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# A comparative study of RKF-based Filon, Levin and Monte Carlo methods for generalized oscillatory integral

Mohsen Alimohammady <sup>a</sup>

Meghdad Mirebrahimi <sup>a\*</sup>

<sup>a</sup> *Department of Mathematics, University of Mazandaran, Babolsar, Iran*

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## Abstract

In the previous works, the authors presented the reproducing kernel function (RKF)-based Filon and Levin methods for solving highly oscillatory integrals and compared their accuracy with some earlier numerical methods. In this paper, we study the RKF-based methods for generalized oscillatory integrals and apply the Monte Carlo integration method in comparison with the RKF-based ones.

**keywords:** Reproducing kernel method, Filon, Levin, Monte Carlo

**MSC (2010):** 46N20, 46E22, 65C05

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## 1 Introduction

In this work, we consider to approximate the integral operator

$$I(u) = \int_a^b u(x) f(\lambda v(x)) dx \quad (1)$$

using RKF-based and probabilistic methods where,  $u(x)$  and  $v(x)$  are real valued functions and  $f$  is complex valued function. The special form of this operator

$$I(u) = \int_a^b u(x) e^{i\lambda v(x)} dx \quad (2)$$

has been studied by [4] using the reproducing kernel function-based Filon and Levin methods.

Reproducing kernel Hilbert space (RKHS) is one of the most important and latest theories for numerical analysis. Recently, the technology of reproducing kernel function (RKF) theory has been attended in many fields such as numerical analysis, differential equations, probability and statistics, learning theory and so on [3–5, 11]. In [4], the authors present two new improved numerical methods for highly oscillatory integrals through combining the RKHS theory and the idea of Filon and Levin methods.

In this work, we apply RKF-based Filon and levin method for a generalized form oscillatory integral (1) in order to have a comparison numerically the accuracy of these two methods with probabilistic integration approximation methods such as Monte Carlo and importance sampling.

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\*Corresponding author: [me.mirebrahimi@gmail.com](mailto:me.mirebrahimi@gmail.com)

This article is organized as follows. A background theory of RKHS is summarized in the subsection 2.1. The RKF-based Filon method and RKF-based Levin method for generalized oscillatory integral (1) are presented in subsections 2.2 and 2.3, respectively. The probabilistic integration approximation including Monte Carlo and importance sampling method are reviewed in Section 3. The numerical results are shown in the last section, 4.

## 2 RKF

In this section, we first summarize an essential background of RKF as preliminaries. In two next parts, we apply RKF-based Filon and Levin methods (based on those were introduced in [4]) for the generalized oscillatory integral form (1).

### 2.1 Preliminaries

Suppose that  $H \subseteq \mathcal{F}(E, \mathbb{C})$  is a Hilbert space such that for any  $x$  in  $E$ , the evaluation functional  $e_x(h) = h(x)$  is continuous, then  $H$  is called an *RKHS*. The Riesz Representation Theorem guarantees for a Hilbert function space, there exists a unique  $K_x \in H$  such that  $(h, K_x) = h(x)$  for all  $h \in H$ . Moreover, the function  $K : E \times E \rightarrow \mathbb{R}$  is known as an *RKF of space H* if

- $K(., t) \in H$  for all  $t \in \Omega$ ,
- $g(t) = (g(.), K(., t))$ , for all  $t \in \Omega$  and  $g \in H$ .

The RKF of an RKHS is positive definite i.e. for a symmetric function  $K : E \times E \rightarrow \mathbb{R}$ , it holds  $\sum_{i,j=1}^n c_i c_j K(x_i, x_j) \geq 0$  for any  $n \in \mathbb{N}$ ,  $x_1, x_2, \dots, x_n \in E$ ,  $c_1, c_2, \dots, c_n \in \mathbb{R}$ . Moreover, every positive definite kernel can define a unique RKHS of which it is the RKF.

**Definition 2.1.** Sobolev space  $H^m[a, b]$  consists of functions  $g(t)$  defined on  $[a, b]$  such that  $g^{(k)}(t)$  is absolutely continuous for  $k = 0, \dots, m-1$  and  $g^{(m)}(t) \in L^2[a, b]$ . The inner product for this space is

$$(g_1, g_2)_m := \sum_{k=1}^{m-1} g_1^{(k)}(a)g_2^{(k)}(a) + \int_a^b g_1^{(m)}(t)g_2^{(m)}(t)dt.$$

**Theorem 2.2.** [11] Space  $H^m[a, b]$  is an RKHS and the RKF of  $H^m[0, b]$  is given by

$$K^m(x, y) = \begin{cases} \xi(x, y), & y \leq x, \\ \xi(y, x), & y > x, \end{cases} \quad (3)$$

where  $\xi(x, y) = \sum_{i=0}^{m-1} \left( \frac{y^i}{i!} + (-1)^{m-1-i} \frac{y^{2m-1-i}}{(2m-1-i)!} \right) \frac{x^i}{i!}$ .

### 2.2 RKF-based Filon method

In this part, following the idea of the RKF-based Filon method in [4], we approximate (1) for  $u(x) \in C^{2m}[a, b]$  with the RKF interpolation  $u_X(x)$  where  $X = \{x_1, x_2, \dots, x_N\}$  is a scattered data subset of  $[a, b]$  such that, for each  $u(x) \in H^m$  (see [7])

$$u_X(x) = \sum_{i=1}^N \alpha_i K^m(x, x_i), \quad (4)$$

where  $u_X(x_i) = u(x_i)$  for  $i = 1, \dots, N$ . Moreover, we assume that  $v(x) = \gamma_0 + \gamma_1 x + \gamma_2 x^2$  in this case.

In the following we compute  $u_X(x)$ .

**Theorem 2.3.** [7] For the interpolation (4) there are constants  $\alpha_i$  satisfying (5) such that

$$\sum_{i=1}^N \alpha_i K(x_j, x_i) = u(x_j) \quad (5)$$

is solvable.

**Theorem 2.4.** [7] If the kernel function  $K^m(x, y)$  is strictly positive definite, the kernel matrix  $K^m(x_j, x_i)$  is guaranteed to be invertible.

**Theorem 2.5.** [4] If the RKHS  $H$  over  $E$  possesses an RKF such  $K(x, y)$ ,  $K(x, x)$  is uniformly bounded in  $E$ ,  $\|u_n(x) - u(x)\|_H \rightarrow 0$  when  $n \rightarrow \infty$ , then  $u_n(x)$  uniformly converges to  $u(x)$ .

**Theorem 2.6.** [4] The interpolant  $u_X \in H_X^m$  to a function  $u$  on  $X$  has the following error bound  $|u(x) - u_x(x)| \leq P_X^m(x) \|u(x)\|_m$ , where  $P_X^m(x) = \|K^m(\cdot, x) - K^m(\cdot, x)_X\|_m$  with respect to the set  $X$  and the RKF  $K^m(x, y)$ .

**Lemma 2.7.** [5] If  $u(x) \in C^{2m}[a, b]$  and  $u_n$  is the obtained approximate solution in RKHS  $H^m[a, b](m \leq 3)$ , then

$$\|u_N(x) - u(x)\|_\infty = \max_{x \in [a, b]} |u_N(x) - u(x)| \leq d_1 h^{2m},$$

where  $d_1$  denotes the appropriate constant and  $h = \max_{1 \leq i \leq N-1} |x_i - x_{i+1}|$ .

**Proposition 2.8.** [4]  $I(u)$  can be approximated as

$$\begin{aligned} I_N(u) &= \int_a^b u_N(x) f(\lambda v(x)) dx \\ &= \sum_{i=1}^N \alpha_i s_i \end{aligned}$$

where,  $s_i = \int_a^b f(\lambda v(x)) K^m(x, x_i) dx$  can be calculated explicitly.

**Theorem 2.9.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a bounded function by  $M > 0$ . The following results hold:

(i) There exists a constant  $d > 0$  such that  $|I(u) - I_N(u)| \leq dh^{2m}$ .

(ii) For a homogeneous function  $f$  of degree  $k$ , there exists  $\bar{d} > 0$  s.t  $|I(u) - I_N(u)| \leq \bar{d}h^{2m}$ .

*Proof.* (i)

$$\begin{aligned} |I(u) - I_N(u)| &= \left| \int_a^b (u - u_N) f(\lambda v) dx \right| \\ &\leq \int_a^b |u - u_N| |f(\lambda v)| dx \\ &\leq \|u - u_N\|_\infty \int_a^b |f(\lambda v)| dx \\ &\leq M \|u - u_N\|_\infty. \end{aligned}$$

(ii) If  $f$  is a homogeneous function of degree  $k$

$$\begin{aligned} |I(u) - I_N(u)| &\leq \|u - u_N\|_\infty \int_a^b |f(\lambda v)| dx \\ &\leq \lambda^k \|u - u_N\|_\infty \int_a^b |f(v)| dx \\ &\leq M \lambda^k \|u - u_N\|_\infty. \end{aligned}$$

□

**Theorem 2.10.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function such that  $f \circ (\lambda v) \in L^2([a, b])$ . The following results hold:*

(i) *There exists a constant  $e > 0$  such that  $|I(u) - I_N(u)| \leq eh^{2m}$ .*

(ii) *For a homogeneous function  $f$  of degree  $k$ , there exists constant  $\bar{e}$  s.t  $|I(u) - I_N(u)| \leq \bar{e}h^{2m}$*

*Proof.* (i) Using Holder inequality

$$\begin{aligned} |I(u) - I_N(u)| &= \left| \int_a^b (u - u_N) f(\lambda v) dx \right| \\ &\leq \int_a^b |u - u_N| |f(\lambda v)| dx \\ &\leq \left( \int_a^b (|u - u_N|)^2 dx \right)^{\frac{1}{2}} \left( \int_a^b (|f(\lambda v)|)^2 dx \right)^{\frac{1}{2}} \\ &\leq M \|u - u_N\|_2 \\ &\leq M \|u - u_N\|_\infty \sqrt{b - a} \end{aligned}$$

(ii) If  $f$  is a homogeneous function of degree  $k$

$$\begin{aligned} |I(u) - I_N(u)| &\leq \int_a^b |u - u_N| |f(\lambda v)| dx \\ &\leq \left( \int_a^b (|u - u_N|)^2 dx \right)^{\frac{1}{2}} \left( \int_a^b (|f(\lambda v)|)^2 dx \right)^{\frac{1}{2}} \\ &\leq \lambda^k M \|u - u_N\|_2 \\ &\leq \lambda^k M \|u - u_N\|_\infty \sqrt{b - a}, \end{aligned}$$

where  $M = \int_a^b (|f(\lambda v)|)^2 dx)^{\frac{1}{2}}$ . □

## 2.3 RKF-based Levin method

Following the idea of Levin method, the calculation of (1) is reduced to finding a solution for the following differential equations

$$w' + w\lambda v' \frac{f'(\lambda v)}{f(\lambda v)} = u(x) \quad (6)$$

If a solution of (6) is obtained, then

$$I(u) = \int_a^b u(x) f(\lambda v) dx = \int_a^b (w f(\lambda v))' dx = w(b) f(\lambda v(b)) - w(a) f(\lambda v(a)). \quad (7)$$

Therefore, the key is how to find the effective approximate solution of (6). In this section, we will solve (6) in the RKHS  $H^m[a, b]$  ( $m > 1$ ). Define operator  $Lw = w' + w\lambda v' \frac{f'(\lambda v)}{f(\lambda v)}$  and functionals  $\varphi_i = \delta_{x_i}$  for  $i = 1, 2, \dots, N$ . We choose  $N$  distinct scattered points  $x_1, x_2, \dots, x_N$  in  $[a, b]$ , and construct basis functions by the RKF  $K^m(x, y)$ . Let  $\psi_i(x) = \varphi_{i,y} K^m(x, y)$  for  $i = 1, 2, \dots, N$ , where  $\varphi_{i,y}$  denotes  $\varphi_i$  acts on the function of  $y$ . The collocation solution  $w_N(x)$  for (6) can be written as  $w_N(x) = \sum_{i=1}^N \beta_i \psi_i(x)$ , where  $\{\beta_i\}_{i=1}^N$  are unknown constants.

We require  $w_N(x)$  to satisfy the governing differential equation at all centers  $x_j$  ( $j = 1, \dots, N$ ), i.e.

$$Lw_N(x_k) = \varphi_k(w_N(x)) = \sum_{i=1}^N \beta_i \varphi_k \psi_i(x) = u(x_k), \quad k = 1, 2, \dots, N. \quad (8)$$

The system (8) can be reduced to the following compact form:

$$\beta \hat{K} = \mathbf{u}. \quad (9)$$

where  $\beta = (\beta_1, \beta_2, \dots, \beta_N)^T$ ,  $\mathbf{u} = (u(x_1), u(x_2), \dots, u(x_N))^T$ . Obviously,  $\hat{K}$  is a symmetric positive semi-definite matrix.

**Theorem 2.11.** [7] *If the functionals  $\varphi_i, i = 1, 2, \dots, N$  are linearly independent, then the matrix  $A$  is invertible. That is, system (9) has a unique solution.*

From Theorem 16.4 and Corollary 16.12 in [7], one obtains  $\varphi_i$ 's,  $i = 1, 2, \dots, N$  are linearly independent. Once the approximate solution of (6) is available, the approximation to HOIs (1) can be obtained as follows:

$$I_N(u) = \int_a^b (w_N(x)f(\lambda v))' dx = w_N(b)f(\lambda v(b)) - w_N(a)f(\lambda v(a)). \quad (10)$$

**Lemma 2.12.** *If  $u(x)$  and  $v'(x) \in C^{2(m-1)}[a, b]$ ,  $w_N(x)$  is the approximate solution obtained in RKHS such  $H^m[a, b]$  ( $2 \leq m \leq 4$ ), then we have the error estimate*

$$\|Lw_N(x) - u(x)\|_\infty = \max_{x \in [a, b]} |u(x) - Lw_N(x)| \leq c_1 h^{2m-2}, \quad (11)$$

where  $c_1$  is a positive real number.

**Theorem 2.13.** *Under the assumptions of the Lemma 2.12 and let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a bounded function by  $M$ , then the following hold:*

- (i) *There exists a constant  $c > 0$  such that  $|I(u) - I_N(u)| \leq ch^{2m-2}$ .*
- (ii) *If  $f$  is a homogeneous function of degree  $k$ , then there exists  $\bar{c} > 0$  such that  $|I(u) - I_N(u)| \leq \bar{c}h^{2m-2}$ .*

*Proof.* (i)

$$\begin{aligned} |I(u) - I_N(u)| &= \left| \int_a^b (u(x) - Lw_N(x)) f(\lambda v) dx \right| \\ &\leq \int_a^b |u(x) - Lw_N(x)| |f(\lambda v)| dx \\ &\leq \|Lw_N(x) - u(x)\|_\infty \int_a^b |f(\lambda v)| dx \\ &\leq M \|Lw_N(x) - u(x)\|_\infty. \end{aligned}$$

From 11, there exists a positive real number  $c$  such that  $|I(u) - I_N(u)| \leq c h^{2m-2}$ .

(ii) For a homogeneous function  $f$  of degree  $k$ ,

$$\begin{aligned} |I(u) - I_N(u)| &= \left| \int_a^b (u(x) - Lw_N(x)) f(\lambda v) dx \right| \\ &\leq \int_a^b |u(x) - Lw_N(x)| |f(\lambda v)| dx \\ &\leq \|Lw_N(x) - u(x)\|_\infty \int_a^b |f(\lambda v)| dx \\ &\leq \lambda^k \|Lw_N(x) - u(x)\|_\infty \int_a^b |f(v)| dx \\ &\leq M \lambda^k \|Lw_N(x) - u(x)\|_\infty. \end{aligned}$$

which completes the proof. □

**Theorem 2.14.** *Under the assumptions of Lemma 2.12 and  $f \circ (\lambda v) \in L^2([a, b])$ , the following results hold:*

(i) *There exist constants  $r > 0$  such that  $|I(u) - I_N(u)| \leq rh^{2m-2}$ .*

(ii) *If  $f$  be a homogeneous function of degree  $k$ , then there exists  $\bar{r} > 0$  such that  $|I(u) - I_N(u)| \leq \bar{r}h^{2m-2}$ .*

*Proof.* Using Holder inequality

$$\begin{aligned} |I(u) - I_N(u)| &= \left| \int_a^b (u - Lw_N) f(\lambda v) dx \right| \\ &\leq \int_a^b |u - Lw_N| |f(\lambda v)| dx \\ &\leq \left( \int_a^b (|u - Lw_N|)^2 dx \right)^{\frac{1}{2}} \left( \int_a^b (|f(\lambda v)|)^2 dx \right)^{\frac{1}{2}} \\ &\leq M \|u - Lw_N\|_2 \\ &\leq M \|u - Lw_N\|_\infty \sqrt{b-a} \end{aligned}$$

(ii) If  $f$  is a homogeneous function of degree  $k$

$$\begin{aligned} |I(u) - I_N(u)| &\leq \int_a^b |u - Lw_N| |f(\lambda v)| dx \\ &\leq \left( \int_a^b (|u - Lw_N|)^2 dx \right)^{\frac{1}{2}} \left( \int_a^b (|f(\lambda v)|)^2 dx \right)^{\frac{1}{2}} \\ &\leq \lambda^k M \|u - Lw_N\|_2 \\ &\leq \lambda^k M \|u - Lw_N\|_\infty \sqrt{b-a} \end{aligned}$$

□

### 3 Probabilistic integration methods

#### 3.1 Monte Carlo integration

In this section we study the application of Monte Carlo simulation in the calculation of integral of (1). The key point of Monte Carlo method is to suppose the integral as an expectation of a random variable. Therefore, the law of large numbers guarantees that average of the outcomes of the random variable is an efficient approximation for this integral

Following the idea of Monte Carlo integration, we consider (1) as follows

$$I = \int_a^b u(x) f(\lambda v) dx = \int_a^b \phi(x) dx$$

Therefore, if  $x_1, \dots, x_N$  are generated independently and identically from an uniform probability distribution in  $[a, b]$ , then  $\phi(x_1), \dots, \phi(x_N)$  are random variables with the expectation  $I$ . Thus, the law of large numbers leads to

$$P\left(\lim_{n \rightarrow \infty} \frac{b-a}{N} \sum_{i=1}^N \phi(x_i) = E(\phi(x_i))\right) = 1$$

i.e.  $I_N = \frac{b-a}{N} \sum_{i=1}^N \phi(x_i) \rightarrow I$  almost surely [10].

The error of this estimator is determined via the rate of convergence of its standard deviation as follows

$$\sigma(I_N) = \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) \quad (12)$$

that is bounded and moreover tends to zero with increasing of sample size, since

$$\begin{aligned} \sigma^2(I_N) &= \sigma^2\left(\frac{b-a}{N} \sum_{i=1}^N \phi(x_i)\right) \\ &= \frac{(b-a)^2}{N^2} \sum_{i=1}^N \sigma^2(\phi(x_i)) \\ &= \frac{(b-a)^2}{N} \sigma^2(\phi(x)) \end{aligned}$$

where  $\sigma^2(\phi(x)) = \frac{1}{b-a} \int_a^b (\phi(x) - E(\phi(x)))^2 dx$ .

### 3.2 Importance sampling

Although, the error of Monte Carlo approximation is reduced with the rate (12), still is proportional of standard deviation of  $\phi(x)$ . Instead, Importance sampling method provides to reduce the standard deviation than uniform law. Following the idea of the importance sampling, suppose that random vector  $y = (y_1, \dots, y_N)$  has a known probability density function  $p_y(\cdot)$ . Hence a modified estimator is given by

$$\tilde{I}_N = \frac{b-a}{N} \sum_{i=1}^N \frac{\phi(y_i)}{p(y_i)}$$

such that  $\tilde{I}_N \rightarrow I$  a.s. when  $N \rightarrow \infty$  [10].

The absolute error in Monte Carlo approximation is considered by

$$\begin{aligned} |I_N - I| &< \varepsilon, \\ |\tilde{I}_N - I| &< \varepsilon. \end{aligned}$$

in which to calculate the solution to desired level of an accuracy  $\varepsilon$ ,

$$N = \frac{z_{1-\alpha/2}^2 \sigma^2(\cdot)}{\varepsilon^2}$$

where

$$\begin{aligned} \sigma^2(I_N) &= \frac{(b-a)}{N} \int_a^b (\phi(x) - E(\phi(x)))^2 dx, \\ \sigma^2(\tilde{I}_N) &= \frac{(b-a)}{N} \int_a^b \left(\frac{\phi(y)}{p(y)} - E(\phi(y))\right)^2 dy. \end{aligned}$$

The preference of the estimator  $\tilde{I}_N$  is that with suitable choice of  $p(\cdot)$  such that  $p(y) = c \phi(y)$ , then we have  $z = \frac{\phi(y_i)}{p(y_i)} = \frac{1}{c}$  that implies  $\sigma^2(z) = 0$ , i.e. getting a less variance with better importance sampling (see [9]).

## 4 Numerical examples

Let the absolute error  $E_N = |I - I_N|$  and choosing  $x_i = \frac{i-1}{N-1}$ . In the following problems we take  $\omega = \frac{1}{\lambda}$  in order to equate the integrals with the form (1). Moreover, note that to apply



Importance sampling we take the beta distribution  $\beta(a, b)$  with parameters  $a = 1$  and  $b = 3$  as a known probability distribution.

**Problem 1.** We consider the oscillatory integral given by [8]

$$I = \int_0^1 \cos(x)e^{i\omega(x^2+x)}dx$$

We apply RKF-based Filon method in RKHS  $H^n$  to solve Problem 1. Moreover we apply Monte Carlo method in comparison with the RKF-based one. The obtained absolute error  $E_N$ 's are listed in Table 1.

In Table 2, we summarize the results using RKF-based Levin method in RKHS  $H^n$  in comparison with Importance sampling. In thees two tables  $N = 11$  is fixed and different  $\omega$ 's are displayed. In order to compare the accuracy of two probabilistic methods, we apply the Monte Carlo and Importance Sampling with different  $N$  in table 3. The numerical results in Table 1 and Table 2 show that RKF-based methods have higher accuracy. Moreover, it can be observed from Table 3 that the larger  $N$  is, the higher accuracy of both probabilistic methods becomes.

**Table 1:** Absolute errors  $E_{11}$  for Problem 1.

$\omega$	Filon $H^1$	Filon $H^2$	Filon $H^3$	MonteCarlo
100	$1.76363 \times 10^{-5}$	$2.95031 \times 10^{-6}$	$1.5383 \times 10^{-7}$	$5.08946 \times 10^{-2}$
300	$4.63999 \times 10^{-7}$	$2.92602 \times 10^{-7}$	$2.05819 \times 10^{-8}$	$7.81025 \times 10^{-2}$
1000	$1.12593 \times 10^{-7}$	$2.73966 \times 10^{-8}$	$1.76987 \times 10^{-9}$	$8.88181 \times 10^{-2}$
$10^4$	$1.64673 \times 10^{-9}$	$2.95038 \times 10^{-10}$	$1.57692 \times 10^{-11}$	$2.27703 \times 10^{-2}$

**Table 2:** Absolute errors  $E_{11}$  for Problem 1.

$\omega$	Levin $H^2$	Levin $H^3$	Levin $H^4$	Importance Sampling
100	$2.46664 \times 10^{-5}$	$6.09622 \times 10^{-6}$	$2.11802 \times 10^{-6}$	$5.31605 \times 10^{-2}$
300	$2.69833 \times 10^{-6}$	$6.95863 \times 10^{-7}$	$2.48263 \times 10^{-7}$	$6.00546 \times 10^{-2}$
1000	$2.45155 \times 10^{-7}$	$6.25975 \times 10^{-8}$	$2.19865 \times 10^{-8}$	$2.2191 \times 10^{-2}$
$10^4$	$2.49266 \times 10^{-9}$	$6.22499 \times 10^{-10}$	$1.2638 \times 10^{-10}$	$1.89282 \times 10^{-2}$

**Table 3:** Absolute errors  $E_N$  for Problem 1 with  $\omega = 300$ .

$N$	MonteCarlo	Importance Sampling
20	$3.10356 \times 10^{-2}$	$1.36432 \times 10^{-2}$
100	$2.04119 \times 10^{-2}$	$1.0154 \times 10^{-2}$
1000	$9.01701 \times 10^{-3}$	$3.71157 \times 10^{-3}$
$10^4$	$3.43552 \times 10^{-3}$	$1.9473 \times 10^{-3}$

**Problem 2.** We consider the oscillatory integral given by [8]

$$I = \int_0^1 \frac{1}{1+25x^2}e^{i\omega x}dx.$$

We apply RKF-based Filon method in RKHS  $H^3$  to solve problem 2 for fixed  $N = 11$ . Moreover we apply Monte Carlo and Importance sampling methods in comparison with the RKF-based

one. The results of are listed in Table 4. In table 5 the absolute error  $E_N$  are listed, using RKF-based Levin method in RKHS  $H^n$  for fixed  $\omega = 300$  and different  $N$  in comparison with Importance sampling method. The numerical results show that still RKF-based methods have higher accuracy. Nevertheless, the superiority of accuracy of RKF-based methods in comparison with the probabilistic methods, is decreased in Problem 2 w.r.t the obtained results in Problem1.

**Table 4:** Absolute errors  $E_N$  for Problem 2 with  $\omega = 300$ .

$N$	Filon $H^3$	MonteCarlo	Importance Sampling
10	$5.74591 \times 10^{-5}$	$4.23122 \times 10^{-2}$	$5.40508 \times 10^{-3}$
100	$7.21492 \times 10^{-5}$	$1.92091 \times 10^{-2}$	$2.91083 \times 10^{-3}$
1000	$7.21418 \times 10^{-5}$	$5.69007 \times 10^{-3}$	$2.89973 \times 10^{-3}$

**Table 5:** Absolute errors  $E_{11}$  for Problem 2.

$\omega$	Levin $H^2$	Levin $H^3$	Levin $H^4$	Importance Sampling
100	$1.66474 \times 10^{-4}$	$1.09386 \times 10^{-4}$	$4.3234 \times 10^{-5}$	$1.06183 \times 10^{-2}$
300	$5.41356 \times 10^{-5}$	$6.04515 \times 10^{-5}$	$6.78041 \times 10^{-5}$	$3.77867 \times 10^{-3}$
1000	$7.16477 \times 10^{-5}$	$7.10809 \times 10^{-5}$	$7.04138 \times 10^{-5}$	$6.42895 \times 10^{-3}$
$10^4$	$1.03669 \times 10^{-4}$	$1.03669 \times 10^{-4}$	$1.03669 \times 10^{-4}$	$1.78491 \times 10^{-3}$

**Problem 3.** We consider a new problem of oscillatory integral given as follows

$$I = \int_0^1 \cos(\sin(x)) e^{i\omega \sin(x)} dx. \quad (13)$$

We apply RKF-based Filon methods in RKHS  $H^n$  to solve [13](#). Moreover we apply Monte Carlo method in comparison with the RKF-based one. In Table 6 the absolute error  $E_N$  are listed for fixed  $N = 11$  and different  $\omega$ 's. The results show that RKF-based Filon has the higher accuracy. In table 7, We apply RKF-based Levin methods in RKHS  $H^n$ . Moreover, the absolute error using Importance sampling method are displayed in comparison with the RKF-based one. It can be observed from Table 7 that RKF-based Levin method in space  $H^3$  and  $H^4$  does not approximate the integral [13](#) as well as problems previously. Indeed, the absolute error  $E_N$  are obtained with the huge values in thees cases. The numerical results show that the Importance sampling has the higher accuracy than the RKF-based Levin in space  $H^3$  and  $H^4$ .

**Table 6:** Absolute errors  $E_{11}$  for Problem 3.

$\omega$	Filon $H^1$	Filon $H^2$	Filon $H^3$	Monte Carlo
100	$1.79117 \times 10^{-5}$	$7.27718 \times 10^{-6}$	$1.07228 \times 10^{-6}$	$1.18554 \times 10^{-1}$
300	$3.14773 \times 10^{-6}$	$5.50166 \times 10^{-7}$	$1.00796 \times 10^{-7}$	$1.95602 \times 10^{-1}$
1000	$1.73829 \times 10^{-7}$	$1.95607 \times 10^{-8}$	$7.25793 \times 10^{-9}$	$1.92572 \times 10^{-1}$
$10^4$	$1.88033 \times 10^{-9}$	$5.05618 \times 10^{-10}$	$9.1844 \times 10^{-11}$	$1.09875 \times 10^{-1}$

**Table 7:** Absolute errors  $E_{11}$  for Problem 3.

$\omega$	Levin $H^2$	Levin $H^3$	Levin $H^4$	Importance Sampling
100	$2.89089 \times 10^{-5}$	316.414	18689.3	$5.08946 \times 10^{-2}$
300	$6.04304 \times 10^{-6}$	15.4763	782.181	$7.81025 \times 10^{-2}$
1000	$2.96585 \times 10^{-7}$	51.0473	2353.03	$2.97788 \times 10^{-3}$
$10^4$	$1.26371 \times 10^{-9}$	34.6904	430.602	$2.27703 \times 10^{-2}$

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