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Complete solution over $\mathbb{F}_{p^n}$ of the equation $X^{p^k+1} + X + a = 0$

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Abstract. Solving equations over finite fields is an important problem from both theoretical and practice points of view. The problem of solving explicitly the equation $P_a(X) = 0$ over the finite field $\mathbb{F}_Q$, where $P_a(X) := X^{q+1} + X + a$, $Q = p^n$, $q = p^k$, $a \in \mathbb{F}_Q^*$ and p is a prime, arises in many different contexts including finite geometry, the inverse Galois problem [1], the construction of difference sets with Singer parameters [9], determining cross-correlation between m-sequences [10] and to construct error correcting codes [5], cryptographic APN functions [6, 7], designs [21], as well as to speed up the index calculus method for computing discrete logarithms on finite fields [11, 12] and on algebraic curves [18].

In fact, the research on this specific problem has a long history of more than a half-century from the year 1967 when Berlekamp, Rumsey and Solomon [2] firstly considered a very particular case with $k = 1$ and $p = 2$.

In this article, we discuss the equation $P_a(X) = 0$ without any restriction on p and $\gcd(n, k)$. In a very recent paper [15], the authors have left open a problem that could definitely solve this equation. More specifically, for the cases of one or two $\mathbb{F}_Q$-zeros, explicit expressions for these rational zeros in terms of a were provided, but for the case of $p^{\gcd(n,k)} + 1$ $\mathbb{F}_Q-$zeros it was remained open to compute explicitly the zeros. This paper solves the remained problem, thus now the equation $X^{p^k+1} + X + a = 0$ over $\mathbb{F}_{p^n}$ is completely solved for any prime p, any integers n and k.

Keywords: Equation · Finite field · Zeros of a polynomial.

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1 Introduction

Let n and k be any positive integers, $Q = p^n$ and $q = p^k$ where p is a prime. We consider the polynomial

$$P_a(X) := X^{q+1} + X + a, a \in \mathbb{F}_Q^* := \mathbb{F}_Q \setminus \{0\}.$$

Notice the more general polynomial forms $X^{q+1} + rX^q + sX + t$ with $s \neq r^q$ and $t \neq rs$ can be transformed into this form by the substitution $X = (s-r^q)^{\frac{1}{q}} X_1 - r$. It is clear that $P_a(X)$ have no multiple roots.

These polynomials have arisen in several different contexts including finite geometry, the inverse Galois problem [1], the construction of difference sets with Singer parameters [9], determining cross-correlation between m-sequences [10] and to construct error correcting codes [5], APN functions [6, 7], designs [21]. These polynomials are also exploited to speed up (the relation generation phase in) the index calculus method for computation of discrete logarithms on finite fields [11, 12] and on algebraic curves [18].

Let N_a denote the number of zeros in $\mathbb{F}_Q$ of polynomial $P_a(X)$ and M_i denote the number of $a \in \mathbb{F}_Q^*$ such that $P_a(X)$ has exactly i zeros in $\mathbb{F}_Q$.

1.1 Previous works

Berlekamp, Rumsey and Solomon [2] in 1967 and Williams [22] in 1975 firstly considered a very particular case of this problem with $k = 1$ and $p = 2$. In this particular case one has $P_a(X) = X^3 + X + a$ and so $N_a \leq 3$. In 2004, Bluher [3] proved that N_a takes either of 0, 1, 2 and $p^d + 1$ where $d := \gcd(n, k)$ and computed M_i for every i. She also stated some criteria for the number of the $\mathbb{F}_Q$-zeros of $P_a(X)$. In 2008 and 2010, Helleseth and Kholosha [13, 14] found new criteria for the number of $\mathbb{F}_{2^n}$-zeros of $P_a(X)$. When there is a unique zero or exactly two zeros, and d is odd, and they explicitly provided the expressions of these zeros as polynomials of a [14]. In 2014, Bracken, Tan, and Tan [6] presented a criterion for $N_a = 0$ in $\mathbb{F}_{2^n}$ when $d = 1$ and n is even. In 2019, Kim and Mesnager [16] completely solved this equation $X^{2^k+1} + X + a = 0$ over $\mathbb{F}_{2^n}$ when $d = 1$. Very recently (2021), new criteria for which $P_a(X)$ has 0, 1, 2 or $p^d + 1$ roots were stated by [15, 19] for any characteristic. In [15], for the cases of one or two $\mathbb{F}_Q$-zeros, explicit expressions for these rational zeros in terms of a are provided. For the case of $p^d + 1$ rational zeros, [15] provides a parametrization of such a's and expresses the $p^d + 1$ rational zeros by using that parametrization, but it was remained open to explicitly represent the zeros.

We highlight that some of the latest important achievements listed above could not be found without the precious advances made by Bluher [3, 4].

1.2 Main contribution and organization of the paper

Following [15], this paper discuss the equation $X^{p^k+1} + X + a = 0, a \in \mathbb{F}_{p^n}$, without any restriction on p and $\gcd(n, k)$. After introducing some prerequisites

from [15] (Sec. 2), we solve the open problem remained in [15] to explicitly represent the $\mathbb{F}_Q-$zeros for the case of $p^{\gcd(n,k)}+1$ rational zeros (Sec. 3). After all, it is concluded that the equation $X^{p^k+1}+X+a=0$ over $\mathbb{F}_{p^n}$ is completely solved for any prime p, any integers n and k.

2 Prerequisites

Throughout this paper, we maintain the following notations.

- p is any prime.
- n and k are any positive integers.
- $d := \gcd(n,k)$.
- $m := n/d$.
- $q := p^k$.
- $Q := p^n$.
- a is any non-zero element of the finite field $\mathbb{F}_Q$.

Given positive integers L and l, define a polynomial

$$T_L^{Ll}(X) := X + X^{p^L} + \cdots + X^{p^{L(l-2)}} + X^{p^{L(l-1)}}.$$

Usually we will abbreviate $T_1^l(\cdot)$ as $T_l(\cdot)$. For $x \in \mathbb{F}_{p^l}$, $T_l(x)$ is the absolute trace $\mathrm{Tr}_1^l(x)$ of x. For $x \in \mathbb{F}_{p^{kl}}$, its norm $\mathrm{Nr}_k^{kl}(x)$ over $\mathbb{F}_{p^k}$ is defined by

$$\mathrm{Nr}_k^{kl}(x) := x^{1+p^k+\cdots+p^{k(l-2)}+p^{k(l-1)}}.$$

In [15], the sequence of polynomials $\{A_r(X)\}$ in $\mathbb{F}_p[X]$ is defined as follows:

$$\begin{aligned} &A_1(X) = 1, A_2(X) = -1, \\ &A_{r+2}(X) = -A_{r+1}^q(X) - X^q A_r^{q^2}(X) \text{ for } r \geq 1. \end{aligned} \tag{1}$$

The following lemma gives another identity that can be used as an alternative definition of $\{A_r(X)\}$ and an interesting property of this polynomial sequence, which will be importantly applied afterward.

Lemma 1 ([15]). *For any $r \geq 1$, the following statements hold.*

1.
$$A_{r+2}(X) = -A_{r+1}(X) - X^{q^r} A_r(X). \tag{2}$$

2.
$$A_{r+1}^{q+1}(X) - A_r^q(X) A_{r+2}(X) = X^{\frac{q(q^r-1)}{q-1}}. \tag{3}$$

The zero set of $A_r(X)$ can be completely determined for all r:

Proposition 2 ([15]). *For any $r \geq 3$,*

$$\{x \in \overline{\mathbb{F}_p} \mid A_r(x) = 0\} = \left\{ \frac{(u-u^q)^{q^2+1}}{(u-u^{q^2})^{q+1}}, \quad u \in \mathbb{F}_{q^r} \setminus \mathbb{F}_{q^2} \right\}.$$

Further, define the following polynomial

$$G(X) := -A_{m+1}(X) - XA_{m-1}^q(X).$$

It can be shown that if $A_m(a) \neq 0$ then the $\mathbb{F}_Q$-zeros of $P_a(X)$ satisfy a quadratic equation and therefore necessarily $N_a \leq 2$.

Lemma 3 ([15]). *Let $a \in \mathbb{F}_Q^*$. Assume $A_m(a) \neq 0$. If $P_a(x) = 0$ for $x \in \mathbb{F}_Q$, then*

$$A_m(a)x^2 + G(a)x + aA_m^q(a) = 0. \tag{4}$$

By exploiting these definitions and facts, the following results have been got.

2.1 $N_a \leq 2$: Odd p

Theorem 4 ([15]). *Let p be odd. Let $a \in \mathbb{F}_Q$ and $E = G(a)^2 - 4aA_m^{q+1}(a)$.*

1. *$N_a = 0$ if and only if E is not a quadratic residue in $\mathbb{F}_{p^d}$ (i.e. $E^{\frac{p^d-1}{2}} \neq 0, 1$).*
2. *$N_a = 1$ if and only if $A_m(a) \neq 0$ and $E = 0$. In this case, the unique zero in $\mathbb{F}_Q$ of $P_a(X)$ is $-\frac{G(a)}{2A_m(a)}$.*
3. *$N_a = 2$ if and only if E is a non-zero quadratic residue in $\mathbb{F}_{p^d}$ (i.e. $E^{\frac{p^d-1}{2}} = 1$). In this case, the two zeros in $\mathbb{F}_Q$ of $P_a(X)$ are $x_{1,2} = \frac{\pm E^{\frac{1}{2}} - G(a)}{2A_m(a)}$, where $E^{\frac{1}{2}}$ represents a quadratic root in $\mathbb{F}_{p^d}$ of E.*

2.2 $N_a \leq 2$: $p = 2$

When $p = 2$, in [15] it is proved that $G(x) \in \mathbb{F}_q$ for any $x \in \mathbb{F}_{q^m}$ and using it

Theorem 5 ([15]). *Let $p = 2$ and $a \in \mathbb{F}_Q$. Let $H = \mathrm{Tr}_1^d\left(\frac{\mathrm{Nr}_d^n(a)}{G^2(a)}\right)$ and $E = \frac{aA_m^{q+1}(a)}{G^2(a)}$.*

1. *$N_a = 0$ if and only if $G(a) \neq 0$ and $H \neq 0$.*
2. *$N_a = 1$ if and only if $A_m(a) \neq 0$ and $G(a) = 0$. In this case, $(aA_m^{q-1}(a))^{\frac{1}{2}}$ is the unique zero in $\mathbb{F}_Q$ of $P_a(X)$.*
3. *$N_a = 2$ if and only if $G(a) \neq 0$ and $H = 0$. In this case the two zeros in $\mathbb{F}_Q$ are $x_1 = \frac{G(a)}{A_m(a)} \cdot T_n\left(\frac{E}{\zeta+1}\right)$ and $x_2 = x_1 + \frac{G(a)}{A_m(a)}$, where $\zeta \in \mu_{Q+1} := \{z \in \mathbb{F}_{Q^2} \mid z^{Q+1} = 1\} \setminus \{1\}$.*

2.3 $N_a = p^d + 1$: auxiliary results

Lemma 6 ([15]). *Let $a \in \mathbb{F}_Q^*$. The following are equivalent.*

1. *$N_a = p^d + 1$ i.e. $P_a(X)$ has exactly $p^d + 1$ zeros in $\mathbb{F}_Q$.*
2. *$A_m(a) = 0$, or equivalently by Proposition 2, there exists $u \in \mathbb{F}_{q^m} \setminus \mathbb{F}_{q^2}$ such that $a = \frac{(u-u^q)^{q^2+1}}{(u-u^{q^2})^{q+1}}$.*

3. *There exists* $u \in \mathbb{F}_Q \setminus \mathbb{F}_{p^{2d}}$ *such that* $a = \frac{(u-u^q)^{q^2+1}}{(u-u^{q^2})^{q+1}}$. *Then the* p^d+1 *zeros in* $\mathbb{F}_Q$ *of* $P_a(X)$ *are* $x_0 = \frac{-1}{1+(u-u^q)^{q-1}}$ *and* $x_\alpha = \frac{-(u+\alpha)^{q^2-q}}{1+(u-u^q)^{q-1}}$ *for* $\alpha \in \mathbb{F}_{p^d}$.

Lemma 7 ([15]). *If* $A_m(a) = 0$, *then for any* $x \in \mathbb{F}_Q$ *such that* $x^{q+1}+x+a = 0$, *it holds*

$$A_{m+1}(a) = \mathrm{Nr}_k^{km}(x) \in \mathbb{F}_{p^d}.$$

Furthermore, for any $t \geq 0$

$$A_{m+t}(a) = A_{m+1}(a) \cdot A_t(a). \tag{5}$$

In [15], it is remained an open problem to explicitly compute the p^d+1 rational zeros.

3 Completing the case $N_a = p^d + 1$

Thanks to Lemma 6, throughout this section we assume $A_m(a) = 0$. Let

$$L_a(X) := X^{q^2} + X^q + aX \in \mathbb{F}_Q[X].$$

Define the sequence of polynomials $\{B_r(X)\}$ as follows:

$$B_1(X) = 0, B_{r+1}(X) = -a \cdot A_r^q(X). \tag{6}$$

From Lemma 7 and the definition (1) it follows

$$B_m(a) = -aA_{m-1}^q(a) = A_{m+1}^{\frac{1}{q}}(a) \in \mathbb{F}_{p^d}. \tag{7}$$

Using (5) and induction on l it is easy to check:

Proposition 8.

$$B_{l\cdot m}(a) = B_m^l(a). \tag{8}$$

for any integer $l \geq 1$.

The first step to solve the open problem is to deduce the following statement.

Lemma 9. *For any integer* $r \geq 2$, *in the ring* $\mathbb{F}_Q[X]$ *it holds*

$$X^{q^r} = \sum_{i=1}^{r-1} \left(A_{r-i}^{q^i}(a) \cdot L_a^{q^{i-1}}(X) \right) + A_r(a) \cdot X^q + B_r(a) \cdot X. \tag{9}$$

Proof. The equality (9) for $r = 2$ is $X^{q^2} = L_a(X) - X^q - aX$ which is valid by the definition of $L_a(X)$. Suppose the equality (9) holds for $r \geq 2$. By raising

$q-$th power to both sides of the equality (9), we get

$$\begin{aligned}
X^{q^{r+1}} &= \sum_{i=1}^{r-1} \left(A_{r-i}^{q^{i+1}}(a) \cdot L_a^{q^i}(X) \right) + A_r^q(a) \cdot X^{q^2} + B_r^q(a) \cdot X^q \\
&= \sum_{i=2}^{r} \left(A_{r+1-i}^{q^i}(a) \cdot L_a^{q^{i-1}}(X) \right) + A_r^q(a) \cdot X^{q^2} + B_r^q(a) \cdot X^q \\
&= \sum_{i=2}^{(r+1)-1} \left(A_{r+1-i}^{q^i}(a) \cdot L_a^{q^{i-1}}(X) \right) + A_r^q(a) \cdot L_a(X) - A_r^q(a) \cdot X^q \\
&- a \cdot A_r^q(a) \cdot X + B_r^q(a) \cdot X^q \\
&= \sum_{i=1}^{(r+1)-1} \left(A_{r+1-i}^{q^i}(a) \cdot L_a^{q^{i-1}}(X) \right) + A_{r+1}(a) \cdot X^q + B_{r+1}(a) \cdot X,
\end{aligned}$$

where the last equality follows from the definitions (6) and (1). This shows that the equality (9) holds also for $r+1$ and so for all $r \geq 2$. □

For $r = m$, under the assumption $A_m(a) = 0$, Lemma 9 gives

$$X^{q^m} = \sum_{i=1}^{m-1} A_{m-i}^{q^i}(a) \cdot L_a^{q^{i-1}}(X) + B_m(a) \cdot X.$$

Now, we define

$$F_1(X) := X^{q^m} - B_m(a) \cdot X = \sum_{i=1}^{m-1} A_{m-i}^{q^i}(a) \cdot L_a^{q^{i-1}}(X) \in \mathbb{F}_{p^d}[X] \tag{10}$$

and

$$G_1(X) = \sum_{i=1}^{m-1} A_{m-i}^{q^i}(a) \cdot X^{q^{i-1}}. \tag{11}$$

Then, evidently,

$$F_1(X) = G_1 \circ L_a(X). \tag{12}$$

Furthermore, we can show

Proposition 10.

$$F_1(X) = L_a \circ G_1(X).$$

Proof. When $m = 3$, $A_3(a) = 0$ is equivalent to $a = 1$. Therefore, one has $F_1(X) = X^{q^3} - X = (X^q - X)^{q^2} + (X^q - X)^q + (X^q - X) = L_a \circ G_1(X)$.

Now, suppose $m \geq 4$. Then, by using Definition (6)

$$\begin{aligned}
&L_a \circ G_1(X) = \\
&\sum_{i=1}^{m-1} A_{m-i}^{q^{i+2}}(a) \cdot X^{q^{i+1}} + \sum_{i=1}^{m-1} A_{m-i}^{q^{i+1}}(a) \cdot X^{q^i} + \sum_{i=1}^{m-1} a A_{m-i}^{q^i}(a) \cdot X^{q^{i-1}} \\
&= \sum_{i=2}^{m} A_{m+1-i}^{q^{i+1}}(a) \cdot X^{q^i} + \sum_{i=1}^{m-1} A_{m-i}^{q^{i+1}}(a) \cdot X^{q^i} + \sum_{i=0}^{m-2} a A_{m-1-i}^{q^{i+1}}(a) \cdot X^{q^i} \\
&= X^{q^m} - B_m(a) \cdot X = F_1(X),
\end{aligned}$$

where Equality (2) was exploited to deduce the last second equality. □

By (5), from $A_m(a) = 0$ it follows $A_{l \cdot m}(a) = 0$ for any $l \geq 1$. Therefore, (8) and (9) for $r = lm$ yield that for any $l \geq 1$

$$X^{q^{l \cdot m}} - B_m^l(a) \cdot X = \sum_{i=1}^{l \cdot m - 1} A_{l \cdot m - i}^{q^i}(a) \cdot L_a^{q^{i-1}}(X). \tag{13}$$

Proposition 11. *Relation (13) can be rewritten by using $F_1(X)$ as follows:*

$$X^{q^{l \cdot m}} - B_m^l(a) \cdot X = \sum_{i=0}^{l-1} B_m^{l-1-i}(a) \cdot F_1^{q^{m \cdot i}}(X). \tag{14}$$

Proof. If $l = 1$, the equality is equivalent to the definition of $F_1(X)$. Suppose that it holds for $l \geq 2$. By raising q^m-th power to both sides of (14), we have

$$\begin{aligned}
X^{q^{(l+1)m}} - B_m^l(a) \cdot X^{q^m} &= \sum_{i=0}^{l-1} B_m^{l-1-i}(a) \cdot F_1^{q^{m \cdot (i+1)}}(X) \\
&= \sum_{i=1}^{(l+1)-1} B_m^{(l+1)-1-i}(a) \cdot F_1^{q^{m \cdot i}}(X).
\end{aligned}$$

Since

$$X^{q^{(l+1)m}} - B_m^l(a) \cdot X^{q^m} = X^{q^{(l+1)m}} - B_m^l(a) \cdot F_1(X) - B_m^{l+1}(a) \cdot X,$$

one has

$$\begin{aligned}
X^{q^{(l+1)m}} - B_m^{l+1}(a) \cdot X &= \sum_{i=1}^{(l+1)-1} B_m^{(l+1)-1-i}(a) \cdot F_1^{q^{m \cdot i}}(X) + B_m^l(a) \cdot F_1(X) \\
&= \sum_{i=0}^{(l+1)-1} B_m^{(l+1)-1-i}(a) \cdot F_1^{q^{m \cdot i}}(X)
\end{aligned}$$

This shows that Equality (14) holds for all $l \geq 1$. □

Define

$$N := (p^d - 1) \cdot m,$$
$$G_2(X) := \sum_{i=0}^{p^d-2} B_m^{p^d-2-i}(a) \cdot X^{q^{m \cdot i}}.$$

Since $F_1(X)$ and $G_2(X)$ are p^d-linearized polynomials over $\mathbb{F}_{p^d}$, they are commutative under the symbolic multiplication "$\circ$" (see e.g. 115 page in [17]). Therefore, regarding Equation (14) and Proposition 10, one has

$$X^{q^N} - X = G_2 \circ F_1(X) = F_1 \circ G_2(X) = L_a \circ G_1 \circ G_2(X) \tag{15}$$

and consequently

$$\ker(F_1) = G_2(\mathbb{F}_{q^N}), \tag{16}$$
$$\ker(L_a) = G_1 \circ G_2(\mathbb{F}_{q^N}). \tag{17}$$

Since $L_a(X) = XP_a(X^{q-1})$, here we can state:

Proposition 12. *For $a \in \mathbb{F}_Q^*$,*

$$\{x \in \overline{\mathbb{F}_p} \mid x^{q+1} + x + a = 0\} = \{x^{q-1} \mid x \in G_1 \circ G_2(\mathbb{F}_{q^N})\} \setminus \{0\}. \tag{18}$$

Our goal now is to determine $S_a := \{x \in \mathbb{F}_Q \mid P_a(x) = 0\}$, the set of all $\mathbb{F}_Q-$zeros to $P_a(X) = X^{q+1} + X + a, a \in \mathbb{F}_Q$.

Remark 13. In order to find the $\mathbb{F}_Q-$zeros of $P_a(X)$ it is not enough to consider the $\mathbb{F}_Q-$zeros of $L_a(X)$. In fact, one can see that $B_m(a) \neq 1$ in general. However, it holds:

Proposition 14. *$L_a(X) = 0$ has a solution in $\mathbb{F}_Q^*$ if and only if $B_m(a) = 1$.*

Proof. If $L_a(x) = 0$ for $x \in \mathbb{F}_Q^*$, then by (12) $F_1(x) = 0$. More specifically, we have

$$x^{q^m} - B_m(a) \cdot x = (1 - B_m(a)) \cdot x = 0.$$

Consequently $B_m(a) = 1$. Conversely, assume $B_m(a) = 1$. Then $F_1(X) = X^{q^m} - X = L_a \circ G_1(X)$ and $\ker(L_a) = G_1(\mathbb{F}_{q^m})$. Assume $G_1(\mathbb{F}_Q) = \{0\}$. Then, since G_1 is $q-$linearized, it holds $G_1(\mathbb{F}_{q^m}) = G_1(\mathbb{F}_q(\mathbb{F}_Q)) = \{0\}$ (where $\mathbb{F}_q(\mathbb{F}_Q)$ denotes, by convention, the smallest field containing both $\mathbb{F}_q$ and $\mathbb{F}_Q$) which contradicts to $\deg(G_1) < q^m$. Thus there exists such a $x_0 \in \mathbb{F}_Q^*$ that $G_1(x_0) \neq 0$. Then $G_1(x_0) \in \ker(L_a) \cap \mathbb{F}_Q^*$.

To achieve the goal, we will further need the following lemmas.

Lemma 15. *Let $L(X)$ be any $q-$linearized polynomial over $\mathbb{F}_Q$. If $x_0^{q-1} \in \mathbb{F}_Q$, then $L(x_0)^{q-1} \in \mathbb{F}_Q$.*

Proof. If $x_0^{q-1} \in \mathbb{F}_Q$ i.e. $x_0^{q-1} = \lambda$ for some $\lambda \in \mathbb{F}_Q$, then $x_0^q = \lambda x_0$ and subsequently $x_0^{q^i} = \prod_{j=0}^{i-1} \lambda^{q^j} x_0$ for every $i \geq 1$. Therefore, when $L(X)$ is a $q-$linearized polynomial over $\mathbb{F}_Q$, one can write $L(x_0) = \overline{\lambda} x_0$ for some $\overline{\lambda} \in \mathbb{F}_Q$. Thus, $L(x_0)^{q-1} = \overline{\lambda}^{q-1} \lambda \in \mathbb{F}_Q$. □

Lemma 16. *Let* $s = \frac{(q^m-1)\cdot(p^d-1)}{(Q-1)\cdot(q-1)}$. *If* $A_m(a) = 0$ *and* $x_0 \in \ker(F_1)$, *then* $x_0^s \in \ker(F_1)$ *and* $(x_0^s)^{q-1} \in \mathbb{F}_Q$.

Proof. For $x_0 = 0$, the statement is trivial. Therefore, we can assume $x_0 \neq 0$. Then, $x_0 \in \ker(F_1)$ implies

$$B_m(a) = x_0^{q^m-1} = (x_0^s)^{(q-1)\cdot\frac{Q-1}{p^d-1}}. \tag{19}$$

Since $B_m(a) \in \mathbb{F}_{p^d}$, therefore $(x_0^s)^{q-1} \in \mathbb{F}_Q$.

Now, we will show

$$B_m(a) = B_m(a)^s.$$

Since $P_a(X)$ has $p^d + 1$ rational solutions when $A_m(a) = 0$, there exists such a non-zero x_1 that

$$L_a(x_1) = 0, x_1^{q-1} \in \mathbb{F}_Q.$$

Then (12) gives $F_1(x_1) = 0$ i.e.

$$x_1^{q^m-1} = B_m(a),$$

and on the other hand

$$x_1^{q^m-1} = (\mathrm{Nr}_d^n(x_1^{q-1}))^s = (\mathrm{Nr}_k^{km}(x_1^{q-1}))^s = (x_1^{q^m-1})^s = B_m^s(a),$$

where the second equality followed from the fact that $\mathrm{Nr}_d^n(y) = \mathrm{Nr}_k^{km}(y)$ for any $y \in \mathbb{F}_Q$. Thus, $B_m(a) = B_m^s(a)$.

Hence, $(x_0^s)^{q^m-1} = (x_0^{q^m-1})^s = B_m^s(a) = B_m(a)$ i.e. $F_1(x_0^s) = 0$. □

Now, take any $x_0 \in \ker(F_1)$. The definition (10) and Lemma 16 shows

$$x_0^s \cdot \mathbb{F}_Q^* := \{x_0^s \cdot \alpha \mid \alpha \in \mathbb{F}_Q^*\} \subset \ker(F_1) = G_2(\mathbb{F}_{p^N})$$

and

$$(x_0^s \cdot \mathbb{F}_Q^*)^{q-1} \subset \mathbb{F}_Q.$$

Subsequently, Lemma 15 and Equality (18) prove

$$G_1(x_0^s \cdot \mathbb{F}_Q^*)^{q-1} \subset S_a.$$

To avoid the trivial zero solution, we need

$$G_1(x_0^s \cdot \mathbb{F}_Q^*) \neq \{0\}.$$

In fact, this is the case. Really, if we assume $G_1(x_0^s \cdot \mathbb{F}_Q^*) = \{0\}$, then $G_1(x_0^s \cdot \mathbb{F}_{q^m}) = \{0\}$ (because G_1 is $\mathbb{F}_q-$linear, and $\mathbb{F}_{q^m}$ is generated by $\mathbb{F}_q$ and $\mathbb{F}_Q$) which contradicts to $\deg(G_1) < q^m$.

Next, in order to explicit all $p^d + 1$ elements in S_a, we need to deduce the following lemma.

Lemma 17. *Let $A_m(a) = 0$ and x_0 be a $\mathbb{F}_Q-$solution to $P_a(X) = 0$. Then, $\frac{x_0^2}{a}$ is a $(q-1)-$th power in $\mathbb{F}_Q$. For $\beta \in \mathbb{F}_Q$ with $\beta^{q-1} = \frac{x_0^2}{a}$,*

$$w^q - w + \frac{1}{\beta x_0} = 0 \tag{20}$$

has exactly p^d solutions in $\mathbb{F}_Q$. Let $w_0 \in \mathbb{F}_Q$ be a $\mathbb{F}_Q-$solution to Equation (20). Then, the $p^d + 1$ solutions in $\mathbb{F}_Q$ to $P_a(X) = 0$ are $x_0, (w_0 + \alpha)^{q-1} \cdot x_0$ where α runs over $\mathbb{F}_{p^d}$.

Proof. We substitute x in $P_a(x)$ with $x_0 - x$ to get

$$(x_0 - x)^{q+1} + (x_0 - x) + a = 0$$

or

$$x^{q+1} - x_0 x^q - x_0^q x - x + x_0^{q+1} + x_0 + a = 0$$

which implies

$$x^{q+1} - x_0 x^q - (x_0^q + 1)x = 0,$$

or equivalently,

$$x^{q+1} - x_0 x^q + \frac{a}{x_0} x = 0.$$

Since $x = 0$ corresponds to x_0 being a zero of $P_a(X)$, we can divide the latter equation by x^{q+1} to get

$$\frac{a}{x_0} y^q - x_0 y + 1 = 0 \tag{21}$$

where $y = \frac{1}{x}$. Now, let $y = tw$ where

$$t^{q-1} = \frac{x_0^2}{a}. \tag{22}$$

Then, Equation (21) is equivalent to

$$w^q - w + \frac{1}{t x_0} = 0. \tag{23}$$

If t_0 is a solution to Equation (22), then the set of all $q-1$ solutions can be represented as $t_0 \cdot \mathbb{F}_q^*$. For every $\lambda \in \mathbb{F}_q^*$, when w_0 is a solution to Equation (23) for $t = t_0$, λw_0 is a solution to Equation (23) for $t = t_0/\lambda$. By the way, (t_0, w_0) and $(t_0/\lambda, \lambda w_0)$ give the same $y_0 = t_0 \cdot w_0 = t_0/\lambda \cdot \lambda w_0$. Therefore, to find all $\mathbb{F}_Q-$solutions to Equation (21) one can consider Equation (23) for any fixed solution t_0 of Equation (22).

Now, we will show that any solution t_0 to Equation (22) lies in $\mathbb{F}_q \cdot \mathbb{F}_Q := \{\alpha \cdot \beta \mid \alpha \in \mathbb{F}_q, \beta \in \mathbb{F}_Q\}$. In fact, we know that Equation (23) has p^d solutions w with $y = wt_0 \in \mathbb{F}_Q$. Let's fix a solution w_0 with $y_0 = w_0 t_0 \in \mathbb{F}_Q$ of Equation (23). Then, the set of all solutions to Equation (23) can be written as $w_0 + \mathbb{F}_q$. Therefore, it follows that there exist $p^d \geq 2$ elements $\lambda \in \mathbb{F}_q$ with

$(w_0+\lambda)t_0 \in \mathbb{F}_Q$. As $w_0 t_0 \in \mathbb{F}_Q$ and $(w_0+\lambda)t_0 \in \mathbb{F}_Q$, we have $\lambda t_0 \in \mathbb{F}_Q$ i.e. $t_0 \in \frac{1}{\lambda}\mathbb{F}_Q \subset \mathbb{F}_q \cdot \mathbb{F}_Q$.

Hence, we can write $t_0 = \alpha \cdot \beta$, where $\alpha \in \mathbb{F}_q, \beta \in \mathbb{F}_Q$, and it follows that the set of all solutions to Equation (22) are $\mathbb{F}_q^* \cdot \beta$. This means that Equation (22) has $p^d - 1$ solutions (i.e. $\mathbb{F}_{p^d}^* \cdot \beta$) in $\mathbb{F}_Q$, i.e., $\frac{x_0^2}{a}$ is a $(q-1)-$th power in $\mathbb{F}_Q$. Moreover, Equation (20) has exactly p^d solutions in $\mathbb{F}_Q$ (because Equation (21) has exactly p^d solutions $y = w\beta$ in $\mathbb{F}_Q$). When $w_0 \in \mathbb{F}_Q$ is such a solution, the set of all p^d solutions in $\mathbb{F}_Q$ is $w_0 + \mathbb{F}_{p^d}$. Since Equation (23) yields $y = wt = \frac{1}{(1-w^{q-1})x_0}$, we have $x_0 - x = x_0 - \frac{1}{y} = x_0 - (1-w^{q-1})x_0 = w^{q-1}x_0$, which completes the proof. □

Finally, all discussion of this section is summed up in the following theorem.

Theorem 18. *Assume $A_m(a) = 0$. Let $N = m(p^d-1)$ and $s = \frac{(q^m-1)\cdot(p^d-1)}{(Q-1)\cdot(q-1)}$. Define two polynomials $G_1(X)$ and $G_2(X)$ as follows*

- $G_1(X) = \sum_{i=0}^{m-2} A_{m-1-i}^{q^{i+1}}(a) \cdot X^{q^i}$;
- $G_2(X) = \sum_{i=0}^{p^d-2} B_m^{p^d-2-i}(a) \cdot X^{q^{mi}}$.

It holds $G_1(G_2(\mathbb{F}_{p^N}^*)^s \cdot \mathbb{F}_q^* \cdot \mathbb{F}_Q^*)^{q-1} \neq \{0\}$. Take a $x_0 \in G_1(G_2(\mathbb{F}_{p^N}^*)^s \cdot \mathbb{F}_q^* \cdot \mathbb{F}_Q^*)^{q-1} \setminus \{0\}$. $\frac{x_0^2}{a}$ is a $(q-1)-$th power in $\mathbb{F}_Q$. For $\beta \in \mathbb{F}_Q$ with $\beta^{q-1} = \frac{x_0^2}{a}$,

$$w^q - w + \frac{1}{\beta x_0} = 0 \tag{24}$$

has exactly p^d solutions in $\mathbb{F}_Q$. Let $w_0 \in \mathbb{F}_Q$ be a $\mathbb{F}_Q-$solution to Equation (24). Then, the p^d+1 solutions in $\mathbb{F}_Q$ of $P_a(X)$ are $x_0, (w_0+\alpha)^{q-1} \cdot x_0$ where α runs over $\mathbb{F}_{p^d}$.

Note that one can also explicit w_0 by an immediate corollary of Theorem 4 and Theorem 5 in [20].

4 Conclusion

In [2, 22, 3, 13, 14, 6, 4, 16, 8, 19, 15], partial results about the zeros of $P_a(X) = X^{p^k+1} + X + a$ over $\mathbb{F}_{p^n}$ have been obtained. In this paper, we provided explicit expressions for all possible zeros in $\mathbb{F}_{p^n}$ of $P_a(X)$ in terms of a and thus at last finalize the study initiated by Berlekamp, Rumsey and Solomon before more than a half-century.

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