



**HAL**  
open science

# Complete solution over $F_p$ of the equation $X^{p^k+1} + X + a = 0$

Kwang Ho Kim, Jong Hyok Choe, Sihem Mesnager

► **To cite this version:**

Kwang Ho Kim, Jong Hyok Choe, Sihem Mesnager. Complete solution over  $F_p$  of the equation  $X^{p^k+1} + X + a = 0$ . *Finite Fields and Their Applications*, 2021, 76, pp.101902. 10.1016/j.ffa.2021.101902 . hal-03960627

**HAL Id: hal-03960627**

**<https://hal.science/hal-03960627v1>**

Submitted on 22 Aug 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

# Complete solution over $\mathbb{F}_{p^n}$ of the equation $$X^{p^k+1} + X + a = 0$$

Kwang Ho Kim<sup>1,2</sup>, Jong Hyok Choe<sup>1</sup>, and Sihem Mesnager<sup>3</sup>

<sup>1</sup> Institute of Mathematics, State Academy of Sciences, Pyongyang, Democratic People's Republic of Korea  
[khk.cryptech@gmail.com](mailto:khk.cryptech@gmail.com)

<sup>2</sup> PGitech Corp., Pyongyang, Democratic People's Republic of Korea

<sup>3</sup> Department of Mathematics, University of Paris VIII, F-93526 Saint-Denis, University Sorbonne Paris Cité, LAGA, UMR 7539, CNRS, 93430 Villetaneuse and Télécom Paris, 91120 Palaiseau, France.  
[smesnager@univ-paris8.fr](mailto:smesnager@univ-paris8.fr)

**Abstract.** Solving equations over finite fields is an important problem from both theoretical and practice points of view. The problem of solving explicitly the equation  $P_a(X) = 0$  over the finite field  $\mathbb{F}_Q$ , where  $P_a(X) := X^{q+1} + X + a$ ,  $Q = p^n$ ,  $q = p^k$ ,  $a \in \mathbb{F}_Q^*$  and  $p$  is a prime, arises in many different contexts including finite geometry, the inverse Galois problem [1], the construction of difference sets with Singer parameters [9], determining cross-correlation between  $m$ -sequences [10] and to construct error correcting codes [5], cryptographic APN functions [6, 7], designs [21], as well as to speed up the index calculus method for computing discrete logarithms on finite fields [11, 12] and on algebraic curves [18].

In fact, the research on this specific problem has a long history of more than a half-century from the year 1967 when Berlekamp, Rumsey and Solomon [2] firstly considered a very particular case with  $k = 1$  and  $p = 2$ .

In this article, we discuss the equation  $P_a(X) = 0$  without any restriction on  $p$  and  $\gcd(n, k)$ . In a very recent paper [15], the authors have left open a problem that could definitely solve this equation. More specifically, for the cases of one or two  $\mathbb{F}_Q$ -zeros, explicit expressions for these rational zeros in terms of  $a$  were provided, but for the case of  $p^{\gcd(n, k)} + 1$   $\mathbb{F}_Q$ -zeros it was remained open to compute explicitly the zeros. This paper solves the remained problem, thus now the equation  $X^{p^k+1} + X + a = 0$  over  $\mathbb{F}_{p^n}$  is completely solved for any prime  $p$ , any integers  $n$  and  $k$ .

**Keywords:** Equation · Finite field · Zeros of a polynomial.

**Mathematics Subject Classification.** 12E05, 12E12, 12E10.

## 1 Introduction

Let  $n$  and  $k$  be any positive integers,  $Q = p^n$  and  $q = p^k$  where  $p$  is a prime. We consider the polynomial

$$P_a(X) := X^{q+1} + X + a, a \in \mathbb{F}_Q^* := \mathbb{F}_Q \setminus \{0\}.$$

Notice the more general polynomial forms  $X^{q+1} + rX^q + sX + t$  with  $s \neq r^q$  and  $t \neq rs$  can be transformed into this form by the substitution  $X = (s-r^q)^{\frac{1}{q}}X_1 - r$ . It is clear that  $P_a(X)$  have no multiple roots.

These polynomials have arisen in several different contexts including finite geometry, the inverse Galois problem [1], the construction of difference sets with Singer parameters [9], determining cross-correlation between  $m$ -sequences [10] and to construct error correcting codes [5], APN functions [6, 7], designs [21]. These polynomials are also exploited to speed up (the relation generation phase in) the index calculus method for computation of discrete logarithms on finite fields [11, 12] and on algebraic curves [18].

Let  $N_a$  denote the number of zeros in  $\mathbb{F}_Q$  of polynomial  $P_a(X)$  and  $M_i$  denote the number of  $a \in \mathbb{F}_Q^*$  such that  $P_a(X)$  has exactly  $i$  zeros in  $\mathbb{F}_Q$ .

### 1.1 Previous works

Berlekamp, Rumsey and Solomon [2] in 1967 and Williams [22] in 1975 firstly considered a very particular case of this problem with  $k = 1$  and  $p = 2$ . In this particular case one has  $P_a(X) = X^3 + X + a$  and so  $N_a \leq 3$ . In 2004, Bluhner [3] proved that  $N_a$  takes either of 0, 1, 2 and  $p^d + 1$  where  $d := \gcd(n, k)$  and computed  $M_i$  for every  $i$ . She also stated some criteria for the number of the  $\mathbb{F}_Q$ -zeros of  $P_a(X)$ . In 2008 and 2010, Helleseth and Kholosha [13, 14] found new criteria for the number of  $\mathbb{F}_{2^n}$ -zeros of  $P_a(X)$ . When there is a unique zero or exactly two zeros, and  $d$  is odd, and they explicitly provided the expressions of these zeros as polynomials of  $a$  [14]. In 2014, Bracken, Tan, and Tan [6] presented a criterion for  $N_a = 0$  in  $\mathbb{F}_{2^n}$  when  $d = 1$  and  $n$  is even. In 2019, Kim and Mesnager [16] completely solved this equation  $X^{2^k+1} + X + a = 0$  over  $\mathbb{F}_{2^n}$  when  $d = 1$ . Very recently (2021), new criteria for which  $P_a(X)$  has 0, 1, 2 or  $p^d + 1$  roots were stated by [15, 19] for any characteristic. In [15], for the cases of one or two  $\mathbb{F}_Q$ -zeros, explicit expressions for these rational zeros in terms of  $a$  are provided. For the case of  $p^d + 1$  rational zeros, [15] provides a parametrization of such  $a$ 's and expresses the  $p^d + 1$  rational zeros by using that parametrization, but it was remained open to explicitly represent the zeros.

We highlight that some of the latest important achievements listed above could not be found without the precious advances made by Bluhner [3, 4].

### 1.2 Main contribution and organization of the paper

Following [15], this paper discuss the equation  $X^{p^k+1} + X + a = 0, a \in \mathbb{F}_{p^n}$ , without any restriction on  $p$  and  $\gcd(n, k)$ . After introducing some prerequisites

from [15] (Sec. 2), we solve the open problem remained in [15] to explicitly represent the  $\mathbb{F}_Q$ -zeros for the case of  $p^{\gcd(n,k)} + 1$  rational zeros (Sec. 3). After all, it is concluded that the equation  $X^{p^k+1} + X + a = 0$  over  $\mathbb{F}_{p^n}$  is completely solved for any prime  $p$ , any integers  $n$  and  $k$ .

## 2 Prerequisites

Throughout this paper, we maintain the following notations.

- $p$  is any prime.
- $n$  and  $k$  are any positive integers.
- $d := \gcd(n, k)$ .
- $m := n/d$ .
- $q := p^k$ .
- $Q := p^n$ .
- $a$  is any non-zero element of the finite field  $\mathbb{F}_Q$ .

Given positive integers  $L$  and  $l$ , define a polynomial

$$T_L^{Ll}(X) := X + X^{p^L} + \cdots + X^{p^{L(l-2)}} + X^{p^{L(l-1)}}.$$

Usually we will abbreviate  $T_1^l(\cdot)$  as  $T_l(\cdot)$ . For  $x \in \mathbb{F}_{p^l}$ ,  $T_l(x)$  is the absolute trace  $\text{Tr}_1^l(x)$  of  $x$ . For  $x \in \mathbb{F}_{p^{kl}}$ , its norm  $\text{Nr}_k^{kl}(x)$  over  $\mathbb{F}_{p^k}$  is defined by

$$\text{Nr}_k^{kl}(x) := x^{1+p^k+\cdots+p^{k(l-2)}+p^{k(l-1)}}.$$

In [15], the sequence of polynomials  $\{A_r(X)\}$  in  $\mathbb{F}_p[X]$  is defined as follows:

$$\begin{aligned} A_1(X) &= 1, A_2(X) = -1, \\ A_{r+2}(X) &= -A_{r+1}^q(X) - X^q A_r^{q^2}(X) \text{ for } r \geq 1. \end{aligned} \tag{1}$$

The following lemma gives another identity that can be used as an alternative definition of  $\{A_r(X)\}$  and an interesting property of this polynomial sequence, which will be importantly applied afterward.

**Lemma 1** ([15]). *For any  $r \geq 1$ , the following statements hold.*

1.

$$A_{r+2}(X) = -A_{r+1}(X) - X^{q^r} A_r(X). \tag{2}$$

2.

$$A_{r+1}^{q+1}(X) - A_r^q(X) A_{r+2}(X) = X^{\frac{q(q^r-1)}{q-1}}. \tag{3}$$

The zero set of  $A_r(X)$  can be completely determined for all  $r$ :

**Proposition 2** ([15]). *For any  $r \geq 3$ ,*

$$\{x \in \overline{\mathbb{F}_p} \mid A_r(x) = 0\} = \left\{ \frac{(u - u^q)^{q^2+1}}{(u - u^{q^2})^{q+1}}, u \in \mathbb{F}_{q^r} \setminus \mathbb{F}_{q^2} \right\}.$$

Further, define the following polynomial

$$G(X) := -A_{m+1}(X) - XA_{m-1}^q(X).$$

It can be shown that if  $A_m(a) \neq 0$  then the  $\mathbb{F}_Q$ -zeros of  $P_a(X)$  satisfy a quadratic equation and therefore necessarily  $N_a \leq 2$ .

**Lemma 3 ([15]).** *Let  $a \in \mathbb{F}_Q^*$ . Assume  $A_m(a) \neq 0$ . If  $P_a(x) = 0$  for  $x \in \mathbb{F}_Q$ , then*

$$A_m(a)x^2 + G(a)x + aA_m^q(a) = 0. \quad (4)$$

By exploiting these definitions and facts, the following results have been got.

### 2.1 $N_a \leq 2$ : Odd $p$

**Theorem 4 ([15]).** *Let  $p$  be odd. Let  $a \in \mathbb{F}_Q$  and  $E = G(a)^2 - 4aA_m^{q+1}(a)$ .*

1.  $N_a = 0$  if and only if  $E$  is not a quadratic residue in  $\mathbb{F}_{p^d}$  (i.e.  $E^{\frac{p^d-1}{2}} \neq 0, 1$ ).
2.  $N_a = 1$  if and only if  $A_m(a) \neq 0$  and  $E = 0$ . In this case, the unique zero in  $\mathbb{F}_Q$  of  $P_a(X)$  is  $-\frac{G(a)}{2A_m(a)}$ .
3.  $N_a = 2$  if and only if  $E$  is a non-zero quadratic residue in  $\mathbb{F}_{p^d}$  (i.e.  $E^{\frac{p^d-1}{2}} = 1$ ). In this case, the two zeros in  $\mathbb{F}_Q$  of  $P_a(X)$  are  $x_{1,2} = \frac{\pm E^{\frac{1}{2}} - G(a)}{2A_m(a)}$ , where  $E^{\frac{1}{2}}$  represents a quadratic root in  $\mathbb{F}_{p^d}$  of  $E$ .

### 2.2 $N_a \leq 2$ : $p = 2$

When  $p = 2$ , in [15] it is proved that  $G(x) \in \mathbb{F}_q$  for any  $x \in \mathbb{F}_{q^m}$  and using it

**Theorem 5 ([15]).** *Let  $p = 2$  and  $a \in \mathbb{F}_Q$ . Let  $H = \text{Tr}_1^d\left(\frac{\text{Nr}_d^q(a)}{G^2(a)}\right)$  and  $E = \frac{aA_m^{q+1}(a)}{G^2(a)}$ .*

1.  $N_a = 0$  if and only if  $G(a) \neq 0$  and  $H \neq 0$ .
2.  $N_a = 1$  if and only if  $A_m(a) \neq 0$  and  $G(a) = 0$ . In this case,  $(aA_m^{q-1}(a))^{\frac{1}{2}}$  is the unique zero in  $\mathbb{F}_Q$  of  $P_a(X)$ .
3.  $N_a = 2$  if and only if  $G(a) \neq 0$  and  $H = 0$ . In this case the two zeros in  $\mathbb{F}_Q$  are  $x_1 = \frac{G(a)}{A_m(a)} \cdot T_n\left(\frac{E}{\zeta+1}\right)$  and  $x_2 = x_1 + \frac{G(a)}{A_m(a)}$ , where  $\zeta \in \mu_{Q+1} := \{z \in \mathbb{F}_{Q^2} \mid z^{Q+1} = 1\} \setminus \{1\}$ .

### 2.3 $N_a = p^d + 1$ : auxiliary results

**Lemma 6 ([15]).** *Let  $a \in \mathbb{F}_Q^*$ . The following are equivalent.*

1.  $N_a = p^d + 1$  i.e.  $P_a(X)$  has exactly  $p^d + 1$  zeros in  $\mathbb{F}_Q$ .
2.  $A_m(a) = 0$ , or equivalently by Proposition 2, there exists  $u \in \mathbb{F}_{q^m} \setminus \mathbb{F}_{q^2}$  such that  $a = \frac{(u-u^q)^{q^2+1}}{(u-u^{q^2})^{q+1}}$ .

3. There exists  $u \in \mathbb{F}_Q \setminus \mathbb{F}_{p^{2d}}$  such that  $a = \frac{(u-u^q)^{q^2+1}}{(u-u^{q^2})^{q+1}}$ . Then the  $p^d + 1$  zeros in  $\mathbb{F}_Q$  of  $P_a(X)$  are  $x_0 = \frac{-1}{1+(u-u^q)^{q-1}}$  and  $x_\alpha = \frac{-(u+\alpha)^{q^2-q}}{1+(u-u^q)^{q-1}}$  for  $\alpha \in \mathbb{F}_{p^d}$ .

**Lemma 7 ([15]).** If  $A_m(a) = 0$ , then for any  $x \in \mathbb{F}_Q$  such that  $x^{q+1} + x + a = 0$ , it holds

$$A_{m+1}(a) = \text{Nr}_k^{km}(x) \in \mathbb{F}_{p^d}.$$

Furthermore, for any  $t \geq 0$

$$A_{m+t}(a) = A_{m+1}(a) \cdot A_t(a). \quad (5)$$

In [15], it is remained an open problem to explicitly compute the  $p^d + 1$  rational zeros.

### 3 Completing the case $N_a = p^d + 1$

Thanks to Lemma 6, throughout this section we assume  $A_m(a) = 0$ . Let

$$L_a(X) := X^{q^2} + X^q + aX \in \mathbb{F}_Q[X].$$

Define the sequence of polynomials  $\{B_r(X)\}$  as follows:

$$B_1(X) = 0, B_{r+1}(X) = -a \cdot A_r^q(X). \quad (6)$$

From Lemma 7 and the definition (1) it follows

$$B_m(a) = -aA_{m-1}^q(a) = A_{m+1}^{\frac{1}{q}}(a) \in \mathbb{F}_{p^d}. \quad (7)$$

Using (5) and induction on  $l$  it is easy to check:

**Proposition 8.**

$$B_{l \cdot m}(a) = B_m^l(a). \quad (8)$$

for any integer  $l \geq 1$ .

The first step to solve the open problem is to deduce the following statement.

**Lemma 9.** For any integer  $r \geq 2$ , in the ring  $\mathbb{F}_Q[X]$  it holds

$$X^{q^r} = \sum_{i=1}^{r-1} \left( A_{r-i}^{q^i}(a) \cdot L_a^{q^{i-1}}(X) \right) + A_r(a) \cdot X^q + B_r(a) \cdot X. \quad (9)$$

*Proof.* The equality (9) for  $r = 2$  is  $X^{q^2} = L_a(X) - X^q - aX$  which is valid by the definition of  $L_a(X)$ . Suppose the equality (9) holds for  $r \geq 2$ . By raising

$q$ -th power to both sides of the equality (9), we get

$$\begin{aligned}
X^{q^{r+1}} &= \sum_{i=1}^{r-1} \left( A_{r-i}^{q^{i+1}}(a) \cdot L_a^{q^i}(X) \right) + A_r^q(a) \cdot X^{q^2} + B_r^q(a) \cdot X^q \\
&= \sum_{i=2}^r \left( A_{r+1-i}^{q^i}(a) \cdot L_a^{q^{i-1}}(X) \right) + A_r^q(a) \cdot X^{q^2} + B_r^q(a) \cdot X^q \\
&= \sum_{i=2}^{(r+1)-1} \left( A_{r+1-i}^{q^i}(a) \cdot L_a^{q^{i-1}}(X) \right) + A_r^q(a) \cdot L_a(X) - A_r^q(a) \cdot X^q \\
&\quad - a \cdot A_r^q(a) \cdot X + B_r^q(a) \cdot X^q \\
&= \sum_{i=1}^{(r+1)-1} \left( A_{r+1-i}^{q^i}(a) \cdot L_a^{q^{i-1}}(X) \right) + A_{r+1}(a) \cdot X^q + B_{r+1}(a) \cdot X,
\end{aligned}$$

where the last equality follows from the definitions (6) and (1). This shows that the equality (9) holds also for  $r + 1$  and so for all  $r \geq 2$ .  $\square$

For  $r = m$ , under the assumption  $A_m(a) = 0$ , Lemma 9 gives

$$X^{q^m} = \sum_{i=1}^{m-1} A_{m-i}^{q^i}(a) \cdot L_a^{q^{i-1}}(X) + B_m(a) \cdot X.$$

Now, we define

$$F_1(X) := X^{q^m} - B_m(a) \cdot X = \sum_{i=1}^{m-1} A_{m-i}^{q^i}(a) \cdot L_a^{q^{i-1}}(X) \in \mathbb{F}_{p^d}[X] \quad (10)$$

and

$$G_1(X) = \sum_{i=1}^{m-1} A_{m-i}^{q^i}(a) \cdot X^{q^{i-1}}. \quad (11)$$

Then, evidently,

$$F_1(X) = G_1 \circ L_a(X). \quad (12)$$

Furthermore, we can show

**Proposition 10.**

$$F_1(X) = L_a \circ G_1(X).$$

*Proof.* When  $m = 3$ ,  $A_3(a) = 0$  is equivalent to  $a = 1$ . Therefore, one has  $F_1(X) = X^{q^3} - X = (X^q - X)^{q^2} + (X^q - X)^q + (X^q - X) = L_a \circ G_1(X)$ .

Now, suppose  $m \geq 4$ . Then, by using Definition (6)

$$\begin{aligned}
L_a \circ G_1(X) &= \\
&\sum_{i=1}^{m-1} A_{m-i}^{q^{i+2}}(a) \cdot X^{q^{i+1}} + \sum_{i=1}^{m-1} A_{m-i}^{q^{i+1}}(a) \cdot X^{q^i} + \sum_{i=1}^{m-1} aA_{m-i}^{q^i}(a) \cdot X^{q^{i-1}} \\
&= \sum_{i=2}^m A_{m+1-i}^{q^{i+1}}(a) \cdot X^{q^i} + \sum_{i=1}^{m-1} A_{m-i}^{q^{i+1}}(a) \cdot X^{q^i} + \sum_{i=0}^{m-2} aA_{m-1-i}^{q^{i+1}}(a) \cdot X^{q^i} \\
&= X^{q^m} - B_m(a) \cdot X = F_1(X),
\end{aligned}$$

where Equality (2) was exploited to deduce the last second equality.  $\square$

By (5), from  $A_m(a) = 0$  it follows  $A_{l \cdot m}(a) = 0$  for any  $l \geq 1$ . Therefore, (8) and (9) for  $r = lm$  yield that for any  $l \geq 1$

$$X^{q^{l \cdot m}} - B_m^l(a) \cdot X = \sum_{i=1}^{l \cdot m - 1} A_{l \cdot m - i}^{q^i}(a) \cdot L_a^{q^{i-1}}(X). \quad (13)$$

**Proposition 11.** *Relation (13) can be rewritten by using  $F_1(X)$  as follows:*

$$X^{q^{l \cdot m}} - B_m^l(a) \cdot X = \sum_{i=0}^{l-1} B_m^{l-1-i}(a) \cdot F_1^{q^{m \cdot i}}(X). \quad (14)$$

*Proof.* If  $l = 1$ , the equality is equivalent to the definition of  $F_1(X)$ . Suppose that it holds for  $l \geq 2$ . By raising  $q^m$ -th power to both sides of (14), we have

$$\begin{aligned}
X^{q^{(l+1)m}} - B_m^l(a) \cdot X^{q^m} &= \sum_{i=0}^{l-1} B_m^{l-1-i}(a) \cdot F_1^{q^{m \cdot (i+1)}}(X) \\
&= \sum_{i=1}^{(l+1)-1} B_m^{(l+1)-1-i}(a) \cdot F_1^{q^{m \cdot i}}(X).
\end{aligned}$$

Since

$$X^{q^{(l+1)m}} - B_m^l(a) \cdot X^{q^m} = X^{q^{(l+1)m}} - B_m^l(a) \cdot F_1(X) - B_m^{l+1}(a) \cdot X,$$

one has

$$\begin{aligned}
X^{q^{(l+1)m}} - B_m^{l+1}(a) \cdot X &= \sum_{i=1}^{(l+1)-1} B_m^{(l+1)-1-i}(a) \cdot F_1^{q^{m \cdot i}}(X) + B_m^l(a) \cdot F_1(X) \\
&= \sum_{i=0}^{(l+1)-1} B_m^{(l+1)-1-i}(a) \cdot F_1^{q^{m \cdot i}}(X)
\end{aligned}$$

This shows that Equality (14) holds for all  $l \geq 1$ .  $\square$



Define

$$N := (p^d - 1) \cdot m,$$

$$G_2(X) := \sum_{i=0}^{p^d-2} B_m^{p^d-2-i}(a) \cdot X^{q^{m \cdot i}}.$$

Since  $F_1(X)$  and  $G_2(X)$  are  $p^d$ -linearized polynomials over  $\mathbb{F}_{p^d}$ , they are commutative under the symbolic multiplication “ $\circ$ ” (see e.g. 115 page in [17]). Therefore, regarding Equation (14) and Proposition 10, one has

$$X^{q^N} - X = G_2 \circ F_1(X) = F_1 \circ G_2(X) = L_a \circ G_1 \circ G_2(X) \quad (15)$$

and consequently

$$\ker(F_1) = G_2(\mathbb{F}_{q^N}), \quad (16)$$

$$\ker(L_a) = G_1 \circ G_2(\mathbb{F}_{q^N}). \quad (17)$$

Since  $L_a(X) = X P_a(X^{q-1})$ , here we can state:

**Proposition 12.** For  $a \in \mathbb{F}_Q^*$ ,

$$\{x \in \overline{\mathbb{F}_p} \mid x^{q+1} + x + a = 0\} = \{x^{q-1} \mid x \in G_1 \circ G_2(\mathbb{F}_{q^N})\} \setminus \{0\}. \quad (18)$$

Our goal now is to determine  $S_a := \{x \in \mathbb{F}_Q \mid P_a(x) = 0\}$ , the set of all  $\mathbb{F}_Q$ -zeros to  $P_a(X) = X^{q+1} + X + a$ ,  $a \in \mathbb{F}_Q$ .

*Remark 13.* In order to find the  $\mathbb{F}_Q$ -zeros of  $P_a(X)$  it is not enough to consider the  $\mathbb{F}_Q$ -zeros of  $L_a(X)$ . In fact, one can see that  $B_m(a) \neq 1$  in general. However, it holds:

**Proposition 14.**  $L_a(X) = 0$  has a solution in  $\mathbb{F}_Q^*$  if and only if  $B_m(a) = 1$ .

*Proof.* If  $L_a(x) = 0$  for  $x \in \mathbb{F}_Q^*$ , then by (12)  $F_1(x) = 0$ . More specifically, we have

$$x^{q^m} - B_m(a) \cdot x = (1 - B_m(a)) \cdot x = 0.$$

Consequently  $B_m(a) = 1$ . Conversely, assume  $B_m(a) = 1$ . Then  $F_1(X) = X^{q^m} - X = L_a \circ G_1(X)$  and  $\ker(L_a) = G_1(\mathbb{F}_{q^m})$ . Assume  $G_1(\mathbb{F}_Q) = \{0\}$ . Then, since  $G_1$  is  $q$ -linearized, it holds  $G_1(\mathbb{F}_{q^m}) = G_1(\mathbb{F}_q(\mathbb{F}_Q)) = \{0\}$  (where  $\mathbb{F}_q(\mathbb{F}_Q)$  denotes, by convention, the smallest field containing both  $\mathbb{F}_q$  and  $\mathbb{F}_Q$ ) which contradicts to  $\deg(G_1) < q^m$ . Thus there exists such a  $x_0 \in \mathbb{F}_Q^*$  that  $G_1(x_0) \neq 0$ . Then  $G_1(x_0) \in \ker(L_a) \cap \mathbb{F}_Q^*$ .

To achieve the goal, we will further need the following lemmas.

**Lemma 15.** Let  $L(X)$  be any  $q$ -linearized polynomial over  $\mathbb{F}_Q$ . If  $x_0^{q-1} \in \mathbb{F}_Q$ , then  $L(x_0)^{q-1} \in \mathbb{F}_Q$ .

*Proof.* If  $x_0^{q-1} \in \mathbb{F}_Q$  i.e.  $x_0^{q-1} = \lambda$  for some  $\lambda \in \mathbb{F}_Q$ , then  $x_0^q = \lambda x_0$  and subsequently  $x_0^{q^i} = \prod_{j=0}^{i-1} \lambda^{q^j} x_0$  for every  $i \geq 1$ . Therefore, when  $L(X)$  is a  $q$ -linearized polynomial over  $\mathbb{F}_Q$ , one can write  $L(x_0) = \bar{\lambda} x_0$  for some  $\bar{\lambda} \in \mathbb{F}_Q$ . Thus,  $L(x_0)^{q-1} = \bar{\lambda}^{q-1} \lambda \in \mathbb{F}_Q$ .  $\square$

**Lemma 16.** Let  $s = \frac{(q^m-1)(p^d-1)}{(Q-1)(q-1)}$ . If  $A_m(a) = 0$  and  $x_0 \in \ker(F_1)$ , then  $x_0^s \in \ker(F_1)$  and  $(x_0^s)^{q-1} \in \mathbb{F}_Q$ .

*Proof.* For  $x_0 = 0$ , the statement is trivial. Therefore, we can assume  $x_0 \neq 0$ . Then,  $x_0 \in \ker(F_1)$  implies

$$B_m(a) = x_0^{q^m-1} = (x_0^s)^{(q-1) \cdot \frac{Q-1}{p^d-1}}. \quad (19)$$

Since  $B_m(a) \in \mathbb{F}_{p^d}$ , therefore  $(x_0^s)^{q-1} \in \mathbb{F}_Q$ .

Now, we will show

$$B_m(a) = B_m(a)^s.$$

Since  $P_a(X)$  has  $p^d + 1$  rational solutions when  $A_m(a) = 0$ , there exists such a non-zero  $x_1$  that

$$L_a(x_1) = 0, x_1^{q-1} \in \mathbb{F}_Q.$$

Then (12) gives  $F_1(x_1) = 0$  i.e.

$$x_1^{q^m-1} = B_m(a),$$

and on the other hand

$$x_1^{q^m-1} = (\text{Nr}_d^n(x_1^{q-1}))^s = (\text{Nr}_k^{km}(x_1^{q-1}))^s = (x_1^{q^m-1})^s = B_m^s(a),$$

where the second equality followed from the fact that  $\text{Nr}_d^n(y) = \text{Nr}_k^{km}(y)$  for any  $y \in \mathbb{F}_Q$ . Thus,  $B_m(a) = B_m^s(a)$ .

Hence,  $(x_0^s)^{q^m-1} = (x_0^{q^m-1})^s = B_m^s(a) = B_m(a)$  i.e.  $F_1(x_0^s) = 0$ .  $\square$

Now, take any  $x_0 \in \ker(F_1)$ . The definition (10) and Lemma 16 shows

$$x_0^s \cdot \mathbb{F}_Q^* := \{x_0^s \cdot \alpha \mid \alpha \in \mathbb{F}_Q^*\} \subset \ker(F_1) = G_2(\mathbb{F}_{p^N})$$

and

$$(x_0^s \cdot \mathbb{F}_Q^*)^{q-1} \subset \mathbb{F}_Q.$$

Subsequently, Lemma 15 and Equality (18) prove

$$G_1(x_0^s \cdot \mathbb{F}_Q^*)^{q-1} \subset S_a.$$

To avoid the trivial zero solution, we need

$$G_1(x_0^s \cdot \mathbb{F}_Q^*) \neq \{0\}.$$

In fact, this is the case. Really, if we assume  $G_1(x_0^s \cdot \mathbb{F}_Q^*) = \{0\}$ , then  $G_1(x_0^s \cdot \mathbb{F}_{q^m}) = \{0\}$  (because  $G_1$  is  $\mathbb{F}_q$ -linear, and  $\mathbb{F}_{q^m}$  is generated by  $\mathbb{F}_q$  and  $\mathbb{F}_Q$ ) which contradicts to  $\deg(G_1) < q^m$ .

Next, in order to explicit all  $p^d + 1$  elements in  $S_a$ , we need to deduce the following lemma.

**Lemma 17.** Let  $A_m(a) = 0$  and  $x_0$  be a  $\mathbb{F}_Q$ -solution to  $P_a(X) = 0$ . Then,  $\frac{x_0^2}{a}$  is a  $(q-1)$ -th power in  $\mathbb{F}_Q$ . For  $\beta \in \mathbb{F}_Q$  with  $\beta^{q-1} = \frac{x_0^2}{a}$ ,

$$w^q - w + \frac{1}{\beta x_0} = 0 \quad (20)$$

has exactly  $p^d$  solutions in  $\mathbb{F}_Q$ . Let  $w_0 \in \mathbb{F}_Q$  be a  $\mathbb{F}_Q$ -solution to Equation (20). Then, the  $p^d + 1$  solutions in  $\mathbb{F}_Q$  to  $P_a(X) = 0$  are  $x_0, (w_0 + \alpha)^{q-1} \cdot x_0$  where  $\alpha$  runs over  $\mathbb{F}_{p^d}$ .

*Proof.* We substitute  $x$  in  $P_a(x)$  with  $x_0 - x$  to get

$$(x_0 - x)^{q+1} + (x_0 - x) + a = 0$$

or

$$x^{q+1} - x_0 x^q - x_0^q x - x + x_0^{q+1} + x_0 + a = 0$$

which implies

$$x^{q+1} - x_0 x^q - (x_0^q + 1)x = 0,$$

or equivalently,

$$x^{q+1} - x_0 x^q + \frac{a}{x_0} x = 0.$$

Since  $x = 0$  corresponds to  $x_0$  being a zero of  $P_a(X)$ , we can divide the latter equation by  $x^{q+1}$  to get

$$\frac{a}{x_0} y^q - x_0 y + 1 = 0 \quad (21)$$

where  $y = \frac{1}{x}$ . Now, let  $y = tw$  where

$$t^{q-1} = \frac{x_0^2}{a}. \quad (22)$$

Then, Equation (21) is equivalent to

$$w^q - w + \frac{1}{tx_0} = 0. \quad (23)$$

If  $t_0$  is a solution to Equation (22), then the set of all  $q-1$  solutions can be represented as  $t_0 \cdot \mathbb{F}_q^*$ . For every  $\lambda \in \mathbb{F}_q^*$ , when  $w_0$  is a solution to Equation (23) for  $t = t_0$ ,  $\lambda w_0$  is a solution to Equation (23) for  $t = t_0/\lambda$ . By the way,  $(t_0, w_0)$  and  $(t_0/\lambda, \lambda w_0)$  give the same  $y_0 = t_0 \cdot w_0 = t_0/\lambda \cdot \lambda w_0$ . Therefore, to find all  $\mathbb{F}_Q$ -solutions to Equation (21) one can consider Equation (23) for any fixed solution  $t_0$  of Equation (22).

Now, we will show that any solution  $t_0$  to Equation (22) lies in  $\mathbb{F}_q \cdot \mathbb{F}_Q := \{\alpha \cdot \beta \mid \alpha \in \mathbb{F}_q, \beta \in \mathbb{F}_Q\}$ . In fact, we know that Equation (23) has  $p^d$  solutions  $w$  with  $y = wt_0 \in \mathbb{F}_Q$ . Let's fix a solution  $w_0$  with  $y_0 = w_0 t_0 \in \mathbb{F}_Q$  of Equation (23). Then, the set of all solutions to Equation (23) can be written as  $w_0 + \mathbb{F}_q$ . Therefore, it follows that there exist  $p^d \geq 2$  elements  $\lambda \in \mathbb{F}_q$  with

$(w_0 + \lambda)t_0 \in \mathbb{F}_Q$ . As  $w_0t_0 \in \mathbb{F}_Q$  and  $(w_0 + \lambda)t_0 \in \mathbb{F}_Q$ , we have  $\lambda t_0 \in \mathbb{F}_Q$  i.e.  $t_0 \in \frac{1}{\lambda}\mathbb{F}_Q \subset \mathbb{F}_q \cdot \mathbb{F}_Q$ .

Hence, we can write  $t_0 = \alpha \cdot \beta$ , where  $\alpha \in \mathbb{F}_q, \beta \in \mathbb{F}_Q$ , and it follows that the set of all solutions to Equation (22) are  $\mathbb{F}_q^* \cdot \beta$ . This means that Equation (22) has  $p^d - 1$  solutions (i.e.  $\mathbb{F}_{p^d}^* \cdot \beta$ ) in  $\mathbb{F}_Q$ , i.e.,  $\frac{x_0^2}{a}$  is a  $(q-1)$ -th power in  $\mathbb{F}_Q$ . Moreover, Equation (20) has exactly  $p^d$  solutions in  $\mathbb{F}_Q$  (because Equation (21) has exactly  $p^d$  solutions  $y = w\beta$  in  $\mathbb{F}_Q$ ). When  $w_0 \in \mathbb{F}_Q$  is such a solution, the set of all  $p^d$  solutions in  $\mathbb{F}_Q$  is  $w_0 + \mathbb{F}_{p^d}$ . Since Equation (23) yields  $y = wt = \frac{1}{(1-w^{q-1})x_0}$ , we have  $x_0 - x = x_0 - \frac{1}{y} = x_0 - (1 - w^{q-1})x_0 = w^{q-1}x_0$ , which completes the proof.  $\square$

Finally, all discussion of this section is summed up in the following theorem.

**Theorem 18.** *Assume  $A_m(a) = 0$ . Let  $N = m(p^d - 1)$  and  $s = \frac{(q^m - 1) \cdot (p^d - 1)}{(Q - 1) \cdot (q - 1)}$ . Define two polynomials  $G_1(X)$  and  $G_2(X)$  as follows*

$$\begin{aligned} - G_1(X) &= \sum_{i=0}^{m-2} A_{m-1-i}^{q^{i+1}}(a) \cdot X^i; \\ - G_2(X) &= \sum_{i=0}^{p^d-2} B_m^{p^d-2-i}(a) \cdot X^{q^{mi}}. \end{aligned}$$

*It holds  $G_1(G_2(\mathbb{F}_{p^N}^*)^s \cdot \mathbb{F}_q^* \cdot \mathbb{F}_Q^*)^{q-1} \neq \{0\}$ . Take a  $x_0 \in G_1(G_2(\mathbb{F}_{p^N}^*)^s \cdot \mathbb{F}_q^* \cdot \mathbb{F}_Q^*)^{q-1} \setminus \{0\}$ .  $\frac{x_0^2}{a}$  is a  $(q-1)$ -th power in  $\mathbb{F}_Q$ . For  $\beta \in \mathbb{F}_Q$  with  $\beta^{q-1} = \frac{x_0^2}{a}$ ,*

$$w^q - w + \frac{1}{\beta x_0} = 0 \tag{24}$$

*has exactly  $p^d$  solutions in  $\mathbb{F}_Q$ . Let  $w_0 \in \mathbb{F}_Q$  be a  $\mathbb{F}_Q$ -solution to Equation (24). Then, the  $p^d + 1$  solutions in  $\mathbb{F}_Q$  of  $P_a(X)$  are  $x_0, (w_0 + \alpha)^{q-1} \cdot x_0$  where  $\alpha$  runs over  $\mathbb{F}_{p^d}$ .*

Note that one can also explicit  $w_0$  by an immediate corollary of Theorem 4 and Theorem 5 in [20].

## 4 Conclusion

In [2, 22, 3, 13, 14, 6, 4, 16, 8, 19, 15], partial results about the zeros of  $P_a(X) = X^{p^k+1} + X + a$  over  $\mathbb{F}_{p^n}$  have been obtained. In this paper, we provided explicit expressions for all possible zeros in  $\mathbb{F}_{p^n}$  of  $P_a(X)$  in terms of  $a$  and thus at last finalize the study initiated by Berlekamp, Rumsey and Solomon before more than a half-century.

## Acknowledgement

The authors deeply thank Professor Dok Nam Lee for his many helpful suggestions and careful checking. Also, the authors deeply thank the Assoc. Edit. and the anonymous reviewers for their valuable comments which have highly improved the quality and the presentation of the paper.

## References

1. S.S. Abhyankar, S.D. Cohen, and M.E. Zieve. Bivariate factorizations connecting Dickson polynomials and Galois theory. *Transactions of the American Mathematical Society*, 352(6), 2871 – 2887, 2000.
2. E.R. Berlekamp, H. Rumsey, G. Solomon. On the solution of algebraic equations over finite fields. *Inf. Control*, 10(6), 553 – 564, 1967.
3. A.W. Blüher. On  $x^{q+1} + ax + b$ . *Finite Fields and Their Applications*, 10(3), pp. 285 – 305, 2004.
4. A.W. Blüher. A New Identity of Dickson Polynomials. *ArXiv:1610.05853 [math.NT]*, 2016.
5. C. Bracken and T. Helleseht. Triple-error-correcting BCH-like codes. in: *IEEE Int. Symp. Inf. Theory*, pp. 1723 – 1725, 2009.
6. C. Bracken, C.H. Tan and Y. Tan. On a class of quadratic polynomials with no zeros and its application to APN functions. *Finite Fields and Their Applications*, 25 : pp. 26 – 36, 2014.
7. L. Budaghyan and C. Carlet. Classes of quadratic APN trinomials and hexanomials and related structures, *IEEE Trans. Inform. Theory*, 54 (5), pp. 2354–2357, 2008.
8. B. Csajbók, G. Marino, O. Polverino, and F. Zullo. A characterization of linearized polynomials with maximum kernel. *Finite Fields and Their Applications*, 56, pp. 109 – 130, 2019.
9. J. Dillon and H. Dobbertin. New cyclic difference sets with singer parameters. *Finite Fields and Their Applications*, 10, pp. 342 – 389, 2004.
10. H. Dobbertin, P. Felke, T. Helleseht and P. Rosenthal. Niho type cross-correlation functions via Dickson polynomials and Kloosterman sums. *IEEE Transactions on Information Theory*, 52(2), pp. 613 – 627, 2006.
11. F. Göloğlu, R. Granger, G. McGuire and J. Zumbärgel. On the function field sieve and the impact of higher splitting probabilities application to discrete logarithms in  $\mathbb{F}_{2^{1971}}$  and  $\mathbb{F}_{2^{3164}}$ . *R. Canetti and J.A. Garay (Eds.): CRYPTO 2013, Part II*, LNCS 8043, pp. 109 – 128, 2013.
12. F. Göloğlu, R. Granger, G. McGuire and J. Zumbärgel. Solving a 6120-bit DLP on a desktop computer. *Cryptology ePrint Archive 2013/306*, 2013.
13. T. Helleseht, and A. Kholosha. On the equation  $x^{2^l+1} + x + a$  over  $GF(2^k)$ . *Finite Fields and Their Applications*, 14(1), pp. 159-176, 2008.
14. T. Helleseht, and A. Kholosha.  $x^{2^l+1} + x + a$  and related affine polynomials over  $GF(2^k)$ . *Cryptogr. Commun.*, 2, pp. 85 – 109, 2010.
15. K.H. Kim, J. Choe and S. Mesnager. Solving  $X^{q+1} + X + a = 0$  over finite fields. *Finite Fields and Their Applications*, 70 : 101797, 2021.
16. K.H. Kim and S. Mesnager. Solving  $x^{2^k+1} + x + a = 0$  in  $\mathbb{F}_{2^n}$  with  $\gcd(n, k) = 1$ . *Finite Fields and Their Applications*, 63 : 101630, 2020.
17. R. Lidl and H. Niederreiter, *Finite Fields*, volume 20 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, second edition, 1997.
18. M. Massierer. Some experiments investigating a possible  $L(1/4)$  algorithm for the discrete logarithm problem in algebraic curves. *Cryptology ePrint Archive 2014/996*, 2014.
19. G. McGuire and J. Sheekey. A characterization of the number of roots of linearized and projective polynomials in the field of coefficients. *Finite Fields and Their Applications*, 57, pp. 68 – 91, 2019.

20. S. Mesnager, K.H. Kim, J. H. Choe and D. N. Lee. Solving some affine equations over finite fields. *Finite Fields and Their Applications*, 68 : 101746 , 2020.
21. C. Tang. Infinite families of 3-designs from APN functions. *Journal of Combinatorial Designs*, 28(2), pp. 97 – 117, 2020.
22. K.S. Williams. Note on cubics over  $\text{GF}(2^n)$  and  $\text{GF}(3^n)$ . *J. Number Theory*, 7(4), pp. 361 – 365, 1975.