



Financial Market Equilibrium with Bounded Awareness 1

Ani Guerdjikova, John Quiggin

► To cite this version:

Ani Guerdjikova, John Quiggin. Financial Market Equilibrium with Bounded Awareness 1. 2023. <hal-03962427>

HAL Id: hal-03962427

<https://hal.science/hal-03962427v1>

Preprint submitted on 30 Jan 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Financial Market Equilibrium with Bounded Awareness¹

Ani Guerdjikova² and John Quiggin³

University of Grenoble

University of Queensland

This version: December 27, 2021

Abstract

We consider an infinite-horizon economy with differential awareness in the form of coarsening. Agents with limited awareness are averse to unfavorable surprises. As a result their optimal trades are measurable w.r.t. their respective awareness partitions. We define an equilibrium with differential awareness and illustrate how the obtained equilibrium allocations observationally differ from those in economies with full awareness. In particular, economies with differential awareness can exhibit *(i)* lack of insurance against idiosyncratic risk; *(ii)* partial insurance against aggregate risk; *(iii)* biased state prices even when beliefs are correct and *(iv)* overpricing of assets which pay on events with low aggregate payoffs. We next adapt the results of Guerdjikova and Quiggin (2019) to show that agents with different levels of awareness can survive and influence prices in the limit. In this sense, the characteristics identified above would persist in the long-run. Moreover, differential awareness can lead to belief heterogeneity even in the limit. This is in contrast with the classical result of Blume and Easley (2006) stating that only agents with beliefs closest to the truth can survive. Finally, we examine the individual welfare implications of bounded awareness. If an increase in awareness comes at the cost of wrong beliefs over the larger state-space, bounded awareness can simultaneously increase individual welfare (with respect to the truth) and help avoid ruin. In this sense, heuristics which constrain agents to invest in "assets they understand" can be both ecologically rational in the sense of Gigerenzer (2007) and improve the stability of financial markets by allowing a larger set of agents to survive.

Keywords: ambiguity, ambiguity-aversion, survival.

JEL Codes: D50, D81.

¹ This research was supported by IUF, ANR-17-CE26-0003, Labex MME-DII.

² GAEL, University Grenoble Alpes, INRAE, CNRS, Grenoble INP, IUF, 1241 rue des Résidences, 38400 Saint Martin d'Hères, France, Tel.: +33 4 56 52 85 78, Email: ani.guerdjikova@univ-grenoble-alpes.fr.

³ University of Queensland, School of Economics, Brisbane 4072 Australia, j.quiggin@uq.edu.au

1 Introduction

The standard model of financial markets is one in which the set of assets spans the space of state-contingent consumption possibilities for all agents. Equilibrium is generated by rational agents who are perfectly cognizant of the state-space and choose state-contingent consumption paths to maximize utility.

In reality, however, the set of financial instruments is too complex for any individual to comprehend. The set of contingencies that might affect consumption allocations is even more complex. The result is that most individuals hold portfolios that do not fully diversify the risks they face, either because the necessary instruments do not exist, or because they choose to hold simple, but apparently suboptimal, portfolios.

Hence, it is necessary to reconsider the notion of financial market equilibrium by replacing the assumption of perfect rationality with some form of bounded rationality. However, while perfectly rational agents are all alike (up to differences in utility functions and discount rates), there are many different concepts of bounded rationality. The problem, therefore is to consider forms of bounded rationality help to explain portfolio choices and financial market outcomes.

Developments in the theory of unawareness⁴ deal with a form of bounded rationality that is relevant to the problem of financial market equilibrium in which some or all agents cannot cope with the complexity involved in the description of state-contingent outcomes and asset payoffs. Given this complexity, boundedly rational agents may adopt a description of the state space which encompasses a set of possibilities more limited than those embodied in the full state space. In particular agents may fail to distinguish between distinct states of nature, and may treat them like a single state. We refer to this as coarse awareness.

In this paper, we address two major questions, which so far have not been analyzed in the literature. First, we are interested in the short- and long-run impact of heterogeneity with respect to awareness on prices and allocations in financial markets. Second, we study the implications of unawareness on long-term survival in financial markets.

An agent with coarse awareness of financial markets might fail to distinguish between two states of the world and might find assets which pay conditionally on the realization of these two states incomprehensible relative to his coarse description of the world. He will thus need to form subjective perceptions about the payoffs of such assets which are expressible in terms of the coarse contingencies he understands. Under the assumption of aversion to unfavorable surprises, the agent behaves as if he were to self-restrict his trades in the financial market to be measurable with respect to his awareness. That is, such an agent would not trade on contingencies he does not understand.

In turn, the resulting financial market equilibrium will look as if agents face financial constraints on their choices, even though such constraints are self-imposed and reflect their limited perception of the world. When awareness varies across agents, this leads to a significant departure from common models of market incompleteness, in that agents behave as if they face different financial constraints. In some, but not all, respects, the results are the same as if agents face differential exogenous constraints on the portfolios available to them. This problem is analyzed in Guerdjikova and Quiggin (2019).

Self-imposed constraints have important implications for survival in financial markets. The

⁴ Schipper (2018) provides a comprehensive bibliography on unawareness and its applications.

bounded awareness of the agents, combined with their aversion to unfavorable surprises implies that equilibrium trades will be measurable with respect to the agents' awareness partitions. Thus, boundedly aware agents will choose not to trade their entire wealth on the financial market. This, in turn, leads to a distinction between two survival concepts: an agent might survive in the sense that his consumption does not converge to 0, while vanishing from the financial market, with his financial wealth and his impact on prices converging to 0.

Our results imply that agents who are not fully aware, but have correct beliefs on their awareness partition, survive in the presence of more aware agents (even when the latter have correct beliefs). Furthermore, with bounded awareness, both heterogeneity with respect to awareness and heterogeneity with respect to beliefs can persist in the limit and affect prices and financial-market allocations in the long-run.

On an individual level, we find a trade-off between individual utility and chances of survival as awareness becomes more refined. Becoming more aware implicitly relaxes the self-imposed trading constraints and thus, *ceteris paribus*, increases the agent's subjective expected utility obtained in equilibrium. However, if the agent's beliefs on the finer contingencies are wrong, his expected utility with respect to the truth might diminish when he becomes more aware. A further interesting insight of our analysis is the following: let agent j have coarser awareness than i , and suppose that they have the same discount factor, and that their beliefs with respect to j 's awareness partition are the same. Then i has a lower survival index than j . Thus, one can produce examples of economies, in which the less aware agent j survives a.s., whereas the more aware agent i a.s. vanishes in the financial market.

Thus, bounded awareness, when combined with correct beliefs over a coarser partition of the state-space may have two positive effects: on the one hand, it may increase the individual expected utility with respect to the true probability distribution, on the other hand, it may enhance the chances of an investor to survive in the financial market. With this in mind, we examine the options available to agents who understand that they may have bounded awareness, but cannot incorporate this understanding in the state-act model in which awareness may be represented formally. Such agents may constrain their choices using heuristics which perform better than naive optimization with potentially wrong beliefs and are, thus, in the terminology of Gigerenzer (2007), 'ecologically rational'. We propose the alternative term 'evolutionarily rational' to describe heuristics that ensure that an agent will survive in the financial market. We show that heuristics based on bounded awareness combined with aversion to unfavorable surprises can be both ecologically rational, in that they allow an agent to obtain a higher payoff by avoiding speculation on potentially wrong beliefs, as well as evolutionary rational in that they ensure that an agent survives even if agents with superior reasoning skills and better information are present in the market.

Finally, we offer some concluding comments and directions for future research.

2 Background

Two forms of unawareness are likely to be relevant in financial markets. Agents may fail to consider some possibilities at all. Grant and Quiggin (2013 a) use the term 'restricted awareness'

to describe the associated state-space representation of bounded awareness, in which the state space considered by the agent is a proper subset $\Omega \subseteq \Sigma$ of the full state space. Alternatively, as in Heifetz, Meier and Schipper (2006) and Epstein, Marinacci and Seo (2007), agents may fail to distinguish between distinct states of nature, and may treat them like a single state. In this paper we consider the latter case, which we refer to as ‘coarse awareness’.

A limited amount of work has been done on the implications of unawareness in financial markets⁵. Modica et al. (1998) and Auster et al. (2020) demonstrate the possibility of a bankruptcy in a financial market equilibrium with unaware agents – the former in the context of a restricted awareness, the latter for the case of coarsening.

When agents’ perceived state space is coarser than the actual one, they have to form perceptions of payoffs conditional on such coarse contingencies. We follow Ghirardato (2001) and Vierø (2009) and propose that a boundedly aware agent will consider the set of possible outcomes on a coarse contingency and evaluate such sets using an α -max-min rule. The case of $\alpha = 0$ corresponds to (extreme) aversion to unfavorable surprises: the agent identifies the payoff of an asset on a coarse contingency to coincide with its minimal payoff on the set. By contrast, Auster et al. (2020) assume that such a perception is formed by taking the average payoff across the finer contingencies.

It might be conjectured that agents with bounded rationality would play only a limited role in financial markets. Indeed, a large literature, beginning with Blume and Easley (1992, 2006) is devoted to the idea that markets favor the best-informed and most rational traders. Trades in a financial market may be seen as ‘betting one’s beliefs’ about the relative probabilities of different states of nature, and the resulting returns on assets. Over time, traders who correctly judge these probabilities and make rational investment choices based on their beliefs will accumulate wealth at the expense of others. In the limit, only these rational well-informed traders will survive, and market prices will reflect their beliefs. Boundedly rational traders with incorrect beliefs will not survive. These results may be expressed in terms of the ‘survival index’, the sum of the log-ratio of the agents’ discount factors and the Kullback-Leibler distance between the agent’s probabilistic beliefs about states and the true probability distribution.

However, by constraining the set of assets in which agents may trade, unawareness may remove some possible portfolio choices that lead to speculative trades and eventually to the disappearance of poorly informed agents. Hence, optimal portfolio choices with bounded awareness can be interpreted as heuristics in the sense of Gigerenzer (2007).

3 The Model

The ‘true’ model of the economy is the same as that of Guerdjikova and Quiggin (2019) and will be restated briefly.

⁵ Guiso and Jappelli (2005) collect data about the awareness of Italian investors of different investment opportunities. Not surprisingly, they find that households fail to invest in assets of which they are unaware.

3.1 The "True" Model of the Economy

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ denote the set of time periods. Uncertainty is modelled through a sequence of random variables $\{\mathcal{S}_t\}_{t \in \mathbb{N}}$ each of which takes values in a finite set S , with $S_0 = \{s_0\}$. Events (subsets of S) are denoted w . Denote by $s_t \in S$ the realization of random variable \mathcal{S}_t . Denote by $\Omega = \prod_{t \in \mathbb{N}} S$ the set of all possible observation paths, with representative element $\sigma = (s_0, s_1, s_2 \dots s_t \dots)$. Finally denote by $\Omega_t = \prod_{\tau=0}^t S$ the collection of all finite paths of length t , with representative element $\sigma_t = (s_0, s_1, s_2 \dots s_t)$. We will write $s(\sigma_t)$ for the state realization on path σ at time t . Define the cylinder with base on $\sigma_t \in \Omega_t$, $t \in \mathbb{N}$ as $Z(\sigma_t) = \{\sigma \in \Omega \mid \sigma = (\sigma_t \dots)\}$. Let $\mathbb{F}_t = \{Z(\sigma_t) : \sigma_t \in \Omega_t\}$ be a partition of the set Ω . Clearly, $\mathbb{F} = (\mathbb{F}_0 \dots \mathbb{F}_t \dots)$ denotes a sequence of finite partitions of Ω such that $\mathbb{F}_0 = \Omega$ and \mathbb{F}_t is finer than \mathbb{F}_{t-1} . Let \mathcal{F}_t be the σ -algebra generated by partition \mathbb{F}_t , and let \mathcal{F} be the σ -algebra generated by $\cup_{t \in \mathbb{N}} \mathcal{F}_t$. It can be shown that $\{\mathcal{F}_t\}_{t \in \mathbb{N}}$ is a filtration. We define on (Ω, \mathcal{F}) a probability distribution π . We will assume that the true process of the economy is i.i.d. and write $\pi(s_{t+1} = s \mid \sigma_t) =: \pi(s)$.

There is a single good consumed in positive quantities. There is a finite set I with $|I| = n$ of infinitely lived agents. Each agent's welfare depends on their consumption stream $c^i : \Omega \rightarrow \prod_{t \in \mathbb{N}} \mathbb{R}_+$. Each agent i is endowed with a consumption plan, denoted e^i . The total endowment of the economy is denoted by $e = \sum_i e^i$.

Agents are assumed to be expected utility maximizers given their knowledge about the economy and their perception of asset payoffs relative to their respective awareness level⁶. Agent i 's utility function for risk is denoted by u_i and his discount factor is β^i . Since we are not concerned with differences in time preference, we will simplify by assuming that all discount factors are the same $\beta^i = \beta < 1$.

We will impose the following assumptions on utility functions and endowments, which are standard in the survival literature:

Assumption 1 Agents' utility functions for risk $u_i : \mathbb{R}_+ \rightarrow \mathbb{R}$ are twice continuously differentiable, strictly concave, and satisfy $\lim_{c \rightarrow 0} u'_i(c) = \infty$ and $\lim_{c \rightarrow \infty} u'_i(c) = 0$.

Assumption 2 Individual endowments are strictly positive, $e^i(\sigma_t) > 0$ for all i and σ_t . Aggregate endowments are uniformly bounded away from zero and uniformly bounded from above. Formally, there is an $m > 0$ such that $\sum_{i \in I} e^i(\sigma_t) > m$ for all σ_t ; moreover, there is an $m' > m > 0$ such that $\sum_{i \in I} e^i(\sigma_t) < m'$ for all σ_t .

3.2 Modelling Unawareness as Coarsening

In this paper, we think of unawareness as the inability of the agent to form a sufficiently fine perception of the state space. A partially aware agent i will perceive a state space W^i coarser than S , in which some states with potentially different consumption allocations are coalesced into a single perceived state. To understand the process, it is helpful to think in syntactic (propositional terms). Each state in S may be described in terms of the truth values of a set of propositions P

⁶ An expected utility representation with a coarse subjective state space has been recently axiomatized by Minardi and Savochkin (2016).

describing relevant contingencies, in this case, related to endowments.

An agent may be less aware than another because the set of descriptions available to them is coarser. For example, a relatively unaware agent might consider the proposition ‘the economy is (or is not) at full employment’, giving rise to a state space with two elements. A more aware agent might distinguish the various phases of the economic cycle, such as ‘peak’, ‘contraction’, ‘trough’ and ‘expansion’. An even more aware agent might consider a state space in which the states were indexed by the rate of growth of gross domestic product.

An alternative form of coarsening arises when some agents display ‘pure unawareness’ of relevant propositions (Li 2009). For example, two agents might have access to the same set of propositions to describe the state of the domestic economy, but only one of them might consider developments in the world economy. The more aware agent would have access to a state space derived as the Cartesian product of the state of the domestic economy and the state of the world economy, while the less aware agent would have access to a coarser quotient space, in which all states of the world economy were treated as indistinguishable. We would expect the less aware agent to display ‘home bias’ (French and Poterba 1991).

We now formalize the idea that some agents perceive a coarser state space than the one given by S . In particular, agent i is assumed to be aware of a partition of S given by $W^i = \{w_1^i \dots w_{K_i}^i\}$, where each $w_k^i \subseteq S$, $w_k^i \cap w_{k'}^i = \emptyset$ for any $k \neq k'$ and $\bigcup_{k=1}^{K_i} w_k^i = S$. This is a specific type of unawareness: the agent’s perception of the world is coarser than reality in that he cannot distinguish between those states which are grouped in a given w_k^i .

We assume that all fully aware agents have identical information and that the information revelation process for them is represented by the sequence \mathbb{F} . A fully aware agent can distinguish any two nodes σ_t and σ'_t . By contrast, a partially aware agent cannot distinguish nodes σ_t and σ'_t if and only if, for every $\tau \leq t$, $s_\tau, s'_\tau \in w_{k_\tau}^i$ for some $w_{k_\tau}^i \in W^i$. Hence, for a partially aware agent, the paths he is aware of can be written as $\Omega^i = \prod_{t \in \mathbb{N}} W^i$ with a representative element $\omega^i = (w_0 = \{s_0\}, w_1^i \dots w_t^i \dots)$. Denote by Ω_t^i the set of paths of length t with representative element ω_t^i . We will write $w(\omega_t^i)$ for the event realization on path ω_t^i at time t .

From the point of view of agent i , the information revelation is described by finite partitions of the set Ω^i , $(\mathbb{F}_t^i)_{t \in \mathbb{N}}$ defined in analogy to $(\mathbb{F}_t)_{t \in \mathbb{N}}$. Note that for each t , \mathbb{F}_t^i is coarser than the corresponding \mathbb{F}_t . We will denote by \mathcal{F}_t^i the σ -algebra generated by partition \mathbb{F}_t^i . $\mathcal{F}_0^i = \mathcal{F}_0$ is the trivial σ -algebra. Let \mathcal{F}^i be the σ -algebra generated by $\bigcup_{t \in \mathbb{N}} \mathcal{F}_t^i$. Just as above, $\{\mathcal{F}_t^i\}_{t \in \mathbb{N}}$ is a filtration.

Agent i ’s beliefs π^i are defined on $(\Omega^i, \mathcal{F}^i)$. The one-step ahead probability distribution $\pi^i(w_{t+1}^i \mid \omega_t^i)$ is defined analogously to $\pi(s_{t+1} \mid \sigma_t)$.

\mathcal{F} is finer than \mathcal{F}^i and hence, the true probability distribution π on (Ω, \mathcal{F}) specifies a probability distribution on $(\Omega^i, \mathcal{F}^i)$ with

$$\pi(\omega_t^i = (w_0 \dots w_t^i)) = \pi\{\sigma_t \mid s_\tau \in w_\tau^i \text{ for all } \tau \in \{1 \dots t\}\}.$$

We will say that i ’s beliefs are correct if they coincide with the restriction of π to $(\Omega^i, \mathcal{F}^i)$.

For most of the paper, we will restrict attention to beliefs which describe an i.i.d. process, $\pi^i(w_{t+1}^i = w^i \mid \omega_t^i) = \pi^i(w^i)$. We assume that all states are non-null under the true one-step-ahead probability and all agents have one-step-ahead beliefs that are absolutely continuous with

respect to the truth.

Assumption 3 $\pi(s) > 0$ for all $s \in S$ and for all $i \in I$, $\pi^i(w^i) > 0$ for all $w^i \in W^i$.

Since in general, $\pi^i(w^i) \neq \pi(w^i)$, we introduce the Kullback-Leibler (K-L) distance as a measure of deviation of agents' beliefs from the truth.

Definition 3.1 For a given partition of S , W^i , the Kullback-Leibler (K-L) distance of agent i 's beliefs π^i with respect to the truth π is given by: $\sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi^i(w^i)}$.

As usual, when i 's beliefs on W^i are correct, the K-L distance is 0, whereas the violation of the absolute-continuity property posited in Assumption 3 would lead to a K-L distance of infinity. The consumption set of i consists of functions $c^i : \Omega \rightarrow \prod_{t \in \mathbb{N}} \mathbb{R}_+$. The initial endowment of agent i is one such consumption stream denoted by e^i .

Trade in financial markets occurs via Arrow securities, which pay 1 unit of the consumption good at a given contingency $\sigma_t \in \Omega$. We will denote by $a^i(\sigma_t)$ the amount of the Arrow security for state σ_t in the portfolio of agent i and by a^i , i 's portfolio. The consumption stream of an agent holding portfolio a^i is thus $c^i = a^i + e^i$.

In general, agents might have initial endowments and portfolios (and thus, consumption streams), which are not measurable with respect to their awareness partition. This means that for given a^i and e^i , consumers would have a perception of their consumption stream $c^i = a^i + e^i$, which depend on their awareness partition. We write $\mathbf{c}_{\Omega^i}^i(a^i, e^i, \omega_t^i)$ for i 's perception of his consumption at node $\omega_t^i \in \Omega^i$ resulting from endowment e^i and portfolio a^i . The functional $\mathbf{c}_{\Omega^i}^i$ is assumed to have a range \mathbb{C} . Examples of functionals $\mathbf{c}_{\Omega^i}^i$ and their respective ranges are presented below. In a second step, the agent uses a mapping $v^i(\mathbf{c}_{\Omega^i}^i(\cdot), u^i)$, which combines the agents' perception of his consumption at some node ω_t^i with his utility function for risk to give the utility derived from consumption at ω_t^i . In general, $v^i : \mathbb{C} \times \mathbb{U} \rightarrow \mathbb{R}$, where \mathbb{U} is the set of utility functions over risk, which satisfy the conditions of Assumption 1 above.

We start by presenting the specifications of $\mathbf{c}_{\Omega^i}^i$ and v^i which will be relevant for this paper and then provide an alternative corresponding to the framework of Auster et al. (2020).

Note that from an agent i 's point of view his endowment e^i is a set-valued mapping, which maps his awareness partition $(\Omega^i, \mathcal{F}^i)$ into sets of consumption levels: $\mathbf{e}_{\Omega^i}^i : \Omega^i \rightrightarrows \mathbb{R}_+$ with

$$\mathbf{e}_{\Omega^i}^i(\omega_t^i) = \{e^i(\sigma_t) \mid \sigma_t \in \omega_t^i\}, \quad (1)$$

that is, $\mathbf{e}^i(\omega_t^i)$ is the set of all levels of consumption which can occur on the states contained in the coarse contingency ω_t^i . Analogously, the payoff stream of a portfolio a^i as perceived by agent i is given by $\mathbf{a}_{\Omega^i}^i : \Omega^i \rightrightarrows \mathbb{R}_+$ with

$$\mathbf{a}_{\Omega^i}^i(\omega_t^i) = \{a^i(\sigma_t) \mid \sigma_t \in \omega_t^i\}. \quad (2)$$

In particular, a partially aware agent is not able to reason about the correlation of payoffs of e^i and a^i on a coarse contingency ω_t^i , nor identify hedging opportunities between the two. Thus, as suggested by Vierø (2009), the appropriate definition of the corresponding consumption stream is given by the Minkowski sum of $\mathbf{e}_{\Omega^i}^i$ and $\mathbf{a}_{\Omega^i}^i$,

$$\mathbf{c}_{\Omega^i}^i(a^i, e^i, \omega_t^i) = \mathbf{a}_{\Omega^i}^i(\omega_t^i) \oplus \mathbf{e}_{\Omega^i}^i(\omega_t^i) = \{a^i(\sigma_t) + e^i(\sigma'_t) \mid \sigma_t, \sigma'_t \in \omega_t^i\}. \quad (3)$$

In this case, the range of $\mathbf{c}_{\Omega^i}^i : \mathbb{C}$ is thus the set of finite-valued subsets of \mathbb{R} .

Ghirardato (2001) and Vierø (2009) axiomatize preferences on acts which for a given state result in a set of possible payoffs, similar to the perception of consumption streams $\mathbf{c}_{\Omega^i}^i$. In both papers, the suggested evaluation of a payoff of an act on a coarse contingencies is determined as a weighted combination of the utility of the best and the worst outcome in the set, with the weight on the best outcome α being interpreted as the agent's degree of optimism:

$$v_{\alpha}^i \left(\mathbf{c}_{\Omega^i}^i (a^i, e^i, \omega_t^i) \right) = \alpha \max_{c \in \mathbf{c}_{\Omega^i}^i (a^i, e^i, \omega_t^i)} u^i(c) + (1 - \alpha) \min_{c \in \mathbf{c}_{\Omega^i}^i (a^i, e^i, \omega_t^i)} u^i(c).$$

The case of $\alpha = 0$ captures agents who are extremely pessimistic and associate a coarse contingency with the worst possible payoff realized on it. This can be interpreted as a heuristic that the agents use in view of their limited understanding of the world: such a heuristic allows them to avoid unfavorable surprises in the sense of Grant and Quiggin (2014). Our next assumption captures such preferences:

Assumption 4 An agent with awareness partition $(\Omega^i, \mathcal{F}^i)$ with initial endowment e^i and portfolio holdings a^i perceives his consumption at $\omega_t^i \in \Omega^i$ as $\mathbf{c}_{\Omega^i}^i (a^i, e^i, \omega_t^i)$ given by (3). The utility of a set-valued consumption $\mathbf{c}_{\Omega^i}^i (a^i, e^i, \omega_t^i)$ conditional on an element of Ω^i , ω_t^i is given by

$$v_{\min}^i \left(\mathbf{c}_{\Omega^i}^i (a^i, e^i, \omega_t^i) \right) = \min_{c \in \mathbf{c}_{\Omega^i}^i (a^i, e^i, \omega_t^i)} u^i(c). \quad (4)$$

This assumption is different from the one made by Auster et al. (2020). They assume that the agent's perceptions are given by

$$\mathbf{c}_{\Omega^i, a}^i (a^i, e^i, \omega_t^i) = \sum_{\sigma_t \in \omega_t^i} \pi(\sigma_t) [a^i(\sigma_t) + e^i(\sigma_t)]$$

that is, the agents estimate correctly both the relative likelihoods of the different states σ_t within an element of the partition ω_t^i , as well as the correlation between a^i and e^i and associate the consumption realization on ω_t^i with the so-computed average. The evaluation of payoffs on coarse contingencies is then given by the utility of the expected consumption:

$$v_a^i \left(\mathbf{c}_{\Omega^i}^i (a^i, e^i, \omega_t^i), u^i \right) = u^i \left(\sum_{\sigma_t \in \omega_t^i} \pi(\sigma_t) [a^i(\sigma_t) + e^i(\sigma_t)] \right).$$

Their framework thus models agents who have correct perceptions on average, and are not sensitive to payoffs variations within a given coarse contingency. In general, different perception functionals and aggregators of utility over perceived outcomes can be construed. Below, we focus on $\mathbf{c}_{\Omega^i}^i$ as defined by (3) and v_{\min}^i given by (4). We discuss alternative approaches in Section 6.

To simplify notation, let $\tilde{e}^i : \Omega \rightarrow \prod_{t \in \mathbb{N}} \mathbb{R}_+$ be defined by $\tilde{e}^i(\sigma_t) = \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t)$ and, similarly, $\tilde{a}^i : \Omega \rightarrow \prod_{t \in \mathbb{N}} \mathbb{R}$ be given by $\tilde{a}^i(\sigma_t) = \min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t)$ for any t , any $\omega_t^i \in \Omega_t^i$ and each $\sigma_t \in \omega_t^i$. \tilde{e}^i and \tilde{a}^i are both measurable with respect to the awareness partition of agent i and coincide on each ω_t^i with the minimal endowment, respectively, portfolio holdings, on this ω_t^i . Let $\tilde{c}^i = \tilde{e}^i + \tilde{a}^i$ be the perceived minimal consumption of i . With a slight abuse of notation,

we write $\tilde{e}^i(\omega_t^i) = \tilde{e}^i(\sigma_t)$ whenever $\sigma_t \in \omega_t^i$, and similarly for $\tilde{a}^i(\omega_t^i)$, $\tilde{c}^i(\omega_t)$. Expressions (3) and (4) then imply that

$$v^i(\mathbf{c}_{\Omega^i}^i(\omega_t^i)) = u^i\left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t)\right) = u^i(\tilde{e}^i(\omega_t) + \tilde{a}^i(\omega_t)) = u^i(\tilde{c}^i(\omega_t)).$$

Under Assumption 4, we can write the utility function of agent i with awareness partition $(\Omega^i, \mathcal{F}^i)$ and a (not necessarily measurable) portfolio a^i as:

$$U_0^i(a^i) = \sum_{t=0}^{\infty} \beta^t \sum_{\omega_t^i \in \Omega_t^i} \pi^i(\omega_t^i) u_i\left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t)\right). \quad (5)$$

The budget constraint of agent i is given by:

$$\sum_{\sigma_t \in \Sigma} p(\sigma_t) a^i(\sigma_t) \leq 0 \quad (6)$$

that is, the agent can in principle trade any portfolio as long as it is self-financing. The resulting consumption stream is given by $c^i = e^i + a^i$ and the corresponding minimal consumption is $\tilde{c}^i = \tilde{e}^i + \tilde{a}^i$.

The agent thus maximizes (5) subject to (6) resulting in the problem

$$\begin{aligned} \max_{a^i} U_0^i(a^i) &= \max_{a^i} \left\{ \sum_{t=0}^{\infty} \beta^t \sum_{\omega_t^i \in \Omega_t^i} \pi^i(\omega_t^i) u_i\left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t)\right) \right. \\ &\quad \left. \mid \sum_{\sigma_t \in \Sigma} p(\sigma_t) a^i(\sigma_t) \leq 0, \min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \geq 0 \forall \omega_t^i \in \Omega_t^i \right\}. \end{aligned}$$

The following Lemma shows that the resulting optimal portfolio is measurable with respect to the agent's awareness partition as long as all prices are strictly positive:

Lemma 3.1 *If $p(\sigma_t) > 0$ for every $\sigma_t \in \Omega$, the solution to the agent's optimization problem $\max_{a^i} U_0^i(a^i)$ subject to the budget constraint in (6) is measurable w.r.t. $(\Omega^i, \mathcal{F}^i)$. Furthermore, the optimal minimal consumption, \tilde{c}^i given by*

$$\tilde{c}^i(\omega_t^i) = \min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t). \quad (7)$$

is the solution to the optimization problem:

$$\max_{\tilde{c}^i: \Omega^i \rightarrow \prod_{t \in \mathbb{N}} \mathbb{R}_+} V_0^i(\tilde{c}^i) = u_i(\tilde{c}^i(\sigma_0)) + \sum_{t=1}^{\infty} \beta^t \sum_{\omega_t^i \in \Omega_t^i} \pi^i(\omega_t^i) u_i(\tilde{c}^i(\omega_t^i)) \quad (8)$$

s.t.

$$\sum_{t \in \mathbb{N}} \sum_{\omega_t^i \in \Omega_t^i} [\tilde{c}^i(\omega_t^i) - \tilde{e}^i(\omega_t^i)] \sum_{\sigma_t \in \omega_t^i} p(\sigma_t) \leq 0 \quad (9)$$

If $p(\sigma_t) \geq 0$ for every $\sigma_t \in \Omega$, the solution to (8) under the constraint (9) is (trivially) measurable w.r.t. $(\Omega^i, \mathcal{F}^i)$ and the corresponding measurable portfolio $a^i(\sigma_t) = \tilde{a}^i(\omega_t^i) = \tilde{c}^i(\omega_t^i) - \tilde{e}^i(\omega_t^i)$ for any $\sigma_t \in \omega_t^i$ is a solution (but not necessarily unique) to the agent's

optimization problem $\max_{a^i} U_0^i(a^i)$ subject to the budget constraint in (6).

The proof of the result is straightforward and therefore omitted. Lemma 3.1 shows that when the agent associates with each coarse contingency the worst possible payoff achievable on this contingency, his behavior will mimic maximizing utility from the minimal perceived consumption in each subjective state ω_t^i under the budget constraint derived from the maximal measurable part of his initial endowment. The equivalence holds, whenever all state prices are strictly positive, but even if some prices equal 0, a measurable solution to the agent's maximization problem exists, potentially with some other, non-measurable solutions.

By Lemma 3.1, the agent will only participate in financial markets with the part of his initial endowment which is measurable with respect to their awareness partition. Thus, the total actual consumption the agent obtains equals his optimal minimal perceived consumption (the solution to (8) subject to (9)) plus the non-measurable part of his initial endowment, that is, for any ω_t^i and any $\sigma_t \in \omega_t^i$,

$$c^i(\sigma_t) = e^i(\sigma_t) - \tilde{e}^i(\omega_t^i) + \tilde{c}^i(\omega_t^i) \quad (10)$$

We will refer to the measurable part of the agent's endowment \tilde{e}^i as the agent's financial wealth, whereas the rest of his endowment $e^i - \tilde{e}^i$ is referred to as "non-financial" wealth. Note that the financial wealth is tradeable, whereas the non-financial wealth is non-tradeable from the agent's point of view. In general, we have $c^i = \tilde{c}^i + e^i - \tilde{e}^i \geq 0$. Thus, whenever agent i 's initial endowment is not measurable with respect to his initial awareness partition, his final consumption will be also non-measurable and will exhibit positive surprises relative to the minimal consumption stream \tilde{c}^i .

It is easy to see that by Assumption 2, individual financial wealth $\tilde{e}^i(\sigma_t)$ is strictly positive at every σ_t and that the aggregate financial wealth $\sum_{i \in I} \tilde{e}^i(\sigma_t)$ is uniformly bounded from above. However, Assumption 2 does not imply that the aggregate financial wealth is uniformly bounded away from 0. We thus impose in addition to Assumption 2, the following:

Assumption 2* Aggregate financial wealth is uniformly bounded away from zero. Formally, there is an $m^* > 0$ such that $\sum_{i \in I} \tilde{e}^i(\sigma_t) > m^*$ for all σ_t .

In the following, unless otherwise stated, Assumptions 1, 2, 2*, 3 and 4 are assumed to hold.

4 Equilibrium in Markets with Differential Awareness

Our main results are derived on the assumption that agents trade their endowments at time 0 with no subsequent opportunity for retrading. Intuitively, this corresponds to the case, in which the agents' awareness remains unchanged and they do not respond to price changes which are conditional on events they are unaware of.

Thus, the approach taken here mimics that of Sandroni (2005), in which there is a single period of trade, but information is subsequently revealed according to the structure presented in Section 3. This assumption greatly simplifies the analysis and allows us to derive a simple criterion for survival in economies with differential awareness. Differently from Sandroni, we allow consumption to occur in time. Appendix A of Guerdjikova and Quiggin (2019), shows that the

analysis can be extended to the case of constant awareness with sequential trading. Even though the definition and the analysis of the equilibrium are substantially different for the two cases, the results derived there show that the main insights of the paper are robust to such a modification.

Definition 4.1 Let $v^i(c_{\Omega^i}^i(\omega_t^i))$ be an aggregator function used to evaluate the utility of a set of outcomes on an element of the awareness partition of agent i , $\omega_t^i \in \Omega^i$. An equilibrium of the economy with differential awareness and aggregator functions v^i , consists of an integrable⁷ price system $(p(\sigma_t))_{\sigma_t \in \Omega}$ and a consumption stream c^i for every agent i such that:

(i) all agents $i \in I$ choose a portfolio a^{*i} so as to maximize their utility from perceived consumption $c_{\Omega^i}^i(a^i, e^i, \cdot)$ given the price system and their budget constraint:

$$a^{*i} = \arg \max_{a^i} \left\{ \begin{array}{l} \sum_{t=0}^{\infty} \beta^t \sum_{\omega_t^i \in \Omega_t^i} \pi^i(\omega_t^i) v^i(c_{\Omega^i}^i(a^i, e^i, \omega_t^i), u^i) \\ \sum_{\sigma_t \in \Sigma} p(\sigma_t) a^i(\sigma_t) \leq 0 \\ c \geq 0 \text{ for all } c \in c_{\Omega^i}^i(a^i, e^i, \omega_t^i), \omega_t^i \in \Omega^i \end{array} \right\};$$

(ii) the actual consumption of agent i satisfies:

$$c^i(\sigma_t) = e^i(\sigma_t) + a^{*i}(\sigma_t) \geq 0 \quad \forall i \in I \text{ and } \forall \sigma_t \in \Omega;$$

(iii) markets clear:

$$\sum_{i \in I} a^{*i}(\sigma_t) = 0 \quad \forall \sigma_t \in \Omega.$$

As we showed in Lemma 3.1 in the previous section, when agents have bounded awareness with perceptions satisfying Assumption 4, their participation in the financial markets will be effectively limited by the part of the initial endowment measurable with respect to their awareness partition. Furthermore, their portfolios, as well as the resulting perceived consumption streams will be measurable with respect to the respective awareness partitions as long as asset prices are strictly positive. We thus define the financial market equilibrium for this economy by restricting it (i) to the measurable parts of the agent's endowments and (ii) to measurable consumption streams.

Definition 4.2 A financial market equilibrium of the economy with differential awareness modelled by Assumption 4 consists of an integrable price system $(p(\sigma_t))_{\sigma_t \in \Omega}$ and for every agent i , a minimal perceived consumption stream \tilde{c}^i measurable w.r.t. $(\Omega^i, \mathcal{F}^i)$ such that

(i) all agents $i \in I$ choose a portfolio \tilde{a}^{*i} measurable with respect to their awareness partition so as to maximize their utility from minimal perceived consumption $\tilde{c}^i = \tilde{e}^i + \tilde{a}^{*i}$:

$$\tilde{a}^{*i} = \arg \max_{\tilde{a}^i: \Omega^i \rightarrow \prod_{t \in \mathbb{N}} \mathbb{R}} \left\{ \begin{array}{l} \sum_{t=0}^{\infty} \beta^t \sum_{\omega_t^i \in \Omega_t^i} \pi^i(\omega_t^i) u_i(\tilde{a}^i(\omega_t^i) + \tilde{e}^i(\omega_t^i)) \\ \sum_{t \in \mathbb{N}} \sum_{\omega_t^i \in \Omega_t^i} \sum_{\sigma_t \in \omega_t^i} p(\sigma_t) \tilde{a}^i(\sigma_t) \leq 0 \\ \tilde{a}^i(\omega_t^i) + \tilde{e}^i(\omega_t^i) \geq 0 \quad \forall \omega_t^i \in \Omega^i \end{array} \right\}; \quad (11)$$

⁷ Integrability of $(p(\sigma_t))_{\sigma_t \in \Omega}$, on $(\Omega; \mathcal{F}; \mu)$, where μ is the counting measure, or equivalently, the requirement that the price system is L^1 on $(\Omega; \mathcal{F}; \mu)$, ensures that the total wealth of an individual agent is finite, i.e., that the sum $\sum_{t \in \mathbb{N}} \sum_{\omega_t^i \in \Omega_t^i} \sum_{\sigma_t \in \omega_t^i} p(\sigma_t) a^i(\sigma_t)$ is well-defined, see Bewley (1972, p. 516).

(ii) markets clear:

$$\sum_{i \in I} \tilde{a}^{*i}(\sigma_t) = 0 \quad \forall \sigma_t \in \Omega.$$

The financial market equilibrium allocates the (tradeable) financial wealth of the agents, \tilde{e}^i , by assigning each agent a minimal perceived consumption stream \tilde{c}^i measurable with respect to the agent's awareness partition. However, since the agent's endowment is generally non-measurable with respect to this awareness partition, his actual consumption will be given by the sum of his financial market equilibrium consumption stream augmented by his non-financial wealth. This in turn, according to Lemma 3.1, results in an equilibrium of the economy according to Definition 4.1:

Corollary 4.1 *Suppose that for all $i \in I$, $\mathbf{c}_{\Omega^i}^i$ and v^i satisfy Assumption 4. If $(p(\sigma_t))_{\sigma_t \in \Omega}$ and $(\tilde{c}^i)_{i \in I}$ constitute a financial market equilibrium of the economy with differential awareness, then $p(\sigma_t)_{\sigma_t \in \Omega}$ and $c^i = \tilde{c}^i + e^i - \tilde{e}^i$ for every agent i constitute a market equilibrium of the economy with differential awareness as in Definition 4.2.*

A financial market equilibrium in an economy with differential awareness is consistent with the fact that different agents have different perceptions of the state space and prefer to avoid unfavorable surprises. They thus behave as if they trade only on their respective (different) partitions of the state space and, hence, effectively optimize over different sets of commodities (consumption on events ω_t^i , rather than σ_t). The price of consumption contingent on a coarse contingency ω_t^i is simply the sum of consumption prices over all nodes $\sigma_t \in \omega_t^i$, that is, $\sum_{\sigma_t \in \omega_t^i} p(\sigma_t)$. Later, as a given contingency σ_t is realized, agents might be positively surprised by learning that they have some additional, non-financial revenue $e^i(\sigma_t) - \tilde{e}^i(\sigma_t)$ in addition to their minimal initial endowment and minimal financial holdings, $\tilde{e}^i(\sigma_t) + \tilde{a}^i(\sigma_t)$. Note that when the initial endowment of each agent i is measurable with respect to the agent's awareness partition $(\Omega^i, \mathcal{F}^i)$, we have $\tilde{e}^i = e^i$ for all $i \in I$ and the equilibrium of the economy coincides with the financial market equilibrium of the economy, $\tilde{c}^i = c^i$ for all $i \in I$. The same is also true in the case of full awareness.

The analysis of the financial market equilibrium is largely parallel, in formal terms, to that of the case, analyzed by Guerdjikova and Quiggin (2019), where fully aware agents face differential constraints on the set of assets in which they can trade. For this reason, where results overlap with those of Guerdjikova and Quiggin (2019), we defer formal statements and proofs to an Appendix.

Proposition 1 in Guerdjikova and Quiggin (2019, p. 1704) implies that under Assumptions 1, 2, 2* and 3, a financial market equilibrium and thus, under Assumption 4, an equilibrium, of the economy with differential unawareness exists. Furthermore, in the financial market equilibrium, for each $i \in I$ and at each $\omega_t^i, \omega_{t+1}^i \in \Omega^i$ such that $\pi(\omega_{t+1}^i) > 0$, prices of perceived contingencies are strictly positive, $\sum_{\sigma_t \in \omega_t^i} p(\sigma_t) = p(\omega_t^i) > 0$ and $\sum_{\sigma_{t+1} \in \omega_{t+1}^i} p(\sigma_{t+1}) = p(\omega_{t+1}^i) > 0$ and

$$\frac{u'_i(\tilde{c}^i(\omega_t^i))}{\beta \pi^i(\omega_{t+1}^i | \omega_t^i) u'_i(\tilde{c}^i(\omega_{t+1}^i))} = \frac{p(\omega_t^i)}{p(\omega_{t+1}^i)} = \frac{\sum_{\sigma_t \in \omega_t^i} p(\sigma_t)}{\sum_{\sigma_{t+1} \in \omega_{t+1}^i} p(\sigma_{t+1})}, \quad (12)$$

where $p(\cdot)$ is the equilibrium price system.

We now consider how differential awareness affects financial market equilibrium allocations. The aggregate financial endowment of the economy is given by $\tilde{e} = \sum_i \tilde{e}^i$. First, consider an economy, in which some agent(s) cannot distinguish between two states (σ'_t and σ''_t) in which aggregate financial endowment differs, but do(es) differentiate between two states (σ_t and σ'_t) in which the aggregate financial endowment is the same. In this case, the agent's own financial endowment is constant across σ'_t and σ''_t , but not across σ_t and σ'_t . The aversion to unfavorable surprises embedded in Assumption 4 implies that the agent will have the same financial equilibrium consumption in σ'_t and σ''_t . This has two effects: first, it prevents financial markets from achieving full insurance against idiosyncratic risk (at the cost of providing insurance against aggregate risk to the agent in question) and second, it leads to prices being biased as compared to the true probabilities of the states, even when all agents hold correct beliefs. The following two results build up on examples 1 and 2 in Guerdjikova and Quiggin (2019):

Proposition 4.2 *Consider an economy with differential awareness and correct beliefs. Suppose that for some $\sigma_t, \sigma'_t, \sigma''_t \in \Omega$ $\tilde{e}(\sigma_t) = \tilde{e}(\sigma'_t) \neq \tilde{e}(\sigma''_t)$. Let furthermore, for some $i \in I$, and some $\omega_t^i \in \Omega^i$, $\sigma'_t, \sigma''_t \in \omega_t^i$. Finally, assume that there are distinct ω_t, ω'_t and ω''_t with $\sigma_t \in \omega_t$, $\sigma'_t \in \omega'_t$ and $\omega''_t = \omega_t^i \setminus \omega'_t$ such that for any $j \in I$, $\omega_t \in \Omega_t^j$ and either $\omega'_t, \omega''_t \in \Omega_t^j$ or $\omega_t^i \in \Omega_t^j$. Then:*

- (i) *the financial market equilibrium of the economy with differential awareness provides insurance to i against the aggregate risk on σ'_t and σ''_t ;*
 - (ii) *the financial market equilibrium of the economy with differential awareness does not provide full insurance against idiosyncratic risk;*
 - (iii) *the price ratio $\frac{p(\omega_t)}{p(\omega_t^i)}$ is biased relative to the probabilities of ω_t and ω_t^i .*
- Propositions 4.2 and 4.3.*

Proposition 4.3 *Suppose that there are states σ_t, σ'_t and $\sigma''_t \in \Omega$ such that $\tilde{e}(\sigma_t) = \tilde{e}(\sigma'_t) \neq \tilde{e}(\sigma''_t)$ and two sets of agents \mathcal{I} and \mathcal{J} with $\mathcal{I} \cup \mathcal{J} = I$ with $\omega_t^i = \{\sigma_t, \sigma''_t\} \in \Omega_t^i$ for all $i \in \mathcal{I}$ and $\omega_t^j = \{\sigma'_t, \sigma''_t\} \in \Omega_t^j$ for all $j \in \mathcal{J}$. Then, in a financial market equilibrium,*

$$\sum_{j \in \mathcal{J}} [\tilde{e}^j(\sigma_t) - \tilde{c}^j(\sigma_t)] = \sum_{j \in \mathcal{J}} [\tilde{e}^j(\sigma'_t) - \tilde{c}^j(\sigma'_t)]$$

that is, agents in \mathcal{J} and in \mathcal{I} cannot mutually insure each other against the idiosyncratic risk between σ_t and σ'_t .

The result of Proposition 4.3 is of particular interest in the case in which agents in \mathcal{I} are consistently poorer (in terms of financial wealth) than those in \mathcal{J} in σ_t and consistently richer than those in \mathcal{J} in σ'_t . A standard equilibrium would optimally predict that agents in \mathcal{I} and in \mathcal{J} would mutually insure each other against the idiosyncratic risk in states σ_t and σ'_t . Yet, when the partitions of the agents in \mathcal{I} and \mathcal{J} intersect in σ''_t , a state with a total initial endowment distinct from both σ_t and σ'_t , a positive transfer from \mathcal{J} to \mathcal{I} in σ_t implies also a positive transfer from \mathcal{J} to \mathcal{I} in σ'_t , implying that such mutual insurance is impossible.

The proposition shows a sense in which trade is limited when awareness partitions are non-nested: in extreme cases, differential awareness can completely eliminate any trade.

We now derive conditions that allow us to "observationally distinguish" between standard financial markets with initial financial endowment $(\tilde{e}^i)_{i \in I}$ and financial markets with differential awareness and the same endowment process.

Proposition 4.4 *Consider an economy with differential awareness. If for some $i \in I$ and some $\sigma_t, \sigma'_t \in \Omega$ with $\tilde{e}(\sigma_t) > \tilde{e}(\sigma'_t)$, $\sigma_t, \sigma'_t \in \omega_t^i$ for some $\omega_t^i \in \Omega^i$, then*

(i) *there is no economy with full awareness and homogenous beliefs satisfying Assumptions 1 and 3 and initial total financial endowment process \tilde{e} such that its equilibrium coincides with the financial market equilibrium of the economy with differential awareness;*

(ii) *if an economy with full awareness and i.i.d. beliefs $(\tilde{\pi}^k)_{k \in I}$ satisfying Assumptions 1 and 3, and with an initial endowment process \tilde{e} has an equilibrium that coincides with the financial market equilibrium of the economy with differential awareness, then there is an agent j such that $\frac{\tilde{\pi}^i(\sigma_t)}{\tilde{\pi}^i(\sigma'_t)} < \frac{\tilde{\pi}^j(\sigma_t)}{\tilde{\pi}^j(\sigma'_t)}$, that is, i underestimates the probability of the "good" state of the economy σ_t relative to j .*

If furthermore, there are $s, s' \in S$ such that $\sigma_t = (\sigma_{t-1}, s)$, $\sigma'_t = (\sigma_{t-1}, s')$, $s, s' \in w^i$ for some $w^i \in W^i$ and if for some $\sigma'_{t'} \in \Omega$, $\tilde{e}(\sigma'_{t'}, s) < \tilde{e}(\sigma'_{t'}, s')$, then

(iii) *there is no economy with full awareness and i.i.d. beliefs satisfying Assumptions 1 and 3 and initial endowment process \tilde{e} such that its equilibrium coincides with the financial market equilibrium of the economy with differential awareness.*

Proposition 4.4 illustrates the distinctive characteristics models of financial markets with differential awareness. In particular, agents with lower awareness levels will appear to be underestimating the probability of good states. As a consequence, a financial market with differential awareness will, in general, exhibit the equity premium puzzle, overpricing bonds. More generally, assets that are measurable with respect to partitions of agents with lower awareness levels and thus do not expose such agents to surprises, will be overpriced. Furthermore, when the state process is known to be i.i.d., but financial endowment reversals across states can occur over time, the behavior of partially aware agents cannot be explained by i.i.d. beliefs, since the state that the agent overweighs will change depending on the financial endowment of the economy. While this behavior is reminiscent of ambiguity aversion, we will see below that the long-run behavior of the economy and in particular, the implications for survival, are very different from that of an economy with ambiguity-averse agents. Indeed, while Condie (2008) shows that agents with max-min preferences a.s. vanish in the presence of expected utility maximizers with correct beliefs, our results below demonstrate that boundedly aware agents can survive and affect prices in the long-run.

Remark 4.1 *The results of Propositions 4.2, 4.3 and 4.4 are stated for the financial market equilibrium, as opposed to the equilibrium of the economy with differential awareness. The*

results clearly hold for the equilibrium of the economy, whenever agents' initial endowments are measurable with respect to their respective awareness partitions. They can also be extended to the comparison of equilibria provided that the non-financial endowments of the agents cannot compensate the bias created by the differential awareness.

Consider, e.g., the case in which both financial wealth and total endowments are equalized across two states, σ_t and σ'_t , that is, $\tilde{e}(\sigma_t) = \tilde{e}(\sigma'_t)$ and $e(\sigma_t) = e(\sigma'_t)$ hold simultaneously. Suppose that the corresponding financial market equilibrium has been computed to be \tilde{c}^i and thus, the actual consumption streams are given by

$$e^i - \tilde{e}^i + \tilde{c}^i$$

Under the conditions of Proposition 4.2, it is possible that each agent i , is fully insured with respect to his total consumption, c^i against idiosyncratic risk across σ_t and σ'_t . This, however, would require that initial endowments e be chosen in a very specific way, that is:

$$e^i(\sigma_t) - e^i(\sigma'_t) = \tilde{c}^i(\sigma_t) - \tilde{c}^i(\sigma'_t) - [\tilde{e}^i(\sigma_t) - \tilde{e}^i(\sigma'_t)]$$

In this sense, the result of Proposition 4.2 holds generically also for the equilibrium of the economy.

As for the result of Proposition 4.3, the lack of mutual insurance through the financial market will not be relevant if the actual endowment of the agents satisfied $\sum_{j \in \mathcal{J}} e^j(\sigma_t) = \sum_{j \in \mathcal{J}} e^j(\sigma'_t)$, that is, if the two sets of agents were not actually in need of mutual insurance against idiosyncratic risk across the two states of the world. This, however, again is a non-generic situation.

Finally, Proposition 4.4 can be extended to the equilibrium of the economy as follows:

Corollary 4.5 *Consider an economy with differential awareness. If for some $i \in I$ and some $\sigma_t, \sigma'_t \in \Omega$ with $\tilde{e}(\sigma_t) > \tilde{e}(\sigma'_t)$, $e(\sigma_t) > e(\sigma'_t)$, $\sigma_t, \sigma'_t \in \omega_t^i$ for some $\omega_t^i \in \Omega^i$, and $e^i(\sigma_t) \leq e^i(\sigma'_t)$ then*

(i) *there is no economy with full awareness and homogenous beliefs satisfying Assumptions 1 and 3 and initial total endowment process e such that its equilibrium coincides with the equilibrium of the economy with differential awareness;*

(ii) *if an economy with full awareness and i.i.d. beliefs $(\tilde{\pi}^k)_{k \in I}$ satisfying Assumptions 1 and 3, and with an initial endowment process e has an equilibrium that coincides with the equilibrium of the economy with differential awareness, then there is an agent j such that $\frac{\tilde{\pi}^j(\sigma_t)}{\tilde{\pi}^j(\sigma'_t)} < \frac{\tilde{\pi}^j(\sigma_t)}{\tilde{\pi}^j(\sigma'_t)}$, that is, j underestimates the probability of the "good" state of the economy σ_t relative to j .*

If furthermore, there are $s, s' \in S$ such that $\sigma_t = (\sigma_{t-1}, s)$, $\sigma'_t = (\sigma_{t-1}, s')$, $s, s' \in w^i$ for some $w^i \in W^i$ and if for some $\sigma'_t \in \Omega$, $\tilde{e}(\sigma'_t, s) < \tilde{e}(\sigma'_t, s')$, $e(\sigma'_t, s) < e(\sigma'_t, s')$ and $e^i(\sigma'_t, s) \geq e^i(\sigma'_t, s')$, then

(iii) *there is no economy with full awareness and i.i.d. beliefs satisfying Assumptions 1 and 3 and initial endowment process e such that its equilibrium coincides with the financial market equilibrium of the economy with differential awareness.*

5 Survival in Financial Markets with Coarse Contingencies

In the previous sections, we showed that differential awareness can have an impact on equilibrium prices and allocations in financial markets. This raises the question of whether the impact of less aware agents on prices and allocations is temporary or permanent. Is it the case that their consumption converges to 0 over time, thus driving the equilibrium allocation to the one that would have obtained had all agents been fully aware? In this section, we will show that partially aware agents can have a long-term impact on prices and risk sharing.

We define survival in the financial markets with respect to the consumption of the agents derived from their trades of financial wealth, \tilde{c}^i :

Definition 5.1 *Agent i vanishes from the financial market on a path σ if $\lim_{t \rightarrow \infty} \tilde{c}^i(\sigma_t) = 0$. Agent i survives in the financial market on σ if $\lim_{t \rightarrow \infty} \sup \tilde{c}^i(\sigma_t) > 0$.*

Note that usually, (Blume and Easley, 2006), survival is defined relative to the consumption of the agent, $c^i(\sigma_t)$. Since $c^i(\sigma_t) - \tilde{c}^i(\sigma_t) \geq 0$ always holds, survival in the financial market implies survival in the usual sense. However, an agent who vanishes from the financial market might not have actual consumption which converges to 0. However, since survival is defined relative to the tradeable wealth of the agent, an agent who vanishes from the financial market will have no positive asset holdings in the limit and his impact on asset prices in the limit will be 0.

If Assumptions 1, 2, 2*,3 and 4 hold, the results we derive in Guerdjikova and Quiggin (2019) can be adapted to and summarized in the context of differential awareness as follows⁸. We recall that discount factors are equal across agents, so survival in financial markets will depend only on agents' awareness structure and beliefs.

Our first set of results concerns the case of agents with "nested" awareness partitions, that is, the case when agents in the economy can be ordered with respect to their awareness from "most" to "least" aware. To formulate these results, we need to define when the unawareness of an agent is "relevant in the limit". Intuitively, the unawareness of an agent is relevant in the limit, if the total financial endowment of the economy is not measurable with respect to his awareness partition even in the limit. The unawareness of agent i who is less aware than j is relevant with respect to that of j if in the limit, the maximal part of the financial endowment measurable with respect to j 's awareness partition is not measurable with respect to i 's awareness partition. The formal definitions of these concepts are stated in the appendix. Adapting the analysis of Guerdjikova and Quiggin (2019), Propositions 6-8 we obtain

Remark 5.1 *Consider an economy with differential awareness:*

- (i) if the agents in the economy have nested awareness partitions and correct beliefs, all agents survive a.s. in the financial market;*
- (ii) if agent i has a (weakly) finer awareness partition Ω^i than agent j , Ω^j and if for the partition W^j , the K-L distance of agent's i beliefs from the truth is strictly smaller than that of j , j vanishes a.s. in the financial market;*

⁸ The proofs to the remarks in this section are contained in Guerdjikova and Quiggin (2019 a).

(iii) if the agents in the economy have nested awareness partitions, Ω^1 strictly finer than $\Omega^2 \dots$ strictly finer than Ω^n and identical beliefs, and if the unawareness of any agent $i \geq 2$ is both relevant in the limit and relevant with respect to that of $i - 1$, all agents survive a.s. in the financial market.

(iv) if the agents in the economy have nested awareness partitions, Ω^1 strictly finer than $\Omega^2 \dots$ strictly finer than Ω^n , if for all $i < j$,

$$\sum_{w^j} \pi(w^j) \ln \frac{\pi(w^j)}{\pi^i(w^j)} > \sum_{w^j} \pi(w^j) \ln \frac{\pi(w^j)}{\pi^j(w^j)}$$

and if the unawareness of any agent $i \geq 2$ is both relevant in the limit and relevant with respect to that of $i - 1$, agents 1 and 2 a.s. survive. If, in addition¹⁰, for every $j \in \{2 \dots n - 1\}$, all $w^{j+1} \in W^{j+1}$ and all $w^j \subseteq w^{j+1}$, $\pi^j(w^j | w^{j+1}) = \pi(w^j | w^{j+1})$, all agents a.s. survive.

Our first result (i) shows that whenever agents have correct beliefs relative to their awareness partitions and the awareness partitions are ordered with respect to inclusion, their level of awareness is irrelevant for survival. In fact, all agents survive in the financial market. We can relate this result to the features identified in Propositions 4.2 and 4.4: recall that in economies with differential awareness, insurance against idiosyncratic risk did not obtain in equilibrium. In contrast, more aware agents would insure less aware agents against some of the aggregate risk. Finally, relative state prices would be biased relative to the state probabilities even if all agents hold correct beliefs. The result in part (i) of Remark 5.1 implies that these features of the economy will persist in the long run, even if all agents have correct beliefs and equal discount factors.

The second insight of Remark 5.1 concerns agents with wrong beliefs. Part (iii) extends the result of Part (i) to the case of identical, but not necessarily correct beliefs. Parts (ii) and (iv) consider heterogeneity with respect to beliefs and awareness. When agents are simultaneously less aware and hold beliefs further from the truth than others, they almost surely vanish, as shown in part (ii). However, more awareness can compensate for wrong beliefs. Part (iv) considers the case in which agents have nested partitions such that the less aware agents have beliefs closer to the truth. It requires that for $i \geq 2$, agents' unawareness is relevant even in the limit. In such a scenario, the less aware agents cannot consume the entire financial wealth of the economy: such a consumption stream would expose him to unfavorable surprises at a strictly positive cost, thus violating optimality. Hence, it is the agents with beliefs further away from the truth, but with higher levels of awareness, who ensure that the financial markets clear. They consume the 'leftovers' of the less aware agents and, thus, the fact that the latter's unawareness is relevant ensures that they survive a.s..

To contrast survival in economies with differential awareness to that in an economy with full awareness, consider an extreme version of the Blume and Easley (2006) model with a continuum of agents. If all beliefs have positive support, then only agents with perfectly accurate beliefs survive. More generally, only agents with maximally accurate beliefs (that, is minimal

⁹ The formal definitions of these terms are provided in the Appendix.

¹⁰ Since $\pi^j(w^{j+1})$ is in general incorrect, this does not imply that $\pi^j(w^j)$ is correct.

K-L distance from the truth) can survive. By contrast, with differential awareness and nested partitions, agents with different beliefs, varying in accuracy, can coexist. This results seems more consistent with observed outcomes.

We next consider economies, in which agents' awareness partitions are not nested. We start with a formal definition of economies with non-nested awareness partitions.

Definition 5.2 *Agents i and j have non-nested awareness partitions if there are states¹¹ $s, s', s'', s''' \in S$ such that:*

- *there are elements of i 's awareness partition $w^i, w^{i'}, w^{i''} \in W^i$ with $w^i \neq w^{i'}$ such that $s \in w^i, s' \in w^{i'}$ and $s'', s''' \in w^{i''}$ and*
- *there are elements of j 's awareness partition $w^j, w^{j'}, w^{j''} \in W^j$ with $w^j \neq w^{j''}$ such that $s, s' \in w^j, s'' \in w^{j'}$ and $s''' \in w^{j''}$.*

If the states s and s' satisfy this definition, then we will say that i can distinguish s and s' and trade between them, whereas j cannot.

We will say that agents in the economy have non-nested awareness partitions if, for each agent i , there are states s and $s' \in S$ between which i can distinguish and trade, but between which no other agent in the economy can distinguish, nor trade.

Remark 5.2 *Consider an economy with differential awareness and assume that for an agent j , there are states $s(j)$ and $s'(j) \in S$ such that the awareness partitions of j and any other agent $k \in I \setminus \{j\}$ are non-nested and j can distinguish and trade between $s(j)$ and $s'(j)$, whereas k cannot. Assume that k 's unawareness over states $s(j)$ and $s'(j)$ is relevant in the limit, that is, that condition (15) in the Appendix holds for $s(j)$ and $s'(j)$.*

(i) Agent j survives a.s..

(ii) If, furthermore, the condition holds for all $j \in I \setminus \{i\}$, i is fully aware and all agents have correct beliefs, then all agents a.s. survive.

Our first result (i) shows that whenever an agent is the only one in the economy capable of distinguishing and thus, trading between some relevant contingencies, he survives regardless of his beliefs, and regardless of the awareness of the other agents. This result is of special interest in view of Proposition 4.3. In particular, consider two sets of agents \mathcal{I} and \mathcal{J} who are exposed to some idiosyncratic risk, such as labor income in two different sectors of the economy. While each type of agent is aware of their own labor income stream, they are not aware of the variation in the income of the other group. Thus, mutual insurance of the idiosyncratic risk, which requires conditioning on the specific variation of income of each of the groups is impossible.

¹¹ The definition does not require the four states to be distinct and thus also applies to economies with only 3 states, where one can set $s' = s''$. However, requiring $s' = s''$ is in general too restrictive for our purposes, since it excludes, for example, an economy in which $W^i = \{\{s\}; \{s'\}; \{s'', s'''\}\}$ and $W^j = \{\{s; s'\}; \{s''\}; \{s'''\}\}$. Indeed, choose any three states (for example, s, s'' and s''') and note that at least one of the agents (here: j) can distinguish among any of the three states and hence, the definition of nonnested partitions would not apply, contrary to intuition. In economies with only two states, agents' partitions are trivially nested.

Despite the lack of insurance against idiosyncratic risk, provided that the difference in payoffs in the relevant states is bounded away from 0 in the limit, the consumption of both types will be strictly positive i.o. on almost every path and hence, they will both survive a.s., regardless of their beliefs.

Finally, part (ii) concerns the case in which a fully aware agent with correct beliefs is present in the economy. By Remark 5.1 (ii), this will cause all partially aware agents with incorrect beliefs to vanish a.s. However, as long as the awareness partitions are non-nested, and the partially aware agents have correct beliefs, they survive a.s..

The survival results in this section were formulated with respect to the agents' financial wealth, as opposed to their total consumption. Since, by definition, total consumption exceeds the agents' financial wealth, we conclude that boundedly aware agents will not see their consumption decline to 0 even in the presence of more aware agents, provided that their beliefs (on their respective awareness partition) are at least as accurate as those of the more aware agents.

In contrast, in the presence of a more aware agent with strictly more accurate beliefs, a boundedly aware agent (as e.g., in Part (ii) of Remark 5.1) will almost surely vanish from financial markets, but might still enjoy strictly positive consumption in certain states, even in the limit. In the special case, in which individual endowment are i.i.d. and agent i 's awareness partition includes an element with two states, $w^i = \{s, s'\}$ such that $e^i(\cdot, s) > e^i(\cdot, s')$, agent i 's consumption will satisfy

$$\lim_{t \rightarrow \infty} c^i(\sigma_t, s) = e^i(\cdot, s) - e^i(\cdot, s') > 0,$$

even though his financial wealth will converge to 0, $\lim_{t \rightarrow \infty} \tilde{c}^i(\sigma_t, s) = 0$. However, from the point of view of the agent, occurrences of consumption strictly bounded away from 0 will be "positive surprises" as compared to his minimal perceived consumption, \tilde{c}^i . Furthermore, since the consumption he enjoys in the limit is exactly the part of his initial endowment, which is not traded in the financial market, this type of survival will not have an effect on asset prices in the limit.

This last statement should be contrasted with the case in which the more aware agents can survive with incorrect beliefs: since they can actively trade the part of the financial endowment which is non-measurable with respect to the partitions of less aware agents, their wrong beliefs have an impact on asset prices even in the limit.

6 Robustness to Preference Specification and Survival

The implications of Assumption 4 might seem rather strong in that they effectively restrict the agents' trades in the financial market to be measurable in their respective awareness partitions. In this section, we relax this assumption and thus allow for the possibility that boundedly aware agents can engage in trades which are not measurable with respect to their awareness partition. We consider three alternative scenarios: first, we consider a relaxation of the assumption that $\alpha = 0$ by allowing for agents with $\alpha > 0$. Second, we consider the specification of Auster et al. (2020) who assume that an agent with a coarsening of the state space associates with each element of his partition the expected payoff (with respect to the true probability π) generated

by the asset across the states constituting the respective coarse contingency. We show that generically, both types of agents a.s. vanish. Finally, we consider an agent, who combines the criterion in Assumption 4 with the specification of Auster et al. (2020). We establish conditions under which such an agent survives in the presence of a fully aware agent with correct beliefs. First consider the following modification of Assumption 4:

Assumption 4' An agent with awareness partition $(\Omega^i, \mathcal{F}^i)$ with initial endowment e^i and portfolio holdings a^i perceives his consumption at $\omega_t^i \in \Omega^i$ as $\mathbf{c}_{\Omega^i}^i(a^i, e^i, \omega_t^i)$ given by (3). The utility of a set-valued consumption $\mathbf{c}_{\Omega^i}^i(a^i, e^i, \omega_t^i)$ conditional on an element of Ω^i , ω_t^i is given by

$$v_\alpha^i(\mathbf{c}_{\Omega^i}^i(\omega_t^i)) = \alpha \max_{c \in \mathbf{c}_{\Omega^i}^i(a^i, e^i, \omega_t^i)} u^i(c) + (1 - \alpha) \min_{c \in \mathbf{c}_{\Omega^i}^i(a^i, e^i, \omega_t^i)} u^i(c).$$

Assumption 4' allows the agents to have an intermediate degree of aversion to unfavorable surprises, $(1 - \alpha) \in (0, 1)$ and thus to be open to positive surprises. Assumption 4' however renders agents' preferences non-convex and thus might endanger the existence of an equilibrium as in Definition 4.1. Our next proposition implicitly assumes that an equilibrium exists and demonstrates that if a fully aware agent with correct beliefs is present in the economy, an agent i satisfying Assumption 4' a.s. vanishes. Such an agent trades-off an investment into a safe portfolio on a given coarse contingency ω_t^i versus investing some of his wealth into the state $\underline{\sigma}_t \in \omega_t^i$ with the minimal price $\underline{\sigma}_t = \arg \min_{\tilde{\sigma}_t \in \omega_t^i} p(\tilde{\sigma}_t)$, because this provides him with an extra utility weighted by α . Whether the agent does that depends in general on the trade-off between the marginal utility of such an investment, driven by α and its relative price, $\frac{\min_{\tilde{\sigma}_t \in \omega_t^i} p(\tilde{\sigma}_t)}{p(\omega_t^i)}$. We show that on paths on which an element $w^i \in W^i$ with at least two elements occurs infinitely often, the relative price becomes arbitrarily low, which, for a fixed α , implies that for a given ω_t^i , the agent eventually places all of his holdings on a single state, that with the minimal price. But since the probability of a path on which exactly the chosen state occurs is 0 according to π , the agent π -a.s. vanishes.

Proposition 6.1 *Consider a consumer i with $\alpha > 0$. Suppose that there is a consumer j who is fully aware and has correct beliefs. In any equilibrium of the economy, i vanishes π -a.s.*

Note that the result holds for any strictly positive α . Thus, even slight preferences for positive surprises lead the agent to vanish in the long-run.

We next consider a perception of coarse contingencies similar to Auster et al. (2020) who assume that an agent with a coarsening of the state space associates with each element of his partition the expected payoff (with respect to the true probability π) generated by the asset across the states constituting the respective coarse contingency. We adopt a similar assumption, replacing the true probability distribution with an agent-specific "imputed belief"¹². We will show in what follows that the results are not very sensitive to the specific choice of weights.

¹² Note that the weights $\pi^i(\sigma_t | \omega_t^i)$ are not agent i 's beliefs. Since i is unaware of σ_t , he cannot possibly entertain beliefs on such states. Instead, they can be interpreted as subjective weights used to represent the payoff of an asset which cannot be described according to his perception of the world.

However, they significantly differ from those obtained using Assumption 4.

Recall that the market equilibrium we defined earlier allows for trades in Arrow securities paying on any of the contingencies σ_t . Assumption 5 specifies the agents' perception of consumption streams based on imputed beliefs:

Assumption 5 For each boundedly aware agent i with awareness partition $(\Omega^i, \mathcal{F}^i) \neq (\Omega, \mathcal{F})$, there exist imputed beliefs, that is, positive weights $(\pi^i(s))_{s \in S}$ adding up to 1 such that agent i with initial endowment e^i and portfolio holdings a^i perceives his consumption at $\omega_t^i \in \Omega^i$ as

$$\mathbf{c}_{\Omega^i, a}^i(a^i, e^i, \omega_t^i) = \sum_{\sigma_t \in \omega_t^i} \pi^i(\sigma_t | \omega_t^i) [a^i(\sigma_t) + e^i(\sigma_t)]$$

where

$$\pi^i(\sigma_t | \omega_t^i) = \frac{\prod_{\tau=1}^t \pi^i(s = s(\sigma_\tau))}{\prod_{\tau=1}^t \pi^i(w = w(\omega_\tau^i))} \text{ if } \sigma_t \in \omega_t^i$$

and $\pi^i(\sigma_t | \omega_t^i) = 0$, else. Furthermore, the utility of perceived consumption at ω_t^i is given by

$$v_a^i(\mathbf{c}_{\Omega^i, a}^i(a^i, e^i, \omega_t^i), u^i) = u^i \left(\sum_{\sigma_t \in \omega_t^i} \pi^i(\sigma_t | \omega_t^i) [a^i(\sigma_t) + e^i(\sigma_t)] \right)$$

According to Assumption 5, even though an agent is not aware of the finer contingencies within an element of his awareness partition, he nevertheless evaluates his initial endowment, portfolios and consumption streams as if he were imputing probabilities to such finer contingencies and calculating the expected consumption with respect to these probabilities on the respective element of his partition.

Substituting the agents' perceptions of his initial endowment and portfolios as defined in Assumption 5, into Definition 4.1, we obtain a natural concept of an equilibrium for the economy with bounded awareness and imputed beliefs. The existence of such an equilibrium is established by Bewley's (1972) theorem. However, the equilibrium of such an economy need no longer be interior.

Proposition 6.2 Under Assumption 5, unless $\frac{p(\sigma_t)}{p(\sigma'_t)} = \frac{\pi^i(\sigma_t | \omega_t^i)}{\pi^i(\sigma'_t | \omega_t^i)}$ holds in equilibrium for every $\sigma_t, \sigma'_t \in \omega_t^i$ and every $\omega_t^i \in \Omega^i$, agent i 's equilibrium consumption will satisfy $c^i(\sigma_t) = 0$ for some $\sigma_t \in \Omega$. Agent i thus vanishes with strictly positive probability in finite time on the set of paths ω^i on which $\frac{p(\sigma_t)}{p(\sigma'_t)} = \frac{\pi^i(\sigma_t | \omega_t^i)}{\pi^i(\sigma'_t | \omega_t^i)}$ for all $\sigma_t, \sigma'_t \in \omega_t^i$ occurs only for a finite number of periods.

The result of Proposition 6.2 is due to the fact that two Arrow securities paying on distinct states σ_t, σ'_t which belong to an element of the agent's awareness partition ω_t^i are effectively conceived as having a certain payoff on ω_t^i (and 0, else). Thus, whenever their prices do not reflect the relative weights assigned by the agent to these states, the agent perceives an arbitrage opportunity. He thus takes an extreme short position in the "more expensive" asset, resulting in a consumption of 0 in the respective state¹³.

¹³ If the non-negativity constraint on consumption is not imposed, the perception of arbitrage by

The condition $\frac{p(\sigma_t)}{p(\sigma'_t)} = \frac{\pi^i(\sigma_t|\omega_t^i)}{\pi^i(\sigma'_t|\omega_t^i)}$ can obtain in equilibrium in the special, but relevant case whenever the economy faces no aggregate risk. If all agents' beliefs are correct and if i 's weights coincide with the correct probabilities on S , $\pi^i(s) = \pi(s)$, the equilibrium of the economy will provide full insurance to everyone at prices $\frac{p(\sigma_t)}{p(\sigma'_t)} = \frac{\pi(\sigma_t|\omega_t^i)}{\pi(\sigma'_t|\omega_t^i)}$. In this case, however, differential awareness does not impact the equilibrium allocation relative to the economy with full awareness and all agents survive. Even in the special case of no aggregate risk, agent i , however will have 0-consumption with strictly positive probability in finite time, if his weights differ from the truth, $(\pi^i(s))_{s \in S} \neq (\pi(s))_{s \in S}$ as long as all other agents have correct beliefs. Our next result shows that even if all agents have correct beliefs and i 's weights $\pi^i(s)$ are correct, in an economy with aggregate risk, i will behave as if he had effectively wrong beliefs and thus will vanish a.s. in the limit.

Proposition 6.3 *Consider an economy with aggregate risk, $e(\sigma_t, s) > e(\sigma_t, s') + \epsilon$ for some $s, s' \in w^i$, every $\sigma_t \in \Omega$ and some $\epsilon > 0$. Suppose that all agents have correct beliefs, all agents other than i are fully aware and agent i 's weights $\pi^i(s)$ are correct. If $\frac{p(\sigma_t, s)}{p(\sigma_t, s')} = \frac{\pi(s|w)}{\pi(s'|w)}$ for all $\sigma_t \in \Omega$, then agent i a.s. vanishes.*

Proposition 6.3 thus shows that even if equilibrium prices were such that¹⁴ i does not perceive arbitrage and thus finds an interior consumption stream to be optimal, i nevertheless vanishes when other agents are fully aware and have correct beliefs. While the proof of this proposition relies on the fact that i has correct weights across s and s' , it is easy to see that unless the weights π^i exactly compensate for the ratio of marginal utilities at each σ_t so that

$$\frac{\pi^i(s) u'_i(c^i(\sigma_t, s') + e(\sigma_t, s) - e(\sigma_t, s'))}{\pi^i(s') u'_i(c^i(\sigma_t, s'))} = \frac{\pi(s)}{\pi(s')}$$

i will vanish whenever the rest of the agents have correct beliefs. Since i is assumed to not be able to reason about contingencies s and s' choosing weights that satisfy this condition would amount to a probability 0 event. Furthermore, in as far as i 's consumption is not i.i.d., such weights cannot be independent of σ_t .

Our results so far demonstrate that engaging in trades which are non-measurable with respect to the agent's awareness partition, that is, trading assets that the boundedly aware agent i does not understand, but is forced to evaluate within the limits of his awareness, eventually leads to the agent's ruin. Either the agent mistakenly perceives arbitrage and vanishes with strictly positive probability in finite time, or he believes that markets value the assets correctly, but behaves as if he has wrong beliefs and vanishes a.s. in the limit.

the agents means that they will be taking infinite positions in certain assets. An equilibrium of the economy will thus fail to exist. Alternatively, if short-sale constraints allow for a negative consumption, the agent who perceives arbitrage will a.s. end up with a negative consumption in finite time, similar to the bankruptcy scenario discussed by Auster et al. (2020).

¹⁴ Note that we do not claim that such an equilibrium exists, merely that if it were to exist, it would specify 0-limit consumption for i .

Our final result in this section exhibits a lexicographic rule, which combines Assumptions 4 and 5.

Assumption 6 An agent with awareness partition $(\Omega^i, \mathcal{F}^i)$ with initial endowment e^i and portfolio holdings a^i evaluates his consumption at $\omega_t^i \in \Omega^i$ as

$$v_L^i(c_{\Omega^i}^i(\omega_t^i)) = \min \left\{ v_{\min}^i(c_{\Omega^i}^i(a^i, e^i, \omega_t^i)), \frac{1}{k} v_a^i(c_{\Omega^i, a}^i(a^i, e^i, \omega_t^i), u^i) \right\},$$

where $k > 1$.

The behavior modelled by Assumption 6 combines the extreme aversion to unfavorable surprises as captured by the functional v_{\min} with the perceived "average" payoff of the portfolio in v_a scaled by a factor $k > 1$. The agent evaluates the utility of his consumption at a coarse contingency using each of the functionals and then takes the minimum of these evaluations v_{\min} and $\frac{1}{k}v_a$, where $k > 1$ to make the lexicographic rule non-trivial.

Proposition 6.4 *Consider an economy consisting of a fully aware agent j with correct beliefs and an agent i with a CRRA utility function $u^i(c) = \frac{c^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, and correct beliefs. Suppose that i satisfies assumption 6 and that $\pi^i(s | w^i) = \pi(s | w^i)$ for each $w^i \in W^i$ and each $s \in S$. In any equilibrium of the economy, and on any path $\sigma \in \Sigma$, agent i survives.*

Proposition 6.4 identifies conditions under which combining the minimum and the average in a lexicographic way allows the agent to survive. Several conditions are necessary for this result. First, just as in the results in Section 5, the agent has to have correct beliefs on his awareness partition. Second, the assumption of a CRRA utility function ensures that i 's marginal rate of substitution between investing an additional unit into average versus minimal consumption remains finite on the optimal consumption path, even when both minimal and average consumption become close to 0. Finally, we require that the subjective weights used to determine the average have to coincide with the correct probabilities. If this condition is not satisfied, the probabilities assigned by i and by j to the state i believes has the lowest price-to-probability ratio will diverge to ∞ . Thus, i will eventually find it optimal to assign almost all of his wealth to a set of paths which have 0 probability with respect to the truth and thus, vanish a.s. with respect to the truth.

Thus, even though Proposition 6.4 establishes survival for a boundedly aware agent who does not satisfy extreme aversion to unfavorable surprises as captured by Assumption 4, it does so under the rather restrictive condition of correct weights $\pi^i(s | w^i) = \pi(s | w^i)$, as assumed by Auster et al. (2020). As we argue in the following sections, boundedly aware agents will in general fail to satisfy such an assumption. Indeed, one of the premises of bounded awareness in the form of coarsening is that the agent finds it difficult to distinguish between and assign weights to the individual states within a coarse contingency.

7 Bounded Awareness and Ecological Rationality

In this section we address two closely related questions. First, can agents, and society as a whole,

be made worse off by an increase in awareness? Second, can boundedly aware agents benefit from constraining their portfolio choices, and if so how can such benefits be characterized.

In addressing the first question, our key result is that the beliefs of a more aware agent are at least as far away from the truth as those of a less aware agent whenever the two sets of beliefs coincide on the partition available to the less aware agent. On the other hand, if beliefs are equally accurate, the welfare of the more aware agent will be at least as high as that of the less aware agent. We consider the notion of ‘ecological rationality’, developed by Gigerenzer and applied in the context of bounded awareness by Grant and Quiggin. We show that aversion to unfavorable surprises as captured by Assumption 4 gives rise to heuristic behavior which can be ecologically rational under certain conditions.

To address the second question, we propose the concept of an ‘evolutionary rational’ heuristic: a heuristic which allows a agent to survive in the presence of agents who are more aware and better informed. Heuristics corresponding to aversion to unfavorable surprises are then shown to be evolutionary rational in this sense.

7.1 The Impact of Increased Awareness on Utility and Beliefs

We start by formalizing in Proposition 7.1 the sense in which a finer awareness partition leads to an increase of the K-L distance between agent’s beliefs and the truth.

Proposition 7.1 *Consider agents i and j such that i ’s awareness is coarser than j ’s, with probability beliefs that coincide on the coarser partition W^i . The K-L distance between j ’s beliefs on the more refined partition W^j and the truth is at least as great as the K-L distance of the beliefs of the two agents on the less refined partition W^i and the truth.*

Proposition 7.1 shows that the survival index with refined awareness is less than the survival index with coarse awareness. In the previous section, we showed that the K-L distance between the agent’s beliefs and the truth is relevant for survival. In particular, an agent whose awareness increases will have to form beliefs over a new set of contingencies he did not consider before. Unless reliable statistical information is easily available and can be directly incorporated into the agent’s decision, this opens room for mistakes. Even if the agent’s beliefs on the coarser partition were correct, increased awareness may lead to wrong beliefs and thus potentially diminish his chances for survival.

This, in turn, has an effect on the agents’ survival. In particular, as we show below in Example 7.1, it is easy to construct economies in which i a.s. survives, but j a.s. vanishes, his consumption converging to 0. Finally, if j ’s beliefs are not correct, $V_0^j(c^j)$ is a biased estimate of j ’s expected utility. Example 7.2 shows that in this case j ’s expected utility with respect to the truth may be strictly lower than that of i .

Example 7.1 *Consider an economy with a state space S . Let the set of agents be $I = \{1, 2, 3\}$. Agents 1 and 2 are fully aware so that $\Omega^1 = \Omega^2 = \Omega$. Agent 3 has a coarser awareness partition, W^3 , which induces the partition Ω^3 of Ω . Suppose that all agents have correct i.i.d. beliefs on the coarsest of the three partitions, W^3 , and thus on Ω^3 . Assume as well that agent 1 has correct i.i.d. beliefs on S , whereas agent’s 2 beliefs on S are wrong. Thus, the K-L distance of*

2's beliefs from the truth is larger than that of 1, while their awareness partitions are identical. By Remark 5.1, part (i), agent 2 vanishes almost surely, both in the financial market and in terms of his total consumption going to 0.

Consider next the issue of survival for agents 1 and 3. Given the assumptions made above, and noting that $\tilde{c}^1 = c^1$, Lemma 3 in Guerdjikova and Quiggin (2019, p. 1731) implies that π -a.s. on Ω^3 ,

$$\lim_{t \rightarrow \infty} \frac{\sum_{\sigma \in \omega^3} \pi(\sigma | \omega_t^3) u'_1(c^1(\sigma))}{u'_3(\tilde{c}^3(\omega_t^3))} = \frac{u'_1(c^1(\sigma_0))}{u'_3(\tilde{c}^3(\sigma_0))} \in (0, \infty) \quad (13)$$

We will show that agent 3 almost surely survives. Indeed, assume in a manner of contradiction that 3 vanishes on some ω , $\lim_{t \rightarrow \infty} \tilde{c}^3(\omega_t^3) = 0$ and thus, by Assumption 1,

$$\lim_{t \rightarrow \infty} u'_3(\tilde{c}^3(\omega_t^3)) = \infty.$$

To ensure that the ratio of marginal utilities in (13) is not 0, it is necessary that

$$\lim_{t \rightarrow \infty} \sum_{\sigma \in \omega^3} \pi(\sigma | \omega_t^3) u'_1(c^1(\sigma)) = \infty$$

and thus, by Assumption 1, that there exists an event $F \subseteq \omega$ with $\pi(F | \omega) > 0$ and $c^1(\sigma) = 0$ for all $\sigma \in F$. It follows that on F , π -a.s., $\lim_{t \rightarrow \infty} [c^1(\sigma_t) + \tilde{c}^3(\sigma_t)] = 0$ and since, by Assumption 2*, the initial financial wealth of the economy is uniformly bounded away from 0 by m^* , it follows that agent 2 cannot vanish π -a.s. on F , since on F , his consumption has to satisfy π -a.s. $\lim_{t \rightarrow \infty} c^2(\sigma_t) = \lim_{t \rightarrow \infty} \tilde{c}^2(\sigma_t) \geq m^* > 0$. But since agent 2 vanishes a.s., it follows that the probability of an ω on which agent 3 vanishes has to be 0.

We conclude that agent 3 survives π -a.s. By equation (13) this implies that π -a.s. on Ω^3 , $\lim_{t \rightarrow \infty} \sum_{\sigma \in \omega^3} \pi(\sigma | \omega_t^3) u'_1(c^1(\sigma)) \neq \infty$ and thus, $c^1(\sigma) > 0$ π -a.s. on ω . Thus, agent 1 π -a.s. survives.

Comparing agents 2 and 3 provides an illustration of Proposition 7.1. Both agents' beliefs on W^3 and thus on Ω^3 are correct. However, agent 2 is more aware than 3 and thus has to also form beliefs on the finer state space S . These beliefs happen to be wrong and thus the K-L distance of 2's beliefs from the truth is larger than that of 3, as stipulated in Proposition 7.1. The presence of an agent who is equally aware as 2, but who has correct beliefs, gives 2 the possibility to trade on events not-measurable with respect to Ω^3 . While, as explained above, these additional trading opportunities in general increase the agent's welfare, trading on incorrect beliefs eventually leads to agent 2 vanishing. In contrast, the less aware agent 3 survives and enjoys strictly positive consumption in the limit.

Finally, note that allowing agent 2 to adopt a prior on a continuous set of probability distributions over the finer set of contingencies and learn the correct probabilities in a Bayesian way would still lead to him vanishing relative to agent 1, who has correct beliefs, see Theorem 5 in Blume and Easley (2006).

Example 7.1 illustrates the trade-off between higher expected utility resulting from higher levels of awareness and survival using two agents who are identical in all other characteristics but their awareness partitions, showing that higher levels of awareness may impede survival.

The arguments above highlight a potential conflict: increasing awareness allows the agent to

expand his trading opportunities and obtain higher welfare, but exposes him to losses due to wrong beliefs and eventually to the risk of a ruin (vanishing), his consumption being reduced to 0 in the long-run.

We next tackle the question of how the actual welfare of a single agent is impacted when the agent's awareness increases. To facilitate understanding, we consider a one-period economy with logarithmic preferences:

Example 7.2 *Consider a one-period economy. Assume that agent i 's endowment is constant across states, $e^i(s) = e^i(s')$ for all $s, s' \in S$ and let all agents have logarithmic preferences. If i is fully aware, i 's demand is given by:*

$$c^i(s) = \frac{e^i \pi^i(s)}{p(s)}$$

where π^i are his beliefs, which are not necessarily correct, and we use the normalization $\sum_{s \in S} p(s) = 1$.

Next consider the case of i being only aware of the trivial partition $W^i = \{\{S\}\}$. In the one-period economy, this means that i will abstain from trading and thus, i 's consumption coincides with his initial endowment e^i .

Comparing i 's equilibrium expected utility with respect to the truth in these two scenarios, we obtain that i will obtain a higher expected utility when more aware if and only if:

$$\sum_{s \in S} \pi(s) \ln c^i(s) > \ln e^i$$

which can be rewritten as:

$$-\sum_{s \in S} \pi(s) \ln \frac{\pi(s)}{\pi^i(s)} + \sum_{s \in S} \pi(s) \ln \frac{\pi(s)}{p(s)} > 0. \quad (14)$$

To understand the condition, note that the first term in (14) is the negative of the K-L distance of i 's beliefs π^i with respect to the truth and is always non-positive. It does not depend on the initial endowment of the economy, nor on the beliefs of the other agents. The second term is the relative entropy of the pricing kernel $p(s)$ with respect to the truth and is always non-negative. Hence, whether i 's expected utility will be higher when he is fully unaware depends on whether the deviation of his beliefs from the truth upon becoming fully aware exceeds the deviation of the equilibrium price kernel from the true probability. In particular, if other traders' beliefs are even further away from the truth than his own, i 's expected utility will increase as he becomes fully aware.

However, if the other traders in the economy have correct beliefs and if the economy faces no aggregate risk, then $p(s)$ will be a convex combination of i 's beliefs and the truth and hence, the entire term will be strictly negative. Hence, in an economy with no aggregate risk, i would be better off remaining unaware when all other agents have correct beliefs.

Furthermore, if the aggregate risk and i 's initial endowment are both relatively small, so will be the second term on the l.h.s.,

$$p(s) \approx \pi(s)$$

and hence, i 's expected utility will be higher if i is unaware.

If i 's beliefs when fully aware assign 0-probability to a possible state, for example if $\pi^i(s_1) \rightarrow 0$,

$$-\sum_{s \in S} \pi(s) \ln \frac{\pi(s)}{\pi^i(s)} \rightarrow -\infty,$$

whereas $p(s_1) \neq 0$ will hold as long as at least one other trader deems s_1 possible. It follows that as a function of i 's beliefs the second term is bounded above, whereas the first term is unbounded. We can conclude that for a given initial endowment of the economy, we can find sufficiently 'wrong' beliefs for the fully aware i such that he obtains higher expected utility when fully unaware. Alternatively, for given beliefs of i , we can choose the beliefs of the other traders sufficiently close to the truth and an initial total endowment of the economy sufficiently close to risk-free so that i would be better off with a lower level of awareness.

Finally, note that in the case of logarithmic preferences in an infinite-horizon economy, the same argument can be applied to each time period. It allows us to identify conditions under which, in each period, i 's expected utility with unawareness exceeds that with full awareness.

Example 7.2 shows that when the agent is not likely to acquire correct beliefs upon becoming aware of finer contingencies, he might be better off at a lower level of awareness.

The following proposition extends the example above to preferences which are more general than \ln and to endowments that are not necessarily measurable.

Proposition 7.2 *Suppose that agent i has a utility function for risk u_i such that $\lim_{c \rightarrow 0} u_i(c) = -\infty$. Then, for any strictly positive initial endowment $e^i(s) \gg 0$ and two strictly positive price systems $(p(s))_{s \in S}, (p'(s))_{s \in S} \gg 0$, there exist beliefs $\pi^i(s)$ and a partition W^i of S such that:*

$$\sum_{s \in S} \pi(s) u_i(e^i(s) + a^i(s)) < \sum_{s \in S} \pi(s) u_i(e^i(s) + a_{W^i}^i(s))$$

where a^i is the optimal portfolio given prices $(p(s))_{s \in S}$ when the agent is fully aware with beliefs π^i :

$$a^i = \arg \max_a \left\{ \sum_{s \in S} \pi^i(s) u_i(e^i(s) + a(s)) \mid \sum_{s \in S} a(s) p(s) \leq 0, e^i + a^i \geq 0 \right\}$$

and $a_{W^i}^i$ is the optimal portfolio of an agent satisfying Assumption 4 with a partition W^i given prices $(p'(s))_{s \in S}$:

$$a_{W^i}^i = \arg \max_a \left\{ \sum_{w \in W^i} \pi^i(w) u_i(\min_{s \in w} e^i(s) + \min_{s \in w} a(s)) \mid \sum_{s \in S} a(s) p'(s) \leq 0, \min_{s \in w} e^i(s) + \min_{s \in w} a(s) \geq 0 \right\}$$

Intuitively, partial awareness combined with aversion to unfavorable surprises as in Assumption 4, restricts the agent's investment opportunities and thus prevents him from trading on wrong beliefs allowing him to obtain a higher discounted expected utility relative to the true probability process. Such implicit restrictions on trade due to bounded awareness might also be beneficial from the point of view of the society as a whole.

7.2 Heuristics, ecological rationality and evolutionary rationality

The link between bounded awareness and heuristic constraints is developed by Grant and Quiggin (2013 b). In the model of Grant and Quiggin (2013 a) agents cannot be aware, in the modal-logical sense, of their own unawareness, but may nonetheless infer on the basis of induction from experience that their model of the world is incomplete and will be subject to unforeseen future surprises. Agents may therefore choose to adopt heuristic constraints on their decisions, such as those associated with the ‘precautionary principle’ (Grant and Quiggin 2013 b).

Gigerenzer (2007) defines the concept of ecological rationality: a heuristic is ecologically rational if in a given environment it yields better results, on average, than optimization based on an incomplete and possibly inaccurate model of that environment. An agent cannot know that a heuristic is ecologically rational, since this would require him to possess a complete and accurate model. Only a fully aware external observer can make a definitive assessment. However, based on induction from experience, agents may adopt heuristics that have previously worked well in similar environments. The notion of ecological rationality adopted by Gigerenzer (2007) concentrates on the comparison of different decision criteria / heuristics according to their actual payoff.

A different criterion, which might be particularly relevant in the context of financial markets and which seems inherent to the notion of ecological rationality is that of avoiding a ruin and ensuring survival. The definition suggested by Gigerenzer misses this aspect. Here, we suggest the notion of "evolutionary rational heuristic" to capture this idea:

Definition 7.1 *A heuristic is "evolutionary rational" if it allows the agent to survive a.s. in the financial market in the presence of a fully aware agent with correct beliefs satisfying Assumptions 1 and 2 and with a discount factor identical to that of the agent under consideration.*

Using an evolutionary rational heuristic ensures that the agent will avoid financial ruin and that he will have an impact on market prices even in the limit. The definition requires that this should be true, even if fully rational, fully aware and fully informed agents are present in the market¹⁵. The distinction between ecological rationality and evolutionary rationality is closely related to the trade-off between individual utility and chances of survival discussed above. As awareness becomes more refined, it may be ecologically rational, under appropriate conditions, to trade in a wider range of assets. However, in the presence of wrong beliefs, a more restrictive heuristic may be evolutionarily rational

The results obtained in the previous section can now be interpreted in terms of a heuristic the agent adopts in view of his limited understanding of the world. Agents can adopt robust heuristic procedures, such as trading on the basis of aversion to unfavorable surprises, rather than attempting to optimize on the basis of beliefs that may be incorrect. The avoidance of unfavorable surprises incorporated in Assumption 4 effectively implies the use of the heuristic ‘do not trade assets you don’t understand’.

¹⁵ As in the rest of the paper, we control for the discount factors across agents, so as to make the definition meaningful. In particular, as is well-known from Blume and Easley (2006), an agent with wrong beliefs could survive in the presence of an agent with correct beliefs and a lower discount factor.

An agent using such a heuristic need not formulate potentially wrong beliefs over contingencies he finds hard to conceive of. Notably, Gigerenzer (2007) finds that subjects in experiments not only use this heuristic, but also that this allows them to outperform other subjects, as well as professional investors. The results of the previous section confirm that the use of such a heuristic might indeed lead to the agent obtaining a higher expected utility with respect to the truth than a wrong belief assignment. As we showed above, imposing a restriction on trades is beneficial, when agents' beliefs over finer contingencies are wrong. A second type of heuristic which can be identified from the results in Section 7 is that if an agent considers that his beliefs might be wrong, he should avoid trading in assets that to him appear to be mispriced. This is particularly easy to see for the case of ln-preferences discussed in Example 7.2. Indeed, when all agents have ln-preferences and if all agents have correct beliefs, the price kernel exactly coincides with the true probability distribution $p^*(s) = \pi(s)$. In contrast, if agent i 's beliefs over some subset $w^i \subset S$ are wrong, whereas the other agents have correct beliefs, agent i will consider that (some of) the Arrow securities paying on $s \in w^i$ are mispriced, $p^*(s) \neq \pi^i(s)$. A fully rational expected utility maximizer would continue to trade in his wrong beliefs to eliminate such mispricing. In contrast, the heuristic: '*do not trade in assets that appear to be mispriced*' would coincide with i restricting his trades to the partition on which his beliefs coincide with the price kernel. Again, such a heuristic will increase the expected utility of the agent with respect to the truth.

A similar argument can be extended to the case of an economy in which all agents in the economy are identical, both with respect to preferences and initial endowments. If all agents have correct beliefs, prices would reflect those: $\frac{p^*(s)}{p^*(s')} = \frac{\pi(s)u'(\bar{e}^i(s))}{\pi(s')u'(\bar{e}^i(s'))}$ would hold for any s and s' , so that the initial endowment is supported as the market equilibrium. Suppose, in contrast as before that all agents except for i have correct beliefs, whereas i 's beliefs are wrong on a subset $w^i \subset S$. In this case, the mere fact that i wishes to trade at the market prices p^* is an indication of the fact that his beliefs are wrong. Once again, the results in Section 7 show that avoiding trade in assets that are mispriced can increase individual expected utility with respect to the true probabilities.

Such a heuristic would be consistent with the efficient market hypothesis: from the point of view of a small individual investor, prices already reveal all available information. Trading on one's own beliefs thus cannot make the agent better-off. Odean (1999), as well as Barber and Odean (2001, 2002), however, find that investors trade too much, engaging in speculation which is not justified by new objective information, and which leads to losses (even without accounting for transaction costs). This suggests that a heuristic which avoids trading in assets subjectively considered to be mispriced can increase the earnings of investors in real markets.

An extreme instance of such heuristics is to completely abstain from trading in risky assets. The heuristic '*do not invest in the share market – there are always people smarter than you*' trivially ensures that the agent's beliefs on the relevant partition are correct. However, in this extreme case, the loss from foregone trading opportunity might exceed the benefits from adopting correct beliefs.

We next define formally the three types of heuristics discussed above:

Definition 7.2 (i) Agent i uses the heuristic ‘do not trade assets you don’t understand’ if his trades are measurable with respect to the partition of the state-space $(\Omega^i, \mathcal{F}^i)$ generated by the finest partition W^i of S on which he has correct beliefs. We will say that the agent ‘trades on assets he does not understand’ if his beliefs on the coarsest partition W^i generating the partition $(\Omega^i, \mathcal{F}^i)$ with respect to to which his trades are measurable, are wrong.

(ii) Agent i uses the heuristic ‘do not invest in the share market – there are always people smarter than you’ if he trades only in bonds, that is, if his trades are measurable with respect to the trivial partition of the state space $(\Omega^i, \mathcal{F}^i)$, generated by $W^i = \{\{S\}\}$.

(iii) Agent i uses the heuristic ‘do not trade in assets that appear to be mispriced’ if his asset holdings satisfy $\tilde{c}_i(\sigma_t) - \tilde{c}_i(\sigma'_t) = \tilde{e}_i(\sigma_t) - \tilde{e}_i(\sigma'_t)$ whenever $\frac{p^*(\sigma_t)}{p^*(\sigma'_t)} \neq \frac{\pi^i(\sigma_t)u_i'(\tilde{e}^i(\sigma_t))}{\pi^i(\sigma'_t)u_i'(\tilde{e}^i(\sigma'_t))}$.

The following corollaries recast the results in the previous sections in terms of heuristics:

Corollary 7.3 Under the conditions stated in Remark 5.1 (i), (iii) or (iv), as well as under the conditions of Remark 5.2, the heuristic ‘do not trade assets you don’t understand’ is evolutionary rational. In contrast, under the conditions of Remark 5.1 (ii), trading on assets an agent does not understand is not evolutionary rational.

Corollary 7.4 Under the conditions stated in Remark 5.1 (i), (iii) or (iv), the heuristic ‘do not invest in the share market – there are always people smarter than you’ is evolutionary rational.

Corollary 7.5 Suppose that the agents in the economy have identical preferences and identical initial endowments. If all agents except for agent i are fully aware and have correct beliefs, then the heuristic ‘do not trade in assets that appear to be mispriced’ is evolutionary rational for agent i .

The results above demonstrate that the heuristic implicitly embedded in Assumption 4: ‘do not trade assets you don’t understand’ is indeed evolutionary rational in the sense of the definition. So is the heuristic ‘do not trade in assets that appear to be mispriced’ provided that all agents have identical preferences, identical endowments and, other than the agent under consideration, correct beliefs.

Note that the "evolutionary rationality" can be thought of as a satisficing criterion. Differently from the "ecological rationality", it does not compare utility from consumption across heuristics, but merely requires that consumption not converge to 0. As a result, even the heuristic ‘do not invest in the share market – there are always people smarter than you’ which might result in rather low payoffs as compared to a less restrictive heuristic, is evolutionary rational. It is also different from economic rationality. For example, a fully aware risk-neutral expected utility maximizer with correct beliefs will vanish from the financial market with strictly positive probability in finite time, as long as there is aggregate risk¹⁶.

¹⁶ These points raise the question of whether rationality should be assessed in terms of the maximization of expected utility based on an inevitably incomplete model of the world or on the criterion of survival, as proposed in

At the same time, evolutionary rationality is not a void requirement. As we showed in Section 6, a decision criterion, which relies on imputed beliefs, such as Assumption 5, is not evolutionary rational (except in the special case of no aggregate risk), even when the weights used coincide with the truth. The same applies to the naive application of expected utility theory to contingencies over which one might entertain wrong beliefs.

To conclude, aversion to unfavorable surprises as captured by Assumption 4, gives rise to a heuristic, which can be not only ecologically rational in that it outperforms on average naive optimization, but also evolutionary rational in that it avoids ruin and ensures survival in the presence of better informed agents.

8 Conclusion

The standard analysis of financial markets begins with the ideal of a complete efficient market, in which rational economic agents, endowed with state-contingent consumption paths, trade in assets which span the relevant state space. This is obviously an idealization - The use of idealized models of financial markets has been justified by the idea that a form of ‘natural selection’ will apply, ensuring that only fully rational traders will survive. The work of Blume and Easley shows that this argument is valid in the case where agents have access to the full set of economically relevant contingencies, but differ in the accuracy of their probability judgements. In this paper, we have shown that these results on survival in financial markets do not extend to the case of bounded awareness. Not only can boundedly aware agents survive, but an increase in awareness may reduce agents’ chances of survival. The current paper emphasizes the fact that the perception of and attitude towards unforeseen outcomes have a significant impact on survival results. In particular, aversion to unfavorable surprises and the so-implied measurability of trades seems to be key to survival of boundedly aware agents. Alternative attitudes to surprises such as preferences for favorable surprises or the preferences introduced in Auster et al. (2000) are less favorable to survival.

This analysis is part of a broader research program seeking to model the operation of financial markets under the realistic assumption that no agent can be fully aware of the possible contingencies that affect returns to financial investments. This means in general that agents will hold only a subset of the assets traded in financial markets.

The analysis here builds on the results of Guerdjikova and Quiggin (2019) who consider agents who are exogenously constrained to invest in only a limited portfolio of assets, whereas, in the current paper, financial constraints are derived endogenously from the bounded awareness of the agents. As we have shown, the shift from exogenous to endogenous constraints has important

the notion of evolutionary rationality developed here. Blume and Easley (1992, 1993, 2006) do not consider survival to be a normative criterion, in as far as it reflects differences in preferences. In contrast, Alchian (1950, p. 218), in the context of entrepreneurs, writes: "The pursuit of profits, and not some hypothetical undefinable perfect situation, is the relevant objective whose fulfillment is rewarded with survival". In as far as lack of survival is not attributable to "wrong" preferences, but to the fact that agents have and actively trade on wrong beliefs (over contingencies they do not understand), the concept of evolutionary rationality might be a useful criterion for the evaluation of heuristics.

implication for issues including the measurability of portfolios with respect to awareness.

The next step in the program will focus on probability judgements. In this paper, as is standard in the literature on survival in financial markets, beginning with Blume and Easley, differences in probability judgements are taken as exogenous and unexplained. The concept of differentially restricted awareness suggests a way to endogenize these differences, deriving them from restricted awareness. To the extent that agents differ in their awareness of the states in a given event (that is, the ways the event can happen), they will differ in their subjective probabilities. Consideration of bounded awareness suggests that financial markets are less informationally inefficient, but potentially more forgiving for investors with inaccurate beliefs. In the models presented above, investors may survive because their limited awareness prevents them from undertaking complex, but mistaken, investment strategies. This in turn implies that sophisticated investors, aware of their own fallibility, may choose to constrain their investment strategies in order to increase their chances of survival.

However, these relatively stable outcomes are not guaranteed. In periods of rising asset prices, widely-publicized financial innovations may lead investors to become aware of a finer set of possible state-contingent payoffs, without necessarily forming accurate beliefs about the relevant probabilities. Such a development may pave the way for a financial bubble and bust. This possibility will be examined in future work.

It is hoped that a model of financial markets in which agents have bounded but differential awareness will be consistent with observed phenomena including the equity premium and risk-free rate puzzles, short-termism and financial market bubbles and busts.

9 Appendix

9.1 Relevant Unawareness

Definition 9.1 *The unawareness of agent i , given by the partition Ω^i , is irrelevant in the limit if for any $\omega^i \in \Omega^i$ and any $\sigma, \sigma' \in \omega^i$, $\lim_{t \rightarrow \infty} [\tilde{e}(\sigma_t) - \tilde{e}(\sigma'_t)] = 0$. The unawareness of agent i , given by the partition Ω^i , is relevant in the limit if for some $w^i \in W^i$, s and $s' \in w^i$, there is an $\epsilon > 0$ such that for any $\sigma, \sigma' \in \omega^i$,*

$$\lim_{t \rightarrow \infty} \sup [\tilde{e}(\sigma_t, s) - \tilde{e}(\sigma'_t, s')] > \epsilon. \quad (15)$$

The unawareness of agent i is considered irrelevant if, in the limit, the total financial endowment of the economy is measurable with respect to agent i 's awareness partition. Such an agent is effectively aware of and thus can trade on the total financial endowment process in the limit. In contrast, agent i 's unawareness is relevant even in the limit, if there are at least two states that i cannot distinguish and in which the total financial endowment of the economy remains distinct. Note that if i 's unawareness is irrelevant in the limit, then so are those of any agent j who is more aware and has a partition Ω^j finer than Ω^i . Similarly if i 's awareness is relevant in the limit, then so is that of a less aware agent j with a partition Ω^j coarser than Ω^i .

Consider agent j and for any $\omega^j \in \Omega^j$ with $\omega^i \subseteq \omega^j$, define the set $\hat{\Omega}_t^i(\omega_t^j) = \{\omega_t^i \in \Omega_t^i \mid \omega_t^i \subseteq$

ω_t^j s.t. $\min_{\sigma_t \in \omega_t^j} \tilde{e}(\sigma_t) = \min_{\sigma_t \in \omega_t^i} \tilde{e}(\sigma_t)$, the set of ω_t^i on which the financial endowment of the economy obtains its minimum with respect to the set ω_t^j . Let $\check{\Omega}_t^i(\omega_t^j) = \{\omega_t^i \subseteq \omega_t^j\} \setminus \hat{\Omega}_t^i(\omega_t^j)$.

Definition 9.2 *Let the awareness partition of agent i , Ω^i be finer than that of agent j , Ω^j . The unawareness of agent j given by the partition Ω^j , is irrelevant in the limit with respect to that of agent i given by partition Ω^i if for any $\omega^i \in \Omega^i$ and $\omega^j \in \Omega^j$ s.t. $\omega^i \subseteq \omega^j$, $\lim_{t \rightarrow \infty} \check{\Omega}_t^i(\omega_t^j) = \emptyset$. The unawareness of agent j is relevant in the limit with respect to that of agent i if there is an $\epsilon > 0$, $w^i \in W^i$ and $w^j \in W^j$, $w^i \subseteq w^j$ such that for any $\omega^j \in \Omega^j$ and every $\omega^i \subseteq \omega^j$, $\omega^i \in \Omega^i$, (i) $\min_{(\sigma_{t^k}, s) \in (\omega_{t^k}^i, w^i)} \tilde{e}(\sigma_{t^k}, s) - \min_{(\sigma_{t^k}, s) \in (\omega_{t^k}^j, w^j)} \tilde{e}(\sigma_{t^k}, s) > \epsilon$ occurs on an infinite set of periods $(t^k)_k$ such that (ii) $\min_{\{\sigma_{t^k+1} \in \omega_{t^k+1}^i | \omega_{t^k+1}^i \in \check{\Omega}_{t^k+1}^i(\omega_{t^k}^j, w^j)\}} \tilde{e}(\sigma_{t^k+1}) - \min_{(\sigma_{t^k}, s) \in (\omega_{t^k}^j, w^j)} \tilde{e}(\sigma_{t^k}, s) > \epsilon$ for all t^k .*

To understand the definition, note that, in general, the financial endowment of the economy is not measurable with respect to Ω^i or Ω^j . The maximum financial wealth that j is aware of at ω_t^j , given the financial endowment of the economy, is $\min_{\sigma_t \in \omega_t^j} \tilde{e}(\sigma_t)$, whereas the maximum financial wealth that i is aware of at $\omega_t^i \subseteq \omega_t^j$ is $\min_{\sigma_t \in \omega_t^i} \tilde{e}(\sigma_t)$. Furthermore, if $\check{\Omega}_t^i(\omega_t^j) = \emptyset$, then these two values coincide for all $\omega_t^i \subseteq \omega_t^j$: even though j 's partition is coarser, his awareness about the maximal possible financial wealth of the economy is the same as that of i on ω_t^j . If this property obtains in the limit, we say that j 's unawareness is irrelevant in the limit with respect to that of i . If, in contrast, $\check{\Omega}_t^i(\omega_t^j) \neq \emptyset$, then i is aware that he can obtain a strictly higher financial wealth on ω_t^i than j on ω_t^j , that is, j 's unawareness is "relevant" with respect to that of i . The condition for j 's unawareness to be relevant with respect to those of i in the limit requires that (i) on every path $\omega^i \subseteq \omega^j$, on which w^i occurs infinitely often (i.o.) the maximal financial wealth of which i is aware exceeds that of which j is aware by ϵ i.o. and (ii) on every path ω^j , on which w^j occurs i.o. the minimal non-zero difference in maximal financial wealth of which i and j respectively are aware on ω^j exceeds ϵ i.o..

9.2 Proofs

Proof of Proposition 4.2:

Since $\tilde{e}(\sigma_t') \neq \tilde{e}(\sigma_t'')$, measurability of initial financial endowments with respect to the individual awareness partitions implies that there is a j such that $\omega_t', \omega_t'' \in \Omega_t^j$, that is, j can distinguish σ_t' and σ_t'' .

(i) follows directly from the requirement that i 's financial equilibrium consumption \tilde{c}^i is measurable with respect to Ω^i .

To show (ii), suppose to the contrary that the financial market equilibrium provided full insurance against idiosyncratic risk. We then have: $\tilde{c}^j(\sigma_t) = \tilde{c}^j(\sigma_t')$ for all $j \in I$. The measurability constraint of i further implies that $\tilde{c}^i(\sigma_t') = \tilde{c}^i(\sigma_t'')$. Furthermore, by the equilibrium f.o.c., and since agents' beliefs are correct, we should have for any j such that $\omega_t', \omega_t'' \in \Omega_t^j$,

$$\frac{\pi^i(\omega_t^i) u_i'(\tilde{c}^i(\sigma_t''))}{\pi^i(\omega_t) u_i'(\tilde{c}^i(\sigma_t))} = \frac{\pi(\omega_t^i)}{\pi(\omega_t)} = \frac{\pi(\omega_t') + \pi(\omega_t'')}{\pi(\omega_t)} = \frac{\pi^j(\omega_t') u_j'(\tilde{c}^j(\sigma_t')) + \pi^j(\omega_t'') u_j'(\tilde{c}^j(\sigma_t''))}{\pi^j(\omega_t) u_j'(\tilde{c}^j(\sigma_t))}.$$

Since however, $\tilde{e}(\sigma'_t) \neq \tilde{e}(\sigma''_t)$, there must be a $k \neq i$ such that $\omega'_t, \omega''_t \in \Omega_t^k$ and $\tilde{c}^k(\sigma'_t) \neq \tilde{c}^k(\sigma''_t)$ and since, by assumption $\tilde{c}^k(\sigma_t) = \tilde{c}^k(\sigma'_t)$, we obtain:

$$\frac{\pi^k(\omega'_t) u'_k(\tilde{c}^k(\sigma'_t)) + \pi^k(\omega''_t) u'_k(\tilde{c}^k(\sigma''_t))}{\pi^k(\omega_t) u'_k(\tilde{c}^k(\sigma_t))} \neq \frac{\pi^k(\omega'_t) + \pi^k(\omega''_t)}{\pi^k(\omega_t)} = \frac{\pi(\omega'_t) + \pi(\omega''_t)}{\pi(\omega_t)}$$

in contradiction to the equilibrium f.o.c. above. Thus, full insurance against idiosyncratic risk cannot obtain in a financial market equilibrium.

(iii) When all agents have correct beliefs, an unbiased price $\frac{p(\omega_t)}{p(\omega_t^i)}$ will satisfy

$$\frac{p(\omega_t)}{p(\omega_t^i)} = \frac{\pi(\omega_t)}{\pi(\omega_t^i)}.$$

However, at such price, all agents would choose to be fully insured against idiosyncratic risk, in contradiction to the result shown in (ii). \square

Proof of Proposition 4.3:

Suppose that in a financial market equilibrium of the economy with differential awareness, the total consumption of the agents in \mathcal{J} in σ'_t is given by $\sum_{j \in \mathcal{J}} \tilde{c}^j(\sigma'_t)$. By measurability of the financial market allocation with respect to the agents in \mathcal{J} , we have

$$\sum_{j \in \mathcal{J}} \tilde{c}^j(\sigma'_t) = \sum_{j \in \mathcal{J}} \tilde{c}^j(\sigma''_t).$$

By market clearing, we have

$$\sum_{i \in \mathcal{I}} [\tilde{e}^i(\sigma'_t) - \tilde{c}^i(\sigma'_t)] = \sum_{i \in \mathcal{I}} [\tilde{e}^i(\sigma''_t) - \tilde{c}^i(\sigma''_t)]$$

and by measurability of initial endowments and consumption of agents in \mathcal{I} ,

$$\sum_{i \in \mathcal{I}} [\tilde{e}^i(\sigma_t) - \tilde{c}^i(\sigma_t)] = \sum_{i \in \mathcal{I}} [\tilde{e}^i(\sigma''_t) - \tilde{c}^i(\sigma''_t)] = \sum_{i \in \mathcal{I}} [\tilde{e}^i(\sigma'_t) - \tilde{c}^i(\sigma'_t)].$$

Therefore, market clearing implies

$$\sum_{j \in \mathcal{J}} [\tilde{e}^j(\sigma_t) - \tilde{c}^j(\sigma_t)] = \sum_{j \in \mathcal{J}} [\tilde{e}^j(\sigma'_t) - \tilde{c}^j(\sigma'_t)]. \square$$

Proof of Proposition 4.4:

(i) Since $\sigma_t, \sigma'_t \in \omega_t^i$, we have $\tilde{c}^i(\sigma_t) = \tilde{c}^i(\sigma'_t)$ in any financial market equilibrium of the economy with differential awareness. For such an allocation to be an equilibrium of the full awareness economy satisfying assumptions 1 and 3 with an endowment process e and the corresponding \tilde{e} and with homogeneous beliefs $\tilde{\pi}$, we need

$$\frac{\tilde{\pi}(\sigma_t) u'_i(\tilde{c}_i(\sigma_t))}{\tilde{\pi}(\sigma'_t) u'_i(\tilde{c}_i(\sigma'_t))} = \frac{\tilde{\pi}(\sigma_t)}{\tilde{\pi}(\sigma'_t)} = \frac{\tilde{\pi}(\sigma_t) u'_j(\tilde{c}_j(\sigma_t))}{\tilde{\pi}(\sigma'_t) u'_j(\tilde{c}_j(\sigma'_t))} \quad (16)$$

for any $j \neq i$. This implies $\tilde{c}_j(\sigma_t) = \tilde{c}_j(\sigma'_t)$ for all $j \neq i$. But since $\tilde{e}(\sigma_t) > \tilde{e}(\sigma'_t)$, this cannot be an equilibrium of the full awareness economy, thus proving the claim.

(ii) When beliefs $\tilde{\pi}^i$ and $\tilde{\pi}^j$ are allowed to be heterogeneous, condition (16) becomes:

$$\frac{\tilde{\pi}^i(\sigma_t) u'_i(\tilde{c}_i(\sigma_t))}{\tilde{\pi}^i(\sigma'_t) u'_i(\tilde{c}_i(\sigma'_t))} = \frac{\tilde{\pi}^i(\sigma_t)}{\tilde{\pi}^i(\sigma'_t)} = \frac{\tilde{\pi}^j(\sigma_t) u'_j(\tilde{c}_j(\sigma_t))}{\tilde{\pi}^j(\sigma'_t) u'_j(\tilde{c}_j(\sigma'_t))}$$

Together with $\tilde{e}(\sigma_t) > \tilde{e}(\sigma'_t)$, this implies that there is a j such that $\tilde{c}^j(\sigma_t) > \tilde{c}^j(\sigma'_t)$ and thus, by Assumption 1, $u'_j(\tilde{c}_j(\sigma_t)) < u'_j(\tilde{c}_j(\sigma'_t))$, or

$$\frac{\tilde{\pi}^i(\sigma_t)}{\tilde{\pi}^i(\sigma'_t)} < \frac{\tilde{\pi}^j(\sigma_t)}{\tilde{\pi}^j(\sigma'_t)}$$

as claimed.

(iii) The existence of an economy with i.i.d. beliefs for all agents implies just as in part (ii) that there exist two agents j and k (not necessarily distinct, but distinct from i) such that $\tilde{c}_j(\sigma_t, s) > \tilde{c}_j(\sigma_t, s')$ and $\tilde{c}_k(\sigma'_t, s) < \tilde{c}_k(\sigma'_t, s')$. We also have:

$$\begin{aligned} \frac{\tilde{\pi}^i(s) u'_i(\tilde{c}_i(\sigma_t, s))}{\tilde{\pi}^i(s') u'_i(\tilde{c}_i(\sigma_t, s'))} &= \frac{\tilde{\pi}^i(s)}{\tilde{\pi}^i(s')} = \frac{\tilde{\pi}^j(s) u'_j(\tilde{c}_j(\sigma_t, s))}{\tilde{\pi}^j(s') u'_j(\tilde{c}_j(\sigma_t, s'))} \\ \frac{\tilde{\pi}^i(s) u'_i(\tilde{c}_i(\sigma'_t, s))}{\tilde{\pi}^i(s') u'_i(\tilde{c}_i(\sigma'_t, s'))} &= \frac{\tilde{\pi}^i(s)}{\tilde{\pi}^i(s')} = \frac{\tilde{\pi}^k(s) u'_k(\tilde{c}_k(\sigma'_t, s))}{\tilde{\pi}^k(s') u'_k(\tilde{c}_k(\sigma'_t, s'))} \end{aligned}$$

If $j = k$, then we obtain

$$\frac{\tilde{\pi}^i(s)}{\tilde{\pi}^i(s')} < \frac{\tilde{\pi}^j(s)}{\tilde{\pi}^j(s')} \text{ and } \frac{\tilde{\pi}^i(s)}{\tilde{\pi}^i(s')} > \frac{\tilde{\pi}^j(s)}{\tilde{\pi}^j(s')},$$

a contradiction.

Suppose next that $j \neq k$ and more specifically that

$$\begin{aligned} \tilde{c}_k(\sigma_t, s) &\leq \tilde{c}_k(\sigma_t, s') \\ \tilde{c}_j(\sigma'_t, s) &\geq \tilde{c}_j(\sigma'_t, s') \end{aligned}$$

with at least one of the inequalities being strict (otherwise, we could restate the argument above for either j or k). Then, we obtain:

$$\frac{\tilde{\pi}^k(s)}{\tilde{\pi}^k(s')} \leq \frac{\tilde{\pi}^j(s)}{\tilde{\pi}^j(s')}$$

and

$$\frac{\tilde{\pi}^k(s)}{\tilde{\pi}^k(s')} \geq \frac{\tilde{\pi}^j(s)}{\tilde{\pi}^j(s')},$$

with at least one of the inequalities being strict, a contradiction.

Proof of Proposition 6.1:

The optimization problem of agent i is given by:

$$\begin{aligned} \max_{a^i} U_0^i(a^i) &= \left\{ \sum_t \beta^t \sum_{\omega_t^i \in \Omega_t^i} \pi^i(\omega_t^i) u_i \left((1-\alpha) u_i(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t)) \right) \right. \\ &\quad \left. + \alpha u_i(\max_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \max_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t)) \right\} \\ &\quad | \sum_{\sigma_t} p(\sigma_t) a^i(\sigma_t) \leq 0, \min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \geq 0 \forall \omega_t^i \in \Omega^i \}. \end{aligned}$$

Denote by $\bar{a}(\omega_t^i) = \max_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t)$, $\underline{a}(\omega_t^i) = \min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t)$ and similarly for $\bar{e}^i(\omega_t^i)$ and $\underline{e}^i(\omega_t^i)$. Let $\underline{p}(\omega_t^i) = \min_{\tilde{\sigma}_t \in \omega_t^i} p(\tilde{\sigma}_t)$. Clearly, $\bar{a}(\omega_t^i) \geq \underline{a}(\omega_t^i)$. The optimization problem then becomes:

$$\begin{aligned} & \max_{\underline{a}^i(\omega_t^i), \bar{a}^i(\omega_t^i)} \left\{ \sum_t \beta^t \sum_{\omega_t^i \in \Omega_t^i} \pi^i(\omega_t^i) (1 - \alpha) u_i(\underline{a}^i(\omega_t^i) + \underline{e}^i(\omega_t^i)) + \alpha u_i(\bar{a}(\omega_t^i) + \bar{e}^i(\omega_t^i)) \right. \\ & \left. \mid \sum_t \sum_{\omega_t^i \in \Omega_t^i} \left[p(\omega_t^i) \underline{a}^i(\omega_t^i) + \min_{\tilde{\sigma}_t \in \omega_t^i} p(\tilde{\sigma}_t) [\bar{a}^i(\omega_t^i) - \underline{a}^i(\omega_t^i)] \right] \leq 0, \underline{a}^i(\omega_t^i) + \underline{e}^i(\omega_t^i) \geq 0 \forall \omega_t^i \in \Omega_t^i \right\}. \end{aligned}$$

Furthermore, if $\bar{a}(\omega_t^i) > \underline{a}(\omega_t^i)$, then the agent chooses

$$\begin{aligned} a(\sigma_t) &= \underline{a}^i(\omega_t^i) \text{ for all } \sigma_t \neq \underline{\sigma}_t \\ a(\underline{\sigma}_t) &= \bar{a}(\omega_t^i) - \underline{a}^i(\omega_t^i) \end{aligned}$$

where $\underline{\sigma}_t = \arg \min_{\tilde{\sigma}_t \in \omega_t^i} p(\tilde{\sigma}_t)$. Thus, the first-order conditions of i are given by:

$$\frac{\pi^i(\omega_t^i) u'_i(\underline{a}^i(\omega_t^i) + \underline{e}^i(\omega_t^i))}{\pi^i(\omega_t'^i) u'_i(\underline{a}^i(\omega_t'^i) + \underline{e}^i(\omega_t'^i))} = \frac{p(\omega_t^i) - \underline{p}(\omega_t^i)}{p(\omega_t'^i) - \underline{p}(\omega_t'^i)}$$

and, whenever $p(\omega_t^i) > \underline{p}(\omega_t^i)$,

$$\frac{(1 - \alpha) u'_i(\underline{a}^i(\omega_t^i) + \underline{e}^i(\omega_t^i))}{\alpha u'_i(\bar{a}(\omega_t^i) + \bar{e}^i(\omega_t^i))} = \frac{p(\omega_t^i)}{\underline{p}(\omega_t^i)}$$

Now note that since prices are integrable, in equilibrium, we have:

$$\sum_{\omega_t^i} p(\omega_t^i) < \infty$$

which implies that there is an L such that

$$\sum_{\sigma_t \in \omega_t^i} p(\sigma_t) = p(\omega_t^i) < L < \infty.$$

Since for t -large, $|\omega_t^i| = \prod_{\tau=1}^t |w_\tau^i(\omega^i)|$ which grows exponentially on any path ω , on which an w with at least 2 elements is observed infinitely often, which happens π -a.s., we have that π -a.s. for any ω , and any N , there is a \bar{t} such that

$$|\omega_t^i| > N \text{ for } t \geq \bar{t}$$

and thus, for any $t > \bar{t}$,

$$\underline{p}(\omega_t^i) < \frac{\sum_{\sigma_t \in \omega_t^i} p(\sigma_t)}{N} = \frac{p(\omega_t^i)}{N} < \frac{L}{N}.$$

It follows that

$$\lim_{t \rightarrow \infty} \frac{p(\omega_t^i)}{p(\omega_t^i)} = 0$$

and thus, the even slightly optimistic, $\alpha > 0$, agent will set

$$\lim_{t \rightarrow \infty} \frac{\alpha u'_i(\bar{a}(\omega_t^i) + \bar{e}^i(\omega_t^i))}{(1 - \alpha) u'_i(\underline{a}^i(\omega_t^i) + \underline{e}^i(\omega_t^i))} = 0$$

Since consumption is uniformly bounded above, $\bar{a}(\omega_t^i) + \bar{e}(\omega_t^i) \leq \bar{m}$, this requires

$$\lim_{t \rightarrow \infty} [\underline{a}^i(\omega_t^i) + \underline{e}^i(\omega_t^i)] = 0.$$

Thus, if $\alpha > 0$, in the long run, implies (almost) all of the agent's holdings will be invested into the state with the minimal price within the coarse contingency $\omega_t^i, \underline{\sigma}_t$, while his minimal consumption on ω_t^i will converge to 0. Thus, for a given ω^i , consumption will not converge to 0 only on the path $\underline{\sigma} = (\underline{\sigma}_t)_t \in \omega^i$. Since $\pi(\underline{\sigma} | \omega^i) = 0$, agent i a.s. vanishes for each ω^i and thus, a.s. vanishes on Σ . \square

Proof of Proposition 6.2:

Consider an element of i 's partition ω_t^i and the set of corresponding Arrow securities, which pay on an element of the partition $\sigma_t \in \omega_t^i$. Agent i associates with each such security a payoff which is certain conditional on ω_t^i and given by $\pi^i(\sigma_t | \omega_t^i)$. Thus, if $p(\sigma_t)$ is the price of such an asset and if for two assets payoff on σ_t and on $\sigma'_t \in \omega_t^i$, respectively, $\frac{p(\sigma_t)}{p(\sigma'_t)} \neq \frac{\pi^i(\sigma_t | \omega_t^i)}{\pi^i(\sigma'_t | \omega_t^i)}$, i

perceives arbitrage. Suppose, w.l.o.g., that $\frac{p(\sigma_t)}{p(\sigma'_t)} > \frac{\pi^i(\sigma_t | \omega_t^i)}{\pi^i(\sigma'_t | \omega_t^i)}$. In this case, i will find it optimal to hold an extreme portfolio position, in which he will short the "more expensive" asset paying in σ_t to buy the less expensive one for state σ'_t . The condition that i 's perceived consumption be non-negative prevents him from buying a portfolio specifying strictly negative consumption in any given state implies that $c^i(\sigma_t | \omega_t^i) = 0$ in i 's optimum. Thus, i will vanish with strictly positive probability at (a finite) time period t . \square

Proof of Proposition 6.3:

Since $\frac{p(\sigma_t, s)}{p(\sigma_t, s')} = \frac{\pi(s|w)}{\pi(s'|w)}$, and since all agents other than i are fully aware and have correct beliefs, it follows that (i) all agents other than i are fully insured across s and s' (otherwise their MRS will differ from the price ratio) and (ii) agent i is the only one exposed to the risk of the "bad" state s' at every σ_t :

$$c^i(\sigma_t, s) = c^i(\sigma_t, s') + e(\sigma_t, s) - e(\sigma_t, s') > c^i(\sigma_t, s') + \epsilon$$

It follows that for any σ_t ,

$$\frac{u'_i(c^i(\sigma_t, s))}{u'_i(c^i(\sigma_t, s'))} = \frac{u'_i(c^i(\sigma_t, s') + e(\sigma_t, s) - e(\sigma_t, s'))}{u'_i(c^i(\sigma_t, s'))} < \frac{u'_i(c^i(\sigma_t, s') + \epsilon)}{u'_i(c^i(\sigma_t, s'))} < \frac{u'_i(m')}{u'_i(m' - \epsilon)} \quad (17)$$

Thus, i behaves as if he had "effective" beliefs given by $\tilde{\pi}^i(\sigma_t, \cdot)$ satisfying

$$\frac{\tilde{\pi}^i(\sigma_t, s) u'_i(c^i(\sigma_t, s') + \epsilon)}{\tilde{\pi}^i(\sigma_t, s') u'_i(c^i(\sigma_t, s'))} = \frac{p(\sigma_t, s)}{p(\sigma_t, s')} = \frac{\pi(s)}{\pi(s')}.$$

Inequality (17) thus implies that i 's effective beliefs satisfy:

$$\frac{\tilde{\pi}^i(\sigma_t, s)}{\tilde{\pi}^i(\sigma_t, s')} > \frac{\pi(s)}{\pi(s')} \frac{u'_i(m' - \epsilon)}{u'_i(m')} > \frac{\pi(s)}{\pi(s')}$$

for every σ_t , that is, that their K-L distance from the truth is strictly positive and bounded away from 0. As we know from Blume and Easley (2006), i thus vanishes a.s. \square

Proof of Proposition 6.4:

Since agent j assigns a strictly positive probability to any node in Σ and since u_j satisfies the Inada conditions, in any equilibrium of the economy, prices $p(\sigma_t)$ are strictly positive and finite at all $\sigma_t \in \Sigma$. This in turn implies that for any finite ω_t^i , the price of increasing $v_{\min}^i(\mathbf{c}_{\Omega^i}^i(a^i, e^i, \omega_t^i))$, (given by $p(\omega_t^i)$) as well as the price of increasing $v_a^i(\mathbf{c}_{\Omega^i, a}^i(a^i, e^i, \omega_t^i), u^i)$ (given, as explained in the proof of Proposition 6.2, by $\min_{\sigma_t \in \omega_t^i} \frac{p(\sigma_t)}{\pi^i(\sigma_t | \omega_t^i)}$) are both strictly positive and finite. Thus, agent i will optimally set

$$v_{\min}^i(\mathbf{c}_{\Omega^i}^i(a^i, e^i, \omega_t^i)) = \frac{1}{k} v_a^i(\mathbf{c}_{\Omega^i, a}^i(a^i, e^i, \omega_t^i), u^i) \quad (18)$$

for each finite ω_t^i . Recall that

$$v_{\min}^i(\mathbf{c}_{\Omega^i}^i(a^i, e^i, \omega_t^i)) = u_i \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right).$$

For a given ω_t^i , let

$$\delta_a^i(\omega_t^i) = \sum_{\tilde{\sigma}_t \in \omega_t^i} \pi^i(\sigma_t | \omega_t^i) [a^i(\tilde{\sigma}_t) + e^i(\tilde{\sigma}_t)] - \min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) - \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t)$$

be the difference between the perceived average and the perceived minimal consumption of i . It follows that:

$$\frac{1}{k} v_a^i(\mathbf{c}_{\Omega^i, a}^i(a^i, e^i, \omega_t^i), u^i) = \frac{1}{k} u_i \left(\delta_a^i(\omega_t^i) + \min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right).$$

Furthermore, as already noted, i will only choose to hold assets in excess of $\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t)$ in those states σ_t for which the corresponding price minimizes

$$p(\sigma_t) = \min_{\tilde{\sigma}_t \in \omega_t^i} \frac{p(\tilde{\sigma}_t)}{\pi^i(\tilde{\sigma}_t | \omega_t^i)}.$$

It follows that the "price" of $\delta_a^i(\omega_t^i)$ is given by $p_a(\omega_t^i) = \min_{\tilde{\sigma}_t \in \omega_t^i} \frac{p(\tilde{\sigma}_t)}{\pi^i(\tilde{\sigma}_t | \omega_t^i)}$. Since i 's weights coincide with the true probabilities, we have:

$$p_a(\omega_t^i) = \min_{\tilde{\sigma}_t \in \omega_t^i} \frac{p(\tilde{\sigma}_t)}{\pi(\tilde{\sigma}_t | \omega_t^i)}.$$

Next observe that (18) implies that

$$\frac{1}{k} u_i \left(\delta_a^i(\omega_t^i) + \min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right) = u_i \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right),$$

or

$$\delta_a^i(\omega_t^i) = u^{-1} \left(k u_i \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right) \right) - \left[\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right] \quad (19)$$

and thus, at the optimal consumption of i , we have"

$$d \frac{\delta_a^i(\omega_t^i)}{d(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t))} = \frac{ku'_i(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t))}{u'_i(u_i^{-1}(ku_i(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t))))} - 1.$$

It follows that in the optimum of agent i , a marginal increase in minimal consumption (and thus, in $\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t)$) "costs"

$$p(\omega_t^i) + p_a(\omega_t^i) \left[\frac{ku'_i(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t))}{u'_i(u_i^{-1}(ku_i(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t))))} - 1 \right].$$

The intertemporal first-order condition of agent i between σ_0 and a node $\omega_t^i \in \Omega^i$ becomes:

$$\begin{aligned} & \beta \pi^i(\omega_t^i) \frac{u'_i(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t))}{u'_i(a^i(\sigma_0) + e^i(\sigma_0))} \\ &= \frac{p(\omega_t^i) + p_a(\omega_t^i) \left[\frac{ku'_i(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t))}{u'_i(u_i^{-1}(ku_i(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t))))} - 1 \right]}{p(\sigma_0)}. \end{aligned}$$

Using the first-order conditions of agent j , who is fully aware and has correct beliefs $\pi^j = \pi$,

$$\begin{aligned} \beta \pi(\omega_t^i) \frac{\sum_{\tilde{\sigma}_t \in \omega_t^i} \pi(\tilde{\sigma}_t | \omega_t^i) u'_j(a^j(\tilde{\sigma}_t) + e^j(\tilde{\sigma}_t))}{u'_j(a^j(\sigma_0) + e^j(\sigma_0))} &= \frac{p(\omega_t^i)}{p(\sigma_0)} \\ \beta \pi(\omega_t^i) \frac{u'_j(a^j(\sigma_t) + e^j(\sigma_t))}{u'_j(a^j(\sigma_0) + e^j(\sigma_0))} &= \frac{p_a(\omega_t^i)}{p(\sigma_0)} \end{aligned}$$

for any $\sigma_t \in \arg \min_{\tilde{\sigma}_t \in \omega_t^i} \frac{p(\tilde{\sigma}_t)}{\pi^i(\tilde{\sigma}_t | \omega_t^i)}$. Combining the first-order conditions of the two agents, we obtain for $\sigma_t \in \arg \min_{\tilde{\sigma}_t \in \omega_t^i} \frac{p(\tilde{\sigma}_t)}{\pi^i(\tilde{\sigma}_t | \omega_t^i)}$,

$$\begin{aligned} & \frac{u'_i(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t))}{u'_i(a^i(\sigma_0) + e^i(\sigma_0))} \\ &= \frac{\sum_{\tilde{\sigma}_t \in \omega_t^i} \pi(\tilde{\sigma}_t | \omega_t^i) u'_j(a^j(\tilde{\sigma}_t) + e^j(\tilde{\sigma}_t))}{u'_j(a^j(\sigma_0) + e^j(\sigma_0))} + \\ & \quad + \frac{u'_j(a^j(\sigma_t) + e^j(\sigma_t)) \left[\frac{ku'_i(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t))}{u'_i(u_i^{-1}(ku_i(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t))))} - 1 \right]}{u'_j(a^j(\sigma_0) + e^j(\sigma_0))}. \end{aligned} \tag{20}$$

Note that since $u_i(0) = 0$ or $u_i(0) = -\infty$, (depending on whether $\gamma < 1$ or $\gamma \geq 1$), we have that if for some $\omega^i \in \Omega^i$,

$$\lim_{t \rightarrow \infty} \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right) = 0, \tag{21}$$

then either

$$\lim_{t \rightarrow \infty} u_i \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right) = \lim_{t \rightarrow \infty} k u_i \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right) = 0$$

or

$$\lim_{t \rightarrow \infty} u_i \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right) = \lim_{t \rightarrow \infty} k u_i \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right) = -\infty.$$

Thus, by (19), and the strict monotonicity of u_i , we have:

$$\lim_{t \rightarrow \infty} u^{-1} \left(k u_i \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right) \right) = 0.$$

In words, if agent i chooses his optimal minimal perceived consumption such that it converges to 0 on a given path ω^i , his optimal average consumption on ω^i also converges to 0. It follows that if i vanishes on a given path σ iff he vanishes on ω^i such that $\sigma \in \omega^i$ and iff

$$\lim_{t \rightarrow \infty} u'_i \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right) = \infty.$$

Using the CRRA specification, it is easy to see that:

$$u^{-1} \left(k u_i \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right) \right) = k^{\frac{1}{1-\gamma}} \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right)^{1-\gamma}$$

and thus,

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{k u'_i \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right)}{u'_i \left(u_i^{-1} \left(k u_i \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right) \right) \right)} \\ &= \frac{k \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right)^{-\gamma}}{k^{\frac{1}{1-\gamma}} \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right)^{-\gamma}} = k^{\frac{\gamma}{\gamma-1}} \end{aligned}$$

whenever $\gamma \neq 1$. When $\gamma = 1$, we have:

$$e \left(k \ln \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right) \right) = \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right)^k$$

and thus,

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{k u'_i \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right)}{u'_i \left(u_i^{-1} \left(k u_i \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right) \right) \right)} \\ &= \frac{k \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right)^k}{\left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right)} = 0 \end{aligned}$$

since $k > 1$.

Suppose that (in contradiction to the statement of the proposition) that i vanishes on some ω^i and thus (21) holds. By condition (20) it follows that the r.h.s. satisfies for $\sigma_t \in \arg \min_{\tilde{\sigma}_t \in \omega_t^i} \frac{p(\tilde{\sigma}_t)}{\pi^i(\tilde{\sigma}_t | \omega_t^i)}$,

and $\gamma \neq 1$,

$$\lim_{t \rightarrow \infty} \sum_{\tilde{\sigma}_t \in \omega_t^i} \pi(\tilde{\sigma}_t | \omega_t^i) u'_j(a^j(\tilde{\sigma}_t) + e^j(\tilde{\sigma}_t)) + \lim_{t \rightarrow \infty} u'_j(a^j(\sigma_t) + e^j(\sigma_t)) \left(k^{\frac{\gamma}{\gamma-1}} - 1\right) = \infty$$

and for $\gamma = 1$

$$\begin{aligned} & \lim_{t \rightarrow \infty} \sum_{\tilde{\sigma}_t \in \omega_t^i} \pi(\tilde{\sigma}_t | \omega_t^i) u'_j(a^j(\tilde{\sigma}_t) + e^j(\tilde{\sigma}_t)) + \\ & + \lim_{t \rightarrow \infty} u'_j(a^j(\sigma_t) + e^j(\sigma_t)) k \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right)^{k-1} = \infty \end{aligned} \quad (22)$$

Since i vanishes on ω^i and by market clearing, we have for any $\tilde{\sigma}_t \in \omega_t^i$,

$$a^j(\tilde{\sigma}_t) = -a^i(\tilde{\sigma}_t) = -\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t)$$

$$\lim_{t \rightarrow \infty} \left[a^j(\tilde{\sigma}_t) - \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right] = \lim_{t \rightarrow \infty} \left[-\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) - \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right] = 0.$$

Since by Assumption 2*, $e^j(\sigma_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \geq m^*$, for any sufficiently small $\epsilon > 0$, there thus exists a \bar{t} such that for any $t \geq \bar{t}$,

$$\begin{aligned} m' + \epsilon & \geq \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) + \epsilon + e^j(\tilde{\sigma}_t) \geq \\ & \geq a^j(\tilde{\sigma}_t) + e^j(\tilde{\sigma}_t) \geq \\ & \geq \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) - \epsilon + e^j(\tilde{\sigma}_t) \geq m^* - \epsilon. \end{aligned} \quad (23)$$

This in turn implies that for $t \geq \bar{t}$ and $\gamma > 1$,

$$\begin{aligned} & \sum_{\tilde{\sigma}_t \in \omega_t^i} \pi(\tilde{\sigma}_t | \omega_t^i) u'_j(a^j(\tilde{\sigma}_t) + e^j(\tilde{\sigma}_t)) + \lim_{t \rightarrow \infty} u'_j(a^j(\sigma_t) + e^j(\sigma_t)) \left(k^{\frac{\gamma}{\gamma-1}} - 1\right) \\ & \in \left[\sum_{\tilde{\sigma}_t \in \omega_t^i} \pi(\tilde{\sigma}_t | \omega_t^i) u'_j(m' + \epsilon) + u'_j(m' + \epsilon) \left(k^{\frac{\gamma}{\gamma-1}} - 1\right), \right. \\ & \quad \left. \sum_{\tilde{\sigma}_t \in \omega_t^i} \pi(\tilde{\sigma}_t | \omega_t^i) u'_j(m^* - \epsilon) + u'_j(m^* - \epsilon) \left(k^{\frac{\gamma}{\gamma-1}} - 1\right) \right] \\ & = \left[k^{\frac{\gamma}{\gamma-1}} u'_j(m' + \epsilon), k^{\frac{\gamma}{\gamma-1}} u'_j(m^* - \epsilon) \right]. \end{aligned}$$

For $\gamma < 1$, $k^{\frac{\gamma}{\gamma-1}}$ is decreasing in k and thus, the r.h.s. can be bounded by

$$\left[k^{\frac{\gamma}{\gamma-1}} u'_j(m^* - \epsilon), k^{\frac{\gamma}{\gamma-1}} u'_j(m' + \epsilon) \right].$$

Finally, for $\gamma = 1$, equation (23) implies that the second term in (22) converges to 0. By the same argument as above, for any $\epsilon > 0$ and $\xi > 0$, there is a \bar{t} such that for $t \geq \bar{t}$ and $\gamma = 1$,

$$\begin{aligned} & \sum_{\tilde{\sigma}_t \in \omega_t^i} \pi(\tilde{\sigma}_t | \omega_t^i) u'_j(a^j(\tilde{\sigma}_t) + e^j(\tilde{\sigma}_t)) + \lim_{t \rightarrow \infty} u'_j(a^j(\sigma_t) + e^j(\sigma_t)) k \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right)^{k-1} \\ & \in \left[u'_j(m^* + \epsilon) - \xi, u'_j(m' + \epsilon) + \xi \right]. \end{aligned}$$

It follows that if i vanishes on ω^i , the l.h.s. of (20) converges to ∞ , whereas the r.h.s. is bounded above, a contradiction.

We conclude that on any ω^i ,

$$\limsup_{t \rightarrow \infty} u_i \left(\min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t) \right) > 0$$

and since for any $\sigma_t \in \omega_t^i$, we have:

$$c^i(\sigma_t) \geq \min_{\tilde{\sigma}_t \in \omega_t^i} a^i(\tilde{\sigma}_t) + \min_{\tilde{\sigma}_t \in \omega_t^i} e^i(\tilde{\sigma}_t)$$

i survives on any path $\sigma \in \Sigma$. \square

Proof of Proposition 7.1:

For any event $w^i \in W^i$ in the coarser partition, and a sub-event $w^{ij} \in W^j$, $w^{ij} \subseteq w^i$ in the finer partition write the true conditional probability as

$$\pi(w^{ij}|w^i) = \frac{\pi(w^{ij})}{\pi(w^i)}$$

and the conditional probability implied by j 's beliefs as

$$\pi^j(w^{ij}|w^i) = \frac{\pi^j(w^{ij})}{\pi^j(w^i)} = \frac{\pi^j(w^{ij})}{\pi^i(w^i)}$$

Taking \ln on both sides of the equation, summing over all $w^{ij} \subseteq w^i$, multiplying by $\pi(w^{ij})$ and summing again over all $w^i \in W^i$, we obtain:

$$\begin{aligned} \sum_{w^j \in W^j} \pi(w^j) \ln \frac{\pi(w^j)}{\pi^j(w^j)} &= \sum_{w^i \in W^i} \sum_{w^{ij} \subseteq w^i} \pi(w^{ij}) \ln \frac{\pi(w^{ij})}{\pi^j(w^{ij})} \\ &= \sum_{w^i \in W^i} \pi(w^i) \sum_{w^{ij} \subseteq w^i} \pi(w^{ij}|w^i) \ln \frac{\pi(w^{ij})}{\pi^j(w^{ij})} \\ &= \sum_{w^i \in W^i} \pi(w^i) \sum_{w^{ij} \subseteq w^i} \pi(w^{ij}|w^i) \left(\ln \frac{\pi(w^{ij}|w^i)}{\pi^j(w^{ij}|w^i)} + \ln \frac{\pi(w^i)}{\pi^j(w^i)} \right) \\ &= \sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi^i(w^i)} + \sum_{w^i \in W^i} \sum_{w^{ij} \subseteq w^i} \pi(w^{ij} | w^i) \left(\ln \frac{\pi(w^{ij}|w^i)}{\pi^j(w^{ij}|w^i)} \right) \\ &\geq \sum_{w^i \in W^i} \pi(w^i) \ln \frac{\pi(w^i)}{\pi^i(w^i)} \end{aligned}$$

which completes the proof. \square

Proof of Proposition 7.2:

Because $\lim_{c \rightarrow 0} u_i(c) = -\infty$, prices, endowments and true probabilities are strictly positive and bounded, for every $B < 0$, there is an ϵ such that whenever $\pi^i(s) < \epsilon$, $c^i(s)$ is sufficiently close to 0 resulting in

$$u^i(c^i(s)) < \frac{B - (1 - \pi(s)) u^i(\max_{s \in S} e^i(s))}{\pi(s)}$$

Note that choosing $W^i = \{S\}$ trivially implies $\pi^i(\{S\}) = 1$ and thus,

$$u_i(e^i(s) + a_{\{S\}}^i(s)) = u_i(e^i(s))$$

regardless of the price system. Thus, choosing $B < u_i(e^i(s))$ implies

$$\sum_{s \in S} \pi(s) u_i(e^i(s) + a^i(s)) < B < \sum_{s \in S} \pi(s) u_i(e^i(s) + a_{\{S\}}^i(s)) = u_i(e^i(s)). \square$$

Proof of Corollaries 7.3 and 7.4:

Consider agent i who uses the heuristic ‘do not trade assets you don’t understand’ and thus has correct beliefs on a partition $(\Omega^i, \mathcal{F}^i)$ and trades measurable with respect to this partition. By condition (i) of Remark 5.1, all agents in the economy have correct beliefs on their respective partitions and thus, everyone, including i , a.s. survives. By condition (iii) of Remark 5.1, all agents in the economy have identical beliefs (and since i ’s beliefs on $(\Omega^i, \mathcal{F}^i)$ are correct, so are all the beliefs of all other agents), but these beliefs need not be correct on finer partitions. Nevertheless, according to the remark, everyone, including i , a.s. survives. According to condition (iv) of Remark 5.1, since agent i ’s beliefs on his partition $(\Omega^i, \mathcal{F}^i)$ are correct, this must be the coarsest partition in the economy. Again, according to the remark, all agents, including i survive a.s.. When $W^i = \{\{S\}\}$, then i ’s beliefs on the corresponding $(\Omega^i, \mathcal{F}^i)$ are trivially correct and this is also, trivially, the coarsest partition in the economy, thus the arguments above apply to the heuristic ‘do not invest in the share market – there are always people smarter than you’.

Note that part (i) of Remark 5.2 does not depend on the accuracy of beliefs of agent i . As long as the partition $(\Omega^i, \mathcal{F}^i)$ identified by the heuristic satisfies the conditions of the Remark, agent i a.s. survives. Furthermore, even if as in part (ii) a fully aware agent with correct beliefs exists, the fact that i trades on a partition on which his beliefs are correct ensures that i a.s. survives. Finally, under the conditions of Remark 5.1 (ii), there is an agent j with a finer partition than i and beliefs on i ’s partition which are closer to the truth. In this case, i a.s. vanishes. \square

Proof of Corollary 7.5:

Note that when all agents are identical in terms of preferences and initial endowments, and agent i uses the heuristic ‘do not trade in assets that appear to be mispriced’, $\tilde{c}_j = \tilde{e}_j$ for all $j \in I$ together with a price system satisfying:

$$\frac{p^*(\sigma_t)}{p^*(\sigma'_{t'})} = \frac{\beta^t \pi(\sigma_t) u'_j(\tilde{e}^j(\sigma_t))}{\beta^{t'} \pi(\sigma'_{t'}) u'_j(\tilde{e}^j(\sigma'_{t'}))} \quad (24)$$

for all $t, t' \in \mathbb{N}$, all $\sigma_t, \sigma'_{t'} \in \Omega$ and all for all $j \in I$ is a financial market equilibrium. This is trivially true if agent i has correct beliefs. Assume thus that i ’s beliefs $(\pi^i(s))_s$ are wrong, $(\pi^i(s))_s \neq (\pi(s))_s$. Let W^i be the finest partition of S on which i ’s beliefs are correct (note that the trivial partition $W = \{\{S\}\}$ always satisfies this requirement) and denote by $(\Omega^i, \mathcal{F}^i)$ the partition of Ω generated by W^i . Assume that prices satisfy (24). Clearly, at these prices, all agents $j \neq i$ find it optimal to consume $\tilde{c}_j = \tilde{e}_j$ and, therefore, setting $\tilde{c}_i = \tilde{e}_i$ clears the markets. We thus have to show that $\tilde{c}_i = \tilde{e}_i$ is optimal for i given prices p^* and given the constraint imposed by the heuristic.

Indeed, consider two nodes, σ_t and σ'_t . Then

$$\frac{p^*(\sigma_t)}{p^*(\sigma'_t)} \neq \frac{\pi^i(\sigma_t) u'_i(\tilde{e}^i(\sigma_t))}{\pi^i(\sigma'_t) u'_i(\tilde{e}^i(\sigma'_t))}$$

can only hold if $\sigma_t, \sigma'_t \in \omega_t^i$ for some $\omega_t^i \in \Omega^i$ and, by the definition of the heuristic, we have that $\tilde{e}^i(\sigma_t) - \tilde{e}^i(\sigma'_t) = \tilde{e}^i(\sigma'_t) - \tilde{e}^i(\sigma_t)$. Suppose, in contradiction to the claim above and w.l.o.g. that $\tilde{e}^i(\sigma_t) - \tilde{e}^i(\sigma'_t) > 0$. Then, in order to satisfy the budget constraint of i , there must be another node σ''_t such that $\tilde{e}^i(\sigma''_t) - \tilde{e}^i(\sigma'_t) < 0$. Note that if $\sigma''_t \in \omega_t^i$, this is in contradiction to the measurability condition imposed on trades by the heuristic. Thus, $\sigma''_t \in \omega_{t'}^i \neq \omega_t^i$. Suppose first that $t' = t$. Then, it must be that i 's beliefs over ω_t^i and $\omega_{t'}^i$ are correct and thus,

$$\frac{\sum_{\tilde{\sigma}_t \in \omega_t^i} p^*(\tilde{\sigma}_t)}{\sum_{\tilde{\sigma}_t \in \omega_{t'}^i} p^*(\tilde{\sigma}_t)} = \frac{\sum_{\tilde{\sigma}_t \in \omega_t^i} \pi(\tilde{\sigma}_t) u'_i(\tilde{e}^i(\tilde{\sigma}_t))}{\sum_{\tilde{\sigma}_t \in \omega_{t'}^i} \pi(\tilde{\sigma}_t) u'_i(\tilde{e}^i(\tilde{\sigma}_t))} = \frac{\sum_{\tilde{\sigma}_t \in \omega_t^i} \pi^i(\tilde{\sigma}_t) u'_i(\tilde{e}^i(\tilde{\sigma}_t))}{\sum_{\tilde{\sigma}_t \in \omega_{t'}^i} \pi^i(\tilde{\sigma}_t) u'_i(\tilde{e}^i(\tilde{\sigma}_t))}$$

If \tilde{e}^i is optimal for i , it has to be that:

$$\frac{\sum_{\tilde{\sigma}_t \in \omega_t^i} p^*(\tilde{\sigma}_t)}{\sum_{\tilde{\sigma}_t \in \omega_{t'}^i} p^*(\tilde{\sigma}_t)} = \frac{\sum_{\tilde{\sigma}_t \in \omega_t^i} \pi(\tilde{\sigma}_t) u'_i(\tilde{e}^i(\tilde{\sigma}_t))}{\sum_{\tilde{\sigma}_t \in \omega_{t'}^i} \pi(\tilde{\sigma}_t) u'_i(\tilde{e}^i(\tilde{\sigma}_t))} = \frac{\sum_{\tilde{\sigma}_t \in \omega_t^i} \pi^i(\tilde{\sigma}_t) u'_i(\tilde{e}^i(\tilde{\sigma}_t))}{\sum_{\tilde{\sigma}_t \in \omega_{t'}^i} \pi^i(\tilde{\sigma}_t) u'_i(\tilde{e}^i(\tilde{\sigma}_t))}$$

but this is impossible, because by the definition of \tilde{e}^i , $\tilde{e}^i(\tilde{\sigma}_t) \leq \tilde{e}^i(\sigma_t)$ for all $\tilde{\sigma}_t \in \omega_t^i$ (with at least one strict inequality), whereas $\tilde{e}^i(\tilde{\sigma}_t) \geq \tilde{e}^i(\sigma_t)$ for all $\tilde{\sigma}_t \in \omega_{t'}^i$ (with at least one strict inequality) and by the strict monotonicity of u'_i we have

$$\frac{\sum_{\tilde{\sigma}_t \in \omega_t^i} \pi(\tilde{\sigma}_t) u'_i(\tilde{e}^i(\tilde{\sigma}_t))}{\sum_{\tilde{\sigma}_t \in \omega_{t'}^i} \pi(\tilde{\sigma}_t) u'_i(\tilde{e}^i(\tilde{\sigma}_t))} > \frac{\sum_{\tilde{\sigma}_t \in \omega_t^i} \pi^i(\tilde{\sigma}_t) u'_i(\tilde{e}^i(\tilde{\sigma}_t))}{\sum_{\tilde{\sigma}_t \in \omega_{t'}^i} \pi^i(\tilde{\sigma}_t) u'_i(\tilde{e}^i(\tilde{\sigma}_t))},$$

a contradiction to the optimality of \tilde{e}^i . It then follows that for an optimal consumption stream \tilde{e}^i , $\tilde{e}^i(\sigma_t) - \tilde{e}^i(\sigma'_t) > 0$ for some $\sigma_t \in \omega_t^i$ implies that for any $\omega_{t'}^i \neq \omega_t^i$, there is a $\tilde{\sigma}_t \in \omega_{t'}^i$ with $\tilde{e}^i(\tilde{\sigma}_t) - \tilde{e}^i(\sigma_t) > 0$.

We thus conclude that ω_t^i and $\omega_{t'}^i$ must be such that $t \neq t'$. Our arguments so far imply that $\tilde{e}^i(\tilde{\sigma}_t) \leq \tilde{e}^i(\sigma_t)$ for all $\tilde{\sigma}_t \in \tilde{\omega}_t^i$ where $\tilde{\omega}_t^i \in \Omega_t^i$ (with at least one strict inequality) and $\tilde{e}^i(\tilde{\sigma}_t) \geq \tilde{e}^i(\sigma_t)$ for all $\tilde{\sigma}_t \in \tilde{\omega}_{t'}^i$ where $\tilde{\omega}_{t'}^i \in \Omega_{t'}^i$ (with at least one strict inequality). But we then have

$$\frac{\sum_{\tilde{\sigma}_t \in \Omega_t} p^*(\tilde{\sigma}_t)}{\sum_{\tilde{\sigma}_t \in \Omega_{t'}} p^*(\tilde{\sigma}_t)} = \frac{\sum_{\tilde{\sigma}_t \in \Omega_t} \pi^i(\tilde{\sigma}_t) u'_i(\tilde{e}^i(\tilde{\sigma}_t))}{\sum_{\tilde{\sigma}_t \in \Omega_{t'}} \pi^i(\tilde{\sigma}_t) u'_i(\tilde{e}^i(\tilde{\sigma}_t))} < \frac{\sum_{\tilde{\sigma}_t \in \Omega_t} \pi(\tilde{\sigma}_t) u'_i(\tilde{e}^i(\tilde{\sigma}_t))}{\sum_{\tilde{\sigma}_t \in \Omega_{t'}} \pi(\tilde{\sigma}_t) u'_i(\tilde{e}^i(\tilde{\sigma}_t))}$$

in contradiction to the conjectured optimality of \tilde{e}^i .

We infer that at p^* , $\tilde{e}^i = \tilde{e}^i$ is indeed optimal for i and thus, $(p^*, (\tilde{e}_j)_{j \in I})$ indeed constituted a financial market equilibrium. Since \tilde{e}^i is uniformly bounded below, in this equilibrium i survives and therefore, the heuristic ‘do not trade in assets that appear to be mispriced’ is evolutionary rational. \square

10 References

Alchian, A. A. (1950). "Evolution, and Economic Theory", Journal of Political Economy 58:

211-221.

Auster, S., Kettering, J., Kochov, A. (2020). "Sequential Trading with Coarse Contingencies", mimeo.

Barber, B. M., Odean, T. (2001). "Boys will Be Boys: Gender, Overconfidence and Common Stock Investment", *Quarterly Journal of Economics* 116: 261-292.

Barber, B. M., Odean, T. (2002). "Online Investors: Do the Slow Die First?", *Review of Financial Studies* 15: 455-488

Bewley, T. (1972): Existence of equilibria in economies with infinitely many commodities, *Journal of Economic Theory*, 4: 514-540.

Blume, L., Easley, D. (1992). "Evolution and Market Behavior," *Journal of Economic Theory* 58: 9-40.

Blume, L., Easley, D. (1993). "Economic Natural Selection", *Economics Letters* 42: 281-289.

Blume, L., Easley, D. (2006). If You Are So Smart, Why Aren't You Rich? Belief Selection in Complete and Incomplete Markets, *Econometrica*, 74: 929-966.

Condie, S. (2008). "Living with Ambiguity: Prices and Survival When Agents Have Heterogeneous Preferences for Ambiguity", *Economic Theory* 36: 81-108.

Epstein, L., Marinacci, M., Seo, K. (2007). "Recursive Smooth Ambiguity Preferences", *Theoretical Economics* 2: 355-394.

French, K., and J. Poterba (1991): Agent diversification and international equity markets, *American Economic Review*, 81: 222-226.

Ghirardato, P. (2001). "Coping with ignorance: unforeseen contingencies and non-additive uncertainty", *Economic Theory* 17: 247-276.

Gigerenzer, G. (2007). *Gut Feelings: The Intelligence of the Unconscious*, Penguin Books.

Grant, S., Quiggin, J. (2013 a). "Inductive Reasoning about Unawareness", *Economic Theory* 54: 717-755.

Grant, S., Quiggin, J., (2013 b). "Bounded Awareness, Heuristics and the Precautionary Principle," *Journal of Economic Behavior and Organization* 93: 17-31.

Grant, S., Quiggin, J., (2014). "A preference model for choice subject to surprise", *Theory and Decision* 79:1-14.

Guerdjikova, A., Quiggin, J. (2019). "Market Selection with Differential Financial Constraints", *Econometrica* 87: 1693-1762.

Guiso, L., Jappelli, T. (2005). "Awareness and Stock Market Participation", CFS Working Paper 2005/29.

Heifetz, A., Meier, M., Schipper, B. C. (2006). "Interactive Unawareness", *Journal of Economic Theory* 130: 78-94.

Li, J. (2009). "Information Structures with Unawareness", *Journal of Economic Theory* 144: 977-993.

Minardi, St., and Savochkin, A. (2016): Subjective Contingencies and Limited Bayesian Updating, HEC Paris Research Paper No. ECO-SCD-2017-1203. Available at SSRN: <https://ssrn.com/abstract=2962260> or <http://dx.doi.org/10.2139/ssrn.2962260>

Modica, S., Rustichini, A., Tallon, J.-M. (1998). "Unawareness and bankruptcy: A general equilibrium model", *Economic Theory* 12: 259-292.

- Odean, T. (1999). "Do Investors Trade Too Much?", *American Economic Review* 89: 1279-1298.
- Sandroni, A. (2005). "Market Selection when Markets are Incomplete", *Journal of Mathematical Economics*, 41, 91-104.
- Schipper, B. (2018). The Unawareness Bibliography, <http://www.econ.ucdavis.edu/faculty/schipper/unaw.htm>
- Vierø, M.-L. (2009). "Exactly what happens after the Anscombe–Aumann race?", *Economic Theory* 41: 175-212.