



HAL
open science

Well-posedness results for hyperbolic operators with coefficients rapidly oscillating in time

Ferruccio Colombini, Daniele Del Santo, Francesco Fanelli

► **To cite this version:**

Ferruccio Colombini, Daniele Del Santo, Francesco Fanelli. Well-posedness results for hyperbolic operators with coefficients rapidly oscillating in time. *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze*, 2023, pp.18. 10.2422/2036-2145.202301_017 . hal-03960829

HAL Id: hal-03960829

<https://hal.science/hal-03960829>

Submitted on 15 Mar 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

WELL-POSEDNESS RESULTS FOR HYPERBOLIC OPERATORS WITH COEFFICIENTS RAPIDLY OSCILLATING IN TIME

Ferruccio Colombini¹, Daniele Del Santo² and Francesco Fanelli³

¹ UNIVERSITÀ DI PISA

Dipartimento di Matematica

Largo Bruno Pontecorvo, 5 – I-56127 Pisa, ITALY

² UNIVERSITÀ DEGLI STUDI DI TRIESTE

Dipartimento di Matematica e Geoscienze

Via Valerio, 12/1 – I-34127 Trieste, ITALY

³ UNIVERSITÉ DE LYON, UNIVERSITÉ CLAUDE BERNARD LYON 1

Institut Camille Jordan – UMR 5208

43 blvd. du 11 novembre 1918, F-69622 Villeurbanne cedex, FRANCE

Email addresses: ¹ ferruccio.colombini@unipi.it, ² delsanto@units.it, ³ fanelli@math.univ-lyon1.fr

January 27, 2023

Abstract

In the present paper, we consider second order strictly hyperbolic linear operators of the form $Lu = \partial_t^2 u - \operatorname{div}(A(t, x)\nabla u)$, for $(t, x) \in [0, T] \times \mathbb{R}^n$. We assume the coefficients of the matrix $A(t, x)$ to be smooth in time on $]0, T] \times \mathbb{R}^n$, but rapidly oscillating when $t \rightarrow 0^+$; they match instead minimal regularity assumptions (either Lipschitz or log-Lipschitz regularity conditions) with respect to the space variable.

Correspondingly, we prove well-posedness results for the Cauchy problem related to L , either with no loss of derivatives (in the Lipschitz case) or with a finite loss of derivatives, which is linearly increasing in time (in the log-Lipschitz case).

2020 Mathematics Subject Classification: 35L10 (primary); 35B65, 35B30, 35L15 (secondary).

Keywords: second order hyperbolic equations; variable coefficients; fast time oscillations; Lipschitz and log-Lipschitz regularity.

1 Introduction

The goal of the present paper is to study the well-posedness of the Cauchy problem related to the second order strictly hyperbolic operator

$$(1) \quad Lu := \partial_t^2 u - \operatorname{div}(A(t, x)\nabla u), \quad \text{with} \quad \lambda_0 \operatorname{Id} \leq A(t, x) \leq \Lambda_0 \operatorname{Id},$$

where λ_0 and Λ_0 are two positive constants and $A(t, x) = (a_{jk}(t, x))_{j,k=1}^n$ is defined on a strip $[0, T] \times \mathbb{R}^n$, for some time $T > 0$, and is symmetric, *i.e.* $a_{jk} = a_{kj}$ for any $j, k = 1 \dots n$.

It has been well-known since the pioneering work [5] (see also [13] for a similar result in the case of weakly hyperbolic equations) that the low regularity in time of the coefficient matrix $A(t, x)$ entails weak well-posedness results, in the sense that a (finite or even infinite) loss of regularity of the solution u is produced in the time evolution as soon as the coefficients are not Lipschitz-continuous with respect to time. There is a broad literature on the subject, keeping into account

also minimal regularity assumptions on the coefficients with respect to the space variable x . We refer, for instance, to [27], [16], [3], [21], [18], [15], [29], [17] and, more recently, to [7], [8], [23], [4], [6]. Similar results for first order hyperbolic systems have also been obtained, see e.g. [9], [10].

On the other hand, when dealing with the construction of counterexamples, it appeared from the beginning that phenomena like loss of derivatives, non-existence or even non-uniqueness of solutions in hyperbolic Cauchy problems may arise when the coefficients are smooth with respect to the time variable, but they highly oscillate when t approaches one point, which we can assume to be 0 without loss of any generality. We refer to e.g. [5], [13], [19], [14] for the construction of such counterexamples.

As a consequence, some works have focused on studying the relation between fast oscillations of the coefficients a_{jk} and the well-posedness of the Cauchy problem. In this direction, Yamazaki proved in [30] that the Cauchy problem for L , defined in (1), is well-posed without any loss of derivatives under the following assumptions on the coefficients a_{jk} : they must be continuous in time on $[0, T] \times \mathbb{R}^n$ and C^2 in time on $]0, T] \times \mathbb{R}^n$, smooth with respect to the space variable together with $\partial_t a_{jk}$ and $\partial_t^2 a_{jk}$, and they have to satisfy the bound

$$(2) \quad |t \partial_t a_{jk}(t, x)| + |t^2 \partial_t^2 a_{jk}(t, x)| \leq C \quad \text{on} \quad]0, T] \times \mathbb{R}^n.$$

The control on the oscillations of the second order time-derivative seems to be necessary for well-posedness without loss of regularity. As a matter of fact, for coefficients $a_{jk} = a_{jk}(t)$ depending only on time, in [11] it was shown that, under the assumption

$$\left| t \frac{d}{dt} a_{jk}(t) \right| \leq C \quad \text{on} \quad]0, T] \times \mathbb{R}^n,$$

the Cauchy problem for L is well-posed with a finite loss of derivatives. In addition, an analogous result was proven in [12] under the weaker condition

$$(3) \quad \left| \frac{t}{\log t} \frac{d}{dt} a_{jk}(t) \right| + \left| \left(\frac{t}{\log t} \right)^2 \frac{d^2}{dt^2} a_{jk}(t) \right| \leq C \quad \text{on} \quad]0, T] \times \mathbb{R}^n.$$

We refer also to [28], [20] and [22] for related results.

In the present paper, we continue the study in this direction and we consider the case of coefficients rapidly oscillating in time, for $t \rightarrow 0^+$. More precisely, we will work under the assumption (2), which is the same one formulated in [30]. Our main contribution to this problem is two-fold. First of all, we recover Yamazaki's result [30] by proposing a different proof, which looks somehow simpler for us (we will give more details about it in a while) than the original one. Secondly, and more importantly, we will be able to consider minimal regularity assumptions on the coefficients with respect to the space variable, namely Lipschitz and log-Lipschitz regularities in x . In the former case (Lipschitz regularity), we will prove an energy estimate for L with *no* loss of derivatives in the space $H^{1-\theta} \times H^{-\theta}$, for any $\theta \in [0, 1[$. From this, it is classical to recover well-posedness of the Cauchy problem related to L in the same space. In the latter case (log-Lipschitz regularity of the coefficients in x), we will prove an energy estimate with a finite, linearly increasing in time, *loss of derivatives*: if the datum (u_0, u_1) belongs to $H^{1-\theta} \times H^{-\theta}$, for some $0 < \theta < 1$, then the norm of the solution at time t is controlled in the space $H^{1-\theta-\beta^*t} \times H^{-\theta-\beta^*t}$, where the constant $\beta^* > 0$ depends only on the coefficient matrix $A(t, x)$.

The proof is inspired by the techniques recently employed in [7], [8] and [6], and consists of two main ingredients. The first ingredient is a definition of the energy functional which presents a lower order corrector; in turn, this idea goes back to work [29] of Tarama. The role of the corrector is to cancel out bad remainders which appear when computing the time derivative of the energy and which would be otherwise out of control. The second ingredient, instead, is the use of paradifferential calculus with parameters, as introduced by Métivier (see [24] and [26]), which provides us with a powerful tool for dealing with the low regularity in space of the coefficients.

To conclude this short introduction, let us mention that, in the case when $a_{jk} = a_{jk}(t)$ does not depend on x , we are able to consider even faster oscillations, in the spirit of condition (3) from [12], recovering the result of [12] with a more elementary proof. For this, it is enough to use exactly the same energy of Tarama [29] in order to carry out energy estimates. However, at present we are not able to extend this approach to the case in which the coefficients also depend on the space variables. This is why we will not devote attention to that case in this work.

Acknowledgements

The first and the second author are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). They acknowledge INdAM for the support.

The work of the third author has been partially supported by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program "Investissement d'Avenir" (ANR-11-IDEX-0007), and by the projects SingFlows (ANR-18-CE40-0027) and CRISIS (ANR-20-CE40-0020-01), both operated by the French National Research Agency (ANR).

2 Statement of the main results

We consider the linear differential operator

$$(4) \quad Lu = \partial_t^2 u - \sum_{j,k=1}^n \partial_j (a_{jk}(t, x) \partial_k u),$$

where $a_{j,k} : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ are such that $a_{jk} = a_{kj}$. We also assume that there exist $\lambda_0, \Lambda_0 > 0$ such that, for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$,

$$(5) \quad \lambda_0 |\xi|^2 \leq \sum_{j,k=1}^n a_{jk}(t, x) \xi_j \xi_k \leq \Lambda_0 |\xi|^2.$$

In particular, under this condition, the operator L becomes strictly hyperbolic.

Suppose moreover that the coefficients $a_{j,k}$ satisfy the following assumptions.

(H1) There exist $\ell \in \{0, 1\}$ and $C_0 > 0$ such that, for any $j, k \in \{1 \dots n\}$, one has

$$|a_{jk}(t, x + y) - a_{jk}(t, x)| \leq C_0 |y| \log^\ell \left(1 + \frac{1}{|y|} \right),$$

for all $t \in [0, T]$ and for all $x, y \in \mathbb{R}^n$, with $0 < |y| < 1$.

(H2) Each $a_{j,k}$ is twice differentiable with respect to t in $]0, T] \times \mathbb{R}^n$; in addition there exist constants $C_1 > 0$, $C_2 > 0$, $C_3 > 0$ for which, for any $j, k \in \{1 \dots n\}$, the controls

$$\begin{aligned} |\partial_t a_{jk}(t, x)| &\leq C_1 \frac{1}{t}, \\ |\partial_t a_{jk}(t, x + y) - \partial_t a_{jk}(t, x)| &\leq C_2 \frac{1}{t} |y| \log^\ell \left(1 + \frac{1}{|y|} \right), \\ |\partial_t^2 a_{jk}(t, x)| &\leq C_3 \frac{1}{t^2} \end{aligned}$$

hold true for all $t \in]0, T]$ and for all $x, y \in \mathbb{R}^n$, with $0 < |y| < 1$, where $\ell \in \{0, 1\}$ is the same exponent fixed in assumption (H1).

Notice that hypothesis (H1) means that the coefficients $a_{j,k}$ are Lipschitz continuous in space when $\ell = 0$, log-Lipschitz continuous when $\ell = 1$, uniformly with respect to the time variable. Requiring a similar control also for $\partial_t a_{j,k}$, see the second condition appearing in (H2), is somehow natural in this kind of study; see e.g. [7], [8], [6].

We are now able to state the main results of the paper. The first one is devoted to the Lipschitz case, namely to the case $\ell = 0$. It is an extension of Yamazaki's result [30] to the case when coefficients have low regularity in the space variable; besides, we propose a proof based on a different technique, which looks to an extent simpler.

Theorem 2.1. *Let L be defined by (4), and assume that the assumptions (5), (H1) and (H2) are in force with $\ell = 0$. Fix $\theta \in [0, 1[$.*

Then, there exists a universal constant $C > 0$ such that, for any $u \in C^2([0, T]; H^\infty(\mathbb{R}^n))$, one has the estimate

$$(6) \quad \sup_{0 \leq t \leq T} \left(\|u(t, \cdot)\|_{H^{1-\theta}} + \|\partial_t u(t, \cdot)\|_{H^{-\theta}} \right) \leq C \left(\|u(0, \cdot)\|_{H^{1-\theta}} + \|\partial_t u(0, \cdot)\|_{H^{-\theta}} + \int_0^T \|Lu(\tau, \cdot)\|_{H^{-\theta}} d\tau \right).$$

From the previous statement, it is classical to derive the well-posedness (without loss of derivatives) of the Cauchy problem associated to L in the energy space $H^{1-\theta}(\mathbb{R}^n) \times H^{-\theta}(\mathbb{R}^n)$, for any given $\theta \in [0, 1[$.

Let us now switch to the log-Lipschitz case, for which $\ell = 1$ in (H1) and (H2) above. As is well-known (see e.g. [16], [17], [7] and references therein), in this case the estimates present a loss of derivatives, which is linearly increasing in time. In addition, because of that and because of the low regularity in space of the coefficients, the estimate becomes local in time. Finally, one has to work in energy spaces which are less regular than $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, i.e. one has to take $\theta \in]0, 1[$.

Theorem 2.2. *Let L be defined by (4), and assume that its coefficients a_{jk} satisfy assumptions (5), (H1) and (H2) with $\ell = 1$. Fix $\theta \in]0, 1[$.*

Then, there exist $\beta^ > 0$, $T_0 \in]0, T[$ and $C > 0$ such that, for all function $u \in C^2([0, T_0]; H^\infty(\mathbb{R}^n))$, one has the energy estimate*

$$(7) \quad \sup_{0 \leq t \leq T_0} \left(\|u(t, \cdot)\|_{H^{1-\theta-\beta^*t}} + \|\partial_t u(t, \cdot)\|_{H^{-\theta-\beta^*t}} \right) \leq C \left(\|u(0, \cdot)\|_{H^{1-\theta}} + \|\partial_t u(0, \cdot)\|_{H^{-\theta}} + \int_0^{T_0} \|Lu(\tau, \cdot)\|_{H^{-\theta-\beta^*\tau}(\mathbb{R}^n)} d\tau \right).$$

From the previous result, the well-posedness of the Cauchy problem related to L follows in the space H^∞ , with a finite loss of derivatives.

The rest of the paper is devoted to the proof of the previous results. As already announced, we propose a different approach than [30], based on Tarama's energy [29] and paradifferential calculus with parameters, as used in [7] and [8] for operators with low regularity coefficients both in time and space variables.

Before moving on, we remark that, in the case that the coefficients $a_{jk} = a_{jk}(t)$ depend only on time, our method enable us to treat the case of even faster oscillations in time. Namely, in assumption (H2) we can admit an additional logarithmic loss: there exists $\delta \in [0, 1]$ such that

$$|\partial_t a_{jk}(t)| \leq C \frac{1 + |\log t|^\delta}{t},$$

$$|\partial_t^2 a_{jk}(t)| \leq C \left(\frac{1 + |\log t|^\delta}{t} \right)^2.$$

By resorting to Tarama's energy (defined in frequencies, as in [5], [29]), we can prove energy estimates with no loss when $\delta = 0$, with finite (fixed) loss of derivatives when $\delta > 0$ and with arbitrarily small loss when $0 < \delta < 1$, in any Sobolev space H^s . This result has already been known (see [12], [28] for the case $\delta = 1$ and [30] for the case $\delta = 0$), but our proof looks much simpler than the previous ones.

3 Approximation and regularisation

The goal of this section is to introduce a suitable approximation/regularisation of the coefficients, both in time and space variables, in order to perform (in Section 4) energy estimates for approximated energies related to the function u .

3.1 Approximation of the coefficients

We start by dealing with the time variable. In order to perform energy estimates, we need to define approximations of the coefficients which are smooth up to the time $t = 0$. The obvious problem is that, under assumption (H2), the coefficients are highly oscillating when t approaches 0. As a consequence, we need to approximate the original coefficients by smooth ones: for this, we will follow [12] and proceed by truncation around $t = 0$.

Thus, let $\varepsilon \in]0, \frac{T}{2}]$. Following [12], we set

$$\tilde{a}_{\varepsilon,jk}(t, x) := \begin{cases} a_{jk}(\varepsilon, x) & \text{for } t < \varepsilon, \\ a_{jk}(t, x) & \text{for } \varepsilon \leq t \leq T, \\ a_{jk}(T, x) & \text{for } t > T. \end{cases}$$

Of course, $\tilde{a}_{jk,\varepsilon}$ previously defined is not smooth in time, so not suitable for energy estimates. Therefore, a more involved approximation procedure is required. For this, consider $(\rho_\eta)_{\eta \in]0,1]}$ a standard family of mollifiers. We then define

$$\tilde{\tilde{a}}_{\varepsilon,jk}(t, x) := (\rho_{\varepsilon/2} *_t \tilde{a}_{\varepsilon,jk})(t, x),$$

where the notation $*_t$ means that the convolution is taken only with respect to the time variable.

This approximation is not satisfactory for our scopes yet, because, in view of the next computations, it would be more convenient to keep the old values of the function a_{jk} when the time t is far from 0. Hence, let us take a function $\theta \in C^\infty(\mathbb{R})$ such that $0 \leq \theta \leq 1$, $\theta(s) = 1$ for $s \leq 1/4$ and $\theta(s) = 0$ for $s \geq 3/4$. Then, we set $\theta_\varepsilon(t) := \theta(\frac{1}{3\varepsilon}t + \frac{1}{12})$ and we define

$$(8) \quad a_{\varepsilon,jk}(t, x) := \tilde{\tilde{a}}_{\varepsilon,jk}(t, x) \theta_\varepsilon(t) + a_{jk}(t, x) (1 - \theta_\varepsilon(t)).$$

We remark that, with this definition, one has

$$\begin{aligned} a_{\varepsilon,jk}(t, x) &= a_{jk}(t, x) & \text{if } 2\varepsilon \leq t \leq T, \\ a_{\varepsilon,jk}(t, x) &= a_{jk}(\varepsilon, x) & \text{if } t \leq \frac{\varepsilon}{2} \end{aligned}$$

for any $x \in \mathbb{R}^n$. In addition, the coefficients $a_{\varepsilon,jk}$ satisfy the following properties:

- (i) for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$, we have

$$\lambda_0 |\xi|^2 \leq \sum_{j,k=1}^n a_{\varepsilon,jk}(t, x) \xi_j \xi_k \leq \Lambda_0 |\xi|^2;$$

- (ii) there exists a constant $C_0 > 0$, possibly different from the one appearing in (H1) in Section 2, but still independent of $\varepsilon \in]0, 1]$, such that

$$|a_{\varepsilon,jk}(t, x + y) - a_{\varepsilon,jk}(t, x)| \leq C_0 |y| \log^\ell \left(1 + \frac{1}{|y|} \right)$$

for all $t \in \mathbb{R}$ and for all $x, y \in \mathbb{R}^n$, with $0 < |y| < 1$;

(iii) there exists a function $\varphi_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ of class C^2 , satisfying

$$\varphi_\varepsilon(t) = 0 \quad \text{for} \quad t \leq \frac{\varepsilon}{2} \quad \text{and} \quad 0 \leq \varphi_\varepsilon(t) \leq \frac{1}{t} \quad \text{for} \quad t \geq \frac{\varepsilon}{2},$$

and there exist constants $C_1, C_2, C_3 > 0$ (possibly different from the ones appearing in (H2) of Section 2, but still independent of $\varepsilon \in]0, 1[$) such that

$$\begin{aligned} |\partial_t a_{\varepsilon,jk}(t, x)| &\leq C_1 \varphi_\varepsilon(t), \\ |\partial_t a_{\varepsilon,jk}(t, x+y) - \partial_t a_{\varepsilon,jk}(t, x)| &\leq C_2 \varphi_\varepsilon(t) |y| \log^\ell \left(1 + \frac{1}{|y|} \right), \\ |\partial_t^2 a_{\varepsilon,jk}(t, x)| &\leq C_3 (\varphi_\varepsilon(t))^2, \end{aligned}$$

for all $t \in \mathbb{R}$ and for all $x, y \in \mathbb{R}^n$, with $0 < |y| < 1$.

The coefficients $a_{\varepsilon,jk}$ constructed in this way are certainly smooth in time on $[0, T] \times \mathbb{R}^n$, but they are not regular with respect to the space variable. Therefore, a further approximation is needed: this is the scope of the next subsections.

3.2 Tools from Littlewood-Paley theory

Here we recall basic facts linked to Littlewood-Paley decomposition and paradifferential calculus with parameters, which will be very useful tools in our proof.

Littlewood-Paley decomposition and Sobolev spaces

Let us start by defining the Littlewood-Paley decomposition of any tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ and the dyadic characterisation of Sobolev spaces. We refer to e.g. [1] and [25] for details.

We fix a smooth radial function $\chi : \mathbb{R}^n \rightarrow \mathbb{R}$, supported in the ball $B(0, 2)$ of center 0 and radius 2, with $\chi \equiv 1$ in a neighbourhood of the unit ball $B(0, 1)$ and χ non-increasing along the radial directions. For $\xi \in \mathbb{R}^n$, we also define $\phi(\xi) := \chi(\xi) - \chi(2\xi)$.

For any $j \in \mathbb{N}$ and $\xi \in \mathbb{R}^n$, we also define

$$(9) \quad \chi_j(\xi) := \chi(2^{-j}\xi) \quad \text{and} \quad \phi_j(\xi) := \phi(2^{-j}\xi).$$

Then, we can introduce the dyadic blocks Δ_j , for $j \in \mathbb{N}$. First of all, we define Δ_0 as the operator

$$\Delta_0 u := \chi(D_x)u = \mathcal{F}^{-1}(\chi(\xi) \widehat{u}(\xi)),$$

where $D_x = -i\partial_x$, \widehat{u} denotes the Fourier transform of a tempered distribution u and the symbol \mathcal{F}^{-1} the inverse Fourier transform. Next, for any $j \geq 1$, we set

$$\Delta_j u := \phi_j(D_x)u = \mathcal{F}^{-1}(\phi_j(\xi) \widehat{u}(\xi)).$$

For $j \geq 0$, we also set

$$S_j u := \chi_j(D_x)u = \sum_{k=0}^j \Delta_k u.$$

Thanks to the previous operators, one can define the *Littlewood-Paley decomposition* of tempered distributions: for any $u \in \mathcal{S}'(\mathbb{R}^n)$, the decomposition

$$u = \sum_{j=0}^{+\infty} \Delta_j u$$

holds true in the sense of $\mathcal{S}'(\mathbb{R}^n)$.

One can also characterise Sobolev classes in terms of Littlewood-Paley decomposition. More precisely, given some $s \in \mathbb{R}$ and $u \in \mathcal{S}'(\mathbb{R}^n)$, one has that

$$u \in H^s(\mathbb{R}^n) \quad \Longleftrightarrow \quad \left(2^{sj} \|\Delta_j u\|_{L^2} \right)_{j \geq 0} \in \ell^2(\mathbb{N}),$$

with equivalence of norms: there exists a constant $C > 0$, independent of u but possibly depending on s , such that

$$\frac{1}{C} \sum_{j=0}^{+\infty} 2^{2sj} \|\Delta_j u\|_{L^2}^2 \leq \|u\|_{H^s}^2 \leq C \sum_{j=0}^{+\infty} 2^{2sj} \|\Delta_j u\|_{L^2}^2.$$

Paradifferential calculus with parameters

In order to deal with the low regularity of the coefficients with respect to the space variable, we will adopt the approach used in [7] and [8], based on a parilinearisation of the elliptic part $\sum_{j,k} \partial_j (a_{jk} \partial_k)$ of the operator L . Of course, L being linear, we only mean that the products by the coefficients a_{jk} will be replaced by paraproduct operators $T_{a_{jk}}$ (see [25], [1]): as a matter of fact, this procedure intrinsically involves a regularisation in space.

On the other hand, in order to guarantee positivity of the parilinearised operator, we have to resort to Métivier's paradifferential calculus with parameters, as introduced in [24], [26]. This technique being by now quite well understood, we will just recall the main ideas of the construction and use it in our computations; we refer the reader to [7] and [8] for details.

Thus, let us fix a symbol $f = f(x, \xi)$, defined for $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$, smooth with respect to the ξ variable and merely bounded in x . The main idea of paradifferential calculus (see [25]) is to associate to f a classical symbol $\sigma_f = \sigma_f(x, \xi)$ which is smooth also with respect to the space variable x . Then, one defines the paradifferential operator T_f associated to the (rough) symbol f as the classical pseudodifferential operator associated to σ_f : we set

$$T_f := \sigma_f(x, D_x).$$

If the symbol f also depend on time, we can treat the time variable as a parameter in the construction.

This having been defined, one can thus develop a symbolic calculus for pseudodifferential operators, based on the smoothness of the original symbol f . We will see in a while how to do that in the case of our hyperbolic operator L . For the time being, let us emphasise a couple of facts.

First of all, all the construction is, in some sense, *canonical*: although the classical symbol σ_f associated to f is not uniquely determined, the paradifferential operator T_f is, up to a remainder of lower order (the precise order depends on the space regularity and frequency decay of f). In particular, one can fix the choice

$$T_f : u \mapsto T_f u := S_0 f S_3 u + \sum_{\nu=1}^{+\infty} S_\nu f \Delta_{\nu+3} u,$$

which coincides (whenever $f = f(x)$ only depends on the space variable x) with the classical Bony paraproduct operator [2] (see also Chapter 2 of [1]). The point is that this construction makes sense for more general symbols $f = f(t, x, \xi)$.

Nonetheless, we remark that, with this construction, for positive symbols one loses positivity of the corresponding paradifferential operator, a property which is very much desirable when dealing with energy functionals in hyperbolic Cauchy problems. As a matter of fact, if $f = f(x)$ is a bounded scalar function defined over \mathbb{R}^n , with $f(x) \geq c > 0$ for a.e. $x \in \mathbb{R}^n$, one has $\|f u\|_{L^2} \geq c \|u\|_{L^2}$, but it is not true, in general, that the property $\|T_f u\|_{L^2} \geq \tilde{c} \|u\|_{L^2}$ holds (even with a new constant $\tilde{c} > 0$).

In order to solve the previous issue, the basic idea of Métivier (see [24], [26]) was to add enough blocks of frequencies to the definition of the paraproduct to recover, in turn, positivity of the modified operator. Notice that the product fu mixes all the frequencies of f and u , whereas the paraproduct $T_f u$ uses only a few of them. More rigorously, one introduces a parameter $\gamma \geq 1$ in the construction, obtaining classical symbols σ_f^γ and operators T_f^γ depending on such γ . Considering larger values of γ allows to take into account more and more frequencies in the definition of the operator T_f^γ : for $u \in \mathcal{S}(\mathbb{R}^n)$, one can consider

$$(10) \quad T_f^\gamma u = S_{\mu-1} f S_{\mu+2} u + \sum_{\nu=\mu}^{+\infty} S_\nu f \Delta_{\nu+3} u,$$

where $\mu = \mu(\gamma) = [\log_2 \gamma]$ is the integer part of $\log_2 \gamma$. Now, given a symbol $f = f(x, \xi)$ of (say) order 0 such that $f \geq c > 0$, there exists a $\gamma_0 = \gamma_0(f)$, depending on the symbol f , such that, for all $\gamma \geq \gamma_0$, one has

$$\forall u \in \mathcal{S}(\mathbb{R}^n), \quad \left\| T_f^\gamma u \right\|_{L^2} \geq \frac{c}{2} \|u\|_{L^2}.$$

In light of the previous considerations, without loss of generality, throughout this paper we can fix the previous choice of paradifferential operators: given a symbol $f = f(t, x, \xi)$ as described above, we define the operator T_f^γ according to formula (10). Observe also that, more generally, one can even deal with symbols $f = f(t, x, \xi, \gamma)$ depending also on the parameter γ .

To conclude this part, let us recall (see Section 3.2 of [7] for details) that we can define also a Littlewood-Paley decomposition depending on the parameter $\gamma \geq 1$, as well as Sobolev spaces H_γ^s depending on γ . For $\gamma \geq 1$ fixed, the space H_γ^s coincides with the classical H^s space, with equivalence of norms (where the multiplicative constants depend on γ , of course). Thus, the dyadic characterisation that we have seen above still applies.

3.3 Symbolic calculus

Let us see how to apply the previous ideas to the case of the strictly hyperbolic operator L , defined in (4). To begin with, we introduce the 0-th order symbol

$$\alpha_\varepsilon(t, x, \xi, \gamma) := (\gamma^2 + |\xi|^2)^{-\frac{1}{2}} \left(\gamma^2 + \sum_{j,k=1}^n a_{\varepsilon,jk}(t, x) \xi_j \xi_k \right)^{\frac{1}{2}}.$$

Following the original idea of [5] and later [16], we immediately choose

$$\varepsilon = 2^{-\nu}.$$

For simplicity of notation, we write α_ν , $a_{\nu,jk}$ and φ_ν instead of $\alpha_{2^{-\nu}}$, $a_{2^{-\nu},jk}$ and $\varphi_{2^{-\nu}}$, respectively.

In order to develop symbolic calculus, we need suitable estimates for the classical symbols associated to the non-smooth symbol α_ν and its time derivatives. Repeating the computations of [7], [8] and [6], it is easy to obtain the following bounds: for any $\alpha, \beta \in \mathbb{N}^n$, with $|\beta| \geq 1$, and for any $(t, x, \xi, \gamma) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times [1, +\infty[$, one has

$$\begin{aligned} \left| \partial_\xi^\alpha \sigma_{\alpha_\nu}(t, x, \xi, \gamma) \right| &\leq C_\alpha (\gamma + |\xi|)^{-|\alpha|}, \\ \left| \partial_x^\beta \partial_\xi^\alpha \sigma_{\alpha_\nu}(t, x, \xi, \gamma) \right| &\leq C_{\beta,\alpha} (\gamma + |\xi|)^{-|\alpha|+|\beta|-1} \log^\ell(1 + \gamma + |\xi|), \\ \left| \partial_\xi^\alpha \sigma_{\partial_t \alpha_\nu}(t, x, \xi, \gamma) \right| &\leq C_\alpha \varphi_\nu(t) (\gamma + |\xi|)^{-|\alpha|}, \\ \left| \partial_x^\beta \partial_\xi^\alpha \sigma_{\partial_t \alpha_\nu}(t, x, \xi, \gamma) \right| &\leq C_{\beta,\alpha} \varphi_\nu(t) (\gamma + |\xi|)^{-|\alpha|+|\beta|-1} \log^\ell(1 + \gamma + |\xi|), \end{aligned}$$

$$\left| \partial_x^\beta \partial_\xi^\alpha \sigma_{\partial_t^2 \alpha_\nu}(t, x, \xi, \gamma) \right| \leq C_\alpha (\varphi_\nu(t))^2 (\gamma + |\xi|)^{-|\alpha|+|\beta|},$$

where the involved multiplicative constants do not depend on ν , nor on (t, x, ξ, γ) . We point out that, whenever an estimate as above holds with $\ell = 1$, we will say that the paradifferential operator associated to the symbol under consideration is of order Log .

Next, we fix $\gamma \geq 1$ so large that, for all $u \in H^\infty$, one has

$$(11) \quad \left\| T_{\alpha_\nu}^{\gamma-1/2} u \right\|_{L^2} \geq \frac{\lambda_0}{2} \|u\|_{L^2} \quad \text{and} \quad \left\| T_{\alpha_\nu}^{\gamma-1/2} (\gamma^2 + |\xi|^2)^{1/2} u \right\|_{L^2} \geq \frac{\lambda_0}{2} \|u\|_{H^1}.$$

From now on, we forget about the index γ (which is now fixed) in the notation and we simply write T_f instead of T_f^γ , for any symbol f which will enter into play in our computations.

4 Energy estimates

We are now in the position of proving the energy estimates claimed in Theorems 2.1 and 2.2. For this, as already announced, we will follow the strategy of [16], [7] and [6].

To begin with, some preliminaries are in order. Firstly, we approximate the operator L by parilinearisation: given $u \in C^2([0, T]; H^\infty(\mathbb{R}^n))$, we have

$$(12) \quad \partial_t^2 u = \sum_{j,k=1}^n \partial_j (a_{jk}(t, x) \partial_k u) + Lu = \sum_{j,k=1}^n \partial_j (T_{a_{jk}} \partial_k u) + \tilde{L}u,$$

where we have defined

$$\tilde{L}u := Lu + \sum_{j,k=1}^n \partial_j \left((a_{jk} - T_{a_{jk}}) \partial_k u \right).$$

Next, we want to localise the previous equation in frequency, getting in this way an equation for each dyadic block $u_\nu := \Delta_\nu u$ associated to u . Applying the operator Δ_ν to equation (12) yields

$$(13) \quad \partial_t^2 u_\nu = \sum_{j,k=1}^n \partial_j (T_{a_{jk}} \partial_k u_\nu) + \sum_{j,k=1}^n \partial_j \left([\Delta_\nu, T_{a_{jk}}] \partial_k u \right) + \left(\tilde{L}u \right)_\nu,$$

where we have set $\left(\tilde{L}u \right)_\nu = \Delta_\nu \left(\tilde{L}u \right)$ and $[\Delta_\nu, T_{a_{jk}}]$ denotes the commutator between the localisation operator Δ_ν and the paramultiplication operator $T_{a_{jk}}$.

Our first goal is to estimate, by the use of (13), the growth of the localised energy, namely of the energy e_ν associated to each dyadic block u_ν . The total energy \mathcal{E} will be defined only afterwards, as a suitable weighted sum of the e_ν 's. Observe that, owing to the fast oscillations of the coefficients in time, defining e_ν in a classical way, namely as the sum of $\|\partial_t u_\nu\|_{L^2}^2 + \|u_\nu\|_{H^1}^2$, would cause the appearing of bad terms in the energy estimates, which are out of control. The key point here is to resort to the idea of Tarama [29] and introduce a lower order corrector in the definition of e_ν , whose role is exactly to cancel those bad terms. We refer to [7], [8] and [9] for further implementations of this idea in the context of hyperbolic Cauchy problems.

4.1 The localised energy

In order to use Tarama's energy in our context, we start by defining

$$v_\nu(t, x) := T_{\alpha_\nu}^{-1/2} \partial_t u_\nu - T_{\partial_t(\alpha_\nu^{-1/2})} u_\nu,$$

$$w_\nu(t, x) := T_{\alpha_\nu}^{1/2} (\gamma^2 + |\xi|^2)^{1/2} u_\nu.$$

Then, we introduce the approximate energy associated to the ν -th component u_ν as

$$e_\nu(t) := \|v_\nu(t, \cdot)\|_{L^2}^2 + \|w_\nu(t, \cdot)\|_{L^2}^2 + \|u_\nu(t, \cdot)\|_{L^2}^2 .$$

We remark that e_ν is equivalent to the classical energy associated to u_ν . As a matter of fact, we can bound

$$\begin{aligned} \|\partial_t u_\nu(t, \cdot)\|_{L^2} &\leq C \left\| T_{\alpha_\nu^{-1/2}} \partial_t u_\nu(t, \cdot) \right\|_{L^2} \\ &\leq C \left(\|v_\nu(t, \cdot)\|_{L^2} + \left\| T_{\partial_t(\alpha_\nu^{-1/2})} u_\nu(t, \cdot) \right\|_{L^2} \right) \\ &\leq C \left(\|v_\nu(t, \cdot)\|_{L^2} + \varphi_\nu(t) \|u_\nu(t, \cdot)\|_{L^2} \right) \\ &\leq C \left(\|v_\nu(t, \cdot)\|_{L^2} + \varphi_\nu(t) 2^{-\nu} \|w_\nu(t, \cdot)\|_{L^2} \right) . \end{aligned}$$

At this point, noticing that

$$(14) \quad 0 \leq \varphi_\nu(t) 2^{-\nu} \leq C ,$$

for a suitable constant $C > 0$, we deduce that

$$(15) \quad \|\partial_t u_\nu(t, \cdot)\|_{L^2} \leq C (e_\nu(t))^{\frac{1}{2}} .$$

From this inequality, together with (11), we easily deduce that

$$(16) \quad \forall t \in [0, T], \quad e_\nu(t) \sim \|\partial_t u_\nu(t, \cdot)\|_{L^2}^2 + \|u_\nu(t, \cdot)\|_{L^2}^2 + \|\nabla_x u_\nu(t, \cdot)\|_{L^2}^2 ,$$

where the (implicit) multiplicative constants do not depend on $t \in \mathbb{R}_+$, nor on u and $\nu \in \mathbb{N}$.

4.2 Time derivative of the energy components

We now proceed to differentiating e_ν with respect to time and estimating the terms thus produced.

4.2.1 The zero-th order term

To begin with, we compute the time derivative of the zero-th order term appearing in the definition of e_ν . This is an easy task: we have

$$\frac{d}{dt} \|u_\nu(t, \cdot)\|_{L^2}^2 = 2 \operatorname{Re} (u_\nu(t, \cdot), \partial_t u_\nu(t, \cdot))_{L^2} .$$

Consequently, in view of the bound (15), we obtain

$$(17) \quad \frac{d}{dt} \|u_\nu(t, \cdot)\|_{L^2}^2 \leq C e_\nu(t) .$$

The computations of the terms involving v_ν and w_ν are a bit more involved: we now switch our attention to them.

4.2.2 The term involving v_ν

We start by considering the term of e_ν depending on v_ν . Since

$$\partial_t v_\nu = T_{\alpha_\nu^{-1/2}} \partial_t^2 u_\nu - T_{\partial_t^2(\alpha_\nu^{-1/2})} u_\nu ,$$

using also equation (13) we deduce that

$$\begin{aligned}
\frac{d}{dt} \|v_\nu(t, \cdot)\|_{L^2}^2 &= 2 \operatorname{Re} (v_\nu(t, \cdot), \partial_t v_\nu(t, \cdot))_{L^2} \\
&= 2 \operatorname{Re} (v_\nu, T_{\alpha_\nu^{-1/2}} \partial_t^2 u_\nu)_{L^2} - 2 \operatorname{Re} (v_\nu, T_{\partial_t^2(\alpha_\nu^{-1/2})} u_\nu)_{L^2} \\
&= -2 \operatorname{Re} (v_\nu, T_{\partial_t^2(\alpha_\nu^{-1/2})} u_\nu)_{L^2} + 2 \operatorname{Re} \left(v_\nu, T_{\alpha_\nu^{-1/2}} \left(\sum_{j,k} \partial_j (T_{a_{jk}} \partial_k u_\nu) \right) \right)_{L^2} \\
&\quad + 2 \operatorname{Re} \left(v_\nu, \sum_{j,k} T_{\alpha_\nu^{-1/2}} \partial_j ([\Delta_\nu, T_{a_{jk}}] \partial_k u) \right)_{L^2} + 2 \operatorname{Re} (v_\nu, T_{\alpha_\nu^{-1/2}} (\tilde{L}u)_\nu)_{L^2}.
\end{aligned}$$

Observe that

$$\left| \operatorname{Re} (v_\nu, T_{\alpha_\nu^{-1/2}} (\tilde{L}u)_\nu)_{L^2} \right| \leq C (e_\nu(t))^{\frac{1}{2}} \left\| (\tilde{L}u)_\nu \right\|_{L^2}.$$

In addition, whenever $\nu \geq 1$ Bernstein's inequalities (see Lemma 2.1 in [1]) allow us to bound

$$\left| \operatorname{Re} (v_\nu, T_{\partial_t^2(\alpha_\nu^{-1/2})} u_\nu)_{L^2} \right| \leq C \|v_\nu\|_{L^2} (\varphi_\nu(t))^2 \|u_\nu\|_{L^2} \leq C (\varphi_\nu(t))^2 2^{-\nu} e_\nu(t);$$

remark that, the previous inequality is trivially verified also in the case $\nu = 0$.

In view of the previous bounds, we obtain

$$\begin{aligned}
(18) \quad \frac{d}{dt} \|v_\nu(t, \cdot)\|_{L^2}^2 &\leq 2 \operatorname{Re} \left(v_\nu, T_{\alpha_\nu^{-1/2}} \left(\sum_{j,k} \partial_j (T_{a_{jk}} \partial_k u_\nu) \right) \right)_{L^2} \\
&\quad + 2 \operatorname{Re} \left(v_\nu, \sum_{j,k} T_{\alpha_\nu^{-1/2}} \partial_j ([\Delta_\nu, T_{a_{jk}}] \partial_k u) \right)_{L^2} \\
&\quad + C (e_\nu(t))^{\frac{1}{2}} \left\| (\tilde{L}u)_\nu \right\|_{L^2} + C (\varphi_\nu(t))^2 2^{-\nu} e_\nu(t).
\end{aligned}$$

4.2.3 Estimate of the term involving w_ν

We now compute the time derivative of the last term appearing in the definition of e_ν , namely the one depending on w_ν . An easy computation shows that

$$\frac{d}{dt} \|w_\nu(t, \cdot)\|_{L^2}^2 = 2 \operatorname{Re} \left(T_{\partial_t(\alpha_\nu^{1/2})(\gamma^2+|\xi|^2)^{1/2}} u_\nu + T_{\alpha_\nu^{1/2}(\gamma^2+|\xi|^2)^{1/2}} \partial_t u_\nu, w_\nu(t, \cdot) \right)_{L^2}.$$

We claim that

$$(19) \quad \frac{d}{dt} \|w_\nu(t, \cdot)\|_{L^2}^2 = 2 \operatorname{Re} (v_\nu, T_{\alpha_\nu^{-1/2}} T_{\alpha_\nu^2(\gamma^2+|\xi|^2)} u_\nu)_{L^2} + Q,$$

where the remainder term Q satisfies the bound

$$(20) \quad |Q| \leq C (\nu^\ell + 1) e_\nu(t),$$

where $\ell \in \{0, 1\}$ has been introduced in Section 2.

In order to prove our claim, we follow the computations of [7], [8] and freely use the facts proven therein, about the principal symbols of the remainder operators and their orders. We start by writing

$$(21) \quad \begin{aligned} \operatorname{Re} \left(T_{\partial_t(\alpha_\nu^{1/2})(\gamma^2+|\xi|^2)^{1/2}} u_\nu, w_\nu \right)_{L^2} \\ = \operatorname{Re} \left(T_{\alpha_\nu(\gamma^2+|\xi|^2)^{1/2}} T_{-\partial_t(\alpha_\nu^{-1/2})} u_\nu, w_\nu \right)_{L^2} + \operatorname{Re} (R_1 u_\nu, w_\nu)_{L^2} , \end{aligned}$$

where the principal symbol of R_1 is

$$\partial_\xi \left(\alpha_\nu(\gamma^2 + |\xi|^2)^{1/2} \right) \partial_x \partial_t \left(\alpha_\nu^{-1/2} \right)$$

and consequently one has

$$\begin{aligned} |\operatorname{Re} (R_1 u_\nu, w_\nu)_{L^2}| &\leq C \varphi_\nu(t) \left(1 + \nu^\ell\right) \|u_\nu\|_{L^2} \|w_\nu\|_{L^2} \leq C \varphi_\nu(t) \left(1 + \nu^\ell\right) 2^{-\nu} \|w_\nu\|_{L^2}^2 \\ &\leq C \left(\nu^\ell + 1\right) e_\nu(t) , \end{aligned}$$

where we have used also estimate (14). Observe that passing from the first to the second inequality relies on the use of the Bernstein inequality, thus requires $\nu \geq 1$; however, for $\nu = 0$ one can directly pass from the first inequality to the last one.

Next, we can compute

$$(22) \quad \begin{aligned} \operatorname{Re} \left(T_{\alpha_\nu^{1/2}(\gamma^2+|\xi|^2)^{1/2}} \partial_t u_\nu, w_\nu \right)_{L^2} \\ = \operatorname{Re} \left(T_{\alpha_\nu(\gamma^2+|\xi|^2)^{1/2}} T_{\alpha_\nu^{-1/2}} \partial_t u_\nu, w_\nu \right)_{L^2} + \operatorname{Re} (R_2 \partial_t u_\nu, w_\nu)_{L^2} , \end{aligned}$$

where, this time, the principal symbol of the remainder R_2 is

$$\partial_\xi \left(\alpha_\nu(\gamma^2 + |\xi|^2)^{1/2} \right) \partial_x \left(\alpha_\nu^{-1/2} \right) .$$

With the help of (15), we can bound

$$|\operatorname{Re} (R_2 \partial_t u_\nu, w_\nu)_{L^2}| \leq C \left(1 + \nu^\ell\right) \|\partial_t u_\nu\|_{L^2} \|w_\nu\|_{L^2} \leq C \left(\nu^\ell + 1\right) e_\nu(t) .$$

Thus, putting together (21) and (22), we find

$$\frac{d}{dt} \|w_\nu(t, \cdot)\|_{L^2}^2 = 2 \operatorname{Re} \left(T_{\alpha_\nu(\gamma^2+|\xi|^2)^{1/2}} v_\nu, w_\nu \right)_{L^2} + \tilde{Q} ,$$

where the remainder \tilde{Q} satisfies the claimed bound (20). In addition, we can write

$$\operatorname{Re} \left(T_{\alpha_\nu(\gamma^2+|\xi|^2)^{1/2}} v_\nu, w_\nu \right)_{L^2} = \operatorname{Re} \left(v_\nu, T_{\alpha_\nu(\gamma^2+|\xi|^2)^{1/2}} w_\nu \right)_{L^2} + \operatorname{Re} (R_3 v_\nu, w_\nu)_{L^2} ,$$

where the principal symbol of R_3 is given by

$$\partial_\xi \partial_x \left(\alpha_\nu(\gamma^2 + |\xi|^2)^{1/2} \right) .$$

Consequently, keeping in mind the estimates of Subsection 3.3, we see that R_3 is an operator of order 0 in the case $\ell = 0$, of order Log if $\ell = 1$. This means that

$$(23) \quad |\operatorname{Re} (R_3 v_\nu, w_\nu)_{L^2}| \leq C \left(1 + \nu^\ell\right) \|v_\nu\|_{L^2} \|w_\nu\|_{L^2} \leq C \left(\nu^\ell + 1\right) e_\nu(t) .$$

Next, we continue to decompose

$$\operatorname{Re} \left(v_\nu, T_{\alpha_\nu(\gamma^2+|\xi|^2)^{1/2}} w_\nu \right)_{L^2} = \operatorname{Re} \left(v_\nu, T_{\alpha_\nu^{-1/2}} T_{\alpha_\nu^{3/2}(\gamma^2+|\xi|^2)^{1/2}} w_\nu \right)_{L^2} + \operatorname{Re} (R_4 v_\nu, w_\nu)_{L^2} ,$$

where, by using again symbolic calculus, it is plain to see that also the new remainder R_4 is an operator of order Log , in the sense that it satisfies an estimate like (23).

Finally, we can write

$$\operatorname{Re} \left(v_\nu, T_{\alpha_\nu^{-1/2}} T_{\alpha_\nu^{3/2}(\gamma^2+|\xi|^2)^{1/2}} w_\nu \right)_{L^2} = \operatorname{Re} \left(v_\nu, T_{\alpha_\nu^{-1/2}} T_{\alpha_\nu^2(\gamma^2+|\xi|^2)} u_\nu \right)_{L^2} + \operatorname{Re} (R_5 v_\nu, u_\nu)_{L^2} ,$$

where the principal symbol of R_5 is

$$\partial_\xi \left(\alpha_\nu^{3/2} (\gamma^2 + |\xi|^2)^{1/2} \right) \partial_x \left(\alpha_\nu^{-1/2} (\gamma^2 + |\xi|^2)^{1/2} \right) .$$

The previous expression implies that R_5 is an operator of order 1 if $\ell = 0$, of order $1 + \operatorname{Log}$ when $\ell = 1$; using the Bernstein inequalities to absorb the additional power of ν coming out in the estimate (with the usual modification when $\nu = 0$ for dealing with the low frequency term), we can bound

$$\begin{aligned} |\operatorname{Re} (R_5 v_\nu, u_\nu)_{L^2}| &\leq C 2^\nu \left(1 + \nu^\ell \right) \|v_\nu\|_{L^2} \|u_\nu\|_{L^2} \leq C \left(1 + \nu^\ell \right) \|v_\nu\|_{L^2} \|w_\nu\|_{L^2} \\ &\leq C \left(\nu^\ell + 1 \right) e_\nu(t) . \end{aligned}$$

In the end, the claimed relations (19)-(20) follow.

4.2.4 Summing up v_ν and w_ν

At this point, we would like to put formulas (17), (18), (19) and (20) together. For this, we first need to handle the sum

$$\operatorname{Re} \left(v_\nu, T_{\alpha_\nu^{-1/2}} \left(\sum_{j,k} \partial_j (T_{a_{jk}} \partial_k u_\nu) \right) \right)_{L^2} + \operatorname{Re} \left(v_\nu, T_{\alpha_\nu^{-1/2}} T_{\alpha_\nu^2(\gamma^2+|\xi|^2)} u_\nu \right)_{L^2} .$$

To begin with, we remark that we can write

$$(24) \quad \sum_{j,k} \partial_j (T_{a_{jk}} \partial_k u_\nu) = \sum_{j,k} R_{jk} u_\nu - T_{\sum_{j,k} a_{jk} \xi_j \xi_k} u_\nu ,$$

where, recalling the definition (10) of the paraproduct operator, we have set

$$R_{jk} u_\nu = \partial_j (S_{\mu-1} a_{jk}) S_{\mu+2} \partial_k u_\nu + \sum_{h=\mu-3}^{+\infty} \partial_j (S_{h-3} a_{jk}) \Delta_h \partial_k u_\nu .$$

Recall that $\mu = \mu(\gamma)$ is now fixed, owing to our choice of γ . Consequently, as done in Paragraph 4.2.4 of [7], we can bound

$$\|R_{jk} u_\nu\|_{L^2} \leq C \left(\nu^\ell + 1 \right) e_\nu(t) ,$$

for a constant $C > 0$ depending on the chosen γ and on the Lipschitz norms (if $\ell = 0$) or log-Lipschitz norms (when $\ell = 1$) of the a_{jk} 's.

On the other hand, by definition of α_ν , we have

$$T_{\alpha_\nu^2(\gamma^2+|\xi|^2)} u_\nu = T_{\gamma^2+\sum_{j,k} a_{\nu,jk} \xi_j \xi_k} u_\nu .$$

From this expression, we deduce the equality

$$T_{\sum_{j,k} a_{\nu,jk} \xi_j \xi_k} u_\nu - T_{\sum_{j,k} a_{jk} \xi_j \xi_k} u_\nu = T_{\sum_{j,k} (a_{\nu,jk} - a_{jk}) \xi_j \xi_k} u_\nu.$$

Observe that, owing to the properties stated in Subsection 3.1, one has

$$\begin{aligned} a_{\nu,jk} - a_{jk} &= 0 & \text{if } t &\geq 2^{-\nu+1}, \\ |a_{\nu,jk} - a_{jk}| &\leq C & \text{if } t &\leq 2^{-\nu+1}. \end{aligned}$$

Thanks to these estimates, we gather (by Bernsteing inequality when $\nu \geq 1$, by simple computation when $\nu = 0$) the bound

$$\begin{aligned} \left\| T_{\sum_{j,k} (a_{\nu,jk} - a_{jk}) \xi_j \xi_k} u_\nu \right\|_{L^2} &\leq C 2^{2\nu} \chi_{[0, 2^{-\nu+1}]}(t) \|u_\nu\|_{L^2} \\ &\leq C 2^\nu \chi_{[0, 2^{-\nu+1}]}(t) (e_\nu(t))^{1/2}. \end{aligned}$$

Consequently, bearing also (24) in mind, we infer

$$(25) \quad \left| \operatorname{Re} \left(v_\nu, T_{\alpha_\nu^{-1/2}} \left(\sum_{j,k} \partial_j (T_{a_{jk}} \partial_k u_\nu) + T_{\alpha_\nu^2 (\gamma^2 + |\xi|^2)} u_\nu \right) \right) \right|_{L^2} \leq C \left(1 + \nu^\ell + 2^\nu \chi_{[0, 2^{-\nu+1}]}(t) \right) e_\nu(t),$$

where the constant $C > 0$ depends on γ , Λ_0 and on the Lipschitz (for $\ell = 0$) or log-Lipschitz (for $\ell = 1$) norms of the a_{jk} 's.

Summing up, putting estimates (17), (18), (19), (20) and (25) together, we deduce

$$(26) \quad \begin{aligned} \frac{d}{dt} e_\nu(t) &\leq K_1 \left(1 + \nu^\ell + (\varphi_\nu(t))^2 2^{-\nu} + 2^\nu \chi_{[0, 2^{-\nu+1}]}(t) \right) e_\nu(t) \\ &\quad + 2 \operatorname{Re} \left(v_\nu, \sum_{j,k} T_{\alpha_\nu^{-1/2}} \partial_j \left([\Delta_\nu, T_{a_{jk}}] \partial_k u \right) \right)_{L^2} + \tilde{K}_1 (e_\nu(t))^{1/2} \left\| (\tilde{L}u)_\nu \right\|_{L^2}, \end{aligned}$$

for two suitable positive constants K_1 and \tilde{K}_1 , only depending on γ , λ_0 , Λ_0 and on the (Lipschitz or log-Lipschitz) functional norms of the coefficients a_{jk} of L .

4.3 Estimate of the commutator term

In order to find an estimate on the time derivative of the localised energy e_ν , it remains us to exhibit a control for the commutator term appearing in (26) above. However, it turns out that it is better to handle the whole sum over ν of all the commutator terms, instead of considering them one by one at ν fixed.

Here the computation is exactly the same as in [6] if $\ell = 0$, as in [7] in the case $\ell = 1$. Indeed, no derivatives with respect to time are involved and only the (Lipschitz or log-Lipschitz) regularity with respect to x of the coefficients is needed.

Let us consider the case $\ell = 1$ for a while, and take a $\theta \in]0, 1[$. Then, for $\beta > 0$ and $T_0 \in]0, T[$ to be chosen later, we obtain, for all $t \in [0, T_0]$, the inequality

$$(27) \quad \begin{aligned} \sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t} 2^{-2\nu\theta} \left| 2 \operatorname{Re} \left(v_\nu, \sum_{j,k} T_{\alpha_\nu^{-1/2}} \partial_j \left([\Delta_\nu, T_{a_{jk}}] \partial_k u \right) \right) \right|_{L^2} \\ \leq K_2 \sum_{\nu=0}^{+\infty} (\nu+1) e^{-2\beta(\nu+1)t} 2^{-2\nu\theta} e_\nu(t), \end{aligned}$$

where the constant $K_2 > 0$ depends on γ , θ , the product βT_0 and the log-Lipschitz norm of the a_{jk} 's. We refer to Subsection 4.3 of [7] for details.

Consider now the case $\ell = 0$. Take any $\theta \in [0, 1[$. Then, according to estimate (37) in [6], for any $t \in [0, T]$ one has instead

$$(28) \quad \sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} \left| 2 \operatorname{Re} \left(v_\nu, \sum_{j,k} T_{\alpha_\nu^{-1/2}} \partial_j \left([\Delta_\nu, T_{a_{jk}}] \partial_k u \right) \right) \right|_{L^2} \leq K_2 \sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} e_\nu(t),$$

for a new positive constant, that we keep calling K_2 for convenience. Notice that, as above, K_2 only depends on γ , θ and on the Lipschitz norms of the coefficients a_{jk} .

4.4 The total energy: end of the estimates

It is now time to define the total energy \mathcal{E} related to u and close the estimate for such \mathcal{E} .

First of all, let us define the function $f_\nu(t)$, for any $t \in [0, T_0]$ (where we agree that $T_0 \equiv T$ when $\ell = 0$), as

$$f_\nu(t) := \int_0^t \left((\varphi_\nu(\tau))^2 2^{-\nu} + 2^\nu \chi_{[0, 2^{-\nu+1}]}(\tau) \right) d\tau.$$

Observe that, for all $t \in [0, T_0]$, one has

$$(29) \quad 0 \leq f_\nu(t) \leq C,$$

where $C > 0$ does not depend on $\nu \in \mathbb{N}$. Consequently, when $\ell = 1$, from (27) we infer

$$(30) \quad \sum_{\nu=0}^{+\infty} e^{-K_1 f_\nu(t)} e^{-2\beta(\nu+1)t} 2^{-2\nu\theta} \left| 2 \operatorname{Re} \left(v_\nu, \sum_{j,k} T_{\alpha_\nu^{-1/2}} \partial_j \left([\Delta_\nu, T_{a_{jk}}] \partial_k u \right) \right) \right|_{L^2} \\ \leq K_3 \sum_{\nu=0}^{+\infty} (\nu+1) e^{-K_1 f_\nu(t)} e^{-2\beta(\nu+1)t} 2^{-2\nu\theta} e_\nu(t),$$

where K_3 depends on the same quantities as K_2 depends on, namely on γ , θ , the product βT_0 and the log-Lipschitz norm of the a_{jk} 's.

When $\ell = 0$, instead, we use (28) and get

$$(31) \quad \sum_{\nu=0}^{+\infty} e^{-K_1 f_\nu(t)} 2^{-2\nu\theta} \left| 2 \operatorname{Re} \left(v_\nu, \sum_{j,k} T_{\alpha_\nu^{-1/2}} \partial_j \left([\Delta_\nu, T_{a_{jk}}] \partial_k u \right) \right) \right|_{L^2} \leq K_3 \sum_{\nu=0}^{+\infty} 2^{-2\nu\theta} e_\nu(t),$$

where in fact K_3 can be taken exactly equal to the constant K_2 appearing in (28).

We are now in the position of defining the total energy \mathcal{E} and closing the estimates. From now on, we will keep our argument general, without specifying each time whether $\ell = 0$ or $\ell = 1$. However, we will tacitly agree on the following conditions:

- if $\ell = 1$, we take $\theta \in]0, 1[$, $\beta > 0$ and $0 < T_0 \leq T$, where both β and T_0 have to be fixed later;
- if instead $\ell = 0$, we take $\theta \in [0, 1[$, $\beta = 0$ and $T_0 = T$.

This having been pointed out, let us define

$$\mathcal{E}(t) := \sum_{\nu=0}^{\infty} e^{-K_1 f_\nu(t)} e^{-2\beta(\nu+1)t} 2^{-2\nu\theta} e_\nu(t).$$

Then, combining (26) with (30) when $\ell = 1$, or with (31) for $\ell = 0$, we deduce that

$$\mathcal{E}'(t) \leq (K_1 + K_3 - 2\beta) \sum_{\nu=0}^{+\infty} (\nu+1) e^{-K_1 f_\nu(t)} e^{-2\beta(\nu+1)t} 2^{-2\nu\theta} e_\nu(t)$$

$$+ K_4 \sum_{\nu=0}^{+\infty} e^{-K_1 f_\nu(t)} e^{-2\beta(\nu+1)t} 2^{-2\nu\theta} (e_\nu(t))^{\frac{1}{2}} \left\| \left(\tilde{L}u \right)_\nu \right\|_{L^2},$$

for another “universal” constant $K_4 > 0$.

At this point, the argument for bounding the term $\tilde{L}u$ is the same as [7] (for $\ell = 1$) and [6] (for $\ell = 0$), the only difference being the presence of the term $e^{-K_1 f_\nu(t)}$ which however, in view of (29), plays the role of a constant. Thus, for $\ell = 1$ we obtain that, for any $\theta \in]0, 1[$, there exist $\beta^* = \beta(\log 2)^{-1}$, $T_0 \in]0, T]$ and $C > 0$ such that

$$\mathcal{E}'(t) \leq C \left(\mathcal{E}(t) + (\mathcal{E}(t))^{1/2} \|Lu(t, \cdot)\|_{H^{-\theta-\beta^*t}} \right)$$

for all $u \in C^2([0, T_0]; H^\infty(\mathbb{R}^n))$. Once again, if $\ell = 0$, then θ can take the value 0 and the previous estimate holds with $\beta^* = 0$ and $T_0 = T$.

In the end, the claimed estimate follows from the fact that

$$\mathcal{E}(0) \leq C_\theta \left(\|u(0, \cdot)\|_{H^{1-\theta}}^2 + \|\partial_t u(0, \cdot)\|_{H^{-\theta}}^2 \right),$$

$$\mathcal{E}(t) \geq C'_\theta \left(\|u(t, \cdot)\|_{H^{1-\theta-\beta^*t}}^2 + \|\partial_t u(t, \cdot)\|_{H^{-\theta-\beta^*t}}^2 \right).$$

Observe that, in fact, the constant C'_θ also depends on K_1 , hence on the quantities γ , λ_0 , Λ_0 and on the functional norms of the coefficients a_{jk} , namely on the constants $C_0 \dots C_3$ appearing in assumptions (H1) and (H2) of Section 2.

This concludes the proof of Theorems 2.1 and 2.2.

References

- [1] H. Bahouri, J.-Y. Chemin, R. Danchin: “Fourier analysis and nonlinear partial differential equations”. Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), Springer, Heidelberg, 2011.
- [2] J.-M. Bony: *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*. Ann. Sci. École Norm. Sup., **14** (1981), n. 2, 209–246.
- [3] M. Cicognani, F. Colombini: *Sharp regularity of the coefficients in the Cauchy problem for a class of evolution equations*. Differential Integral Equations, **16** (2003), n. 11, 1321–1344.
- [4] M. Cicognani, D. Lorenz: *Strictly hyperbolic equations with coefficients low-regular in time and smooth in space*. J. Pseudo-Differ. Oper. Appl., **9** (2018), n. 3, 643–675.
- [5] F. Colombini, E. De Giorgi, S. Spagnolo: *Sur les équations hyperboliques avec des coefficients qui ne dépendent que du temps*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **6** (1979), n. 3, 511–559.
- [6] F. Colombini, D. Del Santo, F. Fanelli: *No loss of derivatives for hyperbolic operators with Zygmund coefficients in time*. In “Anomalies in partial differential equations”, 127–148, Springer INdAM ser., **43**, Springer, Cham., 2021.
- [7] F. Colombini, D. Del Santo, F. Fanelli, G. Métivier: *Time-dependent loss of derivatives for hyperbolic operators with non regular coefficients*. Comm. Partial Differential Equations, **38** (2013), n. 10, 1791–1817.
- [8] F. Colombini, D. Del Santo, F. Fanelli, G. Métivier: *A well-posedness result for hyperbolic operators with Zygmund coefficients*. J. Math. Pures Appl. (9), **100** (2013), n. 4, 455–475.
- [9] F. Colombini, D. Del Santo, F. Fanelli, G. Métivier: *The well-posedness issue in Sobolev spaces for hyperbolic systems with Zygmund-type coefficients*. Comm. Partial Differential Equations, **40** (2015), n. 11, 2082–2121.
- [10] F. Colombini, D. Del Santo, F. Fanelli, G. Métivier: *On the Cauchy problem for microlocally symmetrizable hyperbolic systems with log-Lipschitz coefficients*. Indiana Univ. Math. J., **69** (2020), n. 3, 785–836.

- [11] F. Colombini, D. Del Santo, T. Kinoshita: *Well-posedness of the Cauchy problem for a hyperbolic equation with non-Lipschitz coefficients*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), **1** (2002), n. 2, 327–358.
- [12] F. Colombini, D. Del Santo, M. Reissig: *On the optimal regularity of coefficients in hyperbolic Cauchy problems*. Bull. Sci. Math., **127** (2003), n. 4, 328–347.
- [13] F. Colombini, E. Jannelli, S. Spagnolo: *Well-posedness in the Gevrey classes of the Cauchy problem for a nonstrictly hyperbolic equation with coefficients depending on time*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **10** (1983), n. 2, 291–312.
- [14] F. Colombini, E. Jannelli, S. Spagnolo: *Nonuniqueness in hyperbolic Cauchy problems*. Ann. of Math. (2), **126** (1987), n. 3, 495–524.
- [15] F. Colombini, K. Kajitani: *The Cauchy problem for nonstrictly second order hyperbolic equations with nonregular coefficients*. J. Anal. Math., **97** (2005), 243–255.
- [16] F. Colombini, N. Lerner: *Hyperbolic operators with non-Lipschitz coefficients*. Duke Math. J., **77** (1995), n. 3, 657–698.
- [17] F. Colombini, G. Métivier: *The Cauchy problem for wave equations with non Lipschitz coefficients; application to continuation of solutions to some nonlinear wave equations*. Ann. Sci. Éc. Norm. Supér. (4), **41** (2008), n. 2, 177–220.
- [18] F. Colombini, T. Nishitani: *On finitely degenerate hyperbolic operators of second order*. Osaka J. Math., **41** (2004), n. 4, 933–947.
- [19] F. Colombini, S. Spagnolo: *Hyperbolic equations with coefficients rapidly oscillating in time: a result of nonstability*. J. Differential Equations, **52** (1984), n. 1, 24–38.
- [20] D. Del Santo, T. Kinoshita, M. Reissig: *Energy estimates for strictly hyperbolic equations with low regularity in coefficients*. Differential Integral Equations, **20** (2007), n. 8, 879–900.
- [21] D. Del Santo, M. Prizzi: *A remark on well-posedness for hyperbolic equations with singular coefficients*. Tsukuba J. Math., **27** (2003), n. 1, 189–198.
- [22] D. Del Santo, M. Prizzi: *Well-posedness for hyperbolic equations whose coefficients lose regularity at one point*. Monatsh. Math., **197** (2022), n. 3, 407–417.
- [23] C. Garetto, M. Ruzhansky: *Hyperbolic second order equations with non-regular time dependent coefficients*. Arch. Ration. Mech. Anal., **217** (2015), n. 1, 113–154.
- [24] G. Métivier: *Interaction de deux chocs pour un système de deux lois de conservation, en dimension deux d’espace*. Trans. Amer. Math. Soc., **296** (1986), n. 2, 431–479.
- [25] G. Métivier: “Para-differential calculus and applications to the Cauchy problem for nonlinear systems”. Centro di Ricerca Matematica Ennio De Giorgi (CRM) Series, **5**, Edizioni della Normale, Pisa, 2008.
- [26] G. Métivier, K. Zumbrun: “Large viscous boundary layers for noncharacteristic nonlinear hyperbolic problems”. Mem. Amer. Math. Soc., **175** (2005), n. 826.
- [27] T. Nishitani: *Sur les équations hyperboliques à coefficients höldériens en t et de classe de Gevrey en x* . Bull. Sci. Math. (2), **107** (1983), n. 2, 113–138.
- [28] M. Reissig: *Hyperbolic equations with non-Lipschitz coefficients. Microlocal analysis and related topics*. Rend. Sem. Mat. Univ. Politec. Torino, **61** (2003), n. 2, 135–182.
- [29] S. Tarama: *Energy estimate for wave equations with coefficients in some Besov type class*. Electron. J. Differential Equations (2007), paper n. 85, 12 pp.
- [30] T. Yamazaki: *On the $L^2(\mathbb{R}^n)$ well-posedness of some singular or degenerate partial differential equations of hyperbolic type*. Comm. Partial Differential Equations, **15** (1990), n. 7, 1029–1078.