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RESONANCES AND RESIDUE OPERATORS FOR SYMMETRIC SPACES OF RANK ONE

J. HILGERT (*) AND A. PASQUALE (**)

ABSTRACT. Let $X = G/K$ be a rank-one Riemannian symmetric space of the noncompact type and let $-\Delta$ be the Laplace-Beltrami operator on X . We show that the resolvent operator $R(z)$ of Δ can be meromorphically continued across the spectrum and explicitly determine the poles, i.e. the resonances. Further we describe the residue operators in terms of finite dimensional spherical representations of G . The result answers a question posed by M. Zworski in [13]. The rank of the residue operators is derived from a restricted root version of the Weyl dimension formula for spherical highest weight representations which we prove for arbitrary symmetric spaces of the noncompact type.

Résumé. Soit $X = G/K$ un espace symétrique riemannien de type non-compact et de rang un, et soit $-\Delta$ l'opérateur de Laplace-Beltrami sur X . Nous montrons que la résolvente $R(z)$ de Δ se prolonge méromorphiquement à travers le spectre et nous déterminons explicitement ses pôles, c'est-à-dire les résonances. En outre, nous écrivons les opérateurs résiduels au moyen des représentations sphériques de dimension finie de G . Les résultats de cet article répondent à une question posée par M. Zworski dans [13]. Le rang des opérateurs résiduels est obtenu à partir d'une version de la formule de la dimension de Weyl pour les représentations sphériques qui utilise les racines restreintes ; nous démontrons cette formule pour des espaces symétriques du type non-compact arbitraires.

INTRODUCTION

Let G be a connected noncompact real semisimple Lie group with finite center and let K be a maximal compact subgroup of G . Then the homogeneous space $X = G/K$ is a symmetric space of the noncompact type. As X is complete with respect to its canonical G -invariant Riemannian structure, the Laplace-Beltrami operator L_X of X is a self-adjoint operator on the Hilbert space $L^2(X)$ of square integrable functions on X . We shall consider in the sequel the operator Δ which is the negative of the Laplace-Beltrami operator. In this way, the spectrum of Δ is the half-line $[\langle\rho, \rho\rangle, +\infty[$, where $\langle\rho, \rho\rangle$ is a positive constant depending on the structure of X . We refer to section 1 for the precise definition of $\langle\rho, \rho\rangle$. The resolvent $R(z) = (\Delta - z)^{-1}$ of Δ is thus a holomorphic function on $\mathbb{C} \setminus [\langle\rho, \rho\rangle, +\infty[$ with values in the space of bounded operators on $L^2(X)$. If a meromorphic continuation of $R(z)$ across the spectrum of Δ is possible, then the poles of the meromorphically extended resolvent are called the resonances.

The problem of the meromorphic continuation of the resolvent $R(z)$ has been studied in many frameworks, such as the asymptotically hyperbolic manifolds (see for instance [9], [4], [3] and references therein) and the locally symmetric spaces of rank one (see for instance [11] and [1]). For an arbitrary symmetric space of the noncompact type G/K , Mazzeo and Vasy [10] and Strohmaier [12] have determined a domain where the analytic continuation of the resolvent is possible. This domain

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is related to the singularities of the Plancherel measure occurring in the spectral decomposition of $L^2(G/K)$.

More detailed information on the nature of the resonances is so far available only in very specific cases. In [4], Guillopé and Zworski have studied the resonances for the real hyperbolic space $\mathbb{H}^n = \mathrm{SO}(n, 1)/\mathrm{SO}(n)$. They proved that there are no resonances for n odd; for n even, there are resonances (which are explicitly determined), and the corresponding residue operators are shown to have finite rank. A representation theoretical approach to the study of the resolvent residue operator was presented for the upper half-plane $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$ in [13] by Maciej Zworski. He proved that the resonant states continue analytically to the spherical harmonics, that is to the eigenfunctions of the Laplace operator of the dual compact symmetric space. This shows, in particular, that the rank of the residues operators is finite and related to the dimension of the finite dimensional spherical representations. In [13], the author asked for the general pattern for symmetric spaces. This paper provides the answer for the symmetric spaces G/K of rank one: as in the case of $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$, we prove an explicit formula for the resolvent residue operators. This allows us to show that each resolvent residue operator maps onto the space of a finite dimensional spherical representation of G and it is therefore of finite rank. See Theorem 3.8. Observe that, by complexification and restriction, finite dimensional spherical representations of G can be seen as spherical representations of the compact simply connected semisimple Lie group U so that U/K is the dual compact symmetric space of G/K . Our main tool in proving Theorem 3.8 is the Fourier-Helgason transform on the symmetric space G/K together with a detailed analysis of the singularities of Plancherel measure for $L^2(G/K)$ and of the joint eigenspaces of the G -invariant differential operators on G/K .

Even if the main result of this paper is in rank one, we keep the higher rank notation to underline some complementary results which we expect to be interesting on their own. First of all, because of additional cancellations, the list of singularities of the Plancherel measure is shorter than those given in [10] or in [12]. Consequently, the resolvent of the Laplace-Beltrami operator of general symmetric spaces extends analytically to a larger domain. See Corollary 2.2 and Remark 2.3. Secondly, we prove a formula allowing us to compute the dimension of the finite dimensional spherical representations using the restricted root systems. One can therefore use directly the parametrization of these representations provided by the Cartan-Helgason theorem, without having to reconstruct the root systems and highest weights as required by Weyl's dimension formula. The resulting dimension formula, given in Proposition 3.5, does not seem to appear in the literature. Thirdly, the description of the joint eigenspaces in Theorem 3.2 seems to be new. Finally, we expect that the singularities of the Plancherel measure and the joint eigenspaces will play a role in the description of the singularities of the analytic continuation of the resolvent in the higher rank situation as well. We hope to come back to this point in some future work.

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Notation. We shall use the standard notation \mathbb{Z} , \mathbb{Z}^+ , \mathbb{R} and \mathbb{C} respectively for the integers, the nonnegative integers, the real and complex numbers. If X is a manifold, then $\mathcal{E}(X)$ and $\mathcal{D}(X)$ denote respectively the spaces of C^∞ and compactly supported C^∞ complex valued functions on X with their usual topologies. Their duals $\mathcal{E}'(X)$ and $\mathcal{D}'(X)$ consist respectively of the compactly supported distributions and the distributions on X .

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1. PRELIMINAIRES

1.1. Symmetric spaces of noncompact type and their structure. Let X be a symmetric space of noncompact type. Hence $X = G/K$ where G is a connected noncompact real semisimple Lie group with finite center, and K is a maximal compact subgroup of G . We denote the origin $\{K\}$ of X by o . Let \mathfrak{g} and \mathfrak{k} ($\subset \mathfrak{g}$) be respectively the Lie algebras of G and K , and let \mathfrak{p} be the orthogonal complement of \mathfrak{k} in \mathfrak{g} with respect to the Cartan-Killing form B of \mathfrak{g} . Hence $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The dimension of any maximal abelian subspace of \mathfrak{p} is a constant, called the (real) rank of G . We fix a maximal abelian subspace \mathfrak{a} of \mathfrak{p} . Let \mathfrak{a}^* be the (real) dual space of \mathfrak{a} and let $\mathfrak{a}_{\mathbb{C}}^*$ be its complexification.

The set of (restricted) roots of the pair $(\mathfrak{g}, \mathfrak{a})$ is indicated by Σ . It consists of all $\alpha \in \mathfrak{a}^*$ for which the vector space $\mathfrak{g}_{\alpha} := \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for every } H \in \mathfrak{a}\}$ contains nonzero elements. The dimension m_{α} of \mathfrak{g}_{α} is called the multiplicity of the root α . The subset of \mathfrak{a} on which one of the roots vanishes is a finite union of hyperplanes. We can therefore choose $Y \in \mathfrak{a}$ so that $\alpha(Y) \neq 0$ for all $\alpha \in \Sigma$. The set Σ^+ of the $\alpha \in \Sigma$ with $\alpha(Y) > 0$ is a system of positive roots, and Σ is the disjoint union of Σ^+ and $-\Sigma^+$. Moreover, $\mathfrak{a}^+ := \{H \in \mathfrak{a} : \alpha(H) > 0 \text{ for all } \alpha \in \Sigma^+\}$ is an open polyhedral cone called the positive Weyl chamber. The half-sum of the positive roots counted with multiplicities is denoted ρ : hence

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_{\alpha} \alpha. \tag{1}$$

A root $\alpha \in \Sigma$ is said to be indivisible if $\alpha/2 \notin \Sigma$. We denote by Σ_0 and $\Sigma_0^+ := \Sigma^+ \cap \Sigma_0$ the sets of indivisible and positive indivisible roots, respectively.

The Cartan-Killing form B is positive definite on $\mathfrak{p} \times \mathfrak{p}$, so $\langle X, Y \rangle := B(X, Y)$ defines a Euclidean structure in \mathfrak{p} and in $\mathfrak{a} \subset \mathfrak{p}$. We extend this inner product to \mathfrak{a}^* by duality, that is we set $\langle \lambda, \mu \rangle := \langle H_{\lambda}, H_{\mu} \rangle$ if H_{γ} is the unique element in \mathfrak{a} such that $\langle H_{\gamma}, H \rangle = \gamma(H)$ for all $H \in \mathfrak{a}$. The \mathbb{C} -bilinear extension of $\langle \cdot, \cdot \rangle$ to $\mathfrak{a}_{\mathbb{C}}^*$ will be denoted by the same symbol. For $\alpha \in \Sigma$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we shall employ the notation

$$\lambda_{\alpha} := \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}. \tag{2}$$

The Weyl group W of the pair $(\mathfrak{g}, \mathfrak{k})$ is the finite group of orthogonal transformations of \mathfrak{a} generated by the reflections r_{α} with $\alpha \in \Sigma$, where

$$r_{\alpha}(H) := H - 2 \frac{\alpha(H)}{\langle \alpha, \alpha \rangle} H_{\alpha}, \quad H \in \mathfrak{a}.$$

The Weyl group action extends to \mathfrak{a}^* by duality, to A via the exponential map, and to $\mathfrak{a}_{\mathbb{C}}$ and $\mathfrak{a}_{\mathbb{C}}^*$ by complex linearity.

The restriction of the exponential map of G to \mathfrak{a} is an analytic diffeomorphism onto the abelian subgroup $A := \exp \mathfrak{a}$. The inverse diffeomorphism is denoted by \log .

Set $\mathfrak{n} := \sum_{\alpha \in \Sigma^+} \mathfrak{g}_{\alpha}$. Then $N := \exp \mathfrak{n}$ is a simply connected nilpotent subgroup of G . The map $(k, a, n) \mapsto kan$ is an analytic diffeomorphism of the product manifold $K \times A \times N$ onto G , and the resulting decomposition $G = KAN$ is called the Iwasawa decomposition of G . Thus, for $g \in G$ we have $g \in K \exp H(g)N$ with a uniquely determined $H(g) \in \mathfrak{a}$. Let M be the centralizer of A in K and set $B = K/M$. Then $A : X \times B \rightarrow \mathfrak{a}$ is the map defined by $A(gK, kM) := -H(g^{-1}k)$. We refer to [8], Chapter II, §3, No. 4 for the geometric interpretation of this map.

1.2. Invariant differential operators. Let $\mathbb{D}(X)$ denote the algebra of differential operators on X which are invariant under the action of G by left translations. Then $\mathbb{D}(X)$ is a commutative algebra which contains the Laplace-Beltrami operator $L_X := -\Delta$ of X . See for instance [7], Ch. II, §4, for the definition of L_X . Moreover, let $\mathbb{D}_W(A)$ be the algebra of differential operators on A with constant coefficients and invariant under the action of W . It can be identified with the algebra of W -invariant polynomial functions on $\mathfrak{a}_{\mathbb{C}}^*$. Then there is a surjective isomorphism $\Gamma : \mathbb{D}(X) \rightarrow \mathbb{D}_W(A)$. See [7], Theorem 5.18, p.306.

Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $b \in B$. Then, according to Proposition 3.4, p. 118, of [8], the function $e_{\lambda, b} : X \rightarrow \mathbb{C}$ defined by $e_{\lambda, b}(x) := e^{(i\lambda + \rho)(A(x, b))}$ is a joint eigenfunction of $\mathbb{D}(X)$. In fact, we have

$$De_{\lambda, b} = \Gamma(D)(i\lambda)e_{\lambda, b}. \quad (3)$$

In particular, $\Gamma(\Delta)(i\lambda) = \langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle$, see [7], Ch. II, Corollary 5.20. Hence

$$\Delta e_{\lambda, b} = (\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle)e_{\lambda, b} \quad (4)$$

The spherical function φ_{λ} of spectral parameter $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is given by the formula

$$\varphi_{\lambda}(x) = \int_B e_{\lambda, b}(x) db, \quad x \in X. \quad (5)$$

It is an entire function of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and satisfies $\varphi_{w\lambda} = \varphi_{\lambda}$ for $w \in W$. By (3) we have

$$D\varphi_{\lambda} = \Gamma(D)(i\lambda)\varphi_{\lambda} \quad (6)$$

for all $D \in \mathbb{D}(X)$.

1.3. The Helgason-Fourier transform. The Helgason-Fourier transform of a (sufficiently regular) function f on X is the function $\mathcal{F}f$ defined by

$$\mathcal{F}f(\lambda, b) = \int_X f(x)e_{-\lambda, b}(x) dx \quad (7)$$

for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $b \in B$ for which this integral exists. The Harish-Chandra c -function is the meromorphic function on $\mathfrak{a}_{\mathbb{C}}^*$ given explicitly by the Gindikin-Karpelevich product formula:

$$c(\lambda) = c_0 \prod_{\alpha \in \Sigma_0^+} c_{\alpha}(\lambda) \quad (8)$$

where

$$c_{\alpha}(\lambda) = \frac{2^{-i\lambda_{\alpha}} \Gamma(i\lambda_{\alpha})}{\Gamma\left(\frac{i\lambda_{\alpha}}{2} + \frac{m_{\alpha}}{4} + \frac{1}{2}\right) \Gamma\left(\frac{i\lambda_{\alpha}}{2} + \frac{m_{\alpha}}{4} + \frac{m_{2\alpha}}{2}\right)} \quad (9)$$

and the constant c_0 is given by the condition $c(-i\rho) = 1$. Here and in the following we adopt the usual convention that $m_{2\alpha} = 0$ if 2α is not a root.

Set $\mathfrak{a}_+^* := \{\lambda \in \mathfrak{a}^* : \langle \alpha, \lambda \rangle > 0 \text{ for all } \alpha \in \Sigma^+\}$. We consider the space $L^2(X)$ of square-integrable functions on X with respect to the (suitably normalized) G -invariant measure on X , and the space

$L^2(\mathfrak{a}_+^* \times B)$ of square integrable functions with respect to the measure $d\lambda db/|c(\lambda)|^2$. The Plancherel theorem ([8], Theorem 1.5, p. 227) states that the Helgason-Fourier transform \mathcal{F} extends to an isometry of $L^2(X)$ onto $L^2(\mathfrak{a}_+^* \times B)$: for all $f, h \in L^2(X)$ we have

$$\langle f, h \rangle := \int_X f(x) \overline{h(x)} dx = \frac{1}{|W|} \int_{\mathfrak{a}^* \times B} \mathcal{F}f(\lambda, b) \overline{\mathcal{F}h(\lambda, b)} \frac{d\lambda db}{|c(\lambda)|^2} \quad (10)$$

where $|W|$ denotes the cardinality of the Weyl group W .

A C^∞ function $F : \mathfrak{a}_\mathbb{C}^* \times B \rightarrow \mathbb{C}$ is said to be holomorphic of uniform exponential type if it is holomorphic as a function of $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and if there exists a constant $R \geq 0$ so that for each $N \in \mathbb{Z}^+$

$$\sup_{\lambda \in \mathfrak{a}_\mathbb{C}^*, b \in B} e^{-R|\operatorname{Im} \lambda|} (1 + |\lambda|)^N |F(\lambda, b)| < \infty. \quad (11)$$

We denote by $\mathcal{H}(\mathfrak{a}_\mathbb{C}^* \times B)$ the space of holomorphic function of uniform exponential type and by $\mathcal{H}(\mathfrak{a}_\mathbb{C}^* \times B)_W$ the subspace consisting of the functions F satisfying the additional condition

$$\int_B e_{w\lambda, b}(x) F(w\lambda, b) db = \int_B e_{\lambda, b}(x) F(\lambda, b) db \quad (12)$$

for all $w \in W$ and $x \in X$. The Paley-Wiener theorem (see [8], Theorem 5.1, p. 270) states that the Helgason-Fourier transform \mathcal{F} is a bijection of the space $\mathcal{D}(X)$ of compactly supported C^∞ functions on X onto the space $\mathcal{H}(\mathfrak{a}_\mathbb{C}^* \times B)_W$. For $F = \mathcal{F}f$, $f \in \mathcal{D}(X)$, the constant R appearing in the estimate (11) is related to the size of the support of f .

1.4. The resolvent of Δ . We keep the notation of the previous subsections. Recall from (4) that the functions $e_{\lambda, b}$ appearing in the definition of the Helgason-Fourier transform \mathcal{F} are joint eigenfunctions of the Laplace operator $-\Delta$. Hence, by the Plancherel theorem (10), \mathcal{F} gives a unitary equivalence of Δ with the multiplication operator M on $L^2(\mathfrak{a}^* \times B, \frac{d\lambda db}{|c(\lambda)|^2})$ given by $Mf(\lambda, b) := (\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle) f(\lambda, b)$. It follows in particular that the spectrum of Δ is the half-line $[\langle \rho, \rho \rangle, +\infty[$. Moreover, for all $z \in \mathbb{C} \setminus [\langle \rho, \rho \rangle, +\infty[$ and all $f, h \in L^2(X)$ we have for the resolvent $R(z) = (\Delta - z)^{-1}$:

$$\begin{aligned} \langle R(z)f, h \rangle &= \langle (\mathcal{F}(\Delta - z)^{-1} \mathcal{F}^{-1}) \mathcal{F}f, \mathcal{F}h \rangle_{L^2(\mathfrak{a}^* \times B)} \\ &= \langle (M - z)^{-1} \mathcal{F}f, \mathcal{F}h \rangle_{L^2(\mathfrak{a}^* \times B)} \\ &= \frac{1}{|W|} \int_{\mathfrak{a}^* \times B} (\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle - z)^{-1} \mathcal{F}f(\lambda, b) \overline{\mathcal{F}h(\lambda, b)} \frac{d\lambda db}{|c(\lambda)|^2} \end{aligned}$$

We shall study the meromorphic continuation of R across the spectrum of Δ under the assumption that $f, h \in \mathcal{D}(X)$, that is by considering R as $\mathcal{D}'(X \times X)$ valued function.

When $f, h \in \mathcal{D}(X)$, the Paley-Wiener theorem (see subsection 1.3) allows us to apply Fubini's theorem to the formula for the resolvent:

$$\begin{aligned} \langle R(z)f, h \rangle &= \frac{1}{|W|} \int_{\mathfrak{a}^*} (\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle - z)^{-1} \left[\int_B \mathcal{F}f(\lambda, b) \left(\int_X \overline{h(y)} e_{\lambda, b}(y) dy \right) db \right] \frac{d\lambda}{|c(\lambda)|^2} \\ &= \frac{1}{|W|} \int_X \left[\int_{\mathfrak{a}^*} (\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle - z)^{-1} \left(\int_B \mathcal{F}f(\lambda, b) e_{\lambda, b}(y) db \right) \frac{d\lambda}{|c(\lambda)|^2} \right] \overline{h(y)} dy. \end{aligned}$$

Let f_1, f_2 be sufficiently regular functions on X . The convolution $f_1 \times f_2$ is the function on X defined by $(f_1 \times f_2) \circ \pi = (f_1 \circ \pi) * (f_2 \circ \pi)$. Here $\pi : G \rightarrow X = G/K$ is the natural projection and $*$ denotes the convolution product of functions on G . Then, for $f \in \mathcal{D}(X)$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*$ we have by [8], Lemma 1.2, p. 225:

$$(f \times \varphi_\lambda)(y) = \int_B \mathcal{F}f(\lambda, b) e_{\lambda, b}(y) db, \quad y \in X \quad (13)$$

where φ_λ is the spherical function on X of parameter λ given in (5). Therefore for $z \in \mathbb{C} \setminus [\langle \rho, \rho \rangle, +\infty[$

$$[R(z)f](y) = \frac{1}{|W|} \int_{\mathfrak{a}^*} (\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle - z)^{-1} (f \times \varphi_\lambda)(y) \frac{d\lambda}{|c(\lambda)|^2} \quad (14)$$

as an element of $\mathcal{D}'(X)$. Observe that the Paley-Wiener theorem and the smoothness of Iwasawa projection map A ensure that $R(z)f$ is a C^∞ function on X . Again by the Paley-Wiener theorem, for every $f \in \mathcal{D}(X)$ there is a constant $R \geq 0$ (depending on the size of the support of f), so that for every $N \in \mathbb{Z}^+$ and every compact subset D of X we have

$$\sup_{y \in D, \lambda \in \mathfrak{a}_\mathbb{C}^*} e^{-R|\operatorname{Im} \lambda|} (1 + |\lambda|)^N |(f \times \varphi_\lambda)(y)| \leq \infty. \quad (15)$$

Note also that, as $|c(\lambda)|^2 = c(\lambda)c(-\lambda)$, the integrand of (14) is a meromorphic function of $\lambda \in \mathfrak{a}_\mathbb{C}^*$; it is holomorphic in the region where the functions $\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle - z$ and $c(\lambda)c(-\lambda)$ do not vanish.

Let us introduce the variable $\zeta := \sqrt{z - \langle \rho, \rho \rangle}$, where $\sqrt{\cdot}$ denotes the single-valued holomorphic branch of the square root function determined on $\mathbb{C} \setminus [0, +\infty[$ by the condition $\sqrt{-1} = -i$. With respect to this new variable, we have

$$[R(\zeta)f](y) = \frac{1}{|W|} \int_{\mathfrak{a}^*} (\langle \lambda, \lambda \rangle - \zeta^2)^{-1} (f \times \varphi_\lambda)(y) \frac{d\lambda}{c(\lambda)c(-\lambda)}. \quad (16)$$

By Morera's theorem, $R(\zeta)f$ is a holomorphic $\mathcal{D}'(X)$ -valued functions in the regions $\operatorname{Im} \zeta > 0$ and $\operatorname{Im} \zeta < 0$.

2. MEROMORPHIC CONTINUATION OF THE RESOLVENT OF Δ

2.1. The singularities of the Plancherel density. In this subsection we list the singularities of the Plancherel density

$$\frac{1}{c(\lambda)c(-\lambda)} = \frac{1}{c_0^2} \prod_{\alpha \in \Sigma_0^+} \frac{1}{c_\alpha(\lambda)c_\alpha(-\lambda)} \quad (17)$$

and of its modified version

$$\frac{1}{\Pi(\lambda)c(\lambda)c(-\lambda)} \quad (18)$$

where

$$\Pi(\lambda) := \prod_{\alpha \in \Sigma_0^+} \lambda_\alpha \quad (19)$$

We recall that if both α and 2α are roots, then m_α is even and $m_{2\alpha}$ is odd (see e.g. [6], Ch. X, Ex.F.4., p. 530). For a fixed $\alpha \in \Sigma_0^+$, the singularities of the function $\frac{1}{c_\alpha(\lambda)c_\alpha(-\lambda)}$ are then described by distinguishing the following four cases:

- (a) m_α even, $m_{2\alpha} = 0$;
- (b) m_α odd, $m_{2\alpha} = 0$;
- (c) $m_\alpha/2$ even, $m_{2\alpha}$ odd;
- (d) $m_\alpha/2$ odd, $m_{2\alpha}$ odd.

Observe that, if $m_{2\alpha} = 0$, the duplication formula for the gamma function (see e.g. [2], 1.2.(15)):

$$\sqrt{\pi} \Gamma(2z) = 2^{2z-1} \Gamma(z) \Gamma(z + 1/2) \quad (20)$$

simplifies the function c_α as follows:

$$c_\alpha(\lambda) = 2^{m_\alpha/2-1} \pi^{-1/2} \frac{\Gamma(i\lambda_\alpha)}{\Gamma(i\lambda_\alpha + m_\alpha/2)}. \quad (21)$$

Using the classical relations ([2] 1.2.(1), 1.2.(5) and 1.2.(7)) :

$$\Gamma(z+1) = z\Gamma(z) \quad (22)$$

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z \sin(\pi z)} \quad (23)$$

$$\Gamma(1/2+z)\Gamma(1/2-z) = \frac{\pi}{\cos(\pi z)}, \quad (24)$$

we obtain that

$$\frac{1}{c_\alpha(\lambda)c_\alpha(-\lambda)} = C_\alpha \lambda_\alpha p_\alpha(\lambda) q_\alpha(\lambda) \quad (25)$$

where C_α is a positive constant (depending on α and on the multiplicities), p_α is a polynomial and q_α is a function. With the convention that a product over an empty set is equal to 1, the explicit expressions for p_α and q_α in the four cases listed above are the following:

- (a) $p_\alpha(\lambda) = \lambda_\alpha \prod_{k=1}^{m_\alpha/2-1} (\lambda_\alpha^2 + k^2)$,
 $q_\alpha(\lambda) = 1$;
- (b) $p_\alpha(\lambda) = \prod_{k=0}^{(m_\alpha-3)/2} [\lambda_\alpha^2 + (k+1/2)^2]$,
 $q_\alpha(\lambda) = \tanh(\pi\lambda_\alpha)$;
- (c) $p_\alpha(\lambda) = \prod_{k=0}^{m_\alpha/4-1} [(\lambda_\alpha/2)^2 + (k+1/2)^2] \cdot \prod_{k=0}^{m_\alpha/4+(m_{2\alpha}-1)/2-1} [(\lambda_\alpha/2)^2 + (k+1/2)^2]$,
 $q_\alpha(\lambda) = \tanh(\pi\lambda_\alpha/2)$;
- (d) $p_\alpha(\lambda) = \prod_{k=0}^{(m_\alpha-2)/4} [(\lambda_\alpha/2)^2 + k^2] \cdot \prod_{k=1}^{(m_\alpha+2m_{2\alpha})/4-1} [(\lambda_\alpha/2)^2 + k^2]$
 $q_\alpha(\lambda) = \coth(\pi\lambda_\alpha/2)$.

The constant C_α is equal to $2^{2-m_\alpha}\pi$ in the cases (a) and (b), and to 2π in the cases (c) and (d). Set

$$\rho_\alpha^{(\alpha)} := \frac{1}{2}m_\alpha + m_{2\alpha}. \quad (26)$$

By considering the singularities of the function q_α and their possible cancellation by zeros of p_α , we obtain the following lemma.

Lemma 2.1. *Let $\alpha \in \Sigma_0^+$. The function $[c_\alpha(\lambda)c_\alpha(-\lambda)]^{-1}$ and its modified version $[\lambda_\alpha c_\alpha(\lambda)c_\alpha(-\lambda)]^{-1}$ are holomorphic functions of $\lambda \in \mathfrak{a}_\mathbb{C}^*$ in the case (a) above. In the cases (b)–(d), they are meromorphic, with singularities of first order located along real hyperplanes in $\mathfrak{a}_\mathbb{C}^*$. In the case (b), the singular hyperplanes are described by the equations*

$$\pm i\lambda_\alpha - \rho_\alpha^{(\alpha)} \in \mathbb{Z}^+. \quad (27)$$

In the cases (c) and (d), that is if $m_{2\alpha} \neq 0$, they are described by the equations

$$\pm i\lambda_\alpha - \rho_\alpha^{(\alpha)} \in 2\mathbb{Z}^+. \quad (28)$$

□

Corollary 2.2. *Set*

$$L := \min \{ \rho_\alpha^{(\alpha)} |\alpha| = (m_\alpha/2 + m_{2\alpha}) |\alpha| : \alpha \in \Sigma_0^+ \text{ with } m_\alpha \text{ odd or } m_{2\alpha} \neq 0 \}, \quad (29)$$

the minimum over an empty set being $+\infty$. Then, for every $\omega \in \mathfrak{a}^$ with $|\omega| = 1$, the function $r \mapsto [c(r\omega)c(-r\omega)]^{-1}$ is holomorphic on $\mathbb{C} \setminus i(]-\infty, -L] \cup [L, +\infty[)$.*

Proof. Set $\lambda = r\omega$ with $\omega \in \mathfrak{a}^*$ and $|\omega| = 1$. If $r \in \mathbb{C} \setminus i\mathbb{R}$, then

$$i\lambda_\alpha = ir\omega_\alpha = \begin{cases} 0 & \text{if } \omega_\alpha = 0 \\ \in \mathbb{C} \setminus \mathbb{R} & \text{if } \omega_\alpha \neq 0. \end{cases}$$

Hence $\pm i\lambda_\alpha \notin \rho_\alpha^{(\alpha)} + \mathbb{Z}^+$ in this case. If $r = iy$ with $|y| < L$, then

$$|i\lambda_\alpha| = |y| \frac{|\omega|}{|\alpha|} < \frac{L}{|\alpha|} \leq \rho_\alpha^{(\alpha)}.$$

So $\pm i\lambda_\alpha \notin \rho_\alpha^{(\alpha)} + \mathbb{Z}^+$. □

Remark 2.3. Corollary 2.2 is needed to study the analytic continuation of the resolvent of Δ in (16) by means of a polar coordinate transformation. It is the analogue of Lemma 7.1 in [10], or of the argument on p. 420 of [12]. It shows that the analytic continuation given in Theorem 7.3 of [10] (respectively, in Theorems 3.1 and 3.2 of [12]) can in fact be performed on a larger domain.

2.2. The rank-one case. In this subsection we suppose that $X = G/K$ is a noncompact symmetric space of rank one. In this case, the set of (restricted) roots Σ contains at most two positive roots: α and, possibly, 2α . Let H be the element of \mathfrak{a} satisfying $\alpha(H) = 1$. The choice of H and α allows us to identify \mathfrak{a} and \mathfrak{a}^* with \mathbb{R} . Hence $\mathfrak{a}_\mathbb{C}$ and $\mathfrak{a}_\mathbb{C}^*$ coincide with \mathbb{C} . The Weyl group reduces to $\{\pm 1\}$ acting by multiplication. Observe that the element ρ in (1) is identified with the real number $\rho_\alpha = \frac{1}{2}m_\alpha + m_{2\alpha}$. Hence

$$\rho_\alpha^{(\alpha)} = \rho_\alpha \equiv \rho \tag{30}$$

in this case. The bottom of the spectrum of Δ becomes the positive real number $\rho^2|\alpha|^2$ with $|\alpha| := \sqrt{\langle \alpha, \alpha \rangle}$. Moreover, functions on \mathfrak{a} or on \mathfrak{a}^* (respectively, on $\mathfrak{a}_\mathbb{C}$ or on $\mathfrak{a}_\mathbb{C}^*$) are identified with functions on \mathbb{R} (respectively, on \mathbb{C}). For instance, if f is a function on \mathfrak{a}^* (respectively, on $\mathfrak{a}_\mathbb{C}^*$), then we will denote by the same symbol f the function on \mathbb{R} (respectively, on \mathbb{C}) given by $\lambda \mapsto f(\lambda\alpha)$. Furthermore, under these identifications, the Plancherel measure becomes a measure on \mathbb{R} with density

$$\frac{1}{c(\lambda)c(-\lambda)} = \frac{1}{c_0^2} C_\alpha \lambda p_\alpha(\lambda) q_\alpha(\lambda), \tag{31}$$

with respect to a suitable renormalisation $d\lambda$ of the Lebesgue measure on \mathbb{R} . Here the expressions for the constant C_α , the polynomial p_α and the function q_α are given by the four cases listed in the previous subsection. Furthermore, c_0 is the constant appearing in the definition of the Harish-Chandra's c -function. In the rank-one case, its value is

$$c_0 = 2^\rho \Gamma\left(\frac{m_\alpha}{2} + \frac{m_{2\alpha}}{2} + 1\right). \tag{32}$$

Lemma 2.4. *Keep the above notation. Then, as an element of $\mathcal{D}'(X \times X)$, the resolvent of Δ in (16) is given by the formula*

$$[R(\zeta)f](y) = \frac{C_\alpha}{2c_0^2|\alpha|} \int_{\mathbb{R}} \frac{1}{\lambda|\alpha| - \zeta} (f \times \varphi_\lambda)(y) p_\alpha(\lambda) q_\alpha(\lambda) d\lambda \tag{33}$$

for all $\zeta \in \mathbb{C} \setminus [0, +\infty[$, $f \in \mathcal{D}(X)$ and $y \in X$.

Proof. The formula (16) for the resolvent yields

$$\begin{aligned} [R(\zeta)f](y) &= \frac{C_\alpha}{2c_0^2} \int_{\mathbb{R}} (\lambda^2|\alpha|^2 - \zeta^2)^{-1} (f \times \varphi_\lambda)(y) \lambda p_\alpha(\lambda) q_\alpha(\lambda) d\lambda \\ &= \frac{C_\alpha}{4c_0^2|\alpha|} \int_{\mathbb{R}} \left[\frac{1}{\lambda|\alpha| - \zeta} + \frac{1}{\lambda|\alpha| + \zeta} \right] (f \times \varphi_\lambda)(y) p_\alpha(\lambda) q_\alpha(\lambda) d\lambda. \end{aligned}$$

The result follows because, as functions of λ , the product $p_\alpha(\lambda)q_\alpha(\lambda)$ is odd and φ_λ is even. □

We now determine a meromorphic continuation of $[R(\zeta)f](y)$ from $\text{Im } \zeta > 0$ to \mathbb{C} by shifting the contour of integration in the direction of the negative imaginary axis. The shift is allowed by

the estimate (15) together with the fact that there is a constant $M > 0$ so that for $\text{Im } \lambda < 0$, $|\text{Re } \lambda| \gg 0$ and $\text{Im } \zeta > 0$ we have

$$|(\lambda|\alpha| - \zeta)^{-1} p_\alpha(\lambda) q_\alpha(\lambda)| \leq M (\text{Im } \zeta)^{-1} (1 + |\lambda|)^{\deg p_\alpha}.$$

In the cases (b)–(d), one needs to take into account the simple poles of the function $p_\alpha(\lambda) q_\alpha(\lambda) = [\lambda c(\lambda) c(-\lambda)]^{-1}$ at the points $\lambda_k = -i(\rho + jk)$ with $k \in \mathbb{Z}^+$. Here we have set

$$j = \begin{cases} 1 & \text{in case (b),} \\ 2 & \text{in cases (c) and (d)} \end{cases} \quad (34)$$

Let $N \in \mathbb{Z}^+$. Then for $\text{Im } \zeta > 0$ we have by the residue theorem

$$[R(\zeta)f](y) = [R_N(\zeta)f](y) + \frac{i\pi C_\alpha}{c_0^2 |\alpha|} \sum_{k=0}^N \frac{1}{\lambda_k |\alpha| - \zeta} (f \times \varphi_{\lambda_k})(y) p_\alpha(\lambda_k) \underset{\lambda=\lambda_k}{\text{Res}} q_\alpha(\lambda) \quad (35)$$

where

$$[R_N(\zeta)f](y) := \frac{C_\alpha}{2c_0^2 |\alpha|} \int_{\mathbb{R} - i(\rho + j/2 + jN)} \frac{1}{(\lambda|\alpha| - \zeta)} (f \times \varphi_\lambda)(y) p_\alpha(\lambda) q_\alpha(\lambda) d\lambda. \quad (36)$$

The right-hand side of (35) provides a meromorphic extension of $R(\zeta)f$ as $\mathcal{D}'(X)$ -valued function on $\text{Im } \zeta > -(\rho + j/2 + jN)$. As $N \in \mathbb{Z}^+$ is arbitrary, we obtain the required meromorphic extension to \mathbb{C} . The extension is in fact holomorphic in case (a); it is meromorphic, with simple poles at $\zeta_k = \lambda_k |\alpha| = -i(\rho + jk)|\alpha|$ with $k \in \mathbb{Z}^+$ in cases (b)–(d). In fact, in these cases,

$$\underset{\lambda=\lambda_k}{\text{Res}} q_\alpha(\lambda) = j/\pi.$$

The residue at ζ_k is therefore

$$M_\alpha p_\alpha(\lambda_k)(f \times \varphi_{\lambda_k})(y) \quad (37)$$

where

$$M_\alpha := \frac{ijC_\alpha}{c_0^2 |\alpha|}.$$

The resulting resolvent residue operator $R_k : \mathcal{D}(X) \rightarrow \mathcal{E}(X)$, given by

$$R_k(f) := M_\alpha p_\alpha(\lambda_k)(f \times \varphi_{\lambda_k}), \quad (38)$$

will be studied in the following section.

Remark 2.5. The following table gives the restricted root structure of the irreducible connected simply connected Riemannian symmetric spaces of rank-one G/K . In the table, G_0 denotes the connected component of the identity in the group G .

G	K	Σ^+	m_α	$m_{2\alpha}$	G/K
$SO_0(2n+1, 1)$, $n \geq 1$	$SO(2n+1)$	$\{\alpha\}$	$2n$	0	real hyperbolic space
$SO_0(2n, 1)$, $n \geq 1$	$SO(2n)$	$\{\alpha\}$	$2n-1$	0	real hyperbolic space
$SU(n, 1)$, $n \geq 2$	$S(U(n) \times U(1))$	$\{\alpha, 2\alpha\}$	$2(n-1)$	1	complex hyperbolic space
$Sp(n, 1)$, $n \geq 2$	$Sp(n) \times Sp(1)$	$\{\alpha, 2\alpha\}$	$4(n-1)$	3	quaternionic hyperbolic space
$F_{4(-20)}$	$Spin(9)$	$\{\alpha, 2\alpha\}$	8	7	octonion hyperbolic plane

3. THE RESOLVENT RESIDUE OPERATORS

3.1. Joint eigenspaces and finite-dimensional spherical representations. In this section we come back to setting of a noncompact symmetric space $X = G/K$ of arbitrary rank. Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. According to [5] or to p. 142 in [8], we define the joint eigenspace $\mathcal{E}_\lambda(X)$ for the algebra $\mathbb{D}(X)$ as follows:

$$\mathcal{E}_\lambda(X) = \{f \in \mathcal{E}(X) : Df = \Gamma(D)(i\lambda)f \text{ for all } D \in \mathbb{D}(X)\}. \quad (39)$$

The group G acts on $\mathcal{E}_\lambda(X)$ by left translations:

$$[T_\lambda(g)f](x) := f(g^{-1}x), \quad g \in G, x \in X \quad (40)$$

We observe that $\mathcal{E}_\lambda(X) = \mathcal{E}_{w\lambda}(X)$ (hence $T_\lambda = T_{w\lambda}$) for all $w \in W$. The representation T_λ is irreducible if and only if $e(\lambda)e(-\lambda) \neq 0$ where

$$e(\lambda) := \prod_{\alpha \in \Sigma_0^+} \Gamma\left(\frac{i\lambda_\alpha}{2} + \frac{m_\alpha}{4} + \frac{1}{2}\right) \Gamma\left(\frac{i\lambda_\alpha}{2} + \frac{m_\alpha}{4} + \frac{m_{2\alpha}}{2}\right) \quad (41)$$

is the denominator of Harish-Chandra c -function (see [5], Theorem 9.1, and 8). Recall the notation λ_α from (2). In particular, T_λ is reducible for all λ in the singular set of $[c(\lambda)c(-\lambda)]^{-1}$. In this case, we shall see that T_λ admits a finite dimensional irreducible subrepresentation which consists of G -finite functions. Recall that a function $f \in \mathcal{E}_\lambda(X)$ is said to be G -finite if the vector space spanned by the left translates $T_\lambda(g)f$ of f with $g \in G$, is finite dimensional. Notice that the space $\mathcal{E}_{\lambda,G}(X)$ of G -finite elements in $\mathcal{E}_\lambda(X)$ is a (possibly zero) invariant subspace of $\mathcal{E}_\lambda(X)$. Moreover, $\mathcal{E}_{w\lambda,G}(X) = \mathcal{E}_{\lambda,G}(X)$ for every $w \in W$.

A representation π of G on a locally convex complex vector space V is said to be spherical if V contains a vector $v \neq 0$ (called a spherical vector) that is fixed under all operators $T(k)$ with $k \in K$. We refer to [7], p. 410, for more information on this subject.

An element $\mu \in \mathfrak{a}^*$ is a highest restricted weight if $\mu_\alpha \in \mathbb{Z}^+$ for all $\alpha \in \Sigma^+$. As $\mu_{2\alpha} = \frac{\langle \mu, 2\alpha \rangle}{\langle 2\alpha, 2\alpha \rangle} = \mu_\alpha/2$, this condition in fact implies that $\mu_\alpha \in 2\mathbb{Z}^+$ whenever 2α is a root.

The following proposition will be crucial in the sequel.

Proposition 3.1. *We have $\mathcal{E}_{\lambda,G}(X) \neq \{0\}$ if and only if for some $w \in W$*

$$-i(w\lambda)_\alpha - \rho_\alpha \in \mathbb{Z}^+ \quad \text{for all } \alpha \in \Sigma^+.$$

In this case, $\mathcal{E}_{\lambda,G}(X)$ is finite dimensional and irreducible under G . It is the finite-dimensional spherical representation of highest restricted weight $-iw\lambda - \rho$. All finite-dimensional spherical representations of G arise in this fashion.

Proof. This is Proposition 4.16, p. 142 in [8]. □

We now prove an alternative description for the spaces $\mathcal{E}_{\lambda,G}(X)$. It will provide the relation between these spaces and the resolvent residue operators.

Theorem 3.2. *Suppose $\mathcal{E}_{\lambda,G}(X) \neq \{0\}$ is the finite-dimensional spherical representation of highest restricted weight $-iw\lambda - \rho$ (for some $w \in W$). Then $\mathcal{E}_{\lambda,G}(X) = \{f \times \varphi_\lambda : f \in \mathcal{D}(X)\}$.*

Proof. Set $\mathcal{D}_\lambda := \{f \times \varphi_\lambda : f \in \mathcal{D}(X)\}$. Observe first that $\mathcal{D}_\lambda \neq \{0\}$ as $\varphi_\lambda \neq 0$.

For all $D \in \mathbb{D}(X)$ we have $D(f \times \varphi_\lambda) = f \times D\varphi_\lambda$ (see [7], Theorem 5.5, p. 293). Hence (6) implies $D(f \times \varphi_\lambda) = \Gamma(D)(i\lambda)(f \times \varphi_\lambda)$. So $\mathcal{D}_\lambda \subset \mathcal{E}_\lambda$.

Let $g \in G$, and let $F^{\tau(g)}$ denote the left-translate by g of the function $F : X \rightarrow \mathbb{C}$. Hence $F^{\tau(g)}(h \cdot o) := F(g^{-1}h \cdot o)$ for all $h \in G$. We have

$$T_\lambda(g)(f \times \varphi_\lambda) := (f \times \varphi_\lambda)^{\tau(g)} = f^{\tau(g)} \times \varphi_\lambda.$$

As $f^{\tau(g)} \in \mathcal{D}(X)$, the subspace \mathcal{D}_λ of \mathcal{E}_λ is T_λ -invariant.

We now prove that each element $f \times \varphi_\lambda$ is G -finite, that is so that $\{T_\mu(g)(f \times \varphi_\lambda) = f^{\tau(g)} \times \varphi_\lambda : g \in G\}$ spans a finite dimensional subspace of \mathcal{E}_λ . This will prove that $\mathcal{D}_\lambda \subset \mathcal{E}_{\lambda,G}(X)$, and hence that $\mathcal{D}_\lambda = \mathcal{E}_{\lambda,G}(X)$ by irreducibility.

The G -translates of φ_λ belong to a finite-dimensional space V (see [8], p. 328). If $\varphi_1, \dots, \varphi_d$ is a basis of V , then $\varphi_\lambda^{\tau(g)} = \sum_{i=1}^d a_i(g)\varphi_i$ for uniquely determined constants $a_i(g) \in \mathbb{C}$. The functions $a_i : G \rightarrow \mathbb{C}$ so determined are (at least) continuous, hence locally integrable. Let $f \in \mathcal{D}(X)$. For $g \in G$ we have

$$\begin{aligned} (f \times \varphi_\lambda)(g \cdot o) &= [(f \circ \pi) * (\varphi_\lambda \circ \pi)](g) \\ &= \int_G (f \circ \pi)(h)(\varphi_\lambda \circ \pi)(h^{-1}g) dh \\ &= \int_G (f \circ \pi)(h)(\varphi_\lambda^{\tau(h)} \circ \pi)(g) dh \\ &= \sum_{i=1}^d \left(\int_G (f \circ \pi)(h)a_i(h) dh \right) \varphi_i(g \cdot o). \end{aligned}$$

Set $c_i(g) := \int_G (f^{\tau(g)} \circ \pi)(h)a_i(h) dh$. Then

$$f^{\tau(g)} \times \varphi_\lambda = \sum_{i=1}^d c_i(g)\varphi_i \in V,$$

which proves that $f \times \varphi_\lambda$ is G -finite. □

3.2. The dimension of the spherical representations. The dimension of the finite-dimensional spherical representations can be expressed in terms of the Harish-Chandra c -function.

Lemma 3.3. *The dimension $d(\mu)$ of the finite-dimensional spherical representation of highest restricted weight μ is given by*

$$d(\mu) = \frac{c(\lambda + i\mu)c(-\lambda - i\mu)}{c(\lambda)c(-\lambda)} \Big|_{\lambda = -i(\mu + \rho)}.$$

Proof. This is Theorem 9.10, p. 337, in [8]. □

The singularities appearing in the above formula can be removed by using the expression for $c_\alpha(\lambda)c_\alpha(-\lambda)$ found in (25).

Lemma 3.4. *Let $q_\alpha(\lambda)$ be the function appearing in (25). Then*

$$\frac{q_\alpha(\lambda + i\mu)}{q_\alpha(\lambda)} = 1$$

for every $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and every highest restricted weight μ .

Proof. Recall that q_α assumes different values according to the four cases considered in subsection 2.1:

- (a) $q_\alpha(\lambda) = 1$;
- (b) $q_\alpha(\lambda) = \tanh(\pi\lambda_\alpha)$;
- (c) $q_\alpha(\lambda) = \tanh(\pi\lambda_\alpha/2)$;
- (d) $q_\alpha(\lambda) = \coth(\pi\lambda_\alpha/2)$.

Observe that the value of $\frac{q_\alpha(\lambda+i\mu)}{q_\alpha(\lambda)}$ in case (d) is the reciprocal of the value of this quotient in case (c). It is therefore enough to prove the Lemma for the cases (b) and (c). In the latter cases we have

$$\frac{q_\alpha(\lambda+i\mu)}{q_\alpha(\lambda)} = \frac{\tanh(j\pi(\lambda+i\mu)_\alpha)}{\tanh(j\pi\lambda_\alpha)} = \frac{\tanh(\pi(j\lambda_\alpha+i\mu_\alpha))}{\tanh(j\pi\lambda_\alpha)}$$

where $j = 1$ in case (c) and $j = 1/2$ in case (d). Hence $j\mu_\alpha \in \mathbb{Z}^+$. The result follows because \tanh is $i\pi$ periodic. \square

Proposition 3.5. *In the above notation, we have*

$$d(\mu) = \prod_{\alpha \in \Sigma_0^+} \frac{(\mu_\alpha + \rho_\alpha)}{\rho_\alpha} \frac{p_\alpha(-i(\mu + \rho))}{p_\alpha(-i\rho)},$$

where p_α is the polynomial on \mathfrak{a}^* appearing in (25).

Proof. This is an immediate consequence of (25) and Lemmas 3.3 and 3.4. \square

Corollary 3.6. *For all $\lambda \in \mathfrak{a}^*$ satisfying the condition stated in Proposition 3.1, we have*

$$\dim \mathcal{E}_{\lambda,G}(X) = d(-i\omega\lambda - \rho) = \prod_{\alpha \in \Sigma_0^+} \frac{(-i\omega\lambda)_\alpha}{\rho_\alpha} \frac{p_\alpha(-\omega\lambda)}{p_\alpha(-i\rho)}.$$

3.3. Resolvent residue operators in the rank-one case. In this subsection we restrict ourself to the case of a symmetric space $X = G/K$ of rank one. We keep the notation introduced in subsection 2.2. In particular, we adopt the identification of \mathfrak{a}^* with \mathbb{R} fixed there, under which the element $\rho \in \mathfrak{a}^*$ is identified with the real number $\rho_\alpha^{(\alpha)}$, see (30). We also assume that either m_α is odd (and hence $m_{2\alpha} = 0$), or $m_{2\alpha} \neq 0$. The first situation corresponds to the case (b) of subsection 2.1; the second to the cases (c) and (d) there. These are the situations in which the meromorphically extended resolvent admits simple poles at the points $\zeta_k = \lambda_k|\alpha|$ with $\lambda_k = -i(\rho + jk)$ and $k \in \mathbb{Z}^+$. We now study the corresponding resolvent residue operator $R_k : \mathcal{D}(X) \rightarrow \mathcal{E}(X)$ given by equation (38).

Corollary 3.7. *Let $j = 1$ if $m_{2\alpha} = 0$ and m_α is odd, and $j = 2$ if $m_{2\alpha} \neq 0$. Let $k \in \mathbb{Z}^+$ be fixed. Then the resolvent residue operator R_k is a finite rank operator with image $\mathcal{E}_{\lambda_k\alpha,G}(X)$. Its rank is*

$$\dim \mathcal{E}_{\lambda_k\alpha,G}(X) = \frac{\rho + jk}{\rho} \frac{p_\alpha(\lambda_k)}{p_\alpha(-i\rho)}. \quad (42)$$

Proof. The parameter $\lambda_k = -i(\rho + jk)$ satisfies the conditions of Proposition 3.1 with $w = -1$. Thus $\{f \times \varphi_{\lambda_k} : f \in \mathcal{D}(X)\} = \mathcal{E}_{\lambda_k\alpha,G}(X)$ by Theorem 3.2. Moreover, by Corollary 3.6,

$$\dim \mathcal{E}_{\lambda_k\alpha,G}(X) = d(i\lambda_k - \rho) = \frac{i\lambda_k}{\rho} \frac{p_\alpha(\lambda_k)}{p_\alpha(-i\rho)} = \frac{\rho + jk}{\rho} \frac{p_\alpha(\lambda_k)}{p_\alpha(-i\rho)}.$$

\square

The formula (42) allows us to express $p_\alpha(\lambda_k)$ in terms of the dimension of the space $\mathcal{E}_{\lambda_k\alpha,G}(X)$. In turn, this gives a different way of writing the formula for the resolvent residue operator at ζ_k .

We collect the results on the meromorphic extension of the resolvent of Δ for rank-one symmetric spaces in the following theorem.

Theorem 3.8. *Let $X = G/K$ be a Riemannian symmetric space of (real) rank one. If the system of positive roots consists of a unique root α of even multiplicity m_α , then the resolvent $R(\zeta)$ of the operator Δ given in (33) extends from $\text{Im } \zeta > 0$ to a $\mathcal{D}'(X \times X)$ -valued holomorphic function on \mathbb{C} . In the other cases, the resolvent $R(\zeta)$ admits a meromorphic extension with simple poles at the*

points $\zeta_k = \lambda_k |\alpha|$ with $\lambda_k = -i(\rho + jk)$ and $k \in \mathbb{Z}^+$. Here we use the convention (34) for the variable j and $\rho = \rho_\alpha^{(\alpha)}$ is the constant given in (26). The resolvent residue operator $R_k : \mathcal{D}(X) \rightarrow \mathcal{E}(X)$ at ζ_k is a finite rank operator with image coinciding with the space $\mathcal{E}_{\lambda_k \alpha, G}(X)$ of the finite-dimensional spherical representation of highest restricted weight $i\lambda_k - \rho$. It is given by the explicit formula:

$$R_k(f) = K_\alpha \frac{1}{\rho + jk} \dim \mathcal{E}_{\lambda_k \alpha, G}(X) (f \times \varphi_{\lambda_k}) \quad (43)$$

where φ_{λ_k} is the spherical function on G of spectral parameter λ_k and the constant K_α is given by:

$$K_\alpha = \frac{(-1)^{(m_\alpha + m_{2\alpha} - 1)/2}}{2^{m_\alpha + j m_{j\alpha} - j}} \frac{\pi i}{|\alpha|} \frac{\left(\frac{2\rho}{j}\right)!}{\left(\frac{m_\alpha + m_{2\alpha} - 1}{2}\right)! \left(\frac{m_{j\alpha} - 1}{2}\right)!} \quad (44)$$

Proof. The only part left to prove is the explicit formula for the resolvent residue operator R_k . This formula is a consequence of (37), (42) and (32), together with following values for $p_\alpha(-i\rho)$:

$$p_\alpha(-i\rho) = \begin{cases} (-1)^{(m_\alpha - 1)/2} (m_\alpha - 1)! & \text{if } m_\alpha \text{ is odd} \\ (-1)^{(m_\alpha + m_{2\alpha} - 1)/2} (\rho - 1)! \frac{\left(\frac{m_\alpha + m_{2\alpha} - 1}{2}\right)!}{\left(\frac{m_{2\alpha} - 1}{2}\right)!} & \text{if } m_{2\alpha} \neq 0 \end{cases} .$$

□

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