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Generalized Solutions to Degenerate Dynamical Systems

Philippe JOUAN*, Ulysse SERRES†

January 30, 2023

Abstract

The solutions to Degenerate Dynamical Systems of the form $A(x)\dot{x} = f(x)$ are studied by considering the equation as a differential inclusion. The set $\mathcal{Z} = \{\det(A(x)) = 0\}$, called the singular set, is assumed to have empty interior.

The reasons leading us to the definition of the sets used for the differential inclusion are exposed in detail.

This definition is then applied on the one hand to generic cases, and on the other hand to the particular cases resulting from physics which can be found in [18].

It is shown that generalized solutions may enter, leave, or remain in the singular locus.

Keywords: Degenerate Dynamical Systems, differential inclusions, generalized solutions, genericity.

1 Introduction

The aim of the present paper is to investigate the solutions of degenerate dynamical systems, also called singular differential equations, of the form:

$$A(x)\dot{x} = f(x), \tag{1}$$

where $x \in \mathbb{R}^n$, f is a smooth vector field on $\mathbb{R}^n$, and A: $\mathbb{R}^n \longmapsto \mathcal{M}(n;\mathbb{R})$ is a field of square matrices, smoothly depending on x. From now on, degenerate dynamical systems are referred to as DDSs.

The **singular locus** of (E), denoted by $\mathcal{Z}$, is the set of points where the determinant of A vanishes:

$$\mathcal{Z} = \{x \in \mathbb{R}^n;\ \ \det(A(x)) = 0\}.$$

Assumption. *All along the paper the set $\mathcal{Z}$ is assumed to be non empty but with empty interior.*

We are particularly interested in the problem of the existence of solutions lying in $\mathcal{Z}$ and to the extension in $\mathcal{Z}$ of classical solutions ending in the singular locus. Our motivation comes from a discussion with one of the authors of the papers [18] and [5], where some equations of the previous kind coming from physics are analyzed. In a first step, and contrary to [18], we do not assume any regularity on the set $\mathcal{Z}$ and we allow f to vanish on $\mathcal{Z}$, even if in some cases

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this setting is not related to a physical model. Actually, we are interested in defining general notions of solutions to these systems that remain valid in all situations and we only assume that everything is smooth and that the interior of $\mathcal{Z}$ is empty.

There is an important literature about the kind of equations we consider here, under different names, in particular Differential-Algebraic Equations (DAE) and Implicit Differential Equations.

Differential-Algebraic Equations are actually partially different to the one we consider here because it is generally assumed in the literature about DAEs that the matrix A is either constant, hence singular everywhere, or is singular of constant rank on an open subset of $\mathbb{R}^n$ (see for example the pioneer book [3]). The same kind of assumption is made in the nonlinear case, as explained in the book [11] where an important bibliography about DAEs can be found. Let us quote [4], [15], [13], [14], [19]. Differential-Algebraic Equations can also be controlled, see [4] or [2] on Lie groups.

The paper [16] deals with equations of the form (1) at singular points, but with a point of view close to the one of DAE.

The geometric approach of Implicit Differential Equations goes back to Poincaré. An elementary account can be found in [1] and more complete overview in [17].

The point of view of the present paper is different: it considers generalized solutions to (E) defined as trajectories of a differential inclusion. Assuming that a closed and convex set $\Gamma(x)$ has been assigned to each point $x \in \mathbb{R}^n$ in a way that will be made clear later, we can use the following classical definition of solutions to a differential inclusion:

Definition 1 *A solution to (E) is an absolutely continuous function $x(t)$ from an interval I of $\mathbb{R}$ to $\mathbb{R}^n$ that satisfies*

$$\dot{x}(t) \in \Gamma(x(t)) \qquad \text{for almost every } \ t \in I.$$

Outside the singular locus the equation simply reads $\dot{x} = A(x)^{-1} f(x)$, and every solution to (E) must be a trajectory of the vector field $F(x) = A(x)^{-1} f(x)$. Consequently all possible definitions of Γ must imply $\Gamma(x) = \{A(x)^{-1} f(x)\}$ whenever $x \notin \mathcal{Z}$.

In a first approach (Section 2) the sets Γ are defined by imitating Filippov's solutions to differential equations with discontinuous right-hand side. However, in Filippov's book [7] (see also [6]) the vector field that defines the equation is assumed to be bounded (similarly in the literature about differential inclusions, for instance in [20] and [10], the sets $\Gamma(x)$ are assumed to be bounded). This assumption cannot be made herein since $A(x)^{-1} f(x)$ is generally unbounded in the neighborhood of a singular point. Though the Filippov-like definition works well in some simple cases (Examples 1-3 of Section 2.1) this lack of boundedness causes drawbacks that are examined in Section 4.2 and leads to a new definition, different from Filippov's one (Definition 4 in Section 4.2).

Section 4.2 is preceded by a detailed analysis of generic DDSs, which is the key to understanding the meaning of the possible definitions of the sets Γ (Section 3). It is in particular shown that generically the singular set $\mathcal{Z}$ is the disjoint union of the codimension r^2 submanifolds $\mathcal{Z}_r$ where the rank of A is $n - r$, and then in Section 4.1, that at a generic point p belonging to the stratum $\mathcal{Z}_1$ of a generic system the Filippov-like set $\Gamma_f(p)$ is equal to $\mathbb{R}^n$. This is due to the variation of the kernel of $A(x)$ as the point x moves within $\mathcal{Z}_1$. A consequence is that according to the Filippov-like definition, all absolutely continuous curves lying inside $\mathcal{Z}_1$ are generalized solutions to (E).

It was therefore necessary to modify the definition of the sets Γ in order to reduce their dimension at generic points of $\mathcal{Z}_1$ while preserving the results obtained in the simple cases. This

brings us to Definition 4 according to which the set Γ at a generic point of $\mathcal{Z}_1$ is a dimension 1 affine space. Moreover the trace of these sets on the tangent space of $\mathcal{Z}_1$ provides a smooth vector field (in an open and dense subset of $\mathcal{Z}_1$, there are singularities). The gap between the Filippov-like sets and the ones of Definition 4 and its consequences are exhibited in a generic example postponed to Section 7.

Section 5 is devoted to the computation of the solutions to (E) outside of $\mathcal{Z}$ by reparametrization of the trajectories of a vector field $\widetilde{f}$ that has the double interest to be defined and smooth everywhere, including $\mathcal{Z}$, and to be colinear with F outside the singular locus. It is also shown that in many cases the solutions thus obtained can be extended to solutions inside $\mathcal{Z}$.

These results are adapted in Section 6 to the particular and motivating case of [18] that deals with systems in $\mathbb{R}^2$ of the form

$$(S) \qquad \phi(x,y)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_1(x,y) \\ a_2(x,y) \end{pmatrix}.$$

Assuming that ϕ is generic, hence a Morse function, we study the behaviour of (S) in the three generic cases.

In conclusion the DDSs under consideration behave very differently from the classical ones for various reasons, the main ones being:

1. *It may exist trajectories that enter the singularities in finite time.*
2. *They may be extended to trajectories that live in $\mathcal{Z}$.*
3. *The DDSs are not always deterministic and even if a system is deterministic the time reversed one need not be.*

Notations. The set $\mathbb{R}^n$ is endowed with the Euclidean norm denoted by $\|.\|$. The inner product is denoted by $\langle .,. \rangle$.

The points of $\mathbb{R}^n$ are denoted by x, sometimes by p or q, and their coordinates by $(x_1, x_2, \ldots, x_n)$. In low dimension examples the coordinates are often (x, y) or (x, y, z).

The convex hull of a subset S of $\mathbb{R}^n$ is denoted by co S and the closure of the convex hull of S by $\overline{\text{co}}\ S$.

The open set $\mathbb{R}^n \setminus \mathcal{Z}$ is denoted by Ω.

The vector field F is defined outside of $\mathcal{Z}$ by $F(x) = A(x)^{-1}f(x)$.

2 Basic settings

2.1 The Filippov-like definition

Definition 2 *At all points $x \in \mathbb{R}^n$ the set $\Gamma_f(x)$ is defined by*

$$\Gamma_f(x) = \bigcap_{r>0} \overline{co}\ \{F(x) = A(x)^{-1}f(x); \quad x \in B(x,r) \setminus \mathcal{Z}\}.$$

where $B(x,r)$ stands for the open ball of radius r centered at x.

This definition is equivalent to Filippov's one in the case where the Lebesgue mesure of the set of discontinuities vanishes. The sets $\Gamma_f(x)$ are closed and convex and it could be proved that at least one solution exists for any initial condition if the vector field $A(x)^{-1}f(x)$ were bounded (see [7]). This existence statement also holds for bounded differential inclusions (see [20]).

Unfortunately the existence of solutions is not guaranteed for unbounded equations as shown by the following example.

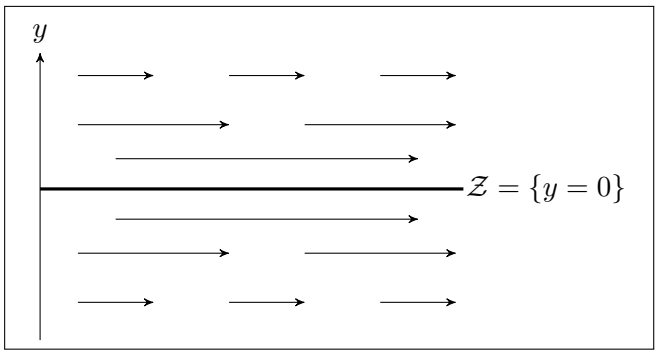

Figure 1: Example 1

Example 1 *Consider in* $\mathbb{R}^2$

$$\begin{cases} y^2\dot{x} &= 1 \\ \dot{y} &= 0 \end{cases}$$

The vector field F *is here* $\dfrac{1}{y^2}\dfrac{\partial}{\partial x}$ *and is everywhere parallel to the singular locus* $\mathcal{Z} = \{y = 0\}$. *A possible solution starting from* $p = (x, 0) \in \mathcal{Z}$ *(or ending to* p*) would consequently remain in* $\mathcal{Z}$ *but* $\dfrac{1}{y^2}\dfrac{\partial}{\partial x}$ *tends to infinity in only one direction, so that this solution should have infinite velocity.*

Actually $\Gamma_f(p) = \emptyset$ *at all* $p \in \mathcal{Z}$ *because* $\overline{co}\ \{A(q)^{-1}f(q);\ \ q \in B(p,r)\setminus\mathcal{Z}\} = \left[\dfrac{1}{r^2}, +\infty\right) \times \{0\}$ *for* $r > 0$.

The conclusion is that no solution exists for an initial point in $\mathcal{Z}$. *It should be noticed that there is no solutions to this equation starting from* $\mathcal{Z}$ *for any definition of solution we know.*

According to the geometry of the equation the behaviour may be completely different, for instance some solutions may cross the singular locus (Example 2) or remain in that last (Example 3).

Example 2 *Consider in* $\mathbb{R}^2$

$$\begin{cases} \dot{x} &= 0 \\ 3y^2\dot{y} &= 1 \end{cases}$$

It is clear that the function $(x_0, t^{\frac{1}{3}})$ *is a solution through the point* $(x_0, 0)$ *for any* x_0, *and that it is unique for this initial state at* $t = 0$.

The vector field F *is here equal to* $\dfrac{1}{3y^2}\dfrac{\partial}{\partial y}$ *and the set* $\Gamma_f(x, 0)$ *is empty at all points* $(x, 0) \in \mathcal{Z}$. *It is not in contradiction with the definition of solutions since it is sufficient that the differential*

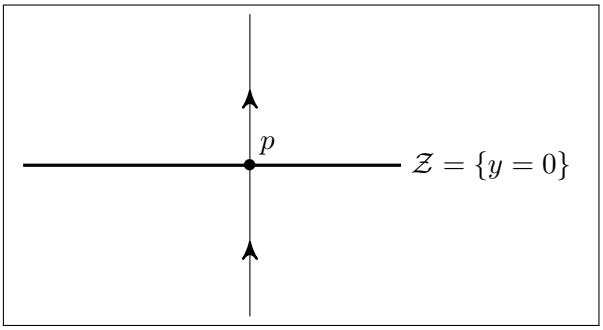

Figure 2: Example 2

inclusion be satisfied for almost all t. On the other hand the emptiness of the sets Γ_f at the points of $\mathcal{Z}$ implies that there are no other solutions than the exhibited ones.

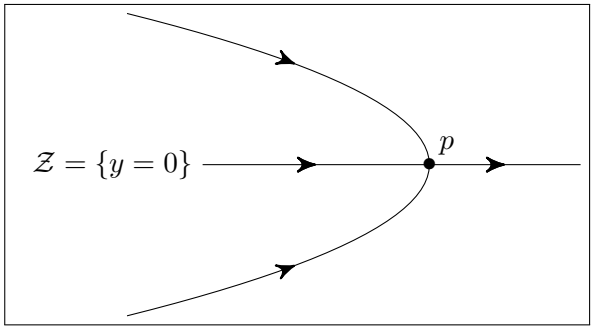

Figure 3: S^-

Example 3 *Consider in* $\mathbb{R}^2$

$$(S_-) \qquad \left\{ \begin{array}{ll} \dot{x} & = 1 \\ 2y\dot{y} & = -1 \end{array} \right. \qquad \textit{Again} \quad \mathcal{Z} = \{y = 0\}.$$

It is clear that $x(t) = x_0 + t - t_0, y(t) = \pm\sqrt{-(t - t_0)}$ is solution on $(-\infty, t_0]$ and ends in $\mathcal{Z}$.

Thanks to Definition 2 we can obtain solutions that remain in $\mathcal{Z}$. Indeed $\Gamma_f(x, 0) = \{1\} \times \mathbb{R}$ and each of these sets contains the vector $(1, 0)$ which is tangent to $\mathcal{Z}$. This defines a constant vector field in $\mathcal{Z}$, the solutions of which are $t \longmapsto (t + a, 0)$, where $a \in \mathbb{R}$, on any time interval.

In conclusion all solution curves starting from a point (x_0, y_0) outside of $\mathcal{Z}$ reach the singular locus at a finite time t_0 and at a point $(a, 0)$. They extend into $t \longmapsto (a + t - t_0, 0)$ on $[t_0, +\infty)$.

We can also consider the similar system

$$(S_+) \qquad \left\{ \begin{array}{ll} \dot{x} & = 1 \\ 2y\dot{y} & = 1 \end{array} \right.$$

As previously the solutions $t \longmapsto (a+t-t_0, 0)$ exist in the singular locus but starting from any point $(a, 0) \in \mathcal{Z}$ at a time t_0 three trajectories are available: the two curves $t \longmapsto (a + t - t_0, \pm\sqrt{t - t_0})$ that escape from $\mathcal{Z}$ and the solution $t \longmapsto (a + t - t_0, 0)$ that remains in $\mathcal{Z}$.

This shows that the equation (S_+) is not deterministic. It is clear that the time-reversed system of (S_-) is neither deterministic, although (S_-) is.

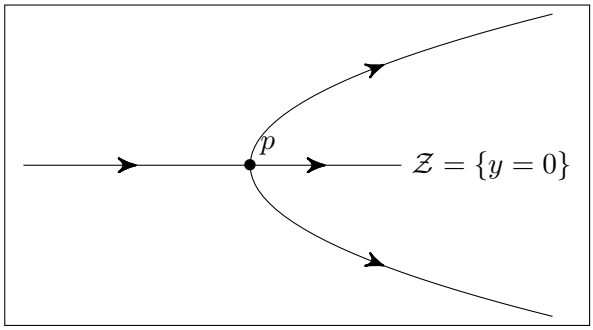

Figure 4: S^+

These three examples show that the Fillippov's like definition may provide acceptable solutions in simple cases. However, we will see in Section 4 that the sets Γ_f are generically too big.

Before looking at what generic systems are, let us turn our attention to other possible definitions of the sets Γ.

2.2 Other possible definitions

In this subsection two other definitions are proposed and analyzed.

Definition 3 *1. The naive definition: at all points $x \in \mathbb{R}^n$ the set $\Gamma(x)$ is equal to*

$$E(x) = \{\xi \in \mathbb{R}^n; \quad A(x)\xi = f(x)\}.$$

2. The set of limit values: at all points $x \in \mathbb{R}^n$ the set $\Gamma(x)$ is the smallest closed and convex set $\mathcal{A}(x)$ containing the limits of all sequences $(F(p_k))_{k\geq 1}$, where $p_k \notin \mathcal{Z}$ and $\lim_{k\mapsto+\infty} p_k = x$.

It is clear that in both cases the set $\Gamma(x)$ is reduced to $\{F(x) = A(x)^{-1}f(x)\}$ whenever $x \notin \mathcal{Z}$.

On the other hand $E(x)$ is an affine subspace of $\mathbb{R}^n$ as soon as it is not empty, it is therefore closed and convex. However, let $x \in \mathcal{Z}$; then $E(x) \neq \emptyset \Longleftrightarrow f(x) \in Im(A(x))$ which not generic since rank $(A(x)) < n$.

The insufficiency of the naive definition is well illustrated by Example 3: the set $E(x, 0)$ is empty at all points $(x, 0)$ belonging to $\mathcal{Z}$ and the solutions that remain in $\mathcal{Z}$ are lost if $\Gamma(x, 0) = E(x, 0)$.

It is possible to state a notion of solutions that encompasses as well the ones obtained by the Filippov-like definition than the naive ones by setting:

$$\Gamma(x) = \bigcap_{r>0} \overline{\text{co}}\ \{\xi;\ \exists x \in B(x, r) \text{ such that } A(x)\xi = f(x)\},$$

but these sets are bigger than the Filippov-like ones and suffer the same defect analyzed in the next section.

The sets $\mathcal{A}(x)$ have the interest to be often easier to compute than the sets $\Gamma_f(x)$, and to be included in both $\Gamma_f(x)$ and $E(x)$. Indeed let $(p_k)_{k\geq 1}$ be a sequence in Ω that tends to a point x in $\mathcal{Z}$ and such that the sequence $F(p_k)$ tends to a limit ξ. Then:

1. The vector ξ belongs to $\Gamma_f(x)$, and since this set is closed and convex, it contains $\mathcal{A}(x)$.

2. The equality $A(p_k)F(p_k) = f(p_k)$ holds for all k, hence $A(x)\xi = f(x)$ and $\xi \in E(x)$.

Though the sets $\mathcal{A}(x)$ are included in the sets $\Gamma_f(x)$ there are many examples where they are not equal. It is easy to prove that these sets would be equal if the vector field F were bounded (it is shown in [7]), but it is not the case in the present more general setting, and as well as the sets $E(x)$ the sets $\mathcal{A}(x)$ are not big enough in many cases. In Example 3 again, the set $\mathcal{A}(x)$ is empty if $x \in \mathcal{Z}$ since $\|F(x)\| \to +\infty$ as x approaches $\mathcal{Z}$.

2.3 A universal vector field

. Let A be a $n \times n$ square matrix. We denote by $\widetilde{A}$ the transpose of its cofactor matrix. It satisfies $A\widetilde{A} = \widetilde{A}A = \det(A)I_n$, and $A^{-1} = \dfrac{1}{\det(A)}\widetilde{A}$ if $\det(A) \neq 0$.

In what follows we consider the smooth vector field $\widetilde{f}$ defined by $\widetilde{f}(x) = \widetilde{A}(x)f(x)$ and we denote by $\delta(x)$ the determinant of $A(x)$.

This vector field has the interest to be always defined and smooth on $\mathbb{R}^n$ (everywhere). Moreover the vector fields F and $\widetilde{f}$ are colinear outside of $\mathcal{Z}$ since for $x \notin \mathcal{Z}$:

$$F(x) = A^{-1}(x)f(x) = \frac{1}{\det(A(x))}\widetilde{A}(x)f(x) = \frac{1}{\delta(x)}\widetilde{f}(x).$$

Proposition 1 *Let $p \in \mathcal{Z}$.*
If rank $A(p) = n-1$, then $f(p) \in ImA(p) \Longleftrightarrow \widetilde{f}(p) = 0$.
If rank $A(p) < n-1$, then $\widetilde{f}(p) = 0$.
Moreover $\widetilde{f}(p) \neq 0 \Longrightarrow E(p) = \mathcal{A}(p) = \emptyset$

Proof. It holds for any square matrix A: $\ker \widetilde{A} = \mathrm{Im}A$ if rank $A = n-1$, and $\widetilde{A} = 0$ if rank $A < n-1$. The two first assertions are proved.

Next $\widetilde{f}(p) \neq 0$ may happen only if rank $A(p) = n-1$ and $f(p) \notin \mathrm{Im}A(p)$. But this last condition implies $E(p) = \emptyset$ and also $\mathcal{A}(p) = \emptyset$ since $\mathcal{A}(p)$ is included in $E(p)$.

■

Remarks.

1. Let $\mathcal{Z}_1$ be the set of points $x \in \mathcal{Z}$ where the rank of $A(x)$ is $n-1$. The set of points of $\mathcal{Z}_1$ where $\widetilde{f}$ does not vanish is open and we will see in the next section that it is generically dense.

2. On the other hand $\widetilde{f}$ vanishes on $\mathcal{Z} \setminus \mathcal{Z}_1$. This case cannot be neglected because there exist generically points where rank $(A) < n-1$ as soon as $n \geq 4$, this is the subject of the next section.

3 Genericity

The systems $A(x)\dot{x} = f(x)$ considered in this paper are completely determined by the mapping:

$$x \longmapsto \Phi(x) = (A(x), f(x)) \quad \text{from} \quad \mathbb{R}^n \quad \text{to} \quad \mathcal{M}_n(\mathbb{R}) \times \mathbb{R}^n,$$

where $\mathcal{M}_n(\mathbb{R})$ stands for the set of real $n \times n$ matrices. More generally we denote by $\mathcal{M}_{n\times m}(\mathbb{R})$ the set of real $n \times m$ matrices and $\mathcal{M}_n(\mathbb{R}) \times \mathbb{R}^n$ can be identified with $\mathcal{M}_{n\times(n+1)}(\mathbb{R})$.

In what follows we will say that a property of degenerate dynamical systems is **generic** if the set of smooth mappings from $\mathbb{R}^n$ to $\mathcal{M}_n(\mathbb{R}) \times \mathbb{R}^n$ for which this property is satisfied is open and dense in the $\mathcal{C}^2$ Whitney topology (see [8] and [9]).

The proof of the next theorem uses several times the product of coranks theorem, that states: let L^r be the set of elements of $\mathcal{M}_{n\times m}(\mathbb{R})$ of corank r. It is a submanifold of $\mathcal{M}_{n\times m}(\mathbb{R})$ of codimension $(n-q+r)(m-q+r)$ where $q = \min\{n, m\}$ (see [8]).

Theorem 1 *Let $\mathcal{Z}_r$ be the set of points where the rank of $A(x)$ is $n-r$ and let R be the largest integer such that $R^2 \leq n$. The following properties are generic:*

1. *For $1 \leq r \leq R$ the set $\mathcal{Z}_r$ is a codimension r^2 submanifold of $\mathbb{R}^n$ and the singular locus $\mathcal{Z}$ is the disjoint union of these submanifolds.*
2. *The submanifold $\mathcal{Z}_{r+1}$ is included in the closure $\overline{\mathcal{Z}}_r$ of $\mathcal{Z}_r$ for $r = 1, \ldots, R-1$.*
3. *The mapping $x \longmapsto \delta(x) = \det(A(x))$ is a submersion at all points $x \in \mathcal{Z}_1$.*
4. *In each stratum $\mathcal{Z}_r$ the set $\mathcal{ZI}_r$ of points x where $f(x)$ belongs to $Im(A(x))$ is a submanifold of codimension $r(r+1)$ in $\mathbb{R}^n$, hence of codimension r in $\mathcal{Z}_r$.*

Proof.
1. For each r in $\{1, \ldots, n\}$ the set L^r of elements of $\mathcal{M}_n(\mathbb{R})$ of corank r is a submanifold of $\mathcal{M}_n(\mathbb{R})$ of codimension r^2. Moreover the union of the L^r for $r = 1, \ldots, n$ is closed. According to [8] (Theorem 4.9, and Exercise 3 page 59) the set of smooth fields of matrices $A(x)$ that are tranversal to all the L^r is open and dense in $\mathcal{C}^\infty(\mathbb{R}^n, \mathcal{M}_n(\mathbb{R}))$ endowed with the $\mathcal{C}^2$ Whitney topology [1].

The codimension of L^r is r^2 and transversality means nonintersection if $r^2 > n$. If $A^{-1}(L^r)$ is not empty transversality implies that it is a submanifold of $\mathbb{R}^n$ of codimension r^2. This proves the first point.
2. Let us consider a generic mapping A from $\mathbb{R}^n$ to $\mathcal{M}_n(\mathbb{R})$ and a point p belonging to $\mathcal{Z}_r$ with $r \geq 2$. The rank of $\Lambda = A(p)$ is $n-r$ and there exists an invertible matrix P such that $P\Lambda P^{-1} = \begin{pmatrix} A_\Lambda & B_\Lambda \\ C_\Lambda & D_\Lambda \end{pmatrix}$ where $A_\Lambda \in \mathcal{M}_{n-r}(\mathbb{R})$ is invertible. As in the proof of the product of coranks Theorem we consider
$T = \begin{pmatrix} I & 0 \\ -C_\Lambda A_\Lambda^{-1} & I \end{pmatrix}$ so that $TP\Lambda P^{-1} = \begin{pmatrix} A_\Lambda & B_\Lambda \\ 0 & D_\Lambda - C_\Lambda A_\Lambda^{-1} B_\Lambda \end{pmatrix}$.

Let the matrix P be fixed. The set Ω of matrices Q such that, with obvious notations, A_Q is invertible is open and the mapping $\Theta(Q) = D_Q - C_Q A_Q^{-1} B_Q$ is a submersion from this open subset of $\mathcal{M}_n(\mathbb{R})$ onto $\mathcal{M}_r(\mathbb{R})$. Clearly rank $(Q) = n - r +$ rank $(\Theta(Q))$ and for $0 \leq s \leq r$ we have $L^s_{\mathcal{M}_n(\mathbb{R})} = \Theta^{-1}(L^s_{\mathcal{M}_r(\mathbb{R})})$ (with obvious notations again).

Since the mapping A is generic we have $A \pitchfork_p L^r$ which implies that $\Theta \circ A$ is a submersion at p (see [8], Lemma 4.3). Let V be an open neighborhood of p and $W = V \bigcap A^{-1}(\Omega)$. Then $\Theta \circ A(W)$ is a neighborhood of 0 in $\mathcal{M}_r(\mathbb{R})$ that encounters $L^s_{\mathcal{M}_r(\mathbb{R})}$ for $0 \leq s \leq r$. Consequently

$$\forall s, \quad 0 < s < r, \qquad V \bigcap \mathcal{Z}_s \neq \emptyset.$$

[1] The Whitney topology is called the strong topology in [9]. The references in this book are Thom transversality Theorem, the pararametric Theorem 2.7 and Exercise 15 p. 84

3. Consider now the determinant mapping, denoted by det, from $\mathcal{M}_n(\mathbb{R})$ to $\mathbb{R}$. The differential of det at $\Lambda \in \mathcal{M}_n(\mathbb{R})$ applied to H is $D\det(\Lambda).H = \mathrm{Trace}(\widetilde{\Lambda}H)$ where $\widetilde{\Lambda}$ stands for the transpose of the matrix of cofactors of Λ. Consequently $D\det(\Lambda) \neq 0$ if and only if rank $(\Lambda) \geq n-1$. This shows that det is a submersion on $\mathcal{M}_n(\mathbb{R}) \setminus \left(\bigcup_{r\geq 2} L^r \right)$. In particular det is a submersion in a neighborhood W of L^1 small enough for $L^1 = \{\det = 0\} \bigcap W$.

Let us consider a generic mapping A from $\mathbb{R}^n$ to $\mathcal{M}_n(\mathbb{R})$ and a point p belonging to $\mathcal{Z}_1$. Since $A \pitchfork_p L^1$ and according to [8] (Lemma 4.3) $\det \circ A$ is a submersion at p. In other words $x \longmapsto \det(A(x))$ is a submersion at p.

4. We can identify $\mathcal{M}_n(\mathbb{R}) \times \mathbb{R}^n$ with $\mathcal{M}_{n\times(n+1)}(\mathbb{R})$. The codimension of the subset L_1^r of elements of $\mathcal{M}_{n\times(n+1)}(\mathbb{R})$ of corank r is $r(r+1)$. Again the set of smooth mappings Φ that are tranversal to all the L_1^r is open and dense in the set $\mathcal{C}^\infty(\mathbb{R}^n, \mathcal{M}_{n\times(n+1)}(\mathbb{R}))$ endowed with the $\mathcal{C}^2$ Whitney topology.

Consider such a generic mapping Φ. Then $\Phi^{-1}(L_1^r)$ is a submanifold of $\mathbb{R}^n$ of codimension $r(r+1)$. At any point x in this submanifold the rank of $A(x)$ is $n-r$ or $n-r-1$. But the set of points x where this rank is $n-r$ is open in $\Phi^{-1}(L_1^r)$ which proves that $\mathcal{ZI}_r = \mathcal{Z}_r \bigcap \Phi^{-1}(L_1^r)$ is a submanifold of $\mathbb{R}^n$ of codimension $r(r+1)$.

■

Notice that generically the singular locus is equal to the codimension 1 submanifold $\mathcal{Z}_1$ if $n = 2, 3$. However the set $\mathcal{ZI}_1$ of points of $\mathcal{Z}_1$ where $f(x) \in Im(A(x))$ is generically not empty even if $n = 2, 3$: it is made of isolated points if $n = 2$ and it is a submanifold of dimension 1 if $n = 3$.

4 Analysis of the sets Γ

4.1 Critique of the Filippov-like definition

Let us consider a generic system and let p be a point in $\mathcal{Z}_1$. In order to determine the Filippov-like set $\Gamma_f(p)$, we first compute $\widetilde{f}$ and δ in a neighborhood of p. For this purpose we consider a point x outside of $\mathcal{Z}$ and it orthogonal projection $\pi(x)$ onto $\mathcal{Z}_1$ (the points x are assumed to belong to a tubular neighborhood of $\mathcal{Z}_1$ around p for the orthogonal projection to exist). Since $\delta = \det(A)$ is a submersion on $\mathcal{Z}_1$ its gradient does not vanish and is orthogonal to $T_q\mathcal{Z}_1$ at all points $q \in \mathcal{Z}_1$. In what follows $\overrightarrow{a}(x)$ stands for $\overrightarrow{\mathrm{grad}\delta}(\pi(x))$ and $a(x)$ for $\|\overrightarrow{a}(x)\|$. To begin with we have:

$$\widetilde{f}(x) = \widetilde{f}(\pi(x)) + D\widetilde{f}(\pi(x)).(x - \pi(x)) + O(\|x - \pi(x)\|^2),$$

where $D\widetilde{f}$ is the differential of $\widetilde{f}$. On the one hand $x - \pi(x)$ is orthogonal to $\mathcal{Z}_1$ at $\pi(x)$ hence equal to $\alpha\overrightarrow{a}(x)$ for some real number α. On the other hand there exists a unique unitary vector ξ in $T_p\mathcal{Z}_1$ such that $\pi(x) - p = \beta\xi + O(\beta^2)$. Consequently:

$$\widetilde{f}(x) = \widetilde{f}(p) + \beta D\widetilde{f}(p).\xi + \alpha D\widetilde{f}(\pi(x)).\overrightarrow{a}(x) + O(\alpha^2) + O(\beta^2).$$

The variation of $D\widetilde{f}(\pi(x)).\overrightarrow{a}(x)$ from $D\widetilde{f}(p).\overrightarrow{a}(p)$ being of order $\beta + \alpha$ we obtain:

$$\widetilde{f}(x) = \widetilde{f}(p) + \beta D\widetilde{f}(p).\xi + \alpha D\widetilde{f}(p).\overrightarrow{a} + O(\alpha^2 + \beta^2),$$

where $\overrightarrow{a}$ stands for $\overrightarrow{a}(p)$. In the same way:

$$\begin{aligned}\delta(x) &= \delta(\pi(x)) + \alpha D\delta(\pi(x)).\overrightarrow{a}(x) + \alpha^2 D^{(2)}\delta(\pi(x)).(\overrightarrow{a}(x), \overrightarrow{a}(x)) + O(\alpha^3)\\ &= 0 + \alpha \left\langle \overrightarrow{\text{grad}}\delta(\pi(x)), \overrightarrow{a}(x) \right\rangle + \alpha^2 D^{(2)}\delta(\pi(x)).(\overrightarrow{a}(x), \overrightarrow{a}(x)) + O(\alpha^3)\\ &= \alpha a^2(x)\left(1 + \frac{\alpha}{a^2(x)} D^{(2)}\delta(\pi(x)).(\overrightarrow{a}(x), \overrightarrow{a}(x)) + O(\alpha^2)\right)\\ &= \alpha a^2(p)\left(1 + \alpha D^{(2)}\delta(p).(\overrightarrow{u}, \overrightarrow{u}) + O(\alpha^2 + \beta^2)\right),\end{aligned}$$

where $\overrightarrow{u}$ stands for $a(p)^{-1}\overrightarrow{a}(p)$. Finally we get

$$\frac{1}{\delta(x)} = \frac{1}{\alpha a^2}\left(1 - \alpha D^{(2)}\delta(p).(\overrightarrow{u}, \overrightarrow{u}) + O(\alpha^2 + \beta^2)\right),$$

and according to $F(x) = \dfrac{1}{\delta(x)}\widetilde{f}(x)$:

$$\begin{aligned}F(x) &= \frac{1}{\alpha a^2}\left(\widetilde{f}(p) + \beta D\widetilde{f}(p).\xi + \alpha D\widetilde{f}(p).\overrightarrow{a} - \alpha D^{(2)}\delta(p).(\overrightarrow{u}, \overrightarrow{u})\widetilde{f}(p) + O(\alpha^2 + \beta^2)\right)\\ &= \frac{1}{\alpha a^2}\widetilde{f}(p) + \frac{1}{a^2}\left(D\widetilde{f}(p).\overrightarrow{a} - D^{(2)}\delta(p).(\overrightarrow{u}, \overrightarrow{u})\widetilde{f}(p)\right)\\ &\qquad + \frac{\beta}{\alpha}\frac{1}{a^2}D\widetilde{f}(p).\xi + O(|\alpha| + |\beta|). \qquad (2)\end{aligned}$$

Consider a point y outside of $\mathcal{Z}$ equal to $-\beta\xi - \alpha\overrightarrow{a}$ at first order. Then

$$\frac{1}{2}(F(x) + F(y)) = \frac{1}{a^2}\left(D\widetilde{f}(p).\overrightarrow{a} - D^{(2)}\delta(p).(\overrightarrow{u}, \overrightarrow{u})\widetilde{f}(p)\right) + \frac{\beta}{\alpha}\frac{1}{a^2}D\widetilde{f}(p).\xi + O(|\alpha| + |\beta|).$$

Since the ratio $\dfrac{\beta}{\alpha}$ can take any real value and ξ is an arbitrary unitary vector in $T_p\mathcal{Z}$ there exists some vector v such that $v + D\widetilde{f}(p)(T_p\mathcal{Z})$ be included in $\Gamma_f(p)$.

We can state:

Proposition 2 *Let p be a point in the stratum $\mathcal{Z}_1$ of a generic DDS. Then $\Gamma_f(p)$ contains an affine space whose direction is $\mathbb{R}\widetilde{f}(p) + D\widetilde{f}(p)(T_p\mathcal{Z}_1)$.*

It is shown in the next proposition that the dimension of $D\widetilde{f}(p)(T_p\mathcal{Z}_1)$ is generically equal to $n-1$. If moreover p belongs to $\mathcal{Z}_1 \setminus \mathcal{ZI}_1$, then $f(p)$ does not belong to the image of $A(p)$, and according to Proposition 1 the vector field $\widetilde{f}$ does not vanish at p. Consequently $\Gamma_f(p) = \mathbb{R}^n$ as soon as $\widetilde{f}(p)$ does not belong to $D\widetilde{f}(p)(T_p\mathcal{Z}_1)$.

Proposition 3 *For a generic point $p \in \mathcal{Z}_1 \setminus \mathcal{ZI}_1$ of a generic system, the dimension of the set $D\widetilde{f}(p)(T_p\mathcal{Z})$ is $n-1$ and $\Gamma_f(p) = \mathbb{R}^n$.*

Sketch of the proof. Since rank $A(p) = n-1$ there exist coordinates such that $A(p) = \begin{pmatrix} 0 & l(p) \\ 0 & \Delta(p) \end{pmatrix}$ where $\Delta(p)$ is an invertible $(n-1)\times(n-1)$ matrix, and there exists a neighborhood V of p where $A(x) = \begin{pmatrix} a(x) & l(x) \\ c(x) & \Delta(x) \end{pmatrix}$ with $\Delta(x)$ invertible. Let $\xi = \begin{pmatrix} \alpha \\ \Lambda \end{pmatrix}$ be a nonzero vector, where

$\alpha \in \mathbb{R}$ and $\Lambda \in \mathbb{R}^{n-1}$. For $x \in V$, $A(x)\xi = 0$ is not possible if $\alpha = 0$ and we can assume that $\alpha = 1$.

Now $A(x)\xi = 0$ is equivalent to $a(x) + l(x)\Lambda = 0$ and $c(x) + \Delta(x)\Lambda = 0$. The second equality implies $\Lambda = -\Delta^{-1}(x)c(x)$, and the first one becomes $a(x) - l(x)\Delta^{-1}(x)c(x) = 0$. If x belongs to $\mathcal{Z}_1$ this equality must be satisfied, so that:

$$\forall x \in \mathcal{Z}_1 \bigcap V, \qquad \ker A(x) = \mathbb{R}\Phi(x) \quad \text{where} \quad \Phi(x) = \begin{pmatrix} 1 \\ -\Delta^{-1}(x)c(x) \end{pmatrix}.$$

It is clear that the set of points of $\mathcal{Z}_1 \bigcap V$ where the rank of $D\Phi(x)$ is $n-1$ is generically open and dense, and that we can assume without lost of generality that p belongs to this set.

But for all $x \in \mathcal{Z}_1$ the vector $\widetilde{f}(x)$ belongs to $\ker A(x)$ and there exists a function θ defined in $\mathcal{Z}_1 \bigcap V$ such that $\widetilde{f}(x) = \theta(x)\Phi(x)$. To finish $\theta(p) \neq 0$ because $p \notin \mathcal{Z}\mathcal{I}_1$ and the rank of the restriction of $D\widetilde{f}(p) = D\theta(p)\Phi(p) + \theta(p)D\Phi(p)$ to $T_p\mathcal{Z}_1$ is $n-1$.

Moreover the vector $\widetilde{f}(p)$ is generically not in the image of $T_p\mathcal{Z}_1$ by $D\widetilde{f}(p)$ and according to Formula (2) the set $\Gamma_f(p)$ is equal to $\mathbb{R}^n$.

■

Before trying to state a better definition of the sets Γ let us consider again a generic system and a point p in $\mathcal{Z}_1 \setminus \mathcal{Z}\mathcal{I}_1$. Since the vector field $\widetilde{f}$ does not vanish at p, it holds for x in a neighborhood of p, but outside of $\mathcal{Z}$:

$$\frac{F(x)}{\|F(x)\|} = \frac{\widetilde{f}(x)}{\left\|\widetilde{f}(x)\right\|} \xrightarrow[x \to p]{} \frac{\widetilde{f}(p)}{\left\|\widetilde{f}(p)\right\|}.$$

It seems therefore natural that the set $\Gamma(p)$ be a one-dimensional affine space of direction $\mathbb{R}\widetilde{f}(p)$.

4.2 Our definition of the sets Γ

To overcome the drawbacks raised in the previous subsection we propose herein a construction of the sets $\Gamma(p)$ that takes into account the vectors $F(q)$ for q in the orthogonal complement to $T_p\mathcal{Z}_r$ only. In order to consider such orthogonal complements in nongeneric cases, it is necessary that the singular locus be locally stratified, which is always the case of generic DDSs according to Theorem 1, but not only see for instance Section 6. We keep otherwise the Filippov-like definition. This leads to:

Definition 4 *Let p be a point in $\mathcal{Z}$.*

1. *If the set $\mathcal{Z}$ can be locally stratified around p in the disjoint union of the codimension k submanifolds $\mathcal{Z}_k$ and p belongs to Z_{k_0}, let N_p be the orthogonal complement to $T_p\mathcal{Z}_{k_0}$ in $\mathbb{R}^n$. Then:*

$$\Gamma(p) = \bigcap_{r>0} \overline{co}\ \{F(x) = A(x)^{-1}f(x); \quad x \in \left(B(p,r) \bigcap N_p\right) \setminus \mathcal{Z}\}.$$

2. *The set $\Gamma(p)$ is otherwise defined as the Filippov-like set:*

$$\Gamma(p) = \Gamma_f(p) = \bigcap_{r>0} \overline{co}\ \{F(x) = A(x)^{-1}f(x); \quad x \in B(p,r) \setminus \mathcal{Z}\}.$$

Remark. This definition makes use of the orthogonal complement to $T_p\mathcal{Z}$. Consequently it depends on the Euclidean structure, or more generally on the Riemannian structure, with which $\mathbb{R}^n$ is endowed.

Consider a generic system and a point p in $\mathcal{Z}_1$. The set N_p is then the line generated by $\overrightarrow{a} = \overrightarrow{\text{grad}}\delta(p)$. Let $x = p + \alpha\overrightarrow{a}$ be a point in this line. According to the computation of the previous section, but with $\pi(x) = p$, we get:

$$F(x) = \frac{1}{\alpha a^2}\widetilde{f}(p) + \frac{1}{a^2}\left(D\widetilde{f}(p).\overrightarrow{a} - D^{(2)}\delta(p).(\overrightarrow{u},\overrightarrow{u})\widetilde{f}(p)\right) + O(|\alpha|) \tag{3}$$

The set $\Gamma(p)$ is therefore equal to $\xi + \mathbb{R}\widetilde{f}(p)$ where ξ stands for $\frac{1}{a^2}D\widetilde{f}(p).\overrightarrow{a}$. The vector ξ need not be orthogonal to $\widetilde{f}(p)$ and could be replaced by the vector ξ_0 of minimal norm in $\Gamma(p)$ which is indeed orthogonal to $\widetilde{f}(p)$ so that:

$$\Gamma(p) = \xi_0 + \mathbb{R}\widetilde{f}(p) \qquad \text{where} \quad \xi_0 \perp \widetilde{f}(p).$$

Remarks.

1. It is clear that a modification of the Euclidean structure would not change the direction of $\Gamma(p)$, given by $\widetilde{f}(p)$, but would modify the vector ξ (or ξ_0). This must be taken into account if a change of coordinates is made.
2. The analysis and conclusions of Examples 1 to 3 are not modified while replacing the Filippov-like sets by the ones of Definition 4.

In order to understand the behaviour of the dynamics inside the singular set it is necessary to determine the elements of $\Gamma(p)$ that belong to $T_p\mathcal{Z}_1$. Let $v = \frac{1}{a^2}D\widetilde{f}(p)(\overrightarrow{a}) + c\widetilde{f}(p)$ be a vector in $\Gamma(p)$. Then

$$v \in T_p\mathcal{Z}_1 \Longleftrightarrow v \perp \overrightarrow{\text{grad}}\delta(p) \Longleftrightarrow \left\langle \frac{1}{a^2}D\widetilde{f}(p).\overrightarrow{a} + c\widetilde{f}(p), \overrightarrow{a} \right\rangle = 0$$

We have consequently two different cases:

1. If $\widetilde{f}(p)$ does not belong to $T_p\mathcal{Z}_1$ then there is an unique element in $T_p\mathcal{Z}_1 \bigcap \Gamma(p)$, it is obtained for $c = -\dfrac{\left\langle D\widetilde{f}(p).\overrightarrow{a}, \overrightarrow{a} \right\rangle}{a^2\left\langle \widetilde{f}(p), \overrightarrow{a} \right\rangle}$.
2. If $\widetilde{f}(p)$ belongs to $T_p\mathcal{Z}_1$, in particular if $\widetilde{f}(p) = 0$, the intersection $T_p\mathcal{Z}_1 \bigcap \Gamma(p)$ is empty except if $D\widetilde{f}(p).\overrightarrow{a} \perp \overrightarrow{a}$. In this last case $\Gamma(p) \subset T_p\mathcal{Z}_1$.

This value of c allows to define a smooth vector field in the open subset of $\mathcal{Z}_1$ where $\widetilde{f}(p)$ does not belong to $T_p\mathcal{Z}_1$. This vector field is defined as:

$$v(p) = \frac{1}{a^2}D\widetilde{f}(p).\overrightarrow{a} - \frac{\left\langle D\widetilde{f}(p).\overrightarrow{a}, \overrightarrow{a} \right\rangle}{a^2\left\langle \widetilde{f}(p), \overrightarrow{a} \right\rangle}\widetilde{f}(p). \tag{4}$$

We can state:

Theorem 2 *For a generic DDS holds:*

1. *At all points $p \in \mathcal{Z}_1$ the set $\Gamma(p)$ is $\frac{1}{a^2} d\widetilde{f}(p).\overrightarrow{a} + \mathbb{R}\widetilde{f}(p)$.*
2. *In the open subset of $\mathcal{Z}_1$ where $\widetilde{f}(p)$ does not belong to $T_p\mathcal{Z}_1$ the intersection of $\Gamma(p)$ with $T_p\mathcal{Z}_1$ is reduced to the smooth vector field v defined by (4).*
3. *All trajectories of v are generalized solutions to the DDS.*

It appears in the proof of Proposition 3 that the dimension of $\Gamma_f(p)$ at a generic point $p \in \mathcal{Z}_1$ depends on the variation of the direction of $\widetilde{f}(z)$, hence on the variation of $\ker A(z)$, when the point z moves in a neighborhood of p inside $\mathcal{Z}$. This is confirmed by the following proposition.

Proposition 4 *Let p be a point in the stratum $\mathcal{Z}_1$ of a generic system such that $\widetilde{f}(p) \neq 0$. If there exists a neighborhood W of p in $\mathcal{Z}_1$ such that the kernel of $A(z)$ does not depend on the point $z \in W$, then $\Gamma_f(p) = \Gamma(p)$.*

Proof. Since $\widetilde{f}$ is contained in the kernel of A at all points of $\mathcal{Z}_1$ there exists a smooth function θ defined on W such that

$$\forall z \in W, \quad \widetilde{f}(x) = \theta(z)\widetilde{f}(p).$$

The differential of $\widetilde{f}$ applied to a vector $\xi \in T_p\mathcal{Z}_1$ is therefore $D\widetilde{f}(p).\xi = (D\theta(p).\xi)\widetilde{f}(p)$ and is directed by $\widetilde{f}(p)$ for all ξ. Consequently Formula (2) becomes:

$$F(x) = \frac{1}{\alpha a^2}\widetilde{f}(p) + \frac{1}{a^2}\left(D\widetilde{f}(p).\overrightarrow{a} - D^{(2)}\delta(p).(\overrightarrow{u},\overrightarrow{u})\widetilde{f}(p)\right) + \frac{\beta}{\alpha}\frac{1}{a^2}(D\theta(p).\xi)\widetilde{f}(p) + O(|\alpha| + |\beta|).$$

From this it is clear that:

$$\Gamma_f(p) = \frac{1}{a^2}D\widetilde{f}(p).\overrightarrow{a} + \mathbb{R}\widetilde{f}(p) = \Gamma(p).$$

■

An example that illustrates all the conclusions of this section is postponed to Section 7. In particular the difference between the sets Γ and Γ_f, and the difference in the behaviour of the system inside $\mathcal{Z}$ that it implies, appear clearly.

5 Computation of the solutions by reparametrization

In this section we make use of the universal vector field $\widetilde{f}$ to compute the solutions to (E) by reparametrization. Recall that

$$\forall x \in \mathbb{R}^n \quad \widetilde{f}(x) = \widetilde{A}(x)f(x), \quad \text{and} \quad \widetilde{f}(x) = \delta(x)F(x) \quad \text{outside of} \quad \mathcal{Z}.$$

The vector field $\widetilde{f}$ is smooth and has well-defined trajectories everywhere in $\mathbb{R}^n$.

Important remark. *The singular locus $\mathcal{Z}$ splits the state space into domains where the sign of δ is constant.*

It is assumed in what follows that $\mathcal{Z}_1$ is a codimension 1 submanifold, which is in particular the case when the system is generic. Locally $\mathcal{Z}_1$ splits the space in two disjoint open subsets and the sign of $\delta = \det(A)$ is constant on each of these sets. In the generic case the sign of δ changes when crossing $\mathcal{Z}_1$ but the case where it is the same on both sides is also of interest because it provides trajectories that cross the singular locus.

5.1 The generic case

Let $p \in \mathcal{Z}_1$ and let us assume that $\widetilde{f}(p)$ does not vanish and points outside of $\mathcal{Z}$ toward a region where $\delta > 0$. Let $\widetilde{x}(s)$ be the solution to $\dfrac{d\widetilde{x}}{ds} = \widetilde{f}(\widetilde{x})$ that satisfies $\widetilde{x}(0) = p$ and let $\theta(s)$ be defined by

$$\theta(s) = \int_0^s \delta(\widetilde{x}(\tau))\, d\tau.$$

Since $\{\delta(\widetilde{x}(s)) > 0\}$, for $s > 0$ and as long as $\widetilde{x}(s)$ does not meet $\mathcal{Z}$, the function θ is strictly increasing and can be inverted. Let $x(t) = \widetilde{x}(\theta^{-1}(t))$. It is defined on an interval $[0, \beta[$ and satisfies for $t > 0$:

$$\frac{dx}{dt}(t) = \frac{d\widetilde{x}}{ds}(\theta^{-1}(t)) \left(\frac{d\theta}{ds}(\theta^{-1}(t))\right)^{-1} = \widetilde{f}(\widetilde{x}(\theta^{-1}(t))) \frac{1}{\delta(\widetilde{x}(\theta^{-1}(t))} = \widetilde{f}(x(t)) \frac{1}{\delta(x(t))} = F(x(t)).$$

In the same way solutions of the DDS that enter or leave the singular locus, according to the sign of δ and the direction of $\widetilde{f}(p)$, can be obtained by reparametrization of the trajectories of $\widetilde{f}$.

Remark. The singularities of an ordinary differential equation $\dot{x} = v(x)$ are the stationnary points p where $v(p) = 0$. They can be reached in infinite time only, forward or backward. On the contrary the points of a DDS that belong to $\mathcal{Z}$ are singularities that can be reached in finite time. The following simplification of Example 3 illustrates this phenomenon.

Example 4 *Consider in* $\mathbb{R}^2$

$$\begin{cases} \dot{x} & = 0 \\ 2y\dot{y} & = -1 \end{cases}$$

The set $\Gamma(p)$ *at a point* $p = (a, 0)$ *in* $\mathcal{Z} = \{y = 0\}$ *is* $\{0\} \times \mathbb{R}$ *and the only vector* $\Gamma(p) \bigcap T_p\mathcal{Z}$ *is the null one.*

This system can be viewed as the equation with discontinuous right-hand side $\dot{q} = F(q)$ *where the vector field* F *is defined by* $F(x, y) = \dfrac{-1}{2y} \dfrac{\partial}{\partial y}$ *if* $y \neq 0$ *and* $F(x, 0) = 0$.

The trajectory issued from a point (x_0, y_0) *outside of* $\mathcal{Z}$ *reaches* $\mathcal{Z}$ *in finite time and then remains at the point* $(x_0, 0)$.

5.2 An important nongeneric case

It is the case where despite the fact that $\mathcal{Z}$ is locally a codimension one submanifold, the sign of δ is the same on both sides of $\mathcal{Z}$. This situation arises typically when locally $\mathcal{Z} = \mathcal{Z}_1$ is a codimension one submanifold but $\delta(x) = \det A(x)$ is the square of some function. Consider for instance in $\mathbb{R}^2$:

$$(NG) \qquad \begin{pmatrix} 1 & 0 \\ 0 & y^2 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a(x, y) \\ b(x, y) \end{pmatrix}$$

which is nothing but a generalization of Exemple 2.

We consider in the present subsection an open and connected subset V of $\mathbb{R}^n$ such that, locally in $V \bigcap \mathcal{Z}_1$, the sign of δ is the same on both sides of $\mathcal{Z}_1$. Since the singular locus splits the state space into domains where the sign of δ does not change, this remains true as long as $\mathcal{Z}_1$ does not meet another part of $\mathcal{Z}$.

The vector field $\widetilde{f}$ may point towards one of the two domains delimited by $V \bigcap \mathcal{Z}_1$ or may be parallel to $\mathcal{Z}_1$. The set of this last kind of points is closed and splits $V \bigcap \mathcal{Z}_1$ into domains where $\widetilde{f}$ point towards the same side.

It holds for such a setting:

Proposition 5 *The sets $\Gamma(p)$ and $\Gamma_f(p)$ are empty at all points $p \in V \bigcap \mathcal{Z}_1$ where $\widetilde{f}(p)$ does not vanish.*

If $\widetilde{f}(p)$ is not parallel to $\mathcal{Z}_1$ then there exists a trajectory of the DDS that crosses $\mathcal{Z}_1$ at p.

Proof. Since $\Gamma(p)$ is always included in $\Gamma_f(p)$ it is enough to prove that $\Gamma_f(p)$ is empty.

Let u stand for the unitary vector $\frac{\widetilde{f}(p)}{\left\|\widetilde{f}(p)\right\|}$ and recall that

$$\frac{F(x)}{\|F(x)\|} = \frac{\widetilde{f}(x)}{\left\|\widetilde{f}(x)\right\|} \xrightarrow[x \to p]{} u \qquad \text{for} \quad x \in V \bigcap \mathcal{Z}_1 \setminus \mathcal{Z}.$$

There exists $r > 0$ such that $\left\| \frac{\widetilde{f}(x)}{\left\|\widetilde{f}(x)\right\|} - u \right\| < \frac{1}{2}$, for all x in $B(p,r)$. If moreover $x \notin \mathcal{Z}$ it holds $\|F(x) - \|F(x)\| u\| < \frac{1}{2} \|F(x)\|$ and:

$$\|F(x) - \|F(x)\| u\|^2 = \|F(x)\|^2 - 2 \|F(x)\| \langle F(x), u \rangle + \|F(x)\|^2 < \frac{1}{4} \|F(x)\|^2 ,$$

which entails $\langle F(x), u \rangle > \frac{7}{8} \|F(x)\|^2$. But $\lim\limits_{x \mapsto p} \|F(x)\| = +\infty$ because $\widetilde{f}(p) \neq 0$, and finally for any $M > 0$ there exists $r > 0$ such that $\langle F(x), u \rangle > M$ for all $x \in B(p,r) \setminus \mathcal{Z}$. This shows that the set $\Gamma_f(p)$ is included in the half-space $\{\xi; \ \langle \xi, u \rangle > M\}$ for any $M > 0$, hence that it is empty.

If $\widetilde{f}(p)$ is not parallel to $\mathcal{Z}$ it is clear that by reparametrization we can obtain a solution that reaches p in finite time, say at $t = 0$, and another one starting from p at $t = 0$ that leaves $\mathcal{Z}$ in finite time. The concatenation of these two solutions provides an absolutely continuous trajectory through p.

■

6 Application to the models of [18]

The paper [18] mainly deals with equations in $\mathbb{R}^2$ of the form:

$$\phi(x,y) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_1(x,y) \\ a_2(x,y) \end{pmatrix},$$

where the function ϕ and the vector field defined by a_1 and a_2 are smooth. In other words it is the particular case where the field of matrices A is:

$$A(x,y) = \begin{pmatrix} 0 & \phi(x,y) \\ -\phi(x,y) & 0 \end{pmatrix}.$$

We can set $f = -a_2$ and $g = a_1$ to obtain the form

$$(E) \qquad \begin{cases} \phi(x,y)\dot{x} &= f(x,y) \\ \phi(x,y)\dot{y} &= g(x,y) \end{cases}$$

In this setting the notion of genericity has a different meaning: as well as in [18] we will say that the system is generic if ϕ is a Morse function (see for instance [12] for this notion).

Let p be a point in $\mathcal{Z}$. Up to a change of coordinates we can assume that p is the point $(0,0)$ and that ϕ being a Morse function has locally one of the three following forms:

1. $\phi(x,y) = y\psi(x,y)$ with $\psi(x,0) \neq 0$. In other words the singular locus is the line $\{y=0\}$. It is possible to divide f and g by ψ and, writing f for f/ψ and g for g/ψ, we get the equation:
$$(E_1) \qquad \begin{cases} y\dot{x} &= f(x,y) \\ y\dot{y} &= g(x,y) \end{cases}$$

2. $\phi(x,y) = (x^2+y^2)\psi(x,y)$ with $\psi(0,0) \neq 0$. Similarly we get the model:
$$(E_2) \qquad \begin{cases} (x^2+y^2)\dot{x} &= f(x,y) \\ (x^2+y^2)\dot{y} &= g(x,y) \end{cases}$$

3. $\phi(x,y) = (x^2-y^2)\psi(x,y)$ with $\psi(0,0) \neq 0$. Thanks to a second change of variable we can write $\phi(x,y) = xy\psi(x,y)$ and obtain:
$$(E_3) \qquad \begin{cases} xy\dot{x} &= f(x,y) \\ xy\dot{y} &= g(x,y) \end{cases}$$

Notice that the singular locus is (locally) a line in the first case, an isolated point in the second one and the union of two intersecting lines in the third one. For general 2-dimensional DDSs the last two cases are not generic. Moreover the singular locus is in the three cases equal to $\mathcal{Z}_2$. It is however possible to compute the trajectories by replacing δ by y (resp. by x^2+y^2, by xy) in the reparametrization.

Let v be the vector field defined by $v(x,y) = (f(x,y), g(x,y))^T$. Its trajectories are denoted by $\widetilde{x}(s)$ for s in some interval $(-\alpha, \beta)$.

1. *First case.* Let F be defined by $\frac{1}{y}v$ outside of $\mathcal{Z}$ as for general DDSs, let p$\in \mathcal{Z}$ and let us assume that $v(p)$ points outside of $\mathcal{Z}$. With the same kind of computations as in Section 5 we get a trajectory that enters $\mathcal{Z}$ if $v(p)$ points to $\{y<0\}$ and a trajectory that leaves $\mathcal{Z}$ if $v(p)$ points to $\{y>0\}$.

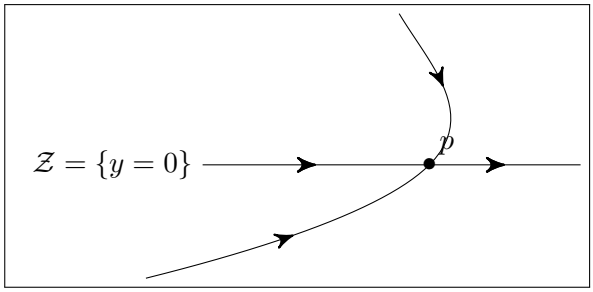

Figure 5: First case, $v(p) \neq 0$

It is clear that a formula similar to Formula (3) holds and that as well as in Section 4.2 or Example 3 trajectories may live in $\mathcal{Z}$. However the direction of the vector field induced

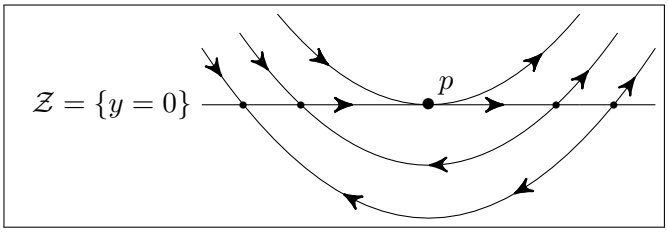

Figure 6: First case. $v(p) = 0$

in $\mathcal{Z}$ depends on the differential of the vector field v and cannot be made more precise in general.

It may happen that $v(p)$ be parallel to $\mathcal{Z}$. We will only discuss here the case where this happens at an isolated point, because it would not be generic that v be parallel to $\mathcal{Z}$ at all points of a segment of $\mathcal{Z}$. Consider a trajectory $\widetilde{x}$ of v that verifies $\widetilde{x}(0) = p$. Except for $s = 0$ the point $\widetilde{x}(s)$ does not belong to $\mathcal{Z}$ and can be reparametrized into solutions to (E_1) on $(-\alpha, 0]$ and $[0, \beta)$ for some $\alpha, \beta > 0$. The solution $x(t)$ of (E_1) obtained in this way may remain in one of the half-planes $\{y < 0\}$ and $\{y > 0\}$ or cross the singular locus.

2. *Second case.* Let us assume that v does not vanish at $(0, 0)$, let $\theta(s) = \displaystyle\int_0^s \|\widetilde{x}(\tau)\|^2 \, d\tau$, and let $x(t) = \widetilde{x}(\theta^{-1}(t))$. Then $x(t)$ is a trajectory through the singular point $(0, 0)$. Notice that the set $\Gamma(0, 0)$ is empty. This is due to the fact that the vector field v does not vanish at $(0, 0)$ and is divided by $x^2 + y^2 > 0$. The trajectory through the point $(0, 0)$ is consequently a generalized solution, not a classical one.

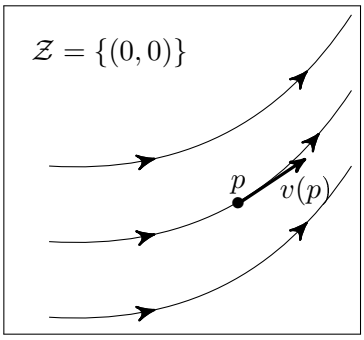

Figure 7: Second case. $v(p) \neq 0$

Though it is not generic the case where $v(0, 0) = 0$ is of interest. We assume of course that $v(x, y)$ cannot be divided by $x^2 + y^2$, it is for example the case if the real parts of the eigenvalues of $DV(0, 0)$ are both negative. The eigenvalues are then both smaller than $-b$ for some $b > 0$ and there exists a neighborhood W of $(0, 0)$ such that any trajectory $\widetilde{x}$ of v starting in W satisfies $\|\widetilde{x}(s)\| \leq \|\widetilde{x}(0)\| \, e^{-bs}$.

Let $\theta(s) = \int_0^s \|\widetilde{x}(\tau)\|^2 \, d\tau \leq \|\widetilde{x}(0)\| \int_0^s e^{-2b\tau} \, d\tau$. Then $\theta(s)$ has a limit smaller than $\frac{1}{2b} \|\widetilde{x}(0)\|$ as $s \mapsto +\infty$, which shows that the trajectory $x(t) = \widetilde{x}(\theta^{-1}(t))$ reaches the point $(0, 0)$ in finite time.

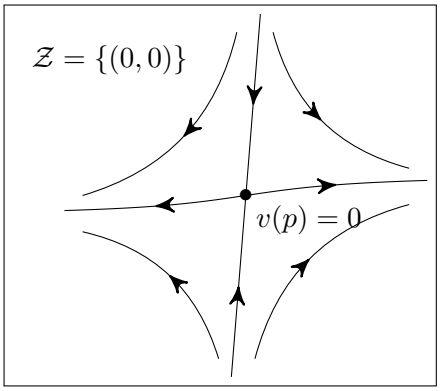

Figure 8: Case 2. Opposite eigenvalues

The set $\Gamma(0,0) = \Gamma_f(0,0)$ is here equal to $\mathbb{R}^2$ and the trajectories that reach the origin in finite time can be extended to the constant solution, which is again a generalized solution.

The time reverse system is clearly not deterministic since a steady state solution equal to $(0,0)$ may leave this point at any time.

Let us have a look to the case where the real parts of the eigenvalues of $DV(0,0)$ have different signs. It is clear that here $\Gamma(0,0) = \mathbb{R}^2$ and that the steady state solution $(0,0)$ is available on any time interval. On the other hand only two trajectories of v enter the point $(0,0)$, in infinite time of course, and two other ones leaves this point. Finally there exist generalized solutions that enter the singular locus in finite time, may remain at $(0,0)$ for any duration and then leave the singular locus at any time. The DDS is not deterministic.

3. *Third case.* The singular locus splits locally $\mathbb{R}^2$ in four domains defined by the signs of x and y. We assume that $v(0,0)$ does not vanish and is neither parallel to $\{x = 0\}$ nor to $\{y = 0\}$.

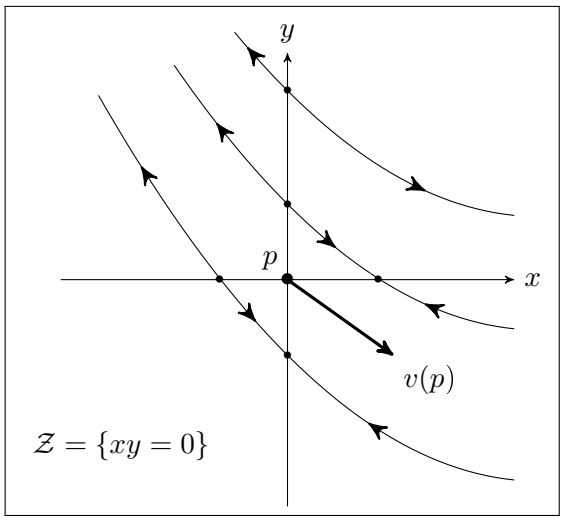

Figure 9: Case 3. $v(p)$ points outside of $\mathcal{Z}$

Consequently any point p of $\mathcal{Z} \setminus (0,0)$, that is any point of the form $(x,0)$, $x \neq 0$ or

$(0,y),\ y\neq 0$, is a point of the first kind where $v(p)$ points outside of $\mathcal{Z}$. This means that there are two trajectories that both enter or both leave the singular locus at p in finite time. There are also trajectories in $\mathcal{Z}$ that may go to the origin, or leave it, or that can be constant.

To finish, let us consider the following nongeneric case:

$$(E_4)\qquad \begin{cases} y^2\dot{x} &= f(x,y)\\ y^2\dot{y} &= g(x,y)\end{cases}$$

It has the interest to provide trajectories that cross the singular locus, at least at all points $p=(x,0)$ where $v(p)$ does not have the direction of the singular locus $\mathcal{Z}=\{y=0\}$.

7 A detailed example

Let in $\mathbb{R}^3$:

$$(E)\qquad \begin{pmatrix}1&0&x\\0&1&y\\0&0&z\end{pmatrix}\begin{pmatrix}\dot{x}\\\dot{y}\\\dot{z}\end{pmatrix}=\begin{pmatrix}\phi(x,y,z)\\\psi(x,y,z)\\1+z\end{pmatrix}.$$

The data of this system are $\mathcal{Z}=\mathcal{Z}_1=\{z=0\}$ because $\delta(x,y,z)=z$, and:

$$\widetilde{A}(x,y,z)=\begin{pmatrix}z&0&-x\\0&z&-y\\0&0&1\end{pmatrix}\quad\text{and}\quad \widetilde{f}(x,y,z)=\begin{pmatrix}z\phi(x,y,z)-x(1+z)\\z\psi(x,y,z)-y(1+z)\\1+z\end{pmatrix}.$$

At points where $z\neq 0$ the vector field F is :

$$F(x,y,z)=\xi(x,y,z)+\frac{1}{z}\begin{pmatrix}-x\\-y\\1\end{pmatrix}\quad\text{where}\quad \xi(x,y,z)=\begin{pmatrix}\phi(x,y,z)-x\\\psi(x,y,z)-y\\1\end{pmatrix}.$$

Let us fix a point $p=(x_0,y_0,0)$ in $\mathcal{Z}$ and let $u=(a,b,0)^T$ be a vector in $T_p\mathcal{Z}$. To simplify the notation we write $F(\beta,z)$ (resp. $\xi(\beta,z)$) for $F((x_0,y_0,z)+\beta u)$ (resp. $\xi((x_0,y_0,z)+\beta u)$). Then:

$$\begin{aligned}\tfrac{1}{2}\left(F(\beta,z)+F(-\beta,-z)\right) &= \tfrac{1}{2}(\xi(\beta,z)+\xi(-\beta,-z))+\tfrac{1}{2z}\begin{pmatrix}-x_0-\beta a\\-y_0-\beta b\\1\end{pmatrix}+\tfrac{1}{-2z}\begin{pmatrix}-x_0+\beta a\\-y_0+\beta b\\1\end{pmatrix}\\ &= \tfrac{1}{2}(\xi(\beta,z)+\xi(-\beta,-z))-\tfrac{\beta}{z}u.\end{aligned}$$

Since $\frac{\beta}{z}$ takes all real values in any neighborhood of p and since the vector $u\in T_p\mathcal{Z}$ is arbitrary, this implies that $\Gamma_f(p)$ contains the affine space $\xi(p)+T_p\mathcal{Z}$.

For $\beta=0$ the vector field F takes the value $F(0,z)=\xi(0,z)+\frac{1}{z}\begin{pmatrix}-x_0\\-y_0\\1\end{pmatrix}$ and taking into account its value for $-z$ we obtain that $\xi(p)+\mathbb{R}\begin{pmatrix}-x_0\\-y_0\\1\end{pmatrix}$ is also included in $\mathcal{Z}$.

The conclusion is that $\Gamma_f(p)$ is equal to $\mathbb{R}^3$ at all points $p \in \mathcal{Z}$. This means that according to the Filippov-like definition all almost continuous curves in $\mathcal{Z}$ would be generalized solutions to (E).

Let us now apply Definition 4: the variations of F around p are restricted to the points (x_0, y_0, z) and it is immediate that $\Gamma(p) = \xi(p) + \mathbb{R} \begin{pmatrix} -x_0 \\ -y_0 \\ 1 \end{pmatrix}$. The intersection of this set with $T_p\mathcal{Z}$ provides the vector field

$$v(x, y) = \begin{pmatrix} \phi(x, y, 0) \\ \psi(x, y, 0) \\ 0 \end{pmatrix} \qquad \text{within } \mathcal{Z}.$$

Unlike the sets Γ_f, the sets Γ provide an unique vector field in $\mathcal{Z}$, hence well-defined trajectories.

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