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# Regulation of a Liquid Propelled Rocket Engine using Contraction Theory<sup>\*</sup>

Jules Gibart<sup>\*,\*\*</sup> H el ene Piet-Lahanier<sup>\*</sup> Fran ois Farago<sup>\*\*</sup>  
Marco Galeotta<sup>\*\*</sup>

<sup>\*</sup> DTIS, ONERA, University Paris-Saclay, F-91123 Palaiseau, France  
<sup>\*\*</sup> CNES, Paris, 75012, FRANCE

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**Abstract:** Liquid propelled rocket engines (LPRE) are highly non-linear systems that require complex stability analysis and regulation. Most often, this is performed by linearizing in the neighborhood of a functioning point which makes it difficult to account for changes of points *e.g.* for reusable launchers. In this paper the objective is to propose a non linear control law to regulate the thrust of the engine, which is represented by the reaction chamber pressure, and the mixture ratio between the fuel and the oxidizer. This control law must provide stability guarantees for the system for a variety of functioning points. The new design is based on Contraction theory and is shown to address both stability and regulation objectives as illustrated with simulation results.

*Keywords:* Rocket Engine, Trajectories, Stability, Contraction Theory

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## 1. INTRODUCTION

Reusable launchers are equipped with liquid propelled rocket engine (LPRE) that are due to function for different points. These engines are highly non-linear systems. Fixed-point engines were used for a single functioning point and the stability analysis was performed on a linearized domain in the neighborhood of this point while in the case of a reusable engine, we have to prove stability along the trajectories between several functioning points. Quite often stability is derived from the definition of a Lyapunov function. However, for such systems, the determination of a suitable Lyapunov function proves to be very complex. An interesting alternative is the use of contraction theory which allows, when a regulation control has been designed, to evaluate the system stability along given trajectories.

Contraction theory has been initially introduced by Lohmiller and Slotine (1998); Lohmiller (1999). For a more historical approach of contraction theory and links to Lyapunov's theory more material is proposed in Jouffroy (2005); Forni and Sepulchre (2013). Recent developments have been suggested in Andrieu and Tarbouriech (2019); Tsukamoto et al. (2021) to propose a Linear Matrix Inequality (LMI) formulation of the contraction problem. Various applications of contraction theory have been considered, *e.g.* coupled oscillators in Wang and Slotine (2005), mechanical systems (Lohmiller and Slotine, 2000; Ijspeert et al., 2013). These applications already integrate feedback combination of systems in the equations and properties of the combination of several systems in contraction theory. Although rocket engine, as a thermodynamic dissipating system, can be considered as a feedback combination of several subsystems, to the best of our knowledge, application of this theory to the stability of LPRE has never been considered. The innovation of this paper consists in the new model of

LPRE and the derivation of a regulation control leading to a contracting behavior and thus providing guarantee on the stability of the system along specified trajectories.

The article is organized as follows. After describing the main features of the LPRE in section 2, basis of contraction theory will be presented in section 3. The design of the control used in this paper is described in section 4. Simulated results are presented in section 4.4 to illustrate the performances of the methodology.

## 2. PROBLEM STATEMENT

We consider here a simplified LPRE model derived from the representation depicted in Perez Roca (2020); P erez-Roca et al. (2018); Galeotta et al. (2019). This model, even if simplified, illustrates the interdependencies of such thermodynamical systems as well as the sensitivity of the system to initial conditions and bias on the value of the parameters. The engine is functioning with dihydrogen as fuel and oxygen as oxidizer. Each tank is connected to a motopump which converts an input power into an increase in outlet pressure, enabling the fluid to enter the combustion chamber. The basic features of the LPRE are illustrated in figure 1. It can be noticed that the dihydrogen branch integrates an additional element, the regenerative circuit that transfers heat from the combustion chamber's structure to the incoming fluid. This element is modeled with a change of temperature in the injection system. To complete the description of the engine, the dissipating effects of the lines and the injectors will be introduced in the state equations of the dynamics. We aim to regulate the inner pressure of the combustion chamber, using the power inputs on both motopumps, while keeping the mixture ratio, *i.e.* the ratio between oxidizer and fuel mass flows injected in the chamber, close to a reference value (P erez-Roca et al. (2019)). One of the main sources of instability is the bias on the parameters that build the model, leading

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<sup>\*</sup> This work is under common ground by ONERA and CNES

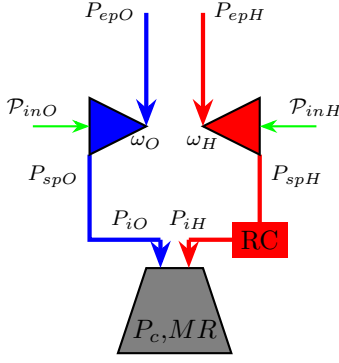


Fig. 1. Model of the LPRE system used

to an other requirement for a regulation independent of initial conditions. For example, a dynamical state feedback regulation proves to be sufficient for regulation, but is unable to withstand the bias of parameters, leading to unstable behavior. As the two branches of the LPRE are almost identical, we focus on the description of the dihydrogen model. This model will be referred to as the subsystem  $H$ , and results will be extended to the complete system in section 4.4.

The following table describes the different notations used in our LPRE model. Note that every variable described for  $H$  can also be used for  $O$ . All results are presented with normalized variables, therefore reduced units.

#### NOMENCLATURE

##### Constant terms

$\rho_H$	Density of the H fluid ( $kg.m^{-3}$ )
$I_H$	Fluidic Inertia ( $m^{-1}$ )
$J_H$	Angular Inertia ( $kg.m^2$ )
$k_1, k_2$	Combustion chamber parameters
$k_{eq}$	Equivalent resistance coefficient ( $kg^{-1}.m^{-1}$ )
$P_{EPH}$	Tank pressure of the H fluid ( $Pa$ )

##### Time varying terms

$\dot{m}_H$	Mass flow in the H branch ( $kg.s^{-1}$ )
$\mathcal{P}_{inH}$	Power input of the motopump H ( $W$ )
$\omega_H$	Rotational speed of the H motopump ( $rad.s^{-1}$ )
$MR$	Mixture ratio in the combustion chamber (-)
$P_c$	Combustion chamber pressure ( $Pa$ )
$P_{iH}$	Injector pressure ( $Pa$ )
$P_{spH}$	Motopump output pressure ( $Pa$ )

#### 2.1 Equations of the model

The motopump is described by a polynomial relation between the resisting torque  $T_{rH}$ , the rotational speed and mass flow:

$$T_{rH} = \left| \frac{a_{cH}}{\rho_H} \dot{m}_H^2 + b_{cH} \omega_H \dot{m}_H + c_{cH} \rho_H \omega_H^2 \right|. \quad (1)$$

We can then find a relation for the rotational speed  $\omega_H$ , by using the conservation of the kinetic moment:

$$\dot{\omega}_H = \frac{1}{J_H} \left( \frac{\mathcal{P}_{inH}}{\omega_H} - T_{rH} \right), \quad (2)$$

with  $\mathcal{P}_{inH}$  being the input power of the motopump, and  $a_{ch}, b_{ch}, c_{ch}$  being specific coefficients of the motopump. We introduce following change of variables  $\Omega_H = \omega_H^2$ , which transforms (2) into the following:

$$\dot{\Omega}_H = \frac{2}{J_H} \left( \mathcal{P}_{inH} - \left| \frac{a_{cH}}{\rho_H} \dot{m}_H^2 \Omega_H^{\frac{1}{2}} - b_{cH} \Omega_H \dot{m}_H - c_{cH} \rho_H \Omega_H^{\frac{3}{2}} \right| \right). \quad (3)$$

The evolution of the reaction chamber pressure can be approximated by a linear equation as in Perez Roca (2020), by:

$$\dot{P}_c = k_1(\dot{m}_H + \dot{m}_O) - k_2 P_c, \quad (4)$$

where  $k_1$  and  $k_2$  are obtained by first order Taylor expansion of the thermodynamics equation in the neighborhood of the steady-state nominal value.

The dissipating effects of the lines are modelled using the equation of conservation of the momentum of the fluid

$$I_H \ddot{m}_H = P_{spH} - P_{iH} - k_{eq} \dot{m}_H^2. \quad (5)$$

A similar expression can be obtained for the injection pressure drop by representing the loss as  $k_{inj} \dot{m}_H^2$  and expressing  $P_c$  as  $P_c = P_{iH} - k_{inj} \dot{m}_H^2$ . The output pressure of the motopump  $P_{spH}$  is expressed as a polynomial function:

$$P_{spH} = P_{epH} + \frac{a_{pH}}{\rho_H} \dot{m}_H^2 + b_{pH} \Omega_H \dot{m}_H + c_{pH} \rho_H \Omega_H, \quad (6)$$

where  $a_{pH}, b_{pH}, c_{pH}$  are specific coefficients of the motopump. We can then use both equation (5) and (6) to obtain the evolution of the mass flow  $\dot{m}_H$ :

$$\ddot{m}_H = \frac{1}{I_H} \left( P_{spH} - P_c \right). \quad (7)$$

From this set of differential equations, the following state representation of the LPRE is obtained, where  $X_1$  corresponds to  $\Omega_H$ ,  $X_2$  to  $\dot{m}_H$ ,  $X_3$  to  $\Omega_O$ ,  $X_4$  to  $\dot{m}_O$  and  $X_5$  to  $P_c$ , and where  $u_H$  designates the control input of the motopump  $\mathcal{P}_{inH}$  and  $u_O, \mathcal{P}_{inO}$ .

$$\begin{aligned} \dot{X}_1 &= \frac{2}{J_H} \left( \mathcal{P}_{inH} - \left| \frac{a_{cH}}{\rho_H} X_2^2 X_1^{\frac{1}{2}} + b_{cH} X_1 X_2 + c_{cH} \rho_H X_1^{\frac{3}{2}} \right| \right) \\ \dot{X}_2 &= \frac{1}{I_H} \left( P_{EPH} - X_5 + \left( \frac{a_{pH}}{\rho_H} - k_{eq} \right) X_2^2 + b_{pH} X_1^{\frac{1}{2}} X_2 + c_{pH} \rho_H X_1 \right) \\ \dot{X}_3 &= \frac{2}{J_O} \left( \mathcal{P}_{inO} - \left| \frac{a_{cO}}{\rho_O} X_4^2 X_3^{\frac{1}{2}} + b_{cO} X_3 X_4 + c_{cO} \rho_O X_3^{\frac{3}{2}} \right| \right) \\ \dot{X}_4 &= \frac{1}{I_O} \left( P_{EPO} - X_5 + \left( \frac{a_{pO}}{\rho_O} - k_{eqO} \right) X_4^2 + b_{pO} X_3^{\frac{1}{2}} X_4 + c_{pO} \rho_O X_3 \right) \\ \dot{X}_5 &= k_1 (X_2 + X_4) - k_2 X_5. \end{aligned} \quad (8)$$

The scenario addressed is the regulation of  $P_{CC}$  along a transition between two functioning points, while insuring stability along the trajectory. Additionally, we aim to maintain the mixture ratio  $MR$  to the normal value. The main source of instability of an LPRE system comes from the bias on parameters, due to residual species in the combustion chamber or variation of internal parameters. The stability analysis around functioning points can leave uncertain zones. To cover these uncertain zones this paper proposes an approach that guarantees stability around a trajectory. Use of contraction theory will be made to cover the stability requirements.

### 3. CONTRACTION THEORY

Contraction theory is a recent tool for analyzing the stability of nonlinear systems. The analysis is performed using the convergence of the different trajectories of a system, i.e. the time evolution of the states given the dynamics and initial conditions. Contracting behavior is illustrated by the ability of a system to forget initial conditions and perturbations. Consider systems of the form:

$$\dot{x} = f(x, t), \quad (9)$$

where  $f$  is a  $n \times 1$  non-linear vector function and  $x$  is a  $n \times 1$  state vector. Additionally, let the virtual displacement between the different points of the flow of trajectories be defined by:

$$\delta\dot{x} = \frac{\partial f}{\partial x}(x, t)\delta x, \quad (10)$$

*Definition 1.* (Lohmiller and Slotine (1998)). Given the system equations  $\dot{x} = f(x, t)$ , a region of the state space is called a contraction region if the Jacobian  $\frac{\partial f}{\partial x}$  is uniformly negative definite in that region.

Similarly, a region of the state space is called a semi-contraction region if the Jacobian is uniformly negative semi-definite, and finally a region is called indifferent if the Jacobian is skew-symmetric. This definition leads to the first result in contraction analysis, which is a sufficient exponential convergence result.

*Theorem 2.* (Lohmiller and Slotine (1998)). Given the system equations  $\dot{x} = f(x, t)$ , any trajectory, which starts in a ball of constant radius centered around a given trajectory and contained at all times in a contraction region, remains in that ball and converges exponentially to this trajectory. Furthermore, global exponential convergence to the given trajectory is guaranteed if the whole state space is a contraction region.

If the LPRE system verifies theorem 2, then exponential convergence of any couple of trajectories and therefore stability are proven. This constitutes an incremental form of stability (Jouffroy and Slotine (2004)), which is stronger than simple exponential stability with respect to the origin. However, as will be shown below such a system is not initially contracting, so we will use another result from Andrieu and Tarbouriech (2019), which enables us to compute an input that makes the considered system contracting. In order to use these results, it is necessary to formulate the system differently. A new representation of a system is used from now on, where we separate the linear and non-linear components of the function  $f$  in (11), as:

$$\dot{x} = Ax + M\phi(y, t), y = Lx, \quad (11)$$

where  $A$  is a square  $n \times n$  matrix, that is built using the linear terms in  $f(x, t)$ , and  $\phi(y, t)$  contains the terms to add to  $Ax$  to reconstruct the equation of  $\dot{x}$ . Both matrices  $M, L$  are defined in order to respect the dimensions of the other elements. With this new formulation of the system, conditions are presented for which we can find LMIs that characterize contracting behavior.

*Assumption 3.* (Andrieu and Tarbouriech (2019)) (Monotonic nonlinearities) The mapping  $\phi : \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^m$  is such that:

$$0 \leq \frac{\partial \phi}{\partial y}(y, t) + \frac{\partial \phi}{\partial y}(y, t)^\top \leq \Gamma, \forall (y, t) \in \mathbb{R}^m \times \mathbb{R}, \quad (12)$$

where  $\Gamma \in \mathbb{R}^{m \times m}$  is a symmetric positive definite matrix.

*Theorem 4.* (Andrieu and Tarbouriech (2019)) Assume that  $\phi$  satisfies the assumption previously mentioned. If there exist a symmetric positive definite matrix  $P \in \mathbb{R}^{n \times n}$  and a positive real number  $v$  such that:

$$\begin{bmatrix} A^\top P + PA + vId_n & L^\top + PM \\ L + M^\top P & -4\Gamma^{-1} \end{bmatrix} \leq 0. \quad (13)$$

The system (11) defines a contraction.

When the system is not contracting, Andrieu and Tarbouriech (2019) search for a control  $u$  under the form:

$$\dot{x} = Ax + M\phi(y, t) + Bu \quad (14)$$

$$u = Kx + N\phi(Lx, t).$$

*Corollary 5.* (Andrieu and Tarbouriech (2019)) Assume that  $\phi$  satisfies Assumption 3. If there exist a symmetric positive definite matrix  $W \in \mathbb{R}^{n \times n}$ , two matrices  $Z \in \mathbb{R}^{q \times n}, N \in \mathbb{R}^{q \times p}$  and a positive real number  $\eta$  such that:

$$\begin{bmatrix} (AW + BZ)^\top + (AW + BZ) & WL^\top + (M + BN) & W \\ LW + (M + BN)^\top & -D\Gamma^{-1} & 0 \\ W & 0 & -\eta Id_n \end{bmatrix} \leq 0, \quad (15)$$

then the closed-loop system (14) is a contraction with the control law defined by  $K = ZW^{-1}$  and  $N$ .

*Remark 6.* If one has

$$0 \leq \frac{\partial \phi}{\partial y}(y, t) + \frac{\partial \phi}{\partial y}(y, t)^\top, \forall (y, t) \in \mathbb{R}^m \times \mathbb{R}, \quad (16)$$

and if the following LMI is verified:

$$\begin{bmatrix} A^\top P + PA + vId_n & L^\top + PM \\ L + M^\top P & 0 \end{bmatrix} \leq 0, \quad (17)$$

it is shown in Andrieu and Tarbouriech (2019) that the same result on the contracting behavior of the system than in theorem 4 applies. This LMI can be rewritten into two constraints:  $A^\top P + PA + vId_n < 0, L^\top = -PM$ . This result can also be extended to the corollary 5, with the matrices previously introduced  $W, M, N$  verifying:

$$\begin{aligned} (AW + BZ)^\top + (AW + BZ) &\leq 0, \\ W > 0, WL^\top &= -(M + BN), \end{aligned} \quad (18)$$

which are LMI conditions derived from equation (15).

The method presented in Andrieu and Tarbouriech (2019) is used to find the control inputs to obtain a contracting system, and will be presented below.

## 4. DESIGN OF A CONTRACTING CONTROL LAW FOR THE LPRE

### 4.1 Preliminaries

A first example of contraction of a system is presented in this section, using the subsystem  $H$  of our LPRE. The

result will then be applied to the global system of the LPRE. To create the subsystem  $H$  we suppose that a control is applied to the oxygen mass flow that allows us to consider  $MR(t)$  constant, which leads to the following system:

$$\begin{aligned}\dot{X}_1 &= \frac{2}{J_H} \left( u_H - \left| \frac{a_{cH}}{\rho_H} X_2^2 X_1^{\frac{1}{2}} + b_{cH} X_1 X_2 + c_{cH} \rho_H X_1^{\frac{3}{2}} \right| \right) \\ \dot{X}_2 &= \frac{1}{I_H} \left( P_{EPH} - X_5 + \left( \frac{a_{pH}}{\rho_H} - k_{eqH} \right) X_2^2 + b_{pH} X_1^{\frac{1}{2}} X_2 \right. \\ &\quad \left. + c_{pH} \rho_H X_1 \right) \\ \dot{X}_5 &= k_1 (1 + MR_d) X_2 - k_2 X_5.\end{aligned}\quad (19)$$

A preliminary analysis of the Jacobian of this system gives:

$$\frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial X_2} & \frac{\partial f_1}{\partial X_3} \\ \frac{\partial f_2}{\partial X_1} & \frac{\partial f_2}{\partial X_2} & \frac{\partial f_2}{\partial X_3} \\ \frac{\partial f_3}{\partial X_1} & \frac{\partial f_3}{\partial X_2} & \frac{\partial f_3}{\partial X_3} \end{pmatrix}, \quad (20)$$

where

$$\begin{aligned}\frac{\partial f_1}{\partial X_1} &= -\frac{2}{J_H} \left( a_{cH} \frac{X_2^2}{2X_1^{\frac{1}{2}}} + b_{cH} X_2 + \frac{3}{2} c_{cH} \rho_H X_1^{\frac{1}{2}} \right), \\ \frac{\partial f_1}{\partial X_2} &= -\frac{2}{J_H} (2a_{cH} X_2 X_1^{\frac{1}{2}} + b_{cH} X_1), \quad \frac{\partial f_1}{\partial X_3} = 0, \\ \frac{\partial f_2}{\partial X_1} &= \frac{1}{I_H} (b_{pH} \frac{X_2}{2X_1^{\frac{1}{2}}} + c_{pH} \rho_H), \\ \frac{\partial f_2}{\partial X_2} &= \frac{1}{I_H} (2 \frac{a_{pH}}{\rho_H} - k_{eqH}) X_2 + b_{pH} X_1^{\frac{1}{2}}, \quad \frac{\partial f_2}{\partial X_3} = -\frac{1}{I_H}, \\ \frac{\partial f_3}{\partial X_1} &= 0, \quad \frac{\partial f_3}{\partial X_2} = k_1 (1 + MR_d), \quad \frac{\partial f_3}{\partial X_3} = -k_2,\end{aligned}$$

which is not uniformly negative definite, as  $\frac{\partial f_2}{\partial X_1}$ ,  $\frac{\partial f_2}{\partial X_2}$  and  $\frac{\partial f_3}{\partial X_2}$ , are positive. To answer this application of the method from Andrieu and Tarbouriech (2019) is made to compute a new input signal that will make the system contracting. Reformulation of the system with (14) gives:

$$\dot{x} = Ax + M\phi(y, t), y = Lx, \quad (21)$$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{I_H} \\ 0 & k_1 & -k_2 \end{pmatrix}, y = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\phi(y, t) = \begin{bmatrix} \frac{2}{J_H} \left( -\frac{a_{cH}}{\rho_H} X_2^2 X_1^{\frac{1}{2}} - b_{cH} X_1 X_2 - c_{cH} \rho_H X_1^{\frac{3}{2}} \right) \\ \frac{1}{I_H} \left( \left( \frac{a_{pH}}{\rho_H} - k_{eqH} \right) X_2^2 + b_{pH} X_1^{\frac{1}{2}} X_2 + c_{pH} \rho_H X_1 \right) \end{bmatrix}. \quad (22)$$

#### 4.2 Design of the control feedback

In this part we use the same model presented in (22).

Analysis of the monotony of the non-linearity as in assumption 3 shows that  $\frac{\delta\phi}{\delta y}(y, t) + \frac{\delta\phi}{\delta y}^\top(y, t)$  is not Lipschitz, as it depends on the states  $X_1$  and  $X_2$ . Indeed, by taking note that:

$$\frac{\delta\phi}{\delta y}(y, t) = \begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial X_2} \\ \frac{\partial f_2}{\partial X_1} & \frac{\partial f_2}{\partial X_2} \end{pmatrix}, \quad (23)$$

we can verify that condition (16) has to be used in this case, as we can not find a suitable  $\Gamma$  mentioned in (12). However, as this equation presents negative eigenvalues, to verify condition (16), it is necessary to add a linear term in the expression of  $\phi(x, t)$ , under the form  $(l_1 X_1 \ 0)^\top$ . The term  $l_1$  is computed from the smallest negative value of  $\frac{\delta\phi}{\delta y}(y, t) + \frac{\delta\phi}{\delta y}^\top(y, t)$ . This term is then subtracted from the matrix  $A$  so that we keep the same equation as in (21). The constraints described in (18) impose some terms for the matrices we search for. With the expressions of the known matrices expressed earlier, we have:

$$WL^\top = -(M + BN),$$

$$W \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} = - \begin{pmatrix} n_1 + 1 & n_2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (24)$$

This constraint fixes six of the terms in  $W$ , which leads to the constraint  $(AW + BZ)^\top + (AW + BZ) \leq 0$  not to be verified. To address this issue, we chose to change the matrix  $M$  into:

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}. \quad (25)$$

Note that the appropriate sign changes need to be made in the  $\phi(y, t)$  function. The new representation of the system is then the following:

$$\dot{X} = \left( A - \begin{pmatrix} l_1 \\ 0 \\ 0 \end{pmatrix} \right) X + M \left( \phi(LX, t) + \begin{pmatrix} l_1 \\ 0 \end{pmatrix} LX \right) + Bu. \quad (26)$$

With this formalism, we have the preliminaries to solve the LMI defined in (18). We solve the LMI with the use of YALMIP (Lofberg, 2004), which gives us the three matrices that enable us to build the feedback control for our system:  $W, N, Z$ . The control law is then defined by:

$$u = KX + N\phi(LX, t), K = ZW^{-1}. \quad (27)$$

In the sense of (Andrieu and Tarbouriech, 2019), our system is now contracting. To regulate the system around a trajectory  $v(t)$ , we add the tracking term into the equation (26).

This leads to the final equation being:

$$\begin{aligned}\dot{X} &= \left( A - \begin{pmatrix} l_1 \\ 0 \\ 0 \end{pmatrix} \right) X + M \left( \phi(Lx, t) + \right. \\ &\quad \left. \begin{pmatrix} l_1 \\ 0 \end{pmatrix} LX \right) + B \left( K(X - v) + N\phi(L(X - v), t) \right).\end{aligned}\quad (28)$$

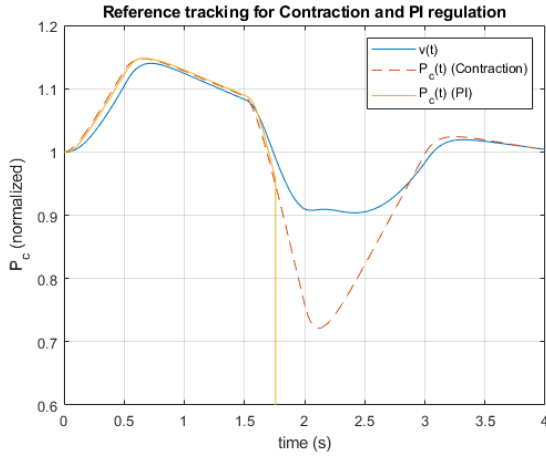


Fig. 2. Reference tracking

#### 4.3 Contracting behavior of the system

To illustrate the contracting behavior of the system, we will provide a few examples of reference tracking and rejection of initial conditions.

*Reference tracking* On figure 2, we illustrate an example of reference tracking. The values have been normalized, and the signal  $v(t)$  has been scaled with a gain as well as the signal  $P_c$  obtained with the PI regulation. An interesting illustration of the contracting behavior is the gap that appears when the control descends under the value  $v(t) = 0.9$ . Indeed, the system by construction can diverge under a certain value of the states. The contracting behavior prevents the system from diverging as mentioned in theorem 2. To illustrate this, we set up a PI regulation, that diverges when it reaches this critical value.

*Initial conditions* Figure 3 illustrates the behavior of several trajectories submitted to different initial conditions, with the same reference  $u(t)$ . For a contracting behavior we should observe the flow of the trajectories converging towards  $u(t)$ , each one progressively "forgetting" its initial state. For this example, we use four different initial states of  $P_c$ :  $P_c = [1, 0.9, 1.1, 1.2]$ . We notice that the convergence behavior is indeed respected, as all trajectories merge into a single one that follows the reference  $u(t)$ . This merge of the trajectories is the single-stream behavior that inspired the contraction theory.

*Additional comments* The regulation performances give satisfying results for perturbations. Simulations invoking time-delays give satisfying results towards stability. However the contracting control law implies violent variations of variables, exceeding most physical systems capacities. Additional saturations have been used to address this issue.

#### 4.4 Global system

The final aspect of this contraction study, is to be able to make both sides of the system contracting.

At this stage, we have the first subsystem  $H$ , that is contracting. The second subsystem, composed of two equations and two states, will be analysed with contraction theory using the same method, this time we will consider

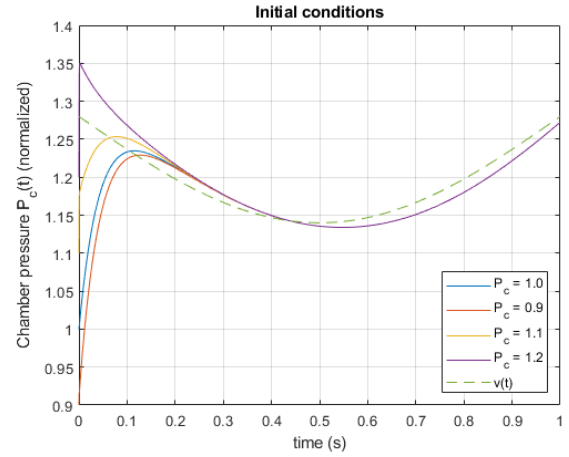


Fig. 3. Convergence of trajectories with different initial conditions

that  $P_c(t)$  is constant. Using contracting systems properties, we can conclude that the system presented in (8), with both controls computed as stated, is also contracting. Indeed, the contraction of a feedback combination of two contracting systems is proved in Jouffroy and Slotine (2004). Another way to visualize this conclusion is to note that the two subsystems are particular solutions of system's equation: in the  $H$  subsystem,  $\dot{X} = f(X, t)$  with  $X_2 = MR_d X_4$  and in the subsystem  $O$   $\dot{X} = f(X, t)$  where  $X_5 = P_{c0}$  with  $P_{c0}$  the initial value of  $P_c$ . Having proved that with our two inputs, the LPRE system is contracting, we have proven stability of the system, towards a reference trajectory.

To regulate both the main chamber pressure and the mixture ratio, two inputs are used. Using the notations defined in 4, the input for the chamber pressure is  $v_H(t) = P_{cd}(t) - P_c(t)$ , and the input for the mixture ratio is  $v_O(t) = MR_d(t) - MR(t)$ .

The system presents a static error during the tracking process, which can prove to be problematic in the case of the mixture ratio tracking. However, the addition of an integrator can be the source of a loss of the contracting behavior. It thus requires careful tuning. First it must be noticed that this effect should be increased on the subsystem  $O$ , as a faster integration on the  $MR(t)$  regulation leads to less perturbations, while maintaining a more efficient combustion. This can be noticed on figure 5, where oscillations appear when the oxidizer mass flow evolves. This retroacting behavior of the system is the main source of instability. The integrator gain on the oxidizer has been selected to be 10 times greater than the one for subsystem  $H$ .

On figure 4, the behavior of the system is illustrated, following a reference and changing functioning points. On figure 6, we see the evolution of the mixture ratio for the same reference. We can observe that the mixture ratio follows the behavior that we wanted, as it deviates of 10% from the desired value  $MR_{stat}$ , and converges to its optimal value.

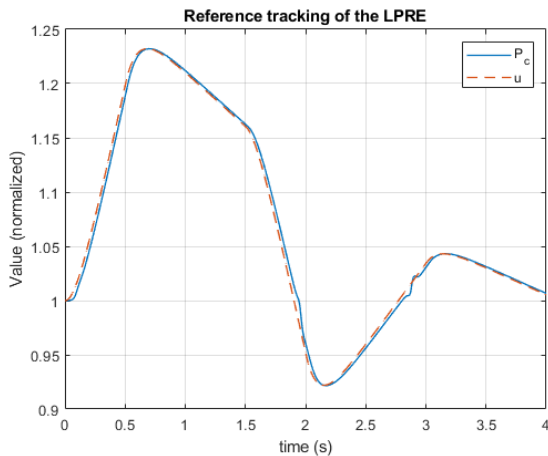


Fig. 4. Evolution of the chamber pressure

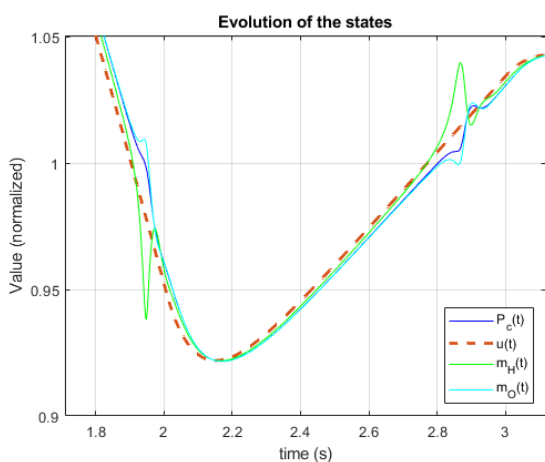


Fig. 5. Zoom on the evolution of the states

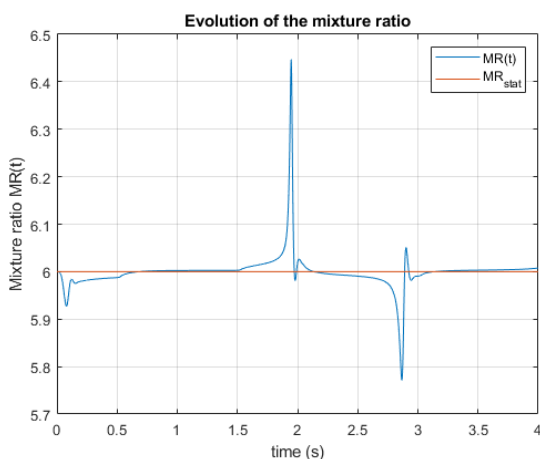


Fig. 6. Evolution of the mixture ratio

## 5. CONCLUSION AND PERSPECTIVES

In this paper we used a transformation of LPRE dynamics to determine a feedback control that makes the system contracting. The solution is obtained via solving LMIs. By using the properties of contracting behavior, we simplified the problem of a two-input two-output system into two separate single-input single-output problems, which allows

us to simplify the LMIs that are being solved. The results obtained on simulation show promising performances. However, such regulation relies on the assumption that all states are measured. In reality, LPRE being complex thermodynamics systems, only the value of the rotational speeds of the pumps and the chamber pressure are available and observers must be designed to reconstruct the remaining states. Future work implies the determination of such observers to reconstruct the states to be coupled with the contracting control law.

## REFERENCES

- Andrieu, V. and Tarbouriech, S. (2019). Lmi conditions for contraction and synchronization. *IFAC-PapersOnLine*, 52(16), 616–621.
- Forni, F. and Sepulchre, R. (2013). A differential lyapunov framework for contraction analysis. *IEEE Transactions on Automatic Control*, 59(3), 614–628.
- Galeotta, M., Ventimiglia, F., Usseglio, G., and Vermes, R. (2019). Carmen, the liquid propulsion rocket engine simulation platform, development status and perspectives. *CNES Direction des Lanceurs*.
- Ijspeert, A.J., Nakanishi, J., Hoffmann, H., Pastor, P., and Schaal, S. (2013). Dynamical movement primitives: learning attractor models for motor behaviors. *Neural computation*, 25(2), 328–373.
- Jouffroy, J. (2005). Some ancestors of contraction analysis. In *Proc. of the 44th IEEE Conf. on Decision and Control*, 5450–5455.
- Jouffroy, J. and Slotine, J.J. (2004). Methodological remarks on contraction theory. In *Proc. of the 43rd IEEE Conf. on Decision and Control*, volume 3, 2537–2543.
- Lofberg, J. (2004). Yalmip: A toolbox for modeling and optimization in matlab. In *IEEE ICRA*, 284–289.
- Lohmiller, W. (1999). *Contraction Analysis of Nonlinear Systems*. Ph.D. thesis, MIT.
- Lohmiller, W. and Slotine, J.J. (2000). Control system design for mechanical systems using contraction theory. *IEEE Trans. on Autom. Contr.*, 45(5), 984–989.
- Lohmiller, W. and Slotine, J.J.E. (1998). On contraction analysis for non-linear systems. *Automatica*, 34(6), 683–696.
- Perez Roca, S. (2020). *Model-based robust transient control of reusable liquid-propellant rocket engines*. Ph.D. thesis, Université Paris-Saclay.
- Pérez-Roca, S., Langlois, N., Marzat, J., Piet-Lahanier, H., Galeotta, M., Farago, F., and Le Gonidec, S. (2018). Derivation and analysis of a state-space model for transient control of liquid-propellant rocket engines. In *9th International Conference on Mechanical and Aerospace Engineering (ICMAE)*, 58–67.
- Pérez-Roca, S., Marzat, J., Piet-Lahanier, H., Langlois, N., Farago, F., Galeotta, M., and Le Gonidec, S. (2019). A survey of automatic control methods for liquid-propellant rocket engines. 107, 63–84.
- Tsukamoto, H., Chung, S.J., and Slotine, J.J.E. (2021). Contraction theory for nonlinear stability analysis and learning-based control: A tutorial overview. *Annual Reviews in Control*, 52, 135–169.
- Wang, W. and Slotine, J.J.E. (2005). On partial contraction analysis for coupled nonlinear oscillators. *Biological cybernetics*, 92(1), 38–53.