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# SWEEPOUTS OF CLOSED RIEMANNIAN MANIFOLDS

ALEXANDER NABUTOVSKY, REGINA ROTMAN, AND STÉPHANE SABOURAU

ABSTRACT. We show that for every closed Riemannian manifold there exists a continuous family of 1-cycles (defined as finite collections of disjoint closed curves) parametrized by a sphere and sweeping out the whole manifold so that the lengths of all connected closed curves are bounded in terms of the volume (or the diameter) and the dimension  $n$  of the manifold, when  $n \geq 3$ . An alternative form of this result involves a modification of Gromov’s definition of waist of sweepouts, where the space of parameters can be any finite polyhedron (and not necessarily a pseudomanifold). We demonstrate that the so-defined polyhedral 1-dimensional waist of a closed Riemannian manifold is equal to its filling radius up to at most a constant factor. We also establish upper bounds for the polyhedral 1-waist of some homology classes in terms of the volume or the diameter of the ambient manifold. In addition, we provide generalizations of these results for sweepouts by polyhedra of higher dimension using the homological filling functions. Finally, we demonstrate that the filling radius and the hypersphericity of a closed Riemannian manifold can be arbitrarily far apart.

## 1. INTRODUCTION

One can define a  $p$ -slicing of an  $n$ -dimensional manifold (or a pseudomanifold)  $N$  as a collection of inverse images  $h^{-1}(t)$  of points under a “nice” mapping  $h : N^n \rightarrow T^{n-p}$  to a lower dimensional (pseudo)manifold, where  $t$  runs over  $T^{n-p}$ . Here, the “niceness” of  $h$  must imply that all  $h^{-1}(t)$  are subpolyhedra of  $N^n$  of dimension  $\leq p$ . (Note, however, that even for a nice map  $h$ , the fibers  $h^{-1}(t)$  need not automatically form a continuous family of subpolyhedra under any reasonable choice of topology.) One can define a  $p$ -sweepout of  $M$  as the image of a slicing of a (pseudo)manifold  $N$  under a topologically nontrivial continuous map  $\varphi : N \rightarrow M$ . Here, one possible meaning of “topologically nontrivial” is that the image under  $\varphi$  of the fundamental homology class of  $N$  with  $\mathbb{Z}$  or  $\mathbb{Z}_2$  coefficients is the fundamental homology class of  $M$ . Obviously, a slicing is a sweepout (corresponding to the identity map of  $M$ ), but not vice versa. More generally, one can define a  $p$ -sweepout of a  $k$ -dimensional homology class  $a$  of  $M$  from a continuous map  $\varphi : N \rightarrow M$  with  $\varphi_*([N]) = a$ . Given two maps  $h : N \rightarrow T^{n-p}$  and  $\varphi : N \rightarrow M$  such that  $\varphi_*([N]) = [M]$  (respectively,  $\varphi_*([N])$  represents a prescribed lower dimensional class  $a$  of  $M$ ), taking the supremum over  $t \in N$  of the  $p$ -volumes of the fibers  $h^{-1}(t)$ , then taking the infimum over all maps  $h : N \rightarrow T^{n-p}$  and  $\varphi : N \rightarrow M$  as above, one arrives to the definition of the  $p$ -waist of  $M$  (respectively, of a homology class  $a$  of  $M$ ). This definition was introduced by Gromov; see [12, §15] and [11, §6].

Among other things we analyze a version of this concept when the space  $T^{n-p}$  is not required to be a pseudomanifold, but can be an arbitrary finite polyhedron. Consider a  $p$ -sweepout of  $M$  (or, more generally, of a homology class  $a$  of  $M$  with coefficients in  $G = \mathbb{Z}$  or  $\mathbb{Z}_2$ ) whose fibers  $h^{-1}(t)$  are  $p$ -dimensional cycles that continuously depend on  $t$ . We can regard  $\varphi(h^{-1}(t))$  as  $p$ -cycles of  $M$ . (If  $h^{-1}(t)$  is empty, then the 1-cycle is, by definition, zero.) Therefore, the maps  $h : N \rightarrow T^{n-p}$  and  $\varphi : N \rightarrow M$  induce a continuous map from the pseudomanifold  $T^{n-p}$  to the space  $\mathcal{Z}_p(M; G)$  of  $p$ -cycles of  $M$  with coefficients in  $G$ . This map sends the fundamental homology class of  $T^{n-p}$  to an  $(n - p)$ -dimensional homology class of  $\mathcal{Z}_p(M; G)$ . If this homology class is

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nontrivial, we can take the minimax value of the  $p$ -volume over  $p$ -cycles, where the supremum is taken over the space of all cycles in a sweepout, and the infimum over all sweepouts  $h : N \rightarrow T^{n-p}$  and  $\varphi : N \rightarrow M$  corresponding to the considered homology class of  $\mathcal{Z}_p(M; G)$ . The resulting quantities (called the *widths* of  $M$ ) were studied in many important papers. Properties of widths were crucial for a recent progress in geometric analysis including proofs of the Willmore conjecture by F. Marques and A. Neves, and the Yau conjecture by A. Song. If  $T^{n-p} = S^{n-p}$ , then the construction gives rise to an  $(n-p)$ -dimensional homotopy class of  $\mathcal{Z}_p(M; G)$ . As  $\pi_{n-p}(\mathcal{Z}_p(M; G))$  is isomorphic to  $H_n(M; G)$  (see [2]), we obtain a corresponding homology class of  $M$ . Looking at Almgren's proof in [2], it is easy to see that this is the same homology class that was used in the definition of the  $p$ -sweepout.

There are many different versions of  $p$ -waists in the literature that are referred to as width, waist, diastole, etc. One can consider all sweepouts, or only slicings, consider different notions of "topologically nontrivial", or impose different restrictions on the spaces of parameters parameterizing sweepouts. Also, one can measure the size of the fibers  $h^{-1}(t)$  not as their volume, but using a different functional, such as the diameter. For example, if we consider only slicings (so  $\varphi$  is the identity map) by arbitrary not necessarily polyhedral fibers  $h^{-1}(t)$  and measure their size using the diameter, we obtain the notion of *Urysohn width*. It was first proven by L. Guth, that if the codimension  $p$  equals 1, then the Urysohn width of a closed Riemannian manifold  $M$  can be majorized in terms of only the volume and the dimension of  $M$ ; see [14], [18], [23], [21], [27] for this and related results. In this paper, we will consider the case, when  $T^{n-p}$  is a pseudomanifold as in Gromov's definition, but the functional of interest is the maximal  $p$ -volume of a connected component of the fiber  $h^{-1}(t)$  (instead of the  $p$ -volume of the whole  $h^{-1}(t)$ ). Also, we are going to do this in the most interesting case, when  $T^{n-p} = S^{n-p}$ , and for maps  $h : N \rightarrow T^{n-p}$  such that  $h^{-1}(t)$  is a continuous family of  $p$ -cycles. Thus, the resulting notion could be regarded as a version of the widths of  $M$  (for a different functional).

There is a number of papers with interesting and important upper and lower bounds for various versions of waists/widths. Here are some of them for the most studied situation when we take the min-max of the volume over all slicings of a Riemannian manifold (slicing waists).

**Upper bounds.** The first bound in this direction applies to closed Riemannian surfaces and involves only the area and the genus of the surface; see [3]. In higher dimension, a similar bound in terms of the volume holds for closed Riemannian  $n$ -manifold  $M$  with nonnegative Ricci curvature, which leads to the existence of a closed minimal hypersurface (with a singular set of Hausdorff dimension at most  $n-8$ ) whose  $(n-1)$ -volume is bounded in terms of the volume of  $M$ ; see [5], [26]. Further estimates for closed Riemannian manifolds with Ricci curvature bounded below can be found in [5]. For closed 3-manifolds  $M^3$  with positive Ricci curvature, the bound in terms of the volume holds for the min-max length of the fibers of the maps from  $M$  to the plane; see [19]. Without any curvature condition, these results fail; see [24] and [23] for counterexamples.

**Lower bounds.** A lower bound for slicing waists of the form  $\text{const}_n \text{FillRad}(M^n)$  can be found in Appendix 1 of [11, §6], where  $\text{FillRad}$  is the filling radius; see Definition 3.1. In a different direction the exact lower bound for the volume of fibers of maps between round spheres of different dimensions is the content of Gromov's famous waist theorem; see [10]. There are many generalizations for other spaces; see [17] for a highly nontrivial generalization of Gromov's waist theorem for cubes and [1] for a survey on the subject.

**1.1. Homology 1-waist bounds.** In this paper, we consider two different notions of sweepout and waist similar but somewhat different to those introduced by Gromov in [11, §6]; see Definitions 1.1 and 1.10. With these notions, universal upper bounds on the waist of one-parameter

families of one-cycles sweeping out essential surfaces of closed Riemannian manifolds were obtained in [26]. Our first theorem extends this result to the waist of multi-parameters families of one-cycles sweeping out any closed Riemannian manifold.

Before stating the precise result, we need to introduce the following notion of  $p$ -waist.

**Definition 1.1.** Let  $M$  be a closed  $n$ -manifold. A polyhedral  $p$ -sweepout of  $M$  is a family

$$\varphi[h^{-1}(t)] \subseteq M$$

with  $t \in T$ , where  $h : N \rightarrow T$  is a continuous map from a closed  $n$ -pseudomanifold  $N$  to a finite  $(n-p)$ -dimensional polyhedron  $T$  such that all fibers  $h^{-1}(t)$  are  $p$ -subpolyhedra of  $N$ , and  $\varphi : N \rightarrow M$  is a continuous degree one map. That is,

$$\varphi_*([N]) = [M] \in H_n(M)$$

where the homology coefficients are in  $\mathbb{Z}$  if  $M$  is orientable, and in  $\mathbb{Z}_2$  otherwise. Define the *homology  $p$ -waist* of a closed Riemannian manifold  $M$  as

$$W_p(M) = \inf_{\varphi, h} \sup_{t \in T} \text{vol}_p(\varphi|_{h^{-1}(t)}) \quad (1.1)$$

where the infimum is taken over all  $n$ -pseudomanifolds  $N$ , all simplicial  $(n-p)$ -complexes  $T$ , and all maps  $\varphi : N \rightarrow M$  and  $h : N \rightarrow T$  defining a polyhedral  $p$ -sweepout of  $M$ . Here, the notation  $\text{vol}_p(\varphi|_{h^{-1}(t)})$  stands for the volume of the map  $\varphi$  restricted to the fiber  $h^{-1}(t) \subseteq N$ , and not merely the volume of its image (which might be smaller). If such sweepouts do not exist, we let  $W_p(M) = 0$ . When  $p = 1$ , we simply write  $W(M) = W_1(M)$ .

The only difference between this definition and Gromov's is that we do not require that  $T$  is a pseudomanifold. This distinction can be illustrated by the following example.

**Example 1.2.** Let  $M$  be a Riemannian 2-sphere that looks like a "thin" 2-dimensional three-legged starfish. We can choose  $T$  as a tripod, that is, the union of three closed intervals intersecting at a common endpoint, and polyhedral 1-sweepout of  $M$  by 1-cycles, most of which are very short closed curves running around individual tentacles. Only one of them, namely, the inverse image of the center of the tripod looks like the  $\theta$ -graph with two vertices and three edges connecting these vertices. (Each pair of these three edges forms a closed curve around one of the tentacles that appears as the limit of closed curve above inner points of the corresponding leg of tripod  $T$ .) Note that this polyhedral sweepout would not be allowed in Gromov's definition. Also, note that if one would consider the inverse images of a point of  $T$  as a function from  $T$  to the space of currents on the 2-sphere, this function will *not* be continuous. Indeed, the inverse image of the center of the tripod will be the  $\theta$ -graph that consists of three arcs. When we approach the center along each ray, the inverse images of points will consist of two arcs, and will converge to a subset of the  $\theta$ -graph that consists of two (out of three) arcs. Note, that this type of discontinuity would be impossible if  $h$  were a map to a manifold (*e.g.*, a sphere) such that for each  $t$ , the fiber  $h^{-1}(t)$  is a cycle.

With this notion of waist, we can prove the following homology 1-sweepout estimates.

**Theorem 1.3.** *Let  $M$  be a closed  $n$ -manifold. Then every Riemannian metric on  $M$  satisfies*

$$\begin{aligned} \text{FillRad}(M) &\geq c_n W(M) \\ \text{vol}(M) &\geq c'_n W(M)^n \\ \text{diam}(M) &\geq c''_n W(M) \end{aligned}$$

for some explicit positive constants  $c_n$ ,  $c'_n$  and  $c''_n$  depending only on  $n$ .

Here,  $\text{FillRad}(M)$  is the filling radius introduced by Gromov in [6]; see Definition 3.1 below. Gromov proved that  $\text{FillRad}(M^n) \leq \text{const}(n) \text{vol}(M)^{\frac{1}{n}}$ . Later, it was established in [21] that one can take  $\text{const}(n) = n$ . Also, M. Katz proved that  $\text{FillRad}(M) \leq \frac{1}{3} \text{diam}(M)$ ; see [16]. Thus, the last two inequalities follow from the first one. This result is geometrically appealing, and can be used to demonstrate that  $\text{FillRad}(M)$  is equal to  $W(M)$  up to at most a constant factor (see Theorem 1.13), thus leading to some geometric intuition about  $\text{FillRad}(M)$ .

The classical approach to obtain lower bounds on the filling radius is to argue by contradiction and construct a retraction, one simplex at a time, from a pseudomanifold  $P$  bounding  $M$  onto its boundary. However, such a construction is not always possible in general. A different path was taken in [25], where a retraction from a different filling  $Q$  was constructed by considering all the simplices lying in the 2-skeleton of  $Q$  at the same time (and not only one at a time) and by proceeding by induction on the dimension of the higher-dimensional skeleta of  $Q$  from there, using a topological assumption on the manifold. In the proof of Theorem 1.3, where the manifold  $M$  is arbitrary, we take yet a different approach. In particular, our construction does not proceed by induction on the skeleta of the filling. Instead, we construct a pseudomanifold  $N$  homologous to  $M$  and a map  $N \rightarrow M$  non-homologous to the identity map by considering all simplices of the filling at the same time without arguing by induction in order to derive a contradiction.

It would be interesting to know whether the one-cycle sweepout estimates of Theorem 1.3 hold for sweepouts made of pairwise disjoint one-cycles, that is, when the maps  $\varphi : N \rightarrow M$  in Definition 1.1 are required to be diffeomorphisms. In the case of surfaces, this would yield a positive answer to Bers' pants decomposition problem, which may or may not be true.

**1.2. Sweepouts and geometric measure theory.** From the point of view of geometric measure theory, one would prefer a situation where

- all inverse images  $h^{-1}(t)$  are 1-cycles on  $N$  (or, more precisely, finite collections of piecewise smooth closed curves on  $N$  so that their images  $\varphi(h^{-1}(t))$  are 1-cycles on  $M$ );
- $T = S^{n-1}$ ;
- the map  $S^{n-1} \rightarrow \mathcal{Z}_1(M; G)$  with  $G = \mathbb{Z}$  if  $M$  is orientable, and  $G = \mathbb{Z}_2$  otherwise, that sends every  $t \in T$  to  $\varphi(h^{-1}(t))$  is continuous with respect to the flat topology on  $\mathcal{Z}_1(M; G)$ .

To achieve this goal in the case when  $h^{-1}(t)$  is already a 1-cycle for each  $t$ , we can first replace the map  $h : N \rightarrow T$  with a continuous map  $\bar{h} : N \rightarrow S^{n-p}$  obtained as the composition of  $h : N \rightarrow T$  with a finite-to-one continuous map  $T \rightarrow S^{n-p}$ . In this case, the  $p$ -sweepouts are parameterized by  $S^{n-p}$  and the  $p$ -waist of  $M$  can be defined by minimizing the maximal volume of the map  $\varphi$  restricted to the connected components of the fibers of  $\bar{h}$ . (Note that we cannot hope to have a control over the cardinality of the inverse images of the many-to-one map to the sphere. Therefore, the best we can hope for is to control the volume (length) of the individual connected components.) That is,

$$\bar{W}_p(M) = \inf_{\varphi, \bar{h}} \sup_{t \in S^{n-p}} \max_{C \subseteq \bar{h}^{-1}(t)} \text{vol}_p(\varphi|_C) \quad (1.2)$$

where the infimum is taken over all maps  $\varphi : N \rightarrow M$  and  $\bar{h} : N \rightarrow S^{n-p}$  as above and the maximum is taken over all connected components  $C$  of  $\bar{h}^{-1}(t)$ .

Note that, vice versa, given a PL-map  $\bar{h} : N \rightarrow S^{n-p}$ , one can define the space  $T^{n-p}$  of connected components of all inverse images  $\bar{h}^{-1}(t)$  for every  $t \in S^{n-p}$ . This gives rise to a map  $h : N \rightarrow T^{n-p}$  sending every point  $x \in N$  to its connected component in  $\bar{h}^{-1}(\bar{h}(x))$ . There exists a map  $\psi : T^{n-p} \rightarrow S^{n-p}$  that sends each point of  $t \in T^{n-p}$  to the corresponding value of  $t$

under  $h$  such that  $\bar{h} = \psi \circ h$ . Obviously, the fibers of  $h$  in  $N$  are connected and coincide with the connected components of the fibers of  $\bar{h}$ . As we will see below, the map that sends each point  $t \in T$  to the 1-cycle  $\varphi(\bar{h}^{-1}(t))$  need not be continuous, so one will need some extra care in constructing the finite-to-one map  $\psi : T^{n-p} \rightarrow S^{n-p}$  to ensure the continuity of this map.

As pointed out before, the first step is to alter  $h : N \rightarrow T^{n-1}$  so that all  $h^{-1}(t)$  become 1-cycles. To illustrate our approach consider the following example

**Example 1.4.** Consider the three-legged starfish 2-sphere  $M = N$  mapped to the tripod  $T$  as described in Example 1.2. The inverse image of the centre  $c$  of the tripod is a collection of three arcs connecting two points on the sphere (a  $\theta$ -graph), which is not a cycle. The inverse images of the points on each ray of the tripod are closed curves hugging the legs of the sphere. As a point on a ray of the tripod approaches the center, its inverse image approaches the union of two of the three arcs forming the  $\theta$ -graph. Replace the three arcs in the inverse image of  $c$  by pairs of arcs with the same endpoints as the original arc, running very close to the original arc. Each pair forms a digon bounding a thin disk. The boundary of each disk can be contracted to a point inside the disk via concentric simple loops. This contraction corresponds to a map of each thin disk to a small interval such that the curves during the contracting homotopy are inverse images of the points of the small interval. Combining these three homotopies, we obtain a map of three thin disks to a small tripod. The inverse image of the center of this small tripod is a collection of six arcs. Now, note that these six arcs can be grouped into three pairs of arcs so that each pair forms a simple closed curve “hugging” one of three long legs of the three-legged star-fish. Each of these three simple closed curves can be also contracted to a point via concentric simple loops along the corresponding long leg of the three-legged starfish. Combining these three contracting homotopies, we obtain a map of the 2-sphere minus the three thin disks to another tripod. (This map will be very close to the original map of the whole 2-sphere to the tripod). The inverse image of the centre of the tripod under this new map is the same collection of six arcs (or, three petals). We can glue these two tripods into one hexapod by identifying their centers and define a map from the 2-sphere to the hexapod by combining the two maps defined on the union of the three thin disks and its complement to the tripods forming the hexapod. The maximal length of a fiber is (almost) twice the maximal length of a fiber in the original map to a tripod and every fiber now is a 1-cycle. (Observe that an elaboration of this idea can be used to enhance Theorem 1.3 by demanding that all fibers  $h^{-1}(t)$  are 1-cycles, if desired. This will follow from an argument used to prove our next theorem below.)

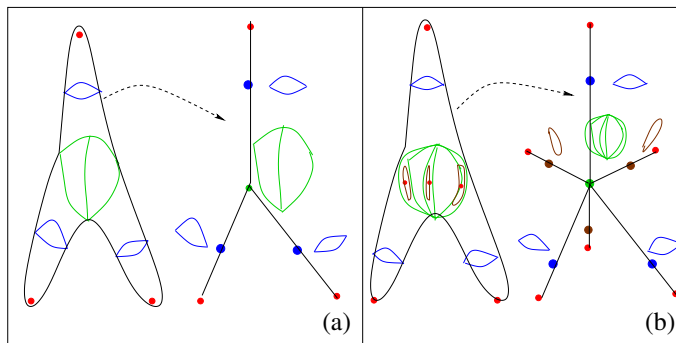


FIGURE 1. Sweeping out a three-legged sphere

In this construction, the inverse images of the points of the hexapod do not form a continuous family of 1-cycles. We observe exactly the same discontinuity near the center of the hexapod as

the discontinuity near the center of the tripod for the original map. Of course, this discontinuity disappears once we take a composition of the map to the hexapod  $H$  with an appropriate continuous finite-to-one map  $H \rightarrow S^1$ . Here is a description of such a finite-to-one map that will be used below. First, consider the following map from the hexapod to the interval  $[0, 1]$  where the center of the hexapod is mapped to  $\frac{1}{2}$ , each ray of the first tripod is (linearly) mapped to  $[\frac{1}{2}, 1]$ , and each ray of the second (small) tripod is mapped to  $[0, \frac{1}{2}]$ . Now, the inverse images of both endpoints of  $[0, 1]$  are finite collections of points, while the inverse image of each point of  $(0, 1) \setminus \{\frac{1}{2}\}$  is a collection of three simple loops. It is easy to see that the inverse images form a continuous family of 1-cycles (for the flat topology). Regarding these 1-cycles as points in  $\mathcal{Z}_1(M; G)$ , we see that both endpoints of  $[0, 1]$  are mapped to the zero cycle, and our map can be factorized through  $S^1 = [0, 1]/\{0, 1\}$ . One can easily check the continuity of the corresponding map of  $S^1$  to the space of 1-cycles on the three-legged starfish.

Now, we are going to give an example illustrating that not every finite-to one map from the hexapod to  $S^1$  yields a continuous map to the space of 1-cycles. Consider a map that sends all vertices of degree 1 of the hexapod  $H$  to a point  $a \in S^1$ , the degree 6 vertex of  $H$  to another point  $b \in S^1$ , and every edge of  $H$  to the same arc of  $S^1$  connecting  $a$  and  $b$ . The points of  $S^1$  not in the image of the map  $H \rightarrow S^1$  correspond to the zero 1-cycle, and so is  $a$ . However, the image of  $b$  is the 1-cycle formed of six arcs on the three-legged star-fish. Hence, a discontinuity at  $b$ .

An elaboration of the ideas involved in Example 1.4 can be combined with our construction in the proof of Theorem 1.6 to turn the family  $h^{-1}(t)$  of 1-cycles, into a continuous family of 1-cycles parameterized by  $S^{n-1}$  in the general case. This leads us to introduce the following definition. (We refer to [20] for a general background in geometric measure theory, including the notions of currents and varifolds.)

**Definition 1.5.** Let  $M$  be a closed Riemannian  $n$ -manifold. Define

$$W'_1(M) = \inf_{\Xi} \sup_{u \in S^{n-1}} \max_{C \subseteq \Xi_u} \text{length}(\varphi|_C). \quad (1.3)$$

In this expression, the infimum is taken over the families  $\Xi$  of 1-cycles (more precisely, of finite collections of closed curves) parameterized by  $S^{n-1}$  on a closed  $n$ -pseudomanifold  $N$ , which are continuous both in the flat topology of the 1-cycle space and in the weak topology of the 1-varifold space, and whose image under a degree-one map  $\varphi : N \rightarrow M$  represents the fundamental class  $[M]$  via the Almgren isomorphism  $\pi_{n-1}(\mathcal{Z}_1(N; G)) \simeq H_n(M; G)$ , see [2]. Furthermore, the maximum is taken over the connected components  $C$  of  $\Xi_u$ , where  $u \in S^{n-1}$ .

Note that since  $\Xi$  is continuous with respect to the weak topology of varifolds, the length of the image of  $\Xi_u$  varies continuously. Clearly,  $W'_1(M) \geq W_1(M)$ .

There is a non-equivalent but equally adequate and more geometric way to define  $\Xi$ . Consider the space  $\Gamma$  of piecewise smooth paths on  $M$  endowed with the Lipschitz distance topology. A finite collection of piecewise smooth closed curves can be parameterized by  $k$ -tuples of paths of  $\Gamma$  whose endpoints match to form a 1-cycle. The distance between two such collections of closed curves can be defined as the infimum of the Lipschitz distance between their parametrizations as  $k$ -tuples of paths of  $\Gamma$ . (Here, the integer  $k$  must be the same for both collections of curves and we take the infimum over all  $k$ .) Formally, we also identify two collections of curves that differ by a union of closed curves reduced to finitely many points. We can modify the definition of  $W'_1(M)$  by taking the infimum over all families  $\Xi$  of finite collections of piecewise smooth closed curves which represent the fundamental class of  $M$  under the Almgren isomorphism.

Our estimates (and their proofs) are valid for both choices of  $\Xi$  in the definition of  $W'_1(M)$ .

The following theorem extends the estimates of Theorem 1.3.

**Theorem 1.6.** *Let  $M$  be a closed  $n$ -manifold. Then every Riemannian metric on  $M$  satisfies*

$$\begin{aligned}\text{vol}(M) &\geq c_n W'_1(M)^n \\ \text{diam}(M) &\geq c'_n W'_1(M)\end{aligned}$$

for some explicit positive constants  $c_n$  and  $c'_n$  depending only on  $n$ .

We will also show that if  $n \geq 3$ , then this theorem can be somewhat improved by changing the definition of  $W'_1(M)$ . In the current definition, we are taking the infimum of the maximal length of the  $\varphi$ -images of the connected components of a 1-cycle. We can instead take the infimum of the maximal length of the connected components of its image under  $\varphi$ . Here is the formal definition of this new invariant

$$W''_1(M) = \inf_{\Lambda} \sup_{u \in S^{n-1}} \max_{C \subseteq \Lambda_u} \text{length}(C), \quad (1.4)$$

where the infimum is taken over all families  $\Lambda$  of 1-cycles (more precisely, finite collections of closed curves) parametrized by  $S^{n-1}$  on  $M$  (not  $N$  as before!), that are continuous in either of the two topologies from the definition of  $W''_1(M)$  and correspond to the fundamental homology class of  $M$  under the Almgren isomorphism.

In addition, one can require that 1-cycle family  $\Lambda$  arises from a slicing of  $N$  as in Definition 1.5. More precisely, one can assume that the 1-cycle family  $\Lambda$  is given by the image of the inverse images  $h^{-1}(u)$  of a continuous map  $h : N \rightarrow S^{n-1}$  defined on a closed  $n$ -pseudomanifold  $N$  under a degree one map  $\varphi : N \rightarrow M$  such that the fibers  $h^{-1}(u)$  with  $u \in S^{n-1}$  define a family of 1-cycles on  $N$  continuous with respect to both topologies involved in Definition 1.5.

So defined invariant  $W''_1(M)$  has a very natural geometric meaning: It measures the maximal length of a connected component in an optimal sweepout of  $M$  by 1-cycles, where the sweepouts are parametrized by the sphere of codimension one.

The following estimates also hold for this invariant.

**Theorem 1.7.** *Let  $M$  be a closed  $n$ -manifold,  $n \geq 3$ . Then every Riemannian metric on  $M$  satisfies*

$$\begin{aligned}\text{vol}(M) &\geq c_n W''_1(M)^n \\ \text{diam}(M) &\geq c'_n W''_1(M)\end{aligned}$$

for some explicit positive constants  $c_n$  and  $c'_n$  depending only on  $n$ .

As a further comment on the definition of  $W''_1(M)$ , we consider the following example.

**Example 1.8.** Let  $M = N = (S^2, \text{can})$  and  $\varphi : S^2 \rightarrow S^2$  be the identity map. Consider a very fine triangulation of  $M$ . Let us construct a map  $h : M \rightarrow [0, 1]$  as follows. The inverse image of 0 under  $h$  is the (finite) collection of the centers of the 2-simplices of the triangulation. When  $t$  grows from 0 to 1, the preimage  $h^{-1}(t)$  is a collection of concentric triangles connecting the center of each simplex to its boundary. When  $t = 1$ , the inverse image of  $t$  is the 1-skeleton of the triangulation. Thus, the fiber  $h^{-1}(t)$  is a 1-cycle, except at  $t = 1$ . Still, the map  $[0, 1) \rightarrow \mathcal{Z}_1(M; \mathbb{Z})$  taking  $t \in [0, 1)$  to  $h^{-1}(t)$  extends by continuity at  $t = 1$  by sending 1 to the zero 1-cycle. The map  $S^1 \rightarrow \mathcal{Z}_1(M; \mathbb{Z})$  thus-defined is continuous in the flat topology of 1-cycles (albeit not in the weak topology of 1-varifolds) and induces the fundamental class of  $M$  under the Almgren isomorphism. Furthermore, the length of the connected components of this family of 1-cycles can be arbitrarily small. This example would tend to show that  $W'_1(M)$  is trivial. Let us recall however that the map  $S^1 \rightarrow \mathcal{Z}_1(M; \mathbb{Z})$  is not continuous in the weak topology of 1-varifolds and that the 1-cycle at  $t = 1$  is not of the form  $\varphi(h^{-1}(t))$  for  $t = 1$ ,



as we changed its value for continuity reasons. Thus, this 1-cycle family does not occur in the definition of  $W'_1(M)$ . Actually, we will show in Theorem 1.13 that  $W'_1(M)$  is always positive.

**Remark 1.9.** Below, we will consider some generalizations and extensions of Theorem 1.3. It will be clear that they also hold true for the analogs of  $W'_1(M)$  and  $W''_1(M)$ . We leave the (rather obvious) details to the reader.

**1.3. Sweeping out lower-dimensional strata.** In the previous theorems, we consider  $(n-1)$ -parameter families of polyhedral 1-chains sweeping out the whole manifold  $M$ . One may wonder whether one can extract  $(k-1)$ -parameter families of polyhedral 1-chains sweeping out nontrivial  $k$ -dimensional homology classes of  $M$  or more generally essential  $k$ -complexes of  $M$  so that these 1-chain sweepouts satisfy the same upper bounds as the (full) 1-sweepout of Theorem 1.3. Though examples can be found in [25], the existence of such sweepouts may not hold in general. In our next result, we give topological conditions which ensure the existence of such sweepouts. The existence of these sweepouts does not follow directly from the proof of Theorem 1.3 and requires some new ideas. In particular, we will need to change our main definition and make various changes in the proof of Theorem 1.3.

First, let us introduce a more general notion of sweepout leading to a different notion of waist.

**Definition 1.10.** Let  $\Phi : M \rightarrow K$  be a continuous map from a closed manifold  $M$  to a CW-complex  $K$ . A  $\Phi$ -homotopy  $(p, k)$ -sweepout of  $M$  is a family

$$\varphi[h^{-1}(t)] \subseteq M$$

with  $t \in T$ , where  $h : X \rightarrow T$  is a continuous map from a finite simplicial  $(k+p)$ -complex  $X$  to a finite simplicial  $k$ -complex  $T$  such that all fibers  $h^{-1}(t)$  are  $p$ -subpolyhedra of  $X$ , and  $\varphi : X \rightarrow M$  is a continuous map whose composition  $\Phi \circ \varphi : X \rightarrow K$  is not homotopic to a map

$$X \xrightarrow{h} T \rightarrow K$$

which factors out through  $h$ . Define the  $\Phi$ -homotopy  $(p, k)$ -waist of a closed Riemannian manifold  $M$  as

$$W_{p,k}(M, \Phi) = \inf_{\varphi, h} \sup_{t \in T} \text{vol}_p(\varphi|_{h^{-1}(t)})$$

where the infimum is taken over all maps  $\varphi : X \rightarrow M$  and  $h : X \rightarrow T$  defining a  $\Phi$ -homotopy  $(p, k)$ -sweepout of  $M$ . If such sweepouts do not exist, we let  $W_{p,k}(M, \Phi) = 0$ .

We will be especially interested in the case where  $p = 1$ . As in (1.2), we can assume that the homotopy  $(p, k)$ -sweepouts are parameterized by the sphere  $S^k$  and that the  $\Phi$ -homotopy  $(1, k)$ -waist of  $M$  is defined by minimizing the maximal length of the map  $\varphi$  restricted to the connected components of the fibers of  $\bar{h} : X \rightarrow S^k$ .

Sweepout estimates also hold with this notion of waist when  $p = 1$ .

**Theorem 1.11.** Fix  $k \leq n - 1$ . Let  $M$  be a closed  $n$ -manifold and  $\Phi : M \rightarrow K$  be a continuous map to a CW-complex  $K$  with  $\pi_i(K) = 0$  for every  $i \geq k + 1$ . Suppose that  $\Phi_*([M]) \neq 0 \in H_n(K; G)$  for some homology coefficient group  $G$ . Then every Riemannian metric on  $M$  satisfies

$$\begin{aligned} \text{vol}(M) &\geq c_n W_{1,k}(M, \Phi)^n \\ \text{diam}(M) &\geq c'_n W_{1,k}(M, \Phi) \end{aligned}$$

for some explicit positive constants  $c_n$  and  $c'_n$  depending only on  $n$ .

The following example illustrates the theorem.

**Example 1.12.** The main examples arise when  $K$  is the Eilenberg-MacLane space  $K(G, m)$ , where  $G$  is an abelian group and  $n = qm$ . Assume that the map  $\Phi : M \rightarrow K$  represents a nonzero cohomology class  $c \in H^m(M; G)$  such that the  $q$ -th cup power of  $c$  is nonzero in  $H^n(M; G)$ . Assume also that the map  $h : X \rightarrow M$  defined on a closed  $k$ -pseudomanifold  $X$  represents a homology class  $a \in H_k(M; G)$  dual to a nonzero multiple of the  $l$ -th cup power of  $c$ , where  $k = lm$ , in the sense that  $\langle c^l, a \rangle \neq 0$ . In this case, the map  $\Phi \circ h : X \rightarrow K$  does not factor through a  $(k - 1)$ -dimensional complex  $T$  since it induces a nontrivial homomorphism between the  $k$ -dimensional homology groups of  $X$  and  $K$ . Thus, Theorem 1.11 yields curvature-free upper bounds for the homotopy  $(1, k)$ -waist of some lower-dimensional homology classes of  $M$ .

**1.4. Intrinsic geometric interpretation of the filling radius.** The filling radius of a closed Riemannian manifold  $M$  is defined in an extrinsic way from the Kuratowski embedding of  $M$  into  $L^\infty(M)$ ; see Definition 3.1. A different (more intrinsic) interpretation of the filling radius can be deduced from the filling radius estimates of Theorems 3.3, 4.7, 4.10, and Proposition 6.2. More specifically, we show that the filling radius of a closed Riemannian manifold is roughly equal to its homology 1-waist.

**Theorem 1.13.** *There exist two explicit constant  $c_n$  and  $C_n$  depending only on  $n$  such that every closed Riemannian  $n$ -manifold  $M$  satisfies*

$$c_n W(M) \leq \text{FillRad}(M) \leq C_n W(M).$$

*We can take  $C_n = \frac{1}{2}$ . The same assertion holds for  $W'_1(M)$  and, if  $n \geq 3$ , also for  $W''_1(M)$  instead of  $W(M)$ .*

**Remark 1.14.** Since  $W(M) \leq W'_1(M) \leq W''_1(M)$ , the upper bound for  $\text{FillRad}(M)$  in terms of  $W(M)$  immediately implies the same upper bounds in terms of  $W'_1(M)$  and  $W''_1(M)$ . The previous theorem implies the following alternative (and also imprecise up to a constant factor) geometric interpretation of the filling radius when  $n \geq 3$ . *Up to at most a dimensional factor  $c(n)$ , the filling radius of a closed Riemannian  $n$ -manifold  $M$  is equal to the maximal length of a connected component in an “optimal” sweepout of  $M$  by a continuous family of closed curves.* Here, “optimal” means that the sweepout (nearly) realizes the infimum of the minimal length. When  $n = 2$ , this is still true, but for a somewhat less geometrically intuitive definition of “connected components of a sweepout” stemming from the definition of  $W'_1(M)$ . (In this case, one looks at the images of connected components of a slicing of  $N$  under a degree one map  $\varphi : N \rightarrow M$ .)

As shown in Proposition 7.2, the hypersphericity of a closed orientable Riemannian manifold is roughly bounded by its filling radius (and so by its Urysohn width). For Riemannian 2-spheres, these Riemannian invariants are roughly the same; see Section 7. Still, there are examples of manifolds where the hypersphericity and the Urysohn width can be arbitrarily far apart; see [13]. Applying the filling radius estimates of Theorem 1.13 to these examples, we can strengthen this result by showing that the same occurs between the hypersphericity and the filling radius.

**Theorem 1.15.** *There exists a sequence  $(g_i)$  of Riemannian metrics on  $S^4$  with arbitrarily small hypersphericity and filling radius bounded away from zero.*

It would be interesting to determine whether we can replace  $W(M)$  with the Urysohn width in Theorem 1.13 or whether the filling radius can be arbitrarily far apart from the Urysohn width as in Theorem 1.15.

**1.5. Homology  $p$ -waist bounds.** The bounds in Theorem 1.3 about homology 1-waist can be extended to homology  $p$ -waist using the notion of homological filling function defined below.

**Definition 1.16.** The  $k$ -homological filling function of a closed Riemannian  $n$ -manifold  $M$  is a function  $\text{FH}_k : [0, \infty) \rightarrow [0, \infty]$  defined as

$$\text{FH}_k(v) = \sup_{\text{vol}_k(\Sigma_0^k) \leq v} \inf\{\text{vol}_{k+1}(\Sigma^{k+1}) \mid \partial\Sigma^{k+1} = \Sigma_0^k\}$$

where the supremum is taken over all closed  $k$ -pseudomanifolds  $\Sigma_0^k$  in  $M$  of volume at most  $v$  and the infimum is taken over all compact pseudomanifolds  $\Sigma^{k+1}$  in  $M$  with boundary  $\partial\Sigma^{k+1} = \Sigma_0^k$ . By convention,  $\inf \emptyset = \infty$ . This means that if the  $k$ -th homology group of  $M$  is nontrivial, then  $\text{FH}_k(v) = \infty$  for all  $v$  greater than some  $v_0$ . This threshold value  $v_0$  can, however, be arbitrarily large in comparison with, say,  $\text{vol}(M)^{\frac{k}{n}}$ . For example, consider  $M = S^1 \times S^2$  endowed with the product metric, where  $S^1$  has a very large length  $L$ , but the area of  $S^2$  is just  $\frac{1}{L}$ . Although  $H_1(M)$  is nontrivial,  $\text{FH}_1(v)$  will be finite for all  $v < L$ .

Observe that the homological filling function  $\text{FH}_k$  is nondecreasing and that  $\text{FH}_k(v) \leq \alpha_k v^{k+1}$  for every  $v$  small enough, where  $\alpha_k$  is some constant depending only on  $k$  involved in the isoperimetric inequality. It will be convenient to introduce  $\overline{\text{FH}}_k(v) = \text{FH}_k(2(k+1)v)$ . Also, we define  $\text{FH}_k(\infty) = \overline{\text{FH}}_k(\infty) = \infty$ .

The notion of homological filling function was considered in [22] to bound the least area of a (possibly singular) minimal surface and, more generally, the least mass of a nontrivial stationary integral  $k$ -varifold in a closed Riemannian manifold whose first  $k-1$  homology groups are trivial. The homological filling functions can be estimated by taking a simplicial approximation and minimizing the volume of a filling using an elementary linear algebra argument in connection with systems of linear equations with integer coefficients given by the boundary operator; see [22] for more details. Strictly speaking the definition of homological filling functions in [22] was stated in terms of singular chains and not pseudomanifolds, but this leads to the same notion after desingularization.

The following result provides an extension of Theorem 1.3 to higher dimensional homology waists; see (1.1).

**Theorem 1.17.** *Let  $M$  be a closed Riemannian  $n$ -manifold. Then, for every positive integer  $p$ ,*

$$\begin{aligned} W_p(M) &< \frac{1}{2^{n-p+1}} \binom{n+1}{p}^{-1} \overline{\text{FH}}_{p-1} \circ \cdots \circ \overline{\text{FH}}_1(C_n \text{vol}(M)^{\frac{1}{n}}) \\ W_p(M) &< \frac{1}{2^{n-p+1}} \binom{n+1}{p}^{-1} \overline{\text{FH}}_{p-1} \circ \cdots \circ \overline{\text{FH}}_1(C'_n \text{diam}(M)) \end{aligned}$$

for some explicit positive constants  $C_n$  and  $C'_n$  depending only on  $n$ .

The constants involved in the filling functions can be improved by following Remark 8.4. Note, that if all homology groups  $H_i(M)$  for  $i \in \{1, \dots, p-1\}$  vanish, then the right-hand sides in both inequalities are always finite. However, if at least one of these homology groups is nontrivial, it is possible that one or both right-hand sides are  $\infty$ , and the inequality(ies) become trivial.

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## 2. NATURAL SWEEPOUTS OF THE STANDARD CUBICAL SIMPLEX

In this section, we describe natural  $p$ -sweepouts of the standard cubical simplex defined as the fibers of a map from the cube to a complex of codimension  $p$ . The reason we consider the

standard cubical simplex instead of the standard simplex is because it is simpler to describe the natural sweepouts in this case. Similar constructions hold for the standard simplex.

Let us start by describing a decomposition of the standard cube. The case when  $p = 1$  is considered in Sections 3, 4, 5, 6 and 7, while the general case is considered in Section 8. At first reading, one can assume that  $p = 1$ .

**Definition 2.1.** Let  $C^{n+1} = [-1, 1]^{n+1}$  be the standard cubical  $(n+1)$ -simplex. Fix an integer  $p \geq 1$ . The  $p$ -skeleton  $(C^{n+1})^{(p)}$  of  $C^{n+1}$  is formed of the points of  $C^{n+1}$  all of whose coordinates except possibly  $p$  of them are equal to  $\pm 1$ . That is,

$$(C^{n+1})^{(p)} = \{x \in C^{n+1} \mid \text{there exist } i_1, \dots, i_{n-p+1} \in \{1, \dots, n+1\} \text{ distinct such that } x_{i_k} = \pm 1\}.$$

The cubical  $(n-p)$ -complex  $Z^{n-p} \subseteq C^{n+1}$  dual to  $(C^{n+1})^{(p)}$  is formed of the points of  $C^{n+1}$  with at least  $p+1$  zero coordinates; see Figure 2.(c) and Figure 6.(c). That is,

$$Z^{n-p} = \{x \in C^{n+1} \mid \text{there exist } i_1, \dots, i_{p+1} \in \{1, \dots, n+1\} \text{ distinct such that } x_{i_k} = 0\}.$$

Fix  $\varepsilon \in (0, 1)$ . The space

$$X_{1,\varepsilon}^{n+1} = \{x \in C^{n+1} \mid \text{there exist } i_1, \dots, i_{n-p+1} \in \{1, \dots, n+1\} \text{ distinct such that } |x_{i_k}| \geq \varepsilon\}$$

formed of the points of  $C^{n+1}$  with at most  $p$  coordinate less than  $\varepsilon$  in absolute value is a tubular neighborhood of  $(C^{n+1})^{(p)}$ . Similarly, the space

$$X_{2,\varepsilon}^{n+1} = \{x \in C^{n+1} \mid \text{there exist } i_1, \dots, i_{p+1} \in \{1, \dots, n+1\} \text{ distinct such that } |x_{i_k}| \leq \varepsilon\}$$

formed of the points of  $C^{n+1}$  with a least  $p+1$  coordinates bounded by  $\varepsilon$  in absolute value is a tubular neighborhood of  $Z^{n-p}$ . Both spaces  $X_{1,\varepsilon}^{n+1}$  and  $X_{2,\varepsilon}^{n+1}$  are endowed with a cubical structure, where the cubical simplices are bounded by the hyperplanes  $x_i = \pm\varepsilon$  and  $x_i = \pm 1$ . The cubical complexes  $X_{1,\varepsilon}^{n+1}$  and  $X_{2,\varepsilon}^{n+1}$  cover the cube  $C^{n+1}$  and intersect along a cubical  $n$ -complex

$$Y_\varepsilon^n = X_{1,\varepsilon}^{n+1} \cap X_{2,\varepsilon}^{n+1}$$

which decomposes into a disjoint union

$$\begin{aligned} Y_\varepsilon^n = \bigsqcup_{k=p+1}^{n+1} \{x \in C^{n+1} \mid \text{there exist } i_1, \dots, i_k \in \{1, \dots, n+1\} \text{ distinct such that} \\ |x_{i_1}| \leq \varepsilon, \dots, |x_{i_p}| \leq \varepsilon \\ |x_{i_{p+1}}| = \dots = |x_{i_k}| = \varepsilon \\ |x_i| > \varepsilon \text{ for every } i \neq i_1, \dots, i_k\} \end{aligned} \quad (2.1)$$

of cubical  $(n+p+1-k)$ -complexes with  $p+1 \leq k \leq n+1$ .

Strictly speaking, the complexes  $X_{i,\varepsilon}^{n+1}$ ,  $Y_\varepsilon^n$  and  $Z^{n-p}$  depend also on  $p$ . In order not to burden the notations, we keep the dependence on  $p$  of these complexes and the following constructions implicit.

Let us define a natural  $p$ -sweepout of the standard cube.

**Definition 2.2.** Let  $\lambda_\varepsilon : [-1, 1] \rightarrow [-1, 1]$  be the odd piecewise linear function defined by

$$\lambda_\varepsilon(t) = \begin{cases} \frac{t-\varepsilon}{1-\varepsilon} & \text{if } t \in [\varepsilon, 1] \\ 0 & \text{if } t \in [-\varepsilon, \varepsilon] \\ \frac{t+\varepsilon}{1-\varepsilon} & \text{if } t \in [-1, -\varepsilon] \end{cases}$$

keeping  $-1, 0, 1$  fixed and sending  $[-\varepsilon, \varepsilon]$  to  $\{0\}$ .

Consider the map  $\theta_\varepsilon : Y_\varepsilon^n \rightarrow Z^{n-p}$  defined by

$$\theta_\varepsilon(x_1, \dots, x_{n+1}) = (\lambda_\varepsilon(x_1), \dots, \lambda_\varepsilon(x_{n+1})). \quad (2.2)$$

By definition, every point of  $Y_\varepsilon^n$  has at least  $p+1$  coordinates bounded by  $\varepsilon$  in absolute value which are sent to 0 by  $\lambda_\varepsilon$ . This shows that the map  $\theta_\varepsilon$  takes values in  $Z^{n-p}$ . The preimage of  $z \in Z^{n-p}$  under  $\theta_\varepsilon$  can be determined as follows. Denote by  $z_{i_1}, \dots, z_{i_k}$  all the zero coordinates of  $z$ . Note that  $k \geq p+1$ . By construction,

$$\begin{aligned} \theta_\varepsilon^{-1}(z) = \{x \in Y_\varepsilon^n \mid & |x_{i_{s_1}}| \leq \varepsilon, \dots, |x_{i_{s_p}}| \leq \varepsilon \text{ for } s_1, \dots, s_p \in \{1, \dots, k\} \text{ distinct} \\ & |x_{i_s}| = \varepsilon \text{ for every } s \neq s_1, \dots, s_p \text{ in } \{1, \dots, k\} \\ & x_j = \lambda_\varepsilon^{-1}(z_j) \text{ for every } j \neq i_1, \dots, i_k\}. \end{aligned}$$

Since  $\theta_\varepsilon^{-1}(z)$  lies in  $Y_\varepsilon^n$ , all the coordinates  $x_{i_1}, \dots, x_{i_k}$  of  $x \in \theta_\varepsilon^{-1}(z)$  are equal to  $\pm\varepsilon$ , except possibly  $p$  of them. Thus, the preimage  $\theta_\varepsilon^{-1}(z)$  is a cubical  $p$ -complex isomorphic to the  $p$ -skeleton of the  $k$ -cube. See Figure 2 for a description of the sweepout of  $Y_\varepsilon^n$ .

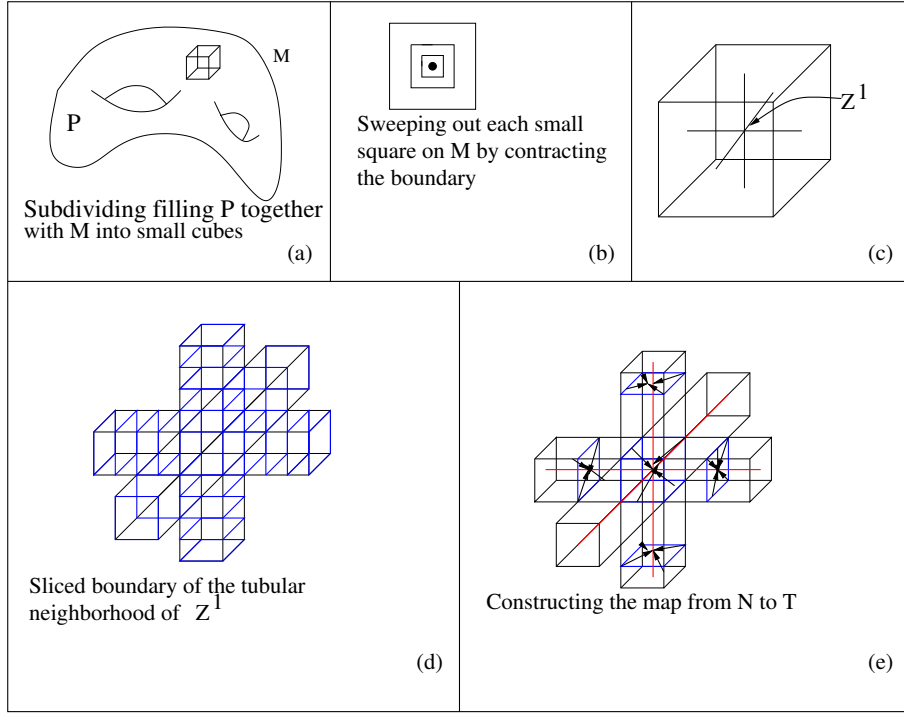


FIGURE 2. Sweeping out 2-dimensional  $N$

The other map  $\Theta$  we need to define will be used only on  $C^n$ . For this reason, we carry our construction on  $C^n$  and not on  $C^{n+1}$ . The subsets  $Y_\varepsilon^{n-1}$  with  $\varepsilon \in (0, 1)$  foliate  $C^n \setminus ((C^n)^{(p)} \cup Z^{n-p-1})$ . More precisely, they are the level sets of the composition of the continuous map

$$\begin{aligned} \Theta : C^n \setminus ((C^n)^{(p)} \cup Z^{n-p-1}) &\longrightarrow Z^{n-p-1} \times (0, 1) \\ x &\longmapsto (\theta_\varepsilon(x), \varepsilon) \end{aligned}$$

where  $\varepsilon$  is given by  $x \in Y_\varepsilon^{n-1}$ , with the projection  $Z^{n-p-1} \times (0, 1) \rightarrow (0, 1)$ . Thus, the map  $\Theta$  is given by  $\theta_\varepsilon$  on each subset  $Y_\varepsilon^{n-1}$ .

Define the simplicial  $(n-1)$ -complex

$$\hat{Z}^{n-p} = \text{Cone}(Z^{n-p-1}) = Z^{n-p-1} \times [0, 1] / Z^{n-p-1} \times \{1\}$$

where  $Z^{n-p-1} \times \{1\}$  is collapsed to a point  $\star$ .

The map  $\Theta$  extends to a continuous map still denoted by

$$\Theta : C^n \rightarrow \hat{Z}^{n-p} \tag{2.3}$$

where  $\Theta(x) = (\theta_0(x), 0)$  for every  $x \in Z^{n-p-1} = Y_0^{n-1}$  and  $\Theta(x) = \star$  for every  $x \in (C^n)^{(p)}$ .

The fibers of  $\Theta$  define a *natural p-sweepout* of the  $n$ -cube which is invariant by the group of symmetries of  $C^n$ .

**Remark 2.3.** Loosely speaking, when  $n = 3$  and  $p = 1$ , the boundary of the neighborhood of the dual to the 1 skeleton of the 3-cube varies with  $\varepsilon$  from two extremes where it collapses to the 1-skeleton of the 3-cube or its dual  $Z^1$ . Deforming the slicing of  $Y_\varepsilon^2$  as  $\varepsilon$  varies induces a natural sweepout of the 3-cube; see Figure 3.(b).

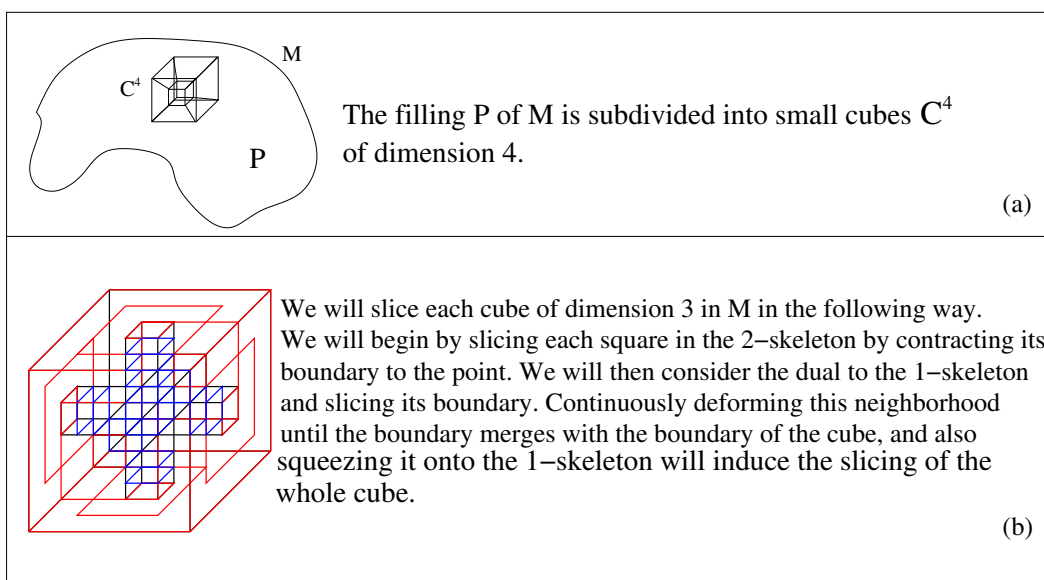


FIGURE 3. Filling and sweeping-out 3-dimensional  $M$

Finally, let us define some deformations on the standard cube.

**Definition 2.4.** Let  $\mu_\varepsilon : [-1, 1] \rightarrow [-1, 1]$  be the odd piecewise linear function defined by

$$\mu_\varepsilon(t) = \begin{cases} 1 & \text{if } t \in [\varepsilon, 1] \\ \frac{t}{\varepsilon} & \text{if } t \in [-\varepsilon, \varepsilon] \\ -1 & \text{if } t \in [-1, -\varepsilon] \end{cases}$$

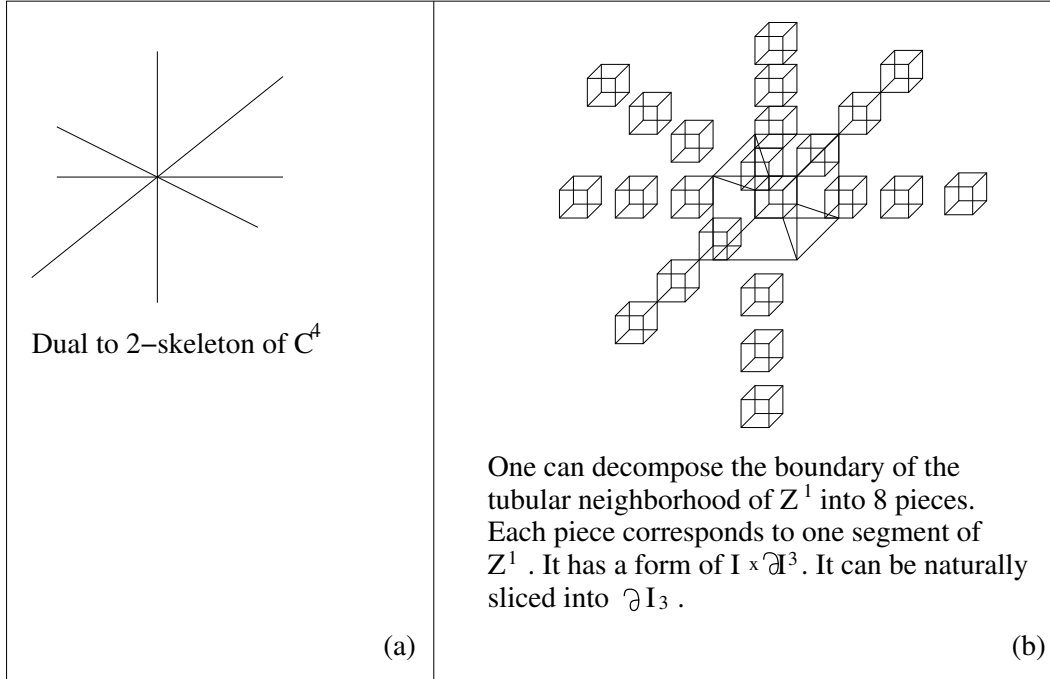
sending  $[\varepsilon, 1]$  to  $\{1\}$  and  $[-1, -\varepsilon]$  to  $\{-1\}$ .

Consider the map  $\rho_\varepsilon : X_{1,\varepsilon}^{n+1} \rightarrow (C^{n+1})^{(p)}$  defined by

$$\rho_\varepsilon(x) = (\mu_\varepsilon(x_1), \dots, \mu_\varepsilon(x_{n+1})). \tag{2.4}$$

By definition, every point of  $X_{1,\varepsilon}^{n+1}$  has at least  $n - p$  coordinates bounded below by  $\varepsilon$  in absolute value, which are sent to  $\pm 1$  by  $\mu_\varepsilon$ . This shows that the map  $\rho_\varepsilon$  takes values in  $(C^{n+1})^{(p)}$ . Observe also that the map  $\rho_\varepsilon$  fixes the vertices of  $C^{n+1}$  and sends every edge of  $C^{n+1}$  to itself. We will refer to  $\rho_\varepsilon$  as the “retraction” of  $X_{1,\varepsilon}^{n+1}$  onto  $(C^{n+1})^{(p)}$ . Note that the “retraction”  $\rho_\varepsilon$  extends to a degree one map

$$\bar{\rho}_\varepsilon : C^{n+1} \rightarrow C^{n+1}. \tag{2.5}$$

FIGURE 4. Sweeping out the 4-cube;  $p=2$ 

In the following sections, we will fix  $\varepsilon = \frac{1}{2}$  and drop the subscripts. For instance, we will write  $Y^n$  for  $Y_\varepsilon^n$ ,  $X_i^{n+1}$  for  $X_{i,\varepsilon}^{n+1}$ ,  $\theta$  for  $\theta_\varepsilon$  and  $\bar{\rho}$  for  $\bar{\rho}_\varepsilon$ .

We conclude this section with the following result.

**Proposition 2.5.** *The cubical  $n$ -complex  $Y_\varepsilon^n$  is an  $n$ -pseudomanifold with boundary lying in  $\partial C^{n+1}$ .*

First recall the general definition of a pseudomanifold.

**Definition 2.6.** An  $n$ -pseudomanifold with boundary is a simplicial  $n$ -complex  $P$  such that

- every simplex of  $P$  is a face of some  $n$ -simplex of  $P$ ;
- every  $(n-1)$ -simplex of  $P$  is the face of at most two  $n$ -simplices of  $P$ ;
- given two  $n$ -simplices of  $P$ , there exists a sequence of  $n$ -simplices of  $P$  with two consecutive  $n$ -simplices having an  $(n-1)$ -face in common that starts at one of them and ends at the other.

The boundary  $\partial P$  of an  $n$ -pseudomanifold  $P$  is the simplicial  $(n-1)$ -subcomplex of  $P$  formed of the  $(n-1)$ -simplices of  $P$  which are the faces of exactly one  $n$ -simplex of  $P$ .

*Proof.* The cubical  $n$ -complex  $Y_\varepsilon^n$  decomposes into a union of cubical  $n$ -simplices of two kinds

$$[-\varepsilon, \varepsilon]^p \times \{\pm\varepsilon\} \times [\varepsilon, 1]^{n-p} \quad \text{and} \quad [-\varepsilon, \varepsilon]^p \times \{\pm\varepsilon\} \times [-1, -\varepsilon]^{n-p}$$

up to factor permutations; see (2.1). The boundary of the first kind of cubical  $n$ -simplices is a union of  $(n-1)$ -faces of the following three forms

$$[-\varepsilon, \varepsilon]^{p-1} \times \{\pm\varepsilon\} \times \{\pm\varepsilon\} \times [\varepsilon, 1]^{n-p}, \quad [-\varepsilon, \varepsilon]^p \times \{\pm\varepsilon\} \times \{\varepsilon\} \times [\varepsilon, 1]^{n-p-1}, \quad [-\varepsilon, \varepsilon]^p \times \{\pm\varepsilon\} \times \{1\} \times [\varepsilon, 1]^{n-p-1}$$

up to factor permutations. The same holds for the boundary of the second kind of cubical  $n$ -simplices. The  $(n-1)$ -faces involved in these unions have exactly two components of the form of a singleton  $\{\pm\varepsilon\}$  or  $\{\pm 1\}$ , with at most one singleton of the form  $\{\pm 1\}$ . The  $(n-1)$ -faces with

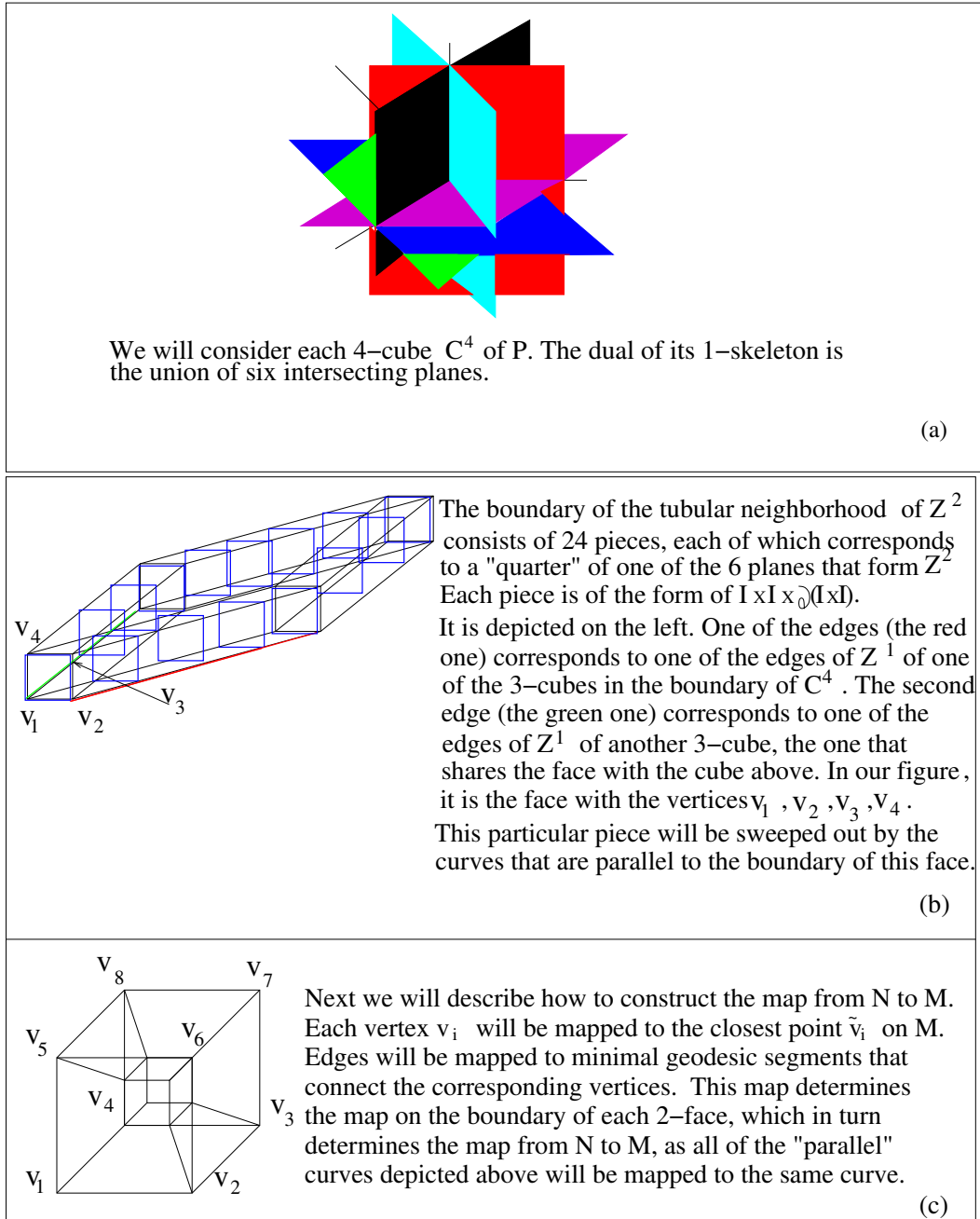


FIGURE 5. Sweeping out the 4-cube;  $\dim M = 3$

no singleton of the form  $\{\pm 1\}$  appear in the boundary decomposition of exactly two cubical  $n$ -simplices. For instance,  $[-\varepsilon, \varepsilon]^{p-1} \times \{\varepsilon\} \times \{-\varepsilon\} \times [\varepsilon, 1]^{n-p}$  appears in the boundary decomposition of  $[-\varepsilon, \varepsilon]^p \times \{-\varepsilon\} \times [\varepsilon, 1]^{n-p}$  and  $[-\varepsilon, \varepsilon]^{p-1} \times \{\varepsilon\} \times [-\varepsilon, \varepsilon] \times [\varepsilon, 1]^{n-p}$ . (Recall that every point in an open  $n$ -face of  $Y_\varepsilon^n$  has exactly  $p + 1$  coordinates bounded by  $\varepsilon$  in absolute value.) The  $(n-1)$ -faces with exactly one singleton of the form  $\{\pm 1\}$  lie in  $\partial C^{n+1}$  and appear in the boundary decomposition of exactly one cubical  $n$ -simplex. For instance,  $[-\varepsilon, \varepsilon]^p \times \{\varepsilon\} \times \{1\} \times [\varepsilon, 1]^{n-p-1}$  appears in the boundary decomposition of  $[-\varepsilon, \varepsilon]^p \times \{\varepsilon\} \times [\varepsilon, 1]^{n-p}$ . Furthermore, each  $k$ -face



of  $Y_\varepsilon^n$  lies in a cubical  $n$ -simplex of  $Y_\varepsilon^n$ . Also, it is not difficult to see that given two cubical  $n$ -simplices of  $Y_\varepsilon^n$ , there exists a sequence of cubical  $n$ -simplices of  $Y_\varepsilon^n$  with two consecutive cubical  $n$ -simplices having one  $(n-1)$ -face in common. This shows that  $Y_\varepsilon^n$  is a cubical  $n$ -pseudomanifold with boundary lying in  $\partial C^{n+1}$ .  $\square$

### 3. FILLING RADIUS AND HOMOLOGY 1-WAIST

We establish a lower bound on the filling radius of a closed Riemannian manifold in terms of its homology 1-waist and derive Theorem 1.3.

Let us recall the notion of filling radius introduced by M. Gromov in [6] to established systolic inequalities on essential manifolds.

**Definition 3.1.** Let  $M$  be a closed  $n$ -manifold with a Riemannian metric  $g$ . Denote by  $d_g$  the distance on  $M$  induced by the Riemannian metric  $g$ . The map

$$i : (M, d_g) \hookrightarrow (L^\infty(M), \|\cdot\|)$$

defined by  $i(x)(\cdot) = d_g(x, \cdot)$  is an embedding from the metric space  $(M, d_g)$  into the Banach space  $L^\infty(M)$  of bounded functions on  $M$  endowed with the sup-norm  $\|\cdot\|$ . This natural embedding, also called the Kuratowski embedding, is an isometry between metric spaces. We will consider  $M$  isometrically embedded into  $L^\infty(M)$ .

The *filling radius* of  $M$  with a Riemannian metric  $g$ , denoted by  $\text{FillRad}(M)$ , is the infimum of the positive reals  $\nu$  such that

$$(i_\nu)_*([M]) = 0 \in H_n(U_\nu(M))$$

where  $i_\nu : M \hookrightarrow U_\nu(M)$  is the inclusion into the  $\nu$ -neighborhood of  $M$  in  $L^\infty(M)$ , and  $[M] \in H_n(M)$  is the fundamental class of  $M$ . Unless specified otherwise, the homology coefficients are in  $\mathbb{Z}$  if  $M$  is orientable, and in  $\mathbb{Z}_2$  otherwise.

The filling radius of a Riemannian manifold satisfies the following fundamental bounds respectively obtained by M. Gromov [6] and M. Katz [16] with an improvement in the constant recently obtained in [21].

**Theorem 3.2** (see [6], [16], [21]). *Let  $M$  be a closed Riemannian  $n$ -manifold. Then*

$$\begin{aligned} \text{FillRad}(M) &\leq n \text{vol}(M)^{\frac{1}{n}} \\ \text{FillRad}(M) &\leq \frac{1}{3} \text{diam}(M) \end{aligned}$$

where  $c_n$  is an explicit constant depending only on  $n$ .

The homology 1-waist is related to the filling radius as follows.

**Theorem 3.3.** *Let  $M$  be a closed  $n$ -manifold. Then every Riemannian metric on  $M$  satisfies*

$$\text{FillRad}(M) \geq c_n W(M)$$

for  $c_n = \frac{1}{(n+1)2^{n+1}}$ .

*Proof.* We will work with cubical complexes instead of simplicial complexes and rely on the construction of Section 2. By definition of the filling radius, the fundamental class  $[M]$  of  $M$  vanishes in the  $\nu$ -neighborhood  $U_\nu(M)$  of  $M$  in  $L^\infty(M)$ , where  $\nu > \text{FillRad}(M)$  is very close to  $\text{FillRad}(M)$ . Therefore, there exists a compact cubical  $(n+1)$ -pseudomanifold  $P \subseteq U_\nu(M)$  with boundary  $\partial P = M$ . (Recall that the finite chains defining the homology of a space can be replaced with compact pseudomanifolds with or without boundary; see [15, p. 108–109]. This explains why we can take a pseudomanifold for the chain  $P$ .) Subdivide  $P$  so that every cubical  $(n+1)$ -simplex of  $P$  has at most one  $n$ -face in  $\partial P$ .

Suppose that  $\nu < \frac{1}{(n+1)2^{n+1}} W(M)$ . The usual argument to obtain a contradiction and derive a lower bound on the filling radius of  $M$  consists in constructing a retraction from  $P$  onto  $\partial P = M$ . However, this may not be possible in our case. Instead, we will construct a continuous extension  $\bar{f}$  of the identity map on  $M$  to a cubical  $(n+1)$ -complex containing an  $n$ -pseudomanifold  $N$  homologous to  $M$  taking the fundamental classes of  $M$  and  $N$  to different homology classes in  $M$ . This will lead to a contradiction as wanted.

It will be convenient to think of  $P$  as an abstract compact cubical  $(n+1)$ -pseudomanifold related to  $M$  through a continuous map  $\sigma : P \rightarrow U_\nu(M)$  whose restriction  $\sigma : \partial P \rightarrow M$  to  $\partial P$  satisfies

$$\sigma_*([\partial P]) = [M] \in H_n(M) \quad (3.1)$$

Deforming the map  $\sigma$ , we can assume that  $\sigma$  takes every edge of  $\partial P$  to a minimizing segment of  $M$ . Denote by  $P^k$  the  $k$ -skeleton of  $P$ . Subdividing  $P$  if necessary, we can further assume that the images by  $\sigma$  of the cycles of the natural sweepout of the cubical simplices of  $\partial P$  are of length less than  $\varepsilon < \min\{\frac{1}{(n+1)2^n} W(M) - 2\nu, \frac{1}{2}\text{inj}(M)\}$ ; see Section 2 where  $p = 1$ .

We first define a map  $f : P^0 \rightarrow M$  which agrees with  $\sigma$  on the vertices of  $\partial P$ , by sending each vertex  $p_i \in P^0$  to a nearest point of  $\sigma(p_i)$  in  $M \subseteq L^\infty(M)$ , as we wish. Since the inclusion  $i : M \hookrightarrow U_\nu(M)$  is distance-preserving, every pair  $p_i, p_j$  of adjacent vertices of  $P$  satisfies

$$d_M(f(p_i), f(p_j)) \leq d_{L^\infty}(f(p_i), \sigma(p_i)) + d_{L^\infty}(\sigma(p_i), \sigma(p_j)) + d_{L^\infty}(\sigma(p_j), f(p_j)) < \delta$$

with  $\delta = 2\nu + \varepsilon < \frac{1}{(n+1)2^n} W(M)$ . We extend the map  $f$  to  $P^1$  by taking the edges of  $P$  to minimizing segments joining the images of their endpoints, as we wish. Observe that the map  $f$  agrees with  $\sigma$  on the edges of  $\partial P$ . By construction, the lengths of the images of the edges of  $P^1$  are less than  $\delta$ .

Let  $Q \subseteq P$  be the neighborhood of  $P^1$  in  $P$  composed of the pieces  $X_1^{n+1} \subseteq C^{n+1}$  corresponding to the cubical  $(n+1)$ -simplices of  $P$ ; see Section 2 where  $p = 1$ . Put together, the “retractions”  $\rho : X_1^{n+1} \rightarrow (C^{n+1})^{(1)}$  defined in (2.4) with  $p = 1$  give rise to a “retraction”  $r : Q \rightarrow P^1$ . Denote by  $\bar{f} : Q \rightarrow M$  the composition of  $r : Q \rightarrow P^1$  with  $f : P^1 \rightarrow M$ , that is,  $\bar{f} = f \circ r$ . Deform  $\sigma : \partial P \rightarrow M$  into  $\bar{\sigma} : \partial P \rightarrow M$  so that the restriction of  $\bar{\sigma}$  to each cubical  $n$ -simplex of  $\partial P$  agrees with  $\sigma \circ \bar{\rho}$ , where  $\bar{\rho}$  is the extension of the “retraction”  $\rho$  defined in (2.5) with  $p = 1$ . By construction, the map  $\bar{f} : Q \rightarrow M$  agrees with  $\bar{\sigma}$  on  $\partial P \cap Q$  and can be extended to a map

$$\bar{f} : Q \cup \partial P \rightarrow M \quad (3.2)$$

which agrees with  $\bar{\sigma}$  on  $\partial P$ ; see Figure 6. Furthermore, the map  $\bar{f}$  takes every edge of  $P^1$  to a segment of length less than  $\delta$ .

The cubical  $n$ -pseudomanifolds  $Y^n \subseteq C^{n+1}$  pasted together according to the assembling pattern of the cubical  $(n+1)$ -simplices  $C^{n+1}$  of the pseudomanifold  $P$ , see Section 2 where  $p = 1$  (and Figure 7), form a compact cubical  $n$ -pseudomanifold  $N' \subseteq Q$  with boundary lying in  $\partial P$ . The map  $\theta : Y^n \rightarrow Z^{n-1}$  defined in (2.2) with  $p = 1$  gives rise to a map  $h' : N' \rightarrow T'$  to the finite cubical  $(n-1)$ -complex  $T'$  formed of the pieces  $Z^{n-1} \subseteq C^{n+1}$ , where  $C^{n+1}$  is a cubical  $(n+1)$ -simplex of  $P$ . More precisely, the restriction of  $h'$  to the pieces  $Y^n$  of  $N'$  is given by  $\theta$ . The cubical  $n$ -complexes  $X_i^n \subseteq C^n$ , where  $C^n$  is a cubical  $n$ -simplex of  $\partial P \simeq M$ , form a compact cubical  $n$ -pseudomanifold  $N''_i \subseteq \partial P$  with the same boundary as  $N'$ . As previously, the map  $\Theta : X_2^n \rightarrow Z^{n-2} \times [0, \frac{1}{2}]$  defined in (2.3) with  $p = 1$  gives rise to a map  $h'' : N''_2 \rightarrow T''$  to the finite cubical  $(n-1)$ -complex  $T''$  formed of the pieces  $Z^{n-2} \times [0, \frac{1}{2}]$  with  $Z^{n-2} \subseteq C^n$ , where  $C^n$  is a cubical  $n$ -simplex of  $\partial P$ . Observe that  $\Theta(x) = (\theta(x), \frac{1}{2})$  for every  $x \in Y^n \cap X_2^n \simeq Y^{n-1} \subseteq C^n$ . Thus, the two maps  $h'$  and  $h''$  so-defined agree on the common boundary of  $N'$  and  $N''_2$  after

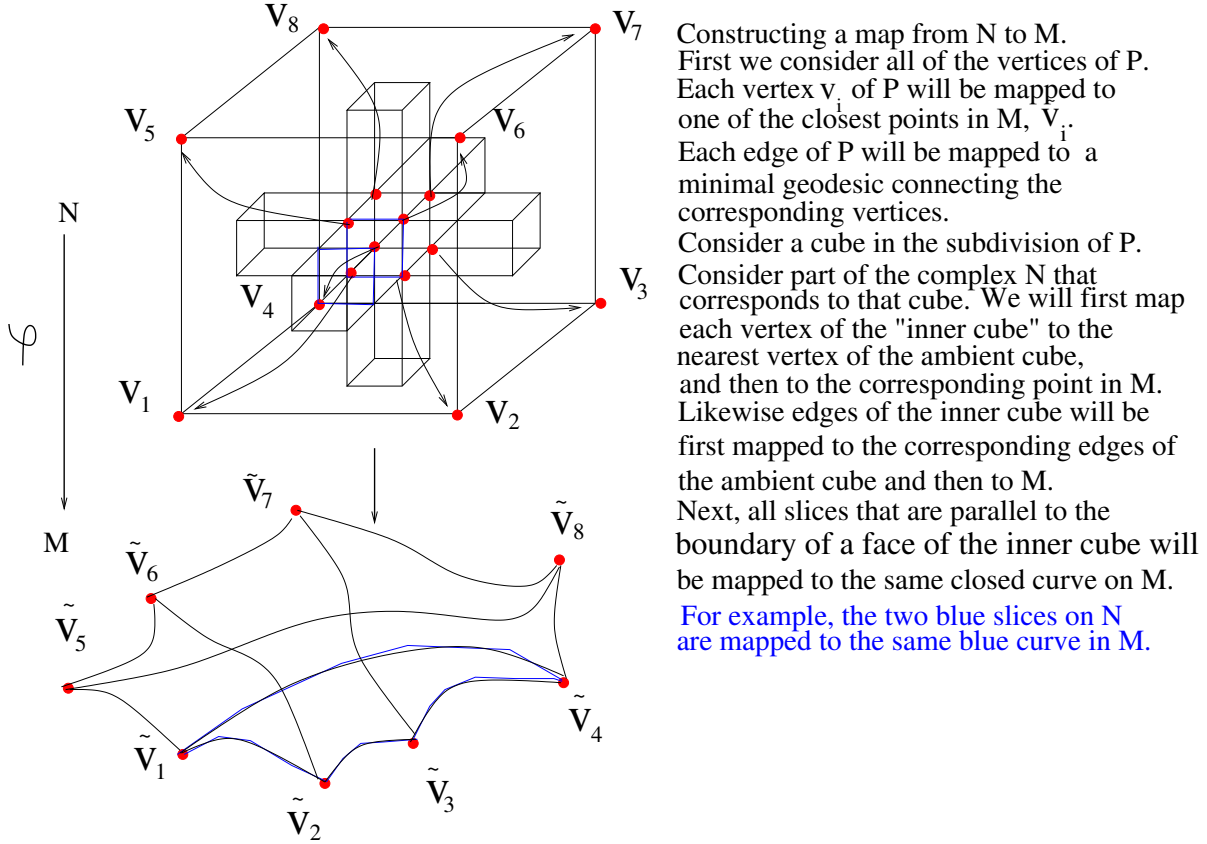


FIGURE 6. Mapping  $Q$  to  $M$

identifying  $Z^{n-2} \subseteq Z^{n-1} \subseteq T'$  and  $Z^{n-2} = Z^{n-2} \times \{\frac{1}{2}\} \subseteq T''$ , where  $Z^{n-2} \subseteq C^n$  lies in  $\partial P$ . Put together, these maps give rise to a continuous map

$$h : N \rightarrow T$$

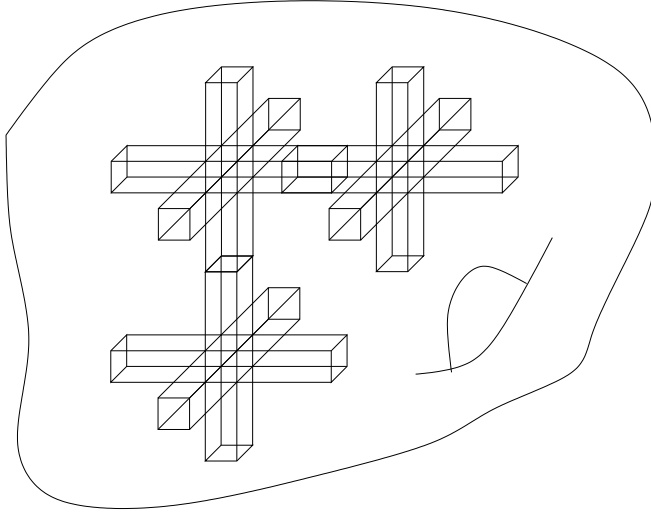
from the closed  $n$ -pseudomanifold  $N = N' \cup N''_2$  lying in  $Q \cup \partial P$  to the cubical  $(n - 1)$ -complex  $T = T' \cup_S T''$  obtained by gluing  $T'$  and  $T''$  along the cubical  $(n - 2)$ -complex  $S$  formed of the pieces  $Z^{n-2} \subseteq C^n$ , where  $C^n$  is a cubical  $n$ -simplex of  $\partial P$ . (As a result of the inclusions  $Z^{n-2} \subseteq Z^{n-1} \subseteq T'$  and  $Z^{n-2} = Z^{n-2} \times \{\frac{1}{2}\} \subseteq T''$ , the complex  $S$  lies both in  $T'$  and in  $T''$ .) See Figure 7 for a representation of  $N$ .

By construction, every fiber  $h^{-1}(t) \subseteq N$  with  $t \in T$  agrees with a fiber of  $\theta$  or  $\Theta$ , and therefore is isomorphic to the 1-skeleton of a cube of dimension at most  $n + 1$ ; see Figure 3. Thus, every fiber  $h^{-1}(t)$  has at most  $(n + 1) 2^n$  edges. Furthermore, every fiber of  $h$  is sent by  $\bar{f}$  to a (possibly degenerate) cubical graph of length at most  $(n + 1) 2^n \delta < W(M)$ . By definition of the homology 1-waist, see Definition 1.1 where  $p = 1$ , the restriction  $\bar{f}_1 N : N \rightarrow M$  of  $\bar{f}$  to  $N$  is not of degree one. That is,

$$\bar{f}_*([N]) \neq [M] \in H_n(M). \tag{3.3}$$

By construction, the pseudomanifolds  $N$  and  $\partial P$  are homologous in  $Q \cup \partial P$ . Specifically, their difference as  $n$ -cycles bounds the pseudomanifold  $Q$ . This implies that

$$\bar{f}_*([N]) = \bar{f}_*([\partial P]) = \bar{\sigma}_*([\partial P]) \tag{3.4}$$



This is a part of  $N$ . It consists of the subdivided manifold together with the boundary of the tubular neighborhood of the inner part of the dual complex to  $P^1$ .

FIGURE 7. Cubical structure of the pseudomanifold  $N$

since  $\bar{f}$  agrees with  $\bar{\sigma}$  on  $\partial P$ . Now, the map  $\bar{\sigma}$  is a deformation of  $\sigma$ . Therefore,

$$\bar{\sigma}_*([\partial P]) = \sigma_*([\partial P]) = [M] \quad (3.5)$$

by (3.1). Thus, the relations (3.3), (3.4) and (3.5) lead to a contradiction.  $\square$

**Remark 3.4.** Working with simplicial complexes instead of cubical complexes yields a better constant in Theorem 3.3, namely,  $c_n = \frac{1}{(n+1)(n+2)}$ . Still, we decided to work with cubical complexes since the constructions of Section 2 are simpler to describe in this context.

Theorem 1.3 follows from Theorem 3.3 and Theorem 3.2.

#### 4. FILLING RADIUS AND MODIFIED HOMOLOGY 1-WAIST

The goal of this section is to present an extension of Theorem 3.3 where the filling radius is bounded from below in terms of a modified homology 1-waist more suited for applications in the calculus of variations from a geometric measure theory point of view.

As a preliminary, we are interested in triangulated compact  $n$ -pseudomanifolds  $M$  with or without boundary such that every vertex is incident to at most  $\kappa_n$  edges for some very large constants  $\kappa_n$  depending only on  $n$ . In this case, we say that  $M$  has  $\kappa_n$ -bounded local complexity, or bounded local complexity.

For example, every smooth manifold can be triangulated with bounded local complexity. This follows, for example, from H. Whitney's proof of the existence of smooth triangulations of manifolds. First, embed a manifold into a high-dimensional Euclidean space and consider a subdivision of the Euclidean space into very small cubes. Making a generic shift of this triangulation, one ensures that the manifold intersects each face of each cube transversely. If the cubes are small enough, the intersections of the manifold with the cubes are very close to the intersections with an  $n$ -plane and, therefore, can be subdivided in a locally bounded number of simplices.

Gromov proved in [8, §5 $\frac{5}{7}$ .II'] that every triangulated closed  $n$ -pseudomanifold such that each vertex is incident to at most  $\kappa$  simplices can be filled by a compact  $(n+1)$ -pseudomanifold such

that each vertex is incident to at most  $f(\kappa, n)$  edges for some function  $f$ . In other words, a closed pseudomanifold with locally bounded complexity is the boundary of a compact pseudomanifold with boundary with locally bounded complexity.

The above discussion can be summarized by the following lemma.

**Lemma 4.1.** *There exist positive integer numbers  $\kappa_1 < \kappa_2 < \dots$  such that*

- (1) *every smooth compact  $n$ -manifold can be triangulated with  $\kappa_n$ -bounded local complexity;*
- (2) *every closed  $n$ -pseudomanifold with  $\kappa_n$ -bounded local complexity bounds a compact  $(n+1)$ -pseudomanifold with  $\kappa_{n+1}$ -bounded local complexity.*

**Remark 4.2.** If a closed pseudomanifold  $P$  with bounded local complexity is contained in a (small) metric ball  $B$  of a Banach space, we can assume that the filling of  $P$  with bounded local complexity given by Lemma 4.1.(2) also lies in  $B$  after projection.

**Remark 4.3.** Bounded local complexity could also be defined for cubical structures instead of simplicial structures. Since every simplex can be decomposed into cubes and every cubical simplex can be decomposed into simplices, the two notions are equivalent. Namely, an  $n$ -pseudomanifold  $M$  admits a triangulation with  $\kappa_n$ -bounded local simplicial complexity if and only if it admits a cubical structure with  $\kappa'_n$ -bounded local cubical complexity.

The following result implies that the filling with bounded local complexity given by Lemma 4.1.(2) can be chosen arbitrarily close to any given filling.

**Proposition 4.4.** *Let  $P$  be a compact  $(n+1)$ -pseudomanifold in a Banach space  $E$ , whose boundary  $\partial P$  has  $\kappa_n$ -bounded local complexity. Then there exists a compact  $(n+1)$ -pseudomanifold  $P'$  in  $E$  with the same boundary as  $P$  such that  $P'$  has  $\kappa_{n+1}$ -bounded local complexity and is at arbitrarily small Hausdorff distance to  $P$ .*

*Proof.* First, we are going to sketch the general idea of the proof. We proceed in the following way. We remove small neighbourhoods of all vertices, creating for each vertex  $p$  a “hole” bounded by the pseudomanifold  $Lk(p)$ , where  $Lk(p)$  denotes the link of  $p$ . Then we remove small neighbourhoods of all edges  $e$  creating “holes”  $Lk(e) \times e$ , and so on. At the end, each  $n$ -simplex will be truncated to a certain polytope (called permutohedron). We can triangulate this polytope into a certain fixed way. The resulting polyhedron will automatically have bounded local complexity. Then we reconstruct a filling  $P'$  of  $\partial P$  starting from faces with the lowest codimension and going up. At each stage, we fill each link by the bounded local complexity filling provided by Lemma 4.1. As all our surgeries can be done arbitrarily closely to the original filling, our construction almost does not affect the distance from the filling  $P'$  to  $\partial P$ .

The actual proof goes as follows. Without loss of generality, we can assume that  $P$  is piecewise linear and that its triangulation  $\mathcal{T}$  is  $\epsilon$ -fine (but still has bounded local complexity). Consider the dual polyhedral decomposition  $\mathcal{P}$  of the triangulation  $\mathcal{T}$  of  $P$ . By definition, the dual polyhedral decomposition  $\mathcal{P}$  is formed of the closed stars of the vertices of  $\mathcal{T}$  in the first barycentric subdivision  $\mathcal{T}'$  of  $\mathcal{T}$ .

Let  $\Delta^{n-i}$  be an  $(n-i)$ -face of  $\mathcal{T}$  with  $i \in \{0, \dots, n\}$ . Since  $P$  is a compact  $(n+1)$ -pseudomanifold with boundary, the link  $Lk(\Delta^{n-i})$  of  $\Delta^{n-i}$  in  $\mathcal{T}'$  is a compact  $i$ -pseudomanifold\* with boundary if  $\Delta^{n-i}$  lies in  $\partial P$  and without boundary otherwise. Define also the closed  $i$ -pseudomanifold

$$Lk_+(\Delta^{n-i}) = Lk(\Delta^{n-i}) \cup \text{cone}(\partial Lk(\Delta^{n-i})) \quad (4.1)$$

---

\*In the proof of Proposition 4.5, we relax the usual definition of a pseudomanifold to a finite disjoint union of pseudomanifolds allowing a pseudomanifold to be non-connected.

where  $\text{cone}(\partial Lk(\Delta^{n-i}))$  is the cone over  $\partial Lk(\Delta^{n-i})$  arising from the center  $\omega(\Delta^{n-i})$  of  $\Delta^{n-i}$ . Note that  $Lk_+(\Delta^{n-i}) = Lk(\Delta^{n-i})$  if  $\Delta^{n-i}$  does not lie in  $\partial P$ , and that  $Lk_+(\Delta^{n-i}) \subseteq \partial P$  if  $\Delta^{n-i}$  lies in  $\partial P$ .

By construction,

$$Lk(\Delta^{n-i}) = \bigcup_{\Delta^{n-i} \subseteq \Delta^{n-i+1}} \text{cone}(Lk_+(\Delta^{n-i+1})) \quad (4.2)$$

where the union is over all the  $(n-i+1)$ -simplices  $\Delta^{n-i+1}$  of  $\mathcal{T}$  containing  $\Delta^{n-i}$  and  $\text{cone}(Lk_+(\Delta^{n-i+1}))$  is the cone over  $Lk_+(\Delta^{n-i+1})$  arising from  $\omega(\Delta^{n-i+1})$ .

Note that

$$\text{cone}(Lk(\{p\})) = \text{Star}(p)$$

for every vertex  $p$  of  $\mathcal{T}$ . Furthermore,  $\text{Star}(p)$  and  $\text{Star}(q)$  intersect each other if and only if  $p$  and  $q$  are adjacent vertices of  $\mathcal{T}$ . In this case,

$$\text{Star}(p) \cap \text{Star}(q) = \text{cone}(Lk_+([p, q])). \quad (4.3)$$

Since the pseudomanifold  $P$  is formed of the stars  $\text{Star}(p)$ , it can be reconstructed from the augmented 1-dimensional links  $Lk_+(\Delta^{n-1})$  and the centers of the faces of  $\mathcal{T}$  by following the pattern (4.2) and the relation (4.1). This has to be done iteratively for  $i$  equals 1 to  $n$  until  $Lk(\{p\})$  and eventually  $\text{cone}(Lk(\{p\}))$  are reconstructed. Observe that the boundary  $\partial P$  of  $P$  is formed of the union of all the cones occurring in (4.1). That is,

$$\partial P = \bigcup_{p \text{ vertex of } \mathcal{T}} \text{cone}(\partial Lk(\{p\}))$$

where the union is over all vertices  $p$  of  $\mathcal{T}$  (lying in  $\partial P$ ).

Now, we want to construct a different filling  $P'$  with bounded local complexity following a similar (re)-construction procedure. Specifically, we want to define a compact  $i$ -pseudomanifold  $\Phi(Lk(\Delta^{n-i}))$  with bounded local complexity lying at distance  $\lesssim \epsilon$  from a vertex of  $\mathcal{T}$  for every  $(n-i)$ -face  $\Delta^{n-i}$  of  $\mathcal{T}$ . Moreover, we require that  $\partial\Phi(Lk(\Delta^{n-i})) = \partial Lk(\Delta^{n-i})$  whenever  $\Delta^{n-i}$  lies in  $\partial P$ . For  $i = 1$ , let  $\Phi(Lk(\Delta^{n-1})) = Lk(\Delta^{n-1})$ . As a union of 1-pseudomanifolds,  $Lk(\Delta^{n-1})$  has 2-bounded local complexity and lies at distance at most  $\epsilon$  from  $\omega(\Delta^{n-1})$ .

Suppose that  $\Phi(Lk(\Delta^{n-i}))$  is defined for every  $(n-i)$ -face  $\Delta^{n-i}$  of  $\mathcal{T}$ . By similarity with (4.1), define

$$\Phi(Lk_+(\Delta^{n-i})) = \Phi(Lk(\Delta^{n-i})) \cup \text{cone}(\partial\Phi(Lk(\Delta^{n-i}))).$$

Recall that  $\partial\Phi(Lk(\Delta^{n-i})) = \partial Lk(\Delta^{n-i})$  is empty if  $\Delta^{n-i}$  does not lie in  $\partial P$  and is contained in  $\partial P$  otherwise. Thus,  $\Phi(Lk_+(\Delta^{n-i}))$  is a closed  $i$ -pseudomanifold with bounded local complexity.

Now, define  $\Phi(\text{cone}(Lk_+(\Delta^{n-i})))$  as the polyhedral pseudomanifold filling of  $\Phi(Lk_+(\Delta^{n-i}))$  with bounded local complexity given by Lemma 4.1.(2). This filling lies within distance  $\lesssim \epsilon$  from a vertex of  $\mathcal{T}$ ; see Remark 4.2. Finally, by similarity with (4.2), let

$$\Phi(Lk(\Delta^{n-i-1})) = \bigcup_{\Delta^{n-i-1} \subseteq \Delta^{n-i}} \Phi(\text{cone}(Lk_+(\Delta^{n-i}))).$$

Note that  $\Phi(Lk(\Delta^{n-i-1}))$  also lies within distance  $\lesssim \epsilon$  from a vertex of  $\mathcal{T}$ .

As in (4.3),  $\Phi(\text{Star}(p))$  and  $\Phi(\text{Star}(q))$  intersect each other if and only if  $p$  and  $q$  are adjacent vertices of  $\mathcal{T}$ . In this case, their intersection

$$\Phi(\text{Star}(p)) \cap \Phi(\text{Star}(q)) = \Phi(\text{cone}(Lk_+([p, q])))$$

is a compact  $n$ -pseudomanifold. This implies that the union

$$P' = \bigcup_{p \text{ vertex of } \mathcal{T}} \Phi(\text{Star}(p))$$

over all vertices of  $\mathcal{T}$  is a compact  $(n+1)$ -pseudomanifold with bounded local complexity lying at distance  $\lesssim \epsilon$  from the vertex set of  $\mathcal{T}$  and so from  $P$ . Furthermore,

$$\partial P' = \bigcup_{p \text{ vertex of } \mathcal{T}} \text{cone}(\partial\Phi(Lk(\{p\}))) = \bigcup_{p \text{ vertex of } \mathcal{T}} \text{cone}(\partial Lk(\{p\})) = \partial P$$

as required.  $\square$

This proposition has the following immediate corollary.

**Proposition 4.5.** *Let  $M$  be a closed Riemannian  $n$ -manifold embedded into  $L^\infty(M)$  by the Kuratowski embedding. Then, for every  $\epsilon > 0$ , the manifold  $M$  bounds a compact  $(n+1)$ -pseudomanifold  $P$  with  $\kappa_{n+1}$ -bounded local complexity that is contained in the  $(\text{FillRad}(M) + \epsilon)$ -neighbourhood of  $M$ .*

**Remark 4.6.** Proposition 4.5 yields a control on the local complexity of the simplicial/cubical structure of the filling, but it does not provide any control on the total number of  $n$ -simplices/cubical  $n$ -simplices of the filling (which may be arbitrarily large).

Now, we can derive the following filling radius estimate extending Theorem 3.3. See (1.3) for the definition of  $W'_1(M)$ .

**Theorem 4.7.** *Let  $M$  be a closed  $n$ -manifold. Then every Riemannian metric on  $M$  satisfies*

$$\text{FillRad}(M) \geq c_n W'_1(M)$$

for  $c_n = \frac{1}{2^{2n-1}(n+1)! \kappa_n}$ .

*Proof.* We argue by contradiction as in the proof of Theorem 3.3 using the same notations. Suppose that there exists a continuous map  $\sigma : P \rightarrow U_\nu(M) \subseteq L^\infty(M)$  defined on a compact cubical  $(n+1)$ -pseudomanifold  $P$  with boundary such that the restriction  $\sigma : \partial P \rightarrow M$  satisfies

$$\sigma_*([\partial P]) = [M] \in H_n(M)$$

with  $\nu < \frac{1}{8\kappa'_n} W'_1(M)$ , where  $\kappa'_n$  is defined in terms of  $\kappa_n$  in (4.5).

From now on, we will consider a cubical structure of  $P$  with bounded local complexity given by Proposition 4.5. As in the proof of Theorem 3.3, we construct a map  $\bar{f} : Q \cup \partial P \rightarrow M$  where  $Q$  is a neighborhood of  $P^1$  in  $P$  which agrees with a deformation  $\bar{\sigma}$  of  $\sigma$ . Recall that the map  $\bar{f}$  takes every edge of  $P^1$  to a segment of length less than  $\delta = 2\nu + \epsilon < \frac{1}{4\kappa'_n} W'_1(M)$ . We also construct a map  $h : N \rightarrow T$  from a closed  $n$ -pseudomanifold  $N$  homologous to  $\partial P$  in  $Q \cup \partial P$  to a cubical  $(n-1)$ -complex  $T = T' \cup_S T''$  formed of  $(n-1)$ -cubes glued together along their  $(n-2)$ -faces, where each fiber  $h^{-1}(t)$  is isomorphic to the 1-skeleton of a cube of dimension at most  $n+1$ . If  $M$  is orientable, the filling  $P$  is also orientable (modulo  $\partial P$ ) and so is the closed pseudomanifold  $N$ .

Let  $C^{n+1}$  be a cubical  $(n+1)$ -simplex of  $P$ . Denote by  $\text{Isom}(C^{n+1})$  the (full) isometry group of  $C^{n+1}$ . Recall that  $T' \cap C^{n+1}$  agrees with

$$Z^{n-1} = \{t \in [-1, 1]^{n+1} \mid \text{there exist } i \neq j \text{ such that } t_i = t_j = 0\}.$$

A fundamental domain for the action of  $\text{Isom}(C^{n+1})$  on  $T' \cap C^{n+1} = Z^{n-1}$  is given by

$$\Delta = \Delta^{n-1} = \{t \in [-1, 1]^{n+1} \mid 0 \leq t_1 \leq \dots \leq t_{n-1} \leq 1 \text{ and } t_n = t_{n+1} = 0\}. \quad (4.4)$$

Its orbit gives rise to a natural triangulation of  $Z^{n-1}$  with

$$|\text{Isom}(C^{n+1})|/|\text{Stab}(\Delta^{n-1})| = 2^{n+1}(n+1)!/2 = 2^n(n+1)!$$

copies of  $\Delta^{n-1}$ . The pieces  $Z^{n-2} \times [0, 1] \subseteq C^{n+1}$  composing  $T''$  also lie in  $Z^{n-1}$  and the action of  $\text{Isom}(C^{n+1}) \cap \text{Stab}(Z^{n-2} \times [0, 1])$  on each of these pieces has  $\Delta^{n-1}$  for fundamental domain too. This gives rise to a triangulation of  $Z^{n-2} \times [0, 1]$  with

$$|\text{Isom}(C^{n+1}) \cap \text{Stab}(Z^{n-2} \times [0, 1])|/|\text{Stab}(\Delta^{n-1})| = 2^{n-1}n!$$

copies of  $\Delta^{n-1}$ , which is compatible with the triangulation of  $Z^{n-1}$ . Denote by  $q' : Z^{n-1} \rightarrow \Delta^{n-1}$  and by  $q'' : Z^{n-2} \times [0, 1] \rightarrow \Delta^{n-1}$  the quotient maps. Since  $T = T' \cup T''$  is made of copies of  $Z^{n-1}$  and  $Z^{n-2} \times [0, 1]$  glued together, there exists a surjective continuous map  $j : T \rightarrow \Delta^{n-1}$  whose restriction to each copy of  $Z_T$  agrees with the quotient maps  $q'$  or  $q''$ , where  $Z_T = Z^{n-1}$  or  $Z^{n-2} \times [0, 1]$ . (It does not matter how the copies of  $Z_T$  are isometrically identified as long as  $Z^{n-2} \times \{1\}$  lies in  $T' \cap T''$  since, at the end, we take the quotient by the isometry group of  $Z^{n-1}$  or  $Z^{n-2}$ .) Note that the restriction of every fiber of  $j : T \rightarrow \Delta$  to any copy of  $Z_T$  coincides with an orbit of the isometry group of  $Z^{n-1}$  or  $Z^{n-2}$ . By construction, the complex  $T$  is tiled with copies  $\Delta_i$  of  $\Delta$  such that  $j|_{\Delta_i} : \Delta_i \rightarrow \Delta$  is a diffeomorphism. Denote by  $\mathcal{T}_T$  the corresponding triangulation of  $T$ .

Since the map  $h : N \rightarrow T$  is a submersion away from the inverse image of the  $(n-2)$ -skeleton of  $\mathcal{T}_T$ , the composition

$$\tilde{h} : N \xrightarrow{h} T \xrightarrow{j} \Delta$$

is a submersion over the interior  $\mathring{\Delta}$  of  $\Delta$ . Moreover, every fiber of  $\tilde{h}$  over  $\mathring{\Delta}$  is composed of exactly  $2^n(n+1)!|P| + 2^{n-1}n!|\partial P|$  disjoint simple loops, where  $|P|$  is the number of cubical  $(n+1)$ -simplices of  $P$  and  $|\partial P|$  is the number of cubical  $n$ -simplices of  $\partial P$ . Furthermore, the image under  $\tilde{f}|_N : N \rightarrow M$  of each of these loops is of length at most  $4\delta$ .

Suppose that  $M$  is orientable. Fix an orientation on  $N$  and  $\Delta$ . Since  $\tilde{h} : N \rightarrow \Delta$  is a submersion away from the inverse image of  $\partial\Delta$ , we can define in a unique way an orientation on the fibers  $\tilde{h}^{-1}(x)$ , with  $x \in \mathring{\Delta}$ , so that  $\tilde{h}^*\omega_\Delta \wedge \omega_{\tilde{h}^{-1}(x)}$  is positive, where  $\omega_\Delta$  is a positive volume form on  $\Delta$  and  $\omega_{\tilde{h}^{-1}(x)}$  is a volume form on  $\tilde{h}^{-1}(x)$  defining its orientation. If  $M$  is nonorientable, we only consider unoriented cycles and there is no need to define an orientation on the fibers of  $\tilde{h}$ .

The family of 1-cycles  $\Xi_x = \tilde{h}^{-1}(x) \subseteq N$  with  $x \in \mathring{\Delta}$  extends by continuity to a family of 1-cycles parameterized by  $\Delta$  as follows. Specifically, we want to define  $\Xi_{x_0}$  for  $x_0 \in \partial\Delta$ . Fix  $t_0 \in j^{-1}(x_0)$ . For a small enough neighborhood  $\mathcal{U}$  of  $t_0$  in  $T$ , the points of  $j^{-1}(x) \cap \mathcal{U}$  converge to  $t_0$  as  $x \in \mathring{\Delta}$  goes to  $x_0$ . Moreover, the cardinality  $k_{t_0}$  of  $j^{-1}(x) \cap \mathcal{U}$  is bounded by  $2^n(n+1)!$  times the number of pieces  $Z_T$  of  $T$  containing  $t_0$ . Since each of these pieces lies in a cube  $C^{n+1}$  corresponding to a cubical  $(n+1)$ -simplex of  $P$  or a cubical  $n$ -simplex of  $\partial P$ , it follows from the  $\kappa_n$ -bounded local complexity of  $P$  that

$$k_{t_0} = |j^{-1}(x) \cap \mathcal{U}| \leq \kappa'_n := 2^n(n+1)! \kappa_n. \quad (4.5)$$

Furthermore, we have the following claim.

**Claim 4.8.** *Suppose that  $t_0 \notin Z^{n-2} \times \{0\}$ . For  $x \in \mathring{\Delta}$  close enough to  $x_0$ , the points of  $j^{-1}(x) \cap \mathcal{U}$  can be partitioned into pairs  $\{t_i, \bar{t}_i\}$  whose inverse images  $h^{-1}(t_i)$  and  $h^{-1}(\bar{t}_i)$  converge to  $h^{-1}(t_0)$  with opposite orientations as  $x$  goes to  $x_0$ .*

*Proof.* By symmetry, without loss of generality, we can assume that  $t_0$  lies in (the boundary of) a copy of  $\Delta \subseteq C^{n+1}$ ; see (4.4). Let us examine three mutually disjoint cases.



*Case 1.* Suppose that  $(t_0)_{n-1} = 1$  and  $0 < (t_0)_1 < \dots < (t_0)_{n-2} < 1$ . Thus,  $t_0$  lies in an open  $(n-2)$ -face  $F$  of  $\partial\Delta$ . Since  $P$  and  $\partial P$  are cubical pseudomanifolds (with and without boundary) and  $t_0 \notin Z^{n-2} \times \{0\}$ , there are exactly two pieces  $Z_T$  and  $\bar{Z}_T$  of  $T$  containing the  $(n-2)$ -face  $F$  of  $\partial\Delta$  with  $\Delta \subseteq Z_T$ . Denote by  $\bar{\Delta} \subseteq \bar{Z}_T$  the symmetric of  $\Delta$  with respect to  $F$ . The set  $j^{-1}(x) \cap \mathcal{U}$  is formed of exactly two points  $t \in \Delta$  and  $\bar{t} \in \bar{\Delta}$  symmetric with respect to  $F$ . By symmetry, the loops  $h^{-1}(t)$  and  $h^{-1}(\bar{t})$  have opposite orientation at the limit when  $x$  goes to  $x_0$ .

*Case 2.* Suppose that  $(t_0)_1 = 0$  (so Case 1 is not satisfied). Denote by  $\sigma$  the symmetry of  $C^{n+1}$  with respect to the hyperplane  $\{t_1 = 0\}$  of  $C^{n+1}$ . The set  $j^{-1}(x) \cap \mathcal{U} \cap C^{n+1}$  decomposes into two subsets  $\Sigma_+ \subseteq \{t_1 > 0\}$  and  $\Sigma_- \subseteq \{t_1 < 0\}$  symmetric with respect to  $\sigma$ . By symmetry, the loops  $h^{-1}(t)$  and  $\sigma(h^{-1}(t))$  have opposite orientation at the limit when  $x$  goes to  $x_0$ .

*Case 3.* Suppose that  $(t_0)_1 > 0$  and that Case 1 is not satisfied. As  $t_0 \in \partial\Delta^{n-1}$ , there exist disjoint subsets  $I_1, \dots, I_k \subseteq \{1, \dots, n-1\}$  with  $|I_i| \geq 2$  such that  $(t_0)_p = (t_0)_q$  for every  $p, q \in I_i$ . The isotropy subgroup of  $t_0$  for the action of  $\text{Isom}(C^{n+1})$  is isometric to the product  $\Gamma = S(I_1) \times \dots \times S(I_k)$ , where the symmetry groups  $S(I_i)$  act by isometries on  $C^{n+1}$  by permuting the coordinates with index in  $I_i$ . Furthermore, the points of  $\Sigma = j^{-1}(x) \cap \mathcal{U} \cap C^{n+1}$  form a free orbit of  $\Gamma$ . Fix a point  $s \in \Sigma$  and a bijection  $\theta : \Gamma_+ \rightarrow \Gamma_-$  between the orientation-preserving and orientation-reversing isometries of  $\Gamma$ . This bijection gives rise to a partition of  $\Sigma$  into pairs of points  $\{t_i, \bar{t}_i\}$ , where  $t_i = \sigma_i(s)$  and  $\bar{t}_i = \theta(\sigma_i)(s)$  for some  $\sigma_i \in \Gamma_+$ . By symmetry, the loops  $h^{-1}(t_i)$  and  $h^{-1}(\bar{t}_i)$  have opposite orientation at the limit when  $x$  goes to  $x_0$ .  $\square$

For every  $x \in \Delta$ , we have the following decomposition of  $\bar{h}^{-1}(x)$  into connected components

$$\bar{h}^{-1}(x) = \bigcup_{t \in j^{-1}(x)} h^{-1}(t). \quad (4.6)$$

Observe also that if  $t_0 \in Z^{n-2} \times \{0\}$ , then the fiber  $h^{-1}(t)$  converges to a point as  $t$  goes to  $t_0$ .

We can now define the 1-cycle  $\Xi_{x_0} \subseteq N$  so that its restriction to  $h^{-1}(t_0)$  agrees with the limit of the simple loops corresponding to the fibers of  $h$  over the  $k_{t_0}$  points of  $j^{-1}(x) \cap \mathcal{U}$  as  $x \in \mathring{\Delta}$  goes to  $x_0$ . This completely characterizes  $\Xi_{x_0}$  for each of its connected components since its support lies in  $\bar{h}^{-1}(x_0)$ . By Claim 4.8 and the observation following (4.6), the 1-cycle  $\Xi_{x_0}$  is equivalent to the zero 1-cycle. See Figure 8 and Figure 9.

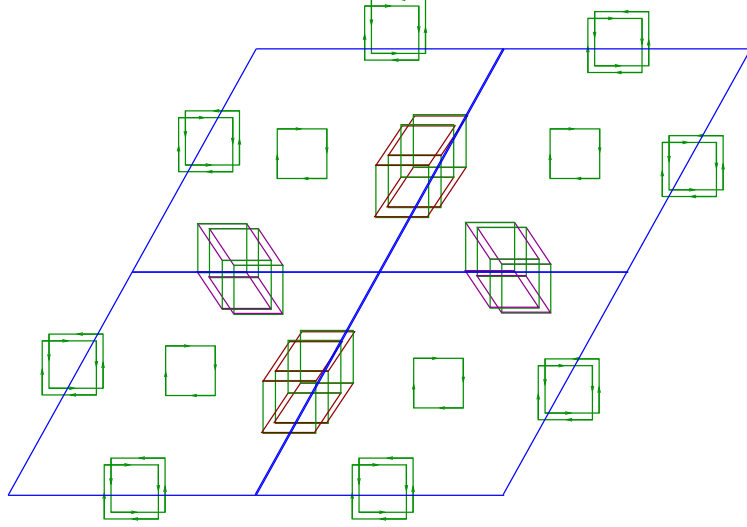
As noticed before, the length of the image under  $\bar{f}|_N : N \rightarrow M$  of each fiber  $h^{-1}(t)$  with  $t \in j^{-1}(x)$  and  $x \in \mathring{\Delta}$  is at most  $4\delta$ . Thus, the maximal length of the image of a connected component of  $\Xi_x$  (counted according to its geometric multiplicity) is at most  $4\delta\kappa'_n$ ; see Figure 10. That is, for every  $x \in \Delta$ , we have

$$\max_{C \subseteq \Xi_x} \text{length}(\bar{f}|_C) \leq 4\delta\kappa'_n$$

where  $C$  runs over the connected components of the 1-cycle  $\Xi_x$ .

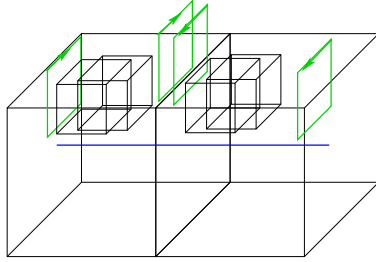
Let us extend this family  $\Xi_x$  of 1-cycles to  $\partial\Delta \times [0, 1]$ . By Claim 4.8 (and the observation following (4.6)), the 1-cycle  $\Xi_x$  with  $x \in \partial\Delta$  can be seen as a graph where the (algebraic) sum of each edge  $[a, b]$  vanishes. Denote by  $m$  the midpoint of  $[a, b]$ . For  $(x, t) \in \partial\Delta \times [0, 1]$ , define the 1-cycle  $\Xi_{x,t}$  by replacing each edge  $[a, b]$  of  $\Xi_x$  with  $[a, a_t] \cup [b_t, b]$  (keeping the same multiplicity), where  $a_t = ta + (1-t)m$  and  $b_t = tb + (1-t)m$  in barycentric coordinates; see Figure 11. Observe that  $\Xi_{x,t}$  is a continuous family of 1-cycles which agrees with  $\Xi_x$  for  $t = 0$  and with a union of points corresponding to the vertices of  $\Xi_x$  for  $t = 1$ . Thus, we obtain a family of 1-cycles  $\Xi_u$

This figure illustrates the map from  $S^2$  to the space of cycles in  $N$  or  $M$  in 3-D case. Here we are assuming that  $P$  is a manifold. The center is mapped to the 1-skeleton of a 4-cube, where each edge appears three times with each of the two possible orientations.



Part of  $T$  that lies in some cubical simplex of dimension 4.

FIGURE 8. The  $S^2$ -family of 1-cycles  $\Xi_x$



This figure illustrates the map from  $S^1$  to the space of cycles on  $N$  or  $M$ , when  $M$  is a surface. Here we see an edge of  $T$  that passes through two cubical simplices. The points that lie at the centers of cubes are mapped to zero cycles formed by taking the 1-skeleton of the corresponding 3-cube with both orientations. The center of an edge that lies in the common face of the two cubes is mapped to the zero cycle formed by two identical, but oppositely oriented curves.

FIGURE 9. The  $S^1$ -family of 1-cycles  $\Xi_x$

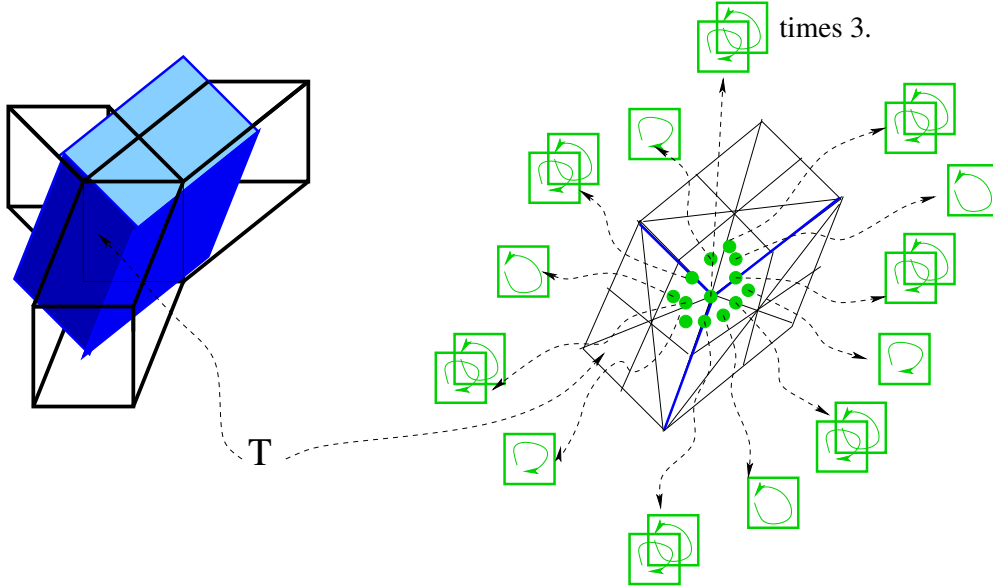
with  $u \in B^{n-1} = \Delta \cup (\partial\Delta \times [0, 1])$ , where the peripheral cycles  $\Xi_u$  for  $u \in \partial B^{n-1}$  are unions of points and

$$\max_{C \subseteq \Xi_u} \text{length}(\bar{f}|_C) \leq 4\delta\kappa'_n < W'_1(M) \tag{4.7}$$

for every  $u \in B^{n-1}$ .

By construction, the inverse image under  $h : N \rightarrow T$  of the interior  $\mathring{\Delta}_i$  of an  $(n-1)$ -simplex  $\Delta_i$  of  $\mathcal{T}_T$  identifies with the product  $\mathring{\Delta}_i \times S^1$ , where each factor  $\{x\} \times S^1$  agrees with the restriction of  $\Xi_x$  to  $h^{-1}(\mathring{\Delta}_i)$  for every  $x \in \mathring{\Delta}_i$ . This implies that the family of 1-cycles  $\Xi_u$  induces a nontrivial class in

$$\pi_{n-1}(\mathcal{Z}_1(N; G), \{0\}) \simeq H_n(N; G) \simeq G$$



When the filling of  $M$  is not a manifold, ( $\dim M=3$ ), it is possible for a large number of 3-cubes to share a face. This number will affect the bound when we consider the map from  $S^2$  to the space of cycles. In our figure the number of 3-cubes sharing the face is 3.

FIGURE 10. Multiple cubes sharing a face

under the Almgren isomorphism [2], where  $G = \mathbb{Z}$  if  $M$  (and so  $N$ ) is orientable and  $G = \mathbb{Z}_2$  otherwise. By definition of  $W'_1(M)$ , see (1.3), we derive from (4.7) that the image by  $\bar{f}$  of the family of 1-cycles  $\Xi_u$  does not represent the fundamental class of  $M$ . Thus,

$$\bar{f}_*([N]) \neq [M] \in H_n(M; G).$$

On the other hand, since the pseudomanifolds  $N$  and  $\partial P$  are homologous in  $Q \cup \partial P$  and the map  $\bar{f} : Q \cup \partial P \rightarrow M$  agrees with a deformation of  $\sigma$  on  $\partial P$ , we deduce that

$$\bar{f}_*([N]) = \bar{f}_*([\partial P]) = \sigma_*([\partial P]) = [M].$$

Hence a contradiction.  $\square$

**Remark 4.9.** In the proof of Theorem 4.7, we did not define the family  $\Xi_x$  of 1-cycles of  $N$  with  $x \in S^{n-1}$  as the inverse images of some map  $N \rightarrow S^{n-1}$ , but as a perturbation/extension of the family given by the inverse images of  $\bar{h} : N \rightarrow \Delta^{n-1}$ . It would be possible to do so by pushing apart the 1-cycles and inserting small new 1-cycles along the lines of the examples given in Subsection 1.2. This would allow us to replace  $W'_1(M)$  in Theorem 4.7 with  $\bar{W}_1(M)$  defined in (1.2).

We can refine the inequality of Theorem 4.7 by considering the invariant  $W''_1(M)$  instead of  $W'_1(M)$ ; see (1.4).

**Theorem 4.10.** *Let  $M$  be a closed  $n$ -manifold with  $n \geq 3$ . Then every Riemannian metric on  $M$  satisfies*

$$\text{FillRad}(M) \geq c_n W''_1(M)$$

for some explicit positive constant  $c_n$  depending only on  $n$ .

*Proof.* We argue as in the proof of Theorem 4.7, using the same notations and pointing out only the differences. Since  $n \geq 3$ , by slightly perturbing  $\bar{f}$ , we can assume that the restriction

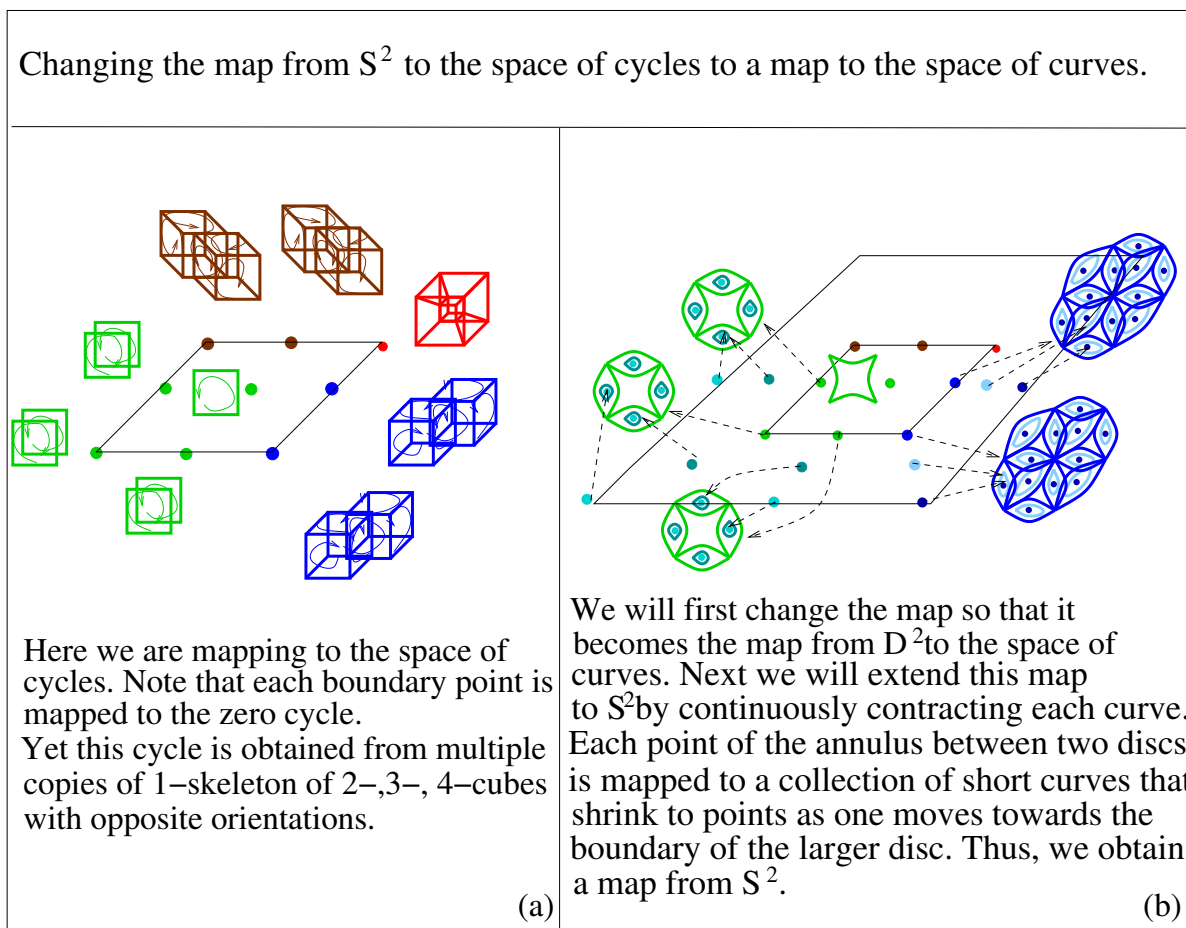


FIGURE 11. Extension of the family of cycles to the collar of  $\Delta$

of  $\bar{f}$  to  $P^1$  is an embedding into  $M$ . By construction, the images by  $\bar{f}$  of the fibers  $h^{-1}(t)$  with  $t \in T \setminus T''$  lie in small cubes  $\bar{f}(C^n)$  of  $M$  with  $C^n \subseteq \partial P$ . Furthermore, these images are pairwise disjoint in  $M$  and do not intersect the graph  $\bar{f}(P^1)$ . Likewise, the fibers  $h^{-1}(t)$  with  $t \in T \setminus T''$  are isomorphic to the 1-skeleton of a cube of dimension at most  $n + 1$ , and their images by  $\bar{f}$  lie in the graph  $\bar{f}(P^1)$ . We can slightly modify  $\bar{f}$  so that the images of the fibers  $h^{-1}(t)$  with  $t \in T \setminus T''$  are disjoint in  $M$ . Perturbing also the map  $j : T \rightarrow \Delta$ , we can further assume the following. For every  $t \in j^{-1}(x)$ , denote by  $C_t^{n-1}$  the  $(n - 1)$ -cube of  $T$  containing  $t$  (or one of them). For every  $t' \in j^{-1}(x)$  not lying in an  $(n - 1)$ -cube of  $T$  intersecting  $C_t^{n-1}$ , the image of  $h^{-1}(t')$  is disjoint from the image of  $h^{-1}(t)$ . Now, by the  $\kappa_n$ -bounded local complexity of  $P$ , the number of fibers  $h^{-1}(t')$  lying in an  $(n - 1)$ -cube of  $T$  intersecting  $C_t^{n-1}$  is bounded by an explicit constant depending only on  $n$ . As a result, we obtain a family  $\Xi'_u$  of 1-cycles of  $N$  with  $u \in S^{n-1}$  such that every connected component  $C'$  of  $\bar{f}(\Xi'_u)$  satisfies

$$\text{length}(C') \leq \kappa_n'' \delta$$

for some explicit constant  $\kappa_n''$  depending only on  $n$ . Since the 1-cycle family  $\Xi'_u$  is a deformation of the original 1-cycle family  $\Xi_u$ , the homotopy class it induces in  $\pi_1(\mathcal{Z}_1(N; G), \{0\})$  is nontrivial. We conclude as in the proof of Theorem 4.7.  $\square$

## 5. FILLING RADIUS AND RELATIVE HOMOTOPY 1-WAIST

We adapt the argument of Section 3 to establish a lower bound on the filling radius of a closed Riemannian manifold in terms of its relative homotopy 1-waist and derive Theorem 1.11.

**Theorem 5.1.** *Fix  $k \leq n - 1$ . Let  $M$  be a closed  $n$ -manifold and  $\Phi : M \rightarrow K$  be a continuous map to a CW-complex  $K$  with  $\pi_i(K) = 0$  for every  $i \geq k + 1$ . Suppose that  $\Phi_*([M]) \neq 0 \in H_n(K; G)$  for some homology coefficient group  $G$ . Then every Riemannian metric on  $M$  satisfies*

$$\text{FillRad}(M) \geq c_n W_{1,k}(M, \Phi)$$

for some explicit positive constant  $c_n$  depending only on  $n$ .

*Proof.* We are going to give a proof by contradiction. Initially, we argue as in the proof of Theorem 3.3 using the same notations. Suppose that there exists a continuous map  $\sigma : P \rightarrow U_\nu(M) \subseteq L^\infty(M)$  defined on a compact cubical  $(n + 1)$ -pseudomanifold such that the restriction  $\sigma : \partial P \rightarrow M$  satisfies

$$\sigma_*([\partial P]) = [M] \in H_n(M; G)$$

with  $\nu < \frac{1}{(k+1)2^{k+1}} W_{1,k}(M, \Phi)$ . As in the proof of Theorem 3.3, we construct a map  $f : P^1 \rightarrow M$  which agrees with  $\sigma$  on the 1-skeleton of  $\partial P$  so that the lengths of the images of the edges of  $P^1$  are less than  $\delta = 2\nu + \varepsilon < \frac{1}{(k+1)2^k} W_{1,k}(M, \Phi)$ .

Let  $P_*^{k+1}$  be the cubical  $(k + 1)$ -complex formed of the cubical  $(k + 1)$ -simplices of  $P$  not lying in  $\partial P$ . Denote by  $Q^{k+1} \subseteq P_*^{k+1}$  the cubical  $(k + 1)$ -complex formed of the pieces  $X_1^{k+1} \subseteq C^{k+1}$ , where  $C^{k+1}$  is a cubical  $(k + 1)$ -simplex of  $P$  not lying in  $\partial P$ . We define a continuous map  $\bar{f} : Q^{k+1} \cup \partial P \rightarrow M$  from  $f : P^1 \rightarrow M$  which coincides with the deformation  $\bar{\sigma} : \partial P \rightarrow M$  of  $\sigma$  and takes every edge of  $P^1$  to a segment of length less than  $\delta$ ; see the proof of Theorem 3.3 for the details of the construction.

Let us extend this map to the  $(k + 1)$ -skeleton  $P^{k+1}$  of  $P$ . Define the cubical  $k$ -complex  $R'$  and the cubical  $(k - 1)$ -complex  $T'$  by respectively pasting together the pieces  $Y^k \subseteq C^{k+1}$  and the pieces  $Z^{k-1} \subseteq C^{k+1}$ , where  $C^{k+1}$  is a cubical  $(k + 1)$ -simplex of  $P$  not lying in  $\partial P$ . Denote by  $h' : R' \rightarrow T'$  the map whose restriction to  $Y^k$  agrees with the map  $\theta : Y^k \rightarrow Z^{k-1}$  defined in (2.2) with  $p = 1$ . Similarly, define the cubical  $k$ -complex  $R''$  and the cubical  $(k - 1)$ -complex  $T''$  by respectively pasting together the pieces  $X_2^k \subseteq C^k$  and the pieces  $Z^{k-2} \times [0, \frac{1}{2}]$  with  $Z^{k-2} \subseteq C^k$ , where  $C^k$  is a cubical  $k$ -simplex of  $\partial P$ . Denote by  $h'' : R'' \rightarrow T''$  the map whose restriction to  $X_2^k$  agrees with the map  $\Theta : X_2^k \rightarrow Z^{k-2} \times [0, \frac{1}{2}]$  defined in (2.3) with  $p = 1$ . The two maps  $h'$  and  $h''$  so-defined agree on the intersection  $R' \cap R''$  formed of the pieces  $Y^{k-1} \subseteq C^k$  lying in  $\partial P$ , after identifying  $Z^{k-2} \subseteq Z^{k-1} \subseteq T'$  and  $Z^{k-2} = Z^{k-2} \times \{\frac{1}{2}\} \subseteq T''$ , where  $Z^{k-2} \subseteq C^k$  lies in  $\partial P$ . Put together, these maps give rise to a continuous map

$$h : R \rightarrow T$$

from the cubical  $k$ -complex  $R = R' \cup R''$  lying in  $P^{k+1}$  to the cubical  $(k - 1)$ -complex  $T = T' \cup_S T''$  obtained by gluing  $T'$  and  $T''$  along the cubical  $(k - 2)$ -complex  $S$  formed of the pieces  $Z^{k-2} \subseteq C^k$  lying in  $\partial P$ .

Consider the composite map

$$F = \Phi \circ \bar{f} : Q^{k+1} \cup \partial P \rightarrow K$$

extending  $\Phi \circ \bar{\sigma} : \partial P \rightarrow K$ . By construction, every fiber  $h^{-1}(t) \subseteq R$  with  $t \in T$  agrees with a fiber of  $\theta$  or  $\Theta$ , and is isomorphic to the 1-skeleton of a cube of dimension at most  $k + 1$ . Thus, every fiber of  $h$  is sent by  $\bar{f}$  to a graph of length at most  $(k + 1)2^k \delta < W_{1,k}(M, \Phi)$ . By

definition of the  $\Phi$ -relative homotopy  $k$ -waist, see Definition 1.1, the restriction  $F|_R : R \rightarrow K$  is homotopic to a map

$$R \xrightarrow{h} T \rightarrow K$$

which factors out through  $h$ . Thus, the map  $F : Q^{k+1} \cup \partial P \rightarrow K$  extends to

$$R \times [0, 1] / \sim$$

where  $(x, 1) \sim (y, 1)$  if and only if  $h(x) = h(y)$ . Since the complement of the interior of  $Q^{k+1}$  in  $P_*^{k+1}$  is homeomorphic to  $R \times [0, 1] / \sim$ , this yields a map  $F : P_*^{k+1} \cup \partial P \rightarrow K$  defined in particular on the  $(k+1)$ -skeleton  $P^{k+1}$  of  $P$ .

Now, since  $\pi_i(K) = 0$  for every  $i \geq k+1$ , the map  $F$  further extends into  $F : P \rightarrow K$ . Recall that the restriction of  $F$  to  $\partial P$  agrees with  $\Phi \circ \bar{\sigma}$ . Therefore, the homology class

$$(\Phi \circ \bar{\sigma})_*([\partial P]) = \Phi_*([M]) \in H_n(K; G)$$

is trivial. Hence a contradiction.  $\square$

Theorem 1.11 follows from Theorem 5.1 and Theorem 3.2.

## 6. FILLING RADIUS, URYSOHN WIDTH AND 1-WAIST

Using the filling estimate established in Theorem 3.3, we show that the filling radius of a closed Riemannian manifold is roughly equal to its homology 1-waist.

We need to introduce the following notion related to the 1-waist.

**Definition 6.1.** The *Urysohn width* of a closed Riemannian  $n$ -manifold  $M$ , denoted by  $\text{UW}(M)$ , is the infimum of the distances  $\delta$  such that there exists a continuous map  $M \rightarrow X$  from  $M$  to a simplicial  $(n-1)$ -complex whose fibers have diameters less than  $\delta$ . Strictly speaking, this definition corresponds to the notion of Urysohn  $(n-1)$ -width.

Though the homology 1-waist  $W(M)$  and the Urysohn width  $\text{UW}(M)$  are defined in similar terms, the two notions present some differences. First, the homology 1-waist measures the 1-polyhedron length, while the Urysohn width measures the 1-polyhedron diameter. Second, the homology 1-waist is concerned with homology 1-sweepouts made of 1-polyhedra which may intersect each other, while, by definition, the fibers involved in the definition of the Urysohn width are disjoint. Still, the two notions are connected through the filling radius estimate of Theorem 3.3 and the general bound

$$\text{FillRad}(M) \leq \frac{1}{2} \text{UW}(M) \tag{6.1}$$

obtained in [6, Appendix 1] satisfied by every closed pseudomanifold.

The following result extends the bound (6.1) to the homology 1-waist  $W(M)$ .

**Proposition 6.2.** *Every closed Riemannian  $n$ -manifold  $M$  satisfies*

$$\text{FillRad}(M) \leq \frac{1}{2} W(M).$$

*Proof.* By definition of the homology 1-waist, see Definition 1.1 where  $p = 1$ , there exist a continuous map  $h : N \rightarrow T$  from a closed  $n$ -pseudomanifold  $N$  to a finite simplicial  $(n-1)$ -complex  $T$  and a degree one map  $\varphi : N \rightarrow M$  such that

$$\text{length } \varphi|_{h^{-1}(t)} < W(M) + 2\varepsilon \tag{6.2}$$

for every  $t \in T$ , where  $\varepsilon > 0$  is any given positive real. Slightly perturbing  $\varphi$  if necessary, we can always assume that  $\varphi$  is piecewise smooth. Consider the metric  $g_N = \varphi^*g_M + \lambda^2g_0$  on  $N$ ,

where  $\varphi^*g_M$  is the pull-back of the metric  $g_M$  on  $M$  under  $\varphi$ , and  $g_0$  is a fixed metric on  $N$  with  $\lambda > 0$  arbitrarily small. The following chain of inequalities holds

$$\text{FillRad}(M) \leq \text{FillRad}(N) \leq \frac{1}{2} \text{UW}(N) \leq \frac{1}{2} \text{W}(M) + \varepsilon$$

where each inequality can be justified as follows. By [6, p. 6], the filling radius does not increase under 1-Lipschitz degree one maps. Applying this result to the contracting map  $\varphi : (N, g_N) \rightarrow (M, g_M)$  yields the first inequality. The second inequality is given by (6.1). By construction, the diameter of the fibers  $h^{-1}(t)$  of  $h : N \rightarrow T$  is less than  $\text{W}(M) + 2\varepsilon$ , see (6.2), which implies the third inequality. Now, letting  $\varepsilon$  go to zero in the previous inequality chain, we obtain the relation  $\text{FillRad}(M) \leq \frac{1}{2} \text{W}(M)$ .  $\square$

Theorem 1.13 follows from Theorem 3.3 and Proposition 6.2.

## 7. HYPERSPHERICITY, FILLING RADIUS AND URYSOHN WIDTH

Using the filling estimate established in Theorem 3.3, we derive that the filling radius and the hypersphericity of a closed orientable Riemannian manifold can be arbitrarily far apart.

**Definition 7.1.** The *hypersphericity* of a closed orientable Riemannian  $n$ -manifold  $M$ , denoted by  $\text{HS}(M)$ , is the supremum of the radii  $R$  such that there exists a 1-Lipschitz map  $M \rightarrow S^n(R)$  of nonzero degree from  $M$  to the standard  $n$ -sphere  $S^n(R)$  of radius  $R$ .

The hypersphericity is related to the filling radius and the Urysohn width through the following inequalities.

**Proposition 7.2.** *Let  $M$  be a closed orientable  $n$ -manifold. Then*

$$\frac{1}{2} \arccos\left(-\frac{1}{n+1}\right) \text{HS}(M) \leq \text{FillRad}(M) \leq \frac{1}{2} \text{UW}(M).$$

*Proof.* The second inequality comes from (6.1). For the first inequality, it is convenient to work with the rational filling radius, which is defined in a similar way as the standard filling radius, see Definition 3.1, except that the homology coefficients are in  $\mathbb{Q}$ . It follows from abstract nonsense using the relation  $H_n(X; \mathbb{Q}) \simeq H_n(X; \mathbb{Z}) \otimes \mathbb{Q}$  for simplicial complexes  $X$  given by the universal coefficient theorem for homology that

$$\text{FillRad}_{\mathbb{Q}}(M) \leq \text{FillRad}(M).$$

By [6, p. 6], the rational filling radius does not increase under 1-Lipschitz maps of nonzero degree. Thus,

$$\text{FillRad}_{\mathbb{Q}}(S^n) \text{HS}(M) \leq \text{FillRad}_{\mathbb{Q}}(M).$$

Now, the filling radius of the standard sphere  $S^n$  has been computed in [16] and the argument extends to the rational filling radius. More specifically,

$$\text{FillRad}_{\mathbb{Q}}(S^n) = \frac{1}{2} \arccos\left(-\frac{1}{n+1}\right).$$

Hence the proposition.  $\square$

**Remark 7.3.** A direct inequality between the hypersphericity and the Urysohn width can be found in [7], [9, 2.12 $\frac{1}{2}_+$ ] and [13].

Proposition 8.3 of [3], combined with the previous proposition or remark, shows that for every Riemannian metric on  $S^2$ , these geometric invariants are roughly the same

$$\text{HS}(S^2) \simeq \text{FillRad}(S^2) \simeq \text{UW}(S^2).$$

In higher dimension, examples showing that the hypersphericity and the Urysohn width can be arbitrarily far apart have first been constructed in [13, §5]. Using the relationship between the filling radius and the homology 1-waist established in Theorem 3.3, we show that a similar phenomenon occurs between the hypersphericity and the filling radius.

**Theorem 7.4.** *There exists a sequence  $(g_i)$  of Riemannian metrics on  $S^4$  with arbitrarily small hypersphericity and filling radius bounded away from zero.*

*Proof.* We argue as in [13, §5]. Let  $\mathbb{H}P^2$  be the quaternionic projective plane (of dimension 8) with the standard homogeneous metric. Consider a sequence  $S_i^4 \subseteq \mathbb{H}P^2$  of 4-spheres with their induced metrics, representing  $\mathbb{H}P^1 \simeq S^4$  in homology, which Gromov-Hausdorff converges to  $\mathbb{H}P^2$ . Such an approximating sequence exists; see [4] and [13, §5]. The girth of the inclusion maps  $f_i : S_i^4 \rightarrow \mathbb{H}P^2$  tend to zero. That is, there exists a finite open cover  $\{U_k^i\}$  of  $\mathbb{H}P^2$  such that every preimage  $f_i^{-1}(U_k^i)$  has an arbitrarily small radius for  $i$  large enough. By [13, Lemma 5.2], every 1-Lipschitz map  $S_i^4 \rightarrow S^4(R)$  to the round sphere of radius  $R$  is homotopic to a map which factors out through

$$S_i^4 \xrightarrow{f_i} \mathbb{H}P^2 \rightarrow S^4(R) \quad (7.1)$$

for  $i$  large enough. This implies that the map  $S_i^4 \rightarrow S^4(R)$  has zero degree. Otherwise, the induced homomorphism  $H^4(S^4(R)) \rightarrow H^4(S_i^4)$  would be nonzero. In particular, the homomorphism  $H^*(S^4(R)) \rightarrow H^*(\mathbb{H}P^2)$  induced by the second map in (7.1) takes the fundamental cohomology class  $\alpha \in H^4(S^4(R))$  to a nonzero class  $\omega \in H^4(\mathbb{H}P^2)$ . By naturality of the cup product, this homomorphism takes the product  $\alpha \cup \alpha \in H^8(S^4(R))$  to the product  $\omega \cup \omega \in H^8(\mathbb{H}P^2)$ , which is impossible since  $\alpha \cup \alpha = 0$  and  $\omega \cup \omega \neq 0$ . Therefore, the hypersphericity of  $S_i^4$  tends to zero.

On the other hand, suppose that the filling radius of  $S_i^4$  is not bounded away from zero. Using the relationship between the filling radius and the homology 1-waist established in Theorem 3.3, there exist a continuous map  $h_i : N_i \rightarrow T_i$  from a closed orientable 4-pseudomanifold  $N_i$  to a finite simplicial 3-complex  $T_i$  and a degree one map  $\varphi : N_i \rightarrow S_i^4$  such that

$$\text{length } \varphi_i|_{h_i^{-1}(t)} < \varepsilon \quad (7.2)$$

for every  $t \in T_i$ , where  $\varepsilon > 0$  is any given positive real. Define a metric  $g_N$  on  $N$  by pulling-back the metric on  $M$  as in the proof of Proposition 6.2 so that the map  $\varphi_i : N_i \rightarrow S_i^4$  is contracting. By construction, the length of the fibers  $h_i^{-1}(t)$  is less than  $\varepsilon$ ; see (7.2). Thus, the girth of  $h_i : N_i \rightarrow T_i$  is less than  $\varepsilon$ . By [13, Lemma 5.2], the contracting map  $f_i \circ \varphi_i : N_i \rightarrow \mathbb{H}P^2$  is homothetic to a map which factors out through

$$N_i \xrightarrow{h_i} T_i \rightarrow \mathbb{H}P^2.$$

Since  $T_i$  is a simplicial 3-complex, this implies that the homology class  $(f_i \circ \varphi_i)_*([N_i])$  vanishes in  $H_4(\mathbb{H}P^2)$ , which contradicts the injectivity of the homomorphisms  $(\varphi_i)_*$  and  $(f_i)_*$  in homology.  $\square$

## 8. FILLING RADIUS AND HOMOLOGY $p$ -WAIST

We establish a lower bound on the filling radius of a closed Riemannian manifold in terms of its homology  $p$ -waist and its homological filling functions. As a consequence, we derive Theorem 1.3.

Let us first introduce the following replacement transformation of a map defined on a cubical complex based on the notion of homological filling functions; see Definition 1.16. For  $\varepsilon > 0$  small enough, define  $\overline{\text{FH}}_k^\varepsilon(v) = \overline{\text{FH}}_k(v) + \varepsilon$ . In order to keep the notations simple and despite the risk of confusion, we will continue to write  $\overline{\text{FH}}_k$  for  $\overline{\text{FH}}_k^\varepsilon$  in the following proposition.

**Proposition 8.1.** *Let  $M$  be a closed Riemannian  $n$ -manifold and  $p$  be a positive number with  $p \leq n$ . For every cubical  $i$ -complex  $K^i$  with  $i \leq p$  and every continuous map  $f : K^1 \rightarrow M$  defined on the 1-skeleton of  $K^i$ , sending every edge of  $K^1$  to a minimizing segment of  $M$  of*



length at most  $\delta > 0$ , there exists a continuous extension  $F : \mathcal{X}^i \rightarrow M$  of  $f : K^1 \rightarrow M$  defined on a cubical  $i$ -complex  $\mathcal{X}^i$  containing  $K^1$  with

$$\text{vol}_j(F|_C) \leq \overline{\text{FH}}_{j-1} \circ \cdots \circ \overline{\text{FH}}_1(\delta) \quad (8.1)$$

for every cubical  $j$ -simplex  $C$  of  $\mathcal{X}^i$  with  $j \leq p$ . The extension  $F : \mathcal{X}^i \rightarrow M$  is called the  $\mathcal{R}$ -transformation<sup>†</sup> of  $f : K^1 \rightarrow M$  modeled on  $K^i$  (or simply the  $\mathcal{R}$ -transformation of  $f$  if the model space  $K^i$  is a cube or is implicit).

Furthermore, the  $\mathcal{R}$ -transformation can be defined so as to satisfy the following properties:

(1) (Triviality) If  $f : K^i \rightarrow M$  is continuous on  $K^i$  and if the volume bound

$$\text{vol}_j(f|_C) \leq \overline{\text{FH}}_{j-1} \circ \cdots \circ \overline{\text{FH}}_1(\delta) \quad (8.2)$$

holds for each cubical  $j$ -simplex  $C$  of  $K^i$  with  $j \leq i$ , then  $\mathcal{X}^i = K^i$  and  $F = f$ .

(2) (Coherence) If  $K_j^{i_j}$  (with  $j = 1, 2$ ) are two cubical  $i_j$ -complexes with  $K_1^{i_1} \subseteq K_2^{i_2}$  and  $f_j : K_j^{i_j} \rightarrow M$  are two continuous maps which coincide on  $K_1^{i_1}$ , where  $i_j \leq p$ , then  $\mathcal{X}_1^{i_1} \subseteq \mathcal{X}_2^{i_2}$  and the two corresponding maps  $F_j : \mathcal{X}_j^{i_j} \rightarrow M$  coincide on  $\mathcal{X}_1^{i_1}$ .

(3) (Commutation with the boundary operator  $\partial$ ) If  $K^i$  is a closed  $i$ -cube and  $e_1, \dots, e_{2i}$  denote its  $(i-1)$ -faces, then  $\mathcal{X}^i = \Sigma^i$  is a compact  $i$ -pseudomanifold whose boundary  $\partial \Sigma^i$  agrees with the union of the  $2i$  cubical  $(i-1)$ -complexes/pseudomanifolds  $\mathcal{Y}_j$  corresponding to the domains of the  $\mathcal{R}$ -transformations of the restrictions  $f|_{e_j^1} : e_j^1 \rightarrow M$  modeled on  $e_j$  for  $j = 1, \dots, 2i$ .

*Proof.* We argue by induction on  $p$ . If  $p = 1$ , we simply take  $\mathcal{X}^1 = K^1$  and  $F : K^1 \rightarrow M$  for  $f : K^1 \rightarrow M$ . The inequality (8.1) and the properties (1)-(3) are satisfied in this case. Suppose that the result of the proposition holds true for  $p \geq 1$ . Let  $K^{p+1}$  be a cubical  $(p+1)$ -complex and  $f : K^1 \rightarrow M$  be a continuous map as in the proposition. Let us define the  $\mathcal{R}$ -transformation of the restriction of  $f$  to the 1-skeleton of each cubical  $(p+1)$ -simplex  $C^{p+1} \subseteq K^{p+1}$ . Denote by  $C_i^p$  the  $p$ -faces of  $C^{p+1}$  with  $1 \leq i \leq 2(p+1)$ . By induction, the  $\mathcal{R}$ -transformation of the restriction of  $f$  to the 1-skeleton of  $C_i^p$  is a map  $F_i : \mathcal{X}_i^p = \Sigma_i^p \rightarrow M$  defined on a compact  $p$ -pseudomanifold  $\Sigma_i^p$  with

$$\text{vol}_p(F_i) \leq \overline{\text{FH}}_{p-1} \circ \cdots \circ \overline{\text{FH}}_1(\delta).$$

By coherence of the  $\mathcal{R}$ -transformation, the maps  $F_i : \Sigma_i^p \rightarrow M$  and  $F_j : \Sigma_j^p \rightarrow M$  coincide on the cubical  $(p-1)$ -complex given by the intersection  $\Sigma_i^p \cap \Sigma_j^p$ . Thus, the maps  $F_i : \Sigma_i^p \rightarrow M$  give rise to a continuous map  $G : \cup_{i=1}^{2(p+1)} \Sigma_i^p \rightarrow M$  with

$$\text{vol}_p(G) \leq 2(p+1) \overline{\text{FH}}_{p-1} \circ \cdots \circ \overline{\text{FH}}_1(\delta).$$

The cubical structures of the  $p$ -faces  $C_i^p$  of  $\partial C^{p+1} = \cup_i C_i^p$  induce compatible natural decompositions of the boundaries  $\partial \Sigma_i^p$  of the pseudomanifolds  $\Sigma_i^p$  into compact  $(p-1)$ -pseudomanifolds corresponding to the  $(p-1)$ -faces of  $C_i^p$ . Every compact  $(p-1)$ -pseudomanifold of these decompositions appears twice with opposite orientations. Therefore, the sum of the boundaries of the pseudomanifolds  $\Sigma_i^p$  vanishes. Thus, the space  $\cup_i \Sigma_i^p$  obtained by replacing the  $p$ -faces of  $\partial C^{p+1} = \cup_i C_i^p$  with the compact  $p$ -pseudomanifolds  $\Sigma_i^p$  is a closed  $p$ -pseudomanifold  $\Sigma^p$ . By definition of the homological filling function, there exists a continuous extension  $F : \Sigma^{p+1} \rightarrow M$  of  $G : \Sigma^p \rightarrow M$  defined on a compact  $(p+1)$ -pseudomanifold  $\Sigma^{p+1}$  with  $\partial \Sigma^{p+1} = \Sigma^p$  such that

$$\text{vol}_{p+1}(F) \leq \overline{\text{FH}}_p \circ \cdots \circ \overline{\text{FH}}_1(\delta). \quad (8.3)$$

Gluing together the compact  $(p+1)$ -pseudomanifolds  $\Sigma^{p+1}$  corresponding to the cubical  $(p+1)$ -simplices  $C \subseteq K^{p+1}$  along their common faces  $\Sigma_i^p$  following the combinatorial structure of  $K^{p+1}$ ,

<sup>†</sup> $\mathcal{R}$  stands for “replacement”.

we obtain a cubical  $(p+1)$ -complex  $\mathcal{X}^{p+1}$  and a continuous extension  $F : \mathcal{X}^{p+1} \rightarrow M$  of  $f : K^1 \rightarrow M$  modeled on  $K^{p+1}$  satisfying (3).

Now, if  $f$  is defined on  $K^{p+1}$  and if each cubical  $j$ -simplex  $C \subseteq K^{p+1}$  satisfies (8.2), then  $\Sigma_i^p = C_i^p$  and  $F_i : C_i^p \rightarrow M$  agrees with  $f|_{C_i^p} : C_i^p \rightarrow M$  by induction. In this case, we can take  $f : K^{p+1} \rightarrow M$  for its  $\mathcal{R}$ -transformation since  $f|_{C^{p+1}} : C^{p+1} \rightarrow M$  satisfies the volume bound (8.3). Thus, the property (1) is satisfied. The property (2) follows by induction (it holds for the  $p$ -skeleton of  $K^{p+1}$ ) and construction.  $\square$

**Remark 8.2.** The  $\mathcal{R}$ -transformation of the map  $f : K^1 \rightarrow M$  modeled on  $K^p$  only depends on the choice of the filling pseudomanifolds involved in the homological filling functions.

The homology  $p$ -waist is related to the filling radius through the homological filling function; see Definition 1.16.

**Theorem 8.3.** *Let  $M$  be a closed Riemannian  $n$ -manifold. Then, for every positive integer  $p$ ,*

$$W_p(M) \leq \frac{1}{2^{n-p+1}} \binom{n+1}{p}^{-1} \overline{\text{FH}}_{p-1} \circ \cdots \circ \overline{\text{FH}}_1(2 \text{FillRad}(M)).$$

*Proof.* We argue as in the proof of Theorem 3.3 using the same notations. Fix a geodesic cubical structure of  $M$  of size at most  $\varepsilon > 0$  such that every cubical simplex of  $M$  is an almost minimal filling of its boundary. Suppose that there exists a continuous map  $\sigma : P \rightarrow U_\nu(M) \subseteq L^\infty(M)$  defined on a compact cubical  $(n+1)$ -pseudomanifold such that the restriction  $\sigma : \partial P \rightarrow M$  is a PL-homeomorphism, and, therefore, satisfies

$$\sigma_*([\partial P]) = [M] \in H_n(M)$$

where  $\nu$  is chosen so small that

$$\frac{1}{2^{n-p+1}} \binom{n+1}{p}^{-1} \overline{\text{FH}}_{p-1} \circ \cdots \circ \overline{\text{FH}}_1(2\nu) < W_p(M). \quad (8.4)$$

Construct a map  $f : P^1 \rightarrow M$  as in the proof of Theorem 3.3 by projecting the images by  $\sigma$  of the vertices of  $P$  to their closest points in  $M$  and by sending every edge of  $P$  to a segment of  $M$ . The length of these segments is at most  $\delta := 2\nu + \varepsilon$ . Taking a sufficiently fine subdivision of  $P$ , we can assume without loss of generality that the inequality (8.4) is still satisfied with  $\delta$  replacing  $2\nu$ . Note that  $f$  agrees with  $\sigma$  on  $P^1 \cap \partial P$ . Now, denote by  $Q \subseteq P$  the neighborhood of the  $p$ -skeleton  $P^{(p)}$  of  $P$  composed of the pieces  $X_1^{n+1} \subseteq C^{n+1}$  corresponding to the cubical  $(n+1)$ -simplices  $C^{n+1}$  of  $P$ ; see Section 2. Define a “retraction”  $r : Q \rightarrow P^{(p)}$  by putting together the “retractions”  $\rho : X_1^{n+1} \rightarrow (C^{n+1})^{(p)}$  described in (2.4). Deform  $\sigma : \partial P \rightarrow M$  into  $\bar{\sigma} : \partial P \rightarrow M$  so that the restriction of  $\bar{\sigma}$  to each cubical  $n$ -simplex of  $\partial P$  agrees with  $\sigma \circ \bar{\rho}$ , where  $\bar{\rho}$  is the extension of the “retraction”  $\rho$  defined in (2.5). Define

$$\bar{f} : P^1 \cup \partial P \rightarrow M$$

which agrees with  $f \circ r : P^1 \rightarrow M$  on  $P^1$  and with  $\bar{\sigma} : \partial P \rightarrow M$  on  $\partial P$ . Contrary to the proof of Theorem 3.3, see (3.2), it may not be possible to extend  $\bar{f} : P^1 \cup \partial P \rightarrow M$  to  $Q \cup \partial P$ , or even to  $P^{(p)} \cup \partial P$ , when  $p > 1$ . Instead, we will consider the  $\mathcal{R}$ -transformation of  $\bar{f} : P^1 \cup \partial P \rightarrow M$  modeled on  $P^{(p)} \cup \partial P$  to carry on the argument.

The compact cubical  $n$ -pseudomanifold  $N' \subseteq Q$  with boundary lying in  $\partial P$  is composed of the cubical  $n$ -pseudomanifolds  $Y^n \subseteq C^{n+1}$ , where  $C^{n+1}$  is a cubical  $(n+1)$ -simplex of  $P$ ; see Section 2. Similarly, the cubical  $(n-p)$ -complex  $T'$  is composed of the pieces  $Z^{n-p} \subseteq C^{n+1}$ , where  $C^{n+1}$  is a cubical  $(n+1)$ -simplex of  $P$ ; see Section 2. Denote by  $h' : N' \rightarrow T'$  the map whose restriction to  $Y^n$  agrees with the map  $\theta : Y^n \rightarrow Z^{n-p}$  defined in (2.2). The cubical  $n$ -complexes  $X_i^n \subseteq C^n$ , where  $C^n$  is a cubical  $n$ -simplex of  $\partial P \simeq M$ , form a cubical  $n$ -pseudomanifold  $N'_i \subseteq \partial P$  with the same boundary as  $N'_i$ . Also, the pieces  $Z^{n-p-1} \times [0, \frac{1}{2}]$

with  $Z^{n-p-1} \subseteq C^n$ , where  $C^n$  is a cubical  $n$ -simplex of  $\partial P$ , form a finite cubical  $(n-p)$ -complex  $T''$ . Denote by  $\tilde{h}'' : N_2'' \rightarrow T''$  the continuous map whose restriction to  $X_2^n$  agrees with the map  $\Theta : X_2^n \rightarrow Z^{n-p-1} \times [0, \frac{1}{2}]$  defined in (2.3). The two maps  $\tilde{h}'$  and  $\tilde{h}''$  so-defined agree on the common boundary of  $N'$  and  $N_2''$  after identifying  $Z^{n-p-1} \subseteq Z^{n-p} \subseteq T'$  and  $Z^{n-p-1} \subseteq Z^{n-p-1} \times \{\frac{1}{2}\} \subseteq T''$ , where  $Z^{n-p-1} \subseteq C^n$  lies in  $\partial P$ . Put together, these maps give rise to a continuous map

$$\tilde{h} : N \rightarrow T$$

from the closed  $n$ -pseudomanifold  $N = N' \cup N_2''$  lying in  $Q \cup \partial P$  to the cubical  $(n-p)$ -complex  $T = T' \cup_S T''$  obtained by gluing  $T'$  and  $T''$  along the cubical  $(n-p-1)$ -complex  $S$  formed of the pieces  $Z^{n-p-1} \subseteq C^n$ , where  $C^n$  is a cubical  $n$ -simplex of  $\partial P$ .

By construction, every fiber  $\tilde{h}^{-1}(t) \subseteq N'$  with  $t \in T'$  agrees with a fiber of  $\theta$ , and therefore is isomorphic to the  $p$ -skeleton of a cube of dimension at most  $n+1$ . Moreover, the retraction  $r : Q \rightarrow P^{(p)}$  sends every fiber  $\tilde{h}^{-1}(t)$  with  $t$  lying in the interior  $\mathring{\tau}$  of a cubical simplex  $\tau$  of  $T'$  to the same cubical  $p$ -complex  $C_\tau \subseteq C^{n+1}$ , preserving the cubical structure. (Note that  $C_\tau$  is only composed of  $p$ -cubes  $K$  glued together.) In particular, the preimage  $\tilde{h}^{-1}(\mathring{\tau}) \subseteq N'$  of the interior of a cubical simplex  $\tau$  of  $T'$  decomposes as  $\tilde{h}^{-1}(\mathring{\tau}) \simeq \mathring{\tau} \times C_\tau$  and the map  $\tilde{h} : N' \rightarrow T'$  takes  $(t, x) \in \mathring{\tau} \times C_\tau \subseteq N'$  to  $\tilde{h}(t, x) = t \in T'$ . Therefore, the compact  $n$ -pseudomanifold  $N'$  with boundary lying in  $\partial P \simeq M$  decomposes as the union

$$N' = \bigcup_{\tau \subseteq T'} \bigcup_{K \subseteq C_\tau} \tau \times K \quad (8.5)$$

over the cubical simplices  $\tau$  of  $T'$  and the  $p$ -cubes  $K$  of  $C_\tau$ , where  $\tau_1 \times K_1$  is attached to  $\tau_2 \times K_2$  along  $(\tau_1 \cap \tau_2) \times (K_1 \cap K_2)$ . Note that if  $\tau_1 \subseteq \tau_2$  then  $C_{\tau_2} \subseteq C_{\tau_1}$ .

Let  $K$  be a  $p$ -cube of  $P$ . By Proposition 8.1, the  $\mathcal{R}$ -transformation  $\bar{F}_K : \Sigma_K \rightarrow M$  of  $\bar{f}|_{K^1} : K^1 \rightarrow M$  modeled on  $K$  is defined on a compact  $p$ -pseudomanifold  $\Sigma_K$  with boundary and satisfies

$$\text{vol}_p(\bar{F}_K) \leq \overline{\text{FH}}_{p-1} \circ \cdots \circ \overline{\text{FH}}_1(\delta). \quad (8.6)$$

Since the image under  $\bar{f} = \bar{\sigma}$  of each cubical simplex of  $\partial P$  is an almost minimal filling of the image of its boundary, it follows from the property (1) of the  $\mathcal{R}$ -transformation that for every  $p$ -cube  $K \subseteq \partial P$ , the pseudomanifold  $\Sigma_K$  is equal to  $K$  and the map  $\bar{F}_K : \Sigma_K = K \rightarrow M$  agrees with  $\bar{f} : K \rightarrow M$ . Replacing every  $p$ -cube  $K$  with  $\Sigma_K$  in (8.5), we obtain a simplicial  $n$ -complex

$$V' = \bigcup_{\tau \subseteq T'} \bigcup_{K \subseteq C_\tau} \tau \times \Sigma_K$$

where  $\tau_1 \times \Sigma_{K_1}$  is attached to  $\tau_2 \times \Sigma_{K_2}$  along  $(\tau_1 \cap \tau_2) \times (\Sigma_{K_1} \cap \Sigma_{K_2})$  whenever  $\tau_1 \times K_1$  is attached to  $\tau_2 \times K_2$  in (8.5). By the coherence property (2) of Proposition 8.1, the pseudomanifold structure of  $N'$  carries over to  $V'$ . More precisely,  $V'$  is a compact  $n$ -pseudomanifold with the same boundary as  $N'$ . Thus, the union  $V = V' \cup N_2''$  is a closed  $n$ -pseudomanifold.

For every cubical simplex  $\tau$  of  $T'$ , define

$$X_\tau = \bigcup_{K \subseteq C_\tau} \Sigma_K \quad (8.7)$$

as the union of the compact  $p$ -pseudomanifolds  $\Sigma_K$  corresponding to the  $p$ -cubes  $K$  of  $C_\tau$ , where  $\Sigma_{K_1}$  is attached to  $\Sigma_{K_2}$  along  $\Sigma_{K_1 \cap K_2}$ ; see Proposition 8.1. The map

$$h : V \rightarrow T$$

which agrees with  $\tilde{h}'' : N_2'' \rightarrow T''$  on  $N_2''$  and sends  $(t, x) \in \mathring{\tau} \times X_\tau$  to  $t \in T$  for every cubical simplex  $\tau$  of  $T'$  is well defined and continuous. By the coherence property (2) of Proposition 8.1,

the previously defined maps  $\bar{F}_K : \Sigma_K \rightarrow M$  put together induce a map  $\bar{F}_\tau : X_\tau \rightarrow M$ . Now, consider the map

$$F : V \rightarrow M$$

which coincides with  $\bar{f} : \partial P \rightarrow M$  on  $N_2'' \subseteq \partial P$  and whose restriction to each fiber  $h^{-1}(t) \simeq \{t\} \times X_\tau$  with  $t \in \hat{\tau}$  agrees with  $\bar{F}_\tau : X_\tau \simeq h^{-1}(t) \rightarrow M$ . By the coherence property and since  $\bar{F}_K = \bar{f}_K$  for every  $p$ -cube  $K \subseteq \partial P$ , the map  $F : V \rightarrow M$  is well defined and continuous.

For every cubical simplex  $\tau$  of  $T'$ , recall that the cubical  $p$ -complex  $C_\tau$  lies in  $C^{n+1}$ . Since  $C^{n+1}$  has at most  $k = 2^{n-p+1} \binom{n+1}{p}$  faces of dimension  $p$ , the cubical complex  $X_\tau \simeq h^{-1}(t)$  for  $t \in \hat{\tau}$  is composed of at most  $k$  pseudomanifolds  $\Sigma_K$ ; see (8.7). Since the volume of the restriction of  $F$  to each of these pseudomanifolds satisfies (8.6), we obtain

$$\text{vol}_p[F(h^{-1}(t))] \leq 2^{n-p+1} \binom{n+1}{p} \overline{\text{FH}}_{p-1} \circ \cdots \circ \overline{\text{FH}}_1(\delta) < W_p(M)$$

for every  $t \in \hat{\tau}$ , where the second inequality follows from the filling radius assumption; see (8.4). Thus, by definition of the homology  $p$ -waist, see Definition 1.1, the map  $F : V \rightarrow M$  satisfies

$$F_*([V]) \neq [M] \in H_n(M). \quad (8.8)$$

In a different direction, the retraction  $r : Q \rightarrow P^{(p)}$  takes every piece  $\tau \times C_\tau$  of  $N$ , where  $\tau$  is a cubical simplex of  $T'$ , see (8.5), to the cubical  $p$ -complex  $C_\tau$ . Thus, the map  $\bar{f} : Q \rightarrow M$  defined as  $\bar{f} = f \circ r$  takes  $\tau \times C_\tau$  to the image of  $C_\tau$  in  $M$  by  $f$ . Similarly, the map  $F : V \rightarrow M$  takes  $\tau \times X_\tau$  to the image of  $X_\tau$  in  $M$  by  $\bar{F}_\tau$ , where  $\tau$  is a cubical simplex of  $T'$ . Therefore, the contribution of the pieces  $\tau \times X_\tau$  to the image by  $F_*$  of the fundamental class of  $V$  is trivial. Now, by construction, the map  $\bar{f} : \partial P \rightarrow M$  induces a degree one map in relative homology between every cubical  $n$ -complex  $X_2^n$  of  $N_2''$  and the cubical  $n$ -simplex  $C^n$  of  $\partial P$  containing  $X_2^n$ . Since the map  $F$  agrees with  $\bar{f}$  on  $N_2''$ , we deduce that

$$F_*([V]) = [M] \in H_n(M).$$

Hence a contradiction with (8.8). □

**Remark 8.4.** Working this simplicial complexes instead of cubical complexes yields a better quantitative estimate in Theorem 8.3. Namely,

$$W_p(M) \leq \binom{n+1}{p}^{-1} \text{FH}_{p-1}(p \text{FH}_{p-2}(\cdots (3 \text{FH}_2(2 \text{FH}_1(6 \text{FillRad}(M)))) \cdots)).$$

Since the homological filling functions are nondecreasing, Theorem 1.17 follows from Theorem 8.3 and Theorem 3.2.

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