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SHARP REVERSE ISOPERIMETRIC INEQUALITIES IN NONPOSITIVELY CURVED CONES

MIKHAIL G. KATZ AND STÉPHANE SABOURAU

ABSTRACT. We prove a pair of sharp reverse isoperimetric inequalities for domains in nonpositively curved surfaces: (1) metric disks centered at the vertex of a Euclidean cone of angle at least 2π have minimal area among all nonpositively curved disks of the same perimeter and the same total curvature; (2) geodesic triangles in a Euclidean (resp. hyperbolic) cone of angle at least 2π have minimal area among all nonpositively curved geodesic triangles (resp. all geodesic triangles of curvature at most -1) with the same side lengths and angles.

1. INTRODUCTION

Isoperimetric inequalities provide upper bounds on the area of domains in a surface with a fixed metric (typically of constant curvature) in terms of their boundary length; see [5] for an account on this classical subject.

Often the metric is fixed (flat, hyperbolic, or spherical), but there are a few instances where isoperimetric inequalities hold for large classes of metrics satisfying curvature bounds. Thus, André Weil ([12], 1926) developed such inequalities for nonpositively curved planes, proving the Cartan-Hadamard isoperimetric conjecture in dimension two.

In this article, we establish a pair of sharp reverse isoperimetric inequalities providing *lower bounds* on the area of some domains in terms of their boundary length.

These geometric inequalities hold for nonpositively curved surfaces. The optimal metrics for such reverse isoperimetric inequalities, as well as the extremal domains, can be described in terms of the total curvature of the domains under consideration. The boundary cases of equality in our optimal inequalities are attained by Euclidean cones with nonpositive total curvature. These inequalities provide bounds that are stronger than Euclidean ones.

It seems that such geometric inequalities should have been known already for some time, but we were unable to find them in the literature. A possible reason is that previous research focused more on isoperimetric inequalities

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for special homogeneous metrics regardless of the total curvature of the domains considered.

We will now present our two main results. As motivation, note that the area of a Euclidean cone of angle less than 2π can be easily decreased among nonnegatively curved metrics by smoothing off the tip of the cone. Our first result shows that this is impossible while preserving the nonpositive curvature condition; see Theorem 1.2 for a more general statement. Namely, one cannot decrease the area of a metric disk centered at the vertex of a Euclidean cone of angle at least 2π among all nonpositively curved disks of the same perimeter and the same total curvature. We will need the following definition.

Definition 1.1. Given a surface with a complete Riemannian metric with Gaussian curvature function K , we define the nonnegative and nonpositive parts of K as

$$K^+ = \max\{K, 0\} \quad \text{and} \quad K^- = \max\{-K, 0\}$$

so that $K = K^+ - K^-$.

Our convention is consistent with the corresponding definitions of nonnegative and nonpositive parts of curvature measures introduced by Yu. Burago in [4]; see Definition 2.1. We can now state our first result.

Theorem 1.2. *Let M be a surface with a complete Riemannian metric. Then every disk D of radius R and boundary length L in M satisfies the bound*

$$\text{area}(D) \geq \text{area}(\hat{D}) \tag{1.1}$$

where \hat{D} is the disk with the same boundary length L as D , centered at the vertex of the Euclidean cone with total curvature $-\int_D K^- dA$.

A formula for the area of \hat{D} is given in Section 2.

Comparison geodesic triangles play an important role in nonpositive curvature geometry. For our second result, we consider a geodesic triangle of Gaussian curvature at most $\lambda_0 \leq 0$ along with its comparison triangle (in the strong sense) in a cone $\mathcal{C}_{\lambda_0}^\theta$ of constant nonpositive curvature λ_0 with angle θ . Here, a comparison triangle is taken in the following strong sense: both the angles and the side lengths of the two triangles are the same. The following theorem asserts that the area of the initial triangle is bounded from below by the area of its comparison triangle in the cone. (For simplicity, one can assume that $\lambda_0 = 0$.)

Theorem 1.3. *Let Δ be a geodesic (two-dimensional) triangle in a surface with a complete Riemannian metric of Gaussian curvature $K \leq \lambda_0$ for some constant $\lambda_0 \leq 0$. Suppose $\bar{\Delta}$ is a geodesic (two-dimensional) triangle with*

the same side lengths and the same angles α, β, γ as Δ in the cone $\mathcal{C}_{\lambda_0}^\theta$ of constant curvature λ_0 with angle $\theta = 3\pi - (\alpha + \beta + \gamma)$. Then

$$\text{area}(\Delta) \geq \text{area}(\bar{\Delta})$$

with equality if and only if Δ is isometric to $\bar{\Delta}$.

Our proof of Theorem 1.3 also shows that the area of a geodesic triangle of Gaussian curvature at most λ_0 is bounded from below by the area of a comparison triangle having the same base length and the same adjacent angles in the plane \mathbb{H}_{λ_0} of constant curvature $\lambda_0 \leq 0$; see Proposition 3.1.

We would like to provide some context for our study of reverse isoperimetric inequalities. We found the geometric inequalities of this paper while working on a proof that systolically extremal nonpositively curved surfaces are flat with finitely many conical singularities; see [7]. In that context, it was clear that it should be impossible to round off a conical singularity of angle greater than 2π in a nonsystolic region in order to decrease the area (while keeping the nonpositive curvature condition) by any cut-and-paste argument with metric disks of the same perimeter. This observation is formalized by our Theorem 1.2. Theorem 1.3 is a variation on this theme, while trying to cut-and-paste triangles instead of disks. Though we did not use these reverse isoperimetric inequalities in our argument, they confirmed our intuition that piecewise flat metrics with conical singularities should play a role in extremal systolic geometry through their local extremal features.

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2. REVERSE ISOPERIMETRIC INEQUALITY FOR METRIC DISKS

The reverse isoperimetric inequality for metric disks established in this section shows the optimality of the nonpositively curved Euclidean cones for the area with respect to compact deformations keeping the same curvature sign.

Before proving this result, we need to extend the notion of curvature to singular spaces.

Definition 2.1. Associated to a surface M with a complete Riemannian metric g of Gaussian curvature K is the *curvature measure* $K dA$, where dA is the area measure of M . The notion of curvature measure extends to piecewise flat surfaces with conical singularities (and more generally to Alexandrov surfaces), where it is denoted by ω . The curvature measure ω is a signed measure which can be decomposed as $\omega = \omega^+ - \omega^-$, where ω^+ and ω^- are the nonnegative and nonpositive parts of ω , and are both nonnegative measures. For smooth metrics, we have $\omega^\pm = K^\pm dA$.

We refer to [1] and [9] for a precise definition of the curvature measure; see also [11] for a modern exposition on Alexandrov surfaces.

We will make use of the following result on bi-Lipschitz metric approximation, announced by Reshetnyak [8] and proved by Yu. Burago [4, Lemma 6], in the more general setting of Alexandrov surfaces; see also [5, Theorem 3.1.1], [11] and [7, §3].

Proposition 2.2 (See [8] and [4, Lemma 6]). *Let M be a compact surface (possibly with boundary) with a Riemannian metric. Then there is a sequence M_i of piecewise flat surfaces with conical singularities converging to M in the Lipschitz topology such that the nonnegative and nonpositive parts ω_i^\pm of the curvature measure ω_i of M_i weakly converge to their counterparts ω^\pm for the curvature measure ω on M .*

Consider a sequence (g_i) of piecewise flat metrics with conical singularities approximating a given complete Riemannian metric g on a surface for every compact domain as in Proposition 2.2. Denote by M the surface with the complete Riemannian metric g , and by M_i the same surface with the piecewise flat metric g_i .

Proposition 2.3. *Fix $p \in M$. Let $D \subseteq M$ and $D_i \subseteq M_i$ be the disks of radius R centered at the same point p . Then*

- (1) *the area of the symmetric difference $D\Delta D_i$ tends to zero (for every area measure). That is, $|D\Delta D_i| \rightarrow 0$;*
- (2) $\lim \text{area}(D_i, g_i) = \text{area}(D, g)$;
- (3) $\lim \omega_i^\pm(D_i) = \omega^\pm(D)$;
- (4) $\liminf \text{length}(\partial D_i, g_i) \geq \text{length}(\partial D, g)$.

Proof. (1) By bilipschitz convergence of the metrics, the symmetric difference $D\Delta D_i$ is contained in the ε -tubular neighborhood $U_\varepsilon(\partial D)$ of ∂D for i large enough. Since ∂D is 1-rectifiable, the area of this tubular neighborhood tends to zero (see [6, Theorem 3.2.39]), and the result follows.

(2) By bilipschitz convergence of the metrics, we have the inclusions

$$D - \varepsilon \subseteq D_i \subseteq D + \varepsilon$$

for i large enough, where $D \pm \varepsilon \subseteq M$ are the balls of radius $R \pm \varepsilon$ centered at p . We also have weak convergence of the area measures. Taking the area with respect to g_i in the previous double inclusion between D_i and $D \pm \varepsilon$, and using the weak convergence of the area measure, we obtain

$$\text{area}(D - \varepsilon, g) - \varepsilon \leq \text{area}(D_i, g_i) \leq \text{area}(D + \varepsilon, g) + \varepsilon$$

for i large enough. Since the area of the ε -tubular neighborhood $U_\varepsilon(\partial D)$ of ∂D tends to zero, the result is immediate.

(3) As in the proof of item (2), using the weak convergence of the curvature measure instead of the area measures, we obtain

$$\omega^\pm(D - \varepsilon) - \varepsilon \leq \omega_i^\pm(D_i) \leq \omega^\pm(D + \varepsilon) + \varepsilon$$

for i large enough. Since the curvature measure $\omega^\pm = K^\pm dg$ is absolutely continuous with respect to the area measure, the curvature measure of the ε -tubular neighborhood $U_\varepsilon(\partial D)$ of ∂D tends to zero. Hence the result.

(4) The flat distance between the one-cycles ∂D and ∂D_i is bounded by the mass of the 2-current defined as the difference $D - D_i$, see [6] for precise definitions. This mass is equal to the area of $D\Delta D_i$. Thus, by (1), the sequence ∂D_i converges to ∂D in the flat topology. The desired result follows from the lower semicontinuity of the mass (here, the length); see [6]. \square

We can now proceed to the proof of the following theorem. Recall that a disk of radius R centered at the vertex of a Euclidean cone of angle θ has perimeter $L = \theta R$ and area $A = \frac{\theta}{2}R^2 = \frac{L^2}{2\theta}$. Thus, we have

$$A = \frac{L^2}{2(2\pi - \mathcal{K})} \quad (2.1)$$

where $\mathcal{K} = 2\pi - \theta < 0$ is the total curvature of the Euclidean cone.

With this formula, Theorem 1.2 can be restated as follows.

Theorem 2.4. *Let M be a surface with a complete Riemannian metric. Then every disk D of radius R of boundary length L in M satisfies*

$$\text{area}(D) \geq \frac{L^2}{4\pi + 2\mathcal{K}_D^-}$$

where $\mathcal{K}_D^- = \int_D K^- dA \geq 0$ is the total mass of the nonpositive part of the curvature measure of D .

Proof. Proposition 2.3 shows that it is sufficient to prove Theorem 2.4 for piecewise flat metrics with conical singularities. The desired result for Riemannian metrics will follow by piecewise flat metric approximation; see Proposition 2.2. This enables us to avoid regularity issues.

Let p be the center of the disk $D \subseteq M$ of radius R . Consider the (closed) disk D_t and the circle \mathcal{C}_t of radius $R - t$ centered at p . Note that $D_0 = D$ and $\mathcal{C}_0 = \partial D$. Since the metric on M is piecewise flat, the circle \mathcal{C}_t is a piecewise smooth curve (possibly with several connected components) bounding the metric disk D_t in M . Moreover, the length function $t \mapsto L(\mathcal{C}_t)$ is differentiable except for a finite number of values of t , and its derivative is given by the first variation formula; see [5, Lemma 3.2.3]. Namely, as long as D_t is nonempty, we have

$$L'(\mathcal{C}_t) = - \int_{\mathcal{C}_t} \kappa(s) ds - S_t$$

for almost every t , where κ is the geodesic curvature of the curve \mathcal{C}_t and S_t is the sum of the angular difference of the tangent vectors at the corner points of \mathcal{C}_t .

By the Gauss–Bonnet formula for polyhedral metrics (see [9, Theorem 5.3.2]), we derive

$$L'(\mathcal{C}_t) = -2\pi\chi(D_t) + \omega(D_t).$$

Since D_t is a connected region with boundary, its Euler characteristic $\chi(D_t)$ is at most 1, that is, $\chi(D_t) \leq 1$. Since $D_t \subseteq D$ and $\omega = \omega^+ - \omega^-$, we have

$$\omega(D_t) \geq -\omega^-(D).$$

Combining these two bounds, we deduce that

$$L'(\mathcal{C}_t) \geq -2\pi - \omega^-(D).$$

Integrating this relation leads to

$$L(\mathcal{C}_t) \geq L(\partial D) - (2\pi + \omega^-(D)) t. \quad (2.2)$$

In particular, the domain D_t is nonempty for every $t < t_0$, where

$$t_0 = \frac{L(\partial D)}{2\pi + \omega^-(D)}.$$

By the coarea formula, integrating the inequality (2.2) between 0 and t_0 , we obtain

$$\text{area}(D) \geq L(\partial D) t_0 - \frac{1}{2} (2\pi + \omega^-(D)) t_0^2.$$

In other words, we have

$$\text{area}(D) \geq \frac{L(\partial D)^2}{2(2\pi + \omega^-(D))}$$

where the right-hand side represents the area of the disk centered at the vertex of the Euclidean cone with total curvature $\mathcal{K} = -\omega^-(D)$ and with the same boundary length $L(\partial D)$ as D ; see formula (2.1). \square

3. AREA COMPARISON FOR TRIANGLES WITH THE SAME BASE

In order to prove Theorem 1.3, we will need the following result, which may be of independent interest. This result provides a lower bound on the area of a geodesic triangle of Gaussian curvature at most λ_0 , in terms of the area of a comparison triangle having the same base length with the same adjacent angles in the plane \mathbb{H}_{λ_0} of constant curvature $\lambda_0 \leq 0$.

Proposition 3.1. *Let Δ be a geodesic (two-dimensional) triangle with vertices A, B, C in a surface M with a complete Riemannian metric of Gaussian curvature $K \leq \lambda_0$ for some constant $\lambda_0 \leq 0$. Let $\bar{\Delta}$ be a geodesic (two-dimensional) triangle with distinct vertices $\bar{A}, \bar{B}, \bar{C}$ in the plane \mathbb{H}_{λ_0} of constant curvature λ_0 such that*

- *the sides $\bar{A}\bar{B}$ and AB have the same length;*
- *the angles at \bar{A} and A are the same;*
- *the angles at \bar{B} and B are the same.*

Then

$$\text{area}(\Delta) \geq \text{area}(\bar{\Delta}) \quad (3.1)$$

with equality if and only if Δ is isometric to $\bar{\Delta}$.

We first establish the proposition when one of the angles at A or B , say B , is greater or equal to $\frac{\pi}{2}$; see Lemma 3.2. We will then derive the general result from this particular case.

Lemma 3.2. *Let Δ and $\bar{\Delta}$ be as in Proposition 3.1. Let α, β, γ be the angles of Δ at A, B, C . Suppose that $\beta \geq \frac{\pi}{2}$. Then*

$$\text{area}(\Delta) \geq \text{area}(\bar{\Delta})$$

with equality if and only if Δ is isometric to $\bar{\Delta}$.

Proof. Let $\hat{\Delta}$ be a geodesic triangle of \mathbb{H}_{λ_0} with the same side lengths as Δ . Denote by $\hat{A}, \hat{B}, \hat{C}$ the vertices of $\hat{\Delta}$ and by $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ their angles. We can (and will) assume that $\hat{A} = \bar{A}$ and $\hat{B} = \bar{B}$. By [2, Proposition II.1.7.(4)], we have

$$\begin{cases} \hat{\alpha} \geq \alpha \\ \hat{\beta} \geq \beta. \end{cases} \quad (3.2)$$

In particular, $\hat{\beta} \geq \frac{\pi}{2}$. Since the sum of the angles of a geodesic triangle in a nonpositively curved surface is at most π ; see [2, Proposition II.1.7.(4)], we derive that $\hat{\gamma} \leq \frac{\pi}{2}$.

The geodesic triangle of \mathbb{H}_{λ_0} with side $\bar{A}\bar{B}$, and angles α and β at \bar{A} and \bar{B} can be isometrically identified to $\bar{\Delta}$. It follows from the relations (3.2) that the triangle $\bar{\Delta}$ lies in $\hat{\Delta}$; see Figure 1. Since $\hat{\gamma} \leq \frac{\pi}{2}$, we deduce that

$$|\bar{A}\bar{C}| \leq |\hat{A}\hat{C}| = |AC|.$$

This relation holds for any point D lying in the segment BC by replacing \hat{C} with \hat{D} and \bar{C} with \bar{D} (note that \bar{D} lies in the segment $\bar{B}\bar{C}$).

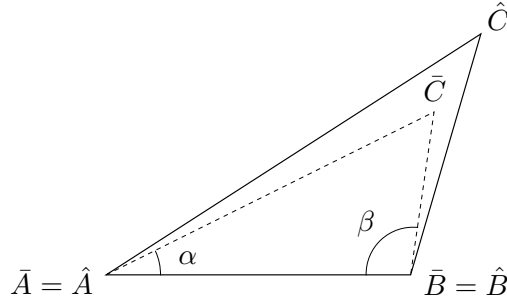


FIGURE 1. The comparison triangles $\bar{\Delta}$ and $\hat{\Delta}$

Consider the two exponential maps $\exp_A : T_A M \rightarrow M$ and $\exp_{\bar{A}} : T_{\bar{A}} \mathbb{H}_{\lambda_0} \rightarrow \mathbb{H}_{\lambda_0}$. Define $\sigma : \mathbb{H}_{\lambda_0} \rightarrow M$ as

$$\sigma = \exp_A \circ I \circ \exp_{\bar{A}}^{-1}$$

where $I : T_{\bar{A}} \mathbb{H}_{\lambda_0} \rightarrow T_A M$ is a linear isometry. We can choose I so that the map σ takes the geodesic rays $[\bar{A}\bar{B})$ and $[\bar{A}\bar{C})$ to the geodesic rays $[AB)$ and $[AC)$. By construction, the map σ sends every segment $\bar{A}\bar{D}$ joining \bar{A} to

a point \bar{D} in the opposite side $\bar{B}\bar{C}$ of $\bar{\Delta}$ to a subarc of the geodesic arc AD . In particular, the map σ sends $\bar{\Delta}$ into Δ .

Since $K \leq \lambda_0$, the Rauch theorem implies that the map σ is distance-nondecreasing; see [3, Theorem 6.5.4]. Therefore,

$$\text{area}(\bar{\Delta}) \leq \text{area}(\Delta)$$

with equality if and only if the map σ is an isometry between $\bar{\Delta}$ and Δ . \square

We can now derive Proposition 3.1 from the previous lemma.

Proof of Proposition 3.1. Let α, β, γ be the angles of Δ at A, B, C . The cases where $\alpha \geq \frac{\pi}{2}$ or $\beta \geq \frac{\pi}{2}$ are covered by Lemma 3.2. Thus, we can assume that $\alpha < \frac{\pi}{2}$ and $\beta < \frac{\pi}{2}$. This implies that the projection H of C to the segment AB strictly lies between A and B . Furthermore, both angles $\angle AHC$ and $\angle BHC$ are equal to $\frac{\pi}{2}$. Thus, the height CH decomposes Δ into two right triangles Δ' and Δ'' . Denote by γ' and γ'' the angles of Δ' and Δ'' at C . Observe that $\gamma = \gamma' + \gamma''$. Since the angles of Δ' and Δ'' at H are right, we can apply Lemma 3.2 to the triangles $\Delta' = AH'C'$ and $\Delta'' = BH''C''$, where $H = H' = H''$ and $C = C' = C''$. Thus, the areas of the triangles Δ' and Δ'' are bounded from below by the areas of the right triangles $\bar{\Delta}' = \bar{A}\bar{H}'\bar{C}'$ and $\bar{\Delta}'' = \bar{B}\bar{H}''\bar{C}''$ in \mathbb{H}_{λ_0} . We can glue together these two right triangles of \mathbb{H}_{λ_0} along their sides $\bar{H}'\bar{C}'$ and $\bar{H}''\bar{C}''$ so that \bar{H}' and \bar{H}'' coincide. Since the angles at \bar{H}' and \bar{H}'' are right, the two segments $\bar{A}\bar{H}'$ and $\bar{H}''\bar{B}$ form a long segment $\bar{A}\bar{B}$ of the same length as AB ; see Figure 2.

If the heights $\bar{H}'\bar{C}'$ and $\bar{H}''\bar{C}''$ have the same length, the union of these two triangles form a large triangle which satisfies the same geometric features as $\bar{\Delta}$ and so can be identified with $\bar{\Delta}$.

If one of these heights, say $\bar{H}'\bar{C}'$, is shorter than the other, we extend the hypotenuse $\bar{A}\bar{C}'$ until it intersects the other hypotenuse $\bar{B}\bar{C}''$ at some point \bar{O} ; see Figure 2. As previously, the triangle $\bar{A}\bar{O}\bar{B}$ can be identified with $\bar{\Delta}$. Moreover, the area of this triangle is bounded by the sum of the two triangles $\bar{\Delta}'$ and $\bar{\Delta}''$.

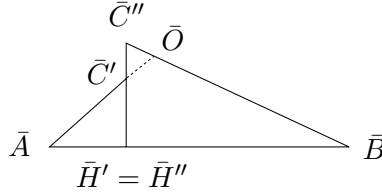


FIGURE 2. The two triangles $\bar{\Delta}'$ and $\bar{\Delta}''$

In either case, we have

$$\text{area}(\bar{\Delta}) \leq \text{area}(\bar{\Delta}') + \text{area}(\bar{\Delta}'') \leq \text{area}(\Delta)$$

with equality if and only if Δ is isometric to $\bar{\Delta}$. \square

4. AREA COMPARISON FOR TRIANGLES WITH THE SAME SIDE LENGTHS
AND ANGLES

We can now prove our second main result.

Theorem 4.1. *Let Δ be a geodesic (two-dimensional) triangle in a surface with a complete Riemannian metric of Gaussian curvature $K \leq \lambda_0$ for some constant $\lambda_0 \leq 0$. Suppose $\bar{\Delta}$ is a geodesic (two-dimensional) triangle with the same side lengths and the same angles α, β, γ as Δ in the cone $\mathcal{C}_{\lambda_0}^\theta$ of constant curvature λ_0 with angle $\theta = 3\pi - (\alpha + \beta + \gamma)$. Then*

$$\text{area}(\Delta) \geq \text{area}(\bar{\Delta}) \quad (4.1)$$

with equality if and only if Δ is isometric to $\bar{\Delta}$.

Remark 4.2. Unlike the situation with Proposition 3.1, the comparison triangle $\bar{\Delta}$ in $\mathcal{C}_{\lambda_0}^\theta$ in Theorem 4.1 do not necessarily exist in general.

Proof of Theorem 4.1. Let A, B and C be the vertices of Δ , with angle α at the vertex A , angle β at B , and angle γ at C . Denote by $\bar{A}, \bar{B}, \bar{C}$ the corresponding vertices of $\bar{\Delta}$.

By [2, Proposition II.1.7.(4)], the angle θ of the conical singularity of the cone is at least 2π . Moreover, we can assume that $\theta > 2\pi$ and that the conical singularity lies in $\bar{\Delta}$. Otherwise, the sum $\alpha + \beta + \gamma$ of the angles of Δ and $\bar{\Delta}$ would be equal to π and so the triangle Δ would be flat isometric to $\bar{\Delta}$ by [2, Proposition II.2.9].

The geodesic rays joining the vertices of $\bar{\Delta}$ to its conical singularity decompose each angle around the vertices of $\bar{\Delta}$ into two angles. In particular, the angle α splits into two angles α' and α'' , where $\alpha = \alpha' + \alpha''$. The same holds with β and γ . Observe that these geodesic rays decompose $\bar{\Delta}$ into three small triangles.

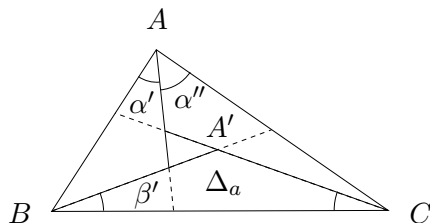
We would like to carry out a similar construction for Δ , except that there is no conical singularity to rely on. Instead, we consider the geodesic rays of Δ emanating from the vertices of Δ and splitting each angle as in $\bar{\Delta}$. These three geodesic rays do not necessarily intersect at a single point as in $\bar{\Delta}$. Nevertheless, they decompose Δ into three triangles $\Delta_a, \Delta_b, \Delta_c$ as in Figure 3. Recall that two geodesic rays in a nonpositively curved surface intersect at most once. There may be a small triangular region

$$X = \Delta \setminus (\Delta_a \cup \Delta_b \cup \Delta_c)$$

lying in Δ which is not covered by the triangles $\Delta_a, \Delta_b, \Delta_c$.

Let A' be the vertex of Δ_a different from B and C . Denote by $\bar{\Delta}_a$ the triangle of $\mathcal{C}_{\lambda_0}^\theta$ with vertices $\bar{A}', \bar{B}, \bar{C}$ such that

- the sides $\bar{B}\bar{C}$ and BC have the same length;
- the angles at \bar{B} and B are the same (equal to β'');
- the angles at \bar{C} and C are the same (equal to γ').

FIGURE 3. Decomposition of Δ into triangles

Similarly, we define $\bar{\Delta}_b$ and $\bar{\Delta}_c$. By construction, the three triangles $\bar{\Delta}_a$, $\bar{\Delta}_b$ and $\bar{\Delta}_c$ are isometric to the three smaller triangles forming $\bar{\Delta}$ and delimited by the geodesic segments joining the conical singularity of $\bar{\Delta}$ to its vertices. By Proposition 3.1, we have

$$\text{area}(\bar{\Delta}_a) \leq \text{area}(\Delta_a)$$

with equality if and only if Δ_a is isometric to $\bar{\Delta}_a$. The same holds with $\bar{\Delta}_b$ and $\bar{\Delta}_c$. Since the triangle $\bar{\Delta}$ is partitioned into $\bar{\Delta}_a$, $\bar{\Delta}_b$ and $\bar{\Delta}_c$, we derive that

$$\begin{aligned} \text{area}(\bar{\Delta}) &= \text{area}(\bar{\Delta}_a) + \text{area}(\bar{\Delta}_b) + \text{area}(\bar{\Delta}_c) \\ &\leq \text{area}(\Delta_a) + \text{area}(\Delta_b) + \text{area}(\Delta_c) \\ &= \text{area}(\Delta) - \text{area}(X) \\ &\leq \text{area}(\Delta) \end{aligned}$$

with equality if and only if Δ is isometric to $\bar{\Delta}$. \square

Remark 4.3. It follows from the proof of Theorem 4.1 that the difference between $\text{area}(\Delta)$ and $\text{area}(\bar{\Delta})$ is bounded from below by the area of the small triangle $X = \Delta \setminus (\Delta_a \cup \Delta_b \cup \Delta_c)$ lying in Δ .

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