Since this equation only depends on the variable z, we may assume that U_0 is only a function of the variable z, that is $U_0(x,t,z) = U_0(z)$. In view of the conditions (18) and (19), we find that U_0 is the unique increasing solution of the following problem

$$\begin{cases} (\varphi(U_0))_{zz} + f(U_0) = 0 \\ U_0(-\infty) = \alpha_-,\ U_0(0) = 0,\ U_0(+\infty) = \alpha_+. \end{cases} \tag{20}$$

In order to understand the nonlinear effect more clearly, we set

$$g(v) := f(\varphi^{-1}(v)),$$

where φ^{-1} is the inverse function of φ and define $V_0(z) := \varphi(U_0(z))$; note that such a transformation is possible by the condition (3). Substituting V_0 into equation (20) yields

$$\begin{cases} V_{0zz} + g(V_0) = 0 \\ V_0(-\infty) = \varphi(\alpha_-),\ V_0(0) = \varphi(0),\ V_0(+\infty) = \varphi(\alpha_+). \end{cases} \tag{21}$$

Condition (4) then implies the existence of the unique increasing solution of (21).

Next we collect the ε^{-1} terms in the asymptotic expansion. In view of the definition of $U_0(z)$ and the condition (18), we obtain the following problem

$$\begin{cases} (\varphi'(U_0)\overline{U_1})_{zz} + f'(U_0)\overline{U_1} = \overline{d}_t U_{0z} - (\varphi(U_0))_z \Delta\overline{d} \\ \overline{U_1}(x,t,0) = 0, \quad \varphi'(U_0)\overline{U_1} \in L^\infty(\mathbb{R}). \end{cases} \tag{22}$$

To prove the existence of solution to (22), we consider the function $\overline{V_1} = \varphi'(U_0)\overline{U_1}$, which satisfies the problem

$$\begin{cases} \overline{V_1}_{zz} + g'(V_0)\overline{V_1} = \dfrac{V_{0z}}{\varphi'(\varphi^{-1}(V_0))}\overline{d}_t - V_{0z}\Delta\overline{d} \\ \overline{V_1}(x,t,0) = 0, \quad \overline{V_1} \in L^\infty(\mathbb{R}). \end{cases} \tag{23}$$

Now, Lemma 2.2 of [1] implies the existence and uniqueness of V_1 provided that

$$\int_{\mathbb{R}} \left(\frac{1}{\varphi'(\varphi^{-1}(V_0))}\overline{d}_t - \Delta\overline{d} \right) V_{0z}^2 = 0.$$

Substituting $V_0 = \varphi(U_0)$ and $V_{0z} = \varphi'(U_0)U_{0z}$ yields

$$\overline{d}_t = \frac{\int_{\mathbb{R}} V_{0z}^2}{\int_{\mathbb{R}} \frac{V_{0z}^2}{\varphi'(\varphi^{-1}(V_0))}} \Delta\overline{d} = \frac{\int_{\mathbb{R}} (\varphi'(U_0)U_{0z})^2}{\int_{\mathbb{R}} \varphi'(U_0)U_{0z}^2} \Delta\overline{d}. \tag{24}$$

It is known that $\overline{d}_t = -V_n$, where V_n is the normal velocity of the interface Γ_t, and $\Delta\overline{d}$ is equal to $(N-1)\kappa$, where κ is the mean curvature of Γ_t. Thus, we obtain the motion equation Γ_t,

$$V_n = -(N-1)\lambda_0\kappa \quad \text{on } \Gamma_t, \tag{25}$$

with

$$\lambda_0 = \frac{\int_{\mathbb{R}} (\varphi'(U_0) U_{0z})^2}{\int_{\mathbb{R}} \varphi'(U_0) U_{0z}^2}. \tag{26}$$

In the appendix, the constant λ_0 is interpreted as the surface tension σ_{AC} multiplied by the mobility μ_{AC} of the interface. In particular, (IP) coincides with the equation (1) in [4].

Finally, we derive an explicit form of λ_0. Indeed, we multiply the equation (20) by $\varphi(U_0)_z$, yielding

$$\varphi(U_0)_{zz}\varphi(U_0)_z + f(U_0)\varphi(U_0)_z = 0\,.$$

Integrating from $-\infty$ to z, we obtain

$$\frac{1}{2}\big[\varphi(U_0)_z\big]^2(z) + \int_{-\infty}^{z} f(U_0)\varphi(U_0)_z dz = 0$$

or alternatively

$$\frac{1}{2}\big[\varphi(U_0)_z\big]^2(z) + \int_{\alpha_-}^{U_0(z)} f(s)\varphi'(s)ds = 0\,.$$

Hence,

$$\varphi(U_0)_z(z) = \sqrt{2}\sqrt{W(U_0(z))}\,, \tag{27}$$

where W is given by

$$W(u) = -\int_{\alpha_-}^{u} f(s)\varphi'(s)ds = \int_{u}^{\alpha_+} f(s)\varphi'(s)ds, \tag{28}$$

where the last equality holds by (4). It follows that

$$\int_{\mathbb{R}} \varphi(U_0)_z U_{0z}(z)dz = \sqrt{2}\int_{\mathbb{R}} \sqrt{W(U_0(z))}U_{0z}(z)dz$$

so that also

$$\int_{\mathbb{R}} \varphi'(U_0)U_{0z}^2(z)dz = \sqrt{2}\int_{\alpha_-}^{\alpha_+} \sqrt{W(u)}du.$$

Similarly, since

$$\int_{\mathbb{R}} (\varphi'(U_0)U_{0z})^2 dz = \sqrt{2}\int_{\mathbb{R}} (\varphi'(U_0)\sqrt{W(U_0(z))}U_{0z})dz,$$

we get

$$\int_{\mathbb{R}} (\varphi'(U_0)U_{0z})^2 dz = \sqrt{2}\int_{\alpha_-}^{\alpha_+} \varphi'(u)\sqrt{W(u)}du, \tag{29}$$

so that we finally obtain the formula

$$\lambda_0 = \frac{\int_{\alpha_-}^{\alpha_+} \varphi'(u)\sqrt{W(u)}du}{\int_{\alpha_-}^{\alpha_+} \sqrt{W(u)}du}. \tag{30}$$

Note that if $\varphi(u) = u$, the case of the linear diffusion Allen-Cahn equation, we recover the value $\lambda_0 = 1$ as expected.

3 Generation of the interface

In this section, we prove Theorem 1.1 on the generation of the interface. The main idea, based on the comparison principle Lemma 1, is to construct suitable sub- and super-solutions. The proof of Theorem 1.1 is given in Section 3.4.

3.1 Comparison principle

Lemma 1. *Let $v \in C^{2,1}(\overline{D} \times \mathbb{R}^+)$ satisfy*

$$(P)\quad \begin{cases} v_t \geq \Delta\varphi(v) + \dfrac{1}{\varepsilon^2} f(v) & \text{in } D \times \mathbb{R}^+ \\ \dfrac{\partial \varphi(v)}{\partial \nu} = 0 & \text{in } \partial D \times \mathbb{R}^+ \\ v(x,0) \geq u_0(x) & \text{for } x \in D. \end{cases}$$

Then, v is a super-solution of Problem (P^ε) and we have

$$v(x,t) \geq u^\varepsilon(x,t), \quad (x,t) \in D \times \mathbb{R}^+.$$

If v satisfies the opposite inequalities in Problem (P), then v is a sub-solution of Problem (P^ε) and we have

$$v(x,t) \leq u^\varepsilon(x,t), \quad (x,t) \in D \times \mathbb{R}^+.$$

Proof. Consider the inequality satisfied for the difference of a super-solution v and a solution u^ε. Apply the maximum principle to the function $w := v - u^\varepsilon$ to deduce that it is positive. □

3.2 Solution of the corresponding ordinary differential equation

In the first stage of development, we expect that the solution behaves as that of the corresponding ordinary differential equation:

$$\begin{cases} Y_\tau(\tau,\zeta) = f(Y(\tau,\zeta)) & \tau > 0 \\ Y(0,\zeta) = \zeta & \zeta \in \mathbb{R}. \end{cases} \tag{31}$$

We deduce the following result from [1].

Lemma 2. *Let $\eta \in (0, \eta_0)$ be arbitrary. Then, there exists a positive constant $C_Y = C_Y(\eta)$ such that the following holds:*

(i) There exists a positive constant $\overline{\mu}$ such that for all $\tau > 0$ and all $\zeta \in (-2C_0, 2C_0)$,

$$e^{-\overline{\mu}\tau} \leq Y_\zeta(\tau,\zeta) \leq C_Y e^{f'(0)\tau}. \tag{32}$$

(ii) For all $\tau > 0$ and all $\zeta \in (-2C_0, 2C_0)$,

$$\left| \frac{Y_{\zeta\zeta}(\tau,\zeta)}{Y_\zeta(\tau,\zeta)} \right| \leq C_Y (e^{f'(0)\tau} - 1).$$

(iii) There exists a positive constants ε_0 such that, for all $\varepsilon \in (0, \varepsilon_0)$, we have

(a) for all $\zeta \in (-2C_0, 2C_0)$

$$\alpha_- - \eta \leq Y(f'(0)^{-1} |\ln \varepsilon|, \zeta) \leq \alpha_+ + \eta; \tag{33}$$

(b) if $\zeta \geq C_Y \varepsilon$, then

$$Y(f'(0)^{-1} |\ln \varepsilon|, \zeta) \geq \alpha_+ - \eta; \tag{34}$$

(c) if $\zeta \le -C_Y\varepsilon$, *then*

$$Y(f'(0)^{-1}|\ln\varepsilon|,\zeta) \le \alpha_- + \eta.$$

Proof. These results can be found in Lemma 4.7 and Lemma 3.7 of [1], except for (32). To prove (32), we follow similar computations as in Lemma 3.2 of [1]. Differentiating (31) by ζ, we obtain

$$\begin{cases} Y_{\zeta\tau}(\tau,\zeta) = f'(Y(\tau,\zeta))Y_\zeta, \quad \tau > 0 \\ Y_\zeta(0,\zeta) = 1, \end{cases}$$

which yields the following equality,

$$Y_\zeta(\tau,\zeta) = \exp\left[\int_0^\tau f'(Y(s,\zeta))\right]. \tag{35}$$

Hence, for $\zeta = 0$,

$$Y_\zeta(\tau,0) = \exp\left[\int_0^\tau f'(Y(s,0))\right] = e^{f'(0)\tau},$$

where the last equality follows since $Y(\tau,0) = 0$. Also, for $\zeta = \alpha_\pm$, by (1), we have

$$Y_\zeta(\tau,\alpha_\pm) \le e^{f'(0)\tau}.$$

For $\zeta \in (\alpha_- + \eta, \alpha_+ - \eta) \setminus \{0\}$, Lemma 3.4 of [1] guarantees the upper bound of Y_ζ in (32). We only need to consider the case that $\zeta \in (-2C_0, 2C_0) \setminus (\alpha_- + \eta, \alpha_+ - \eta)$. It follows from (1) that we can choose a positive constant η and $\overline{\eta}$ such that

$$f'(s) < 0 \ , s \in I, \tag{36}$$

where $I := (\alpha_- - \overline{\eta}, \alpha_- + \eta) \cup (\alpha_+ - \eta, \alpha_+ + \overline{\eta})$. Moreover, (1) and (2) imply

$$Y(\tau,\zeta) \in J, \tag{37}$$

for $\zeta \in J$ where $J := (\min\{-2C_0, \alpha_- - \overline{\eta}\}, \alpha_- + \eta) \cup (\alpha_+ - \eta, \max\{2C_0, \alpha_+ + \overline{\eta}\})$. Thus, (35), (36) and (37) guarantee the upper bound of (32) for $\zeta \in I$, which only leaves the case $\zeta \in (-2C_0, 2C_0) \setminus I$.

We consider now the case $\zeta \in (\alpha_+ + \overline{\eta}, 2C_0)$; the case of $\zeta \in (-2C_0, \alpha_- - \overline{\eta})$ can be analysed in a similar way. By (3.13) in [1], we have

$$\ln Y_\zeta(\tau,\zeta) = f'(\alpha_+)\tau + \int_\zeta^{Y(\tau,\zeta)} \tilde{f}(s)ds, \quad \text{and } \tilde{f}(s) = \frac{f'(s) - f'(\alpha_+)}{f(s)}. \tag{38}$$

Note that $\tilde{f}(s) \to \dfrac{f''(\alpha_+)}{f'(\alpha_+)}$ as $s \to \alpha_+$, so that $\tilde{f}$ may be extended as a continuous function. We define

$$\tilde{F} := \|\tilde{f}\|_{L^\infty(\alpha_+, \max\{2C_0, \alpha_+ + \overline{\eta}\})}.$$

Since (2) yields $Y(\tau,\zeta) > \alpha_+$ for $\zeta \in (\alpha_+ + \overline{\eta}, 2C_0)$, by (38) we can find a constant C_Y large enough such that

$$Y_\zeta(\tau,\zeta) \le C_Y e^{f'(\alpha_+)\tau} \le C_Y e^{f'(0)\tau}.$$

Thus, we obtain the upper bound of (32).
For the lower bound, we first define

$$\overline{\mu} := -\min_{s\in I'} f'(s), \ I' = [-2C_0, 2C_0] \cup [\alpha_-, \alpha_+].$$

Note that $\overline{\mu} > 0$ by (1). Thus, by (35), we obtain

$$Y_\zeta(\tau,\zeta) \ge e^{-\overline{\mu}\tau}.$$

□

3.3 Construction of sub- and super-solutions

We now construct sub- and super-solutions for the proof of Theorem 1.1. For simplicity, we first consider the case where

$$\frac{\partial u_0}{\partial \nu} = 0 \text{ on } \partial D. \tag{39}$$

In this case, we define sub- and super- solution as follows:

$$w_\varepsilon^\pm(x,t) = Y\left(\frac{t}{\varepsilon^2}, u_0(x) \pm \varepsilon^2 C_2\left(e^{\mu t/\varepsilon^2} - 1\right)\right) = Y\left(\frac{t}{\varepsilon^2}, u_0(x) \pm P(t)\right)$$

for some the constant C_2. In the general case, where (39) does not necessarily hold, we need to modify $w_\varepsilon^\pm$ near the boundary ∂D. This will be discussed later in the proof of Theorem 1.1; see after equation (41).

Lemma 3. *Assume (39). Then, there exist positive constants ε_0 and $C_2, \overline{C}_2$ independent of ε such that, for all $\varepsilon \in (0, \varepsilon_0)$, $w_\varepsilon^\pm$ satisfies*

$$\begin{cases} \mathcal{L}(w_\varepsilon^-) < -\overline{C}_2 e^{-\frac{\overline{\mu} t}{\varepsilon^2}} < \overline{C}_2 e^{-\frac{\overline{\mu} t}{\varepsilon^2}} < \mathcal{L}(w_\varepsilon^+) & \text{in } \overline{D} \times [0, t^\varepsilon] \\ \dfrac{\partial w_\varepsilon^-}{\partial \nu} = \dfrac{\partial w_\varepsilon^+}{\partial \nu} = 0 & \text{on } \partial D \times [0, t^\varepsilon]. \end{cases} \tag{40}$$

Proof. We only prove that w_ε^+ is the desired super-solution; one can show that w_ε^- is a sub-solution in a similar way. The assumption (39) implies

$$\frac{\partial w_\varepsilon^\pm}{\partial \nu} = 0 \text{ on } \partial D \times \mathbb{R}^+.$$

Define the operator $\mathcal{L}$ by

$$\mathcal{L}u = u_t - \Delta\varphi(u) - \frac{1}{\varepsilon^2} f(u).$$

Then, direct computation with $\tau = t/\varepsilon^2$ gives

$$\begin{aligned} \mathcal{L}(w_\varepsilon^+) &= \frac{1}{\varepsilon^2} Y_\tau + P'(t) Y_\zeta - \left(\varphi''(w_\varepsilon^+)|\nabla u_0|^2 (Y_\zeta)^2 + \varphi'(w_\varepsilon^+)\Delta u_0 Y_\zeta + \varphi'(w_\varepsilon^+)|\nabla u_0|^2 Y_{\zeta\zeta} + \frac{1}{\varepsilon^2} f(Y)\right) \\ &= \frac{1}{\varepsilon^2}(Y_\tau - f(Y)) + Y_\zeta \left(P'(t) - \left(\varphi''(w_\varepsilon^+)|\nabla u_0|^2 Y_\zeta + \varphi'(w_\varepsilon^+)\Delta u_0 + \varphi'(w_\varepsilon^+)|\nabla u_0|^2 \frac{Y_{\zeta\zeta}}{Y_\zeta}\right)\right). \end{aligned}$$

By the definition of Y, the first term on the right-hand-side vanishes. By choosing ε_0 sufficiently small, for $0 \le t \le t_\varepsilon$, we have

$$P(t) \le P(t^\varepsilon) = \varepsilon^2 C_2 (e^{\mu t^\varepsilon/\varepsilon^2} - 1) = \varepsilon^2 C_2 (\varepsilon^{-1} - 1) < C_0.$$

Hence, $|u_0 + P(t)| < 2C_0$. Applying Lemma 2, (5) and (6) gives

$$\begin{aligned} \mathcal{L}w_\varepsilon^+ &\ge Y_\zeta \left(C_2 \mu e^{\mu t/\varepsilon^2} - (C_0^2 C_1 C_Y e^{\mu t/\varepsilon^2} + C_0 C_1 + C_0^2 C_1 C_Y (e^{\mu t/\varepsilon^2} - 1))\right) \\ &= Y_\zeta \left((C_2\mu - C_0^2 C_1 C_Y - C_0^2 C_1 C_Y) e^{\mu t/\varepsilon^2} + C_0^2 C_1 C_Y - C_0 C_1\right). \end{aligned}$$

By (32), for C_2 large enough, we can find a positive constant $\overline{C}_2$ independent of ε such that

$$\mathcal{L}w_\varepsilon^+ \ge \overline{C}_2 e^{-\frac{\overline{\mu} t}{\varepsilon^2}}.$$

Thus, w_ε^+ is a super-solution for Problem (P^ε). □

3.4 Proof of Theorem 1.1

We deduce from the comparison principle Lemma 1 and the construction of the sub- and super-solutions that

$$w_\varepsilon^-(x,t^\varepsilon) \le u^\varepsilon(x,t^\varepsilon) \le w_\varepsilon^+(x,t^\varepsilon) \tag{41}$$

under the condition (39).

If (39) does not hold, one can modify the functions $w^\pm$ as follows: from condition (8), there exist positive constants d_0 and ρ such that (i) the distance function $d(x,\partial D)$ is smooth enough on $\{x \in D : d(x,\partial D) < 2d_0\}$ and (ii) $u_0(x) \ge \rho$ if $d(x,\partial D) \le d_0$. Let ξ be a smooth cut-off function defined on $[0,+\infty)$ such that $0 \le \xi \le 1, \xi(0) = \xi'(0) = 0$ and $\xi(z) = 1$ for $z \ge d_0$. Define

$$\begin{aligned} u_0^+ &:= \xi(d(x,\partial D))u_0(x) + [1-\xi(d(x,\partial D))] \max_{\overline{D}} u_0 \\ u_0^- &:= \xi(d(x,\partial D))u_0(x) + [1-\xi(d(x,\partial D))]\, \rho. \end{aligned}$$

Then, $u_0^- \le u_0 \le u_0^+$ and $u_0^\pm$ satisfy the homogeneous Neumann boundary condition (39). Thus, by using a similar argument as in the proof of Lemma 3, we may find sub- and super-solutions as follows,

$$w_\varepsilon^\pm(x,t) = Y\left(\frac{t}{\varepsilon^2}, u_0^\pm(x) \pm \varepsilon^2 C_2\left(e^{f'(0)t/\varepsilon^2} - 1\right)\right).$$

We now show (10), (11) and (12). By the definition of C_0 in (5), we have

$$-C_0 \le \min_{x\in\overline{D}} u_0(x) < \rho.$$

Thus, for ε_0 small enough, we have that

$$-2C_0 \le u_0^\pm(x) \pm (C_2\varepsilon - C_2\varepsilon^2) \le 2C_0 \quad \text{for } x \in D$$

holds for any $\varepsilon \in (0,\varepsilon_0)$. Thus, the assertion (10) is a direct consequence of (33) and (41).

For (11), first we choose M_0 large enough so that $M_0\varepsilon - C_2\varepsilon + C_2\varepsilon^2 \ge C_Y\varepsilon$. Then, for any $x \in D$ such that $u_0^-(x) \ge M_0\varepsilon$, we have

$$u_0^-(x) - \varepsilon^2 C_2\left(e^{f'(0)t/\varepsilon^2} - 1\right) \ge u_0^-(x) - (C_2\varepsilon - C_2\varepsilon^2) \ge M_0\varepsilon - C_2\varepsilon + C_2\varepsilon^2 \ge C_Y\varepsilon.$$

Therefore, with (34) and (41), we see that

$$u^\varepsilon(x,t^\varepsilon) \ge \alpha_+ - \eta$$

for any $x \in D$ such that $u_0^-(x) \ge M_0\varepsilon$, which implies (11). Note that (12) can be shown in the same way. This completes the proof of Theorem 1.1. □

4 Propagation of the interface

The main idea of the proof of Theorem 1.2 is that we proceed by imbrication: By the comparison principle Lemma 1, we show at the generation time that $u^+(x,0) \ge w^+(x,t^\varepsilon)$ and $u^-(x,0) \le w^-(x,t^\varepsilon)$ so that we can pass continuously from the generation of interface sub- and super-solutions to the propagation of interface sub- and super-solutions.

To this end, we first introduce a modified signed distance function, and several estimates on the functions U_0 and U_1 useful in the sub- and super-solution construction, before showing Theorem 1.2 in Section 4.4.

4.1 A modified signed distance function

We introduce a useful cut off signed distance function d as follows. Recall the signed distance function $\overline{d}$ defined in (16), and interface Γ_t satisfying (13). Choose $d_0 > 0$ small enough so that the signed distance function $\overline{d}$ is smooth in the set

$$\{(x,t) \in \overline{D} \times [0,T], |\overline{d}(x,t)| < 3d_0\}$$

and that

$$dist(\Gamma_t, \partial D) \geq 3d_0 \text{ for all } t \in [0,T].$$

Let $h(s)$ be a smooth non-decreasing function on $\mathbb{R}$ such that

$$h(s) = \begin{cases} s & \text{if } |s| \leq d_0 \\ -2d_0 & \text{if } s \leq -2d_0 \\ 2d_0 & \text{if } s \geq 2d_0. \end{cases}$$

We then define the cut-off signed distance function d by

$$d(x,t) = h(\overline{d}(x,t)), \quad (x,t) \in \overline{D} \times [0,T].$$

Note, as d coincides with $\overline{d}$ in the region

$$\{(x,t) \in D \times [0,T] : |d(x,t)| < d_0\},$$

that we have

$$d_t = \lambda_0 \Delta d \text{ on } \Gamma_t.$$

Moreover, d is constant near ∂D and the following properties hold.

Lemma 4. *There exists a constant $C_d > 0$ such that*

(i) $|d_t| + |\nabla d| + |\Delta d| \leq C_d$,

(ii) $|d_t - \lambda_0 \Delta d| \leq C_d |d|$

in $\overline{D} \times [0,T]$.

4.2 Estimates for the functions U_0, U_1

Here, we give estimates for the functions which will be used to construct the sub- and super-solutions. Recall that U_0 (cf. (20)) is a solution of the equation

$$(\varphi(U_0))_{zz} + f(U_0) = 0.$$

We have the following results.

Lemma 5. *There exists constants $\hat{C}_0, \lambda_1 > 0$ such that for all $z \in \mathbb{R}$,*

(i) $|U_0|,\ |U_{0z}|,\ |U_{0zz}| \leq \hat{C}_0$,

(ii) $|U_{0z}|,\ |U_{0zz}| \leq \hat{C}_0 \exp(-\lambda_1 |z|)$.

Proof. Recall that $V_0 = \varphi(U_0)$ satisfies the equation (21) with $\varphi \in C^4(\mathbb{R})$. Lemma 2.1 of [1] implies that there exist some positive constants $\overline{C}_0$ and λ_1 such that, for all $z \in \mathbb{R}$,

$$|V_0|,\ |V_{0z}|,\ |V_{0zz}| \le \overline{C}_0;$$
$$|V_{0z}|,\ |V_{0zz}| \le \overline{C}_0 \exp(-\lambda_1|z|),$$

and therefore similar bounds hold for U_0. □

In terms of the cut-off signed distance function $d = d(x,t)$, for each $(x,t) \in \overline{D} \times [0,T]$, we define $U_1(x,t,\cdot) : \mathbb{R} \to \mathbb{R}$ as the solution of the following problem:

$$\begin{cases} (\varphi'(U_0)U_1)_{zz} + f'(U_0)U_1 = (\lambda_0 U_{0z} - (\varphi(U_0))_z)\Delta d \\ U_1(x,t,0) = 0, \quad \varphi'(U_0)U_1 \in L^\infty(\mathbb{R}). \end{cases} \tag{42}$$

Existence of the solution U_1 can be shown in the same way as that for $\overline{U_1}$ in (22). Finally, we give the following estimates for $U_1 = U_1(x,t,z)$.

Lemma 6. *There exist positive constants $\hat{C}_1$ and λ_1 such that for all $z \in \mathbb{R}$*

(i) $|U_1|,\ |U_{1z}|,\ |U_{1zz}|,\ |\nabla U_{1z}|,\ |\nabla U_1|,\ |\Delta U_1|,\ |U_{1t}| \le \hat{C}_1$,

(ii) $|U_{1z}|,\ |U_{1zz}|,\ |\nabla U_{1z}| \le \hat{C}_1 \exp(-\lambda_1|z|)$.

Here, the operators ∇ and Δ act on the variable x.

Proof. Define $V_1(z) := \varphi'(U_0(z))U_1(z)$. As in (22), we obtain an equation for V_1:

$$\begin{cases} V_{1zz} + g'(V_0)V_1 = \left[\lambda_0 \dfrac{V_{0z}}{\varphi'(\varphi^{-1}(V_0))} - V_{0z}\right]\Delta d \\ V_1(x,t,0) = 0, \quad V_1 \in L^\infty(\mathbb{R}). \end{cases} \tag{43}$$

Applying Lemmas 2.2 and 2.3 of [1] to (43) implies the boundedness of V_1, V_{1z}, V_{1zz}. Moreover, since d is smooth in $\overline{D} \times [0,T]$, we can apply Lemma 2.2 of [1] to obtain the boundedness of $\nabla V_1, \Delta V_1$. The desired estimates for the function U_1 now follows via the smoothness of φ as in the proof of Lemma 5. □

4.3 Construction of sub- and super-solutions

We construct candidates sub- and super-solutions as follows: Given $\varepsilon > 0$, define

$$u^\pm(x,t) = U_0\left(\frac{d(x,t) \pm \varepsilon p(t)}{\varepsilon}\right) + \varepsilon U_1\left(x,t,\frac{d(x,t) \pm \varepsilon p(t)}{\varepsilon}\right) \pm q(t) \tag{44}$$

where

$$p(t) = -e^{-\beta t/\varepsilon^2} + e^{Lt} + K,$$
$$q(t) = \sigma\left(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{Lt}\right),$$

in terms of positive constants $\varepsilon, \beta, \sigma, L, K$. Next, we give specific conditions for these constants which will be used to show that indeed $u^\pm$ are sub- and super-solutions. We assume that the positive constant ε_0 obeys

$$\varepsilon_0^2 L e^{LT} \le 1, \quad \varepsilon_0 \hat{C}_1 \le \frac{1}{2}. \tag{45}$$

We first give a result on the boundedness of $f'(U_0(z)) + (\varphi'(U_0(z))_{zz}$.

Lemma 7. *There exists $b > 0$ such that $f'(U_0(z)) + (\varphi'(U_0))_{zz} < 0$ on $\{z : U_0(z) \in [\alpha_-,\ \alpha_- + b] \cup [\alpha_+ - b,\ \alpha_+]\}$.*

Proof. We can choose $b_1, \mathcal{F} > 0$ such that

$$f'(U_0(z)) < -\mathcal{F}$$

on $\{z : U_0(z) \in [\alpha_-,\ \alpha_- + b_1] \cup [\alpha_+ - b_1,\ \alpha_+]\}$.

Note that $(\varphi'(U_0))_{zz} = \varphi'''(U_0)U_{0z}^2 + \varphi''(U_0)U_{0zz}$. From Lemma 5, we can choose $b_2 > 0$ small enough so that

$$|(\varphi'(U_0))_z| < \mathcal{F}, \quad |(\varphi'(U_0))_{zz}| < \mathcal{F}$$

on $\{z : U_0(z) \in [\alpha_-,\ \alpha_- + b_2] \cup [\alpha_+ - b_2,\ \alpha_+]\}$. Define $b := \min\{b_1, b_2\}$. Then, we have

$$f'(U_0(z)) + (\varphi'(U_0))_{zz} < \mathcal{F} - \mathcal{F} = 0.$$

□

Fix $b > 0$ which satisfies the result of Lemma 7. Denote $J_1 := \{z : U_0(z) \in [\alpha_-,\ \alpha_- + b] \cup [\alpha_+ - b,\ \alpha_+]\}, J_2 = \{z : U_0(z) \in [\alpha_- + b,\ \alpha_+ - b]\}$. Let

$$\beta := -\sup\left\{\frac{f'(U_0(z)) + (\varphi'(U_0(z)))_{zz}}{3} : z \in J_1\right\}. \tag{46}$$

The following result plays an important role in verifying sub- and super-solution properties.

Lemma 8. *There exists a constant σ_0 small enough such that for every $0 < \sigma < \sigma_0$, we have*

$$U_{0z} - \sigma(f'(U_0) + (\varphi'(U_0))_{zz}) \geq 3\sigma\beta.$$

Proof. To show the assertion, it is sufficient to show that there exists σ_0 such that, for all $0 < \sigma < \sigma_0$,

$$\frac{U_{0z}}{\sigma} - (f'(U_0) + (\varphi'(U_0))_{zz}) \geq 3\beta. \tag{47}$$

We prove the result on each of the sets J_1, J_2.

On the set J_1, note that $U_{0z} > 0$ on $\mathbb{R}$. If $z \in J_1$, for any $\sigma > 0$ we have

$$\frac{U_{0z}}{\sigma} - (f'(U_0) + (\varphi'(U_0))_{zz}) > -\sup_{z \in J_1}(f'(U_0) + (\varphi'(U_0))_{zz}) = 3\beta.$$

On the set J_2, which is compact in $\mathbb{R}$, there exist positive constants c_1, c_2 such that

$$U_{0z} \geq c_1, \quad |f'(U_0) + (\varphi'(U_0))_{zz}| \leq c_2.$$

Therefore, we have

$$\frac{U_{0z}}{\sigma} - (f'(U_0) + (\varphi'(U_0))_{zz}) \geq \frac{c_1}{\sigma} - c_2 \to \infty \text{ as } \sigma \downarrow 0,$$

which implies (47) on J_2 for σ small enough. □

Before we present the rigorous proof that $u^{\pm}$ are sub- and super-solutions, we first give detailed computations needed in the sequel. Recall (44). First, note, with U_0 and U_1 corresponding to u^+, that

$$\begin{aligned}
\varphi(u^+) &= \varphi(U_0) + (\varepsilon U_1 + q)\varphi'(U_0) + (\varepsilon U_1 + q)^2 \int_0^1 (1-s)\varphi''(U_0 + (\varepsilon U_1 + q)s)ds \\
f(u^+) &= f(U_0) + (\varepsilon U_1 + q)f'(U_0) + \frac{(\varepsilon U_1 + q)^2}{2} f''(\theta(x,t)),
\end{aligned} \tag{48}$$

where θ is a function satisfying $\theta(x,t) \in (U_0, U_0 + \varepsilon U_1 + q(t))$. Straightforward computations yield

$$\begin{aligned}
(u^+)_t &= U_{0z}\left(\frac{d_t + \varepsilon p_t}{\varepsilon}\right) + \varepsilon U_{1t} + U_{1z}(d_t + \varepsilon p_t) + q_t \\
\Delta\varphi(u^+) &= \nabla\cdot\left((\varphi(U_0))_z\frac{\nabla d}{\varepsilon} + U_{1z}\varphi'(U_0)\nabla d + \varepsilon\nabla U_1\varphi'(U_0) + (\varepsilon U_1 + q)(\varphi'(U_0))_z\frac{\nabla d}{\varepsilon} + \nabla R\right) \\
&= (\varphi(U_0))_{zz}\frac{|\nabla d|^2}{\varepsilon^2} + (\varphi(U_0))_z\frac{\Delta d}{\varepsilon} \\
&+ (U_{1z}\varphi'(U_0))_z\frac{|\nabla d|^2}{\varepsilon} + U_{1z}\varphi'(U_0)\Delta d + 2\nabla U_{1z}\varphi'(U_0)\cdot\nabla d + \nabla U_1(\varphi'(U_0))_z\cdot\nabla d + \varepsilon\Delta U_1\varphi'(U_0) \\
&+ (U_1\varphi'(U_0)_z)_z\frac{|\nabla d|^2}{\varepsilon} + q(\varphi'(U_0))_{zz}\frac{|\nabla d|^2}{\varepsilon^2} + \nabla U_1(\varphi'(U_0))_z\cdot\nabla d \\
&+ (\varepsilon U_1 + q)(\varphi'(U_0))_z\frac{\Delta d}{\varepsilon} + \Delta R
\end{aligned} \tag{49}$$

where $R(x,t) = (\varepsilon U_1 + q)^2\int_0^1(1-s)\varphi''(U_0 + (\varepsilon U_1 + q)s)ds$. Define $r(x,t) = \int_0^1(1-s)\varphi''(U_0 + (\varepsilon U_1 + q)s)ds$. Then, we have

$$\begin{aligned}
\Delta R(x,t) &= \nabla\cdot\nabla\Big[\Big((\varepsilon U_1)^2 + 2\varepsilon q U_1 + q^2\Big)r\Big] \\
&= \nabla\cdot\Big[\Big(2\varepsilon U_1\left(U_{1z}\nabla d + \varepsilon\nabla U_1\right) + 2q\left(U_{1z}\nabla d + \varepsilon\nabla U_1\right)\Big]r(x,t) + \Big((\varepsilon U_1)^2 + 2\varepsilon q U_1 + q^2\Big)\nabla r(x,t)\Big] \\
&= \left[2\left(U_{1z}\nabla d + \varepsilon\nabla U_1\right)^2 + 2\varepsilon U_1\left(U_{1zz}\frac{|\nabla d|^2}{\varepsilon} + U_{1z}\Delta d + 2\nabla U_{1z}\cdot\nabla d + \varepsilon\Delta U_1\right)\right]r(x,t) \\
&+ 2q\left(U_{1zz}\frac{|\nabla d|^2}{\varepsilon} + U_{1z}\Delta d + 2\nabla U_{1z}\cdot\nabla d + \varepsilon\Delta U_1\right)r(x,t) \\
&+ 2\Big[2\varepsilon U_1\left(U_{1z}\nabla d + \varepsilon\nabla U_1\right) + 2q\left(U_{1z}\nabla d + \varepsilon\nabla U_1\right)\Big]\nabla r(x,t) \\
&+ \Big((\varepsilon U_1)^2 + 2\varepsilon q U_1 + q^2\Big)\Delta r(x,t)
\end{aligned} \tag{50}$$

where

$$\begin{aligned}
\nabla r(x,t) &= \int_0^1 (1-s)\varphi'''(U_0 + (\varepsilon U_1 + q)s)\left((U_0 + \varepsilon U_1 s)_z\frac{\nabla d}{\varepsilon} + \varepsilon\nabla U_1 s\right)ds \\
\Delta r(x,t) &= \int_0^1 (1-s)\varphi'''(U_0 + (\varepsilon U_1 + q)s)\left((U_0 + \varepsilon U_1 s)_z\frac{\Delta d}{\varepsilon}\right. \\
&\qquad \left. + (U_0 + \varepsilon U_1 s)_{zz}\frac{|\nabla d|^2}{\varepsilon^2} + (2\nabla U_{1z}\cdot\nabla d + \varepsilon\Delta U_1)s\right)ds \\
&\qquad + \int_0^1 (1-s)\varphi^{(4)}(U_0 + (\varepsilon U_1 + q)s)\left((U_0 + \varepsilon U_1 s)_z\frac{\nabla d}{\varepsilon} + \varepsilon\nabla U_1 s\right)^2 ds.
\end{aligned}$$

Define $l(x,t), r_i(x,t)$ for $i = 1,2,3$ as follows:

$$
\begin{aligned}
l(x,t) &= U_{1zz}\frac{|\nabla d|^2}{\varepsilon} + U_{1z}\Delta d + 2\nabla U_{1z}\cdot\nabla d + \varepsilon\Delta U_1 \\
r_1(x,t) &= \left[2\left(U_{1z}\nabla d + \varepsilon\nabla U_1\right)^2 + 2\varepsilon U_1 l(x,t)\right] r(x,t) + 4\varepsilon U_1\left(U_{1z}\nabla d + \varepsilon\nabla U_1\right)\nabla r(x,t) + (\varepsilon U_1)^2\Delta r(x,t) \\
r_2(x,t) &= 2ql(x,t)r(x,t) + 4q\left(U_{1z}\nabla d + \varepsilon\nabla U_1\right)\nabla r(x,t) + 2\varepsilon q U_1\Delta r(x,t) \\
r_3(x,t) &= q^2\Delta r(x,t).
\end{aligned}
$$

Thus,

$$\Delta R = r_1 + r_2 + r_3. \tag{51}$$

We have the following properties for r_i.

Lemma 9. *There exists $C_r > 0$ independent of ε such that*

$$|r_1| \le C_r, \quad |r_2| \le \frac{q}{\varepsilon}C_r, \quad |r_3| \le \frac{q^2}{\varepsilon^2}C_r. \tag{52}$$

Proof. Note that, by Lemmas 5, 6 and (45) the term $U_a := U_0 + (\varepsilon U_1 + q)s$ is uniformly bounded. Hence, the terms $\varphi''(U_a), \varphi'''(U_a), \varphi^{(4)}(U_a)$ are uniformly bounded, and in particular r is bounded. By a similar reasoning for ∇r and Δr, it follows that there exists some positive constants c_∇, c_Δ such that

$$|\nabla r| \le \frac{c_\nabla}{\varepsilon}, \quad |\Delta r| \le \frac{c_\Delta}{\varepsilon^2}.$$

Moreover, by Lemmas 5, 6 there exists a positive constant c_l such that

$$|l(x,t)| \le \frac{c_l}{\varepsilon}.$$

Combining these estimates yields (52).

□

Let σ a fixed constant satisfying

$$0 < \sigma \le \min\{\sigma_0, \sigma_1, \sigma_2\}, \tag{53}$$

where σ_0 is the constant defined in Lemma 8, and σ_1 and σ_2 are given by

$$\sigma_1 = \frac{1}{2(\beta+1)}, \quad \sigma_2 = \frac{\beta}{(F+C_r)(\beta+1)}, \quad F = ||f''||_{L^\infty(\alpha_- - 1, \alpha_+ + 1)}. \tag{54}$$

Note that, since $\sigma < \sigma_1$ and (45), we have

$$\alpha_- - 1 \le |u^\pm| \le \alpha_+ + 1.$$

Lemma 10. *Let β be given by (46) and let σ satisfy (53). Then, there exists $\varepsilon_0 > 0$ and a positive constant C_p, which does not depend on ε, such that*

$$\begin{cases} \mathcal{L}(u^-) < -C_p < C_p < \mathcal{L}(u^+) & \text{in } \overline{D}\times[0,T] \\ \dfrac{\partial u^-}{\partial \nu} = \dfrac{\partial u^+}{\partial \nu} = 0 & \text{on } \partial D\times[0,T] \end{cases} \tag{55}$$

for every $\varepsilon \in (0, \varepsilon_0)$.

Proof. In the following, we only show that u^+ is a super solution; one can show that u^- is a sub-solution in a similar way.

Combining the computations above in (48), (49), (50) and (51), we obtain

$$\begin{aligned}\mathcal{L}u^+ &= (u^+)_t - \Delta(\varphi(u^+)) - \frac{1}{\varepsilon^2}f(u^+)\\ &= E_1 + E_2 + E_3 + E_4 + E_5 + E_6,\end{aligned}$$

where

$$\begin{aligned}
E_1 &= -\frac{1}{\varepsilon^2}\left((\varphi(U_0))_{zz}|\nabla d|^2 + f(U_0)\right) - \frac{|\nabla d|^2-1}{\varepsilon^2}q(\varphi'(U_0))_{zz} - \frac{|\nabla d|^2-1}{\varepsilon}(U_1\varphi'(U_0))_{zz}\\
E_2 &= \frac{1}{\varepsilon}U_{0z}d_t - \frac{1}{\varepsilon}\left((\varphi(U_0))_z\Delta d + (U_{1z}\varphi'(U_0))_z + (U_1\varphi'(U_0)_z)_z + U_1f'(U_0)\right)\\
E_3 &= [U_{0z}p_t + q_t] - \frac{1}{\varepsilon^2}\left[qf'(U_0) + q(\varphi'(U_0))_{zz} + \frac{q^2}{2}f''(\theta)\right] - r_3(x,t)\\
E_4 &= \varepsilon U_{1z}p_t - \frac{q}{\varepsilon}\Big[(\varphi'(U_0))_z\Delta d + U_1f''(\theta)\Big] - r_2(x,t)\\
E_5 &= \varepsilon U_{1t} - \varepsilon\Delta U_1\varphi'(U_0)\\
E_6 &= U_{1z}d_t - 2\nabla U_{1z}\varphi'(U_0)\cdot\nabla d - 2\nabla U_1(\varphi'(U_0))_z\cdot\nabla d - (U_1\varphi'(U_0))_z\Delta d - r_1(x,t) - \frac{(U_1)^2}{2}f''(\theta).
\end{aligned}$$

Estimate of the term E_1. Using (20) we write E_1 in the form

$$E_1 = -\frac{|\nabla d|^2-1}{\varepsilon^2}\big((\varphi(U_0))_{zz} + q(\varphi'(U_0))_{zz}\big) - \frac{|\nabla d|^2-1}{\varepsilon}(U_1\varphi'(U_0))_{zz}.$$

We only consider the term $e_1 := \dfrac{|\nabla d|^2-1}{\varepsilon}(U_1\varphi'(U_0))_{zz}$; the other terms can be bounded similarly. In the region where $|d| \le d_0$, we have $|\nabla d| = 1$ so that $e_1 = 0$. If, however $|\nabla d| \neq 1$, we have

$$\frac{|(U_1\varphi'(U_0))_{zz}|}{\varepsilon} \le \frac{\hat{C}_1}{\varepsilon}e^{-\lambda_1\left|\frac{d}{\varepsilon}+p(t)\right|} \le \frac{\hat{C}_1}{\varepsilon}e^{-\lambda_1\left[\frac{d_0}{\varepsilon}-p(t)\right]} \le \frac{\hat{C}_1}{\varepsilon}e^{-\lambda_1\left[\frac{d_0}{\varepsilon}-(1+e^{LT}+K)\right]}.$$

Choosing ε_0 small enough such that

$$\frac{d_0}{2\varepsilon_0} - \Big(1 + e^{LT} + K\Big) \ge 0,$$

we deduce

$$\frac{|(U_1\varphi'(U_0))_{zz}|}{\varepsilon} \le \frac{\hat{C}_1}{\varepsilon}e^{-\lambda_1\frac{d_0}{2\varepsilon}} \to 0 \text{ as } \varepsilon \downarrow 0.$$

Thus, $\frac{1}{\varepsilon}|(U_1\varphi'(U_0))_{zz}|$ is uniformly bounded, so that there exists $\hat{C}_2$ independent of ε, L such that

$$|e_1| \le \hat{C}_2.$$

Finally, as a consequence, we deduce that there exists $\tilde{C}_1$ independent of ε, L such that

$$|E_1| \le \tilde{C}_1. \tag{56}$$

Estimate of the term E_2. Using (42), we write E_2 in the form

$$E_2 = \frac{1}{\varepsilon}U_{0z}d_t - \frac{1}{\varepsilon}\lambda_0U_{0z}\Delta d = \frac{U_{0z}}{\varepsilon}(d_t - \lambda_0\Delta d).$$

Applying Lemma 4, 5 and 6 gives

$$|E_2| \leq C_d \hat{C}_0 \frac{|d|}{\varepsilon} e^{-\lambda_1 |\frac{d}{\varepsilon}+p|} \leq C_d \hat{C}_0 \max_{\xi \in \mathbb{R}} |\xi| e^{-\lambda_1 |\xi+p|}.$$

Note that $\max_{\xi\in\mathbb{R}} |\xi| e^{-\lambda_1|\xi+p|} \leq |p| + \frac{1}{\lambda_1}$ (cf. [13]). Thus, there exists $\tilde{C}_2$ such that

$$|E_2| \leq \tilde{C}_2(1 + e^{LT}). \tag{57}$$

Estimate of the term E_3. Substituting $p_t = \dfrac{q}{\varepsilon^2\sigma}$ and then replacing q by its explicit form (cf. (44)) gives

$$\begin{aligned}
E_3 &= \frac{q}{\varepsilon^2\sigma}\left[U_{0z} - \sigma(f'(U_0) + (\varphi'(U_0))_{zz}) - \sigma q\left(\frac{1}{2}f''(\theta) + \frac{\varepsilon^2}{q^2}r_3\right)\right] + q_t \\
&= \frac{1}{\varepsilon^2}\left(\beta e^{-\frac{\beta t}{\varepsilon^2}} + \varepsilon^2 L e^{Lt}\right)\left[U_{0z} - \sigma(f'(U_0) + (\varphi'(U_0))_{zz}) - \sigma^2(\beta e^{-\frac{\beta t}{\varepsilon^2}} + L\varepsilon^2 e^{Lt})\left(\frac{1}{2}f''(\theta) + \frac{\varepsilon^2}{q^2}r_3\right)\right] \\
&- \frac{1}{\varepsilon^2}\sigma\beta^2 e^{-\frac{\beta t}{\varepsilon^2}} + \varepsilon^2\sigma L^2 e^{Lt} \\
&= \frac{1}{\varepsilon^2}\beta e^{-\frac{\beta t}{\varepsilon^2}}(I - \sigma\beta) + L e^{Lt}[I + \varepsilon^2\sigma L]
\end{aligned}$$

where

$$I := U_{0z} - \sigma(f'(U_0) + (\varphi'(U_0))_{zz}) - \sigma^2(\beta e^{-\frac{\beta t}{\varepsilon^2}} + L\varepsilon^2 e^{Lt})\left(\frac{1}{2}f''(\theta) + \frac{\varepsilon^2}{q^2}r_3\right).$$

Applying Lemma 8, using (45) and (53), yields

$$\begin{aligned}
I &\geq 3\sigma\beta - \sigma\sigma_2\left(\beta + L\varepsilon^2 e^{Lt}\right)\left(|f''(\theta)| + \frac{\varepsilon^2}{q^2}|r_3|\right) \\
&\geq 3\sigma\beta - \sigma\sigma_2\left(\beta + 1\right)\left(|f''(\theta)| + \frac{\varepsilon^2}{q^2}|r_3|\right) \\
&\geq 2\sigma\beta,
\end{aligned}$$

where the last inequality follows from (54). This implies that

$$E_3 \geq \frac{\sigma\beta^2}{\varepsilon^2} e^{-\frac{\beta t}{\varepsilon^2}} + 2\sigma\beta L e^{Lt}. \tag{58}$$

Estimate of the term E_4. Substituting again $p_t = \dfrac{q}{\varepsilon^2\sigma}$, with q in its explicit form (44) gives

$$\begin{aligned}
E_4 &= \frac{q}{\varepsilon\sigma}\left(U_{1z} - \sigma((\varphi'(U_0))_z\Delta d + U_1 f''(\theta)) - \sigma\frac{\varepsilon}{q}r_2\right) \\
&= \frac{1}{\varepsilon}\left(\beta e^{-\frac{\beta t}{\varepsilon^2}} + \varepsilon^2 L e^{Lt}\right)\left(U_{1z} - \sigma((\varphi'(U_0))_z\Delta d + U_1 f''(\theta)) - \sigma\frac{\varepsilon}{q}r_2\right).
\end{aligned}$$

Applying Lemma 4, 5, 6 and 9 gives the uniform boundedness of the last factor in parenthesis. Thus, there exists a constant $\tilde{C}_4$ such that

$$|E_4| \leq \tilde{C}_4 \frac{1}{\varepsilon}\left(\beta e^{-\frac{\beta t}{\varepsilon^2}} + \varepsilon^2 L e^{Lt}\right). \tag{59}$$

Estimate of the terms E_5 and E_6. Applying Lemma 4, 5 and 6, it follows that there exists $\tilde{C}_5$ such that

$$|E_5| + |E_6| \leq \tilde{C}_5. \tag{60}$$

Combination of the above estimates. Collecting the estimates (56),(57),(58),(59),(60), we obtain

$$\begin{aligned}\mathcal{L}(u^+) &\geq \left[\frac{\sigma\beta^2}{\varepsilon^2} - \tilde{C}_4\frac{\beta}{\varepsilon}\right] e^{-\frac{\beta t}{\varepsilon^2}} + \left[2\sigma\beta L - \varepsilon\tilde{C}_4 L - \tilde{C}_2\right] e^{Lt} - \tilde{C}_1 - \tilde{C}_2 - \tilde{C}_5 \\ &\geq \left[\frac{\sigma\beta^2}{\varepsilon^2} - \tilde{C}_4\frac{\beta}{\varepsilon}\right] e^{-\frac{\beta t}{\varepsilon^2}} + \left[\frac{2\sigma\beta L}{3} - \varepsilon\tilde{C}_4 L\right] e^{Lt} \\ &+ \left[\frac{2\sigma\beta L}{3} - \tilde{C}_2\right] e^{Lt} + \left[\frac{2\sigma\beta L}{3} - \tilde{C}_6\right]\end{aligned}$$

where $\tilde{C}_6 = \tilde{C}_1 + \tilde{C}_2 + \tilde{C}_5$. Choose ε_0 small enough and L large enough so that

$$\sigma\beta > 3\tilde{C}_4\varepsilon_0, \ \ \sigma\beta L > 3\max\{\tilde{C}_2, \tilde{C}_6\}.$$

Then, we deduce that there exists a positive constant C_p, independent of ε, such that $\mathcal{L}(u^+) \geq C_p$. □

4.4 Proof of Theorem 1.2

The proof of Theorem 1.2 is divided in two steps: (i) For large enough $K > 0$, we prove that $u^-(x,t) \leq u^\varepsilon(x, t+t^\varepsilon) \leq u^+(x,t)$ for $x \in \overline{D}, t \in [0, T - t^\varepsilon]$ and (ii) we employ (i) to show the desired result.

Step 1. Let $\eta \in (0, \eta_0)$ be arbitrary. Fix σ, β that satisfy (46), (53) and

$$\sigma\beta < \frac{\eta}{2}.$$

Theorem 1.1 implies the existence of constants ε_0 and M_0 such that (10)-(12) are satisfied with the constant η replaced by $\sigma\beta/4$. Conditions (7) and (8) imply that there exists a positive constant M_1 such that

$$\begin{aligned}&\text{if } d(x,0) \leq -M_1\varepsilon, \quad \text{then } u_0(x) \leq -M_0\varepsilon, \\ &\text{if } d(x,0) \geq M_1\varepsilon, \quad \text{then } u_0(x) \geq +M_0\varepsilon.\end{aligned}$$

Hence, we deduce, by applying (10), (12), that

$$u^\varepsilon(x,t^\varepsilon) \leq H^+(x) := \begin{cases} \alpha_+ + \dfrac{\sigma\beta}{4} & \text{if } d(x,0) \geq -M_1\varepsilon \\ \alpha_- + \dfrac{\sigma\beta}{4} & \text{if } d(x,0) < -M_1\varepsilon. \end{cases}$$

Also, by applying (10), (11),

$$u^\varepsilon(x,t^\varepsilon) \geq H^-(x) := \begin{cases} \alpha_+ - \dfrac{\sigma\beta}{4} & \text{if } d(x,0) > M_1\varepsilon \\ \alpha_- - \dfrac{\sigma\beta}{4} & \text{if } d(x,0) \leq M_1\varepsilon. \end{cases}$$

Next, we fix a sufficient large constant K such that

$$U_0(M_1 - K) \leq \alpha_- + \frac{\sigma\beta}{4} \quad \text{and} \quad U_0(-M_1 + K) \geq \alpha_+ - \frac{\sigma\beta}{4}.$$

For such a constant K, Lemma 10 implies the existence of coefficients ε_0 and L such that the inequalities in (55) holds. We claim that

$$u^-(x,0) \leq H^-(x) \leq H^+(x) \leq u^+(x,0). \tag{61}$$

We only prove the last inequality since the first inequality can be proved similarly. By Lemma 6, we have $|U_1| \le \hat{C}_1$. Thus, we can choose ε_0 small enough so that, for $\varepsilon \in (0, \varepsilon_0)$, we have $\varepsilon \hat{C}_1 \le \dfrac{\sigma\beta}{4}$. Then, noting (44),

$$\begin{aligned} u^+(x,0) &\ge U_0\left(\frac{d(x,0)+\varepsilon p(0)}{\varepsilon}\right) - \varepsilon\hat{C}_1 + \sigma\beta + \varepsilon^2\sigma L \\ &> U_0\left(\frac{d(x,0)}{\varepsilon} + K\right) + \frac{3}{4}\sigma\beta. \end{aligned}$$

In the set $\{x \in D : d(x,0) \ge -M_1\varepsilon\}$, the inequalities above, and the fact that U_0 is an increasing function imply

$$u^+(x,0) > U_0(-M_1 + K) + \frac{3}{4}\sigma\beta \ge \alpha_+ + \frac{\sigma\beta}{2} > H^+(x).$$

Moreover, since $U_0 \ge \alpha_-$ in the set $\{x \in D : d(x,0) < -M_1\varepsilon\}$, we have

$$u^+(x,0) > \alpha_- + \frac{3}{4}\sigma\beta > H^+(x).$$

Thus, we proved the first inequality in (61) above.

The inequalities (61) and Lemma 10 now permit to apply the comparison principle Lemma 1, so that we have

$$u^-(x,t) \le u^\varepsilon(x, t+t^\varepsilon) \le u^+(x,t) \quad \text{for} \quad x \in \overline{D}, t \in [0, T - t^\varepsilon]. \tag{62}$$

Step 2. Choose $C > 0$ so that

$$U_0(C - e^{LT} - K) \ge \alpha_+ - \frac{\eta}{2} \ \text{ and } \ U_0(-C + e^{LT} + K) \le \alpha_- + \frac{\eta}{2}.$$

Then, we deduce from (62), noting (44), that

$$\begin{aligned} &\text{if } d(x,t) \ge \varepsilon C, \text{ then } u^\varepsilon(x, t+t^\varepsilon) \ge \alpha_+ - \eta \\ &\text{if } d(x,t) \le -\varepsilon C, \text{ then } u^\varepsilon(x, t+t^\varepsilon) \le \alpha_- + \eta \end{aligned}$$

and since $\alpha_\pm \pm \eta$ are respectively sub- and super-solutions of (P^ε), we conclude that

$$u^\varepsilon(x, t+t^\varepsilon) \in [\alpha_- - \eta, \alpha_+ + \eta]$$

for all $(x,t) \in D \times [0, T - t^\varepsilon], \varepsilon \in (0, \varepsilon_0)$. □

Remark 2. *These sub and super solutions guarantee that $u^\varepsilon \simeq \alpha_+$ (respectively, $u^\varepsilon \simeq \alpha_-$) for $d(x,t) \ge c$ (respectively, $d(x,t) \le -c$) with $t > \rho t^\varepsilon, \rho > 1$ and $\varepsilon > 0$ small enough. In fact, by the definition of $q(t)$, we expect*

$$\varepsilon U_1 \pm q(t) = \mathcal{O}(\varepsilon)$$

for $t > (\rho - 1)t^\varepsilon$. Also, by Lemma 5, we expect

$$0 < U_0(z) - \alpha_- < \tilde{c}\varepsilon \text{ for } z > \frac{c}{\varepsilon}, \quad 0 < \alpha_+ - U_0(z) < \tilde{c}\varepsilon \text{ for } z < -\frac{c}{\varepsilon}.$$

These estimates yield that there exists a positive constant c' such that

$$|u^\varepsilon(x,t) - \alpha_+| \le c'\varepsilon \text{ for } d(x,t) > c, \quad |u^\varepsilon(x,t) - \alpha_-| \le c'\varepsilon \text{ for } d(x,t) < -c$$

for $t > \rho t^\varepsilon$.

5 Proof of Theorem 1.3

We now introduce the concept of an eternal solution. A solution of an evolution equation is called *eternal* if it is defined for all positive and negative times. In our problem, we study the nonlinear diffusion problem

$$w_\tau = \Delta\varphi(w) + f(w), \quad ((z', z^{(N)}), \tau) \in \mathbb{R}^N \times \mathbb{R}, \tag{63}$$

where $z' \in \mathbb{R}^{N-1}$ and $z^{(N)} \in \mathbb{R}$. In order to prove Theorem 1.3, we first present two lemmas.

Lemma 11. *Let S be a domain of $\mathbb{R}^N \times \mathbb{R}$ and let u be a bounded function on S satisfying*

$$u_t = \Delta\varphi(u) + f(u), \quad (x,t) \in S, \tag{64}$$

where φ, f satisfy conditions (1), (3). Then, for any smooth bounded subset $S' \subset S$ separated from ∂S by a positive constant $\tilde{d}$ we have

$$\|u\|_{C^{2+\theta,1+\theta/2}(\overline{S'})} \leq C', \tag{65}$$

for any positive constants $0 < \theta < 1$ and C' which depends on $\|u\|_{L^\infty(S)}$, φ, f, $\tilde{d}, \theta$ and the size of S', where

$$\|u\|_{C^{k+\theta,k'+\theta'}(\overline{S'})} = \|u\|_{C^{k,k'}(\overline{S'})} + \sum_{i,j=1}^{N} \sup_{(x,t),(y,t)\in S', x\neq y} \left\{ \frac{|D_x^k u(x,t) - D_x^k u(y,t)|}{|x-y|^\theta} \right\}$$
$$+ \sup_{(x,t),(x,t')\in S', t\neq t'} \left\{ \frac{|D_t^{k'} u(x,t) - D_t^{k'} u(x,t')|}{|t-t'|^{\theta'}} \right\}$$

where k, k' are non-negative integers and $0 < \theta, \theta' < 1$.

Proof. Since S' is separated from ∂S by a positive distance, we can find subsets S_1, S_2 such that $S' \subset S_2 \subset S_1 \subset S$ and such that $\partial S, \partial S', \partial S_i$ are separated by a positive distance less than $\tilde{d}$. By condition (3) the regularity of $u(x,t)$ is the same as the regularity of $v(x,t) = \varphi(u(x,t))$. Note that by (64) v satisfies

$$v_t = \varphi'(\varphi^{-1}(v))[\Delta v + g(v)] \ , g(s) = f(\varphi^{-1}(s))$$

on S. By Theorem 3.1 p. 437-438 of [15], there exists a positive constant c_1 such that

$$|\nabla v| \leq c_1 \text{ in } S_1$$

where c_1 depends only on $N, \varphi, ||u||_{L^\infty(S)}$ and the distance between S and S_1. This, together with Theorem 5, p 122 of [10], imply that

$$\|v\|_{W_p^{2,1}(S_2)} \leq c_2(\|v\|_{L^p(S_1)} + \|\varphi'(\varphi^{-1}(v))g(v)\|_{L^p(S_1)})$$

for any $p > N+2$ where c_2 is a constant that depends on c_1, p, N, φ. With this, by fixing p large enough, the Sobolev embedding theorem in chapter 2, section 3 of [15] yields

$$\|v\|_{C^{1+\theta,(1+\theta)/2}(S_2)} \leq c_3 \|v\|_{W_p^{2,1}(S_2)}$$

where $0 < \theta < 1 - \frac{N+2}{p}$ and c_3 depends on c_2 and p. This implies that $\varphi'(\varphi^{-1}(v)), g(v)$ are bounded uniformly in $C^{1+\theta,(1+\theta)/2}(S_2)$. Therefore, by Theorem 10.1 p 351-352 of [15] we obtain

$$\|v\|_{C^{2+\theta,1+\theta/2}(S')} \leq c_4 \|v\|_{C^{1+\theta,(1+\theta)/2}(S_2)}$$

where c_4 depends on c_2, f and φ. □

Remark 3. *Lemma 11 implies the uniform $C^{2,1}$ boundedness of the entire solution w in the whole space. This can be derived as follows: Let*

$$S_{(a,b)} = \{(x,t) \in \mathbb{R}^N \times \mathbb{R}, |x-a|^2 + (t-b)^2 \le 2\},\ S'_{(a,b)} = \{(x,t) \in \mathbb{R}^N \times \mathbb{R}, |x-a|^2 + (t-b)^2 \le 1\}$$

where $(a,b) \in \mathbb{R}^N \times \mathbb{R}$. Then, Lemma 11 implies the uniform $C^{2,1}$ boundedness of w within $S'_{(a,b)}$ where the upper bound is fixed by (65)*. Since this upper bound is independent to the choice of (a,b), we have uniform $C^{2,1}$ bound of w in the whole space.*

Next, we present a result inspired by a similar one in [5].

Lemma 12. *Let $w((z', z^{(N)}), \tau)$ be a bounded eternal solution of* (63) *satisfying*

$$\liminf_{z^{(N)} \to -\infty} \inf_{z' \in \mathbb{R}^{N-1}, \tau \in \mathbb{R}} w((z', z^{(N)}), \tau) = \alpha_-, \quad \limsup_{z^{(N)} \to \infty} \sup_{z' \in \mathbb{R}^{N-1}, \tau \in \mathbb{R}} w((z', z^{(N)}), \tau) = \alpha_+, \tag{66}$$

where $z' = (z^{(1)}, z^{(2)}, \cdots z^{(N-1)})$. Then, there exists a constant $z^* \in \mathbb{R}$ such that

$$w((z', z^{(N)}), \tau) = U_0(z^{(N)} - z^*).$$

Proof. We prove the lemma in two steps. First we show w is an increasing function with respect to the $z^{(N)}$ variable. Then, we prove that w only depends on $z^{(N)}$, which means that there exists a function $\psi : \mathbb{R} \to (\alpha_-, \alpha_+)$ such that

$$w((z', z^{(N)}), \tau) = \psi(z^{(N)}), \quad ((z', z^{(N)}), \tau) \in \mathbb{R}^N \times \mathbb{R}.$$

From the increasing property with respect to $z^{(N)}$, this allows us to identify ψ as the unique standing wave solution U_0 of the problem (20) up to a translation factor z^*.

We deduce from (66) that there exist $A > 0$ and $\eta \in (0, \eta_0)$ such that

$$\begin{cases} \alpha_+ - \eta \le w((z', z^{(N)}), \tau) \le \alpha_+ + \eta, & z^{(N)} \ge A \\ \alpha_- - \eta \le w((z', z^{(N)}), \tau) \le \alpha_- + \eta, & z^{(N)} \le -A \end{cases} \tag{67}$$

where η_0 is defined in (9).

Let $\tilde{\tau} \in \mathbb{R}, \rho \in \mathbb{R}^{N-1}$ be arbitrary. Define

$$w^s((z', z^{(N)}), \tau) := w((z' + \rho, z^{(N)} + s), \tau + \tilde{\tau})$$

where $s \in \mathbb{R}$. Fix $\chi \ge 2A$ and define

$$b^* := \inf \left\{ b > 0 : \varphi(w^\chi) + b \ge \varphi(w) \text{ in } \mathbb{R}^N \times \mathbb{R} \right\}. \tag{68}$$

We will prove that $b^* = 0$, which will imply that $w^\chi \ge w$ in $\mathbb{R}^N \times \mathbb{R}$ since φ is a strictly increasing function. To see this, we assume that this does not hold, that is $b^* > 0$. Note, by (66) and (67) we have

$$w^\chi \ge \alpha_+ - \eta > \alpha_- + \eta \ge w \text{ if } z^{(N)} = -A, \ \lim_{z^{(N)} \to \pm\infty} \varphi(w^\chi) - \varphi(w) \to 0. \tag{69}$$

Let $E = \{(x,t) \in \mathbb{R}^N \times \mathbb{R}, \varphi(w) - \varphi(w^\chi) > 0\}$. Define a function Z on E as follows

$$Z((z', z^{(N)}), \tau) := e^{-C_Z \tau}[\varphi(w) - \varphi(w^\chi)]((z', z^{(N)}), \tau),$$
$$C_Z := \max\left(\sup_{x \in E} \frac{[\varphi'(w) - \varphi'(w^\chi)]\Delta\varphi(w) + [\varphi'(w)f(w) - \varphi'(w^\chi)f(w^\chi)]}{\varphi(w) - \varphi(w^\chi)}, 0 \right) \ge 0.$$

Note that C_Z is bounded, since w^χ is bounded uniformly in $C^{2,1}(E)$ by Remark 3 and in view of (1) and (3) we have

$$\lim_{x\to y}\frac{\varphi'(x)-\varphi'(y)}{\varphi(x)-\varphi(y)}=\frac{\varphi''(y)}{\varphi'(y)}<\infty, \lim_{x\to y}\frac{\varphi'(x)f(x)-\varphi'(y)f(y)}{\varphi(x)-\varphi(y)}=\frac{(\varphi' f)'(y)}{\varphi'(y)}<\infty. \tag{70}$$

Direct computations give

$$\begin{aligned} Z_\tau-\varphi'(w^\chi)\Delta Z &= e^{-C_Z\tau}\varphi'(w)[\Delta\varphi(w)+f(w)]-e^{-C_Z\tau}\varphi'(w^\chi)[\Delta\varphi(w^\chi)+f(w^\chi)]\\ &\quad -C_ZZ-e^{-C_Z\tau}\varphi'(w^\chi)[\Delta\varphi(w)-\Delta\varphi(w^\chi)]\\ &= \Big([\varphi'(w)-\varphi'(w^\chi)]\Delta\varphi(w)+[\varphi'(w)f(w)-\varphi'(w^\chi)f(w^\chi)]\Big)e^{-C_z\tau}-C_ZZ\\ &\le C_ZZ-C_ZZ=0 \end{aligned}$$

in E. Then, the maximum principle [17] Theorem 5 p.173 yields that the maximum of Z is located at the boundary of E. By the definition of E, $Z=0$ on the boundary of E which implies $Z\le 0$ in E. This contradicts the definition of E. Thus, we conclude that $b^*=0$.

Next, we prove that $w\le w^\chi$ for any $\chi>0$ (see (73) below). For this purpose, define

$$\chi^*:=\inf\left\{\chi\in\mathbb{R}, w^{\tilde\chi}\ge w \text{ for all } \ \tilde\chi\ge\chi\right\}. \tag{71}$$

Then, our goal can be obtained by proving that $\chi^*\le 0$. By the previous argument, we already know that $\chi^*\le 2A$. Since $w((z',-\infty),\tau)=\alpha_-$, it follows from (66) that $\chi^*>-\infty$, since otherwise we would have

$$\alpha_-=w^{-\infty}((z',z^{(N)}),\tau)\ge w,$$

leading to a contradiction since $w((z',+\infty),\tau)=\alpha_+>\alpha_-$. Thus, we conclude that $-\infty<\chi^*\le 2A$.

Assume that $\chi^*>0$, and define $E':=\{((z',z^{(N)}),\tau)\in\mathbb{R}^N\times\mathbb{R};\ |z^{(N)}|\le A\}$. If $\inf_{E'}(w^{\chi^*}-w)>0$, then there exists $\delta_0\in(0,\chi^*)$ such that $w\le w^{\chi^*-\delta}$ in E' for all $\delta\in(0,\delta_0)$. Since $w\le w^{\chi^*-\delta}$ on $\partial E'$, we deduce from a similar argument as above that $w\le w^{\chi^*-\delta}$ in $\{((z',z^{(N)}),\tau)\in\mathbb{R}^N\times\mathbb{R};\ |z^{(N)}|\ge A\}$. This contradicts the definition of χ^* in (71) so that $\inf_{E'}(w^{\chi^*}-w)=0$. Thus, we must have a sequence $((z_n',z_n),t_n)$ and $\tilde z_\infty\in[-A,A]$ such that

$$w((z_n',z_n),t_n)-w^{\chi^*}((z_n',z_n),t_n)\to 0,\quad z_n\to z_\infty \text{ as } n\to\infty.$$

Define $w_n((z',z^{(N)}),\tau):=w((z'+z_n',z^{(N)}),\tau+t_n)$. Since w_n is bounded uniformly in $C^{2+\theta,1+\theta/2}(\mathbb{R}^N\times\mathbb{R})$ by Lemma 11, w_n converges in $C^{2,1}_{loc}$ to a solution w_∞ of (63). Define $\tilde Z$ by

$$\tilde Z((z',z^{(N)}),\tau):=[\varphi(w_\infty^{\chi^*})-\varphi(w_\infty)]((z',z^{(N)}),\tau).$$

Since φ is strictly increasing, by (71) we have

$$\begin{cases}\tilde Z((z',z^{(N)}),\tau)\ge 0 \text{ in } \mathbb{R}^N\times\mathbb{R}\\ \tilde Z((0,z_\infty),0)=\lim_{n\to\infty}[\varphi(w_n^{\chi^*})-\varphi(w_n)]((0,z_n),0)=\lim_{n\to\infty}[\varphi(w^{\chi^*})-\varphi(w)]((z_n',z_n),t_n)=0.\end{cases} \tag{72}$$

Then, direct computation gives

$$\begin{aligned}\tilde Z_\tau-\varphi'(w_\infty^{\chi^*})\Delta\tilde Z &= \varphi'(w_\infty^{\chi^*})[\Delta\varphi(w_\infty^{\chi^*})+f(w_\infty^{\chi^*})]-\varphi'(w_\infty)[\Delta\varphi(w_\infty)+f(w_\infty)]\\ &\quad -\varphi'(w_\infty^{\chi^*})[\Delta\varphi(w_\infty^{\chi^*})-\Delta\varphi(w_\infty)]\\ &= [\varphi'(w_\infty^{\chi^*})-\varphi'(w_\infty)]\Delta\varphi(w_\infty)+[\varphi'(w_\infty^{\chi^*})f(w_\infty^{\chi^*})-\varphi'(w_\infty)f(w_\infty)],\end{aligned}$$

If $\tilde{Z}=0$ we obtain $\tilde{Z}_\tau-\varphi'(w_\infty^{\chi^*})\Delta\tilde{Z}=0$. If $\tilde{Z}>0$, we obtain

$$\begin{aligned}\tilde{Z}_\tau-\varphi'(w_\infty^{\chi^*})\Delta\tilde{Z}&=\left(\frac{[\varphi'(w_\infty^{\chi^*})-\varphi'(w_\infty)]\Delta\varphi(w_\infty)+[\varphi'(w_\infty^{\chi^*})f(w_\infty^{\chi^*})-\varphi'(w_\infty)f(w_\infty)]}{\varphi(w_\infty^{\chi^*})-\varphi(w_\infty)}\right)\tilde{Z}\\&\geq -C\tilde{Z},\end{aligned}$$

for some positive constant C, where the last inequality follows from (70) and the fact that $\Delta\varphi(w_\infty)$ is uniformly bounded in the whole space. Since by (72) $\tilde{Z}$ attains a non-positive minimum at $((0,z_\infty),0)$, we deduce from the maximum principle applied on the domain $\mathbb{R}^N\times(-\infty,0]$ that $\tilde{Z}=0$ for all $(z',z^{(N)})\in\mathbb{R}^N,\ \tau\leq 0$. Hence, $\tilde{Z}\equiv 0$ in $\mathbb{R}^N\times\mathbb{R}$. This implies that

$$w_\infty((0,0),0)=w_\infty((\rho,\chi^*),\tilde{\tau})=w_\infty((2\rho,2\chi^*),2\tilde{\tau})=\cdots=w_\infty((k\rho,k\chi^*),k\tilde{\tau})$$

for all $k\in\mathbb{Z}$, contradicting the fact that $w_\infty((k\rho,k\chi^*),k\tilde{\tau})\to\alpha_+$ as $k\to\infty$ and $w_\infty((k\rho,k\chi^*),k\tilde{\tau})\to\alpha_-$ as $k\to-\infty$.

Thus, we have $\chi^*\leq 0$, and therefore

$$w((z',z^{(N)}),\tau)\leq w^0((z',z^{(N)}),\tau)=w((z'+\rho,z^{(N)}),\tau+\tilde{\tau}) \tag{73}$$

hold for any $\rho\in\mathbb{R}^{N-1},\tilde{\tau}\in\mathbb{R}$.

We now show that w only depends on $z^{(N)}$. Suppose w depends on z' and τ. Then, there exist $z_1',z_2'\in\mathbb{R}^{N-1},z^{(N)}\in\mathbb{R}$ and $t_1',t_2'\in\mathbb{R}$ such that

$$w((z_1',z^{(N)}),t_1')<w((z_2',z^{(N)}),t_2'). \tag{74}$$

Then, by letting $z'=z_2',\ \rho=z_1'-z_2'$ and $\tau=t_2',\ \tilde{\tau}=t_1'-t_2'$ in the inequality (73), we deduce

$$w((z_2',z^{(N)}),t_2')\leq w((z_1',z^{(N)}),t_1'),$$

contradicting (74). This implies that w only depends on $z^{(N)}$, namely $w((z',z^{(N)}),\tau)=\psi(z^{(N)})$. Finally, from the definition of χ^*, we have that ψ is increasing. □

Proof of Theorem 1.3. We first prove (i). Recall that $d(x,t)$ is the cut-off signed distance function to the interface Γ_t moving according to equation (13), and $d^\varepsilon(x,t)$ is the signed distance function corresponding to the interface

$$\Gamma_t^\varepsilon:=\{x\in D,\ u^\varepsilon(x,t)=0\}.$$

Let T_1 be an arbitrary constant such that $\frac{T}{2}<T_1<T$. Assume by contradiction that (15) does not hold. Then, there exist $\eta>0$ and sequences $\varepsilon_k\downarrow 0,\ t_k\in[\rho t^{\varepsilon_k},T],\ x_k\in D$ such that $\alpha_+-\eta>0>\alpha_-+\eta$ and

$$\left|u^{\varepsilon_k}(x_k,t_k)-U_0\left(\frac{d^{\varepsilon_k}(x_k,t_k)}{\varepsilon_k}\right)\right|\geq\eta. \tag{75}$$

For the inequality (75) to hold, by Theorem 1.2 and $U_0(\pm\infty)=\alpha_\pm$, we need

$$d^{\varepsilon_k}(x_k,t_k)=\mathcal{O}(\varepsilon_k).$$

With these observations, and also by Theorem 1.2, there exists a positive constant $\tilde{C}$ such that

$$|d(x_k,t_k)|\leq\tilde{C}\varepsilon_k \tag{76}$$

for ε_k small enough.

If $x_k \in \Gamma_{t_k}^{\varepsilon_k}$, then the left-hand side of (75) vanishes, which contradicts this inequality. Since the sign can either be positive or negative, by extracting a subsequence if necessary we may assume that

$$u^{\varepsilon_k}(x_k, t_k) > 0 \quad \text{for all} \quad k \in \mathbb{N}, \tag{77}$$

which is equivalent to

$$d^{\varepsilon_k}(x_k, t_k) > 0 \quad \text{for all} \quad k \in \mathbb{N}.$$

By (76), each x_k has a unique orthogonal projection $p_k := p(x_k, t_k) \in \Gamma_{t_k}$. Let y_k be a point on $\Gamma_{t_k}^{\varepsilon_k}$ that has the smallest distance from x_k, and therefore $u^{\varepsilon_k}(y_k, t_k) = 0$. Moreover, we have

$$u^{\varepsilon_k}(x, t_k) > 0 \quad \text{if} \quad \|x - x_k\| < \|y_k - x_k\|. \tag{78}$$

We now rescale u^{ε_k} around (p_k, t_k). Define

$$w^k(z, \tau) := u^{\varepsilon_k}(p_k + \varepsilon_k \mathcal{R}_k z, t_k + \varepsilon_k^2 \tau), \tag{79}$$

where $\mathcal{R}_k$ is a orthogonal matrix in $SO(N, \mathbb{R})$ that rotates the $z^{(N)}$ axis, namely the vector $(0, \cdots, 0, 1) \in \mathbb{R}^N$ onto the unit normal vector to Γ_{t_k} at $p_k \in \Gamma_{t_k}$, say $\dfrac{x_k - p_k}{\|x_k - p_k\|}$. To prove our result, we use Theorem 1.2 which gives information about u^{ε_k} for $t_k + \varepsilon_k^2 \tau \geq t^{\varepsilon_k}$. Then, since Γ_t is separated from ∂D by some positive distance, w^k is well-defined at least on the box

$$B_k := \left\{ (z, \tau) \in \mathbb{R}^N \times \mathbb{R} : |z| \leq \frac{c}{\varepsilon_k}, \quad -(\rho - 1)\frac{|\ln \varepsilon_k|}{f'(0)} \leq \tau \leq \frac{T - T_1}{\varepsilon_k^2} \right\},$$

for some $c > 0$. We remark that $B_k \subset B_{k+1}, k \in \mathbb{N}$ and $\lim_{k\to\infty} B_k = \mathbb{R}^N \times \mathbb{R}$. Writing $\mathcal{R}_k = (r_{ij})_{1 \leq i,j \leq N}$, we remark that $\mathcal{R}_k^{-1} = \mathcal{R}_k^T$, which implies that

$$\sum_{i=1}^{N} r_{\ell i}^2 = 1, \quad \sum_{i=1, j \neq m}^{N} r_{ji} r_{\ell i} = 0. \tag{80}$$

Since

$$\partial_{z_i}^2 \varphi(w^k) = \varepsilon_k^2 \sum_{j=1}^{N} \sum_{\ell=1}^{N} r_{ji} r_{\ell i} \partial_{x_\ell x_j} \varphi(u^{\varepsilon_k}),$$

we have

$$\begin{aligned}
\Delta \varphi(w^k) &= \varepsilon_k^2 \sum_{i=1}^{N} \partial_{z_i}^2 \varphi(w^k) \\
&= \varepsilon_k^2 \sum_{i=1}^{N} \sum_{\ell=1}^{N} r_{\ell i}^2 \partial_{x_\ell}^2 \varphi(u^{\varepsilon_k}) + \varepsilon_k^2 \sum_{i=1}^{N} \sum_{j,\ell=1, j \neq \ell}^{N} r_{ji} r_{\ell i} \partial_{x_\ell x_j} \varphi(u^{\varepsilon_k}) \\
&= \varepsilon_k^2 \Delta \varphi(u^{\varepsilon_k}).
\end{aligned}$$

Thus, we obtain

$$w_\tau^k = \Delta \varphi(w^k) + f(w^k) \quad \text{in} \quad B_k.$$

From the propagation result in Theorem 1.2 and the fact that the rotation matrix $\mathcal{R}_k$ maps the $z^{(N)}$ axis to the unit normal vector of Γ_t at p_k, there exists a constant $C > 0$ such that

$$z^{(N)} \geq C \Rightarrow w^k(z, \tau) \geq \alpha_+ - \eta > 0, \quad z^{(N)} \leq -C \Rightarrow w^k(z, \tau) \leq \alpha_- + \eta < 0 \tag{81}$$

as long as $(z,\tau) \in B_k$.

It follows from the first line of (14) that $\alpha_- - \eta_0 \le w^k \le \alpha_+ + \eta_0$ for k large enough. Then, by Lemma 11 we can find a subsequence of (w^k) converging to some $w \in C^{2,1}(\mathbb{R}^N \times \mathbb{R})$ which satisfies

$$w_\tau = \Delta\varphi(w) + f(w) \quad on \quad \mathbb{R}^N \times \mathbb{R}.$$

From Remark 2 we can deduce (66). Then, by Lemma 12, there exists $z^* \in \mathbb{R}$ such that

$$w(z,\tau) = U_0(z^{(N)} - z^*). \tag{82}$$

Define sequences of points $\{z_k\}, \{\tilde{z}_k\}$ by

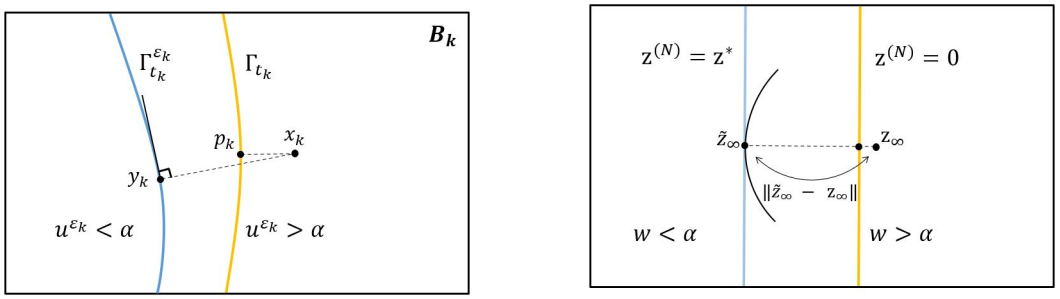
$$z_k := \frac{1}{\varepsilon_k}\mathcal{R}_k^{-1}(x_k - p_k), \quad \tilde{z}_k := \frac{1}{\varepsilon_k}\mathcal{R}_k^{-1}(y_k - p_k). \tag{83}$$

From (76) and Theorem 1.2, we have

$$\begin{aligned} &|d(x_k,t_k)| = \|x_k - p_k\| = \mathcal{O}(\varepsilon_k), \\ &\|y_k - p_k\| \le \|y_k - x_k\| + \|x_k - p_k\| = |d^{\varepsilon_k}(x_k,t_k)| + |d(x_k,t_k)| = \mathcal{O}(\varepsilon_k) \end{aligned}$$

(see Figure 2), which implies that the sequences z_k and $\tilde{z}_k$ are bounded. Thus, there exist subsequences of $\{z_k\}, \{\tilde{z}_k\}$ and $z_\infty, \tilde{z}_\infty \in \mathbb{R}^N$ such that

$$z_{k_n} \to z_\infty, \quad \tilde{z}_{k_n} \to \tilde{z}_\infty, \quad \text{as} \quad k \to \infty.$$



(a) Points x_k, y_k, p_k and interfaces $\Gamma_{t_k}, \Gamma_{t_k}^{\varepsilon_k}$ inside the box B_k.

(b) Points z_∞ and $\tilde{z}_\infty$ and hyperplanes $z^{(N)} = z^*, z^{(N)} = z_\infty^{(N)}$.

Figure 2: In (a) the distance between Γ_{t_k} and $\Gamma_{t_k}^{\varepsilon_k}$ is of $\mathcal{O}(\varepsilon_k)$. In (b), since we rescale space by ε^{-1}, the distance between two hyperplanes is of $\mathcal{O}(1)$.

Since the normal vector to Γ_{t_k} at p_k is equal to $x_k - p_k$, and the mapping $\mathcal{R}_k^{-1}$ sends the unit normal vector to Γ_{t_k} at p_k to the vector $(0,\cdots 0,1) \in \mathbb{R}^N$, we conclude z_∞ must lie on the $z^{(N)}$ axis so that we can write

$$z_\infty = (0,\cdots,0,z_\infty^{(N)}).$$

Since, by (77),

$$w(z_\infty,0) = \lim_{k_n\to\infty} w^{k_n}(z_{k_n},0) = \lim_{k_n\to\infty} u^{\varepsilon_{k_n}}(x_{k_n},t_{k_n}) \ge 0,$$

we deduce from (82) and the fact that U_0 is an increasing function that

$$w(z_\infty,0) = U_0(z_\infty^{(N)} - z^*) \ge 0 \Rightarrow z_\infty^{(N)} \ge z^*.$$

From the definition of y_{k_n} and (79), we have

$$w(\tilde{z}_\infty, 0) = \lim_{k\to\infty} w^{k_n}(\tilde{z}_{k_n}, 0) = \lim_{k\to\infty} u^{\varepsilon_{k_n}}(y_{k_n}, t_{k_n}) = 0. \tag{84}$$

Next, we show that

$$w(z, 0) \geq 0 \text{ if } \|z - z_\infty\| \leq \|\tilde{z}_\infty - z_\infty\|. \tag{85}$$

Choose $z \in \mathbb{R}^N$ satisfying $\|z - z_\infty\| \leq \|\tilde{z}_\infty - z_\infty\|$ and a sequence $a_{k_n} \in \mathbb{R}^+$ such that $a_{k_n} \to \|z - z_\infty\|$ and $\varepsilon_{k_n} a_{k_n} \leq \|x_{k_n} - y_{k_n}\|$ as $k \to \infty$. Then, we define sequences n_{k_n} and b_{k_n} by

$$n_{k_n} = \frac{z - z_{k_n}}{\|z - z_{k_n}\|}, \; b_{k_n} = a_{k_n} n_{k_n} + z_{k_n}.$$

Note that $b_{k_n} \to z$ as $k \to \infty$. Then, by (83), we obtain

$$\begin{aligned} w(z, 0) &= \lim_{k_n\to\infty} w^{k_n}(b_{k_n}, 0) = \lim_{k_n\to\infty} u^{\varepsilon_{k_n}}(p_{k_n} + \varepsilon_{k_n}\mathcal{R}_{k_n}(a_{k_n}k_n + z_{k_n}), t_{k_n}) \\ &= \lim_{k_n\to\infty} u^{\varepsilon_{k_n}}(\varepsilon_{k_n} a_{k_n}\mathcal{R}_{k_n} n_{k_n} + x_{k_n}, t_{k_n}) \geq 0, \end{aligned}$$

where the last inequality holds by (78).

Note that (82) implies $\{w = 0\} = \{(z, \tau) \in \mathbb{R}^N \times \mathbb{R}, z^{(N)} = z^*\}$. Thus, we have either $z_\infty = \tilde{z}_\infty$ or, in view of (82) , (84) and (85), that the ball of radius $||\tilde{z}_\infty - z_\infty||$ centered at z_∞ is tangent to the hyperplane $z^{(N)} = z^*$ at $\tilde{z}_\infty$. Hence, $\tilde{z}_\infty$ is a point on $z^{(N)}$ axis. With this observation and (82), we have

$$\tilde{z}_\infty = (0, \cdots, 0, z^*).$$

This last property implies

$$\frac{d^{\varepsilon_{k_n}}(x_{k_n}, t_{k_n})}{\varepsilon_{k_n}} = \frac{\|x_{k_n} - y_{k_n}\|}{\varepsilon_{k_n}} = \|\mathcal{R}_{k_n}(z_{k_n} - \tilde{z}_{k_n})\| = \|z_{k_n} - \tilde{z}_{k_n}\| \to \|z_\infty - \tilde{z}_\infty\| = z_\infty^{(N)} - z^*. \tag{86}$$

We have therefore reached a contradiction since, by (76), (82) and (86),

$$\begin{aligned} 0 &= |w(z_\infty, 0) - U_0(z_\infty^{(N)} - z^*)| \\ &= \left| \lim_{k_n\to\infty} \left[w^{k_n}(z_{k_n}, 0) - U_0\left(\frac{d^{\varepsilon_{k_n}}(x_{k_n}, t_{k_n})}{\varepsilon_{k_n}}\right)\right]\right| \\ &= \left| \lim_{k_n\to\infty} \left[u^{\varepsilon_{k_n}}(x_{k_n}, t_{k_n}) - U_0\left(\frac{d^{\varepsilon_{k_n}}(x_{k_n}, t_{k_n})}{\varepsilon_{k_n}}\right)\right]\right|, \end{aligned}$$

contradicting (75).

For the proof of (ii), we use the same method as in [3]. □

Appendix: Mobility and surface tension

Mobility is defined as a linear response of the speed of traveling wave to the external force. More precisely, motivated by (4.1) and (4.2) in [18], let us consider the nonlinear Allen-Cahn equation with external force δ on $\mathbb{R}$ for small enough $|\delta|$:

$$u_t = \varphi(u)_{zz} + f(u) + \delta, \quad z \in \mathbb{R}, \tag{87}$$

and the corresponding traveling wave solution $U = U_\delta(z)$ with speed $c(\delta)$:

$$\begin{aligned} &\varphi(U_\delta)_{zz} + c(\delta)U_{\delta z} + f(U_\delta) + \delta = 0, \quad z \in \mathbb{R}, \\ &U_\delta(\pm\infty) = \alpha_{\pm,\delta}, \end{aligned} \tag{88}$$

where $\alpha_{\pm,\delta}$ are the two stable solutions of $f(u)+\delta=0$. Then, we define the mobility by

$$\mu_{AC} := -\frac{c'(0)}{\alpha_+ - \alpha_-},$$

with a normalization factor $\alpha_+ - \alpha_-$ as in [18]; compare (4.6) and (4.7) in [18] noting that the boundary conditions at $\pm\infty$ are switched so that we have a negative sign for μ_{AC}.

To derive the formula for μ_{AC}, we multiply $\varphi(U_\delta)_z$ to (88) and integrate it over $\mathbb{R}$ to obtain

$$c(\delta)\int_{\mathbb{R}} U_{\delta z}\varphi(U_\delta)_z dz + \delta(\varphi(\alpha_+)-\varphi(\alpha_-)) = O(\delta^2), \tag{89}$$

by noting that

$$\begin{aligned}
&\int_{\mathbb{R}} \varphi(U_\delta)_{zz}\varphi(U_\delta)_z dz = \frac{1}{2}\int_{\mathbb{R}} \left\{\left(\varphi(U_\delta)_z\right)^2\right\}_z dz = 0,\\
&\int_{\mathbb{R}} \varphi(U_\delta)_z dz = \varphi(\alpha_{+,\delta}) - \varphi(\alpha_{-,\delta}) = \varphi(\alpha_+)-\varphi(\alpha_-)+O(\delta),\\
&\int_{\mathbb{R}} f(U_\delta)\varphi(U_\delta)_z dz = \int_{\mathbb{R}} f(U_\delta)\varphi'(U_\delta)U_{\delta z}dz = -\int_{\alpha_{-,\delta}}^{\alpha_{+,\delta}} W'(u)du = O(\delta^2).
\end{aligned}$$

The last line follows by the change of variable $u=U_\delta(z)$, $W'(u) = -f(u)\varphi'(u)$ (recall (28)), $\int_{\alpha_-}^{\alpha_+} W'(u)du = 0$ and $W'(\alpha_\pm)=0$, $W'\in C^1$. However, since one can at least formally expect $U_\delta = U_0 + O(\delta)$ (recall (20) for U_0), by (27),

$$\begin{aligned}
\int_{\mathbb{R}} U_{\delta z}\varphi(U_\delta)_z dz &= \int_{\mathbb{R}} U_{0z}\varphi(U_0)_z dz + O(\delta)\\
&= \int_{\mathbb{R}} U_{0z}\sqrt{2W(U_0(z))}dz + O(\delta)\\
&= \int_{\alpha_-}^{\alpha_+}\sqrt{2W(u)}du + O(\delta),
\end{aligned}$$

by the change of variable $u=U_0(z)$ again. This combined with (89) leads to

$$c'(0) = -\frac{\varphi(\alpha_+)-\varphi(\alpha_-)}{\int_{\alpha_-}^{\alpha_+}\sqrt{2W(u)}du}.$$

Thus, the mobility is given by the formula

$$\mu_{AC} = \frac{\varphi_\pm^*}{\int_{\alpha_-}^{\alpha_+}\sqrt{2W(u)}du} = \frac{\varphi_\pm^*}{\int_{\mathbb{R}}\varphi'(U_0)U_{0z}^2(z)dz}, \tag{90}$$

where

$$\varphi_\pm^* = \frac{\varphi(\alpha_+)-\varphi(\alpha_-)}{\alpha_+-\alpha_-}.$$

On the other hand, surface tension is defined as a gap of the energy of the microscopic transition surface from α_- to α_+ in the normal direction and that of the constant profile α_- or α_+. More precisely, define the energy of a profile $u=\{u(z)\}_{z\in\mathbb{R}}$ by

$$\mathcal{E}(u) = \int_{\mathbb{R}}\left\{\frac{1}{2}\left(\varphi(u)_z\right)^2 + W(u)\right\}dz.$$

Recall that the potential W is defined by (28), and $W \geq 0$ and $W(\alpha_\pm) = 0$ hold. In particular, W is normalized as $\min_{u\in\mathbb{R}} W(u) = 0$ so that $\min_{u=u(\cdot)} \mathcal{E}(u) = 0$. Then, the surface tension is defined as

$$\sigma_{AC} := \frac{1}{\varphi^*_\pm} \min_{u:u(\pm\infty)=\alpha_\pm} \mathcal{E}(u),$$

by normalizing the energy by $\varphi^*_\pm$. We observe that $\mathcal{E}$ is defined through φ.

Note that the nonlinear Allen-Cahn equation, that is (87) with $\delta = 0$, is a distorted gradient flow associated with $\mathcal{E}(u)$:

$$u_t = -\frac{\delta \mathcal{E}(u)}{\delta \varphi(u)}, \quad z \in \mathbb{R},$$

where the right hand side is defined as the functional derivative of $\mathcal{E}(u)$ in $\varphi(u)$, which is given by

$$\frac{\delta \mathcal{E}(u)}{\delta \varphi(u)} = -\varphi(u)_{zz} - f(u(z)).$$

Indeed, to see the second term $-f(u(z))$, setting $v = \varphi(u)$, one can rewrite $W(u) = W(\varphi^{-1}(v))$ as a function of v so that

$$\begin{aligned}\big(W(\varphi^{-1}(v))\big)' &= W'(\varphi^{-1}(v))\big(\varphi^{-1}(v)\big)' \\ &= -f(\varphi^{-1}(v))\varphi'(\varphi^{-1}(v))\frac{1}{\varphi'(\varphi^{-1}(v))} \\ &= -f(\varphi^{-1}(v)) = -f(u).\end{aligned}$$

We call the flow "distorted", since the functional derivative is taken in $\varphi(u)$ and not in u. One can rephrase this in terms of the change of variables $v(z) = \varphi(u(z))$. Indeed, we have $\mathcal{E}(u) = \widetilde{\mathcal{E}}(v)$ under this change, where

$$\widetilde{\mathcal{E}}(v) = \int_{\mathbb{R}} \Big\{ \frac{1}{2} v_z^2 + W(\varphi^{-1}(v)) \Big\} dz,$$

and

$$\frac{\delta \widetilde{\mathcal{E}}}{\delta v} = -v_{zz} - f(\varphi^{-1}(v)) = -v_{zz} - g(v).$$

Therefore, in the variable $v(z)$, the nonlinear Allen-Cahn equation can be rewritten as

$$v_t = \varphi'(u)u_t = -\varphi'(\varphi^{-1}(v)) \cdot \frac{\delta \widetilde{\mathcal{E}}}{\delta v} = \varphi'(\varphi^{-1}(v))\big\{v_{zz} + g(v)\big\}.$$

This type of distorted equation for v is sometimes called Onsager equation; see [16].

Now we come back to the computation of the surface tension σ_{AC}. In fact, it is given by

$$\sigma_{AC} = \frac{1}{\varphi^*_\pm} \int_{\mathbb{R}} V_{0z}^2 dz = \frac{1}{\varphi^*_\pm} \int_{\alpha_-}^{\alpha_+} \varphi'(u)\sqrt{2W(u)}du, \tag{91}$$

where $V_0 = \varphi(U_0)$ and satisfies (21). Indeed, the second equality follows from (29). To see the first equality, by definition,

$$\sigma_{AC} = \frac{1}{\varphi^*_\pm} \min_{u:u(\pm\infty)=\alpha_\pm} \mathcal{E}(u) = \frac{1}{\varphi^*_\pm} \min_{v:v(\pm\infty)=\varphi(\alpha_\pm)} \widetilde{\mathcal{E}}(v)$$

and the minimizers of $\widetilde{\mathcal{E}}$ under the condition $v(\pm\infty) = \varphi(\alpha_\pm)$ are given by V_0 and its spatial shifts. Thus,

$$\sigma_{AC} = \frac{1}{\varphi^*_\pm} \widetilde{\mathcal{E}}(V_0) = \frac{1}{\varphi^*_\pm} \int_{\mathbb{R}} \Big\{ \frac{1}{2} V_{0z}^2 + W(\varphi^{-1}(V_0)) \Big\} dz.$$

However, since $V_{0z} = \sqrt{2W(U_0(z))}$ by (27), we have $\int_{\mathbb{R}} W(\varphi^{-1}(V_0))dz = \int_{\mathbb{R}} \frac{1}{2} V_{0z}^2 dz$. In particular, this implies the first equality of (91).

By (30) combined with (90) and (91), we see that $\lambda_0 = \mu_{AC}\sigma_{AC}$.

Remark 4. *The linear case $\varphi(u) = \mathfrak{K}u$ is discussed by Spohn [18], in which $\mathfrak{K}$ is denoted by κ. In this case, since $\varphi' = \mathfrak{K}$ and $\varphi_{\pm}^{*} = \mathfrak{K}$, by* (90) *and* (91)*, we have $\mu_{AC} = \left[\int_{\mathbb{R}} U_{0z}^2 dz\right]^{-1}$ and $\sigma_{AC} = \mathfrak{K}\int_{\mathbb{R}} U_{0z}^2 dz$. These formulas coincide with (4.7) and (4.8) in [18] by noting that U_0 is the same as w in [18] in the linear case except that the direction is switched due to the choice of the boundary conditions.*

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